Backward stochastic partial differential equations driven by infinite dimensional martingales and applications

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Abstract

This paper studies first a result of existence and uniqueness of the solution to a backward stochastic differential equation driven by an infinite dimensional martingale. Then, we apply this result to find a unique solution to a backward stochastic partial differential equation in infinite dimensions. The filtration considered is an arbitrary right-continuous filtration, not necessarily the natural filtration of a Wiener process. This, in particular, allows us to study more applications, for example the maximum principle for a controlled stochastic evolution system. Some examples are discussed in the paper as well.

Keywords: Backward stochastic differential equation; backward stochastic partial differential equation; Martingale representation theorem; strong orthogonality and Galerkin's approximation method

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a right continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Fix $0 < T < \infty$ and denote by $\mathcal{P}$ the *predictable* σ - algebra generated by the algebra $\mathcal{R}$ of subsets $D \times (t, s] \subseteq \Omega \times (0, T]$, where $D \in \mathcal{F}_t$ and $0 \leq t \leq s \leq T$. Suppose that H is a separable Hilbert space and $L_1(H)$ is the space of nuclear operators on H. An H - valued process is said to be predictable if it is $\mathcal{P}/\mathcal{B}(H)$ measurable.

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For a continuous square integrable martingale M in H let $< M >$ be the predictable quadratic variation of M and $\tilde{\mathcal{Q}}_M$ be the predictable process taking values in the space $L_1(H)$, which is associated with the Doléans measure of $M \otimes M$. Proposition 2.1 below contain more details about these processes. Denote by $<< M >>_t = \int_0^t \tilde{\mathcal{Q}}_M(s)\, d < M >_s$. Assuming that there exists a predictable process $\mathcal{Q}$ such that $\mathcal{Q}(t, \omega)$ is a symmetric, positive definite nuclear operator on H and $<< M >>_t = \int_0^t \mathcal{Q}(s)\, ds$, we shall study first the existence and uniqueness of solutions to some *backward stochastic differential equations* (BSDEs for short), which are driven by martingales and which take the following form:

$$(BSDE) \begin{cases} -\,dY(t) = F(t, Y(t), Z(t)\mathcal{Q}^{1/2}(t))\; dt - Z(t)\; dM(t) - dN(t), \\ \;\; Y(T) = \xi, \end{cases}$$

where $0 \leq t \leq T$. Here ξ $(\xi(\omega) \in H)$ is the terminal value and the mapping F satisfies the following properties.
(i) $F : [0, T] \times \Omega \times H \times L_2(H) \to H$ is $\mathcal{P} \otimes \mathcal{B}(H) \otimes \mathcal{B}(L_2(H))/\mathcal{B}(H)$ - measurable.
(ii) $\mathbb{E}\, [\int_0^T |F(t, 0, 0)|_H^2\; dt\,] < \infty$.
(iii) $\exists\; const > 0$ such that $\forall\; y, y' \in H$ and $\forall\; z, z' \in L_2(H)$

$$|F(t, \omega, y, z) - F(t, \omega, y', z')|_H^2 \leq const\; (\; |y - y'|_H^2 + |z - z'|_{L_2(H)}^2\,),$$

uniformly in (t, ω). The space $L_2(H; K)$ (or shortly $L_2(H)$ when $K = H$) denotes the space of Hilbert-Schmidt operators from H to the Hilbert space K.

A solution of this BSDE is a triple (Y, Z, N) of predictable processes that satisfy the integral form of this BSDE for each t, in addition to some integrability conditions as in the Definition 3.1 below, where N is a square integrable martingale required to be very strongly orthogonal to M. The process Z is actually a predictable operator satisfying $Z(t, \omega)\tilde{\mathcal{Q}}_M^{1/2}(t, \omega) \in L_2(H)$, for each (t, ω).

We mention here that among those who studied BSDEs driven by martingales are [18], [16] in finite dimensions and [5] in infinite dimensions. In fact [18] and [5] consider the case when $< M >_t$ is absolutely continuous with respect to a continuous, adapted and increasing process c_t, $t \geq 0$, while in [16] a BMO-martingale is taken as a driving noise of the BSDE, where BMO stands for bounded mean oscillation. On the other hand, Pardoux in [30] studies a BSDE with a right continuous filtration which is generated by a d - dimensional

Brownian motion and and a Poisson random measure and allows the terminal condition T to be a possibly infinite stopping time.

In the second part of the present paper we shall consider the *backward stochastic partial differential equations (BSPDEs)* of the following type:

$$(BSPDE)\begin{cases} -\,dY(t) = (\,A(t)\,Y(t) + F(t,Y(t),Z(t)\mathcal{Q}^{1/2}(t))\,)\,dt \\ \qquad\qquad -Z(t)\,dM(t) - dN(t), \quad 0 \le t \le T, \\ \;Y(T) = \xi, \end{cases}$$

with $A(t,\omega)$ being a predictable linear operator on H that belongs to $L(V;V')$, where (V,H,V') is a rigged Hilbert space or the called Gelfand's triple. Now assuming that $A(t,\omega)$ is coercive for a.a. $(t,\omega) \in [0,T]\times\Omega$ and F satisfies a similar condition to the ones given earlier, we shall show that this BSPDE admits a unique solution (Y,Z,N) of predictable processes taking values in $V \times L_2(H) \times \mathcal{M}^{2,c}(H)$ and that Y is a continuous semimartingale. This space $\mathcal{M}^{2,c}(H)$ consists of square integrable continuous martingales which take values in H.

These results will be applied to the following situation when, for example, we are given an SPDE driven by a Brownian motion β in $\mathbb{R}$ (see the equation (4.30) of Section 4) and having the following data: (i) bounded real valued processes $a_{ij}(\omega,t,x)$, $(i,j=1,\ldots,d)$, defined on $\Omega\times[0,T]\times\mathbb{R}^d$, that are $\mathcal{P}$-predictable, measurable in the x-variable and satisfy a uniform parabolicity condition; (ii) measurable mappings $f_j\, : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $(j=1,\ldots,d)$, each of which satisfies a Lipschitz condition with respect to its third and fourth indices. With these information and the equation (4.30) we consider the operator $A(t,\omega)$ so that, for $\eta,\nu \in V \equiv \mathbb{H}^1(\mathbb{R}^d)$,

$$[A(t,\omega)\,\eta\,,\nu] := -\int_{\mathbb{R}^d}\Big(\sum_{i,j=1}^{d} a_{ij}(\omega,t,x)\,\frac{\partial}{\partial x_j}\,\eta(x)\,\frac{\partial}{\partial x_i}\,\nu(x)\Big)\,dx,$$

and the mapping

$$F(\omega,t,\zeta,\varsigma)(x) := \sum_{j=1}^{d} f_j\,(\omega,t,\zeta(x),\tilde{\varsigma}(x)\,),$$

where $\zeta \in H \equiv L^2(\mathbb{R}^d), \varsigma \in L_2(H), x \in \mathbb{R}^d$ and $\tilde{\varsigma} = \varsigma(h) \in H$ for some fixed $h \in H$ satisfying $|h|_H = 1$. The martingale $M(t)$ here is $h\,\beta(t)$.

BSPDEs have proved also to be very useful in stochastic control theory. Precisely they appear as the adjoint equations of the control problem of minimizing the cost functional

$$J(x,v(\cdot)) := \mathbb{E}\,[\, \int_0^T g(X^{v(\cdot)}(t), v(t)\,)\,dt + \phi(X^{v(\cdot)}(T))\,]$$

over the set of all admissible controls $v(\cdot)$. In [6] a maximum principle for the following system of stochastic evolution equation

$$(SEE) \begin{cases} dX^{v(\cdot)}(t) = (A(t)\,X^{v(\cdot)}(t) + f(X^{v(\cdot)}(t), v(t))\,)\,dt + \,G(X^{v(\cdot)}(t))\,dM(t), \\ X^{v(\cdot)}(0) = x \in H, \end{cases}$$

is derived by using its adjoint BSPDE to show that J attains its infimum at an optimal pair $(X^{v^*(\cdot)}, v^*(\cdot))$. Here f, G, g and ϕ are some given C_b^1 mappings, the last two functions take values in $\mathbb{R}$ and ϕ is convex.

It would be convenient also to know that the history of linear BSDEs goes back to Bismut [10]. It is shown there that linear BSDEs may arise from some stochastic control problem and can be regarded as the adjoint equations in such a problem. Later Peng in [32] studied the nonlinear BSDEs in order to study the stochastic maximum principle.

We recall here that in [35, P. 114, 116] the authors study finite dimensional controlled SDEs similar to this SEE when e.g. A is bounded. They claim that the condition
" $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration generated by a Wiener process $W(\cdot)$, argumented by all the $\mathbb{P}$- null sets in $\mathcal{F}$ "
cannot be omitted if one wants to involve adjoint equations of the concerned SDE. In fact one reason beyond this assumption is that because Pardoux and Peng in [29] originally considered such a condition in order to achieve a result of the existence of solutions to BSDEs by using the martingale representation theorem. See also [23] in this respect. Another reason is that the authors in [35, Chapter 5] wanted to give a relationship between the maximum principle and dynamic programming for their controlled system.

On the other hand, as we know that dynamic programming requires a Markov property to be satisfied by the solution, which does not hold in general when the noise is a martingale, it seems that the maximum principle remains the suitable tool to study such control problems. In particular our results here provide the required way to study the stochastic maximum principle for infinite dimensional controlled stochastic systems like the above (SEE).

Jin Ma et al. [23] studied sort of BSPDEs with respect to a finite dimensional Wiener process. We mention here also that Øksendal et al. [26] studied BSPDEs with a filtration generated by a Wiener process and a Poisson random measure. An interested reader can see also the work of Fuhrman et al. [19] and Pardoux and Răşcanu [31].

We shall deal here with an arbitrary filtration, and moreover, as in the case of Example 4.9 in Section 4 below, this filtration can be larger than the Wiener filtration. Similar cases are treated in details in [3]; cf. Remark 2.3 below. These applications together with the works in finite dimensions of [18], [11], [7] and [16] show that by considering a general filtration (i.e. not necessarily the Wiener filtration) one can deal with more equations than those focused on just the Wiener filtration as for instance in [1], [4] or [35].

It is well known that in many situations strong solutions of SEEs do not exist, and so one has to look for solutions of a weaker sense. In the literature, e.g. [14], [15], [12] and [13] and the references therein, there is an extensive work on the semigroup approach to find what are called mild solutions. For the backward case (i.e. for BSEEs or BSPDEs) one can see [21] and [4]. However, when M is a cylindrical Wiener process on H and the filtration considered is its canonical filtration, by adding more regularity conditions on the BSPDE mappings ξ and F and A (when A is time independent), we can see from [2] that the solution (Y, Z) can also be strong.

Now since our operator $A(t, \omega)$ in the above (BSPDE) depends on t and ω, we expect to have a random propagator $U(t, s)$ associated with A as mentioned in [13, P. 156], that is

$$\frac{d}{dt}\, U(t, s) = A(t)U(t, s), \;\; U(s, s) = id_H, \;\; 0 \leq s \leq t \leq T.$$

This propagator needs not be $\mathcal{F}_*$-adapted, so it becomes difficult if not impossible to use the semigroup approach for these BSPDEs as in [21] or [4]. But in fact if $A(t)$ is not random we can still work with the semigroup approach. For this purpose we refer the reader to [4]. It is quiet useful to know that for the case of the above (BSPDE) where it is given a general filtration we do not know if one is able to use the semigroup approach despite our ability to treat a special case as in Example 4.3 of Section 4 below. This difficulty includes also the autonomous case, i.e. if the operator $A(t, \omega) = A$ is independent of t and ω.

The approach we shall therefore be following for studying the above (BSPDE) is the use of Galerkin's approximation method.

The rest of the paper is organized as follows. Section 2 consists of two subsections and is devoted to giving some necessary information about martingales taking their values in Hilbert spaces and stochastic integration with respect to such martingales. In Section 3 we introduce the spaces of solutions of BSDEs and BSPDEs and establish the proof of the existence and uniqueness of the solutions to the equation (BSDE). Finally Section 4 is concerned with studying the above (BSPDE) and giving some applications.

2 Preliminaries

2.1 Square integrable martingales

Denote by $\mathcal{M}^2_{[0,T]}(H)$ the vector space of cadlag square integrable martingales $\{M(t), 0 \leq t \leq T\}$, taking values in H, that is $\mathbb{E}\,[\,|M(t)|^2_H] < \infty$ for each $t \in [0,T]$. It is a Hilbert space with respect to the inner product $(M,N) \mapsto \mathbb{E}\,[\,\big\langle M(T)\,, N(T)\big\rangle_H\,]$, if we identify $\mathbb{P}$-equivalence classes. Two elements M and N of $\mathcal{M}^2_{[0,T]}(H)$ are said to be *very strongly orthogonal (VSO)* if $\mathbb{E}\,[M(u)\otimes N(u)] = \mathbb{E}\,[M(0)\otimes N(0)]$, for all $[0,T]$-valued stopping times u. For example, if moreover $N(0) = 0$, then M and N are VSO if and only if $\mathbb{E}\,[M(u)\otimes N(u)] = 0$, for all such stopping times.

Let us now recall the definition of *Doléans measure* associated with $|M|^2_H$. Define $d_{|M|^2_H}(A) := \mathbb{E}\ [1_D(|M(s)|^2_H - |M(t)|^2_H)]$, where $A = D \times (t,s] \in \mathcal{R}$. This function can be extended uniquely to a measure α_M on $\mathcal{P}$. This measure is called the Doléans measure associated with $|M|^2_H$ (see [25] or [24]). Analogously, we associate on $\mathcal{P}$ the $H\hat{\otimes}_1 H$-valued σ-additive Doléans measure μ_M of $M \otimes M$. Here the space $H\hat{\otimes}_1 H$ is the completed nuclear tensor product, that is the completion of $H \otimes H$ for the nuclear norm. Recall that the linear form *trace,* denoted here by tr, is defined as the unique continuous extension to $H\hat{\otimes}_1 H$ of the mapping $x \otimes y \mapsto \big\langle x\,, y\big\rangle_H$.

For a square integrable martingale M we write $< M, M >$ (or shortly $< M >$) for the increasing Meyer process associated with the Doléans measure of the submartingale $|M|^2_H$, that is the unique predictable cadlag increasing process such that $|M|^2_H - < M >$ is a martingale. This process $< M >$ is related to the tensor quadratic variation $<< M >>$ through the following proposition.

Proposition 2.1 ([25]) *(i) There is one predictable $H\hat{\otimes}_1 H$ - valued process $\mathcal{Q}_M$, defined up to α_M - equivalence such that for every $G \in \mathcal{P}$*

$$\mu_M(G) = \int_G \mathcal{Q}_M \, d\alpha_M.$$

Moreover, $\mathcal{Q}_M$ takes its value in the set of positive symmetric elements of $H\hat{\otimes}_1 H$ and

$$\mathrm{tr}\ \mathcal{Q}_M(\omega, t) = 1, \ \ \alpha_M \ a.e.$$

(ii) The $H\hat{\otimes}_1 H$ - valued process

$$<< M >>_t := \int_{(0,t]} \mathcal{Q}_M \, d < M >$$

has finite variation, is predictable, admits μ_M as its Doléans measure, and is such that $M \otimes M - << M >>$ is a martingale.

In a similar way we can define $<< M, N >>$. It is obvious that M and N are VSO if and only if $<< M, N >> = 0$.

Let $\tilde{\mathcal{Q}}_M$ be the identification of $\mathcal{Q}_M$ in $L_1(H)$. For example, if $\mathcal{Q}$ is a symmetric nonnegative nuclear operator on H and $\{W(t), 0 \leq t \leq T\}$ is a $\mathcal{Q}$- Wiener process in H (see [14]), then we have $< W >_t = t \ \mathrm{tr}(\mathcal{Q})$ for each t, and the Doléans measure α_W associated with $|W|_H^2$ is the product measure $(l \otimes \mathbb{P}) \ \mathrm{tr}(\mathcal{Q})$, where l is the Lebesgue measure on $([0, T], \mathcal{B}([0, T]))$.

2.2 Stochastic integration

In this section we introduce the definition of stochastic integration with respect to martingales in $\mathcal{M}^2_{[0,T]}(H)$ by following [25] and [24]. Let $L^*(H; \mathcal{P}, M)$ denote the space of processes Φ, the values of which are (possibly non-continuous) linear operators from H into itself with the following properties:
(i) the domain of $\Phi(\omega, t)$ contains $\tilde{\mathcal{Q}}_M^{1/2}(\omega, t)(H)$ for every (ω, t),
(ii) for every $h \in H$, the H- valued process $\Phi \circ \tilde{\mathcal{Q}}_M^{1/2}(h)$ is predictable,
(iii) for every $(\omega, t) \in \Omega \times (0, T]$, $\Phi(\omega, t) \circ \tilde{\mathcal{Q}}_M^{1/2}(\omega, t)$ is a Hilbert-Schmidt operator and

$$\int_{\Omega \times (0,T]} |\Phi \circ \tilde{\mathcal{Q}}_M^{1/2}|^2_{L_2(H)} \, d\alpha_M < \infty.$$

This is sometimes written as

$$\mathbb{E}\,[\int_0^T |\Phi \circ \tilde{\mathcal{Q}}_M^{1/2}|^2_{L_2(H)}\; d < M >_t] < \infty,$$

as we shall do here.

This space is complete with respect to the scalar product

$$(X, Y) \mapsto \int_{\Omega\times(0,T]} \mathrm{tr}\;(X \circ \tilde{\mathcal{Q}}_M \circ Y^*)\, d\alpha_M$$

as seen in the proof of [24, Proposition 22.2, P.142].

Next denote by $\mathcal{E}(L(H))$ the space of $\mathcal{R}$- simple processes and $\Lambda^2(H;\mathcal{P}, M)$ the closure of $\mathcal{E}(L(H))$ in $L^*(H;\mathcal{P}, M)$. Then $\Lambda^2(H;\mathcal{P}, M)$ becomes a Hilbert subspace of $L^*(H;\mathcal{P}, M)$. Now assume that

$$\Phi = \sum_{i=1}^{n} 1_{D_i\times(r_i,s_i]}\; B_i\,, \quad B_i \in L(H), \quad D_i \in \mathcal{F}_{r_i}.$$

Define

$$\int_{(0,t]} \Phi\; dM = \sum_{i=1}^{n} 1_{D_i}\; (\,B_i(M(s_i \wedge t)) - B_i(M(r_i \wedge t))\,), \quad t \in [0, T].$$

This gives an isometric linear mapping from $\mathcal{E}(L(H))$ into $\mathcal{M}^2_{[0,T]}(H)$, $\Phi \mapsto \int \Phi\; dM$. Extend this mapping to $\Lambda^2(H;\mathcal{P}, M)$. The image $\int \Phi\; dM$ of Φ in $\mathcal{M}^2_{[0,T]}(H)$ by this mapping is called the *stochastic integral of* Φ *with respect to* M. For such $\Phi \in \Lambda^2(H;\mathcal{P}, M)$ the stochastic integral $N = \int \Phi\; dM$ satisfies, for every $t \geq 0$, the following properties:

$$<< N >>_t = \int_{(0,t]} (\Phi \circ \tilde{\mathcal{Q}}_M \circ \Phi^*)\; d << M >>,$$

$$< N >_t = \int_{(0,t]} \mathrm{tr}\;(\Phi \circ \tilde{\mathcal{Q}}_M \circ \Phi^*)\; d < M > .$$

The following martingale representation property will be applied in the next section.

Theorem 2.2 ([27]) *Let $M \in \mathcal{M}^2_{[0,T]}(H)$ and $\mathcal{H}_1 := \{\int X \; dM \;\; : \;\; X \in \Lambda^2(H;\mathcal{P},M)\} \subset \mathcal{M}^2_{[0,T]}(H)$. Let $\mathcal{H}_2$ be the orthogonal complement of $\mathcal{H}_1$ in $\mathcal{M}^2_{[0,T]}(H)$. Then every element of $\mathcal{H}_2$ is VSO to every element of $\mathcal{H}_1$. In particular, every $L \in \mathcal{M}^2_{[0,T]}(H)$ can be written uniquely as*

$$L = \int X \; dM + N, \; X \in \Lambda^2(H;\mathcal{P},M), \; N \in \mathcal{H}_2 \, . \tag{2.1}$$

Note that since $M \in \mathcal{H}_1$, the martingales M and N are VSO Note also that if $M, L \in \mathcal{M}^{2,c}_{[0,T]}(H)$, the martingale N has a continuous modification. In such a case, we shall consider this continuous modification.

Remark 2.3 *As a result of Theorem 2.2 it is obvious that a similar representation property holds when M is an infinite dimensional genuine Wiener process W with covariance operator $\mathcal{Q}$. However, if we only have a cylindrical Wiener process W on the Hilbert space H (i.e. when the covariance operator is the identity id_H), we will not be able to apply this theorem directly. So it becomes worthy to record that such a representation also holds in this latter case. Actually the uniqueness is easy. But for existence one may consider the space $\mathcal{H} := \{\int X \; dW \;\; : \;\; X \in \Lambda^2(H;\mathcal{P},W)\}$, where $\Lambda^2(H;\mathcal{P},W)$ agrees with $L^2_{\mathcal{F}}(0,T;L_2(H))$, defined in the next section, and then shows that it a closed subspace of $\mathcal{M}^2_{[0,T]}(H)$. This gives the required decomposition $L = \int X \; dW + N$, because N is VSO to W in the sense that $\mathbb{E}\,[\,W^h(u)\cdot N^g(u)\,] = 0$, for every h and $g \in H$, where $W^h := \big\langle W\,, h \big\rangle_H$ and $N^g := \big\langle N\,, g \big\rangle_H$, which implies that N is VSO to every element of $\mathcal{H}$. This fact can be shown in a similar way to the proof Theorem 3.1 in [27] or by mimicking the finite dimensional case in Lemma 4.2 of [33, Chapter 4] as follows.*

$$\begin{aligned}
\mathbb{E}\,[\,W^h(u)\cdot N^g(u)\,] &= \mathbb{E}\,[\,W^h(u)\cdot \mathbb{E}\,(\,N^g(T)|\mathcal{F}_u\,)\,] \\
&= \mathbb{E}\,[\,W^h(u)\cdot N^g(T)\,] \\
&= \mathbb{E}\,\big[\,\big\langle W^h(u)\,g\,, N(T)\big\rangle_H\,\big] \\
&= \mathbb{E}\,\big[\,\big\langle (W^h(u\wedge\cdot)\,g)(T)\,, N(T)\big\rangle_H\,\big] \\
&= 0,
\end{aligned}$$

since the stopped martingale $W^h(u\wedge\cdot)\,g = \int_0^\cdot 1_{]0,u]}(s)\,(\widetilde{g\otimes h})\,dW(s) \in \mathcal{H}$, where $\widetilde{g\otimes h}$ is the mapping $H \ni h' \mapsto g\cdot\big\langle h\,, h'\big\rangle_H$, which lies in $L_1(H) \subset L_2(H)$.

This corrects the proof of [3, Theorem 3.1] and so our results here are also valid when the martingale M is replaced by a cylindrical Wiener process.

3 Backward stochastic differential equations

Consider the following spaces.
$L^2_{\mathcal{F}}(0,T;K) := \{\phi : [0,T]\times\Omega \to K, \text{predictable and} \, \mathbb{E}\,[\int_0^T |\phi(t)|^2_K dt\,] < \infty \,\}$, where K is a separable Hilbert space.
$\mathcal{S}^2(H) := \{\phi : [0,T]\times\Omega \to H$ cadlag, adapted and $\mathbb{E}\,[\sup_{0\leq t\leq T} |\phi(t)|^2_H\,] < \infty\,\}$.
$\mathcal{B}^2(H) := L^2_{\mathcal{F}}(0,T;H)\times\Lambda^2(H;\mathcal{P},M)$.

Then $\mathcal{S}^2(H)$ is a separable Banach space equipped with the norm:

$$||\phi||^2_{\mathcal{S}^2(H)} \;=\; \mathbb{E}\Big[\sup_{0\leq t\leq T} |\phi(t)|^2_H\Big].$$

Also $\mathcal{B}^2(H)$ is a separable Hilbert space with the norm:

$$\begin{aligned}||(\phi_1,\phi_2)||^2_{\mathcal{B}^2(H)} \;&=\; \mathbb{E}\Big[\int_0^T |\phi_1(t)|^2_H\, dt\Big] \\ &\quad + \mathbb{E}\Big[\int_0^T |\phi_2(t)\tilde{\mathcal{Q}}^{1/2}_M(t)|^2_{L_2(H)}\, d<M>_t\Big].\end{aligned}$$

Let $M\in\mathcal{M}^{2,c}_{[0,T]}(H)$ be such that $M(0)=0$ and consider the following BSDE:

$$\begin{cases} -\,dY(t) = F(t,Y(t),Z(t)\mathcal{Q}^{1/2}(t))\,)\,dt - Z(t)\,dM(t) - dN(t),\; t\in[0,T], \\ \quad Y(T)=\xi. \end{cases} \tag{3.1}$$

The process $\mathcal{Q}$ is shown in the assumption (A3) below. We shall impose the following conditions.

- (A1) $F : [0,T]\times\Omega\times H\times L_2(H)\to H$ is a mapping such that the following properties are verified.
  - (i) F is $\mathcal{P}\otimes\mathcal{B}(H)\otimes\mathcal{B}(L_2(H))/\mathcal{B}(H)$ - measurable.
  - (ii) $\mathbb{E}\,[\int_0^T |F(t,0,0)|^2_H\, dt\,] < \infty$, where $F(t,0,0)=F(t,\omega,0,0)$.
  - (iii) $\exists\, k_1>0$ such that $\forall\, y,y'\in H$ and $\forall\, z,z'\in L_2(H)$

    $$|F(t,\omega,y,z)-F(t,\omega,y',z')|^2_H \leq k_1\,(\,|y-y'|^2_H + |z-z'|^2_{L_2(H)}\,),$$

    uniformly in (t,ω).

- (A2) $\xi\in L^2(\Omega,\mathcal{F}_T,\mathbb{P};H)$.

- (A3) There exists a predictable process $\mathcal{Q}$ such that $\mathcal{Q}(t,\omega)$ is a symmetric, positive definite nuclear operator on H and $<< M >>_t = \int_0^t \mathcal{Q}(s)\, ds$.
- (A4) Every square integrable H-valued martingale with respect $\{\mathcal{F}_t\,,\ 0 \leq t \leq T\}$ has a continuous version.

The process $\mathcal{Q}(t)$ in (A3) is called *the local characteristic operator* or *the local covariation operator* of the martingale $M(t)$.

We note that if (A3) holds then $\tilde{\mathcal{Q}}_M(t) = \frac{\mathcal{Q}(t)}{q(t)}$ and $< M >_t = \int_0^t q(s)ds$, where $q(t) := \mathrm{tr}\,(\mathcal{Q}(t))$. Thus, in particular, if $g \in \Lambda^2(H;\mathcal{P},M)$,

$$\begin{aligned}\mathbb{E}\Big[\,|\int_0^T g(s)dM(s)|_H^2\Big] &= \mathbb{E}\Big[\int_0^T |g(s)\frac{\mathcal{Q}^{1/2}(s)}{\sqrt{q(s)}}|^2_{L_2(H)}\, q(s)\, ds\Big] \\ &= \mathbb{E}\Big[\int_0^T |g(s)\mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds\Big].\end{aligned}$$

Therefore the norm of $\mathcal{B}^2(H)$ becomes

$$||(\phi_1,\phi_2)||^2_{\mathcal{B}^2(H)} = \mathbb{E}\Big[\int_0^T |\phi_1(t)|_H^2\, dt\Big] + \mathbb{E}\Big[\int_0^T |\phi_2(t)\mathcal{Q}^{1/2}(t)|^2_{L_2(H)}\, dt\Big].$$

An example of a filtration satisfying (A4) is that which is generated by two independent cylindrical Wiener processes on H. This (A4) is a technical assumption considered in the book of Durrett [17, P. 92] for dealing with Girsanov's formula and is needed here to reduce the difficulties in the computations below when using Itô's formula and especially in the next section; see also Pardoux et al. [7]. Also one has to be aware of the sentence proceeding the equation (4.16) below.

Definition 3.1 *A solution of (3.1) is a triple* $(Y,Z,N) \in \mathcal{B}^2(H) \times \mathcal{M}^2_{[0,T]}(H)$ *such that for all* $t \in [0,T]$ *the following equality holds a.s.*

$$\begin{aligned}Y(t) &= \xi + \int_t^T F(s,Y(s),Z(s)\mathcal{Q}^{1/2}(s))\, ds \\ &\quad - \int_t^T Z(s)\, dM(s) - \int_t^T dN(s), \qquad (3.2)\end{aligned}$$

with $N(0) = 0$ *and* N *is VSO to* M.

The existence and uniqueness of such solutions are achieved through the following theorem.

Theorem 3.2 *Assume that Assumptions (A1)–(A4) hold. Then there exists a unique solution* $(Y, Z, N) \in \mathcal{B}^2(H) \times \mathcal{M}^2_{[0,T]}(H)$ *of (3.1). And moreover* $Y \in \mathcal{S}^2(H)$.

Proof. We shall use the fixed point theorem for the contraction principle. But before doing this let us indicate that given $(y, z) \in \mathcal{B}^2(H)$ implies that the following local martingale $K(t) := \mathbb{E}\left[\xi + \int_0^T F(s, y(s), z(s))\, ds \mid \mathcal{F}_t\right]$, $0 \leq t \leq T$, belongs to $\mathcal{M}^{2,c}_{[0,T]}(H)$. Indeed, (A1)(iii, ii) implies that

$$\begin{aligned} \mathbb{E}[|K(T)|^2] &\leq 2\,\mathbb{E}\,[\,|\xi|^2\,] + 2\,T\,\mathbb{E}\,[\,\int_0^T |F(s, y(s), z(s)\mathcal{Q}^{1/2}(s))|^2 ds\,] \\ &< \infty. \end{aligned} \tag{3.3}$$

Now define the mapping U on $\mathcal{B}^2(H)$ by $U(y, z) = (Y, Z)$, where

$$Y(t) := \mathbb{E}\left[\xi + \int_t^T F(s, y(s), z(s)\mathcal{Q}^{1/2}(s))\, ds \mid \mathcal{F}_t\right], \tag{3.4}$$

$0 \leq t \leq T$, and Z is given by using the representation of the martingale K in Theorem 2.2 as:

$$K(t) = Y(0) + \int_0^t Z(s)\, dM(s) + N(t), \tag{3.5}$$

$0 \leq t \leq T$, such that N is an H-valued cadlag local martingale very strongly orthogonal to M. From the definition of Y in (3.4) we see that Y is predictable.

Next we apply Doob's inequality, (A1)(iii, ii) and (A2) to find that

$$\begin{aligned} ||Y||^2_{\mathcal{S}^2(H)} &\leq \mathbb{E}\,[\,\sup_{t\in[0,T]} \big(\,\big|\,\mathbb{E}\,[\xi + \int_t^T F(s, y(s), z(s)\mathcal{Q}^{1/2}(s))\, ds \mid \mathcal{F}_t\,]\,\big|^2\,\big)\,] \\ &\leq 2\,\mathbb{E}\,[\,\sup_{t\in[0,T]} (\,\mathbb{E}\,[\,|\xi| \mid \mathcal{F}_t\,]\,)^2\,] \\ &\quad + 2\,\mathbb{E}\,[\,\sup_{t\in[0,T]} \big(\mathbb{E}\,[\int_0^T |F(s, y(s), z(s)\mathcal{Q}^{1/2}(s))|\, ds|\, \mathcal{F}_t\,]\big)^2\,] \\ &\leq 8\,\mathbb{E}\,[|\xi|^2] + 8\,T\,\mathbb{E}\,[\,\int_0^T |F(s, y(s), z(s)\mathcal{Q}^{1/2}(s))|^2\, ds\,] \\ &< \infty. \end{aligned} \tag{3.6}$$

In particular, $Y \in \mathcal{S}^2(H)$ and so

$$\begin{aligned}\mathbb{E}\Big[\int_0^T |Y(t)|^2\, dt\Big] &\leq T\,||Y||^2_{\mathcal{S}^2(H)}\\ &< \infty.\end{aligned} \tag{3.7}$$

On the other hand, we observe that

$$\begin{aligned}&\mathbb{E}\Big[\int_0^T |Z(t)\,\mathcal{Q}^{1/2}(t)|^2\, dt\Big] + \mathbb{E}\,[\,|N(T)|^2\,]\\ &= \mathbb{E}\Big[\int_0^T |Z(t)\,\tilde{\mathcal{Q}}_M^{1/2}(t)|^2\, d < M >_t\Big] + \mathbb{E}\,[\,|N(T)|^2\,]\\ &= \mathbb{E}\Big[\,|\int_0^T Z(t)\, dM(t) + N(T)|^2\Big]\\ &= \mathbb{E}\,[\,|K(T) - K(0)\,|^2\,]\\ &\leq 2\Big(\mathbb{E}\,[\,|K(T)|^2\,] + \mathbb{E}\,[\,|K(0)|^2\,]\Big)\\ &\leq 4\,\mathbb{E}\,[\,|K(T)|^2\,]\\ &< \infty\,,\end{aligned} \tag{3.8}$$

by using Jensen's inequality and (3.3). We have thus concluded that $Y \in L^2_{\mathcal{F}}(0,T;H)$, $Z \in \Lambda^2(H;\mathcal{P},M)$ and $N \in \mathcal{M}^2_{[0,T]}(H)$. So, in particular, U maps $\mathcal{B}^2(H)$ into itself.

Note that $(Y,Z,N) \in \mathcal{B}^2(H) \times \mathcal{M}^2_{[0,T]}(H)$ is a solution of the BSDE (3.1) if and only if its component (Y,Z) is a fixed point of U.

Let us now take two elements $(y_i, z_i) \in \mathcal{B}^2(H)$, $i = 1,2$, with the corresponding image (Y_i, Z_i, N_i) in $\mathcal{B}^2(H) \times \mathcal{M}^2_{[0,T]}(H)$, $i = 1,2$, by using the mapping U as done earlier. Denote by $(\delta y\,,\ \delta z)$, $(\delta Y\,,\ \delta Z\,,\ \delta N)$ the processes $(y_1 - y_2\,, z_1 - z_2)$, $(Y_1 - Y_2\,, Z_1 - Z_2\,, N_1 - N_2)$, respectively. Let γ be a real number. The corresponding equations for these processes and Itô's formula imply that

$$\begin{aligned}&\mathbb{E}\,[\,e^{\gamma\, t}\,|\,\delta Y(t)|^2\,] + \gamma\,\mathbb{E}\,[\int_t^T e^{\gamma\, s}\,|\,\delta Y(s)|^2\, ds\,]\\ &+ \mathbb{E}\,[\int_t^T e^{\gamma\, s}\,|\,\delta Z(s)\mathcal{Q}^{1/2}(s)|^2\, ds\,] + \mathbb{E}\,[\int_t^T e^{\gamma\, s}\, d < \delta N >_s]\\ &= 2\,\mathbb{E}\,[\int_t^T e^{\gamma\, s}\big\langle \delta Y(s), F(s, y_1(s), z_1(s)\mathcal{Q}^{1/2}(s)) - F(s, y_2(s), z_2(s)\mathcal{Q}^{1/2}(s))\big\rangle ds].\end{aligned}$$

Thus, by applying (A1)(iii) and then choosing $\gamma = 2k_1 + 1$, we find that

$$\begin{aligned} & \mathbb{E}\,[\int_0^T e^{\gamma\, s}\,|\,\delta Y(s)|^2\,ds\,] \\ & + \mathbb{E}\,[\int_0^T e^{\gamma\, s}\,|\,\delta Z(s)\mathcal{Q}^{1/2}(s)|^2\,ds\,] + \,\mathbb{E}\,[\int_0^T e^{\gamma\, s}\,d < \delta N >_s] \\ & \leq \frac{1}{2}\,\Big(\,\mathbb{E}\,[\int_0^T e^{\gamma\, s}\,|\,\delta y(s)|^2\,ds\,] + \,\mathbb{E}\,[\int_0^T e^{\gamma\, s}\,|\,\delta z(s)\mathcal{Q}^{1/2}(s)|^2\,ds\,]\,\Big). \end{aligned}$$

In particular $||(\delta Y, \delta Z)||^2_{\mathcal{B}^2(H)} \leq \frac{1}{2}\; ||(\delta y, \delta z)||^2_{\mathcal{B}^2(H)}$, which shows that U is a strict contraction on $\mathcal{B}^2(H)$, equipped with the norm

$$\begin{aligned} ||(Y,Z)||_{\mathcal{B}^2(H)} \;\; := \;\; & \Big(\mathbb{E}\,\Big[\,\int_0^T e^{\gamma_0\, s}\,|Y(s)|^2\,ds \\ & + \int_0^T e^{\gamma_0\, s}\,|Z(s)\mathcal{Q}^{1/2}(s)|^2\,ds\Big]\Big)^{1/2}, \end{aligned}$$

where $\gamma_0 = 2k_1 + 1$. Hence it has a unique fixed point. ■

4 Backward stochastic partial differential equations

In the preceding section we proved the existence of solutions to the BSDEs. We shall apply these results in this section to study the following type of BSPDE:

$$\begin{cases} -\,dY(t) = (\,A(t)\,Y(t) + F(t, Y(t), Z(t)\mathcal{Q}^{1/2}(t))\,)\,dt \\ \qquad\qquad -Z(t)\;dM(t) - dN(t), \quad 0 \leq t \leq T, \\ \;\; Y(T) = \xi. \end{cases} \tag{4.1}$$

The operator $A(t,\omega)$ in this equation is a predictable unbounded linear operator on the Hilbert space H.

Our aim is to find a unique solution (Y, Z, N) to the equation (4.1); to be explained below. But before we do that we need to present some information which we shall need. Let (V, H, V') be a *rigged* Hilbert space (see [36] or [34]), that is V is a separable Hilbert space embedded continuously and densely in H. Hence by identifying H with its dual, we obtain the following continuous and dense two inclusions: $V \subseteq H \subseteq V'$, where V' is the dual space of V. In fact this is seen as follows. For every $h \in H$, there corresponds $\bar{h} : V \to \mathbb{R}$, defined by $\bar{h}(v) := \langle h, v \rangle_H, v \in V$, which is a linear continuous functional since $|\bar{h}(v)| \leq |h|_H |v|_H \leq const\ |h|_H |v|_V$. I.e. $\bar{h} \in V'$. The mapping $h \mapsto \bar{h}$ from H to V' is linear, injective and continuous. The injectivity of this mapping comes from the definition of $\bar{h}$ above and the density of $V \subseteq H$. Thus we may and we will identify $\bar{h}$ with h. We then have $|h|_{V'} \leq const\ |h|_H, \ \forall\, h \in H$. Thus the embedding $H \subseteq V'$ has a meaning and, moreover, it is continuous and dense.

Denoting by $[\cdot\,,\cdot]$ the duality between V and V' we observe that: $|\,[v\,,x]\,| \leq const\ |v|_V \cdot |x|_{V'}, \ \forall\, v \in V$ and $x \in V'$ and $[v\,,x] = \langle v\,,x \rangle_H$ if $x \in H$.

Definition 4.1 *A solution of (4.1) is a triple* $(Y, Z, N) \in L^2_{\mathcal{F}}(0,T;V) \times \Lambda^2(H;\mathcal{P},M) \times \mathcal{M}^{2,c}_{[0,T]}(H)$ *such that the following equation holds a.s. for all* $t \in [0,T]$, $N(0) = 0$ *and* N *is VSO to* M.

$$\begin{aligned} Y(t) &= \xi + \int_t^T (\, A(s)\, Y(s) + F(s, Y(s), Z(s)\mathcal{Q}^{1/2}(s))\,)\, ds \\ &\quad - \int_t^T Z(s)\, dM(s) - \int_t^T dN(s). \end{aligned} \tag{4.2}$$

Note that from this formula (4.2) it follows that Y is a semimartingale.

Let us now keep the assumptions (A1)-(A4) in Section 3 and add also the following.

- (A5) $A(t,\omega)$ is a linear operator on H, $\mathcal{P}$-measurable, belongs to $L(V;V')$ uniformly in (t,ω) and satisfies the following conditions:

  (i) $A(t,\omega)$ satisfies the *coercivity* condition in the sense that

  $$2\,[A(t,\omega)\,y\,,y] + \alpha\,|y|_V^2 \leq \lambda\,|y|_H^2 \quad a.e.\ t \in [0,T]\,, \quad a.s.\ \ \forall\, y \in V,$$

  for some $\alpha, \lambda > 0$.

  (ii) $A(t,\omega)$ is uniformly continuous, i.e. $\exists\, k_3 \geq 0$ such that for all (t,ω)

  $$|\,A(t,\omega)\,y\,|_{V'} \leq k_3\,|y|_V\,,$$

  for every $y \in V$.

Our aim now is to prove the following theorem.

Theorem 4.2 *If Assumptions (A1)–(A5) hold, then there exists a unique solution* (Y, Z, N) *of the equation (4.1) in* $L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}_{[0,T]}(H)$.

Before starting proving the theorem we give the following example.

Example 4.3 *Let* H *be a separable Hilbert space. Assume that* $A : \mathcal{D}(A) \to H$ *is an unbounded linear operator that generates a* C_0 *- semigroup* $\{S(t)\}_{t \geq 0}$ *on* H*. For example one can consider the Laplacian* Δ *on the space* $H = L^2(\mathcal{O})$*, where* $\mathcal{O}$ *is a bounded subset of* $\mathbb{R}^d$ *with a smooth boundary* $\partial\mathcal{O}$.

Consider the following BSPDE:

$$\begin{cases} -\,dY(t) = A\,Y(t)\,dt - Z(t)\,dM(t) - dN(t), \quad 0 \leq t \leq T, \\ \;\; Y(T) = \xi \in L^2(\Omega, \mathcal{F}_T\,, \mathbb{P};\, H). \end{cases} \tag{4.3}$$

This equation can actually be solved directly using the semigroup approach as follows. First we obtain from Theorem 2.2 the following representation identity:

$$\mathbb{E}\,[\,\xi \,|\, \mathcal{F}_t\,] = \mathbb{E}\,[\,\xi\,] + \int_0^t R(s)\,dM(s) + K(t),\; 0 \leq t \leq T,$$

with $R \in \Lambda^2(H; \mathcal{P}, M),\; K \in \mathcal{M}^2_{[0,T]}(H)$ *and* K *is VSO to* M*. Next we let*

$$Y(t) = \mathbb{E}\,[\,S(T-t)\,\xi \,|\, \mathcal{F}_t\,],\; 0 \leq t \leq T.$$

Then it follows that

$$\begin{aligned} Y(t) &= S(T-t)\Big(\,\mathbb{E}\,[\,\xi\,] + \int_0^t R(s)\,dM(s) + K(t)\,\Big) \\ &= S(T-t)\Big(\,\xi - \int_t^T R(s)\,dM(s) - \int_t^T dK(s)\,\Big) \\ &= S(T-t)\xi - \int_t^T S(s-t)\,Z(s)\,dM(s) - \int_t^T S(s-t)\,dN(s), \end{aligned}$$

where $Z(s) := S(T-s)\,R(s)$*, and* $N(s) := \int_0^s S(T-r)\,dK(r)$*, where* $0 \leq s \leq T$*. It is obvious that* N *belongs to* $\mathcal{M}^2_{[0,T]}(H),\; N(0) = 0$ *and is VSO to* M*. Also* (Y, Z, N) *can easily be seen to be a solution to (4.3).*

Remark 4.4 *Note that if we are given a continuous linear operator* $A : V \to V'$*, which satisfies the coercivity condition in (A5), then Theorem 4.2 implies* *that (4.3) has a unique solution* (Y, Z, N) *in* $L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}_{[0,T]}(H)$.

On the other hand, by setting $\mathcal{D} = \{h \in V \mid Ah \in H\} \subset V$ *and* $A_H : \mathcal{D} \to H$, $A_H(h) = A(h)$ *if* $h \in \mathcal{D}$, *it is known ([22]) that* A_H *generates a* C_0 *- semigroup* $\{S(t)\}_{t\geq 0}$ *on* H. *Now the above example shows that this solution is also mild solution to (4.3). In particular, this remark together with Example 4.3 shows in some sense the relationship between our present approach and the semigroup one.*

An alternative way to see this is the study of the relation between these two types of solutions as in [2].

Recall the discussion in the introduction regarding the inability of using the semigroup approach in our case which involves the predictable unbounded linear operator $A(t, \omega)$. So we shall use the method of *Galerkin's* finite dimensional approximation following [28] and [34]. In fact this method is an extension to the stochastic case of that used by J. Lions [22] for the deterministic case. It was used in the stochastic case by many mathematicians, e.g. Bensoussan [8], [9], Rozovskiĭ [34], Pardoux [28] and Gyöngy and Krylov [20].

We shall divide the proof of Theorem 4.2 into different cases starting by the next lemma, which considers a simple version of the equation (4.1). For the convenience of the reader, since we are dealing with different spaces, we prefer to indicate to the space under each norm.

Lemma 4.5 *Suppose that* $F \in L^2_{\mathcal{F}}(0, T; H)$ *and (A2)–(A5) hold. Then*

$$\begin{aligned} Y(t) &= \xi + \int_t^T (\, A(s)\, Y(s) + F(s)\,)\, ds \\ &\quad - \int_t^T Z(s)\, dM(s) - \int_t^T dN(s), \quad 0 \leq t \leq T. \end{aligned} \tag{4.4}$$

attains a unique solution $(Y, Z, N) \in L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}_{[0,T]}(H)$.

Proof. Uniqueness: Let (Y, Z, N) and (Y', Z', N') be two solutions of (4.4). Itô's formula and (A5)(i) show that

$$\begin{aligned} &\mathbb{E}\,[\, |Y(t) - Y'(t)|^2_H \,] + \mathbb{E}\,\Big[\, \int_t^T |(Z(s) - Z'(s))\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \,\Big] \\ &+ \mathbb{E}\,\Big[\, \int_t^T d < N - N' >_s \,\Big] + \alpha\, \mathbb{E}\,\Big[\, \int_t^T |Y(s) - Y'(s)|^2_V\, ds \,\Big] \\ &\leq \lambda\, \mathbb{E}\,\Big[\, \int_t^T |Y(s) - Y'(s)|^2_H\, ds \Big], \quad 0 \leq t \leq T. \end{aligned} \tag{4.5}$$

In particular,

$$\mathbb{E}\,[\,|Y(t)-Y'(t)|_H^2\,] \leq \lambda\,\mathbb{E}\,\Big[\int_t^T |Y(s)-Y'(s)|_H^2\,ds\Big], \quad 0\leq t\leq T.$$

Gronwall's inequality and the continuity of Y and Y', imply that $Y(t) = Y'(t),\ \forall\, t\in[0,T]$ *a.s.* This together with (4.5) gives the uniqueness of Z and N.

Existence: Let $\{e_i\}_{i\geq 1}$ be an orthonormal basis of H. Since V is dense in H, we may suppose that $e_i \in V$ for each $i \geq 1$. Let $H_n := span(e_1,e_2,\dots,e_n)$, $n\geq 1$. Consider the following system of equations in $H_n \cong \mathbb{R}^n$:

$$\begin{aligned} Y_n^i(t) &= \big\langle e_i\,,\xi\big\rangle_H + \int_t^T [\,e_i\,,A(s)\,(\sum_{j=1}^n Y_n^j(s)\cdot e_j)\,]\,ds \\ &\quad + \int_t^T \big\langle e_i\,,F(s)\big\rangle_H\,ds \\ &\quad - \int_t^T Z_n^i(s)\,dM_n(s) - \int_t^T dN_n^i(s), \end{aligned} \tag{4.6}$$

$i=1,2,\dots,n$, where

$$M_n(s)=\sum_{j=1}^n m_j(s)\cdot e_j \ \text{ and } \ m_j(s)=\big\langle M(s)\,,e_j\big\rangle_H \in \mathcal{M}^{2,c}_{[0,T]}(\mathbb{R}),$$

for $j=1,2,\dots,n$. Thus M_n lies in $\mathcal{M}^{2,c}_{[0,T]}(H_n)$.

In fact if we write

$$Y_n(t) := \sum_{i=1}^n Y_n^i(t)\,e_i\,,\ Z_n(t) := \sum_{i=1}^n Z_n^i(t)(\cdot)\,e_i$$

and

$$N_n(t) := \sum_{i=1}^n N_n^i(t)\,e_i\,,$$

for $0\leq t\leq T$, we can rewrite (4.6) as the following finite dimensional BSDE:

$$\begin{aligned} Y_n(t) &= \pi_n\,\xi + \int_t^T (\,\Pi_n\,A(s)\,Y_n(s)\,)\,ds \\ &\quad + \int_t^T \pi_n\,F(s)\,ds - \int_t^T Z_n(s)\,dM_n(s) - \int_t^T dN_n(s), \end{aligned} \tag{4.7}$$

$0 \leq t \leq T$. Here $\Pi_n : V' \to H_n$ and $\pi_n : H \to H_n$ are the orthogonal projection operators.

Now, for a fixed n, the equation (4.7) is actually a BSDE in H_n, of the type we studied in Section 3, and satisfies the assumptions in Theorem 3.2. Thus it has a unique solution $(Y_n, Z_n, N_n) \in \mathcal{B}^2(H_n) \times \mathcal{M}^{2,c}_{[0,T]}(H_n)$.

On the other hand, Itô's formula gives

$$\begin{aligned}\mathbb{E}\,[\,|Y_n(t)|^2_H\,] \;&=\; \mathbb{E}\,[\,|\,\pi_n\,\xi|^2_H\,] + 2\,\Big(\int_t^T \big\langle Y_n(s)\,,\Pi_n\,A(s)\;Y_n(s)\big\rangle_H\,ds\Big)\\ &\quad + 2\,\mathbb{E}\,\Big[\int_t^T \big\langle Y_n(s)\,,\pi_n\;F(s)\big\rangle_H\,ds\Big]\\ &\quad - \mathbb{E}\,\Big[\int_t^T |\tilde{Z}_n(s)\,\mathcal{Q}^{1/2}(s)|_{L_2(H)}\,ds\Big] - \mathbb{E}\,\Big[\int_t^T d < N_n >_s\Big],\end{aligned}$$

where $\tilde{Z}_n(s) := Z(s)\,\pi_n$, which belongs to $\Lambda^2(H;\mathcal{P},M)$. By using the properties preceding Definition 4.1, we realize that

$$\big\langle Y_n(s)\,,\Pi_n\,A(s)\;Y_n(s)\big\rangle_H = [Y_n(s)\,,\Pi_n\,A(s)\;Y_n(s)],$$

since $Y_n(s) \in V$ and $\Pi_n\,A(s)\;Y_n(s) \in H_n \subset H$. Hence by applying (A5), we get

$$\begin{aligned}&\mathbb{E}\,[\,|Y_n(t)|^2_H\,] + \;\alpha\,\mathbb{E}\,\Big[\int_t^T |Y_n(s)|^2_V\,ds\Big]\\ &+ \mathbb{E}\,\Big[\int_t^T |\tilde{Z}_n(s)\,\mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\,ds\Big] + \mathbb{E}\,\Big[\int_t^T d < N_n >_s\Big]\\ &\leq\; \mathbb{E}\,[\,|\xi|^2_H\,] + (\lambda+1)\,\mathbb{E}\,\Big[\int_t^T |Y_n(s)|^2_H\,ds\Big] + \mathbb{E}\,\Big[\int_t^T |F(s)|^2_H\,ds\Big].\end{aligned} \tag{4.8}$$

As a result Gronwall's inequality gives

$$\mathbb{E}\,[\,|Y_n(t)|^2_H\,] \leq e^{(\lambda+1)\,T}\,\Big(\mathbb{E}\,[\,|\xi|^2_H\,] + \mathbb{E}\,\Big[\int_0^T |F(s)|^2_H\,ds\Big]\Big),$$

or in particular

$$\mathbb{E}\,\Big[\int_0^T |Y_n(t)|^2_H\,dt\Big] \leq\; T\,e^{(\lambda+1)\,T}\,\Big(\mathbb{E}\,[\,|\xi|^2_H\,] + \mathbb{E}\,\Big[\int_0^T |F(s)|^2_H\,ds\Big]\Big). \tag{4.9}$$

Consequently (4.9) and (4.8) imply

$$\sup_{n\geq 1}\,\mathbb{E}\,\Big[\int_0^T |Y_n(s)|^2_H\,ds\Big] < \infty\,,$$

$$\sup_{n\geq 1}\ \mathbb{E}\,\Big[\int_0^T |Y_n(s)|_V^2\ ds\Big] < \infty\,, \tag{4.10}$$

$$\sup_{n\geq 1}\ \mathbb{E}\,\Big[\int_0^T |\tilde{Z}_n(s)\,\mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\ ds\Big] < \infty\,,$$

and

$$\sup_{n\geq 1}\ \mathbb{E}\,[\,|N_n(T)|_H^2\,] < \infty\,.$$

From these estimates it follows that for some subsequence $\{n_k\ ,\ k\geq 1\}$, $(Y_{n_k}\,, \tilde{Z}_{n_k}\,, N_{n_k})$ converge weakly in $L^2_{\mathcal{F}}(0,T;V)\times\Lambda^2(H;\mathcal{P},M)\times\mathcal{M}^{2,c}_{[0,T]}(H)$ as $k\to\infty$. Denote their limits by (Y,Z,N).

The proof of the lemma finishes once we show that (Y,Z,N) is a solution to (4.4). For this end we take first ϕ, so that $\phi(t)=\int_0^t\sigma(s)ds$, each $t\in[0,T]$, with σ lying in $L^2([0,T];\mathbb{R})$. And let $\phi_i := \phi\, e_i$ for an arbitrary fixed i. We then apply Itô's formula to (4.7) to see that

$$\begin{aligned}
&\big\langle \xi\,,\phi_i(T)\big\rangle_H + \int_0^T [A(s)\,Y_n(s)\ ,\phi_i(s)\,]\ ds \\
&+\int_0^T \big\langle F(s)\,,\phi_i(s)\big\rangle_H\ ds \\
&-\int_0^T \big\langle \phi_i(s)\,,\tilde{Z}_n(s)\ dM(s)\big\rangle_H - \int_0^T \big\langle \phi_i(s)\,,dN_n(s)\big\rangle_H \\
&= \int_0^T \big\langle Y_n(s)\,,e_i\big\rangle_H\,\sigma(s)\,ds.
\end{aligned} \tag{4.11}$$

Note that the integral $\int_0^T [A(s)\,Y_n(s)\,,\phi_i(s)\,]\,ds$ exists in $L^2(\Omega,\mathcal{F}_T\,,\mathbb{P};\ \mathbb{R})$ since

$$\mathbb{E}\,\Big(\,\Big|\int_0^T [A(s)\,Y_n(s)\,,\phi_i(s)\,]\,ds\,\Big|\,\Big)^2 \ \leq\ const\,\mathbb{E}\,[\int_0^T |Y_n(s)|_V^2\,ds\,],$$

by using the properties of $[\cdot\,,\cdot]$ preceding Definition 4.1 together with (A5)(ii) and (4.10).

Next note that the mapping $\Psi_1(M) = \int_0^T \langle \phi_i(s)\,, dM(s)\rangle_H$ is continuous from $\mathcal{M}^{2,c}(H)$ to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Indeed this comes from the following inequality

$$\begin{aligned}\mathbb{E}\,[\,\Psi_1(M)|^2\,] &\leq \mathbb{E}\Big[\int_0^T |\phi_i(s)|_H^2 \, d < M >_s \Big] \\ &\leq K_1\,\mathbb{E}\,[\,|M(T)|_H^2\,],\end{aligned}$$

where K_1 is a positive constant.

On the other hand, $\Psi_2 : \Lambda^2(H; \mathcal{P}, M) \to L^2(\Omega, \mathcal{F}_T, \mathbb{P};\ \mathbb{R})$, defined by $\Psi_2(R) = \int_0^T \langle \phi_i(s)\,, R(s)\; dM(s)\rangle_H$ is continuous since there exists a positive constant K_2 such that

$$\begin{aligned}\mathbb{E}\,[\,|\Psi_2(R)|^2\,] &= \Big[\int_0^T |\phi(s)|^2 \, d < \Big(\Big\langle \int_0^{\cdot} R(r)\, dM(r)\,, e_i\Big\rangle_H\Big) >_s \Big] \\ &\leq K_2\,\mathbb{E}\Big[\,|\,\Big\langle \int_0^T R(s)\, dM(s)\,, e_i\Big\rangle_H\,|^2\Big] \\ &\leq K_2\,\mathbb{E}\Big[\int_0^T |R(s)\,\mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds\Big],\end{aligned}$$

by using Cauchy-Schwartz inequality.

Similarly but by using the properties of $[\cdot,\cdot]$ and (A5)(ii) we can deduce easily that Ψ_3

$$Y \mapsto \int_0^T [A(s)\,Y(s)\,, \phi_i(s)]\, ds,$$

is a continuous mapping from $L^2_{\mathcal{F}}(0,T;V)$ to $L^2(\Omega, \mathcal{F}_T, \mathbb{P};\ \mathbb{R})$.

But since Ψ_1 and Ψ_2 and Ψ_3 are linear as well, it follows that Ψ_1, Ψ_2 and Ψ_3 are continuous with respect to the weak topologies. So by replacing n by n_k in (4.11) we can pass to the weak limit in $L^2(\Omega, \mathcal{F}_T\,, \mathbb{P}; \mathbb{R})$, as $k \to \infty$ to conclude

$$\begin{aligned}&\langle \xi\,, \phi_i(T)\rangle_H + \int_0^T [A(s)\,Y(s)\,, \phi_i(s)\,]\, ds \\ &+ \int_0^T \langle F(s)\,, \phi_i(s)\rangle_H\, ds \\ &- \int_0^T \langle \phi_i(s)\,, Z(s)\; dM(s)\rangle_H - \int_0^T \langle \phi_i(s)\,, dN(s)\rangle_H \\ &= \int_0^T \langle Y(s)\,, e_i\rangle_H\, \sigma(s)\, ds.\end{aligned}$$

This equation holds true for every $i \geq 1$. Therefore, for any $v \in V$,

$$\begin{aligned}
&\langle \xi , v \rangle_H \, \phi(T) + \int_0^T [A(s)\, Y(s) \, , v \,] \; \phi(s) \; ds \\
&+ \int_0^T \langle F(s) \, , v \rangle_H \, \phi(s) \, ds \\
&- \int_0^T \phi(s) \, \langle v \, , Z(s) \; dM(s) \rangle_H - \int_0^T \phi(s) \, \langle v \, , dN(s) \rangle_H \\
&= \int_0^T \langle Y(s) \, , v \rangle_H \, \sigma(s) \, ds. \qquad (4.12)
\end{aligned}$$

Hence if for $t \in (0, T)$ we choose

$$\phi_m(s) := \begin{cases} 1 & \text{if } \; s \geq t + \frac{1}{2\,m} \; , \\ \frac{1}{2} - m \; (t - s) & \text{if } \;\; t - \frac{1}{2\,m} < s < t + \frac{1}{2\,m} \; , \\ 0 & \text{if } \; s \leq t - \frac{1}{2\,m} \; , \end{cases} \qquad (4.13)$$

for any $m \geq 1$, in (4.12), then

$$\begin{aligned}
&\langle \xi , v \rangle_H + \int_0^T [A(s)\, Y(s) \, , v \,] \; \phi_m(s) \; ds \\
&+ \int_0^T \langle F(s) \, , v \rangle_H \; \phi_m(s) \; ds \\
&- \int_0^T \phi_m(s) \; \langle v \, , Z(s) \; dM(s) \rangle_H - \int_0^T \phi_m(s) \; \langle v \, , dN(s) \rangle_H \\
&= \; m \int_{t-\frac{1}{2\,m}}^{t+\frac{1}{2\,m}} \langle Y(s) \, , v \rangle_H \; ds. \qquad (4.14)
\end{aligned}$$

Since holds for all $m \geq 1$, by applying the continuity of the mappings $\Psi_i, \; i = 1, 2, 3$, we can let $m \to \infty$ in (4.14) to get the following equality

$$\begin{aligned}
&\langle \xi , v \rangle_H + \int_t^T [A(s)\, Y(s) \, , v \,] \; ds + \int_t^T \langle F(s) \, , v \rangle_H \; ds \\
&- \int_t^T \langle v \, , Z(s) \; dM(s) \rangle_H - \int_t^T \langle v \, , dN(s) \rangle_H \\
&= \; \langle Y(t) \, , v \rangle_H, \qquad (4.15)
\end{aligned}$$

for almost all $t \in [0, T]$.

But since V is separable, this equality implies that, for $a.e.\ t \in [0,T]$,

$$Y(t) = \xi + \int_t^T (\, A(s)\, Y(s) + F(s)\,)\, ds \\ - \int_t^T Z(s)\, dM(s) - \int_t^T dN(s)\,. \tag{4.16}$$

We note that the process defined by the right hand side of this equation (4.16) has a continuous modification. Hence we define Y to be this process.

The rest of the proof is devoted to proving that N is VSO to M in the sense that for any $[0,T]$ - valued stopping time u,

$$\mathbb{E}\,[\, M(u) \otimes N(u)\,] = 0.$$

Recall that N_n and M_n are VSO, which means that

$$\mathbb{E}\,[\, M_{n_k}(u) \otimes N_{n_k}(u)\,] = 0, \ \ \text{each } k.$$

But since

$$M_m(u) = \int_0^u \pi_m^{n_k}\, dM_{n_k}(s),$$

where $\pi_m^{n_k} := \pi_m \circ \pi_{n_k}$, this implies that

$$\mathbb{E}\,[\, M_m(u) \otimes N_{n_k}(u)\,] = 0,$$

for all $m \leq n_k$. It follows that

$$\mathbb{E}\,[\, M_m(T) \otimes N_{n_k}(u)\,] = \mathbb{E}\,[\, M_m(u) \otimes N_{n_k}(u)\,] = 0, \tag{4.17}$$

for all $m \leq n_k$.

Now, since N_{n_k} converges weakly to N in $\mathcal{M}^{2,c}_{[0,T]}(H)$ as $k \to \infty$, then $N^g_{n_k}$ $(= \langle N_{n_k}\,, g \rangle)$ converges weakly to N^g in $\mathcal{M}^{2,c}_{[0,T]}(\mathbb{R})$ as $k \to \infty$, for any $g \in H$. This implies, by using the optional stopping theorem, that $N^g_{n_k}(u \wedge \cdot)$ converges weakly to $N^g(u \wedge \cdot)$ in $\mathcal{M}^{2,c}_{[0,T]}(\mathbb{R})$ as $k \to \infty$. Precisely, if $x \in \mathcal{M}^{2,c}_{[0,T]}(\mathbb{R})$, then $x^u := x(u \wedge \cdot) \in \mathcal{M}^{2,c}_{[0,T]}(\mathbb{R})$ and hence by using the weak convergence of $N^g_{n_k}$,

$$\begin{aligned} \lim_{k\to\infty} \mathbb{E}\,[\, x(T) \cdot N^g_{n_k}(u \wedge T)\,] &= \lim_{k\to\infty} \mathbb{E}\,[\, x^u(T) \cdot N^g_{n_k}(T)\,] \\ &= \mathbb{E}\,[\, x^u(T) \cdot N^g(T)\,] \\ &= \mathbb{E}\,[\, x(T) \cdot N^g(u)\,]. \end{aligned}$$

Therefore

$$\lim_{k\to\infty} \mathbb{E}\left[\left\langle\, M_m(T)\otimes N_{n_k}(u)\,,h\otimes g\right\rangle_{H\hat{\otimes}_1 H}\,\right] = \mathbb{E}\left[\,\left\langle M_m(T)\otimes N(u)\,,h\otimes g\right\rangle_{H\hat{\otimes}_1 H}\,\right]$$

for any $h, g \in H$. As a result

$$\lim_{k\to\infty} \mathbb{E}\,[\,M_m(T)\otimes N_{n_k}(u)\,] = \mathbb{E}\,[\,M_m(T)\otimes N(u)\,]. \tag{4.18}$$

Next by arguing as above we can let $m \to \infty$ in this result and use the strong and so the weak convergence of M_m^h to find that

$$\lim_{m\to\infty} \mathbb{E}\,[\,M_m(T)\otimes N(u)\,] = \mathbb{E}\,[\,M(T)\otimes N(u)\,].$$

Finally, by using this result with (4.18) and (4.17) we deduce that

$$\mathbb{E}\,[\,M(u)\otimes N(u)\,] = \mathbb{E}\,[\,M(T)\otimes N(u)\,] = 0.$$

This completes the proof. ∎

Remark 4.6 *The case where the mapping F takes values in the space V', if it is needed, can be treated exactly as it was done in the previous proof. An example would be when F is only continuous as a mapping from H to V'.*

Let us now consider the following BSPDE in which we allow the function F to depend on the variables t and Z but not on Y.

$$\begin{aligned} Y(t) \;&=\; \xi + \int_t^T (\,A(s)\,Y(s) + F(s, Z(s)\mathcal{Q}^{1/2}(s))\,)\,ds \\ &\quad - \int_t^T Z(s)\,dM(s) - \int_t^T dN(s), \quad 0 \le t \le T. \end{aligned} \tag{4.19}$$

For this equation we need to modify the assumption (A1) and in particular we set the following.

- (A1)′ $F : [0,T]\times\Omega\times L_2(H) \to H$ is a mapping such that the following properties are verified.
  - (i) F is $\mathcal{P}\otimes\mathcal{B}(L_2(H))/\mathcal{B}(H)$ - measurable.
  - (ii) $\mathbb{E}\,[\,\int_0^T |F(t,0,0)|_H^2\,dt\,] < \infty$.
  - (iii) $\exists\, k_2 > 0$ such that $\forall\, z, z' \in L_2(H)$

    $$|F(t,\omega,z) - F(t,\omega,z')|_H^2 \le k_2\,|z - z'|_{L_2(H)}^2,$$

    uniformly in (t,ω).

Lemma 4.7 *Assume that F satisfies (A1)$'$ and Assumptions (A2)–(A5) hold. There exists a unique solution (Y, Z, N) of (4.19) in $L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}(H)$.*

Proof. The uniqueness can be shown as in the proof of Lemma 4.5. So we shall only prove the existence. We proceed by approximation using Lemma 4.5 as follows. Let first $Z_0 \equiv 0$ and consider the following BSPDE:

$$\begin{aligned} Y_n(t) &= \xi + \int_t^T (\, A(s)\, Y_n(s) + F(s, Z_{n-1}(s)\mathcal{Q}^{1/2}(s))\,)\, ds \\ &\quad - \int_t^T Z_n(s)\, dM(s) - \int_t^T dN_n(s), \quad 0 \le t \le T, \end{aligned} \tag{4.20}$$

for $n \geq 1$. Lemma 4.5 tells us that this equation attains a unique solution $(Y_n, Z_n, N_n) \in L^2_{\mathcal{F}}(0, T; V) \times \Lambda^2(H; \mathcal{P}, M) \times \mathcal{M}^{2,c}(H)$, for any $n \geq 1$. By using Itô's formula, (A1)$'$(iii) and (A5)(i) we find that

$$\begin{aligned} &\mathbb{E}\,[\,|Y_{n+1}(t) - Y_n(t)|^2_H \,+ \mathbb{E}\,\Big[\int_t^T |(Z_{n+1}(s) - Z_n(s))\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \\ &+ \mathbb{E}\,\Big[\int_t^T d < N_{n+1} - N_n >_s \Big] \\ &\le\; (\lambda + 2\,k_2)\, \mathbb{E}\,\Big[\int_t^T |Y_{n+1}(s) - Y_n(s)|^2_H\, ds \Big] \\ &+ \frac{1}{2}\, \mathbb{E}\,\Big[\int_t^T |(Z_n(s) - Z_{n-1}(s))\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \\ &- \alpha\, \mathbb{E}\,\Big[\int_t^T |Y_{n+1}(s) - Y_n(s)|^2_V\, ds \Big]. \end{aligned} \tag{4.21}$$

Hence by multiplying both sides of (4.21) by $e^{(\lambda+2k_2)\,t}$ and integrating with respect to $t \in [0, T]$, it follows that

$$\begin{aligned} &\mathbb{E}\,\Big[\int_0^T |Y_{n+1}(t) - Y_n(t)|^2_H\, dt \Big] \\ &\quad + \int_0^T e^{(\lambda+2k_2)\,t} \Big(\mathbb{E}\,\Big[\int_t^T |(Z_{n+1}(s) - Z_n(s))\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \Big)\, dt \\ &\quad + \int_0^T e^{(\lambda+2\,k_2)\,t} \Big(\mathbb{E}\,\Big[\int_t^T d < N_{n+1} - N_n >_s \Big] \Big)\, dt \\ &\le \frac{1}{2} \int_0^T e^{(\lambda+2\,k_2)\,t} \Big(\mathbb{E}\,\Big[\int_t^T |(Z_n(s) - Z_{n-1}(s))\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \Big) dt. \end{aligned} \tag{4.22}$$

But this implies

$$\int_0^T e^{(\lambda+2k_2)\,t} \Big(\mathbb{E} \Big[\int_t^T |(Z_{n+1}(s) - Z_n(s))\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \Big)\, dt$$
$$\leq \frac{1}{2} \int_0^T e^{(\lambda+2\,k_2)\,t} \Big(\mathbb{E} \Big[\int_t^T |(Z_n(s) - Z_{n-1}(s))\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \Big)\, dt.$$

And so iterating in n this inequality yields

$$\int_0^T e^{(\lambda+2k_2)\,t} \Big(\mathbb{E} \Big[\int_t^T |(Z_{n+1}(s) - Z_n(s)) \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \Big) dt \leq \Big(\frac{1}{2} \Big)^n K_3\,,$$

where $K_3 := \frac{1}{\lambda+2\,k_2}\, e^{(\lambda+2\,k_2)\,T}\, \mathbb{E} \Big[\int_0^T |Z_1(s)\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big]$. Thus in particular from (4.22) we get

$$\mathbb{E} \Big[\int_0^T |Y_{n+1}(t) - Y_n(t)|^2_H\, dt \Big] \leq \Big(\frac{1}{2} \Big)^n K_3\,. \tag{4.23}$$

Moreover by using (4.21) repeatedly and (4.23) it follows therefore that

$$\mathbb{E} \Big[\int_0^T |(Z_{n+1}(s) - Z_n(s))\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big]$$
$$\leq \Big(\frac{1}{2} \Big)^n \Big(n\, (\lambda + 2k_2)\, K_3 + \mathbb{E} \Big[\int_0^T |Z_1(s)\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \Big). \tag{4.24}$$

On the other hand, we obtain from (4.21), (4.23) and (4.24) that

$$\mathbb{E} \Big[\int_0^T |Y_{n+1}(s) - Y_n(s)|^2_V\, ds \Big]$$
$$\leq \frac{1}{\alpha} \Big(\frac{1}{2} \Big)^n \Big(n\, (\lambda + 2k_2)\, K_3 + \mathbb{E} \Big[\int_0^T |Z_1(s)\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \Big) \tag{4.25}$$

and

$$\mathbb{E}\, [\, |\, (N_{n+1} - N_n)(T)\, |^2_H\,]$$
$$\leq \Big(\frac{1}{2} \Big)^n \Big(n\, (\lambda + 2k_2)\, K_3 + \mathbb{E} \Big[\int_0^T |Z_1(s)\, \mathcal{Q}^{1/2}(s)|^2_{L_2(H)}\, ds \Big] \Big). \tag{4.26}$$

The results in (4.25), (4.24) and (4.26) show that the sequences $\{Y_n\}$, $\{Z_n\}$ and $\{N_n\}$ are Cauchy sequences in $L^2_{\mathcal{F}}(0,T;V)$, $\Lambda^2(H;\mathcal{P},M)$ and $\mathcal{M}^{2,c}(H)$, respectively. Hence they converge to some limits Y, Z, N, respectively.

Note that here N is VSO to M. The proof of this uses the fact that N_n is VSO to and M for each $n \geq 1$, the weak convergence of N_n to N as $n \to \infty$, and can be actually achieved as in the preceding proof; but here it is much simpler.

From (A5)(ii) there is a positive constant $K_4 = T\, k_3$ such that

$$\begin{aligned} & \mathbb{E}\Big[\, \Big| \int_t^T A(s)(Y_n(s) - Y(s))\, ds\, \Big|_{V'}^2 \Big] \\ & \leq\ T\, \mathbb{E}\Big[\int_0^T \big|\, A(s)(Y_n(s) - Y(s))\, \big|_{V'}^2\, ds \Big] \\ & \leq\ K_4\, \mathbb{E}\Big[\int_0^T \big|\, Y_n(s) - Y(s)\, \big|_V^2\, ds \Big] \\ & \to 0 \ \text{ as } \ n \to \infty. \end{aligned}$$

Thus we can pass to the limit in (4.20) as $n \to \infty$ to derive the BSPDE (4.19) showing that (Y, Z, N) is a solution to (4.19). ■

We are now ready to complete the proof of Theorem 4.2.

Proof of Theorem 4.2. The uniqueness part can be done as in the proof of Lemma 4.5.

We shall then establish the existence of solutions to the BSPDE (4.1). Let $Y_0 \equiv 0$ and define recursively by using Lemma 4.7 the following BSPDE:

$$\begin{aligned} Y_n(t) \ &=\ \xi + \int_t^T \big(\, A(s)\, Y_n(s) + F(s, Y_{n-1}(s), Z_n(s)\mathcal{Q}^{1/2}(s))\, \big)\, ds \\ & \quad - \int_t^T Z_n(s)\, dM(s) - \int_t^T dN_n(s), \quad 0 \leq t \leq T, \end{aligned} \tag{4.27}$$

for $n \geq 1$. The solutions (Y_n, Z_n, N_n) lie in $L^2_{\mathcal{F}}(0,T;V) \times \Lambda^2(H;\mathcal{P}, M) \times \mathcal{M}^{2,c}(H)$ for each $n \geq 1$.

By applying Itô's formula, (A1)(iii) and (A5) it follows that

$$\begin{aligned} & \mathbb{E}\,[\, |Y_{n+1}(t) - Y_n(t)|_H^2\,] + \frac{1}{2}\, \mathbb{E}\Big[\int_t^T |(Z_{n+1}(s) - Z_n(s))\, \mathcal{Q}^{1/2}(s)|^2\, ds \Big] \\ & + \mathbb{E}\Big[\int_t^T d < N_{n+1} - N_n >_s \Big] +\ \alpha\, \mathbb{E}\Big[\int_t^T |Y_{n+1}(s) - Y_n(s)|_V^2\, ds \Big] \\ & \leq\ (\lambda + 2k_1 + 1)\, \mathbb{E}\Big[\int_t^T \big(\, |Y_{n+1}(s) - Y_n(s)|_H^2 \\ & \qquad\qquad + |Y_n(s) - Y_{n-1}(s)|_H^2\, \big)\, ds \Big], \end{aligned} \tag{4.28}$$

where $0 \leq t \leq T$. Denote by

$$P_n(t) := \mathbb{E}\Big[\int_t^T |Y_n(s) - Y_{n-1}(s)|_H^2 \, ds \Big],$$

where $t \in [0,T]$ and $n \geq 1$. Observe from (4.28) that

$$- \frac{d}{dt} P_{n+1}(t) - (\lambda + 2k_1 + 1) \, P_{n+1}(t) \leq (\lambda + 2k_1 + 1) \, P_n(t),$$

for each t, or in particular

$$- \frac{d}{dt} \big(P_{n+1}(t) \cdot e^{(\lambda+2k_1+1)\, t} \big) \leq (\lambda + 2k_1 + 1) \, e^{(\lambda+2k_1+1)\, t} \, P_n(t).$$

Let $t \in [0,T]$ and integrate to get

$$P_{n+1}(t) \leq (\lambda + 2k_1 + 1) \int_t^T e^{(\lambda+2k_1+1)\, s} \, P_n(s) \, ds.$$

Hence iterating this inequality gives

$$P_{n+1}(t) \leq [\, (\lambda + 2k_1 + 1) \, e^{(\lambda+2k_1+1)\, T} \,]^n \, \frac{(T-t)^n}{n!} \, P_1(0).$$

But this implies $\sum_{n=1}^{\infty} P_{n+1}(0)$ is convergent and as a result from this, the definition of $P_{n+1}(0)$ and (4.28), we conclude that $\{Y_n\}$, $\{Z_n\}$ and $\{N_n\}$ are Cauchy sequences in $L^2_{\mathcal{F}}(0,T;V)$, $\Lambda^2(H;\mathcal{P},M)$ and $\mathcal{M}^{2,c}(H)$, respectively, and so they are convergent. Let Y, Z and N denote the limits of these sequences.

The very strong orthogonality between N and M is derived as in the proof of the previous lemmas.

Now this convergence together with (A1) and (A5)(ii) allows us to let $n \to \infty$ in (4.27) and see that the following equality holds a.s.

$$\begin{aligned} Y(t) &= \xi + \int_t^T \big(A(s)\, Y(s) + F(s, Y(s), Z(s)) \big) \, ds \\ &- \int_t^T Z(s) \, dM(s) - \int_t^T dN(s), \quad 0 \leq t \leq T. \end{aligned}$$

It follows that (Y,Z,N) is a solution of (4.1). ∎

Remark 4.8 *We note that the condition (A5)(ii) in the theorem is necessary as we saw in the proof. However if we replace it by the following:*
$\exists$ *a predictable nonnegative stochastic process* $k_4'(t,\omega)$ *such that* $\sup_{0\leq t\leq T} k_3'(t) < \infty$ *and for all* (t,ω)

$$| A(t,\omega)\, y \,|_{V'} \leq k_3'(t,\omega)\, |y|_V ,$$

for every $y \in V$*, we can still see that a solution of (4.1) is unique. But the existence of the solution remains an interesting research problem. We refer the reader to [34, Section 3.3, P. 108] for a similar discussion.*

The following example is an application to the above theorem.

Example 4.9 *Let* $V = \mathbb{H}^1(\mathbb{R}^d)$*,* $H = L^2(\mathbb{R}^d;\mathbb{R})$ *and* $V' = \mathbb{H}^{-1}(\mathbb{R}^d)$*, where* $\mathbb{H}^1(\mathbb{R}^d)$ *is the completion of* $C_0^\infty(\mathbb{R}^d)$ *under the norm:*

$$||\varphi||_{\mathbb{H}^1(\mathbb{R}^d)} := \Big(\int_{\mathbb{R}^d} |\varphi(x)|^2\, dx + \int_{\mathbb{R}^d} |\nabla\, \varphi(x)|^2\, dx \Big)^{\frac{1}{2}}$$

and $\mathbb{H}^{-1}(\mathbb{R}^d)$ *is the dual space of* $\mathbb{H}^1(\mathbb{R}^d)$*.*

Then (V,H,V') *is a rigged Hilbert space; see e.g. [34].*

We shall assume as in Section 1 that $(\Omega,\mathcal{F},\mathbb{P})$ *is a complete probability space equipped with a right continuous filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *and* $\mathcal{P}$ *is the predictable* σ *- algebra of subsets* $\Omega \times (0,T]$*.*

Suppose that β *is a one-dimensional Brownian motion with respect to* $\{\mathcal{F}_t\}_{t\geq 0}$*,* $a_{ij}(\omega,t,x)$*,* $(i,j = 1,\dots,d)$*, are bounded real valued processes, defined on* $\Omega \times [0,T] \times \mathbb{R}^d$*, that are* $\mathcal{P}$*-predictable, measurable in the* x *- variable and satisfy the following* uniform parabolicity *condition:* $\exists\, \delta > 0$ *such that*

$$2 \sum_{i,j=1}^{d} a_{ij}(\omega,t,x)\, \eta_i\, \eta_j \geq \delta \sum_{i,j=1}^{d} \eta_j^2 , \tag{4.29}$$

for all $(\omega,t,x) \in \Omega \times [0,T] \times \mathbb{R}^d$ *and* $\eta_1,\dots,\eta_d \in \mathbb{R}$*.*

Consider the following problem:

$$\begin{cases} -\,dy(t,x) = \Big[\displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial\, x_i} \Big(a_{ij}(\omega,t,x)\, \frac{\partial}{\partial\, x_j}\, y(t,x) \Big) \\ \qquad\qquad + \displaystyle\sum_{j=1}^{d} f_j\, (\omega,t,y(t,x),z(t,x)\,) \Big]\, dt \\ \qquad\qquad -\, z(t,x)\, d\beta(t) - d\mathcal{N}(t,x), \\ y(T,x) = \varphi(x), \end{cases} \tag{4.30}$$

where $(t,x)\in[0,T]\times\mathbb{R}^d$, $\mathbb{E}\,[\int_{\mathbb{R}^d}\varphi^2(x)\,dx\,]<\infty$ *and* $f_j\,:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\,,\ j=1,\dots,d$, *satisfies the following two conditions:*

- f_j *is* $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ *- measurable and*

$$\mathbb{E}\,[\int_0^T |\,f_j(t,0,0)\,|^2\,dt\,]<\infty\,,$$

- $\exists\, b_j>0$ *such that* $\forall\,\sigma,\sigma'\in\mathbb{R},\ \forall\,\rho,\rho'\in\mathbb{R}$

$$|\,f_j(t,\sigma,\rho)-f_j(t,\sigma',\rho')\,|^2\le b_j\,(\,|\sigma-\sigma'|^2+|\rho-\rho'|^2\,),$$

  uniformly in (t,ω).

A solution of (4.30) is a triple $(y(t,x),z(t,x),\mathcal{N}(t,x))$ *of processes satisfying the following properties:*
(i) for each $x\in\mathbb{R}^d,\ y(t,x),z(t,x)$ *and* $\mathcal{N}(t,x)$ *are* $\mathcal{P}/\mathcal{B}(\mathbb{R})$ *- measurable,*
(ii)

$$\mathbb{E}\,[\int_0^T\int_{\mathbb{R}^d}(y(t,x))^2\,dxdt\,]<\infty,\ \mathbb{E}\,[\int_0^T\int_{\mathbb{R}^d}(z(t,x))^2\,dxdt\,]<\infty,$$

$$\mathbb{E}\,[\int_{\mathbb{R}^d}(\mathcal{N}(T,x))^2\,dx\,]<\infty,$$

(iii) $\mathbb{E}\,[\int_{\mathbb{R}^d}\mathcal{N}(u,x)\cdot h(x)\,\beta(u)\,dx\,]=0$, *for any* $[0,T]$ *- valued stopping times u and for any* $h\in H$,
(iv) for all $v\in C_0^\infty(\mathbb{R}^d)$ *and* $t\in[0,T]$,

$$\begin{aligned}\int_{\mathbb{R}^d} y(t,x)\;v(x)\;dx &= \int_{\mathbb{R}^d}\varphi(x)\;v(x)\;dx\\ &-\int_t^T\int_{\mathbb{R}^d}\Big(\sum_{i,j=1}^d a_{ij}(\omega,s,x)\,\frac{\partial}{\partial x_j}\,y(s,x)\,\frac{\partial}{\partial x_i}\,v(x)\Big)\,dx\;ds\\ &-\int_t^T\int_{\mathbb{R}^d}\Big(\,\sum_{j=1}^d f_j\,(\omega,s,y(s,x),z(s,x)\,)\Big)\,v(x)\;dx\;ds\\ &-\int_t^T\int_{\mathbb{R}^d} v(x)\;z(s,x)\;dx\;d\beta(s)-\int_{\mathbb{R}^d}\int_t^T v(x)\;d\mathcal{N}(s,x)\;dx.\end{aligned}$$

We now want to relate (4.30) to the BSPDE (4.1). First let $A(t,\omega)$ be defined such that

$$[A(t,\omega)\,\eta\,,\nu] := -\int_{\mathbb{R}^d}\Big(\sum_{i,j=1}^{d} a_{ij}(\omega,t,x)\,\frac{\partial}{\partial x_j}\,\eta(x)\,\frac{\partial}{\partial x_i}\,\nu(x)\Big)\,dx,$$

where $\eta,\nu\in V$. Then from the condition (4.29) we have

$$\begin{aligned} 2\,[A(t,\omega)\,\nu\,,\nu] &\le -\,\delta\int_{\mathbb{R}^d}\sum_{i=1}^{d}\Big(\frac{\partial}{\partial x_i}\,\nu(x)\Big)^2 dx \\ &= \delta\int_{\mathbb{R}^d}\nu^2(x)\,dx - \delta\Big(\int_{\mathbb{R}^d}\nu^2(x)\,dx + \int_{\mathbb{R}^d}\sum_{i=1}^{d}\big(\frac{\partial}{\partial x_i}\,\nu(x)\,\big)^2\,dx\Big), \end{aligned}$$

which means that A satisfies the conditions in (A5).

Secondly, on one hand letting $M(t) = h\,\beta(t)$, where h is some fixed $h\in H$ satisfying $|h|_H = 1$, shows that $M\in\mathcal{M}^{2,c}_{[0,T]}(H)$, satisfies (A3) and its local covariation operator $\mathcal{Q}$ is the identification of $h\otimes h$ in $L_1(H)$, i.e. it is the mapping $g\mapsto\langle h\,,g\rangle_H\,h$. Since $|h|_H = 1$ we observe that $\mathcal{Q}^{1/2}=\mathcal{Q}$ and $\mathcal{Q}^{1/2}(h)=h$. In particular $h\in\mathcal{Q}^{1/2}(H)$.

On the other hand, we define the mapping $F:\Omega\times[0,T]\times H\times L_2(H)\to H$ on elements $(\omega,t,\zeta,\varsigma)$ of $\Omega\times[0,T]\times H\times L_2(H)\to H$ by

$$F(\omega,t,\zeta,\varsigma)(x) := \sum_{j=1}^{d} f_j\,(\omega,t,\zeta(x),\tilde{\varsigma}(x)\,),$$

where $x\in\mathbb{R}^d$ and $\tilde{\varsigma}=\varsigma(h)\in H$.

Note that the above shows that F verifies the conditions in (A1) and the equation (4.30) can be rewritten as the following BSPDE with values in H :

$$\begin{cases} -\,dY(t) = (\,A(t)\,Y(t) + F(t,Y(t),Z(t)\mathcal{Q}^{1/2})\,)\,dt \\ \qquad\qquad -Z(t)\,dM(t) - dN(t),\quad 0\le t\le T, \\ Y(T)=\varphi. \end{cases} \tag{4.31}$$

Assuming that (A4) holds we conclude from Theorem 4.2 that there exists a unique solution (Y,Z,N) of (4.31) in $L^2_{\mathcal{F}}(0,T;H)\times\Lambda^2(H;\mathbb{P},M)\times\mathcal{M}^{2,c}_{[0,T]}(H)$, such that $Y(t)\in V$ for a.e. (ω,t) and N is VSO to M. As a result, by defining $y(t,x) := Y(t)(x), z(t,x) := (Z(t)(h))(x) = (Z(t)\mathcal{Q}^{1/2}(h))(x),\ \mathcal{N}(t,x) =$

$N(t)(x)$, $x \in \mathbb{R}^d$, the above properties (i)–(iv) are fulfilled. Indeed (i) and (ii) are obvious, and (iv) follows directly from the density of $C_0^\infty(\mathbb{R}^d)$ in V and the fact that the integral form of (4.31) holds in V'; see (4.2). But for (iii) we argue as follows.

$$\begin{aligned} \mathbb{E}\,[\int_{\mathbb{R}^d} \mathcal{N}(u,x)\cdot h(x)\,\beta(u)\,dx\,] &= \mathbb{E}\,[\int_{\mathbb{R}^d} N(u)(x)\cdot h(x)\,\beta(u)\,dx\,] \\ &= \mathbb{E}\,[\,\langle N(u)\,,M(u)\rangle_H\,] \\ &= \mathbb{E}\,[\,\mathrm{tr}\,(N(u)\otimes M(u))\,] = 0, \end{aligned}$$

for any $[0,T]$ - valued stopping times u, since N is VSO to M. The fact that (iii) holds for arbitrary h in H follows easily from this result.

Remark 4.10 *(i) A special case of the above functions f_j in Example 4.9 is the following.*

$$f_j(t,\sigma,\rho) = \alpha_t\,\sigma + \beta_t\,\rho\,, \quad \textit{each } j\,,$$

where α and β are two bounded predictable process with values in $\mathbb{R}$. This implies that

$$F(\omega,t,\zeta,\varsigma)(x) := d\cdot(\,\alpha_t\,\zeta(x) + \beta_t\,\tilde{\varsigma}(x)\,),$$

where $x \in \mathbb{R}^d$. For example if $\zeta = h$ and $\varsigma = \mathcal{Q}^{1/2}$, then

$$F(\omega,t,\zeta,\varsigma)(x) = d\cdot h(x)(\alpha_t + \beta_t).$$

(ii) In (i) if one takes $\alpha_t = \beta_t = 0$ for each t, which yields that $F = 0$, he shall end up with a BSPDE of the type of the Example 4.3.

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