Generalisations and Applications of the Clark-Ocone Formula

by

Yang, Yuxin

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Contents

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Declarations

I declare that no portion of this thesis has been previously submitted for any degree at any other university. The contents are based on collaborative research with Professor David Elworthy, except for materials otherwise acknowledged.

Chapter 1

Introduction

The classical Clark-Ocone formula [6; 52] represents a functional on the Wiener space as the sum of its expectation and a stochastic integral of the conditional expectation of its H-derivative. This is the result of the adjoint relationship between the gradient and divergence operators defined on the classical Wiener space, and is closely related to the integration by parts formula, an essential part of Malliavin calculus.

As one of the basic tools in stochastic analysis, the Clark-Ocone representation has many important applications and generalisations. One of its crucial consequences is the spectral gap inequality on the path spaces over compact Riemannian manifolds, proved by S. Fang [30]. We are interested in extending Fang's result to the study of the spectral gap of the Hodge-Kodaira Laplacian on differential forms, and in proving the (possible) vanishing of the L^2 de Rham cohomology classes on the based path spaces. This is part of a long-range goal of developing a Hodge theory on infinite dimensional manifolds, first set by L. Gross [32] almost half a century ago in his pioneering work on infinite dimensional potential theory.

One of our main results is the derivation of a generalised Clark-Ocone formula for one-forms on Riemannian path spaces, which proves the spectral gap for the Hodge Laplacian Δ^1 and the vanishing of the first L^2 de Rham cohomology. Kusuoka [42] studied the L^2 Hodge theory in the setting of hypersurfaces in the Wiener space, and obtained a related result using a very different approach. Our idea is modelled on a generalisation of the classical Clark-Ocone formula for functions to those for differential forms on the classical Wiener space. These generalised formulae imply directly the closedness of the range of the exterior derivatives and the existence of spectral gap for the Laplacian operators on differential forms. They also give an alternative proof of the result by I. Shigekawa [55] of the triviality of the de Rham cohomology groups on the Wiener space. Our approach has the benefits of providing explicit expressions for the components in the Hodge decomposition, and perhaps more importantly, of showing promise to carry over to curved path spaces, as indicated by the aforementioned result on one-forms. Higher-order forms on path spaces, however, involve further complications which still evade our full comprehension, and hence remain a part of the ongoing research.

Since the existence of Clark-Ocone-type formulae has such significance for the L^2 de Rham cohomology, we have explored a few different techniques of formulating them in more general settings, especially where there is no natural concept of time, nor any intrinsically defined filtration. Such filtrations play a principal role in the standard Itô integration theory, since they give rise to the fundamental notions of measurability and adaptedness. Noteworthy examples where the standard theory on the classical Wiener space does not directly apply include abstract Wiener spaces (where there is no intrinsic temporal structure), and the loop spaces (where there is ambiguity in the definition of time and filtration, since the end point coincides with the start point; usually an enlargement of filtration is required).

One of such techniques, first proposed by Üstünel [57] and further developed by Üstünel and Zakai [61], is to construct filtrations using resolutions of the identity of the Cameron-Martin space. Such constructions unify the treatment of the forward and backward Itô integrals, and open up the possibility of even more unconventional flows of time, lending hope to a new approach to the stochastic analysis on loop spaces. There are known obstructions to log-Sobolev inequality and spectral gap inequality on loop spaces [17], so it is natural to inquire into the possible topological or geometrical obstructions to the existence of certain classes of random resolutions of the identity. Although we do not address this question here, it motivated our investigation into random resolutions of the identity.

Another technique, initiated by L. Wu [63], aims at establishing a more intrinsic Clark-Ocone formula by replacing all such concepts of time, filtrations, adaptedness, etc., with the simple yet powerful idea of the Itô isometry. Wu's framework consolidates many of the other Clark-Ocone-type formulae, including those for the functionals on the Brownian sheets and the above resolution method. His concept of subspaces of the isometries is an important ingredient of the basic framework where our generalised Clark-Ocone formulae are derived.

The organisation of the thesis is as follows. After a quick review of the basic definitions and some well-known results in this chapter, we proceed to explore, in Chapter 2, the use of filtrations induced by a random (path-dependent) resolution of the identity, extending the existing results on non-random resolutions of the identity in [57; 61]. We show that the characterisations of measurability and adaptedness by

Üstünel and Zakai [61] remain valid on the path spaces (Theorems 2.2.10 and 2.2.11), and the whole construction fits naturally with the existing Clark-Ocone formula on the path spaces. Chapter 3 discusses the commutation formula, an essential tool for manipulating the derivative and divergence operators, which is to be used repeatedly in the sequel.

The rest of the thesis is devoted to the study of differential forms and de Rham cohomology groups. In Chapter 4, we develop Clark-Ocone-type formulae for differential forms on the Wiener space (Theorems 4.3.8 and 4.3.14), which lead to the vanishing of the de Rham cohomology classes and establish the Hodge decomposition, as an alternative to Shigekawa's original proof [55]. Although the case of one-forms is similar to that of higher-order forms in the Wiener-space setting, we present the former first and separately in order to delineate the basic arguements, and also to mirror our later result for one-forms on the Riemannian path spaces in Chapter 6. We conclude the chapter with some of our attempts at adapting the same approach to abstract Wiener spaces.

Chapter 5 illustrates a representation-theoretic approach of the subject, leveraging on the Itô-Wiener chaos expansion [36] as a correspondence between the Fock space of symmetric tensors and the L^2 space with respect to the Gaussian measure. The Clark-Ocone proof in Chapter 4 for one-forms can be easily restated in terms of chaos expansion, but it becomes intractable for higher-order forms. The solution is to apply the representation theory of symmetric groups to obtain a direct sum decomposition of certain S_n-invariant subspaces of mixed tensor products (Lemma 5.1.2), which then leads to the trivialty of the L^2 de Rham cohomologies on abstract Wiener spaces (Corollary 5.2.2).

In Chapter 6 we present the generalised Clark-Ocone formula for one-forms on Riemannian path spaces (Proposition 6.2.1), emulating the derivation in Chapter 4. The direct consequence is the spectral gap for Δ^1 and the Hodge decomposition for L^2 $\mathcal{H}$-one forms (Theorem 6.2.6). The progress with higher-order forms is explained, together with some partial results. We also explain another approach, which consists of pulling back the one-forms on the path spaces via the Itô map to the Wiener space, and produces expressions in terms of weak derivatives. This gives an earlier result for one-forms, as announced in [21]. We believe that the two different approaches can complement each other, and possibly work together to resolve the difficulty we have encountered with the higher-order forms.

1.1 Notions and Notation

Throughout the thesis, we try to introduce notation only close to where it is used. Therefore, only basic and global notation is given in this section.

Let (E, H, μ) be an abstract Wiener space, i.e., E is a separable Banach space with the norm $\|\cdot\|_E$, H is a separable Hilbert space that is densely and continuously embedded in E with the inner product $< \cdot, \cdot >_H$ and the norm $\|\cdot\|_H = \sqrt{< \cdot, \cdot >_H}$, and μ is the Wiener measure on E with the characteristic functional given by

$$\int_E \exp\{\sqrt{-1}(l, w)\}\mu(dw) = \exp\{-\frac{1}{2}|l|^2_{H^*}\}, \quad l \in E^* \subseteq H^* = H,$$

where E^* is the dual space of E, H^* is the dual space of H and identified with H by the Riesz representation theorem, and $(\cdot, \cdot)$ denotes the natural bilinear pairing on $E^* \times E$. As an example, we have in the case of the classical Wiener space, for a fixed $T > 0$,

$$E = C_0([0, T]; \mathbb{R}^m) \stackrel{\text{def}}{=} \{\text{continuous functions from [0,T] to } \mathbb{R}^m\text{, starting at } 0\},$$

with its norm $\|\sigma\|_{C_0} = \sup_{t\in[0,T]} \|\sigma(t)\|_{\mathbb{R}^m}$,

$$H = L^{2,1}_0([0, T]; \mathbb{R}^m) \stackrel{\text{def}}{=} \{\text{paths starting at 0 with first distributional derivative in } L^2\},$$

with its inner product

$$\langle h_1, h_2\rangle_{L^{2,1}_0} = \int_0^T \langle \dot{h}_1(t), \dot{h}_2(t)\rangle_{\mathbb{R}^m}\, \mathrm{d}t,$$

where

$$\dot{h}_i(t) = \frac{d}{dt}h_i(t), \quad i = 1, 2,$$

and $\mu = \gamma$, the classical Wiener measure. For ease of typography, we frequently put parameters as subscripts, or suppress them altogether if the context is clear.

Given a real-valued smooth cylindrical function F on E, expressed as

$$F(w) = f((l_1, w), (l_2, w), ...(l_n, w)), \tag{1.1}$$

where $w \in E$, $n \in \mathbb{N}$, $l_1, ..., l_n \in E^*$, and $f : \mathbb{R}^m \to \mathbb{R}$ is C^∞ with compact support, we define the gradient, ∇F, of F by

$$< \nabla F(w), h >_H = DF(w)(h), \quad w \in E, h \in H, \tag{1.2}$$

where $DF : E \to H^*$ is the H-directional derivative, i.e.,

$$DF(w)(h) = \lim_{\epsilon \to 0} \frac{F(w + \epsilon h) - F(w)}{\epsilon}.$$

We regard $DF(w)$ as an element of $\mathcal{L}_2(H; \mathbb{R})$, the space of $\mathbb{R}$-valued Hilbert-Schmidt operators on H with the Hilbert-Schmidt norm $|A|_{HS}^2 = \sum_{i=1}^{\infty} |A(e_i)|^2 < \infty$, where $\{e_i\}$ is some (=any) orthonormal basis of H. We also write $\nabla_h F$ for $< \nabla F, h >_H$.

More generally, for any separable Hilbert space X, we have X-valued cylindrical functions of the form

$$F(w) = \sum_{i=1}^{n} F_i(w) x_i, \quad w \in E,$$

where $x_i \in X$, and F_i are real-valued cylindrical functions of the form (1.1) for $i = 1$ to n, $n \in \mathbb{N}$. In the same way as in (1.2) we define the gradient $\nabla F : E \to X \otimes H$, and we have the H-derivative $DF : E \to \mathcal{L}_2(H; X)$, where $\mathcal{L}_2(H; X)$ is the space of Hilbert-Schmidt operators from H to X.

The quasi-invariance of the Wiener measure (Cameron-Martin Theorem) suggests that H-derivatives are the more natural object to study than the usual Fréchet-derivatives. By the integration by parts formula, ∇F is closable on $L^p(\mu)$ for all $1 \le p < \infty$, and can be iterated to define higher powers ∇^k, $k \in \mathbb{N}$. So we have $D^k F \in \mathcal{L}_2^k(H; X)$, the space of all k-linear operators of Hilbert-Schmidt class from $H^k = \underbrace{H \times \cdots \times H}_{k \text{ times}}$ to X. The Sobolev spaces $\mathbb{D}^{p,k}(X)$ are defined as the completions of the cylindrical functions using the H-derivatives and Hilbert-Schmidt norms. For $k = 0$, we put $\mathbb{D}^{p,0}(X) = L^p(E; X)$.

The dual space of $\mathbb{D}^{p,k}(X)$ for $p > 1$ is denoted $\mathbb{D}^{q,-k}(X)$, where $1/p + 1/q = 1$. Set

$$\mathbb{D}^{p,\infty}(X) = \bigcap_{k \in \mathbb{Z}} \mathbb{D}^{p,k}(X), \quad \mathbb{D}^{p,-\infty}(X) = \bigcup_{k \in \mathbb{Z}} \mathbb{D}^{p,k}(X),$$

$$\mathbb{D}^{\infty}(X) = \bigcap_{p > 1} \mathbb{D}^{p,\infty}(X), \quad \text{and} \quad \mathbb{D}^{-\infty}(X) = \bigcup_{p > 1} \mathbb{D}^{p,-\infty}(X).$$

We call $\mathbb{D}^{-\infty}(X)$ the space of X-valued generalised Wiener functionals, or X-valued distributions. Where $X = \mathbb{R}$, we write simply $\mathbb{D}^{p,k}$, $\mathbb{D}^{\infty}$, $\mathbb{D}^{-\infty}$, etc.

The adjoint of the gradient operator, denoted by δ, is the Skorohod integral and can be interpreted as a divergence; we write sometimes $\delta = -\text{div}$. Its L^2 domain $\text{Dom}(\delta)$ is the set of $u \in L^2(E; H)$ such that for all $F \in \mathbb{D}^{2,1}$

$$|\mathbb{E} < \nabla F(w), u >_H | \le c \|F\|_{L^2},$$

where c is a constant depending on u. If $u \in \mathrm{Dom}(\delta)$, we have $\delta(u)$ as the element of $L^2(E)$ characterised by

$$\mathbb{E}F(w)\delta(u) = \mathbb{E}DF(w)(u) = \mathbb{E} < \nabla F(w), u >_H, \quad F \in \mathbb{D}^{2,1}.$$

Corresponding to the H-differentiability, it is natural to study the H-one-forms (one forms defined as H^*-valued functions on E), rather than the usual one forms (as E^*-valued functions on E). More generally, we define H-n-forms as follows. For $n \in \mathbb{N}$, let the alternating map $A^n : \mathcal{L}_2^n(H;\mathbb{R}) \to \mathcal{L}_2^n(H;\mathbb{R})$ be defined by

$$A^n\phi(h_1, \ldots, h_n) = \frac{1}{n!} \sum_{p \in S_n} \mathrm{sgn}(p)\phi(h_{p(1)}, \ldots, h_{p(n)}), \tag{1.3}$$

where $\phi \in \mathcal{L}_2^n(H;\mathbb{R})$, $h_1, \ldots, h_n \in H$, and the summation is over all $n!$ elements of the symmetric group S_n, the set of all permutations of $\{1, \ldots, n\}$. We call an element $\phi \in \mathcal{L}_2^n(H;\mathbb{R})$ *alternating* if $A^n\phi = \phi$, and denote the the set of all alternating elements of $\mathcal{L}_2^n(H;\mathbb{R})$ by $\mathcal{A}\mathcal{L}_2^n(H;\mathbb{R})$, which is a closed subspace of $\mathcal{L}_2^n(H;\mathbb{R})$. Setting $\Lambda_{p,\infty}^n(E) = \mathbb{D}^{p,\infty}(\mathcal{A}\mathcal{L}_2^n(H;\mathbb{R}))$, we call an element of $\Lambda_{p,\infty}^n(E)$ a differential form of degree n, or an H-n-form for short. Similarly, we have $\Lambda_p^n(E) = L^p(E; \mathcal{A}\mathcal{L}_2^n(H;\mathbb{R}))$, the L^p H-n-forms of degree n.

The exterior derivative $d^n : \mathrm{Dom}(d^n) \subset \Lambda_p^n(E) \to \Lambda_p^{n+1}(E)$, $n \in \mathbb{N} \cup \{0\}$, is defined by

$$d^n\phi(w) = A^{n+1}(D\phi(w)),$$

with $\mathrm{Dom}(d^n)$ obtained by taking the completion of cylindrical forms. The dual operator $d^{n*} = D^*$, as $(A^{n+1})^*$ is just the inclusion map $\mathcal{A}\mathcal{L}_2^n(H;\mathbb{R}) \to \mathcal{L}_2^n(H;\mathbb{R})$. In what follows, we denote the closure of the closable operators ∇ and d^n by the same symbols.

1.2 Classical Results

On the classical Wiener space $C_0 = C_0([0,T];\mathbb{R}^m)$, there is the natural filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ generated by the Brownian motion on $\mathbb{R}^m$, denoted by $B_t : C_0 \to \mathbb{R}^m$, with $B_t(\sigma) = \sigma(t)$ for any $\sigma \in C_0$ and $t \in [0,T]$. We assume that all the sigma-algebras on the Wiener space are completed with respect to γ.

The integral representation theory states that any square integrable functional of the Brownian motion can be represented as

$$F = \mathbb{E}F + \int_0^T < u_t, dB_t >_{\mathbb{R}^m},$$

where u is a process adapted to $\mathcal{F}_\cdot$, i.e., u_t is $\mathcal{F}_t$-measurable for all $t \in [0, T]$ and $\mathbb{E}\int_0^T |u_t|^2 dt < \infty$.

Clark [6] gave the following explicit expression of u for F Fréchet-differentiable

$$u_t = \mathbb{E}[\lambda_F((t, 1], \cdot)|\mathcal{F}_t],$$

where $\lambda_F(s, w)$ is the measure induced via the Riesz representation by the Fréchet derivative of F at the point $w \in C_0$, which is a continuous linear operator on C_0. Ocone [52] generalised this result to $F \in \mathbb{D}^{2,1}$ to obtain

$$u_t = \mathbb{E}[\frac{d}{dt}(\nabla F)_t|\mathcal{F}_t]. \tag{1.4}$$

Karatzas, Ocone and Li [40] extend it further to $F \in \mathbb{D}^{1,1}$.

One direct consequence of (1.4) is that the operator ∇ has a closed range. Indeed, for a sequence of functions $F_i \in \mathbb{D}^{2,1}$, $i \in \mathbb{N}$, such that $\mathbb{E}F_i = 0$ and $\nabla F_i \to v$ in L^2 as $i \to \infty$, we can define a function $F = \int_0^T < \mathbb{E}(\frac{d}{dt}v_t|\mathcal{F}_t), dB_t >_{\mathbb{R}^m}$, therefore

$$F_i = \int_0^T < \mathbb{E}[\frac{d}{dt}(\nabla F_i)_t|\mathcal{F}_t], dB_t >_{\mathbb{R}^m} \quad \to \quad \int_0^T < \mathbb{E}(\frac{d}{dt}v_t|\mathcal{F}_t), dB_t >_{\mathbb{R}^m} = F.$$

Since ∇ is closed, we see $F \in \mathbb{D}^{2,1}$ and $v = \nabla F$.

From this, we deduce immediately that the Laplacian $\Delta = \delta\nabla$ on the Wiener space has a spectral gap, based on a result by Donnelly [14] (that the existence of the spectral gap of Δ is equivalent to its having a closed range). Alternatively, we can take the L^2 norm of $|F - \mathbb{E}F|$ and use the Itô isometry to deduce the spectral gap inequality (see [30; 5]):

$$\begin{aligned}
\mathbb{E}|F - \mathbb{E}F|^2 &= \mathbb{E}\left| \int_0^T < \mathbb{E}[\frac{d}{dt}(\nabla F)_t|\mathcal{F}_t], dB_t >_{\mathbb{R}^m} \right|^2 \\
&= \mathbb{E}\int_0^T |\mathbb{E}[\frac{d}{dt}(\nabla F)_t|\mathcal{F}_t]|^2 dt \\
&\leq \mathbb{E}\int_0^T |\frac{d}{dt}(\nabla F)_t|^2 dt = \mathbb{E}|\nabla F|_H^2.
\end{aligned}$$

Further implications include the logarithmic Sobolev inequality and the isoperimetric inequality; see [5] for details.

Of the many existing generalisations of the Clark-Ocone formula, we mention here only a few that relate to the problems we are interested in. Üstünel [58] generalised the Clark-Ocone representation to distributions on the Wiener space, using his extended Itô integral. See [64; 1] for further development in this direction. On a different note, Nualart and Pardoux [48] showed that, for $F \in \mathbb{D}^{2,1}$,

$$F = \mathbb{E}(F|\mathcal{F}_s \vee \mathcal{F}^t) + \int_s^t < \mathbb{E}[\frac{d}{dr}(\nabla F)_r | \mathcal{F}_r \vee \mathcal{F}^t], dB_r >_{\mathbb{R}^m}, \quad \forall 0 \leq s \leq t \leq T, \tag{1.5}$$

where $\mathcal{F}^t = \sigma\{B_T - B_r, r \geq t\}$ and $\mathcal{F}_s \vee \mathcal{F}^t$ denotes $\sigma(\mathcal{F}_s, \mathcal{F}^t)$. This conditioned formulation reduces to Ocone's version if we take $s = 0$ and $t = T$.

1.3 Resolutions of the Identity

On a general abstract Wiener space (E, H, μ), there exists no intrinsic temporal structure, nor any canonically defined filtration to which we could associate the notions of measurability and adaptedness. Üstünel [57] suggested the construction of a filtration through a continuous resolution of the identity on the Cameron-Martin space H, which gives meaning to measurability and adaptedness, allows for an analogue of the integral representation, and provides a generalisation of the Clark-Ocone formula. This idea ties in with his earlier work on distributions in [58].

The same idea of using a resolution of the identity is explored later by Wu [63] and Üstünel and Zakai [61]. We review this technique in this section, with a view to extend it to more general cases in Chapter 2. As usual, fix $T > 0$. For brevity we write $s \vee t = \max(s, t)$, and $s \wedge t = \min(s, t)$.

Definition 1.3.1. A family of orthogonal projections $\pi = \{\pi_t\}_{t \in [0,T]}$ on H is called a continuous resolution of the identity of H if it satisfies

$$\pi_t \pi_s = \pi_{t \wedge s}, \quad \lim_{t \to s} \pi_t = \pi_s, \quad \lim_{t \downarrow 0} \pi_t = 0, \quad \text{and} \quad \lim_{t \uparrow T} \pi_t = \mathrm{Id}_H, \quad \forall t, s \in [0, T],$$

where the limits here are taken in the strong sense.

A filtration can be now defined by setting

$$\mathcal{F}_t^\pi = \sigma\{\delta(\pi_t h),\, h \in H\}, \tag{1.6}$$

completed with respect to the measure μ. Üstünel [57] showed that $\mathcal{F}_T^\pi$ coincides with the Borel sigma-algebra on E. An element u of $L^2(E; H)$ is called adapted if $< u, \pi_t h >_H$ is $\mathcal{F}_t^\pi$-measurable for all $h \in H$, $t \in [0, T]$. The space of L^2 functions is shown to consist of constants and the image of all such adapted elements of $L^2(E;H)$ under the map δ ([57] Theorem 1), thus furnishing the abstract Wiener space with an analogue of the integral representation on the classical Wiener space.

In this setting, Üstünel and Zakai [61] characterised the $\mathcal{F}_t^\pi$-measurability of a random variable $F \in \mathbb{D}^{2,1}$ by

$$\nabla F = \pi_t \nabla F, \quad a.s., \tag{1.7}$$

and the adaptedness of $u \in \mathbb{D}^{2,1}(H)$ by

$$\pi_t \nabla u = \pi_t \nabla u \pi_t, \quad a.s., \forall t \in [0,T], \tag{1.8}$$

i.e., $< \nabla < u, \pi_t k >_H, h >_H = < \nabla < u, \pi_t k >_H, \pi_t h >_H$, a.s., $t \in [0,T]$, $h,k \in H$. Equivalently, (1.7) can be written as

$$DF(w)(h) = DF(w)(\pi_t h), \quad a.s., \tag{1.9}$$

and (1.8) as

$$\pi_t Du(w)(h) = \pi_t Du(w)(\pi_t h), \quad a.s., \forall t \in [0,T].$$

We postpone the generalisation of the Clark-Ocone formula to the next section, and conclude this section with a few examples of resolutions from [63; 61].

Example 1.3.2. [63; 61] The canonical resolution of the identity, π_t, on the Cameron-Martin space $L_0^{2,1} = L_0^{2,1}([0,T];\mathbb{R}^m)$ of the classical Wiener space is defined by

$$(\pi_t h)_s = \int_0^{t\wedge s} \dot{h}_r dr = \int_0^s \mathbf{1}_{(0,t]}(r)\dot{h}_r dr, \quad h \in L_0^{2,1}, t \text{ and } s \in [0,T]. \tag{1.10}$$

Üstünel [57] has shown that $\{\mathcal{F}_t^\pi\}_{t\in[0,T]}$ coincides with $\{\mathcal{F}_t\}_{t\in[0,T]}$, the Brownian filtration defined in Section 1.2, so the classical case fits into this framework seamlessly. In particular, we note here that the $\mathcal{F}_t$-measurability condition (1.7) is reduced to

$$\frac{d}{ds}(\nabla F)_s = 0, \quad \text{a.e. } s \in (t,T].$$

This captures the essence of what it means by saying a function F is $\mathcal{F}_t$-measurable, i.e., it depends only on the restriction of each sample path to the part over the time interval $[0,t]$. Accordingly, the adaptedness condition (1.8) means

$$\frac{d}{ds}(\nabla \frac{d}{dt} u_t)_s = 0, \quad \text{a.e. } s \in (t,T], \forall t \in [0,T],$$

i.e., a vector field u on C_0 is adapted if the map $\sigma \mapsto u(\sigma)_t$ is $\mathcal{F}_t$-measurable for each t.

Example 1.3.3. [63] A slight variation of the above is to define

$$(\pi_t^b h)_s = \int_0^s \mathbf{1}_{[T-t,T]}(r)\dot{h}_r dr, \quad h \in L_0^{2,1}, t \text{ and } s \in [0,T].$$

Here $\{\mathcal{F}_t^{\pi^b}\}_{t\in[0,T]}$ coincides with the backward Brownian filtration $\{\mathcal{F}_t^b\}_{t\in[0,T]}$, which is generated by $B_t^b = B_T - B_{T-t}$ and completed with respect to γ. This construction leads to the backward Itô integral, as well as the following version of the Clark-Ocone formula

$$F = \mathbb{E}F + \int_0^T < \mathbb{E}[\frac{d}{dt}(\nabla F)_t|\mathcal{F}_t^b], dB_t^b >_{\mathbb{R}^m}, \quad F \in \mathbb{D}^{2,1}. \tag{1.11}$$

In particular, the measurability condition (1.7) is reduced to

$$\frac{d}{ds}(\nabla F)_s = 0, \qquad \text{a.e. } s \in [0, T-t),$$

i.e., a function F is $\mathcal{F}_t^b$-measurable iff it depends only on the restriction of each path to the interval $[T-t, T]$. Similarly, the adaptedness condition (1.8) means simply

$$\frac{d}{ds}(\nabla \frac{d}{dt}u_t)_s = 0, \qquad \text{a.e. } s \in [0, T-t), \forall t \in [0,T].$$

This construction shows directly that Skorohod integral reduces to the backward Itô integral for u adapted to $\mathcal{F}_\cdot^b$, pointed out earlier by Nualart [46].

Example 1.3.4. [63] A combination of the above two examples gives a resolution

$$(\pi_t^m h)_s = \int_0^s \mathbf{1}_{[0,t/2]\cup[T-t/2,T]}(r)\dot{h}_r dr, \quad h \in L_0^{2,1}, t \text{ and } s \in [0,T].$$

Here $\{\mathcal{F}_t^{\pi^m}\}_{t\in[0,T]}$ coincides with the filtration $\{\mathcal{F}_t^m\}_{t\in[0,T]}$, where $\mathcal{F}_t^m = \mathcal{F}_{t/2} \vee \mathcal{F}_{t/2}^b$. We can define a Brownian motion B^m by $dB_t^m = dB_{t/2} + dB_{t/2}^b$ with respect to this filtration, and the Clark-Ocone formula now takes the form of

$$F = \mathbb{E}F + \int_0^T < \mathbb{E}[\frac{d}{dt}(\nabla F)_t|\mathcal{F}_t^m], dB_t^m >_{\mathbb{R}^m}, \quad F \in \mathbb{D}^{2,1}. \tag{1.12}$$

This gives what Üstünel and Zakai [61] call a resolution of rank two, the rank of the resolution being the smallest of the dimensions of the reproducing subspaces of H, which equals the cardinality of the independent Wiener processes that generate the filtration induced by the resolution of the identity. It is worth noting that changing the resolution as above has changed the number of the independent Wiener processes generating the induced filtration.

Example 1.3.5. [61] A fancier variation defines the resolution as

$$(\pi_t^f h)_s = \int_0^s \mathbf{1}_{[(T-t)/2,(T+t)/2]}(r)\dot{h}_r dr, \quad h \in L_0^{2,1}, t \text{ and } s \in [0,T].$$

Here the filtration $\{\mathcal{F}_t^{\pi^f}\}_{t\in[0,T]} = \{\mathcal{F}_t^f\}_{t\in[0,T]}$, which is defined by

$$\mathcal{F}_t^f = \sigma\{B_{(T+r)/2} - B_{(T-r)/2}, r \in [0,t]\},$$

completed with respect to γ. With respect to this new filtration we have a Brownian motion defined by $B_t^f = B_{(T+t)/2} - B_{(T-t)/2}$, and a corresponding Clark-Ocone formula

$$F = \mathbb{E}F + \int_0^T < \mathbb{E}[\frac{d}{dt}(\nabla F)_t|\mathcal{F}_t^f], dB_t^f >_{\mathbb{R}^m}, \quad F \in \mathbb{D}^{2,1}. \tag{1.13}$$

This is again a rank-two resolution.

The above examples might seem trivial as they are defined on the classical Wiener space where a natural temporal structure already exists, as do filtrations. For an abstract Wiener space, we have the following natural candidate.

Example 1.3.6. [61] Let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal basis of H, and P_n the projection of H onto the subspace spanned by $\{e_1, \cdots, e_n\}$. Define

$$\pi_t^o = P_{\lfloor \tan(\pi t/2T)\rfloor}, t \in [0,T],$$

where $\lfloor s \rfloor$ denotes the largest integer not exceeding s. This resolution, unlike the previous examples, is only right-continuous, and not continuous, so it does not, strictly speaking, fit the definition 1.3.1. It is not clear what the Brownian motion with respect to this filtration might look like, or how we can formulate the Clark-Ocone formula in this case. An explanation is given in the next section.

1.4 Maximal Subspaces of Isometries

In order to develop an intrinsic Clark-Ocone formula without the use of filtrations, Wu [63] considered replacing adaptedness by the projection onto a maximal subspace of all L^2 processes that preserve the Itô isometry.

Following Wu's notation, we define the collection of isometries under δ as

$$IM(\delta) = \{u \in \text{Dom}(\delta) \subset L^2(E;H) : \mathbb{E}(\delta u)^2 = \mathbb{E}|u|_H^2\},$$

and let V be a linear subspace of $L^2(E;H)$ contained in $IM(\delta)$. The necessary condition for a Clark-Ocone representation is now equivalent to the condition on V that every zero-mean L^2 random variable Y can be represented as $Y = \delta u$, for some $u \in V$, i.e., an integral representation theorem.

Note that $IM(\delta)$ is not a vector space. Define the collection

$$\Phi = \{V : V \subset IM(\delta) \text{ and is a vector subspace of } L^2(E;H)\}, \tag{1.14}$$

which can be partially ordered by the inclusion map, so by Zorn's lemma any element of Φ is contained in some maximal element of Φ. Denote by $L_0^2(E)$ the set of all elements of $L^2(E)$ with zero expectation.

Since δ is a closed operator, the image under δ of any closed subspace V of $L^2(E;H)$ contained in $IM(\delta)$ is also a closed subspace of $L_0^2(E)$. If P_X denotes the projection operator onto a subspace X, Wu [63] showed that

$$P_{\delta(V)}F = \delta(P_V \nabla F), \quad \forall F \in \mathbb{D}^{2,1}, \tag{1.15}$$

which follows easily from

$$\mathbb{E}[\delta(u)\delta(P_V\nabla F)] = \mathbb{E} < u, P_V\nabla F >_H = \mathbb{E} < u, \nabla F >_H = \mathbb{E}[\delta(u)F], \quad \forall u \in V.$$

It is worth noting that the composed operators $P_V D$ and δP_V are dual to each other, hence make themselves good candidates for further study in this set-up. At this stage we do not pursue this direction further, except to mention in passing the work of Cont and Fournie [7; 8], inspired by Dupire [16], on their functional Itô calculus where a certain weak derivative is developed as a non-anticipative version of the H-derivative, not unsimilar to $P_V D$ here.

From (1.15), it is now trivial to derive a Clark-Ocone formula in the form of

$$F = \mathbb{E}(F) + \delta(P_V \nabla F), \quad \forall F \in L^2(E), \tag{1.16}$$

as long as our V satisfies the criterion (an analogue of the integral representation)

$$\delta(V) = L_0^2(E). \tag{1.17}$$

In this formulation, the Clark-Ocone representation is generalised to all square integrable functions since (1.15) itself holds for any $F \in L^2(E)$ in the following sense:

the composed operator $P_V\nabla$ satisfies

$$|(P_V\nabla)(F)|_{L^2(E;H)} = |P_{\delta(V)}(F)|_{L^2(E)} \leq |F|_{L^2(E)}, \tag{1.18}$$

so it extends to a bounded linear operator from $L^2(E)$ to V.

It is then important to study the conditions under which (1.17) holds. A necessary but insufficient condition given in [63] is that V itself is a maximal element of Φ. Interestingly, one way to construct such a maximal subspace of isometries that satisfies (1.17) is through a continuous resolution of the identity π, where V is taken as the closure of the simple adapted elements of $L^2(E;H)$, i.e., $V = \bar{S}$ with

$$S = \left\{ \begin{array}{c} u = \sum_{k=1}^n F_k h_k : n \in \mathbb{N}, h_k \in \pi_{t_{k+1}}(H) \cap \pi_{t_k}(H)^\perp, \\ F_k \in \mathcal{F}_{t_k}, 0 \leq t_1 < \cdots < t_n < t_{n+1} = T \end{array} \right\}. \tag{1.19}$$

Wu [63] showed that (1.17) is equivalent to the condition that $\{\pi_t\}_{t\in[0,T]}$ is continuous (rather than merely *right-continuous* as initially formulated in his paper, following the classical definition from functional analysis as given by Yosida [65]). This means, the contruction in Section 1.3 leads to the Clark-Ocone formula of the form (1.16).

Applying the criterion (1.17) to the examples from the previous section makes an illuminating exercise.

Example 1.3.2. The canonical resolution π_t is clearly continuous. In this case,

$$F = \mathbb{E}(F) + \delta(P_V\nabla F) = \mathbb{E}(F) + \int_0^T < \frac{d}{dt}(P_V\nabla F)_t, dB_t >_{\mathbb{R}^m}, \quad \forall F \in L^2(C_0),$$

where the last term is an Itô integral, and we use the fact that $P_V\nabla$ extends to an operator on $L^2(E)$, as discussed above. If $F \in \mathbb{D}^{2,1}$, this reduces to the classical Clark-Ocone formula (1.4) with

$$P_V\nabla F = \int_0^\cdot \mathbb{E}(\frac{d}{dt}(\nabla F)_t|\mathcal{F}_t)dt.$$

The map P_V here is the projection onto the subspace V of the usual adapted processes inside $L^2(C_0; L_0^{2,1})$, defined by

$$(P_V u)_t = \int_0^t \mathbb{E}[\frac{d}{ds}u_s|\mathcal{F}_s]ds, \quad u \in L^2(C_0; L_0^{2,1}), \tag{1.20}$$

and is a continuous map from $\mathbb{D}^{2,1}(L_0^{2,1})$ to $\mathbb{D}^{2,1}$ (see [58]).

Example 1.3.3. The resolution π_t^b is again continuous, and gives us the Clark-

Ocone formula in terms of the backward Itô integral with respect to the backward Brownian motion B_t^b and backward filtration $\mathcal{F}_t^b$, just as in equation (1.11).

Similarly, the resolution π_t^m leads to (1.12), and π_t^f leads to (1.13), which we do not detail here.

Example 1.3.6. The resolution π_t^o is not continuous, but only right-continuous, therefore condition (1.17) is not satisfied, and we do not have a Clark-Ocone formula.

Chapter 2

Random Resolutions of the Identity

In this chapter, we extend the results of Üstünel and Zakai [61] on non-random resolutions of the identity (see Section 1.3) to certain random resolutions. Such a generalisation is natural and necessary for applying their technique to the study of the based path space on a smooth compact Riemannian manifold, where the tangent spaces themselves are random (path-dependent). This special case is our primary example of study in this chapter.

We start by introducing the basic set-up on Riemannian path spaces. We then verify the validity of Üstünel and Zakai's characterisations of measurability and adaptedness in path spaces, and show that the existing Clark-Ocone formula on the path spaces fits into this framework naturally.

Throughout this thesis, we focus on the non-degenerate case using the Levi-Civita connection and the Brownian motion measure. Although greater generality is possible in terms of the choice of the connection and the underlying diffusion measure on the path spaces, it does not seem worth pursuing at this stage. This is mainly to avoid introducing additional notation, and we do not foresee any difficulty in extending the current results to the case of more general (possibly degenerate) diffusion measures and other connections on the tangent bundle (in the case of degenerate diffusions on the manifold, it would be connections on some subbundle). It is, however, worth mentioning that even on the classical Wiener space, there are different possible choices of metric connections on $\mathbb{R}^n$, and each of them gives rise to a different tangent bundle structure and a different Clark-Ocone formula, which deserves to be compared and perhaps unified in a general framework. Therefore, random resolutions on the flat spaces are already interesting subjects of study.

2.1 Notions and Notation

Given an n-dimensional smooth compact Riemannian manifold M, we fix $x_0 \in M$ and $T > 0$, and denote by $C_{x_0}M = C_{x_0}([0,T];M)$ the space of continuous paths starting at x_0, with its usual C^∞ Banach manifold structure inherited from that of M (see [18; 19]) and the Brownian motion measure μ_{x_0}. The standard filtration $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$ on $C_{x_0}M$ is the one induced by the evaluation map, i.e., denoting by $ev_s : C_{x_0}M \to M$ the evaluation map at time s, we have $ev_s(\sigma) = \sigma_s$ for $\sigma \in C_{x_0}M$, and

$$\mathcal{F}_t^{x_0} = \sigma\{ev_s : C_{x_0}M \to M, s \in [0,t]\}, \tag{2.1}$$

completed with respect to μ_{x_0}. The tangent space $T_\sigma C_{x_0}M$ to $C_{x_0}M$ at a path σ is given by

$$T_\sigma C_{x_0}M \overset{\text{def}}{=} \{v \in C([0,T];TM) : v(0) = 0,\ v(t) \in T_{\sigma_t}M, \forall t \in [0,T]\},$$

with the uniform norm induced by the Riemannian metric of M. We also have the L^2 tangent space at σ,

$$L^2T_\sigma C_{x_0}M \overset{\text{def}}{=} \{v \in L^0([0,T];TM) : (/\!/_\cdot^\sigma)^{-1}v_\cdot \in L^2([0,T];T_{x_0}M)\},$$

with its inner product

$$< u, v >_{L^2_\sigma} = \int_0^T < u_t, v_t >_{\sigma_t} dt,$$

where $L^0([0,T];TM)$ stands for the space of measurable functions from $[0,T]$ to TM, and $/\!/_t^\sigma$ denotes the stochastic parallel translation of $T_{x_0}M$ to $T_{\sigma_t}M$ using the Levi-Civita connection on M. These L^2 tangent spaces form the fibres of a smooth Hilbert bundle $L^2TC_{x_0}M$ over $C_{x_0}M$; see [25] for a more detailed description.

The analogue of the Cameron-Martin space $H = L_0^{2,1}$ is the Bismut tangent space, defined for almost all $\sigma \in C_{x_0}M$ by

$$\mathcal{H}_\sigma = \{v \in T_\sigma C_{x_0}M : (/\!/_\cdot^\sigma)^{-1}v_\cdot \in L_0^{2,1}(T_{x_0}M)\}, \tag{2.2}$$

where $L_0^{2,1}(T_{x_0}M) = L_0^{2,1}([0,T];T_{x_0}M)$ refers to the finite energy paths in the tangent space to M at the base point x_0. Note that $\mathcal{H}_\sigma$ is a Hilbert space whose inner product is given by

$$< u, v >_{\tilde{\mathcal{H}}_\sigma} = \int_0^T < \frac{D}{dt}u_t, \frac{D}{dt}v_t >_{\sigma_t} dt, \tag{2.3}$$

where the operator $\frac{D}{dt}$ on vector fields along σ is defined by

$$\frac{D}{dt}u_t = /\!\!/_t^\sigma \frac{d}{dt}[(/\!\!/_t^\sigma)^{-1}u_t]. \tag{2.4}$$

We can, and will, from now on, endow $\mathcal{H}_\sigma$ with a different inner product

$$< u, v >_{\mathcal{H}_\sigma} = \int_0^T < \frac{\mathbb{D}}{dt}u_t, \frac{\mathbb{D}}{dt}v_t >_{\sigma_t} dt, \tag{2.5}$$

where the operator $\frac{\mathbb{D}}{dt}$ is defined similarly to (2.4), but using the damped version of $/\!\!/_t^\sigma$, so as to take into account of the effect of the Ricci curvature (see [25; 26] for more details). More precisely, the damped parallel translation $W_t : T_{x_0}M \to T_{\sigma_t}M$ is given by the solution of

$$\frac{D}{dt}W_t(V) = -\frac{1}{2}\mathrm{Ric}^\sharp_{\sigma_t}(W_t(V)), \quad 0 \le t \le T,$$

where $\mathrm{Ric}^\sharp : TM \to TM$ is defined by the Ricci curvature Ric corresponding to the connection ∇:

$$< \mathrm{Ric}^\sharp(u), v >_x = \mathrm{Ric}_x(u, v) = \mathrm{Trace} < \mathcal{R}(u, -)-, v >_x, \quad \forall x \in M,$$

and $\mathcal{R}$ is the Riemann curvature tensor. We write

$$\frac{\mathbb{D}}{dt}u_t = W_t \frac{d}{dt}(W_t^{-1}u_t),$$

and note its relationship with (2.4) through

$$\frac{\mathbb{D}}{ds} = \frac{D}{ds} + \frac{1}{2}\mathrm{Ric}^\sharp. \tag{2.6}$$

The operator $\frac{\mathbb{D}}{dt}$ determines an isometry between the Hilbert bundles $\mathcal{H} = \coprod \mathcal{H}_\sigma$ and $L^2TC_{x_0}M$, at least over the subset on which $\frac{\mathbb{D}}{dt}$ is defined, a subset of full measure in $C_{x_0}M$. Its inverse is denoted $\boldsymbol{W} : L^2TC_{x_0}M \to \mathcal{H}$, given by

$$\boldsymbol{W}_t(h) = W_t \int_0^t W_s^{-1}h_s ds, \quad h \in L^2T_\sigma C_{x_0}M.$$

As in the case of the classical Wiener space, differentiation on $C_{x_0}M$ should be restricted to be along the admissible directions of some special Hilbert space, here the Bismut tangent space. We take as initial domain the smooth cylindrical functions, and note that $D_\mathcal{H} : \mathrm{Dom}(D_\mathcal{H}) \subset L^2(C_{x_0}M) \to L^2\Gamma\mathcal{H}^*$ is closable (see

[15] or [25]). We denote its closure by the same symbol $D_{\mathcal{H}}$, and its domain by $\mathbb{D}^{2,1}$ equipped with the graph norm. As in Section 1.1, for functions with values in a separable Hilbert space K, we write $\mathbb{D}^{2,1}(K)$ as a shorthand for $\mathbb{D}^{2,1}(C_{x_0}M;K)$, and omit K when $K=\mathbb{R}$. We also have the gradient operator $\nabla_{\mathcal{H}}:\mathbb{D}^{2,1}\to L^2\Gamma\mathcal{H}$, and the divergence operator $\text{div}=-\nabla_{\mathcal{H}}^*:\text{Dom}(\text{div})\subset L^2\Gamma\mathcal{H}\to L^2(C_{x_0}M;\mathbb{R})$, such that, for $f\in\mathbb{D}^{2,1}$ and $V\in\text{Dom}(\text{div})$,

$$\mathbb{E}D_{\mathcal{H}}f(\sigma)(V(\sigma))=\mathbb{E}<\nabla_{\mathcal{H}}f(\sigma),V(\sigma)>_{\mathcal{H}_\sigma}=-\mathbb{E}f(\sigma)\text{div}(V)(\sigma).$$

Note that $\nabla_{\mathcal{H}}$ corresponds to the damped inner product (2.5). We also have a different gradient operator $\tilde{\nabla}_{\mathcal{H}}$ using the undamped inner product (2.3), so

$$D_{\mathcal{H}}f(\sigma)(V(\sigma))=<\nabla_{\mathcal{H}}f(\sigma),V(\sigma)>_{\mathcal{H}_\sigma}=<\tilde{\nabla}_{\mathcal{H}}f(\sigma),V(\sigma)>_{\tilde{\mathcal{H}}_\sigma}.$$

For brevity we may suppress the subscript $\mathcal{H}$ when the context is clear.

Let $\{x_t\}_{t\in[0,T]}\subset M$ be the solution, starting at x_0, to a Stratonovich stochastic differential equation (SDE)

$$dx_t=X(x_t)\circ dB_t, \tag{2.7}$$

where $B_t:C_0\to\mathbb{R}^m$ is the canonical Brownian motion on $\mathbb{R}^m$, the vector bundle morphism $X:M\times\mathbb{R}^m\to TM$ gives surjective linear maps $X(x):\mathbb{R}^m\to T_xM\cong\mathbb{R}^n$, and is smooth in $x\in M$. Obviously, we have $m\geq n$.

The surjectivity of X implies that we can define an inner product $<,>_x$ on T_xM by

$$<X(x)e_1,X(x)e_2>_x=<e_1,e_2>_{\mathbb{R}^m},\quad \forall e_1,e_2\in\text{Ker}(X(x))^\perp,$$

where $\text{Ker}(X(x))^\perp$ is the orthogonal complement of the kernel of $X(x)$ in $\mathbb{R}^m$. Therefore X induces a Riemannian metric on M.

For this metric, X also induces a metric connection, the LeJan-Watanabe connection $\breve{\nabla}$ (see, for example, [22]), which is defined, for any smooth vector field U on M, by

$$\breve{\nabla}_vU=X(x)D[x\mapsto Y_xU(x)](v),\quad \forall x\in M,v\in T_xM,$$

where $Y_x:T_xM\to\mathbb{R}^m$ is the adjoint of $X(x)$. Note that $X(x)Y_x(v)=v$, for all

$v \in T_x M$. The torsion of this connection is given by

$$\breve{T}(u,v) = X(x)dY(u,v), \quad u, v \in T_x M.$$

where dY is the exterior derivative of Y, considered as an $\mathbb{R}^m$-valued differential one-form on M. If $\breve{\nabla}$ is the Levi-Civita connection ∇ on M, the SDE (2.7) induces the Brownian motion measure μ_{x_0} on M, as the law of its solution starting from x_0.

The following property is essential ([22] Proposition 1.1.1):

$$\breve{\nabla}_v X(x)(e) = 0, \quad \forall v \in T_x M, e \in \mathrm{Ker}(X(x))^{\perp}. \tag{2.8}$$

With the discussion above in mind, we make the following standing assumption throughout this thesis.

Assumption 2.1.1. X induces the Riemannian metric and the Levi-Civita connection ∇ on M. Hence (2.8) holds for ∇.

The Itô map $\mathcal{I} : C_0 \to C_{x_0}M$ of the SDE (2.7) is given by

$$\mathcal{I}_t(w) = \xi_t(x_0, w) = x_t(w),$$

where $\{\xi_t\}_{t\in[0,T]}$ is the solution flow, and $\{x_t\}_{t\in[0,T]}$ is the solution to (2.7) starting at x_0. The Itô map is a measurable measure-preserving map between $(C_0, \mathcal{F}, \gamma)$ and $(C_{x_0}M, \mathcal{F}^{x_0}, \mu_{x_0})$, with $\mathcal{I}_*\gamma = \mu_{x_0}$. The filtration generated by $\mathcal{I}$ is denoted by $\{\mathcal{F}_t^{\mathcal{I}}\}_{t\in[0,T]}$.

Bismut [4] gives the following formula for the H-derivative of $\mathcal{I}_t$,

$$T\mathcal{I}(h)_t = T_{x_0}\xi_t \int_0^t (T_{x_0}\xi_s)^{-1} X(x_s)\dot{h}_s ds, \quad t \in [0,T], h \in H, \tag{2.9}$$

where $T_{x_0}\xi_t : T_{x_0}M \to T_{x_t}M$ is the derivative of ξ_t at x_0. This shows that the H-derivative of $\mathcal{I}$ is a continuous linear map $T_w\mathcal{I} : H \to T_{x_\cdot}C_{x_0}M$ for almost all $w \in C_0$.

For almost all $\sigma \in C_{x_0}M$ and $h \in H$, we define

$$\overline{T\mathcal{I}}_\sigma(h) = \mathbb{E}[T_w\mathcal{I}(h)|\mathcal{I}(w) = \sigma].$$

In general, we denote by $\overline{f}(\sigma)$ the conditional expectation of a function f on C_0 given $\mathcal{I} = \sigma$, which gives a function on $C_{x_0}M$ obtained by factorisation. For a discussion of the conditional expectation of vector bundle valued processes, see [20; 22].

If the connection defined by the SDE (2.7) is the same as the one defining $\mathcal{H}$

and its inner product, as is the case here, the map $\overline{T\mathcal{I}}_\sigma : H \to \mathcal{H}_\sigma$ gives a projection for almost all $\sigma \in C_{x_0}M$, defined by

$$\overline{T\mathcal{I}}_\sigma(h)_t = W_t \int_0^t W_s^{-1} X(\sigma_s)\dot{h}_s ds, \tag{2.10}$$

with an isometric right inverse $v \mapsto \int_0^\cdot Y_{\sigma_s} \frac{\mathbb{D}}{ds} v_s ds$ ([25] Property 3.1).

Relatedly, we have the push-forward map $\overline{T\mathcal{I}(-)}_\sigma$, mapping any L^2 H-vector field h on C_0 to an $\mathcal{H}$-vector field $\overline{T\mathcal{I}(h)}$ on $C_{x_0}M$, given by

$$\overline{T\mathcal{I}(h)}_\sigma = \mathbb{E}\{T_w\mathcal{I}(h(w))|\mathcal{I}(w) = \sigma\}, \quad \text{a.e. } \sigma \in C_{x_0}M.$$

This is a continuous linear map from $L^2(C_0; H)$ to $L^2\Gamma\mathcal{H}$ ([23] Theorem 2.2), and if $h \in \mathcal{F}_T^{x_0}$ with $h = \bar{h} \circ \mathcal{I}$, where $\bar{h}(\sigma) = \mathbb{E}\{h|\mathcal{I} = \sigma\}$, we have

$$\overline{T\mathcal{I}(h)}_\sigma = \overline{T\mathcal{I}}_\sigma(\bar{h}). \tag{2.11}$$

It is remarkable that if the H-vector field h is adapted with respect to $\mathcal{F}_t$ on C_0, we again have (2.11) ([26] Lemma 9.2).

Following [25], we use the map $X : M \times \mathbb{R}^m \to TM$ in the SDE (2.7) to define $\tilde{X} : C_{x_0}M \times L^2([0,T];\mathbb{R}^m) \to L^2TC_{x_0}M$ by

$$(\tilde{X}(\sigma)h)_t = X(\sigma_t)(h_t), \quad \forall \sigma \in C_{x_0}M, t \in [0,T], h \in L^2([0,T];\mathbb{R}^m),$$

and its right inverse, $\tilde{Y}_\sigma : L^2T_\sigma C_{x_0}M \to L^2([0,T];\mathbb{R}^m)$, by

$$\tilde{Y}_\sigma(k)_t = Y_{\sigma_t}(k_t), \quad \forall k \in L^2T_\sigma C_{x_0}M.$$

Also define $\boldsymbol{X} : C_{x_0}M \times H \to \mathcal{H}$ by

$$\boldsymbol{X}(\sigma)(h) = \overline{T\mathcal{I}}_\sigma(h) = \boldsymbol{W}\tilde{X}(\sigma)(\dot{h}), \quad \forall \sigma \in C_{x_0}M, h \in H.$$

with the right inverse $\boldsymbol{Y}_\sigma : \mathcal{H}_\sigma \to H$ given by

$$\boldsymbol{Y}_\sigma(k)_t = \int_0^t Y_{\sigma_s}(\frac{\mathbb{D}}{ds}k_s)ds, \quad \forall k \in \mathcal{H}_\sigma.$$

As assumed, the connection ∇ induced by X is defined for any C^1 vector field V on M by

$$\nabla_v V = X(x)D[x \mapsto Y_x V(x)](v), \quad \forall x \in M, v \in T_xM.$$

It induces on $C_{x_0}M$ the pointwise connection $\tilde{\nabla}$, defined similarly for vector fields $W \in \mathrm{Dom}(\tilde{\nabla}) = \mathbb{D}^{2,1}(L^2TC_{x_0}M)$ by

$$\tilde{\nabla}_u W = \tilde{X}(\sigma)D[\sigma \mapsto \tilde{Y}_\sigma W(\sigma)](u), \quad \forall \sigma \in C_{x_0}M, u \in T_\sigma C_{x_0}M.$$

The pointwise connection is metric for the L^2 metric, and torsion-free if ∇ is chosen to be torsion-free, as is assumed here. We can use the almost surely defined map $\frac{\mathbb{D}}{d.} : \mathcal{H} \to L^2TC_{x_0}M$ to pull back $\tilde{\nabla}$ and obtain a metric connection $\boldsymbol{\nabla}$ on $\mathcal{H}$,

$$\boldsymbol{\nabla} = (\frac{\mathbb{D}}{d.})^{-1}\tilde{\nabla}\frac{\mathbb{D}}{d.}. \tag{2.12}$$

This coincides with the Markovian connection introduced by Cruzeiro and Fang [9]. We can check that for $U \in \mathrm{Dom}(\boldsymbol{\nabla}) = \mathbb{D}^{2,1}(\mathcal{H})$,

$$\boldsymbol{\nabla}_u U = \boldsymbol{X}(\sigma)D[\sigma \mapsto \boldsymbol{Y}_\sigma U(\sigma)](u), \quad \forall \sigma \in C_{x_0}M, u \in T_\sigma C_{x_0}M.$$

Since $X(x)$ is surjective, we have the splitting $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$ with independent Brownian motions $\tilde{B} : [0,T] \times C_0 \to \mathbb{R}^n$ and $\beta : [0,T] \times C_0 \to \mathbb{R}^{m-n}$, as described in [22], such that

$$dB_t = \tilde{/\!/}_t^{\sigma} d\tilde{B}_t + \tilde{/\!/}_t^{\sigma} d\beta_t, \tag{2.13}$$

where $\tilde{B}_.$ and $x_.$ have the same filtration, the map $\tilde{/\!/} : [0,T] \times C_{x_0}M \to O(m)$ is sample continuous and adapted to $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$, with $O(m)$ being the orthogonal group of $\mathbb{R}^m$, such that $\tilde{/\!/}_0^{\sigma} = \mathrm{Id}_{\mathbb{R}^m}$, and the orthogonal transformation $\tilde{/\!/}_t^{\sigma}$ maps $\mathrm{Ker}X(x_0)$ to $\mathrm{Ker}X(x_t)$. Let $K(x)$ be the projection of $\mathbb{R}^m$ onto $\mathrm{Ker}X(x)$, and $K^\perp(x) = \mathrm{Id}_{\mathbb{R}^m} - K(x)$, the orthogonal projection onto the orthogonal complement of $\mathrm{Ker}X(x)$, then we have

$$\tilde{B}_t = \int_0^t (\tilde{/\!/}_s^{\sigma})^{-1}K^\perp(x_s)dB_s,$$

and

$$\beta_t = \int_0^t (\tilde{/\!/}_s^{\sigma})^{-1}K(x_s)dB_s,$$

as Brownian motions on $\mathrm{Ker}(X(x_0))^\perp$ and $\mathrm{Ker}X(x_0)$, respectively.

We can also define

$$\breve{B}_t = \int_0^t (/\!/_s^{\sigma})^{-1}X(x_s)dB_s,$$

which gives the stochastic anti-development of $\{x_t\}_{t\in[0,T]}$ using the given connection. We sometimes write the martingale part of $\int \alpha \circ dx_t$ as $\int \alpha\, d\{x\}_t$, so $/\!\!/_t^\sigma d\breve{B}_t = d\{x\}_t$. Note that $\breve{B}_t = X(x_0)\tilde{B}_t$, so $\breve{B}_\cdot$ generates the same filtration as $\tilde{B}_\cdot$, as well as $x_\cdot$ ([22] Theorem 3.1.2).

For brevity, we sometimes suppress the superscript $^\sigma$ in the parallel translations and write simply $/\!\!/_t$ and $\tilde{/\!\!/}_t$.

2.2 Canonical Resolutions of the Identity on Riemannian Path Spaces

In the classical Wiener space, we have the canonical resolution of the identity $\{\pi_t\}_{t\in[0,T]}$ on the Cameron-Martin space $H = L_0^{2,1}$, as described in Example 1.3.2. Correspondingly, in the Riemannian path space $C_{x_0}M$, we should have a *random* resolution of the identity $\{\pi_t^\sigma\}_{t\in[0,T]}$ for each $\mathcal{H}_\sigma$, random in the sense that it is dependent on the path $\sigma \in C_{x_0}M$ (hence the superscript $^\sigma$). A natural way to define π^σ is

$$(\pi_t^\sigma h)_s = /\!\!/_s^\sigma \int_0^{s\wedge t} \frac{d}{dr}[(/\!\!/_r^\sigma)^{-1} h_r] dr, \quad \forall h \in \mathcal{H}_\sigma,$$

where $/\!\!/_t^\sigma$ is, as in Section 2.1, the stochastic parallel translation along σ. We can also use the damped parallel translation instead to obtain

$$(\boldsymbol{\pi}_t^\sigma h)_s = W_s \int_0^{s\wedge t} \frac{d}{dr}(W_r^{-1} h_r) dr, \quad \forall h \in \mathcal{H}_\sigma.$$

Alternatively, we can define on $L^2T_\sigma C_{x_0}M$ a resolution of the identity $\{\rho_t^\sigma\}_{t\in[0,T]}$ by

$$(\rho_t^\sigma h)_s = \begin{cases} h_s, & 0 \le s \le t, \\ 0, & s > t, \end{cases} \qquad \forall h \in L^2T_\sigma C_{x_0}M,$$

which is related to $\boldsymbol{\pi}^\sigma$ and π^σ through

$$(\rho_t^\sigma h)_s = \frac{\mathbb{D}}{ds}\boldsymbol{\pi}_t^\sigma (\frac{\mathbb{D}}{ds})^{-1} h_s = \frac{D}{ds}\pi_t^\sigma (\frac{D}{ds})^{-1} h_s, \quad \forall h \in L^2T_\sigma C_{x_0}M. \tag{2.14}$$

The resolution ρ^σ has the advantage that $L^2TC_{x_0}M$ is a C^∞ vector bundle, so we can require ρ^σ to be smooth in σ, as is the case here. In what follows, we will switch freely between ρ^σ, π^σ, and $\boldsymbol{\pi}^\sigma$, as ρ^σ tends to make our computations easier, while π^σ and $\boldsymbol{\pi}^\sigma$, being defined on the Bismut tangent space, seem to be the more natural objects to work with. It is also easy to verify that these random resolutions are related to the canonical resolution $\{\pi_t\}_{t\in[0,T]}$ on H (see Example 1.3.2) via

$$\rho_t^\sigma[\tilde{X}(\sigma)(h)]_s = \frac{D}{ds}\pi_t^\sigma(\frac{D}{ds})^{-1}[\tilde{X}(\sigma)(h)]_s = [\tilde{X}(\sigma)(\pi_t h)]_s, \quad h \in H,$$

and

$$\boldsymbol{\pi}_t^\sigma[\boldsymbol{X}(\sigma)h]_s = [\boldsymbol{X}(\sigma)(\pi_t h)]_s, \quad h \in H.$$

To check that ρ^σ is indeed continuous for each σ, we observe that, for $t_1 < t_2$,

$$(\rho_{t_1}^\sigma h - \rho_{t_2}^\sigma h)_s = \begin{cases} h_s, & t_1 < s \le t_2 \\ 0, & s \le t_1 \text{ or } s > t_2 \end{cases} \qquad \forall h \in L^2 T_\sigma C_{x_0} M,$$

and by the monotone convergence theorem

$$\|\rho_{t_1}^\sigma h - \rho_{t_2}^\sigma h\|_{L_\sigma^2}^2 = \int_{t_1}^{t_2} |h_s|_{\sigma_s}^2 ds \to 0, \qquad \text{as } |t_1 - t_2| \to 0.$$

The self-adjointness of ρ^σ is clear, since for all $h, k \in L^2 T_\sigma C_{x_0} M$,

$$< \rho_t^\sigma h, k >_{L_\sigma^2} = \int_0^T < \rho_t^\sigma h_s, k_s >_{\sigma_s} ds = \int_0^T \mathbf{1}_{[0,t]}(s) < h_s, k_s >_{\sigma_s} ds = < h, \rho_t^\sigma k >_{L_\sigma^2},$$

and this extends to π^σ and $\boldsymbol{\pi}^\sigma$ via (2.14). Or, we can calculate directly, say, for π^σ, that given $h, k \in \mathcal{H}_\sigma$,

$$\begin{aligned} < \pi_t^\sigma h, k >_{\mathcal{H}_\sigma} &= \int_0^T < \frac{D}{ds}\pi_t^\sigma h_s, \frac{D}{ds}k_s >_{\sigma_s} ds \\ &= \int_0^T < /\!/_s^\sigma \frac{d}{ds}(/\!/_s^\sigma)^{-1} /\!/_s^\sigma \int_0^{s\wedge t} \frac{d}{dr}[(/\!/_r^\sigma)^{-1} h_r] dr, \frac{D}{ds}k_s >_{\sigma_s} ds \\ &= \int_0^T \mathbf{1}_{[0,t]}(s) < \frac{D}{ds}h_s, \frac{D}{ds}k_s >_{\sigma_s} ds, \end{aligned}$$

which is symmetric in h and k.

All the other conditions of Definition 1.3.1 can be verified similarly, and we conclude that $\{\pi_t^\sigma\}_{t\in[0,T]}$ is indeed a family of orthogonal projections on $\mathcal{H}_\sigma$ (so is $\{\boldsymbol{\pi}^\sigma\}_{t\in[0,T]}$; and also $\{\rho_t^\sigma\}_{t\in[0,T]}$ on $L^2 T_\sigma C_{x_0} M$), satisfying

$$\pi_t^\sigma \pi_s^\sigma = \pi_{t\wedge s}^\sigma, \quad \lim_{t\to s} \pi_t^\sigma = \pi_s^\sigma, \quad \lim_{t\downarrow 0} \pi_t^\sigma = 0, \quad \text{and } \lim_{t\uparrow T} \pi_t^\sigma = \mathrm{Id}_{\mathcal{H}_\sigma}, \quad \forall t, s \in [0, T].$$

2.2.1 Filtration

According to the recipe (1.6), the canonical resolution $\{\pi_t\}_{t\in[0,T]}$ on $H = L_0^{2,1}$, defined in Example 1.3.2, induces a filtration $\{\mathcal{F}_t^\pi\}_{t\in[0,T]}$ on the Wiener space C_0, and $\mathcal{F}_t^\pi = \mathcal{F}_t$ (see [57]). An analogue of (1.6) allows us to define on $C_{x_0}M$ a filtration induced by $\{\boldsymbol{\pi}_t^\sigma\}_{t\in[0,T]}$,

$$\mathcal{G}_t^\pi = \sigma\{\mathrm{div}(\boldsymbol{\pi}_t W h), h \in L_0^{2,1}(T_{x_0}M)\}.$$

As noted in Section 2.1, we have $\mathcal{F}_t^{x_0} = \mathcal{F}_t^{\tilde{B}} = \mathcal{F}_t^{\breve{B}}$.

Lemma 2.2.1. *$\mathcal{F}_t^{x_0} = \mathcal{G}_t^\pi$ for all $t \in [0,T]$.*

Proof. By Corollary 5.2 of [25], we have, for any $t \in [0,T]$,

$$\begin{aligned} \mathrm{div}(\boldsymbol{\pi}_t W h) \circ \mathcal{I} &= -\mathbb{E}[\int_0^T < \frac{\mathbb{D}}{ds}\boldsymbol{\pi}_t^x W_s h_s, X(x_s)dB_s >_{x_s} |\mathcal{F}_T^{x_0}] \\ &= -\int_0^t < W_s \dot{h}_s, X(x_s)dB_s >_{x_s}, \end{aligned}$$

so $\mathrm{div}(\boldsymbol{\pi}_t W h) = -\int_0^t < \dot{h}_s, W_s^* /\!\!/_s d\breve{B}_s >_{x_s}$, which is clearly $\mathcal{F}_t^{x_0}$-measurable, as the parallel translations $/\!\!/_s$ and W_s are both $\mathcal{F}_s^{x_0}$-measurable, and the integral above is an Itô integral.

Such Itô integrals in fact generate the filtration $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$, since, writing $z_t = \int_0^t W_s^* /\!\!/_s d\breve{B}_s$, we have $d\breve{B}_t = /\!\!/_t^{-1}(W_t^*)^{-1}dz_t$, and

$$\sigma\{z_s : s \in [0,t]\} = \sigma\{\breve{B}_s : s \in [0,t]\} = \mathcal{F}_t^{\breve{B}} = \mathcal{F}_t^{x_0},$$

so indeed $\mathcal{F}_t^{x_0} = \mathcal{G}_t^\pi$. □

The measurability of the parallel translations also shows that we can use π_t^σ in the definition of $\mathcal{G}_t^\pi$.

We are now in a position to characterise the measurability and adaptedness with respect to $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$ in terms of the random resolutions defined above.

2.2.2 Measurability

The analogue of the measurability condition (1.9) is

$$D_{\mathcal{H}}F(\sigma)(k) = D_{\mathcal{H}}F(\sigma)(\boldsymbol{\pi}_t^\sigma k), \quad \text{a.s.}, \forall k \in \mathcal{H}_\sigma. \tag{2.15}$$

We could express (2.15) in terms of π^σ or ρ^σ instead, and we show below that these formulations are equivalent.

For an analogue of (1.7) in terms of the gradient operator, we observe first that, for all $k \in \mathcal{H}_\sigma$,

$$< \nabla_{\mathcal{H}} F(\sigma), k >_{\mathcal{H}_\sigma} = \int_0^T < \frac{\mathbb{D}}{ds}(\nabla_{\mathcal{H}} F)_s, \frac{\mathbb{D}}{ds} k_s >_{\sigma_s} ds,$$

and

$$< \boldsymbol{\pi}_t^\sigma \nabla_{\mathcal{H}} F(\sigma), k >_{\mathcal{H}_\sigma} = \int_0^T \mathbf{1}_{[0,t]}(s) < \frac{\mathbb{D}}{ds}(\nabla_{\mathcal{H}} F)_s, \frac{\mathbb{D}}{ds} k_s >_{\sigma_s} ds,$$

so (cf. the comment in Example 1.3.2)

$$\nabla_{\mathcal{H}} F(\sigma) = \boldsymbol{\pi}_t^\sigma \nabla_{\mathcal{H}} F(\sigma) \iff \frac{\mathbb{D}}{ds}(\nabla_{\mathcal{H}} F)_s = 0, \quad \forall s > t. \tag{2.16}$$

Similarly,

$$\tilde{\nabla}_{\mathcal{H}} F(\sigma) = \pi_t^\sigma \tilde{\nabla}_{\mathcal{H}} F(\sigma) \iff \frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}} F)_s = 0, \quad \forall s > t. \tag{2.17}$$

Since, for any $k \in \mathcal{H}_\sigma$,

$$D_{\mathcal{H}} F(\sigma)(k) = \int_0^T < \frac{\mathbb{D}}{ds}(\nabla_{\mathcal{H}} F)_s, \frac{\mathbb{D}}{ds} k_s >_{\sigma_s} ds = \int_0^T < \frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}} F)_s, \frac{D}{ds} k_s >_{\sigma_s} ds,$$

the condition (2.15) is equivalently to

$$\nabla_{\mathcal{H}} F(\sigma) = \boldsymbol{\pi}_t^\sigma \nabla_{\mathcal{H}} F(\sigma), \tag{2.18}$$

and also to

$$\tilde{\nabla}_{\mathcal{H}} F(\sigma) = \pi_t^\sigma \tilde{\nabla}_{\mathcal{H}} F(\sigma).$$

Hence the choice of inner products on $\mathcal{H}$ does not affect our measurability condition. Applying (2.14), we obtain another equivalent formulation

$$\nabla_{\mathcal{H}} F(\sigma) = (\frac{\mathbb{D}}{d.})^{-1} \rho_t^\sigma \frac{\mathbb{D}}{d.} \nabla_{\mathcal{H}} F(\sigma) = (\frac{D}{d.})^{-1} \rho_t^\sigma \frac{D}{d.} \tilde{\nabla}_{\mathcal{H}} F(\sigma).$$

In the sequel, we do not venture to give complete lists of such equivalent statements and only state the versions relevant to us.

Lemma 2.2.2. *For any $F \in \mathbb{D}^{2,1}$, we have, $\forall s \in [0,T]$,*

$$\frac{\mathbb{D}}{ds}(\nabla_{\mathcal{H}} F)_s = \frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}} F)_s - \frac{1}{2}(W_s^{-1})^* \int_s^T W_r^* \mathrm{Ric}^\sharp \frac{D}{dr}(\tilde{\nabla}_{\mathcal{H}} F)_r dr, \tag{2.19}$$

and

$$\frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}}F)_s = \frac{\mathbb{D}}{ds}(\nabla_{\mathcal{H}}F)_s + \frac{1}{2}(/\!/_s^{-1})^* \int_s^T /\!/_r^* \mathrm{Ric}^\sharp \frac{\mathbb{D}}{dr}(\nabla_{\mathcal{H}}F)_r dr. \tag{2.20}$$

Proof. Recall the equation (2.6), which relates $\frac{\mathbb{D}}{ds}$ to $\frac{D}{ds}$ in our setting:

$$\frac{\mathbb{D}}{ds} = \frac{D}{ds} + \frac{1}{2}\mathrm{Ric}^\sharp. \tag{2.6}$$

So by definition, for any $k \in \mathcal{H}_\sigma$,

$$\begin{aligned} D_{\mathcal{H}}F(\sigma)(k) &= \int_0^T < \frac{\mathbb{D}}{ds}(\nabla_{\mathcal{H}}F)_s, \frac{\mathbb{D}}{ds}k_s >_{\sigma_s} ds &(2.21)\\ &= \int_0^T < \frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}}F)_s, \frac{D}{ds}k_s >_{\sigma_s} ds \\ &= \int_0^T < \frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}}F)_s, \frac{\mathbb{D}}{ds}k_s - \frac{1}{2}\mathrm{Ric}^\sharp k_s >_{\sigma_s} ds. &(2.22) \end{aligned}$$

Since Ric is symmetric,

$$\begin{aligned} &\int_0^T < \frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}}F)_s, \mathrm{Ric}^\sharp k_s >_{\sigma_s} ds \\ = &\int_0^T < \mathrm{Ric}^\sharp \frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}}F)_s, k_s >_{\sigma_s} ds \\ = &\int_0^T < W_s^* \mathrm{Ric}^\sharp \frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}}F)_s, W_s^{-1}k_s >_{x_0} ds \\ = &\int_0^T < \int_s^T W_r^* \mathrm{Ric}^\sharp \frac{D}{dr}(\tilde{\nabla}_{\mathcal{H}}F)_r dr, \frac{d}{ds}W_s^{-1}k_s >_{x_0} ds \\ = &\int_0^T < (W_s^{-1})^* \int_s^T W_r^* \mathrm{Ric}^\sharp \frac{D}{dr}(\tilde{\nabla}_{\mathcal{H}}F)_r dr, W_s \frac{d}{ds}W_s^{-1}k_s >_{\sigma_s} ds \\ = &\int_0^T < (W_s^{-1})^* \int_s^T W_r^* \mathrm{Ric}^\sharp \frac{D}{dr}(\tilde{\nabla}_{\mathcal{H}}F)_r dr, \frac{\mathbb{D}}{ds}k_s >_{\sigma_s} ds. \end{aligned}$$

A comparison of (2.21) and (2.22) shows

$$\begin{aligned} &\int_0^T < \frac{\mathbb{D}}{ds}(\nabla_{\mathcal{H}}F)_s, \frac{\mathbb{D}}{ds}k_s >_{\sigma_s} ds \\ = &\int_0^T < \frac{D}{ds}(\tilde{\nabla}_{\mathcal{H}}F)_s - \frac{1}{2}(W_s^{-1})^* \int_s^T W_r^* \mathrm{Ric}^\sharp \frac{D}{dr}(\tilde{\nabla}_{\mathcal{H}}F)_r dr, \frac{\mathbb{D}}{ds}k_s >_{\sigma_s} ds, \end{aligned}$$

and consequently (2.19) holds. The derivation of (2.20) is similar, and hence omitted. □

Remark 2.2.3. The expressions (2.19) and (2.20) reinforce our earlier observation that (2.18) can be equivalently expressed in terms of π^σ.

To verify the $\mathcal{F}_t^{x_0}$-measurability of a function $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$ against (2.18), we use the Itô map $\mathcal{I} : C_0 \to C_{x_0}M$ of our SDE (2.7) to pull back to the Wiener space. Recall the following filtrations we have defined on C_0: the standard Brownian filtration $\mathcal{F}_t = \sigma\{B_s : 0 \le s \le t\}$, the backward Brownian filtration (see Section 1.2) $\mathcal{F}^t = \sigma\{B_T - B_s, s \ge t\}$, the filtration generated by $\mathcal{I}$ (see Section 2.1) $\mathcal{F}_t^{\mathcal{I}} = \sigma\{\mathcal{I}_s : 0 \le s \le t\}$, and the filtration induced by the canonical resolution π_t of the identity on the Cameron-Martin space $H = L_0^{2,1}$ (see Example 1.3.2) $\mathcal{F}_t^\pi = \sigma\{\delta\pi_t h, h \in H\}$. Note that $\mathcal{F}^t \perp\!\!\!\perp \mathcal{F}_t$, and $\mathcal{F}_t = \mathcal{F}_t^\pi$ (see [57]).

The following general lemma will be useful.

Lemma 2.2.4. *Given a probability space $\{\Omega, \mathcal{A}, \mathbb{P}\}$, a measurable space $\{S, \mathcal{G}\}$, functions $\theta : \Omega \to S$ and $f : S \to \mathbb{R}$ both measurable, and sub-sigma-algebras $\mathcal{A}^\theta \subset \mathcal{A}$ and $\mathcal{G}^\theta \subset \mathcal{G}$ such that $\mathcal{A}^\theta$ is the sigma-algebra generated by θ from $\mathcal{G}^\theta$, we have*

$$\mathbb{E}(f|\mathcal{G}^\theta) \circ \theta = \mathbb{E}(f \circ \theta|\mathcal{A}^\theta).$$

Proof. Define the push-forward measure $\theta_*\mathbb{P}$ on $\{S, \mathcal{G}\}$ by $\theta_*\mathbb{P}(G) = \mathbb{P}(\theta^{-1}(G))$, for $G \in \mathcal{G}$. Since any element of $\mathcal{A}^\theta$ has the form $\theta^{-1}(G)$ for some $G \in \mathcal{G}^\theta$, we have

$$\int_{\theta^{-1}(G)} \mathbb{E}(f|\mathcal{G}^\theta) \circ \theta d\mathbb{P} = \int_G \mathbb{E}(f|\mathcal{G}^\theta) d\theta_*\mathbb{P} = \int_G f d\theta_*\mathbb{P} = \int_{\theta^{-1}(G)} f \circ \theta d\mathbb{P}. \qquad \square$$

Lemma 2.2.5. *Given $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$, we have $F \circ \mathcal{I} \in \mathcal{F}_t \iff F \in \mathcal{F}_t^{x_0}$.*

Proof. Since $\mathcal{I}$ is a measurable map, $F \in \mathcal{F}_t^{x_0}$ implies $F \circ \mathcal{I} \in \mathcal{F}_t$.

Conversely, suppose $F \circ \mathcal{I}$ is $\mathcal{F}_t$-measurable. Since $\mathcal{F}_t^{\mathcal{I}} \subset \mathcal{F}_t$, $\mathcal{F}_T^{\mathcal{I}} \subset \mathcal{F}_t^{\mathcal{I}} \vee \mathcal{F}^t$, $F \circ \mathcal{I} \in \mathcal{F}_T^{\mathcal{I}}$, and $\mathcal{F}^t \perp\!\!\!\perp \mathcal{F}_t$, we have, by Lemma 2.2.4,

$$\begin{aligned} F \circ \mathcal{I} &= \mathbb{E}[\mathbb{E}(F \circ \mathcal{I}|\mathcal{F}_t)|\mathcal{F}_T^{\mathcal{I}}] \\ &= \mathbb{E}\{\mathbb{E}[\mathbb{E}(F \circ \mathcal{I}|\mathcal{F}_t)|\mathcal{F}_t^{\mathcal{I}} \vee \mathcal{F}^t]|\mathcal{F}_T^{\mathcal{I}}\} \\ &= \mathbb{E}\{\mathbb{E}[\mathbb{E}(F \circ \mathcal{I}|\mathcal{F}_t)|\mathcal{F}_t^{\mathcal{I}}]|\mathcal{F}_T^{\mathcal{I}}\} \\ &= \mathbb{E}(F \circ \mathcal{I}|\mathcal{F}_t^{\mathcal{I}}) \\ &= \mathbb{E}(F|\mathcal{F}_t^{x_0}) \circ \mathcal{I}, \end{aligned}$$

i.e., $F \circ \mathcal{I}$ is $\mathcal{F}_t^{\mathcal{I}}$-measurable, and F is $\mathcal{F}_t^{x_0}$-measurable. $\square$

On the classical Wiener space, the result of Nualart and Pardoux (Lemma 2.4 of [48]) shows that, if $F \in \mathbb{D}^{2,1}(C_0;\mathbb{R})$, we have $\mathbb{E}(F|\mathcal{F}_A) \in \mathbb{D}^{2,1}(C_0;\mathbb{R})$ for any

Borel set A of $[0,T]$, and

$$\frac{d}{dt}\nabla\mathbb{E}(F|\mathcal{F}_A)_t = \mathbb{E}[\frac{d}{dt}(\nabla F)_t|\mathcal{F}_A]\mathbf{1}_A(t), \quad \text{a.e. in } [0,T]\times C_0. \tag{2.23}$$

We give the analogous statement on the path spaces in the next lemma.

To simplify notation, we extend the definition (1.10) of the canonical resolution $\{\pi_t\}_{t\in[0,T]}$ for the classical Wiener space to

$$(\pi_A h)_t = \int_0^t \mathbf{1}_A(r)\dot{h}_r dr, \quad \forall A\in\mathcal{B}([0,T]), h\in H, t\in[0,T],$$

so π is considered to be indexed by Borel subsets of $[0,T]$. Obviously, $\pi_t = \pi_{[0,t]}$, and $\pi_{(s,t]} = \pi_t - \pi_s$. Note that the first equation in Definition 1.3.1 implies orthogonality, which shows that $\pi_{(s,t]}$ is a projection for $s\le t$, and this extends to any Borel subset $A\in\mathcal{B}([0,T])$, with π_A again a projection on H. Most part of the theory can be generalised this way, although we do not pursue it further here.

Lemma 2.2.6. *If $F\in\mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$ and A is any Borel set of $[0,T]$, we have $\mathbb{E}(F|\mathcal{F}_A^{x_0})\in\mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$, and*

$$\frac{\mathbb{D}}{dt}[\nabla_{\mathcal{H}}\mathbb{E}(F|\mathcal{F}_A^{x_0})]_t = \mathbb{E}[\frac{\mathbb{D}}{dt}(\nabla_{\mathcal{H}}F)_t|\mathcal{F}_A^{x_0}]\mathbf{1}_A(t), \quad \textit{a.e. in } [0,T]\times C_{x_0}M. \tag{2.24}$$

Proof. We assume first $F\in\mathbb{D}^{2,2}(C_{x_0}M;\mathbb{R})$, so Corollary 4.3 in [25] shows that $F\circ\mathcal{I}\in\mathbb{D}^{2,2}(C_0;\mathbb{R})$. Apply Lemma 2.2.5 and equation (2.23), we have

$$\mathbb{E}(F|\mathcal{F}_A^{x_0})\circ\mathcal{I} = \mathbb{E}(F\circ\mathcal{I}|\mathcal{F}_A)\in\mathbb{D}^{2,2}(C_0;\mathbb{R}),$$

and Proposition 7.3 in [25] imply $\mathbb{E}(F|\mathcal{F}_A^{x_0})\in\mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$. The intertwining result from Theorem 6.1 in [25] and Lemma 2.2.5 allow us to calculate, for any $h\in H$,

$$\begin{aligned} D_{\mathcal{H}}\mathbb{E}(F|\mathcal{F}_A^{x_0})\circ T\mathcal{I}(h) &= D[\mathbb{E}(F|\mathcal{F}_A^{x_0})\circ\mathcal{I}](h) \\ &= D[\mathbb{E}(F\circ\mathcal{I}|\mathcal{F}_A)](h) \\ &= \mathbb{E}[D(F\circ\mathcal{I})|\mathcal{F}_A](\pi_A h) \\ &= \mathbb{E}[D_{\mathcal{H}}F\circ T\mathcal{I}(\pi_A h)|\mathcal{F}_A], \end{aligned}$$

so taking conditional expectation with respect to $\mathcal{F}_T^{x_0}$, we obtain, using (2.11),

$$D_{\mathcal{H}}\mathbb{E}(F|\mathcal{F}_A^{x_0})\circ\overline{T\mathcal{I}}(h) = \mathbb{E}[D_{\mathcal{H}}F\circ\overline{T\mathcal{I}(\pi_A h)}|\mathcal{F}_A^{x_0}] = \mathbb{E}[D_{\mathcal{H}}F\circ\overline{T\mathcal{I}}(\pi_A h)|\mathcal{F}_A^{x_0}].$$

This shows that, give any $h \in H$,

$$\begin{aligned} & \int_0^T < \frac{\mathbb{D}}{dt}[\nabla_{\mathcal{H}} \mathbb{E}(F|\mathcal{F}_A^{x_0})]_t, X(x_t)\dot{h}_t >_{x_t} dt \\ = \ & \mathbb{E}[\int_0^T < \frac{\mathbb{D}}{dt} \nabla_{\mathcal{H}} F, X(x_t)\dot{h}_t \mathbf{1}_A(t) >_{x_t} dt | \mathcal{F}_A^{x_0}] \\ = \ & \int_0^T < \mathbb{E}[\frac{\mathbb{D}}{dt}(\nabla_{\mathcal{H}} F)_t | \mathcal{F}_A^{x_0}] \mathbf{1}_A(t), X(x_t)\dot{h}_t >_{x_t} dt, \end{aligned}$$

which proves the result for $F \in \mathbb{D}^{2,2}(C_{x_0}M;\mathbb{R})$, since X is onto.

For any general function $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$, we can take a sequence of functions $F_i \in \mathbb{D}^{2,2}(C_{x_0}M;\mathbb{R})$ such that $F_i \to F$ in $\mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$. This means $\mathbb{E}(F_i|\mathcal{F}_A^{x_0}) \to \mathbb{E}(F|\mathcal{F}_A^{x_0})$ and $\nabla_{\mathcal{H}} F_i \to \nabla_{\mathcal{H}} F$ in L^2. The above arguments also imply that $\mathbb{E}(F_i|\mathcal{F}_A^{x_0}) \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$, and

$$\frac{\mathbb{D}}{dt}[\nabla_{\mathcal{H}} \mathbb{E}(F_i|\mathcal{F}_A^{x_0})]_t = \mathbb{E}[\frac{\mathbb{D}}{dt}(\nabla_{\mathcal{H}} F_i)_t | \mathcal{F}_A^{x_0}] \mathbf{1}_A(t) \to \mathbb{E}[\frac{\mathbb{D}}{dt}(\nabla_{\mathcal{H}} F)_t | \mathcal{F}_A^{x_0}] \mathbf{1}_A(t)$$

in L^2. The fact that $\nabla_{\mathcal{H}}$ is a closed operator implies now $\mathbb{E}(F|\mathcal{F}_A^{x_0}) \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$, and (2.24) indeed holds for $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$. □

Lemma 2.2.7. *Suppose $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$ is $\mathcal{F}_t^{x_0}$-measurable. Then there exists a sequence of cylindrical functions $F_n(\sigma) = f_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_m}})$, where $n_1, \cdots, n_m \in \mathbb{N}$, $0 \le t_{n_1} < \cdots < t_{n_m} \le t$, and each $f_n : M^{n_m} = \underbrace{M \times \cdots \times M}_{n_m \text{ times}} \to \mathbb{R}$ is smooth, such that $F_n \to F$ in $\mathbb{D}^{2,1}$, as $n \to \infty$.*

Proof. By definition, any $F \in \mathbb{D}^{2,1}$ can be approximated by a sequence of smooth cylindrical functions, i.e., $F = \lim_{n\to\infty} G_n$, with $G_n(\sigma) = g_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_l}})$, where $n_1, \cdots, n_l \in \mathbb{N}$, $0 \le t_{n_1} < \cdots < t_{n_{k-1}} \le t < t_{n_k} < \cdots < t_{n_l} \le T$, and g_n is smooth with compact support for each $n \in \mathbb{N}$.

Since $F \in \mathcal{F}_t^{x_0}$, we have

$$F(\sigma) = \mathbb{E}(F(\sigma)|\mathcal{F}_t^{x_0}) = \lim_{n\to\infty} \mathbb{E}(G_n(\sigma)|\mathcal{F}_t^{x_0}) = \lim_{n\to\infty} \mathbb{E}[g_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_l}})|\mathcal{F}_t^{x_0}],$$

where the limit is in L^2 as well as in $D^{2,1}$, by Lemma 2.2.6. Denote by $p(s, x, y)$ the heat kernel on M, for $s \in [0, T]$, $x, y \in M$, so we have

$$\begin{aligned} & \mathbb{E}[g_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_l}})|\mathcal{F}_t^{x_0}] \\ = & \int_{M^{n_l - n_k + 1}} g_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_l}}) p(t_{n_k} - t, \sigma_t, \sigma_{t_{n_k}}) d\sigma_{t_{n_k}} \prod_{i=n_k+1}^{n_l} p(t_i - t_{i-1}, \sigma_{t_{i-1}}, \sigma_{t_i}) d\sigma_{t_i} \end{aligned}$$

which can be simply written as a function of n_k variables, say, $\tilde{g}_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_{k-1}}}, \sigma_t)$. So we set $F_n(\sigma) = \mathbb{E}[G_n(\sigma)|\mathcal{F}_t^{x_0}]$, $n_m = n_k$, and $f_n = \tilde{g}_n$ to conclude the proof. □

Lemma 2.2.8. *Suppose $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$ is $\mathcal{F}_t^{x_0}$-measurable. Then (2.18) holds.*

Proof. For F as given, we have by Lemma 2.2.7 a sequence of cylindrical functions $F_n(\sigma) = f_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_m}})$, where $n_1, \cdots, n_m \in \mathbb{N}$, $0 \le t_{n_1} < \cdots < t_{n_m} \le t$, and each $f_n : M^{n_m} = \underbrace{M \times \cdots \times M}_{n_m \text{ times}} \to \mathbb{R}$ is smooth, such that $F_n \to F$ in $\mathbb{D}^{2,1}$, as $n \to \infty$.

Since $\boldsymbol{\pi}^\sigma$ is continuous, we only need to verify that (2.15) holds for F_n. This is true indeed, since, for any $k \in \mathcal{H}_\sigma$, we have

$$\begin{aligned} D_{\mathcal{H}}F_n(\sigma)(k) &= \sum_{j=1}^{n_m} D_j f_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_m}})(k_{t_{n_j}}) \\ &= \sum_{j=1}^{n_m} D_j f_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_m}})(\boldsymbol{\pi}_t^\sigma k_{t_{n_j}}) \\ &= D_{\mathcal{H}}F_n(\sigma)(\boldsymbol{\pi}_t^\sigma k), \end{aligned}$$

where $D_j f_n(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_m}})$ denotes the partial derivative of f_n at $(\sigma_{t_{n_1}}, \cdots, \sigma_{t_{n_m}})$ with respect to the j-th variable. □

Lemma 2.2.9. *If (2.18) holds for $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$, $t \in [0,T]$, we have $F \in \mathcal{F}_t^{x_0}$.*

Proof. Corollary 4.3 in [25] shows that $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$ implies $F \circ \mathcal{I} \in \mathbb{D}^{2,1}(C_0, \mathbb{R})$. Recall that $\mathcal{F}_t$-measurability on the Wiener space is equivalent to (1.7), so $F \circ \mathcal{I}$ is $\mathcal{F}_t$-measurable iff

$$\nabla(F \circ \mathcal{I}) = \pi_t \nabla(F \circ \mathcal{I}), \quad a.s. \tag{2.25}$$

Therefore, Lemma 2.2.5 implies that we only need to prove that (2.18) implies (2.25).

By the definition of π_t in (1.10), we have

$$\frac{d}{ds}(\pi_t h)_s = \begin{cases} \dot{h}_s, & s \le t \\ 0, & s > t \end{cases}$$

so Bismut's formula (2.9) implies

$$T\mathcal{I}_s(\pi_t h) = T\xi_s \int_0^s (T\xi_r)^{-1} X(x_r) \frac{d}{dr}(\pi_t h)_r dr = T\mathcal{I}_s(h), \quad s \le t.$$

Now applying Theorem 3.4 in [25], we have, for any $h \in H$,

$$
\begin{aligned}
D(F\circ\mathcal{I})(\pi_t h) &= D_{\mathcal{H}}F \circ T\mathcal{I}(\pi_t h)\\
&= \int_0^T < \frac{\mathbb{D}}{ds}(\nabla F)_s, \nabla_{T\mathcal{I}_s(\pi_t h)} X(x_s)\tilde{/\!/}_s{}^{\sigma} d\beta_s + X(x_s)\frac{d}{ds}(\pi_t h)_s ds >_{x_s}\\
&= \int_0^t < \frac{\mathbb{D}}{ds}(\nabla F)_s, \nabla_{T\mathcal{I}_s(\pi_t h)} X(x_s)\tilde{/\!/}_s{}^{\sigma} d\beta_s + X(x_s)\frac{d}{ds}(\pi_t h)_s ds >_{x_s}\\
&= \int_0^t < \frac{\mathbb{D}}{ds}(\nabla F)_s, \nabla_{T\mathcal{I}_s(h)} X(x_s)\tilde{/\!/}_s{}^{\sigma} d\beta_s + X(x_s)\dot{h}_s ds >_{x_s},
\end{aligned}
$$

where the third line follows from the equivalence (2.16). Using (2.16) again,

$$
\begin{aligned}
D(F\circ\mathcal{I})(h) &= \int_0^T < \frac{\mathbb{D}}{ds}(\nabla F)_s, \nabla_{T\mathcal{I}_s(h)} X(x_s)\tilde{/\!/}_s{}^{\sigma} d\beta_s + X(x_s)\dot{h}_s ds >_{x_s}\\
&= \int_0^t < \frac{\mathbb{D}}{ds}(\nabla F)_s, \nabla_{T\mathcal{I}_s(h)} X(x_s)\tilde{/\!/}_s{}^{\sigma} d\beta_s + X(x_s)\dot{h}_s ds >_{x_s},
\end{aligned}
$$

so $D(F\circ\mathcal{I})(\pi_t h) = D(F\circ\mathcal{I})(h)$. This shows (2.18) implies (2.25), so $F \in \mathcal{F}_t^{x_0}$. □

We have thus proved

Theorem 2.2.10. *Given* $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$, $F \in \mathcal{F}_t^{x_0} \iff (2.15) \iff (2.18)$.

2.2.3 Adaptedness

For the characterisation of the adaptedness, we verify the following anologue of (1.8):

$$
\boldsymbol{\pi}_t^{\sigma}\boldsymbol{\nabla} U = \boldsymbol{\pi}_t^{\sigma}\boldsymbol{\nabla} U \boldsymbol{\pi}_t^{\sigma} \quad a.s., \forall t \in [0,T], \tag{2.26}
$$

where $\boldsymbol{\nabla}$ is the Markovian covariant derivative operator introduced in Section 2.1, and U is a vector field in $\mathrm{Dom}(\boldsymbol{\nabla}) = \mathbb{D}^{2,1}(\mathcal{H})$. Recall that a vector field U is said to be adapted to $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$ if the map $\sigma \mapsto \frac{\mathbb{D}}{dt}U(\sigma)_t$ is $\mathcal{F}_t^{x_0}$-measurable for all t.

Equation (2.14) gives the corresponding expression for (2.26) in terms of ρ^{σ},

$$
\rho_t^{\sigma}\frac{\mathbb{D}}{ds}(\boldsymbol{\nabla}_v U)_s = \rho_t^{\sigma}\frac{\mathbb{D}}{ds}(\boldsymbol{\nabla}_{\boldsymbol{\pi}_t^{\sigma} v} U)_s, \quad \forall t \in [0,T], \forall v \in \mathcal{H}_{\sigma}.
$$

This means that (2.26) is equivalent to (cf. the comment in Example 1.3.2)

$$
\frac{\mathbb{D}}{ds}(\boldsymbol{\nabla}_v U)_s|_{[0,t]} = \frac{\mathbb{D}}{ds}(\boldsymbol{\nabla}_{\boldsymbol{\pi}_t^{\sigma} v} U)_s|_{[0,t]}, \quad \forall t \in [0,T], \forall v \in \mathcal{H}_{\sigma}, \tag{2.27}
$$

and (2.12) gives the equivalent expression in terms of the pointwise connection,

$$\tilde{\nabla}_v \frac{\mathbb{D}}{ds} U(\sigma)_s = \tilde{\nabla}_{\boldsymbol{\pi}_t^\sigma v} \frac{\mathbb{D}}{ds} U(\sigma)_s, \quad \forall s \leq t, \forall v \in \mathcal{H}_\sigma. \tag{2.28}$$

It is also clear that (2.26) can be equivalently expressed in terms of π^σ.

Theorem 2.2.11. *A vector field $U \in \mathbb{D}^{2,1}$ is adapted to $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$ iff (2.26) holds.*

Proof. By the relation (2.12) and the definition of our derivative operators,

$$\begin{aligned}
\frac{\mathbb{D}}{ds}(\boldsymbol{\nabla}_{\boldsymbol{\pi}_t^\sigma v} U)_s &= \tilde{\nabla}_{\boldsymbol{\pi}_t^\sigma v} \frac{\mathbb{D}}{ds} U_s \\
&= \tilde{X}(\sigma) D[\sigma \mapsto \tilde{Y}_\sigma \frac{\mathbb{D}}{ds} U(\sigma)_s](\boldsymbol{\pi}_t^\sigma v) \\
&= X(\sigma_s) D[\sigma \mapsto Y_{\sigma_s} \frac{\mathbb{D}}{ds} U(\sigma)_s](\boldsymbol{\pi}_t^\sigma v)_s.
\end{aligned}$$

Suppose U is adapted, then $Y_{\sigma_s} \frac{\mathbb{D}}{ds} U(\sigma)_s \in \mathcal{F}_s^{x_0}$ for all $s \in [0,T]$, so by Theorem 2.2.10,

$$\nabla Y_{\sigma_s} \frac{\mathbb{D}}{ds} U(\sigma)_s = \boldsymbol{\pi}_t^\sigma \nabla Y_{\sigma_s} \frac{\mathbb{D}}{ds} U(\sigma)_s, \quad \text{a.e. } s \leq t.$$

That is,

$$D[\sigma \mapsto Y_{\sigma_s} \frac{\mathbb{D}}{ds} U(\sigma)_s](v_s) = D[\sigma \mapsto Y_{\sigma_s} \frac{\mathbb{D}}{ds} U(\sigma)_s](\boldsymbol{\pi}_t^\sigma v_s), \quad \text{a.e. } s \leq t, \forall v \in \mathcal{H}_\sigma,$$

from which (2.27), as well as (2.28), follows, therefore (2.26) holds.

Conversely, we suppose that (2.26) holds; therefore, so does (2.28). Note first that the map $\tilde{Y}_\sigma : L^2 TC_{x_0} M \to L^2([0,T];\mathbb{R}^m)$ is adapted, in the sense that given any $V \in \Gamma L^2 TC_{x_0} M$ that is adapted, so is $\tilde{Y} \circ V$. Applying Theorem 2.2.10 to

$$(\tilde{\nabla}_v \tilde{Y})_{\sigma_s} = D[\sigma \mapsto \tilde{Y}_\sigma \tilde{X}(\sigma)](v)\Big|_{\sigma_s} Y_{\sigma_s}, \quad \forall v \in \mathcal{H}_\sigma,$$

we see $(\tilde{\nabla}_{\boldsymbol{\pi}_t^\sigma v} \tilde{Y})_{\sigma_s} = (\tilde{\nabla}_v \tilde{Y})_{\sigma_s}$, a.e. $s \leq t$. This and (2.28) together show that, for $v \in \mathcal{H}_\sigma$ and $0 \leq s \leq t \leq T$,

$$\begin{aligned}
\tilde{\nabla}_{\boldsymbol{\pi}_t^\sigma v}[\tilde{Y}_\sigma \frac{\mathbb{D}}{ds} U(\sigma)_s] &= (\tilde{\nabla}_{\boldsymbol{\pi}_t^\sigma v} \tilde{Y}_\sigma) \frac{\mathbb{D}}{ds} U(\sigma)_s + \tilde{Y}_\sigma [\tilde{\nabla}_{\boldsymbol{\pi}_t^\sigma v} \frac{\mathbb{D}}{ds} U(\sigma)_s] \\
&= (\tilde{\nabla}_v \tilde{Y}_\sigma) \frac{\mathbb{D}}{ds} U(\sigma)_s + \tilde{Y}_\sigma [\tilde{\nabla}_v \frac{\mathbb{D}}{ds} U(\sigma)_s] \\
&= \tilde{\nabla}_v [\tilde{Y}_\sigma \frac{\mathbb{D}}{ds} U(\sigma)_s],
\end{aligned}$$

that is, for all $t \in [0,T]$,

$$D[\sigma \mapsto Y_{\sigma_s} \frac{\mathbb{D}}{ds} U(\sigma)_s](v_s) = D[\sigma \mapsto Y_{\sigma_s} \frac{\mathbb{D}}{ds} U(\sigma)_s](\boldsymbol{\pi}_t^\sigma v_s), \quad \text{a.e. } s \in [0,t], \forall v \in \mathcal{H}_\sigma.$$

Theorem 2.2.10 now implies that the map $\sigma \mapsto \tilde{Y}_\sigma \frac{\mathbb{D}}{ds} U(\sigma)_s$ is $\mathcal{F}_s^{x_0}$-measurable, for any $s \in [0,T]$, and therefore, so is $\sigma \mapsto \tilde{X}(\sigma)\tilde{Y}_\sigma \frac{\mathbb{D}}{ds} U(\sigma)_s = \frac{\mathbb{D}}{ds} U(\sigma)_s$, as $\tilde{Y}$ is the right inverse of $\tilde{X}$. We have now, for all $s \in [0,T]$,

$$U_s = W_s \int_0^s W_r^{-1} \frac{\mathbb{D}}{dr} U(\sigma)_r dr \in \mathcal{F}_s^{x_0},$$

i.e., U is adapted indeed. □

Our previous discussion (or a calculation similar to that in Lemma 2.2.2) shows that Theorem 2.2.11 can be stated using the undamped resolution π^σ.

2.2.4 The Clark-Ocone Formula

The Clark-Ocone formula for $F \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$ reads (see [30; 22])

$$F(\sigma) = \mathbb{E}F + \int_0^T < \mathbb{E}[\frac{\mathbb{D}}{dt}(\nabla_{\mathcal{H}} F)_t | \mathcal{F}_t^{x_0}], d\{\sigma\}_t >_{\sigma_t}, \quad \mu_{x_0}\text{-a.e. } \sigma \in C_{x_0}M. \quad (2.29)$$

As in Example 1.3.2, the canonical resolution $\boldsymbol{\pi}^\sigma$ here corresponds naturally to the projection onto the subspace $\mathcal{V}$ of adapted processes in $L^2\Gamma\mathcal{H}$, i.e.,

$$(P_{\mathcal{V}} u)_t = W_t \int_0^t W_s^{-1} \mathbb{E}(\frac{\mathbb{D}}{ds} u_s | \mathcal{F}_s^{x_0}) ds, \quad u \in L^2\Gamma\mathcal{H}. \quad (2.30)$$

The Clark-Ocone formula (2.29) can take the same form as in (1.16), i.e.,

$$F = \mathbb{E}(F) - \operatorname{div}(P_{\mathcal{V}} \nabla_{\mathcal{H}} F), \quad \forall F \in L^2(C_{x_0}M;\mathbb{R}).$$

Equations (2.6) and (2.19) give us the undamped version of (2.29). It seems, however, that we need to change the norm to retain the isometry property, or else refine the definition of $\mathcal{V}$ for the undamped Clark-Ocone formula. Although we do not attempt further in this direction, we comment that the generalisation of $\mathcal{V}$ into higher dimensions turns out to be of fundamental importance in our study of differential forms on the path spaces (see Chapter 6).

Chapter 3

The Commutation Formula

The well-known commutative relationship between the derivative and divergence operators can be most concisely expressed as $[\nabla, \delta] = \mathrm{Id}_H$. Or, as Nualart put it ([47] Propositions 1.3.2 and 1.3.8):

$$[\nabla_v, \delta]u = < u, v >_H, \tag{3.1}$$

where $v \in H = L_0^{2,1}$, and $u \in \mathbb{D}^{2,1}(C_0; H)$ such that $\nabla_v u = < \nabla u, v >_H \in \mathrm{Dom}(\delta)$. The notation $\nabla_v u$ highlights the fact that we only need to consider the directional derivative of u in the direction of v; in fact, we can follow the notation of Nualart [47] to replace the requirement $u \in \mathbb{D}^{2,1}(C_0; H)$ by the weaker condition $u \in \mathbb{D}^{2,v}(C_0; H)$ (which means u only needs to be H-differentiable in the direction of v).

Formula (3.1) can be extended to the case of v being a random H-valued process rather than just an element of H, but that produces an extra term $\mathrm{Trace}(\nabla u, \nabla v)$. More generally, this also applies to any abstract Wiener space (E, H, μ); see [50; 60], for example. The extra term shows its importance in the energy identity of Skorohod integrals (see, e.g., [56; 50]):

$$\mathbb{E}\left[\delta(u)\delta(v)\right] \;=\; \mathbb{E} < u, v >_H + \mathbb{E}\,\mathrm{Trace}(\nabla u, \nabla v), \quad \forall u, v \in L^2(E; H), \tag{3.2}$$

which differs from the Itô isometry by the expectation of the trace. Üstünel and Zakai [61] have shown that the trace term vanishes if u and v are both adapted to the same filtration induced by a resolution of the identity.

Given its significance, we review the commutation formula for abstract Wiener spaces, and state a version involving resolutions of the identity, which follows from the above result of Üstünel and Zakai [61]. Cruzeiro and Fang [9] have extended the formula to the Riemannian path spaces, for u and v adapted vector fields. We give an alternative proof with weakened conditions. These results are essential for the derivation of Clark-Ocone formulae for differential forms, which we prove in Chapters 4 and 6.

3.1 On Abstract Wiener Spaces

A version of the commutation formula for abstract Wiener spaces is given by Lemma 1.1 of Üstünel and Zakai [60], which we include below but state with slightly different conditions. Our proof follows that of Nualart for the commutation formula on the classical Wiener space [47] (Proposition 1.3.2). Denote by τ the transposition operator on $H \otimes H$, i.e., $\tau(g \otimes h) = h \otimes g$. Our convention is that δ acts on the last component of tensor products.

Lemma 3.1.1 (Üstünel and Zakai [60], Nualart [47])**.** *Suppose $u \in \mathbb{D}^{2,1}(E;H)$, and $\tau\nabla u \in \mathrm{Dom}(\delta)$. Then we have $\delta u \in \mathbb{D}^{2,1}$ and*

$$\nabla(\delta u) = u + \delta\tau\nabla u.$$

In other words, if $u \in \mathbb{D}^{2,1}(E;H)$, $v \in L^2(E;H)$, and $\nabla_v u \in \mathrm{Dom}(\delta)$, we have $\delta u \in \mathbb{D}^{2,1}$ and

$$\nabla_v(\delta u) =< u, v >_H + < \delta\tau\nabla u, v >_H \tag{3.3}$$

If in addition $u \in \mathbb{D}^{2,2}(E;H)$, and $v \in \mathbb{D}^{2,1}(E;H) \cap L^\infty(E;H)$, we have also

$$\nabla_v(\delta u) =< u, v >_H + \delta\nabla_v u + \mathrm{Trace}(\nabla u, \nabla v). \tag{3.4}$$

Remark 3.1.2. The original statement of this lemma in [60] (Lemma 1.1) imposes the following conditions: $v \in \mathbb{D}^{p,2}(E;H)$ and $u \in \mathbb{D}^{q,2}(E;H)$, where $p^{-1} + q^{-1} < 1$. We are only interested in the L^2 case. Note that $u \in \mathbb{D}^{2,2}(E;H)$ implies that $u \in \mathbb{D}^{2,1}(E;H)$ and $\tau\nabla u \in \mathrm{Dom}(\delta)$, the conditions we need for (3.3).

Remark 3.1.3. For any two Hilbert-Schmidt operators A and B in H, the trace of A and B given by $\mathrm{Trace}(A, B) = \mathrm{Trace}(A^*B)$ is well-defined, since A^*B is trace class. Note that, given any orthonormal basis $\{e_i\}$ of H, we can express the Hilbert-Schmidt inner product as $< A, B >_{HS} = \sum_{i=1}^{\infty} < Ae_i, Be_i >_H = \mathrm{Trace}(A^*B)$.

Remark 3.1.4. Equation (3.3) reduces to (3.1) for $v \in H$, since $\nabla v = 0$ and the trace term vanishes.

Proof. For any $v \in \mathbb{D}^{2,1}(E;H) \subset \mathrm{Dom}(\delta)$, we apply (3.2) and the adjoint relationship between δ and ∇ to calculate

$$\begin{aligned}\mathbb{F}[\delta(u)\delta(v)] &= \mathbb{E} < u, v >_H + \mathbb{E} < \tau\nabla u, \nabla v >_{HS} \\ &= \mathbb{E} < u, v >_H + \mathbb{E} < \delta\tau\nabla u, v >_H \\ &\leq \|u\|_{L^2}\|v\|_{L^2} + \|\delta\tau\nabla u\|_{L^2}\|v\|_{L^2} \\ &\leq c\|v\|_{L^2},\end{aligned}$$

where c is some contant. Since $\mathbb{D}^{2,1}(E;H)$ is densely contained in the domain of δ, we see $\delta u \in \mathbb{D}^{2,1}$ and (3.3) holds. The proof of (3.4) is the same as in [60]. □

Remark 3.1.5. From the proof above, it is easy to see the following equivalence:

$$u \in \mathrm{Dom}(\delta), \text{ and } \delta u \in \mathbb{D}^{2,1} \iff u \in \mathbb{D}^{2,1}, \text{ and } \tau\nabla u \in \mathrm{Dom}(\delta).$$

In the classical Wiener space, when both u and v are adapted (to the standard Brownian filtration), Trace$(\nabla u, \nabla v)$ vanishes by the very definition of Itô's integrals, reflecting the fundamental Itô isometry. By virtue of Lemma 4.1 of [61], this also happens in an abstract Wiener space when u and v are adapted to the filtration induced by a resolution of the identity. So in both cases we obtain a commutative relationship similar to (3.1). This is typical of the relationship, or rather difference, between Itô and Skorohod integrals: when we have adaptedness, less regularity is required. Therefore we can state the following

Lemma 3.1.6. *On an abstract Wiener space (E, H, μ), suppose $u \in \mathbb{D}^{2,1}(E;H)$ and $v \in L^2(E;H)$ are adapted to the* same *filtration induced by a resolution of the identity. Then we have $\nabla_v u \in \mathrm{Dom}(\delta)$, $\delta(u) \in \mathbb{D}^{2,1}$, and*

$$\nabla_v(\delta u) =< u, v >_H + \delta\nabla_v u. \tag{3.5}$$

Remark 3.1.7. Adopting Nualart's notation in [47], we can replace $u \in \mathbb{D}^{2,1}(E;H)$ by the weaker condition $u \in \mathbb{D}^{2,v}(E;H)$.

Proof. Note that $\nabla_v u \in L^2(E;H)$ is adapted. The claim that $\delta(u) \in \mathbb{D}^{2,1}$ follows the general theme that an Itô integral is differentiable iff its integrand is differentiable (see, e.g., Lemma 1.3.4 in [47], Proposition II.1 of [58], and Lemma III.1 of [59]). To finish the proof: observe that simple adapted processes of the form shown in (1.19) are dense in the subspace of adapted processes, and we can approximate these simple adapted processes by smooth adapted processes, so indeed (3.5) holds. □

For a general abstract Wiener space where the maximal isometry subspace (see Section 1.4) may not necessarily come from a continuous resolution of the identity, we can only obtain the vanishing of expectation of the trace from the energy identity (3.2), i.e., for any isometry subspace V of $L^2(E;H)$,

$$\mathbb{E}\,\mathrm{Trace}(\nabla u, \nabla v) = 0, \quad \forall u, v \in V. \tag{3.6}$$

3.2 On Riemannian Path Spaces

The following lemma generalises the commutation formula of Cruzeiro and Fang [9] (Theorem 3.2).

Lemma 3.2.1. *Suppose a vector field $U \in \mathbb{D}^{2,1}(\mathcal{H})$ is adapted to $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$. Then $\boldsymbol{\nabla}_{-}U$ is also adapted and in the domain of* div, $\mathrm{div}(U) \in \mathbb{D}^{2,1}$, *and for any $\mathcal{H}$-vector field V on $C_{x_0}M$,*

$$D_{\mathcal{H}}\,\mathrm{div}(U)(V) = \mathrm{div}(\boldsymbol{\nabla}_{-}U)(V) - < U, V >_{\mathcal{H}}. \tag{3.7}$$

If, in addition, V is also adapted, we have

$$D_{\mathcal{H}}\,\mathrm{div}(U)(V) = \mathrm{div}(\boldsymbol{\nabla}_{V}U) - < U, V >_{\mathcal{H}}. \tag{3.8}$$

Remark 3.2.2. The map $\boldsymbol{\nabla}_{-}U \in L(\mathcal{H};\mathcal{H})$ is adapted in the sense that its composition with the evaluation map is adapted, i.e., $ev_t \circ \boldsymbol{\nabla}_{-}U \in L(\mathcal{H}; T_{x_t}M)$ is $\mathcal{F}_t^{x_0}$-measurable.

Proof. Since U is adapted to $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$, the map $\boldsymbol{Y}U \circ \mathcal{I} : C_0 \to H$ is adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$ by Lemma 2.2.5. Therefore we apply Corollary 5.2 in [25] to calculate

$$\begin{aligned}
(\mathrm{div}(U)) \circ \mathcal{I} &= -\mathbb{E}[\int_0^T < \frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, X(x_t)dB_t >_{x_t} |\mathcal{F}_T^{x_0}] \\
&= -\int_0^T < \frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, X(x_t)dB_t >_{x_t} \\
&= -\int_0^T < Y\frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, K^{\perp}(x_t)dB_t >_{\mathbb{R}^m},
\end{aligned}$$

where the second line follows from the adaptedness of U, the operator $K^{\perp}(x)$ on the third line is the orthogonal projection of $\mathbb{R}^m$ onto the orthogonal complement of $\mathrm{Ker}X(x)$ (see Section 2.1), and $Y_xX(x) = K^{\perp}(x)$.

Since $U \in \mathbb{D}^{2,1}(C_{x_0}M;\mathcal{H})$, Corollary 4.3 in [25] shows $U \circ \mathcal{I} \in \mathbb{D}^{2,1}(C_0;\mathcal{H})$, hence $K^{\perp}(x_t)Y\frac{\mathbb{D}}{dt}U_t \circ \mathcal{I} \in \mathbb{D}^{2,1}(C_0;\mathbb{R}^m)$, so we can apply the commutation formula for the Wiener space to obtain $(\mathrm{div}(U)) \circ \mathcal{I} \in \mathbb{D}^{2,1}(C_0;\mathbb{R})$. However, we only get $\mathrm{div}(U) \in \mathbb{W}^{2,1}(C_{x_0}M;\mathbb{R})$ from Theorem 6.1 in [25], where the weak Sobolev space $\mathbb{W}^{2,1}$ is defined as the domain of the adjoint of the restriction of $D^*_{\mathcal{H}}$ to $\mathbb{D}^{2,1}(\mathcal{H}^*)$, i.e.,

$$\mathbb{W}^{2,1} = \mathrm{Dom}((D^*_{\mathcal{H}}|_{\mathbb{D}^{2,1}(\mathcal{H}^*)})^*),$$

and $\mathbb{D}^{2,1} \subset \mathbb{W}^{2,1}$ (see [25]). Although there doesn't seem to be any result to show directly $\mathrm{div}(U) \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$, we can take a sequence of $U_j \in \mathbb{D}^{2,2}(\mathcal{H})$ such that $U_j \to U$ in $\mathbb{D}^{2,1}$. The same argument above tells us that $U_j \circ \mathcal{I} \in \mathbb{D}^{2,2}(C_0;\mathcal{H})$, $(\mathrm{div}(U_j)) \circ \mathcal{I} \in \mathbb{D}^{2,2}(C_0;\mathbb{R})$, and $(\mathrm{div}(U_j)) \circ \mathcal{I} \to (\mathrm{div}(U)) \circ \mathcal{I}$ in $\mathbb{D}^{2,1}(C_0;\mathbb{R})$. Proposition 7.3 in [25] shows that $(\mathrm{div}(U_j)) \circ \mathcal{I} \in \mathbb{D}^{2,2}(C_0;\mathbb{R})$ implies $\mathrm{div}(U_j) \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$. Corollary 4.3 of [25] shows that the set

$$\{f \circ \mathcal{I} | f \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})\}$$

is closed in $\mathbb{D}^{2,1}(C_0;\mathbb{R})$, so the convergence of $(\mathrm{div}(U_j)) \circ \mathcal{I}$ to $(\mathrm{div}(U)) \circ \mathcal{I}$ in $\mathbb{D}^{2,1}(C_0;\mathbb{R})$ implies, indeed, $\mathrm{div}(U) \in \mathbb{D}^{2,1}(C_{x_0}M;\mathbb{R})$.

We now make use of the splitting (2.13) and our standing assumption 2.1.1 to calculate, for any $h \in H$,

$$\begin{aligned}
& D[(\mathrm{div}(U)) \circ \mathcal{I}](h) \\
= \; & -\int_0^T < D[Y\frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}], K^\perp(x_t)dB_t >_{\mathbb{R}^m}(h) - \int_0^T < Y\frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, K^\perp(x_t)\dot{h}_t >_{\mathbb{R}^m} dt \\
& -\int_0^T < Y\frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, D(K^\perp(x_t))dB_t >_{\mathbb{R}^m}(h) \\
= \; & -\int_0^T < XD(Y\frac{\mathbb{D}}{dt}U_t)T\mathcal{I}(-)_t, X(x_t)dB_t >_{x_t}(h) - \int_0^T <\frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, X(x_t)\dot{h}_t >_{\mathbb{R}^m} dt \\
& -\int_0^T < \frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, XD[YX]T\mathcal{I}(-)_t dB_t >_{x_t}(h) \\
= \; & -\int_0^T < \tilde{\nabla}_{T\mathcal{I}(-)_t}\frac{\mathbb{D}}{dt}U_t, X(x_t)dB_t >_{x_t}(h) - \int_0^T < \frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, X(x_t)\dot{h}_t >_{\mathbb{R}^m} dt \\
& -\int_0^T < \frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, \tilde{\nabla}_{T\mathcal{I}(-)_t}X(x_t)dB_t >_{x_t}(h) \\
= \; & -\int_0^T < \tilde{\nabla}_{T\mathcal{I}(-)_t}\frac{\mathbb{D}}{dt}U_t, X(x_t)\tilde{/\!/}_t^{\sigma} d\tilde{B}_t >_{x_t}(h) - \int_0^T < \frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, X(x_t)\dot{h}_t >_{\mathbb{R}^m} dt \\
& -\int_0^T < \frac{\mathbb{D}}{dt}U_t \circ \mathcal{I}, \tilde{\nabla}_{T\mathcal{I}(-)_t}X(x_t)\tilde{/\!/}_t^{\sigma} d\beta_t >_{x_t}(h).
\end{aligned} \tag{3.9}$$

Recall the intertwining formula from [25]:

$$D_{\mathcal{H}} f[\overline{T\mathcal{I}(h)}] = \overline{D(f \circ \mathcal{I})(h)}, \tag{3.10}$$

where $f : C_{x_0}M \to \mathbb{R}$ is in $\mathbb{D}^{2,1}$ and $h : C_0 \to H$. If $V = \overline{T\mathcal{I}}(h)$ for a constant $h \in H$, we can apply (3.10) to $f = \mathrm{div}(U)$ to arrive at

$$D_{\mathcal{H}}(\mathrm{div}(U))[V(\sigma)] = \mathbb{E}\{D[(\mathrm{div}(U) \circ \mathcal{I}](h)|\mathcal{I} = \sigma\}. \tag{3.11}$$

To calculate this, first observe that taking conditional expectation of the right-hand side of (3.9) annihilates the last term, since $\beta_{\cdot}$, as a Brownian motion on $\mathrm{Ker}X(x_0)$, is independent of $\mathcal{F}_{\cdot}^{x_0}$. Therefore, using (2.12), we obtain

$$\begin{aligned}
& D_{\mathcal{H}}(\mathrm{div}(U))[V(\sigma)] \\
= \quad & -\int_0^T < \tilde{\nabla}_{ev_t(-)} \frac{\mathbb{D}}{dt} U, d\{x\}_t >_{x_t} [\overline{T\mathcal{I}}(h)] - \int_0^T < \frac{\mathbb{D}}{dt} U, X(x_t)\dot{h}_t >_{\mathbb{R}^m} dt \\
= \quad & -\int_0^T < \tilde{\nabla}_{ev_t(-)} \frac{\mathbb{D}}{dt} U, d\{x\}_t >_{x_t} (V) - \int_0^T < \frac{\mathbb{D}}{dt} U, \frac{\mathbb{D}}{dt} V >_{\mathbb{R}^m} dt \\
= \quad & -\int_0^T < \frac{\mathbb{D}}{dt}(\boldsymbol{\nabla}_{-} U), d\{x\}_t >_{x_t} (V) - < U, V >_{\mathcal{H}} .
\end{aligned}$$

This finishes the proof of (3.7) for V a constant $\mathcal{H}$-vector field, i.e., one of the form $V = \overline{T\mathcal{I}}(h)$ with $h \in H$. Since each term in (3.7) is linear and continuous in V, we can immediately extend this result to a general $\mathcal{H}$-vector field V.

Note that, for a constant $\mathcal{H}$-vector field V, equation (3.8) holds since in the computation of (3.9), we can apply the commutation result for the classical Wiener space to put the constant vector h inside the stochastic integral. Our discussion in Section 3.1 shows that (3.8) actually extends to an adapted $\mathcal{H}$-vector field V, since V being adapted to $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$ implies that $h = \boldsymbol{Y} V \circ \mathcal{I}$ is adapted to $\{\mathcal{F}_t\}_{t\in[0,T]}$, by Lemma 2.2.5. So again in (3.9), we can put the adapted vector field h inside the stochastic integral. □

It would be an interesting exercise to remove the adaptedness condition here, which might give some clue to the Weitzenböck identity on the path spaces; see the works of Cruzeiro and Fang [9; 10; 11] for related discussions.

In the next chapter, we rewrite our commutation formula in terms of exterior derivatives on the Wiener space, which reveals the connection with the Weitzenböck identity. In Chapter 6, we make similar attempts on the path spaces.

Chapter 4

The Clark-Ocone Approach to the Hodge Theory on the Wiener Spaces

Shigekawa [55] has given a complete L^2 de Rham theory for abstract Wiener spaces, using a Weitzenböck formula for the Hodge Laplacian with positive curvature to prove the triviality of the de Rham cohomology. With a view to extend his result to more general settings, we present two alternative proofs: one in this chapter for the special case of the classical Wiener space, based on generalised Clark-Ocone formulae for differential forms; and the other in Chapter 5 for general abstract Wiener spaces, based on the representation theory of symmetric groups in conjunction with the Wiener chaos expansion.

We start by giving some motivation for the first approach. Denote by $\mathfrak{h}_q$ the set of all the harmonic forms of degree q, i.e., $\phi \in L^2\Gamma(\wedge^q H)^*$ is in $\mathfrak{h}_q$ if $\phi \in \mathrm{Dom}(\Delta)$ and $\Delta\phi = 0$, where $\Delta = dd^* + d^*d$ is the de Rham-Hodge-Kodaira Laplacian mapping L^2 forms to L^2 forms. Note that

$$\phi \in \mathfrak{h}_q \iff d^q\phi = 0 \text{ and } d^{(q-1)*}\phi = 0.$$

Theorem (Shigekawa [55])**.** *$L^2\Gamma(\wedge^q H)^* = \mathrm{Image}(d^{q-1}) \oplus \mathrm{Image}(d^{q*}) \oplus \mathfrak{h}_q$. Moreover,*

1. $\mathrm{Image}(d^{q-1}) = \mathrm{Ker}(d^q)$,

2. $\mathrm{Image}(d^{q*}) = \mathrm{Ker}(d^{(q-1)*})$, *and*

3. $\mathfrak{h}_q = \{0\}$ *for* $q \geq 1$, *and* $\mathfrak{h}_0 = \{$*constant functionals*$\}$.

Observe that the Clark-Ocone formula

$$F = \mathbb{E}F + \int_0^T < \mathbb{E}[\frac{d}{dt}(\nabla F)_t|\mathcal{F}_t], dB_t >_{\mathbb{R}^m}, \quad \forall F \in \mathbb{D}^{2,1},$$

implies immediately that ∇ has closed range, as well as

1_0. $\nabla F = 0 \iff F = \text{constant}(= \mathbb{E}F)$; and

2_0. $\mathbb{E}F = 0 \iff F \in \text{Image}(\delta) = \text{Image}(\text{div})$,

which are precisely the statements of the above theorem for L^2 functions, considered as zero-forms, on the classical Wiener space. In fact, not only does the Clark-Ocone formula give an explicit Hodge decomposition for zero-forms in the form of

$$F = \text{constant} + \delta(v),$$

but it also provides the expressions for v and the constant in terms of F.

Nualart and Zakai [50] pointed out the direct consequence of ∇ having a closed range: any H-valued L^2 process u has a unique orthogonal decomposition

$$u = \nabla F + v, \tag{4.1}$$

where $F \in \mathbb{D}^{2,1}$, $v \in \text{Dom}(\delta)$, and $\delta(v) = 0$. This would be the proto-type for a Clark-Ocone formula for one-forms (considered as the Riesz dual of H-valued processes). However, Nualart and Zakai [50] give no explicit solution (F, v) to the above decomposition of u, and we explain below the difficulty of expressing an explicit Hodge decomposition in the form of (4.1). We propose, instead, a Clark-Ocone-type formula for a one-form ϕ in terms of two separate equations,

$$\phi = DF + M(d^1\phi), \quad \forall \phi \in \text{Dom}(d^1),$$

and

$$\phi = d^{1*}\theta + N(d^{0*}\phi), \quad \forall \phi \in \text{Dom}(d^{0*}),$$

where M and N are some nice linear functions. These equations imply, respectively, the statements that

1_1. $d^1\phi = 0 \iff \phi = DF, \quad$ for some $F \in \mathbb{D}^{2,1} = \text{Dom}(\nabla)$; and

2_1. $d^{0*}\phi = 0 \iff \phi = d^{1*}\theta, \quad$ for some $\theta \in \text{Dom}(d^{1*}) \subset L^2\Gamma(\wedge^2 H)^*$, which are equivalent to Shigekawa's theorem for $q = 1$. We mention in passing that Üstünel [58] also gives a direct sum representation for H-valued distributions

$$D'_H = \nabla(D') + \mathrm{Ker}(\delta),$$

where D' is the space of distributions defined as the continuous dual of $\mathbb{D}^\infty$, D'_H consists of H-valued distributions defined as the continuous dual of $\mathbb{D}^\infty(H)$. In fact, he writes more explicitly in terms of the Ornstein-Uhlenbeck operator L

$$u = \nabla(-\frac{1}{2}L^{-1}\delta u) + \eta, \quad \forall u \in D'_H.$$

with $\eta \in \mathrm{Ker}(\delta)$. Again, there is no explicit expression for η. What we show below should be generalisable to distributions, following Üstünel's work, but we do not pursue this direction here.

In general, for $q \in \mathbb{N}$, Shigekawa's result is that, for any L^2 H-q-form ϕ,

$\mathbf{1_q}$. $d^q\phi = 0 \iff \phi = d^{q-1}\psi, \quad$ for some $\psi \in \mathrm{Dom}(d^{q-1}) \subset L^2\Gamma(\wedge^{q-1}H)^*$; and

$\mathbf{2_q}$. $d^{(q-1)*}\phi = 0 \iff \phi = d^{q*}\theta, \quad$ for some $\theta \in \mathrm{Dom}(d^{q*}) \subset L^2\Gamma(\wedge^{q+1}H)^*$.

Instead of seeking a representation for a q-form ϕ in the form of

$$\phi = d^{q-1}\psi + d^{q*}\theta + h, \tag{4.2}$$

and showing the harmonic component $h = 0$, we have a combination of two equations:

$$\phi = d^{q-1}\psi + M_q(d^q\phi), \quad \forall \phi \in \mathrm{Dom}(d^q), \tag{4.3}$$

and

$$\phi = d^{q*}\theta + N_q(d^{(q-1)*}\phi), \quad \forall \phi \in \mathrm{Dom}(d^{(q-1)*}), \tag{4.4}$$

where M_q and N_q are some linear functions. We give the expression for M_q and N_q in Section 4.3 below.

Unlike the situation on Riemannian path spaces, the calculation for higher-order forms in the classical Wiener space is not substantially more difficult than that of one-forms. However, we choose to explain the case of one-forms separately, in Section 4.2, so as to present a clearer outline of the basic arguments, which serves as a direct analogue of the result for one-forms on Riemannian path spaces in Chapter 6. If not for such comparative reasons, Section 4.2 can be skipped as Section 4.3 gives the general proof for q-forms. We conclude the chapter with some results from our attempts at adapting the Clark-Ocone approach to abstract Wiener spaces.

4.1 Notions and Notation

We introduce the following bracket notation on a Hilbert space, say, X, with inner product $<,>_X$, as a generalisation of the interior product:

$$ {}_{(i)}< x_1 \otimes \cdots \otimes x_k, x_{k+1} >_X = < x_i, x_{k+1} >_X \, x_1 \otimes \cdots \otimes \hat{x}_i \otimes \cdots \otimes x_k, $$

where $x_j \in X$ for $j = 1$ to $k+1$, and $\hat{x}_i$ indicates the omission of x_i. As usual, for $n \in \mathbb{N}$, we denote by $X^{\otimes n}$, or interchangeably $\otimes^n X$, the standard Hilbert space completion of the algebraic tensor product $\underbrace{X \otimes \cdots \otimes X}_{n \text{ times}}$, so $X^{\otimes n}$ is a Hilbert space with a complete orthonormal basis, say, $\{e_{k_1} \otimes \cdots \otimes e_{k_n}\}_{k_1,\ldots,k_n=1}^{\infty}$, if $\{e_k\}_{k=1}^{\infty}$ is a complete orthonormal basis of X. We use $\wedge^n X$, and sometimes $X^{\wedge n}$ interchangeably, to denote the subspaces of n-fold skew-symmetric tensor products, completed using the Hilbert space cross norm inherited from $X^{\otimes n}$. In other words,

$$ X^{\wedge n} = \{x \in X^{\otimes n} : A^n(x) = x\}, $$

where A^n is the alternating map defined in (1.3). Similarly, we have $X^{\odot n}$ denoting the completed n-fold symmetric tensor products, and the convention that $X^{\otimes 0} = \mathbb{R}$.

We also need the transposition operator $\tau_{i,j} : X^{\otimes n} \to X^{\otimes n}$, which acts by exchanging the i-th and j-th components of a tensor product: that is, given any $x_1, \cdots, x_n \in X$,

$$ \tau_{i,j}(x_1 \otimes \cdots \otimes x_i \otimes \cdots \otimes x_j \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_j \otimes \cdots \otimes x_i \otimes \cdots \otimes x_n. \quad (4.5) $$

We write simply τ when it acts on a two-tensor. By the definition of exterior product, we have

$$ \begin{aligned} \tau_{i,j}(x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_n) &= x_1 \wedge \cdots \wedge x_j \wedge \cdots \wedge x_i \wedge \cdots \wedge x_n \\ &= -x_1 \wedge \cdots \wedge x_i \wedge \cdots \wedge x_j \wedge \cdots \wedge x_n. \end{aligned} $$

In terms of the standard interior product ι, we can also write

$$ \begin{aligned} {}_{(i)}< x_1 \wedge \cdots \wedge x_n, x_{n+1} >_X &= (-1)^{i-1} {}_{(1)}< x_i \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n, x_{n+1} >_X \\ &= (-1)^{i-1} \iota_{x_{n+1}}(x_i \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n). \end{aligned} $$

For example,

$$_{(1)}< x_1 \wedge x_2, x_3 >_X = \frac{1}{2}[< x_1, x_3 >_X x_2 - < x_2, x_3 >_X x_1] = \iota_{x_3}(x_1 \wedge x_2),$$

so this bracket notation is nothing unusual but convenient for our purpose, as will be obvious in the sequel.

Recall the exterior derivative $d^q : \mathrm{Dom}(d^q) \subset L^2\Gamma(H^{\wedge q})^* \to L^2\Gamma(H^{\wedge(q+1)})^*$, $q \in \mathbb{N}$, defined in Chapter 1 by

$$d^q \phi_\sigma = A^{q+1}(D\phi_\sigma). \tag{4.6}$$

Its dual operator $d^{q*} : \mathrm{Dom}(d^{q*}) \subset L^2\Gamma(H^{\wedge(q+1)})^* \to L^2\Gamma(H^{\wedge q})^*$, is given by

$$d^{q*} = (A^{q+1}D)^* = D^*(A^{q+1})^* = D^*, \tag{4.7}$$

where $(A^{q+1})^*$ is just the inclusion map $L^2\Gamma(H^{\otimes(q+1)})^* \to L^2\Gamma(H^{\wedge(q+1)})^*$, and we restrict d^{q*} to act on $L^2\Gamma(H^{\wedge(q+1)})^*$.

In anticipation of our later discussion on path spaces, we mention that the definition (4.6) is consistent, up to the convention for the constant coefficient $(q+1)$, with the standard invariant formula for d^q (see, e.g., Lang [43]), which reads

$$\begin{aligned}(q+1)d^q\phi_\sigma(h_0, h_1, \cdots, h_q) &= \sum_{j=0}^{q}(-1)^j \mathcal{L}_{h_j}[\phi(h_0, \cdots, \hat{h}_j, \cdots, h_q)](\sigma) \\ &\quad + \sum_{i<j}(-1)^{i+j}\phi_\sigma([h_i, h_j], h_0, \cdots, \hat{h}_i, \cdots, \hat{h}_j, \cdots, h_q),\end{aligned}$$

where $\phi \in L^2\Gamma(H^{\wedge q})^* \cap \mathrm{Dom}(d^q)$, and $h_0, h_1, \cdots, h_q$ are smooth H-vector fields. On the flat Wiener spaces, the second term disappears for $h_0, h_1, \cdots, h_q \in H$, so we have, as in (4.6),

$$(q+1)d^q\phi_\sigma(h_0, h_1, \cdots, h_q) = \sum_{j=0}^{q}(-1)^j < \nabla\phi_\sigma(h_0, \cdots, \hat{h}_j, \cdots, h_q), h_j >_H . \tag{4.8}$$

In particular, for $q = 1$, we have

$$\begin{aligned}2(d^1\phi)_\sigma(h_0, h_1) &= < \nabla\phi(h_1)](\sigma), h_0 >_H - < \nabla[\phi(h_0)](\sigma), h_1 >_H \\ &= D[\phi(h_1)]_\sigma(h_0) - D[\phi(h_0)]_\sigma(h_1).\end{aligned}$$

It is sometimes convenient to write, for $h_1, \cdots, h_{q+1} \in H$,

$$
\begin{aligned}
& (q+1) d^q \phi_\sigma(h_1, \cdots, h_{q+1}) \\
= \; & (-1)^q < \nabla \phi_\sigma(h_1, \cdots, h_q), h_{q+1} >_H \\
& + \sum_{j=1}^{q} (-1)^{j-1} < \nabla \phi_\sigma(h_1, \cdots, h_{j-1}, \hat{h}_j, h_{j+1}, \cdots, h_{q+1}), h_j >_H \\
= \; & (-1)^q [< \nabla \phi_\sigma(h_1, \cdots, h_q), h_{q+1} >_H \\
& - \sum_{j=1}^{q} < \nabla \phi_\sigma(h_1, \cdots, h_{j-1}, h_{q+1}, h_{j+1}, \cdots, h_q), h_j >_H]. \qquad (4.9)
\end{aligned}
$$

By the Riesz representation theorem, we have a natural correspondence between L^2 differential H-forms and L^2 skew-symmetric H-vector fields, so corresponding to d^q, we can define an operator $d^{q\sharp} : \mathrm{Dom}(d^{q\sharp}) \subset L^2\Gamma H^{\wedge q} \to L^2\Gamma H^{\wedge(q+1)}$ by

$$d^{q\sharp} u_\sigma = (d^q u_\sigma^\sharp)^\sharp, \quad u \in L^2\Gamma H^{\wedge q},\ \sigma \in E,$$

where $u^\sharp \in L^2\Gamma(\wedge^q H)^*$ is given by

$$u_\sigma^\sharp(h) = < u_\sigma, h >_{H^{\wedge q}},$$

and clearly $u \in \mathrm{Dom}(d^{q\sharp})$ iff $u^\sharp \in \mathrm{Dom}(d^q)$. Similarly, we have the corresponding $d^{q*\sharp} : \mathrm{Dom}(d^{q*\sharp}) \subset L^2\Gamma H^{\wedge(q+1)} \to L^2\Gamma H^{\wedge q}$, defined by

$$d^{q*\sharp} u_\sigma = (d^{q*} u_\sigma^\sharp)^\sharp, \quad u \in L^2\Gamma H^{\wedge(q+1)},\ \sigma \in E,$$

where $u \in \mathrm{Dom}(d^{q*\sharp})$ iff $u^\sharp \in \mathrm{Dom}(d^{q*})$. It is obvious that $d^{q*\sharp} = d^{q\sharp*}$.

Using this $\sharp$ notation, we switch freely between vector fields and differential forms throughout. In fact, we will state most of our results in terms of vector fields. For example, we write (4.8) as follows

$$
\begin{aligned}
& (q+1) < d^{q\sharp} \phi^\sharp, h_0 \wedge \cdots \wedge h_q >_{H^{\wedge(q+1)}} \\
= \; & < \nabla \phi^\sharp, \sum_{j=0}^{q} (-1)^j (h_0 \wedge \cdots \wedge \hat{h}_j \wedge \cdots \wedge h_q) \otimes h_j >_{H^{\otimes(q+1)}} \\
= \; & (-1)^q (q+1) < \nabla \phi^\sharp, h_0 \wedge \cdots \wedge h_q >_{H^{\wedge(q+1)}} .
\end{aligned}
$$

We can also understand the action of d^{q*} by looking at $d^{q*\sharp}$: for any $u \in L^2\Gamma H^{\wedge q}$

and $h_0, h_1, \cdots, h_q \in H$, we apply (4.8) to obtain

$$\begin{aligned}
& \mathbb{E} < u, (q+1)d^{q*\sharp}(h_0 \wedge \cdots \wedge h_q) >_{H^{\wedge q}} \\
= \; & \mathbb{E} < (q+1)d^{q\sharp}u, h_0 \wedge \cdots \wedge h_q >_{H^{\wedge(q+1)}} \\
= \; & \mathbb{E} < \sum_{j=0}^{q} (-1)^j \nabla u^\sharp (h_0 \wedge \cdots \wedge \hat{h}_j \wedge \cdots \wedge h_q), h_j >_H \\
= \; & \mathbb{E} < u, \sum_{j=0}^{q} (-1)^j \delta(h_j)(h_0 \wedge \cdots \wedge \hat{h}_j \wedge \cdots \wedge h_q) >_{H^{\wedge q}} \qquad (4.10) \\
= \; & \mathbb{E} < u, (-1)^q (q+1)\delta(h_0 \wedge \cdots \wedge h_q) >_{H^{\wedge q}} .
\end{aligned}$$

In short, we verify that

$$d^{q\sharp} = (-1)^q A^{q+1}\nabla, \quad \text{and} \quad d^{q*\sharp} = (-1)^q \delta. \qquad (4.11)$$

The extra sign $(-1)^q$, cf. (4.6) and (4.7), comes from our convention of regarding the gradient, say, ∇u, of a vector-valued function, $u \in \mathbb{D}^{2,1}(X)$, as an element of $X \otimes H$, i.e.,

$$D < u, x >_X (h) = < \nabla u, x \otimes h >_{X \otimes H}, \quad \forall x \in X, h \in H,$$

and having δ acting on the last component of a H-n-tensor, since we have, for $X = H^{\otimes n-1}$, and $h_0, h_1, \cdots, h_q \in H$,

$$\begin{aligned}
\mathbb{E}\, D < u, h_1 \otimes \cdots \otimes h_{n-1} >_{H^{\otimes n-1}} (h_n) &= \mathbb{E} < \nabla u, h_1 \otimes \cdots \otimes h_n >_{H^{\otimes n}} \\
&= \mathbb{E} < u, \delta(h_1 \otimes \cdots \otimes h_{n-1} \otimes h_n) >_{H^{\otimes n-1}} \\
&= \mathbb{E} < u, h_1 \otimes \cdots \otimes h_{n-1} \delta(h_n) >_{H^{\otimes n-1}} .
\end{aligned}$$

On the classical Wiener space $E = C_0([0,T];\mathbb{R}^m)$, let $ev_s : E \to \mathbb{R}^m$ be the evaluation map at time $s \in [0,T]$, i.e, for any $u \in E$, we have $ev_s(u) = u_s$. For a tensor $u \in \otimes_\epsilon^q E$, where $\otimes_\epsilon^q E$ is the space of injective tensor products, i.e., the completion of the algebraic tensor products using the injective cross norm, we can make use of the isometry

$$i : \otimes_\epsilon^q E \to C_0([0,T]^q; \otimes^q \mathbb{R}^m),$$

where $C_0([0,T]^q; \otimes^q \mathbb{R}^m)$ consists of continuous functions $\sigma : [0,T]^q \to \otimes^q \mathbb{R}^m$ such that $\sigma(t_1, \cdots, t_q) = 0$ if $t_j = 0$ for any $j = 1$ to q (see [26] for a more detailed description). So we have,

$$u_{s_1,\cdots,s_q} = i(u)(s_1,\cdots,s_q) = (ev_{s_1}\otimes\cdots\otimes ev_{s_q})u.$$

We also have the isomorphism between the two Hilbert spaces $L^2([0,T];\mathbb{R}^m)$ and $H = L_0^{2,1}([0,T];\mathbb{R}^m)$ given by the indefinite integral

$$\int_0^{\cdot} : L^2([0,T];\mathbb{R}^m) \to H,$$

with the inverse map

$$\frac{d}{d.} : H \to L^2([0,T];\mathbb{R}^m).$$

This gives rise to an isometry between the Hilbert spaces of the tensor powers $\otimes^q H$ and $\otimes^q L^2([0,T];\mathbb{R}^m) \cong L^2([0,T]^q;\otimes^q\mathbb{R}^m)$. Therefore, $u \in \otimes^q H$ iff we can write

$$u_{s_1,\cdots,s_q} = \int_0^{s_1}\cdots\int_0^{s_q} \frac{\partial^q}{\partial r_1\cdots\partial r_q} u_{r_1,\cdots,r_q} dr_1\cdots dr_q,$$

where $\frac{\partial^q}{\partial r_1\cdots\partial r_q} u_{r_1,\cdots,r_q}$ is the weak derivative.

In long equations, we use the following abbreviations for some differential operations on the classical Wiener space:

$$\partial^q_{s_1,\cdots,s_q} u = \frac{\partial^q}{\partial s_1\cdots\partial s_q} u_{s_1,\cdots,s_q}, \quad \forall u \in \wedge^q H,$$

$$D_r u = \frac{d}{dr}(\nabla u)_r, \quad \forall u \in \mathbb{D}^{2,1},$$

$$d^{(q-1)\sharp}_{s_1,\cdots,s_q} u = \frac{\partial^q}{\partial s_1\cdots\partial s_q}(d^{(q-1)\sharp}u)_{s_1,\cdots,s_q}, \quad \forall u \in \mathrm{Dom}(d^{(q-1)\sharp}),$$

and

$$d^{q*\sharp}_{s_1,\cdots,s_q} u = \frac{\partial^q}{\partial s_1\cdots\partial s_q}(d^{q*\sharp}u)_{s_1,\cdots,s_q}, \quad \forall u \in \mathrm{Dom}(d^{q*\sharp}).$$

Note that the second convention above is standard, as in Nualart [47], using the isomorphism between $L^2([0,T];\mathbb{R}^m)$ and H mentioned above.

With these abbreviations, equations (4.8) and (4.9) above can be expressed, for a.e. $s_1, \cdots, s_{q+1} \in [0,T]$, as

$$\begin{aligned} & (q+1) d^{q\sharp}_{s_1,\cdots,s_{q+1}} u \\ = & \sum_{j=1}^{q+1} (-1)^{j-1} D_{s_j} \partial^q_{s_1,\cdots,\hat{s}_j,\cdots,s_{q+1}} u \qquad (4.12) \\ = & (-1)^q \left[D_{s_{q+1}} \partial^q_{s_1,\cdots,s_q} u - \sum_{j=1}^{q} \tau_{j,q+1} D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},s_{q+1},s_{j+1},\cdots,s_q} u \right]. \qquad (4.13) \end{aligned}$$

We often omit the transposition operators when the indices give clear indication of the ordering. In particular, for $q = 1$, we have

$$2\, d^{1\sharp}_{s_1,s_2} u = \tau D_{s_1} \dot{u}_{s_2} - D_{s_2} \dot{u}_{s_1}. \qquad (4.14)$$

Similarly, equation (4.10) above gives, for $h_0, h_1, \cdots, h_q \in H$,

$$\begin{aligned} & (q+1) d^{q*\sharp}_{s_1,\cdots,s_q} (h_0 \wedge \cdots \wedge h_q) \\ = & \sum_{j=0}^{q} (-1)^j \Big(\int_0^T < (\dot{h_j})_r, dB_r >_{\mathbb{R}^m} \Big) \partial^q_{s_1,\cdots,s_q} (h_0 \wedge \cdots \wedge \hat{h}_j \wedge \cdots \wedge h_q) \\ = & \sum_{j=0}^{q} (-1)^j \int_0^T {}_{(1)}< \partial^{q+1}_{r,s_1,\cdots,s_q} \Big(h_j \wedge h_0 \wedge \cdots \wedge \hat{h}_j \wedge \cdots \wedge h_q \Big), dB_r >_{\mathbb{R}^m} \\ = & (q+1)(-1)^i \int_0^T {}_{(i+1)}< \partial^{q+1}_{s_1,\cdots,s_{i-1},r,s_i,\cdots,s_q} (h_0 \wedge \cdots \wedge h_q), dB_r >_{\mathbb{R}^m}, \qquad (4.15) \end{aligned}$$

where i in (4.15) above can be any integer between 0 and q.

4.2 One-Forms on the Classical Wiener Space

On the classical Wiener space $(E = C_0, H = L^{2,1}_0, \gamma)$, we define

$$CO(\phi) = \int_0^T < \mathbb{E}(\dot{\phi}^\sharp_t | \mathcal{F}_t), dB_t >_{\mathbb{R}^m}, \quad \phi \in L^2\Gamma H^*, \qquad (4.16)$$

where $\phi^\sharp : E \to H$ is given by $\phi_\sigma(h) =< \phi^\sharp_\sigma, h >_H$, for any $h \in H$. Note that CO is continuous as a map from $L^2\Gamma H^*$ to $L^2(E;\mathbb{R})$.

From the discussion in Section 4.1, it follows that, for $\phi \in \mathrm{Dom}(d^1)$ and $a, b \in H$,

$$(d^1\phi)_\sigma(a,b) =< (d^1\phi)^\sharp_\sigma, a \wedge b >_{\wedge^2 H} =< d^{1\sharp}\phi^\sharp_\sigma, a \wedge b >_{\wedge^2 H},$$

and

$$\mathbb{E}(d^1\phi)_\sigma(a,b) = \mathbb{E} < \phi^\sharp_\sigma, (d^{1\sharp})^*(a\wedge b) >_H .$$

Recall that d^1 is a closed operator, and Dom(d^1) is the closure of $\mathbb{D}^{2,1}$ in the L^2 graph norm. Since $\mathbb{D}^{2,1}(H^*)$ is dense in $L^2\Gamma H^*$, so is Dom(d^1). If $\phi \in \mathbb{D}^{2,1}(H^*)$, we can write, as in equations (4.8) and (4.9),

$$\begin{aligned}
& (d^1\phi)_\sigma(a,b) \\
= \ & \frac{1}{2}[D(\phi(b))_\sigma(a) - D(\phi(a))_\sigma(b)] \\
= \ & \frac{1}{2}\int_0^T\int_0^T \Big(< D_t\dot{\phi}^\sharp_s, \dot{b}_s \otimes \dot{a}_t >_{\mathbb{R}^m\otimes\mathbb{R}^m} - < D_t\dot{\phi}^\sharp_s, \dot{a}_s\otimes\dot{b}_t >_{\mathbb{R}^m\otimes\mathbb{R}^m}\Big)\, dsdt \\
= \ & \frac{1}{2}\int_0^T\int_0^T < \tau(D_s\dot{\phi}^\sharp_t) - D_t\dot{\phi}^\sharp_s,\ \dot{a}_s\otimes\dot{b}_t >_{\mathbb{R}^m\otimes\mathbb{R}^m} dsdt,
\end{aligned}$$

that is

$$\frac{\partial^2}{\partial s\partial t}(d^1\phi)^\sharp_{s,t} = \frac{1}{2}[\tau(D_s\dot{\phi}^\sharp_t) - D_t\dot{\phi}^\sharp_s]. \tag{4.17}$$

We also have, for $a, b \in H$,

$$\begin{aligned}
\mathbb{E}(d^1\phi)(a,b) &= \mathbb{E} < \phi^\sharp, (d^{1\sharp})^*(a\wedge b) >_H \\
&= \mathbb{E}\ \frac{1}{2}[D(\phi(b))(a) - D(\phi(a))(b)] \\
&= \mathbb{E} < \phi^\sharp, \frac{1}{2}(b\delta a - a\delta b) >_H,
\end{aligned}$$

that is, as in equations (4.10) and (4.15),

$$\begin{aligned}
& (d^{1\sharp})^*(a\wedge b) \\
= \ & \frac{1}{2}(b\delta a - a\delta b) \\
= \ & \frac{1}{2}\int_0^\cdot \dot{b}_s\int_0^T < \dot{a}_r, dB_r >_{\mathbb{R}^m} ds - \int_0^\cdot \dot{a}_s\int_0^T < \dot{b}_r, dB_r >_{\mathbb{R}^m} ds \\
= \ & \frac{1}{2}\int_0^\cdot\int_0^T {}_{(2)}\!< \dot{b}_s\otimes\dot{a}_r, dB_r >_{\mathbb{R}^m} - \int_0^\cdot\int_0^T {}_{(2)}\!< \dot{a}_s\otimes\dot{b}_r, dB_r >_{\mathbb{R}^m} ds \\
= \ & -\int_0^\cdot\int_0^T {}_{(2)}\!< (\dot{a}\wedge\dot{b})_{s,r}, dB_r >_{\mathbb{R}^m} ds.
\end{aligned} \tag{4.18}$$

Proposition 4.2.1 (Clark-Ocone Formula for One-Forms: I)**.** *If $\phi \in L^2\Gamma H^*$ is in*

Dom(d^1), *we have* $CO(\phi) \in \mathbb{D}^{2,1}$, *and*

$$\frac{d}{dt}[\nabla CO(\phi)]_t - \dot{\phi}_t^\sharp = 2\int_t^T {}_{(2)}< \mathbb{E}[\frac{\partial^2}{\partial t \partial s}(d^1\phi)^\sharp_{t,s}|\mathcal{F}_s], dB_s >_{\mathbb{R}^m}, \quad a.e.\ t \in [0,T]. \tag{4.19}$$

If $\phi \in \mathbb{D}^{2,k}$, *we also have* $CO(\phi) \in \mathbb{D}^{2,k}$.

Remark 4.2.2. In general, the two terms $\frac{d}{dt}[\nabla CO(\phi)]_t$ and

$$2\int_t^T {}_{(2)}< \mathbb{E}[\frac{\partial^2}{\partial t \partial s}(d^1\phi)^\sharp_{t,s}|\mathcal{F}_s], dB_s >_{\mathbb{R}^m}$$

are not orthogonal to each other, so equation (4.19) should not be naively taken as the usual Hodge decomposition.

Proof. If $\phi \in \mathbb{D}^{2,1}$, we can apply the Clark-Ocone formula to write, for almost all $t \in [0,T]$,

$$\dot{\phi}_t^\sharp = \mathbb{E}\dot{\phi}_t^\sharp + \int_0^T {}_{(2)}< \mathbb{E}(D_s\dot{\phi}_t^\sharp|\mathcal{F}_s), dB_s >_{\mathbb{R}^m} . \tag{4.20}$$

Observe that by our convention $\nabla\dot{\phi}_t^\sharp : E \to \mathbb{R}^m \otimes H$, and $D_s\dot{\phi}_t^\sharp : E \to \mathbb{R}^m \otimes \mathbb{R}^m$, so the bracket notation in (4.20) rightfully denotes the pairing of the second $\mathbb{R}^m$-component with dB_s.

Taking conditional expectation with respect to $\mathcal{F}_t$, we obtain

$$\mathbb{E}(\dot{\phi}_t^\sharp|\mathcal{F}_t) \quad = \quad \mathbb{E}\dot{\phi}_t^\sharp + \int_0^t {}_{(2)}< \mathbb{E}(D_s\dot{\phi}_t^\sharp|\mathcal{F}_s), dB_s >_{\mathbb{R}^m}, \tag{4.21}$$

hence

$$\dot{\phi}_t^\sharp - \mathbb{E}(\dot{\phi}_t^\sharp|\mathcal{F}_t) = \int_t^T {}_{(2)}< \mathbb{E}(D_s\dot{\phi}_t^\sharp|\mathcal{F}_s), dB_s >_{\mathbb{R}^m} . \tag{4.22}$$

Since $\phi \in \mathbb{D}^{2,1}$ implies $\dot{\phi}_t^\sharp \in \mathbb{D}^{2,1}$, by Lemma 2.4(ii) of [48] we have $\mathbb{E}(\dot{\phi}_t^\sharp|\mathcal{F}_t) \in \mathbb{D}^{2,1}$, and

$$D_s\mathbb{E}(\dot{\phi}_t^\sharp|\mathcal{F}_t) = \mathbb{E}(D_s\dot{\phi}_t^\sharp|\mathcal{F}_t)\mathbf{1}_{[0,t]}(s), \quad \text{a.e. in } [0,T] \times E. \tag{4.23}$$

This also shows that $\nabla\mathbb{E}(\dot{\phi}_t^\sharp|\mathcal{F}_t) \in \text{Dom}(\delta)$, since it is adapted. By the commutation formula in Lemma 3.1.1, we have $CO(\phi) \in \mathbb{D}^{2,1}$, and

$$
\begin{aligned}
& \frac{d}{dt}[\nabla CO(\phi)]_t - \dot{\phi}_t^\sharp \\
= \quad & \int_0^T {}_{(1)}< D_t \mathbb{E}(\dot{\phi}_s^\sharp | \mathcal{F}_s), dB_s >_{\mathbb{R}^m} + \mathbb{E}(\dot{\phi}_t^\sharp | \mathcal{F}_t) - \dot{\phi}_t^\sharp \qquad (4.24) \\
= \quad & \int_t^T {}_{(1)}< \mathbb{E}(D_t \dot{\phi}_s^\sharp | \mathcal{F}_s), dB_s >_{\mathbb{R}^m} - \int_t^T {}_{(2)}< \mathbb{E}(D_s \dot{\phi}_t^\sharp | \mathcal{F}_s), dB_s >_{\mathbb{R}^m} \\
= \quad & \int_t^T {}_{(2)}< \mathbb{E}[\tau(D_t \dot{\phi}_s^\sharp) - D_s \dot{\phi}_t^\sharp | \mathcal{F}_s], dB_s >_{\mathbb{R}^m} \qquad (4.25) \\
= \quad & 2 \int_t^T {}_{(2)}< \mathbb{E}[\frac{\partial^2}{\partial t \partial s}(d^1 \phi)_{t,s}^\sharp | \mathcal{F}_s], dB_s >_{\mathbb{R}^m},
\end{aligned}
$$

where the second equality follows from equations (4.23) and (4.22), and the last one from (4.17). So (4.19) holds for $\phi \in \mathbb{D}^{2,1}$. It is also clear from the above calculation, or from the commutation formula directly, that $CO(\phi) \in \mathbb{D}^{2,k}$ if $\phi \in \mathbb{D}^{2,k}$.

A general L^2 one-form $\phi \in \mathrm{Dom}(d^1)$ can be approximated by cylindrical one-forms $\phi_j \in \mathbb{D}^{2,1}$ such that $\phi_j \to \phi$ and $d^1\phi_j \to d^1\phi$ in L^2, then the above computation shows that $D(CO(\phi_j))$ converges in L^2, in the sense that, for $t \in [0, T]$,

$$
\begin{aligned}
\frac{d}{dt}[\nabla CO(\phi_j)]_t \quad = \quad & (\dot{\phi}_j^\sharp)_t + 2 \int_t^T {}_{(2)}< \mathbb{E}[\frac{\partial^2}{\partial t \partial s}(d^1 \phi_j)_{t,s}^\sharp | \mathcal{F}_s], dB_s >_{\mathbb{R}^m} \\
\to \quad & \dot{\phi}_t^\sharp + 2 \int_t^T {}_{(2)}< \mathbb{E}[\frac{\partial^2}{\partial t \partial s}(d^1 \phi)_{t,s}^\sharp | \mathcal{F}_s], dB_s >_{\mathbb{R}^m} .
\end{aligned}
$$

Since CO is continuous, we have $CO(\phi_j) \to CO(\phi)$, and by the closedness of D we get $CO(\phi) \in \mathbb{D}^{2,1}$, with $D_t CO(\phi)$ given by the limit above, and we are done. □

Corollary 4.2.3. *If $\phi \in L^2\Gamma H^*$ is in* $\mathrm{Dom}(d^1)$*, we have*

$$
\|D(CO(\phi)) - \phi\|_{L^2(E;H^*)} \leq \sqrt{2}\, \|d^1\phi\|_{L^2\Gamma(H^{\wedge 2})^*}.
$$

Proof. Applying the Itô isometry to equation (4.19), we get

$$\begin{aligned}
\int_0^T \|\frac{d}{dt}[\nabla CO(\phi)]_t - \dot{\phi}_t^\sharp\|^2_{L^2(E;\mathbb{R}^m)} dt &= 4\int_0^T \mathbb{E}\int_t^T \{\mathbb{E}[\frac{\partial^2}{\partial t \partial s}(d^1\phi)^\sharp_{t,s}|\mathcal{F}_s]\}^2 ds dt \\
&\leq 4\int_0^T \mathbb{E}\int_t^T \mathbb{E}[\frac{\partial^2}{\partial t \partial s}(d^1\phi)^\sharp_{t,s}]^2 ds dt \\
&= 4\int_0^T \mathbb{E}\int_0^s \mathbb{E}[\frac{\partial^2}{\partial t \partial s}(d^1\phi)^\sharp_{t,s}]^2 dt ds \\
&= 2\int_0^T \int_0^T \mathbb{E}[|\frac{\partial^2}{\partial t \partial s}(d^1\phi)^\sharp_{t,s}|^2] dt ds \\
&= 2\,\|(d^1\phi)^\sharp\|^2_{L^2(E;\wedge^2 H)} \\
&= 2\|d^1\phi\|^2_{L^2\Gamma(H^{\wedge 2})^*},
\end{aligned}$$

where the second line uses a basic property of conditional expectation, and the fourth uses the symmetry of the integrand. □

Corollary 4.2.4. *$\phi \in \mathrm{Dom}(d^1)$, $d^1\phi = 0 \implies \phi = DF$, with $F = CO(\phi) \in \mathbb{D}^{2,1}$. Moreover, if ϕ is smooth, so is F.*

Remark 4.2.5. The proof of Proposition 4.2.1 shows that the apparently weaker condition

$$\mathbb{E}[\frac{\partial^2}{\partial s \partial t}(d^1\phi)^\sharp_{s,t}|\mathcal{F}_s] = 0, \qquad \text{a.e. } 0 \leq t \leq s \leq T \tag{4.26}$$

implies the stronger condition $d^1\phi = 0$. Indeed, we have actually shown that

$$\mathbb{E}[\frac{\partial^2}{\partial s \partial t}(d^1\phi)^\sharp_{s,t}|\mathcal{F}_s] = 0, \text{ a.e. } 0 \leq t \leq s \leq T \iff D(CO(\phi)) = \phi \iff d^1\phi = 0.$$

This, however, isn't surprising if we observe that the Clark-Ocone formula itself implies that

$$\mathbb{E}[\frac{d}{ds}(\nabla F)_s|\mathcal{F}_s] = 0, \text{ a.e. } s \in [0,T] \iff F = \text{ constant } \iff \nabla F = 0.$$

This observation plays an important role in the study of path spaces on manifolds.

Remark 4.2.6. The condition (4.26) can be stated in a more symmetric form:

$$\mathbb{E}[\frac{\partial^2}{\partial s \partial t}(d^1\phi)^\sharp_{s,t}|\mathcal{F}_{s\vee t}] = 0, \qquad \text{a.e. } s,t \in [0,T], \tag{4.27}$$

where $s \vee t = \max(s,t)$. Sigma-algebras of this form have appeared in the work of Nualart and Zakai [51] in their generalisation of multiple integrals for complete symmetric tensor products. We refer to the their paper (in particular, Lemma 4.2) for the measure-theoretic niceties of such constructions.

Remark 4.2.7. Recall the subspace of isometry V of adapted processes inside $L^2(E;H)$, defined in Example 1.3.2. To extend the idea of Wu [63] (see Section 1.4), we can define the following subspace of $L^2\Gamma H^{\otimes 2}$,

$$V^{(2)} = \{u \in L^2\Gamma H^{\otimes 2} : u_{s,t} \in \mathcal{F}_{s\vee t},\ \text{a.e. } s,t \in [0,T]\}.$$

Let $P_{V^{(2)}}$ be the projection onto $V^{(2)}$ defined by

$$\begin{aligned} P_{V^{(2)}}u &= \int_0^\cdot \int_0^\cdot \mathbb{E}[\frac{\partial^2}{\partial s \partial t} u_{s,t}|\mathcal{F}_{s\vee t}]dsdt \\ &= \int_0^\cdot \int_t^\cdot \mathbb{E}[\frac{\partial^2}{\partial s \partial t} u_{s,t}|\mathcal{F}_{s}]dsdt + \int_0^\cdot \int_0^{t\wedge\cdot} \mathbb{E}[\frac{\partial^2}{\partial s \partial t} u_{s,t}|\mathcal{F}_{t}]dsdt, \end{aligned}$$

and $P^i_{V^{(2)}}$ given by the i-th term in the above sum for $i = 1,2$; e.g., for $i = 1$,

$$P^1_{V^{(2)}}u = \int_0^\cdot \int_t^\cdot \mathbb{E}[\frac{\partial^2}{\partial s \partial t} u_{s,t}|\mathcal{F}_{s}]dsdt.$$

Since $CO(\phi) = \delta P_V \nabla \phi^\sharp$, we can state our Clark-Ocone formula for H-one-forms (4.19) in the following form:

$$\nabla \delta P_V \nabla \phi^\sharp - \phi^\sharp = 2\delta P^2_{V^{(2)}}(d^1\phi)^\sharp.$$

From now on, we make regular use of Wu's interpretation and the smoothing effect of the such projections. We write $[P_V\nabla]$ for the composed operator $P_V\nabla$ as a continuous linear map from $L^2(C_0,\mathbb{R})$ to $L^2\Gamma H$. Similarly, we have $[P_V D]$, $[\delta P_V]$, etc. From equation (2.23) ([48] Lemma 2.4), we have $\mathbb{E}(\dot{u}_t|\mathcal{F}_t) \in \mathbb{D}^{2,1}$ for any process $u \in \mathbb{D}^{2,1}$, and

$$\frac{d}{ds}\nabla[\mathbb{E}(\dot{u}_t|\mathcal{F}_t)]_s = \mathbb{E}[\frac{d}{ds}\nabla(\dot{u}_t)_s|\mathcal{F}_t]\mathbf{1}_{[0,t]}(s), \quad \text{a.e. in } [0,T]\times C_0, \tag{4.28}$$

This can be written as

$$\nabla P_V = [P^2_{V^{(2)}}\nabla], \tag{4.29}$$

linking the two subspaces of the isometries V and $V^{(2)}$ for the classical Wiener space. This turns out to be an important condition for our later generalisation to the abstract Wiener space in Section 4.4.

Lemma 4.2.8. *Given $u \in \mathrm{Dom}(\delta)$, we have, for a.e. $t \in [0,T]$,*

$$\mathbb{E}(\int_0^T < \dot{u}_r, dB_r >_{\mathbb{R}^m} |\mathcal{F}_t) = \int_0^t < \mathbb{E}(\dot{u}_r|\mathcal{F}_t), dB_r >_{\mathbb{R}^m} .$$

Proof. This can be taken as an easy consequence of Lemma 1.2.5 in [47] based on the chaos expansion. Here we give a direct proof using the definition of conditional expectations. For any $\mathcal{F}_t$-measurable function $G \in \mathbb{D}^{2,1}$, the discussion in Section 1.3 shows $\frac{d}{dr}(\nabla G)_r = 0$ for a.e. $r \in [t,T]$, so integration-by-parts formula gives

$$\begin{aligned}\mathbb{E}(G\int_0^T < \dot{u}_r, dB_r >_{\mathbb{R}^m}) &= \mathbb{E}\int_0^T < \frac{d}{dr}(\nabla G)_r, \dot{u}_r >_{\mathbb{R}^m} dr \\ &= \mathbb{E}\int_0^t < \frac{d}{dr}(\nabla G)_r, \mathbb{E}(\dot{u}_r|\mathcal{F}_t) >_{\mathbb{R}^m} dr \\ &= \mathbb{E}G\int_0^t < \mathbb{E}(\dot{u}_r|\mathcal{F}_t), dB_r >_{\mathbb{R}^m} . \quad \square\end{aligned}$$

We state the next propositions in terms of vector fields rather than one-forms.

Proposition 4.2.9 (Clark-Ocone Formula for One-Forms: II)**.** *If $u \in L^2\Gamma H$ is in the domain of δ, the skew-symmetric two-vector-field $S(u) \in L^2\Gamma H^{\wedge 2}$ defined by*

$$S(u) = \int_0^\cdot \int_0^\cdot \mathbb{E}[\mathbf{1}_{(r_1,T]}(r)D_r\dot{u}_{r_1} - \mathbf{1}_{(r,T]}(r_1)\tau(D_{r_1}\dot{u}_r)|\mathcal{F}_{r\vee r_1}]drdr_1, \tag{4.30}$$

is in the domain of $d^{1*\sharp}$, and

$$\dot{u}_t + d_t^{1*\sharp}S(u) = \mathbb{E}(D_t\delta u|\mathcal{F}_t), \quad a.e.\ t \in [0,T]. \tag{4.31}$$

Remark 4.2.10. As in Remark 4.2.2, we can check that $d_t^{1*\sharp}S(u)$ is not orthogonal to $\mathbb{E}(D_t\delta u|\mathcal{F}_t)$ in general.

Remark 4.2.11. As in Remark 4.2.7, we can state the formula (4.31) in the following form:

$$u = -2\, d^{1*\sharp}A_2[P^2_{V^{(2)}}\nabla]u + [P_V\nabla]\delta u.$$

The right-hand side of (4.31) is again interpreted in the sense of (1.18) and (1.20): $u \in \mathrm{Dom}(\delta)$ implies $\delta u \in L^2(E;\mathbb{R})$, and the projected process

$$[P_V\nabla]\delta u = \int_0^\cdot \mathbb{E}(D_t\delta u|\mathcal{F}_t)dt$$

is a well-defined element of $L^2(E;H)$.

Proof. To see that S is skew-symmetric, we can check that $S_{s,t}(u) = -\tau S_{t,s}(u)$, or, equivalently, by checking the weak derivatives

$$\frac{\partial^2}{\partial s \partial t} S(u) = -\tau \left(\frac{\partial^2}{\partial t \partial s} S(u) \right),$$

which is obvious since

$$\begin{aligned} -\tau \left(\frac{\partial^2}{\partial t \partial s} S(u) \right) &= -\tau \left(\mathbb{E}[\mathbf{1}_{(s,T]}(t) D_t \dot{u}_s - \mathbf{1}_{(t,T]}(s) \tau (D_s \dot{u}_t) | \mathcal{F}_{s \vee t}] \right) \\ &= \mathbb{E}[-\mathbf{1}_{(s,T]}(t) \tau (D_t \dot{u}_s) + \mathbf{1}_{(t,T]}(s) D_s \dot{u}_t | \mathcal{F}_{s \vee t}] \\ &= \frac{\partial^2}{\partial s \partial t} S(u). \end{aligned}$$

To prove the proposition, first assume $u \in \mathbb{D}^{2,2}$, so $\nabla u \in \mathbb{D}^{2,1} \subset \mathrm{Dom}(\delta)$, and we calculate directly, using (4.18),

$$\begin{aligned} & -d_t^{1*\sharp} S(u) \\ = \ & \frac{d}{dt} \int_0^{\cdot} \int_0^T {}_{(2)}< \mathbb{E}[\mathbf{1}_{(r_1,T]}(r) D_r \dot{u}_{r_1} - \mathbf{1}_{(r,T]}(r_1) \tau (D_{r_1} \dot{u}_r) | \mathcal{F}_{r \vee r_1}], dB_r >_{\mathbb{R}^m} dr_1 \\ = \ & \int_t^T {}_{(2)}< \mathbb{E}(D_r \dot{u}_t | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} - \int_0^t {}_{(1)}< \mathbb{E}(D_t \dot{u}_r | \mathcal{F}_t), dB_r >_{\mathbb{R}^m} . \end{aligned} \tag{4.32}$$

On the other hand, $u \in \mathbb{D}^{2,2}$ satisfies the conditions for the commutation formula (3.1), so we apply Lemma 4.2.8 to calculate

$$\begin{aligned} \mathbb{E}(D_t \delta u | \mathcal{F}_t) &= \mathbb{E}(D_t \int_0^T < \dot{u}_r, dB_r >_{\mathbb{R}^m} | \mathcal{F}_t) \\ &= \mathbb{E}(\int_0^T {}_{(1)}< D_t \dot{u}_r, dB_r >_{\mathbb{R}^m} + \dot{u}_t | \mathcal{F}_t) \\ &= \int_0^t {}_{(1)}< \mathbb{E}(D_t \dot{u}_r | \mathcal{F}_t), dB_r >_{\mathbb{R}^m} + \mathbb{E}(\dot{u}_t | \mathcal{F}_t). \end{aligned}$$

Therefore, we obtain, by conditioning the Clark-Ocone formula as in (4.21),

$$-d_t^{1*\sharp} S(u) + \mathbb{E}(D_t \delta u | \mathcal{F}_t) = \int_t^T {}_{(2)}< \mathbb{E}(D_r \dot{u}_t | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} + \mathbb{E}(\dot{u}_t | \mathcal{F}_t) = \dot{u}_t.$$

For a general $u \in \mathrm{Dom}(\delta)$, we can approximate u by a sequence of cylindrical $u_j \in \mathbb{D}^{2,2}$, such that $u_j \to u$ and $\delta u_j \to \delta u$ in L^2. Remark 4.2.11 also implies that $\mathbb{E}(D_t \delta u_j | \mathcal{F}_t) \to \mathbb{E}(D_t \delta u | \mathcal{F}_t)$ in L^2. The computation above shows that,

$$-d_t^{1*\sharp} S(u_j) = (\dot{u}_j)_t - \mathbb{E}(D_t \delta u_j | \mathcal{F}_t) \to \dot{u}_t - \mathbb{E}(D_t \delta u | \mathcal{F}_t).$$

Since the map $u_j \mapsto S(u_j)$ is continuous in L^2, we also have $S(u_j) \to S(u)$. Using the fact that d^{1*} is a closed operator, and therefore so is $d^{1*\sharp}$, we see $S(u) \in \mathrm{Dom}(d^{1*\sharp})$, with $d_t^{1*\sharp}S(u)$ given by the limit above, so we are done. □

Corollary 4.2.12. *$u \in \mathrm{Dom}(\delta) \subset L^2\Gamma H$, $\delta u = 0 \implies u = -d^{1*\sharp}S(u)$, with $S(u) \in \mathrm{Dom}(d^{1*\sharp})$ defined in (4.30). Equivalently, any $u \in \mathrm{Ker}(\delta)$ can be expressed as*

$$\dot{u}_t = \int_t^T {}_{(2)}\!< \mathbb{E}(D_s\dot{u}_t|\mathcal{F}_s), dB_s >_{\mathbb{R}^m} - \int_0^t {}_{(1)}\!< \mathbb{E}(D_t\dot{u}_s|\mathcal{F}_t), dB_s >_{\mathbb{R}^m}, \qquad a.e.\ t \in [0,T]. \tag{4.33}$$

Moreover, if u is smooth, so is $S(u)$.

Proof. This result follows clearly from Proposition 4.2.9 and equation (4.32). □

Alternative proof. It is worth noting that Corollary 4.2.12 can also be taken as a direct consequence of Proposition 4.2.1, by a simple duality argument.

Suppose $u \in \mathrm{Ker}(\delta)$, so for any $\phi \in \mathbb{D}^{2,1}H^*$, we have

$$0 = \mathbb{E}[(\delta u)CO(\phi)] = \mathbb{E} < u, \nabla CO(\phi) >_H .$$

By (4.19), with $\phi \in \mathrm{Dom}(d^1)$,

$$\begin{aligned}
& \mathbb{E} < u, \phi^\sharp >_H \\
= \ & \mathbb{E} \int_0^T < \dot{u}_t, \dot{\phi}_t^\sharp - D_t CO(\phi) >_{\mathbb{R}^m} dt \\
= \ & \mathbb{E} \int_0^T < \dot{u}_t, -2 \int_t^T {}_{(2)}\!< \mathbb{E}(d^1_{t,s}\phi^\sharp|\mathcal{F}_s), dB_s >_{\mathbb{R}^m} >_{\mathbb{R}^m} dt \\
= \ & -\mathbb{E}\, 2 \int_0^T \int_0^T < \mathbf{1}_{(t,T]}(s)\mathbb{E}(D_s\dot{u}_t|\mathcal{F}_s), d^1_{t,s}\phi^\sharp >_{\wedge^2\mathbb{R}^m} ds dt \\
= \ & \mathbb{E} \int_0^T \int_0^T < \tau[\mathbf{1}_{(s,T]}(t)\mathbb{E}(D_t\dot{u}_s|\mathcal{F}_t)] - \mathbf{1}_{(t,T]}(s)\mathbb{E}(D_s\dot{u}_t|\mathcal{F}_s), d^1_{t,s}\phi^\sharp >_{\wedge^2\mathbb{R}^m} ds dt \\
= \ & -\mathbb{E} \int_0^T < \int_0^T {}_{(1)}\!< \mathbf{1}_{[0,t]}(s)\mathbb{E}(D_t\dot{u}_s|\mathcal{F}_{s\vee t}), dB_s >_{\mathbb{R}^m}, \dot{\phi}_t^\sharp >_{\mathbb{R}^m} dt \\
& +\mathbb{E} \int_0^T < \int_0^T {}_{(2)}\!< \mathbf{1}_{[0,s]}(t)\mathbb{E}[D_s\dot{u}_t|\mathcal{F}_{s\vee t}], dB_s >_{\mathbb{R}^m}, \dot{\phi}_t^\sharp >_{\mathbb{R}^m} dt \\
= \ & -\mathbb{E} < d^{1*\sharp}S(u), \phi^\sharp >_H,
\end{aligned}$$

using (4.32), so we obtain the result by the density of $\mathrm{Dom}(d^1)$ in $L^2\Gamma H^*$.

On the other hand, if we follow the calculation in Proposition 4.2.1 only up to (4.25), we derive the following more explicit expression for $\phi^\sharp \in \mathbb{D}^{2,1}(H)$,

$$\begin{aligned}
\mathbb{E} < u, \phi^\sharp >_H &= \mathbb{E} \int_0^T < \dot{u}_t, \int_t^T {}_{(2)}< \mathbb{E}[D_s \dot{\phi}_t^\sharp - \tau(D_t \dot{\phi}_s^\sharp) | \mathcal{F}_s], dB_s >_{\mathbb{R}^m} >_{\mathbb{R}^m} dt \\
&= \mathbb{E} \int_0^T < \nabla \dot{u}_t, \int_0^{\cdot} \mathbf{1}_{[t,T]}(s) \mathbb{E}[D_s \dot{\phi}_t^\sharp - \tau(D_t \dot{\phi}_s^\sharp) | \mathcal{F}_s] ds >_{\mathbb{R}^m \otimes H} dt \\
&= \mathbb{E} \int_0^T \int_0^T \mathbf{1}_{[t,T]}(s) < \mathbb{E}(D_s \dot{u}_t | \mathcal{F}_s), D_s \dot{\phi}_t^\sharp - \tau(D_t \dot{\phi}_s^\sharp) >_{\mathbb{R}^m \otimes \mathbb{R}^m} dsdt \\
&= \mathbb{E} \int_0^T < \int_t^T {}_{(2)}< \mathbb{E}(D_s \dot{u}_t | \mathcal{F}_s), dB_s >_{\mathbb{R}^m}, \dot{\phi}_t^\sharp >_{\mathbb{R}^m} dt \\
&\quad - \mathbb{E} \int_0^T < \int_0^s {}_{(1)}< \mathbb{E}(D_s \dot{u}_t | \mathcal{F}_s), dB_t >_{\mathbb{R}^m}, \dot{\phi}_s^\sharp >_{\mathbb{R}^m} ds,
\end{aligned}$$

therefore by the density of $\mathbb{D}^{2,1}(H)$ in $L^2\Gamma H$, we obtain (4.33). □

Remark 4.2.13. Note that the first integral in (4.33) is a standard Itô integral, but the second one is a Skorohod integral, which is orthogonal to the first part since, by applying the Skorohod energy identity (3.2), or more directly the equations (1.54) and (1.45) from [47], we get

$$\mathbb{E} \int_t^T < \mathbb{E}(D_s \dot{u}_t | \mathcal{F}_s), dB_s >_{\mathbb{R}^m} \int_0^t < \mathbb{E}(D_t \dot{u}_s | \mathcal{F}_t), dB_s >_{\mathbb{R}^m} = 0.$$

4.3 Higher Order Forms on the Classical Wiener Space

We first state two commutation formulas involving the exterior derivatives and their adjoints. Recall quickly (4.11), our convention for $d^{q\sharp}$ and $d^{q*\sharp}$ here:

$$d^{q\sharp} = (-1)^q A^{q+1} \nabla, \quad \text{and} \quad d^{q*\sharp} = (-1)^q \delta.$$

Lemma 4.3.1 (Commutation Formula for d^q). *Suppose $u \in \mathbb{D}^{2,1}(H^{\wedge q})$ satisfies $\tau_{i,q+1} \nabla u \in \mathrm{Dom}(\delta)$, for all $i = 1$ to q. Then we have $\delta u \in \mathrm{Dom}(d^{(q-1)\sharp})$, and*

$$(-1)^{q-1} q \, d^{(q-1)\sharp} \delta u = \sum_{j=1}^q \delta(\tau_{j,q+1} \nabla u) + q \, u,$$

that is, for a.e. $s_1, \cdots, s_q \in [0,T]$,

$$\begin{aligned}
&(-1)^{q-1} q\, d^{(q-1)\sharp}_{s_1,\cdots,s_q}[\int_0^{\cdot} \cdots \int_0^{\cdot} \int_0^T {}_{(q)}< \partial^q_{r_1,\cdots,r_q} u, dB_{r_q} >_{\mathbb{R}^m} dr_1 \cdots dr_{q-1}] \\
= & \sum_{j=1}^q \int_0^T {}_{(j)}< D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r_q,s_{j+1},\cdots,s_q} u, dB_{r_q} >_{\mathbb{R}^m} + q\, \partial^q_{s_1,\cdots,s_q} u. \qquad (4.34)
\end{aligned}$$

Remark 4.3.2. For $q = 1$, the above reduces to the commutation formula (3.1).

Remark 4.3.3. We can equivalently state the above as, for any $i = 1$ to q,

$$\begin{aligned}
&(-1)^{i-1} q\, d^{(q-1)\sharp}_{s_1,\cdots,s_q}[\int_0^{\cdot} \cdots \int_0^{\cdot} \int_0^T {}_{(i)}< \partial^q_{r_1,\cdots,r_q} u, dB_{r_i} >_{\mathbb{R}^m} dr_1 \cdots dr_{i-1} dr_{i+1} \cdots dr_q] \\
= & \sum_{j=1}^q \int_0^T {}_{(j)}< D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r_i,s_{j+1},\cdots,s_q} u, dB_{r_i} >_{\mathbb{R}^m} + q\, \partial^q_{s_1,\cdots,s_q} u.
\end{aligned}$$

The same goes for many other expressions in this section. It is a result of the skew-symmetry of $u \in L^2\Gamma \wedge^q H$. This points to the many equivalent ways of writing the Clark-Ocone formulae for differential forms.

Proof. This is an easy consequence of (4.12) and (3.1), since

$$\begin{aligned}
&(-1)^{q-1} q\, d^{(q-1)\sharp}_{s_1,\cdots,s_q}[\int_0^{\cdot} \cdots \int_0^{\cdot} \int_0^T {}_{(q)}< \partial^q_{r_1,\cdots,r_q} u, dB_{r_q} >_{\mathbb{R}^m} dr_1 \cdots dr_{q-1}] \\
= & \sum_{j=1}^q (-1)^{q-j} D_{s_j} [\int_0^T {}_{(q)}< \partial^q_{s_1,\cdots,\hat{s}_j,\cdots,s_q,r_q} u, dB_{r_q} >_{\mathbb{R}^m}] \\
= & \sum_{j=1}^q D_{s_j} [\int_0^T {}_{(j)}< \partial^q_{s_1,\cdots,s_{j-1},r_q,s_{j+1},\cdots,s_q} u, dB_{r_q} >_{\mathbb{R}^m}] \\
= & \sum_{j=1}^q \int_0^T {}_{(j)}< D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r_q,s_{j+1},\cdots,s_q} u, dB_{r_q} >_{\mathbb{R}^m} + q\, \partial^q_{s_1,\cdots,s_q} u. \quad \square
\end{aligned}$$

Lemma 4.3.4 (Commutation Formula for d^{q*})**.** *If* $u \in \mathbb{D}^{2,1}(H^{\wedge q})$ *and* $\nabla u \in \mathrm{Dom}(\delta)$, *we have* $d^{(q-1)*\sharp} u \in \mathbb{D}^{2,1}$, *and*

$$(-1)^{q-1} \nabla d^{(q-1)*\sharp} u = \delta(\tau_{q,q+1} \nabla u) + u,$$

that is, for a.e. $s_1, \cdots, s_q \in [0,T]$,

$$(-1)^{q-1} D_{s_q} d^{(q-1)*\sharp}_{s_1,\cdots,s_{q-1}} u = \int_0^T {}_{(q)}< D_{s_q} \partial^q_{s_1,\cdots,s_{q-1},r} u, dB_r >_{\mathbb{R}^m} + \partial^q_{s_1,\cdots,s_q} u. \qquad (4.35)$$

Remark 4.3.5. For $q = 1$, the above formula again reduces to (3.1).

Remark 4.3.6. Just as in Remark 4.3.3, we can equivalently state (4.35) as, for any $i = 1$ to q,

$$\begin{aligned} & (-1)^{i-1} D_s d^{(q-1)*\sharp}_{s_1,\cdots,s_{q-1}} u \\ = & \int_0^T {}_{(i)}< D_s \partial^q_{s_1,\cdots,s_{i-1},r,s_i,\cdots,s_{q-1}} u, dB_r >_{\mathbb{R}^m} + \partial^q_{s_1,\cdots,s_{i-1},s,s_i,\cdots,s_{q-1}} u. \end{aligned}$$

Proof. This is an easy consequence of (4.15) and (3.1), since

$$\begin{aligned} (-1)^{q-1} D_{s_q} d^{(q-1)*\sharp}_{s_1,\cdots,s_{q-1}} u &= (-1)^{q-1} D_{s_q} (-1)^{q-1} \int_0^T {}_{(q)}< \partial^q_{s_1,\cdots,s_{q-1},r} u, dB_r >_{\mathbb{R}^m} \\ &= \int_0^T {}_{(q)}< D_{s_q} \partial^q_{s_1,\cdots,s_{q-1},r} u, dB_r >_{\mathbb{R}^m} + \partial^q_{s_1,\cdots,s_q} u. \quad \square \end{aligned}$$

Using the operators $d^{q\sharp}$ and $d^{q*\sharp}$, we can define the analogue of Shigekawa's Hodge Laplacian on skew-symmetric H-vector fields by

$$\Delta^{q\sharp} = (q+1) d^{q*\sharp} d^{q\sharp} + q d^{(q-1)\sharp} d^{(q-1)*\sharp}.$$

The unusual coefficients here are due to our definition of $d^{q\sharp}$ and $d^{q*\sharp}$ in (4.11), which follows the convention of Kobayashi and Nomizu [41] for exterior product and exterior derivative.

Lemma 4.3.7 ([Shigekawa [55]: Weitzenböck Formula)**.** $\Delta^{q\sharp} = \delta\nabla + q\mathrm{Id}_{H^{\wedge q}}$.

Proof. We follow the calculation as in the proof of Lemma 4.3.1 to derive

$$\begin{aligned} \Delta^{q\sharp} &= (q+1)\delta A^{q+1}\nabla + q d^{(q-1)\sharp} (-1)^{(q-1)} \delta \\ &= \delta(\nabla - \sum_{j=1}^q \tau_{j,q+1}\nabla) + \delta \sum_{j=1}^q \tau_{j,q+1}\nabla + q\mathrm{Id}_{H^{\wedge q}} \\ &= \delta\nabla + q\mathrm{Id}_{H^{\wedge q}}. \quad \square \end{aligned}$$

Theorem 4.3.8 (Clark-Ocone Formula for q-Forms: I)**.** *For all* $u \in \mathrm{Dom}(d^{q\sharp})$*, the skew-symmetric* $(q-1)$*-vector-field* $T_{q-1}(u) \in L^2\Gamma H^{\wedge(q-1)}$*, defined by*

$$T_{q-1}(u) = \int_0^{\cdot} \cdots \int_0^{\cdot} \int_{\max_{i=2}^q r_i}^T {}_{(1)}< \mathbb{E}(\partial^q_{r_1,\cdots,r_q} u | \mathcal{F}_{r_1}), dB_{r_1} >_{\mathbb{R}^m} dr_2 \cdots dr_q, \quad (4.36)$$

is in the domain of $d^{(q-1)\sharp}$, *and, for a.e.* $s_1, \cdots, s_q \in [0,T]$,

$$\partial^q_{s_1,\cdots,s_q} u = q\, d^{(q-1)\sharp}_{s_1,\cdots,s_q} T_{q-1}(u) + (q+1) \int_{\max_{i=1}^q s_i}^T {}^{(1)}< \mathbb{E}(d^{q\sharp}_{r,s_1,\cdots,s_q} u|\mathcal{F}_r), dB_r >_{\mathbb{R}^m} . \tag{4.37}$$

Moreover, if $u \in \mathbb{D}^{2,k}(H^{\wedge q})$, *we also have* $T_{q-1}(u) \in \mathbb{D}^{2,k}(H^{\wedge(q-1)})$.

Remark 4.3.9. The case of $q = 1$ reduces to Proposition 4.2.1.

Remark 4.3.10. As noted in Remark 4.3.3, we can define $T_{q-1}(u)$ in (4.36) as

$$\int_0^{\cdot} \cdots \int_0^{\cdot} \int_{\max_{i=1,i\neq j}^q r_i}^T {}^{(j)}< \mathbb{E}(\partial^q_{r_1,\cdots,r_q} u|\mathcal{F}_{r_j}), dB_{r_j} >_{\mathbb{R}^m} dr_1 \cdots dr_{j-1} dr_{j+1} \cdots dr_q,$$

for j any integer between 1 and q, with the corresponding change in (4.37).

Remark 4.3.11. Note that our statements here hold almost everywhere, so we ignore all sets such that $s_i = s_j$ for $i \neq j$.

Proof. We prove the result first for $u \in \mathbb{D}^{2,1}$, and then use an approximation argument to extend to any general $u \in \mathrm{Dom}(d^{q\sharp})$.

We apply the Clark-Ocone formula to write, for a.e. $s_1, \cdots, s_q \in [0,T]$,

$$\partial^q_{s_1,\cdots,s_q} u = \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u) + \int_0^T {}^{(q+1)}< \mathbb{E}[D_r \partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_r], dB_r >_{\mathbb{R}^m} .$$

Taking conditional expectation with respect to $\mathcal{F}_{\max_{i=1}^q s_i}$, we obtain

$$\mathbb{E}(\partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_{\max_{i=1}^q s_i}) = \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u) + \int_0^{\max_{i=1}^q s_i} {}^{(q+1)}< \mathbb{E}[D_r \partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_r], dB_r >_{\mathbb{R}^m}, \tag{4.38}$$

hence

$$\partial^q_{s_1,\cdots,s_q} u = \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_{\max_{i=1}^q s_i}) + \int_{\max_{i=1}^q s_i}^T {}^{(q+1)}< \mathbb{E}[D_r \partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_r], dB_r >_{\mathbb{R}^m} .$$

For $u \in \mathbb{D}^{2,1}$, equation (2.23) implies $\mathbb{E}(\partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u|\mathcal{F}_r) \in \mathbb{D}^{2,1}$, and

$$D_{s_j} \mathbb{E}(\partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u|\mathcal{F}_r) = \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u|\mathcal{F}_r) \mathbf{1}_{(s_j,T]}(r), \quad \text{a.e.}$$

Therefore, the process $\int_0^{\cdot} D_{s_j} \mathbb{E}(\partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u|\mathcal{F}_r) dr$ is adapted, hence Itô-integrable. We can apply Lemma 4.3.1 (in particular, equation (4.34) and Remark

4.3.3 for $i = 1$), to show that $T_{q-1}(u) \in \mathrm{Dom}(d^{(q-1)\sharp})$, and for a.e. $s_1, \cdots, s_q \in [0, T]$,

$$
\begin{aligned}
& q\, d^{(q-1)\sharp}_{s_1,\cdots,s_q} T_{q-1}(u) \\
= \; & \sum_{j=1}^{q} \int_{\max_{i=1, i\neq j}^{q} s_i}^{T} {}^{(j)}\!< D_{s_j} \mathbb{E}(\partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} \\
& + \sum_{j=1}^{q} \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u | \mathcal{F}_{s_j}) \mathbf{1}_{(\max_{i=1,i\neq j}^{q} s_i, T]}(s_j) \\
= \; & \sum_{j=1}^{q} \int_{\max_{i=1}^{q} s_i}^{T} {}^{(j)}\!< \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} \\
& + \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u | \mathcal{F}_{\max_{i=1}^{q} s_i}). \qquad (4.39)
\end{aligned}
$$

Subtracting (4.39) from (4.38), we obtain, by equation (4.13),

$$
\begin{aligned}
& \partial^q_{s_1,\cdots,s_q} u - q\, d^{(q-1)\sharp}_{s_1,\cdots,s_q} T_{q-1}(u) \\
= \; & \int_{\max_{i=1}^{q} s_i}^{T} {}^{(q+1)}\!< \mathbb{E}(D_r \partial^q_{s_1,\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} \\
& - \sum_{j=1}^{q} \int_{\max_{i=1}^{q} s_i}^{T} {}^{(j)}\!< \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} \\
= \; & (q+1) \int_{\max_{i=1}^{q} s_i}^{T} {}^{(1)}\!< \mathbb{E}(d^{q\sharp}_{r,s_1,\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m},
\end{aligned}
$$

so (4.37) holds for $u \in \mathbb{D}^{2,1}$. It is also clear from the above calculation, or from the commutation formula directly, that $T_{q-1}(u) \in \mathbb{D}^{2,k}(H^{\wedge(q-1)})$ if $u \in \mathbb{D}^{2,k}(H^{\wedge q})$.

A general $u \in \mathrm{Dom}(d^{q\sharp})$ can be approximated by a sequence of $u_j \in \mathbb{D}^{2,1}$ such that $u_j \to u$ and $d^{q\sharp} u_j \to d^{q\sharp} u$ in L^2. The above computation shows that

$$
\begin{aligned}
q\, d^{(q-1)\sharp}_{s_1,\cdots,s_q} T_{q-1}(u_j) \; &= \; \partial^q_{s_1,\cdots,s_q} u_j - (q+1) \int_{\max_{i=1}^{q} s_i}^{T} {}^{(1)}\!< \mathbb{E}(d^{q\sharp}_{r,s_1,\cdots,s_q} u_j | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} \\
& \to \; \partial^q_{s_1,\cdots,s_q} u - (q+1) \int_{\max_{i=1}^{q} s_i}^{T} {}^{(1)}\!< \mathbb{E}(d^{q\sharp}_{r,s_1,\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} .
\end{aligned}
$$

Since the map $u_j \mapsto T_{q-1}(u_j)$ is continuous in L^2, we also have $T_{q-1}(u_j) \to T_{q-1}(u)$. As d^{q-1} is a closed operator, so is $d^{(q-1)\sharp}$, therefore $T_{q-1}(u) \in \mathrm{Dom}(d^{(q-1)\sharp})$, and (4.37) holds for $u \in \mathrm{Dom}(d^{q\sharp})$. □

Corollary 4.3.12. *$u \in \mathrm{Ker}(d^{q\sharp}) \Longrightarrow u = q\, d^{q\sharp} T_{q-1}(u)$, where the skew-symmetric $(q-1)$-vector-field $T_{q-1}(u) \in \mathrm{Dom}(d^{q\sharp})$ is defined by equation (4.36). That is, any*

$u \in \mathrm{Ker}(d^{q\sharp})$ can be expressed as, for a.e. $s_1, \cdots, s_q \in [0,T]$,

$$\begin{aligned}
&\partial^q_{s_1,\cdots,s_q} u \\
= \;& q\, d^{(q-1)\sharp}_{s_1,\cdots,s_q} T_{q-1}(u) \qquad (4.40)\\
= \;& \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u | \mathcal{F}_{\max_{i=1}^q s_i}) \\
& + \sum_{j=1}^q \int_{\max_{i=1}^q s_i}^T {}_{(j)}< \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} \\
= \;& \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u) + \int_0^{\max_{i=1}^q s_i} {}_{(q+1)}< \mathbb{E}(D_r \partial^q_{s_1,\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} \\
& + \sum_{j=1}^q \int_{\max_{i=1}^q s_i}^T {}_{(j)}< \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} . \qquad (4.41)
\end{aligned}$$

Moreover, if $u \in \mathbb{D}^{2,k}(H^{\wedge q})$, we have $T_{q-1}(u) \in \mathbb{D}^{2,k}(H^{\wedge(q-1)})$.

Proof. With $d^{q\sharp}u = 0$, the first statement is a direct consequence of (4.37), the second is the last equation in (4.39), and the last step follows from the conditioned Clark-Ocone formula as in (4.38). □

Remark 4.3.13. Following the discussions in Section 1.4 and Remark 4.2.7, we can define the following subspace of $L^2\Gamma H^{\otimes q}$,

$$V^{(q)} = \{u \in L^2\Gamma H^{\otimes q} : u_{s_1,\cdots,s_q} \in \mathcal{F}_{s_1 \vee \cdots \vee s_q}, \text{ a.e. } s_1, \cdots, s_q \in [0,T]\}.$$

Let $P_{V^{(q)}}$ be the projection onto $V^{(q)}$ defined by

$$\begin{aligned}
P_{V^{(q)}} u &= \int_0^{\cdot} \cdots \int_0^{\cdot} \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u | \mathcal{F}_{s_1 \vee \cdots \vee s_q}) ds_1 \cdots ds_q \\
&= \sum_{j=1}^q \int_0^{\cdot} \cdots \int_0^{\cdot} \mathbf{1}_{s_j = \max_{i=1}^q s_i} \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u | \mathcal{F}_{s_j}) ds_1 \cdots ds_q,
\end{aligned}$$

and $P^j_{V^{(q)}}$ given by the j-th term in the above sum, $j = 1$ to q,

$$P^j_{V^{(q)}} u = \int_0^{\cdot} \cdots \int_0^{\cdot} \mathbf{1}_{s_j = \max_{i=1}^q s_i} \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u | \mathcal{F}_{s_j}) ds_1 \cdots ds_q,$$

then we can state our Clark-Ocone formula for H-q-forms (4.37) in the following form:

$$u = q\, d^{(q-1)\sharp}[\delta P^1_{V^{(q)}}]u + (q+1)[\delta P^1_{V^{(q+1)}}] d^{q\sharp} u.$$

Theorem 4.3.14 (Clark-Ocone Formula for q-Forms: II)**.** *For $u \in \mathrm{Dom}(d^{(q-1)*\sharp})$, the skew-symmetric $(q+1)$-vector-field $S_{q+1}(u) \in L^2\Gamma H^{\wedge(q+1)}$ defined by*

$$\begin{aligned} & S_{q+1}(u) \\ = & \int_0^\cdot \cdots \int_0^\cdot \mathbb{E}[\mathbf{1}_{(\max_{i=1}^q r_i, T]}(r) D_r \partial^q_{r_1,\cdots,r_q} u \\ & - \sum_{j=1}^q \mathbf{1}_{(r \vee \max_{i=1, i\neq j}^q r_i, T]}(r_j) \tau_{j,q+1} D_{r_j} \partial^q_{r_1,\cdots,r_{j-1},r,r_{j+1},\cdots,r_q} u | \mathcal{F}_{r \vee \max_{i=1}^q r_i}] \, dr dr_1 \cdots dr_q \end{aligned} \quad (4.42)$$

is in the domain of $d^{q*\sharp}$, and

$$\begin{aligned} \partial^q_{s_1,\cdots,s_q} u &= (-1)^q d^{q*\sharp}_{s_1,\cdots,s_q} S_{q+1}(u) \\ & \quad + \sum_{j=1}^q (-1)^{j-1} \mathbf{1}_{s_j = \max_{i=1}^q s_i} \mathbb{E}(D_{s_j} d^{(q-1)*\sharp}_{s_1,\cdots,s_{j-1},s_{j+1},\cdots,s_q} u | \mathcal{F}_{s_j}). \end{aligned} \quad (4.43)$$

Remark 4.3.15. The case of $q = 1$ reduces to Proposition 4.2.9.

Remark 4.3.16. As in Remark 4.2.11, the right-hand sides of (4.43) and (4.42) are to be interpreted in the sense of (1.18). That is, the condition $u \in \mathrm{Dom}(d^{(q-1)*\sharp})$ implies $d^{(q-1)*\sharp} u \in L^2\Gamma H^{\wedge(q-1)}$, so the projected process, for each $j = 1$ to q,

$$[P^j_{V(q)} \nabla] d^{(q-1)*\sharp} u = \int_0^\cdot \cdots \int_0^\cdot \mathbf{1}_{s_j = \max_{i=1}^q s_i} \mathbb{E}(D_{s_j} d^{(q-1)*\sharp}_{s_1,\cdots,s_{j-1},s_{j+1},\cdots,s_q} u | \mathcal{F}_{s_j}) ds_1 \cdots ds_q$$

is a well-defined element in $L^2\Gamma H^{\wedge q}$. So as in Remark 4.3.13, we state (4.43) as

$$u = (-1)^q (q+1) d^{q*\sharp} A^{q+1} [P^{q+1}_{V(q+1)} \nabla] u + q A^q [P^q_{V(q)} \nabla] d^{(q-1)*\sharp} u.$$

Proof. We prove first for $u \in \mathbb{D}^{2,2}$, and then use the standard approximation argument to extend the result to any general $u \in \mathrm{Dom}(d^{(q-1)*\sharp})$.

From the skew-symmetry of $u \in L^2\Gamma H^{\wedge q}$, we can see that $S_{q+1}(u)$ is the full skew-symmetrisation of the $(q+1)$-tensor

$$G(u) = (q+1) \int_0^\cdot \cdots \int_0^\cdot \mathbb{E}[\mathbf{1}_{(\max_{i=1}^q r_i, T]}(r) D_r \partial^q_{r_1,\cdots,r_q} u | \mathcal{F}_r] \, dr dr_1 \cdots dr_q,$$

so indeed $S_{q+1}(u) \in L^2\Gamma H^{\wedge(q+1)}$. We compute

$$\begin{aligned}
& (-1)^q d^{q*\sharp}_{s_1,\cdots,s_q} S_{q+1}(u) \qquad (4.44)\\
= & \int_0^T {}_{(q+1)}< \mathbb{E}[\mathbf{1}_{(\max_{i=1}^q s_i, T]}(r) D_r \partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_r], dB_r >_{\mathbb{R}^m}\\
& - \sum_{j=1}^q \int_0^{s_j} {}_{(q+1)}< \mathbb{E}[\mathbf{1}_{s_j=\max_{i=1}^q s_i} \tau_{j,q+1} D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u|\mathcal{F}_{s_j}], dB_r >_{\mathbb{R}^m}\\
= & \int_{\max_{i=1}^q s_i}^T {}_{(q+1)}< \mathbb{E}(D_r \partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_r), dB_r >_{\mathbb{R}^m}\\
& - \sum_{j=1}^q \mathbf{1}_{s_j=\max_{i=1}^q s_i} \int_0^{s_j} {}_{(j)}< \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u|\mathcal{F}_{s_j}), dB_r >_{\mathbb{R}^m} .
\end{aligned}$$

On the other hand, from our assumption $u \in \mathbb{D}^{2,2}$, we have $u \in \text{Dom}(d^{(q-1)*\sharp})$ and $d^{(q-1)*\sharp}u \in \mathbb{D}^{2,1}$, so applying the commutation formula (4.35), we obtain

$$\begin{aligned}
& \sum_{j=1}^q (-1)^{j-1} \mathbf{1}_{s_j=\max_{i=1}^q s_i} \mathbb{E}(D_{s_j} d^{(q-1)*\sharp}_{s_1,\cdots,s_{j-1},s_{j+1},\cdots,s_q} u|\mathcal{F}_{s_j}) \qquad (4.45)\\
= & \sum_{j=1}^q (-1)^{j-1} \mathbf{1}_{s_j=\max_{i=1}^q s_i} \mathbb{E}[(-1)^{j-1} D_{s_j} \int_0^T {}_{(j)}< \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u, dB_r >_{\mathbb{R}^m} |\mathcal{F}_{s_j}]\\
= & \sum_{j=1}^q \mathbf{1}_{s_j=\max_{i=1}^q s_i} \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u + \int_0^T {}_{(j)}< D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u, dB_r >_{\mathbb{R}^m} |\mathcal{F}_{s_j})\\
= & \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_{\max_{i=1}^q s_i})\\
& + \sum_{j=1}^q \mathbf{1}_{s_j=\max_{i=1}^q s_i} \int_0^{s_j} {}_{(j)}< \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u|\mathcal{F}_{s_j}), dB_r >_{\mathbb{R}^m} .
\end{aligned}$$

Now summing up (4.44) and (4.45), we obtain

$$\begin{aligned}
& (-1)^q d^{q*\sharp}_{s_1,\cdots,s_q} S_{q+1}(u) + \sum_{j=1}^q (-1)^{j-1} \mathbf{1}_{s_j=\max_{i=1}^q s_i} \mathbb{E}(D_{s_j} d^{(q-1)*\sharp}_{s_1,\cdots,s_{j-1},s_{j+1},\cdots,s_q} u|\mathcal{F}_{s_j})\\
= & \int_{\max_{i=1}^q s_i}^T {}_{(q+1)}< \mathbb{E}(D_r \partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_r), dB_r >_{\mathbb{R}^m} + \mathbb{E}(\partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_{\max_{i=1}^q s_i})\\
= & \partial^q_{s_1,\cdots,s_q} u,
\end{aligned}$$

by the conditioned Clark-Ocone formula (4.38), which proves (4.43) for $u \in \mathbb{D}^{2,2}$.

For a general $u \in \text{Dom}(d^{(q-1)*\sharp}) \subset L^2\Gamma H^{\wedge q}$, we can approximate by a sequence of cylindrical $u_j \in \mathbb{D}^{2,2}$ such that $u_j \to u$ and $d^{(q-1)*\sharp}u_j \to d^{(q-1)*\sharp}u$ in L^2.

Remark 4.3.16 also implies that

$$\mathbb{E}(D_{s_j} d^{(q-1)*\sharp}_{s_1,\cdots,s_{j-1},s_{j+1},\cdots,s_q} u_j|\mathcal{F}_{s_j}) \to \mathbb{E}(D_{s_j} d^{(q-1)*\sharp}_{s_1,\cdots,s_{j-1},s_{j+1},\cdots,s_q} u|\mathcal{F}_{s_j})$$

in L^2. The above computation shows that, in L^2,

$$\begin{aligned}
& (-1)^q d^{q*\sharp}_{s_1,\cdots,s_q} S_{q+1}(u_j) \\
= \quad & \partial^q_{s_1,\cdots,s_q} u_j - \sum_{j=1}^{q} (-1)^{j-1} \mathbf{1}_{s_j=\max_{i=1}^q s_i} \mathbb{E}(D_{s_j} d^{(q-1)*\sharp}_{s_1,\cdots,s_{j-1},s_{j+1},\cdots,s_q} u_j|\mathcal{F}_{s_j}) \\
\to \quad & \partial^q_{s_1,\cdots,s_q} u - \sum_{j=1}^{q} (-1)^{j-1} \mathbf{1}_{s_j=\max_{i=1}^q s_i} \mathbb{E}(D_{s_j} d^{(q-1)*\sharp}_{s_1,\cdots,s_{j-1},s_{j+1},\cdots,s_q} u|\mathcal{F}_{s_j}).
\end{aligned}$$

Since the map $u \mapsto S_{q+1}(u)$ is continuous in L^2, we also have $S_{q+1}(u_j) \to S_{q+1}(u)$. Using the fact that d^{q*} is a closed operator, and therefore so is $d^{q*\sharp}$, we have indeed $S_{q+1}(u) \in \mathrm{Dom}(d^{q*\sharp})$, and (4.42) holds.

Corollary 4.3.17. $u \in \mathrm{Ker}(d^{(q-1)*\sharp}) \implies u = (-1)^q d^{q*\sharp} S_{q+1}(u)$, *where the skew-symmetric vector field* $S_{q+1}(u) \in \mathrm{Dom}(d^{q*\sharp})$ *is defined by (4.42). That is, any* $u \in \mathrm{Ker}(d^{(q-1)*\sharp})$ *can be expressed as*

$$\begin{aligned}
& \partial^q_{s_1,\cdots,s_q} u \\
= \quad & (-1)^q d^{q*\sharp}_{s_1,\cdots,s_q} S_{q+1}(u) && (4.46)\\
= \quad & \int_{\max_{i=1}^q s_i}^{T} {}_{(q+1)}< \mathbb{E}(D_r \partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_r), dB_r >_{\mathbb{R}^m} && (4.47)\\
& - \sum_{j=1}^{q} \mathbf{1}_{s_j=\max_{i=1}^q s_i} \int_0^{s_j} {}_{(j)}< \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u|\mathcal{F}_{s_j}), dB_r >_{\mathbb{R}^m} .
\end{aligned}$$

To compare the various versions of the Clark-Ocone equations we have derived so far, we write

$$A(u) = \mathbb{E}u,$$

$$B(u) = \int_0^{\cdot} \cdots \int_0^{\cdot} \int_0^{\max_{i=1}^q s_i} {}_{(q+1)}< \mathbb{E}(D_r \partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_r), dB_r >_{\mathbb{R}^m} ds_1 \cdots ds_q,$$

$$B_1(u) = \sum_{j=1}^{q} \mathbf{1}_{s_j=\max_{i=1}^q s_i} \int_0^{\cdot} \cdots \int_0^{\cdot} \int_0^{s_j} {}_{(j)}< \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u|\mathcal{F}_{s_j}), dB_r >_{\mathbb{R}^m} ds_1 \cdots ds_q,$$

$$C(u) = \int_0^{\cdot} \cdots \int_0^{\cdot} \int_{\max_{i=1}^q s_i}^{T} {}_{(q+1)}< \mathbb{E}(D_r \partial^q_{s_1,\cdots,s_q} u|\mathcal{F}_r), dB_r >_{\mathbb{R}^m} ds_1 \cdots ds_q,$$

and

$$C_1(u) = \sum_{j=1}^{q} \int_0^{\cdot} \cdots \int_0^{\cdot} \int_{\max_{i=1}^{q} s_i^{(j)}}^{T} < \mathbb{E}(D_{s_j} \partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q} u | \mathcal{F}_r), dB_r >_{\mathbb{R}^m} ds_1 \cdots ds_q.$$

The classical Clark-Ocone expression for $u \in L^2\Gamma \wedge^q H$, or rather, for $\partial^q_{s_1,\cdots,s_q} u \in L^2\Gamma \wedge^q \mathbb{R}^m$, takes the form

$$u = A(u) + B(u) + C(u), \tag{4.48}$$

The proof of Proposition 4.3.8 shows

$$u = \underbrace{A(u) + [B(u) + C_1(u)]}_{\in \text{Image}(d^{(q-1)\sharp})} + \underbrace{[C(u) - C_1(u)]}_{M_q(d^{q\sharp} u)},$$

while the proof of Proposition 4.3.14 gives

$$u = \underbrace{A(u) + [B(u) + B_1(u)]}_{N_q(d^{(q-1)*\sharp} u)} + \underbrace{[C(u) - B_1(u)]}_{\in \text{Image}(d^{q*\sharp})}.$$

The presence of the two new terms $B_1(u)$ and $C_1(u)$, both of which are absent from (4.48) and involve skew-symmetrisation of the differentiation operator intertwined with conditioning and divergence, lies behind the difficulty of obtaining a Hodge-decomposition-type of representation of the form (4.2), for $u \in L^2\Gamma \wedge^q H$:

$$u = d^{(q-1)\sharp} v + d^{q*\sharp} w, \quad \text{for some } v \in \text{Dom}(d^{(q-1)\sharp}) \text{ and } w \in \text{Dom}(d^{q*\sharp}).$$

In general, there is no reason to expect $B_1(u)$ and $C_1(u)$ to be equal, and as mentioned in Remarks 4.2.2 and 4.2.10, the two terms in each of our generalised Clark-Ocone formulae (i.e., $A + B + B_1$ and $C - B_1$, and $A + B + C_1$ and $C - C_1$) are not expected to be orthogonal to each other. So we have to be satisfied with a less ambitious representation in terms of a combination of two equations (4.3) and (4.4), i.e., as in equation (4.41),

$$u \in \text{Ker}(d^{q\sharp}) \iff u = \underbrace{A(u) + [B(u) + C_1(u)]}_{\in \text{Image}(d^{(q-1)\sharp})},$$

and in (4.47)

$$u \in \text{Ker}(d^{(q-1)*\sharp}) \iff u = \underbrace{[C(u) - B_1(u)]}_{\in \text{Image}(d^{q*\sharp})}.$$

In the next section, we give another explanation of the above difficulty from a representation-theoretical point of view.

To summarise, our generalised Clark-Ocone formulae in Propositions 4.3.8 and 4.3.14 still enable us to recover the full results of Shigekawa [55], with extra details. We formulate this in terms of two separate sets of equivalent conditions.

Corollary 4.3.18. *The following conditions are equivalent for* $u \in L^2\Gamma \wedge^q H$*:*

1-a. $d^{q\sharp}u = 0$*, i.e.,* $u^\sharp \in L^2\Gamma(\wedge^q H)^*$ *is closed;*

1-b. *for a.e.* $s_1, \cdots, s_q \in [0,T]$ *and* $r \in (\max_{i=1}^q s_i, T]$*,*

$$\mathbb{E}(d^{q\sharp}_{r,s_1,\cdots,s_q}u|\mathcal{F}_r) = 0,$$

i.e.,

$$\mathbb{E}(D_r\partial^q_{s_1,\cdots,s_q}u|\mathcal{F}_r) - \mathbb{E}(\sum_{j=1}^q D_{s_j}\partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q}u|\mathcal{F}_r) = 0; \qquad (4.49)$$

1-c. $u = d^{(q-1)\sharp}v$*, where* $v \in L^2\Gamma \wedge^{q-1} H$ *has an expression as in (4.40), i.e.,* $u^\sharp \in L^2\Gamma(\wedge^q H)^*$ *is exact.*

Corollary 4.3.19. *The following conditions are equivalent for* $u \in L^2\Gamma \wedge^q H$*:*

2-a. $d^{(q-1)*\sharp}u = 0$*;*

2-b. *for a.e.* $s_1, \cdots, s_q \in [0,T]$*,*

$$\mathbb{E}(\partial^q_{s_1,\cdots,s_q}u) = 0,$$

and

$$\begin{aligned} &\int_0^{\max_{i=1}^q s_i} {}_{(q+1)}\!< \mathbb{E}(D_r\partial^q_{s_1,\cdots,s_q}u|\mathcal{F}_r), dB_r >_{\mathbb{R}^m} \qquad (4.50)\\ = \quad & -\sum_{j=1}^q \mathbf{1}_{s_j=\max_{i=1}^q s_i}\int_0^{s_j} {}_{(j)}\!< \mathbb{E}(D_{s_j}\partial^q_{s_1,\cdots,s_{j-1},r,s_{j+1},\cdots,s_q}u|\mathcal{F}_{s_j}), dB_r >_{\mathbb{R}^m}; \end{aligned}$$

2-c. $u = d^{q*\sharp}w$*, where* $w \in L^2\Gamma \wedge^{q+1} H$ *has an expression as in (4.46) .*

Remark 4.3.20. For the case of $q = 0$, the statements **1-b.** and **2-b.** are vacuously true, and the rest could be made sense of, if we define, following Shigekawa [55], $d^{-1} = \iota$, with $\iota : \mathbb{R} \to L^2(E;\mathbb{R})$ defined by $\iota(c)(x) = c$ for $c \in \mathbb{R}$, $x \in E$, whose adjoint $\iota^* : L^2(E;\mathbb{R}) \to \mathbb{R}$ is defined by $\iota^* f = \mathbb{E}f$, $f \in L^2(E;\mathbb{R})$.

Note also that Condition **1-b** is simpler than Condition **2-b**: both terms in equation (4.49) are adapted; on the other hand, while the left-hand side of (4.50) is an Itô integral, the right-hand side involves Skorohod integrals, so we cannot compare the integrands directly.

4.4 Generalisation to More General Settings

The above reasoning can be extended, in the language of Wu [63] introduced in Section 1.4, to abstract Wiener spaces and to more general situations. One particularly interesting case is the Carathéory-Finsler manifolds studied by Elworthy and Ma [29; 28]. These are Finsler manifolds where the Carathéory metric, defined by

$$\rho_M(x, y) = \sup\{f(y) - f(x) : f \in C^1_1(M; \mathbb{R})\},$$

is an admissible metric (i.e., it generates the original topology of M) and complete. Here $C^1_1(M; \mathbb{R})$ is the set of all C^1 maps $f : M \to \mathbb{R}$ such that $\|Df\| \leq 1$, where

$$\|Df\| = \sup_{x \in M} \|Df(x)\|_{L(T_x M; \mathbb{R})}.$$

The mapping space $E = C(S; M)$ over a Carathéory-Finsler manifold M, where S is a compact metric space, is again a Carathéory-Finsler manifold with the natural Finsler structure induced from M

$$\|v\|_\sigma = \sup_{s \in S} |v(s)|_{\sigma(s)}, \quad \sigma \in E, v \in T_\sigma E,$$

as are all its closed submanifolds ([29] Proposition 2.6). Thus the path spaces over Riemannian manifolds $C([0, T]; M)$ and their submanifolds, such as based path spaces and based loop spaces, are natural examples in this generality. We refer to [29; 28] for a more detailed description of such Carathéory-Finsler manifolds.

Denote by $\mathcal{B}$ the Borel sets on E, and by $C^1_b(M; \mathbb{R})$ the set of C^1 maps $f : M \to \mathbb{R}$ such that $\|Df\| < \infty$. Let μ be a finite measure on $(E, \mathcal{B})$, and $\mathcal{D}$ a linear subspace of $C^1_b(M; \mathbb{R}) \cap L^2(E, \mu)$ such that $\mathcal{D}$ is dense in $L^2(E, \mu)$ and is an algebra with pointwise multiplication. A vector field v on E is called $\mathcal{D}$-admissible if the Fréchet derivative in direction v, denoted D_v, as a linear operator on the space $\mathcal{D}$ is closable as an operator in $L^2(E, \mu)$. A sufficient condition for v to be $\mathcal{D}$-admissible, when $D_v f \in L^2(E, \mu)$ for all $f \in \mathcal{D}$, is that v has a divergence: that is, there exists an element $\text{Div}(v)$ of $L^2(E,\mu)$ such that

$$\int_E D_v f d\mu = -\int_E f\text{Div}(v)d\mu, \quad \forall f \in \mathcal{D}.$$

In this case, we call v strongly $\mathcal{D}$-admissible. The set AV of all strongly $\mathcal{D}$-admissible vector fields v satisfying

$$\|v\|_{AV}^2 := \int_E |v(\sigma)|_\sigma^2 \mu(d\sigma) + \int_E |\text{Div}(v)|^2 \mu(d\sigma) < \infty,$$

is a Banach space with the norm $\|\cdot\|_{AV}$ ([29] Theorem 3.7).

Assume for each $\sigma \in E$ there is a Hilbert space H_σ continuously embedded in $T_\sigma E$, and there is a $\mathcal{B}$-measurable function $\Psi \geq 1$ such that $\|v\|_\sigma \leq \Psi(\sigma)\|v\|_{H_\sigma}$ for all $v \in H_\sigma$. Denote by AVH the subset of AV consisting of vector fields v such that $v(\sigma) \in H_\sigma$ for almost all σ and

$$\|v\|_{AVH}^2 := \int_E |v(\sigma)|_{H_\sigma}^2 \Psi^2(\sigma)\mu(d\sigma) + \int_E |\text{Div}(v)|^2 \mu(d\sigma) < \infty.$$

This gives a Hilbert space with the inner product from $\|\cdot\|_{AVH}$ ([29] Theorem 3.9).

For instance, AVH can be obtained by taking the image of a separable Hilbert space H under a linear map $T : H \to AV$, such that $|Th|_\sigma \leq \Psi\|h\|_H$ with $\Psi \in L^2(E,\mu)$. If, in addition, for some constant C,

$$\int_E |\text{Div}(Th)|^2 \mu(d\sigma) \leq C\|h\|_H^2,$$

we have $\|Th\|_{AV}^2 \leq C\|h\|_H^2$ for all $h \in H$, and in this case, we can define

$$AVH = \{v \in AV : v = Th \text{ for some } h \in (\ker T)^\perp\},$$

with the inner product given by

$$< v_1, v_2 >_{AVH} = < h_1, h_2 >_H, \quad v_i = Th_i, h_i \in (\ker T)^\perp, i = 1, 2.$$

If we define a measurable $\boldsymbol{X} : E \times H \to TE$ by $\boldsymbol{X}(\sigma, h) = T(h)(\sigma)$, we see immediately the connection of this example with the path space (see Section 2.1, and Corollary 4.9 of [29]). Note that for μ-almost all σ, $\boldsymbol{X}(\sigma, \cdot) : H \to T_\sigma E$ is continuous linear, $|\boldsymbol{X}(\sigma, h)|_\sigma \leq \Psi(\sigma)\|h\|_H$, and $\boldsymbol{X}(\cdot, h)$ is $\mathcal{D}$-admissible for all $h \in H$. We can define the gradient operator and its adjoint, the divergence operator, and using $\boldsymbol{X}$ and its right inverse we can define covariant derivatives to continue the programme.

We will, however, focus on the flat case here: we take (E, H, μ) to be an abstract Wiener space, and continue the discussion from Section 1.4. Assuming that a given subspace of isometries $V \in \Phi$ satisfies the integral representation (1.17), we have a Clark-Ocone representation as in (1.16), and for $\phi \in L^2\Gamma H^*$, equations (4.16) and (4.17) take the forms of

$$CO(\phi) = \delta P_V \phi^\sharp, \tag{4.51}$$

and

$$(d^1\phi)^\sharp = \frac{1}{2}[\tau(\nabla\phi^\sharp) - \nabla\phi^\sharp], \tag{4.52}$$

respectively. To be able to deal with vector fields and one-forms, we need an explicit vector-valued Clark-Ocone formula. As noted by Wu [63], equation (1.15) remains true for vector-valued L^2 functions, i.e., for $F \in L^2(E; X)$,

$$P_{\delta(V)}F = \delta(P_V \nabla F),$$

where X is a Hilbert space, P_V and $P_{\delta(V)}$ are understood in the sense that,

$$< P_V u, x >_X = P_V < u, x >_X, \quad \forall x \in X, u \in L^2(E; X),$$

and

$$< P_{\delta(V)}F, x >_X = P_{\delta(V)} < F, x >_X, \quad \forall x \in X, F \in L^2(E; \mathbb{R}),$$

and the Skorohod integral is regarded as a densely defined operator

$$\delta : L^2(E; X \otimes H) \to L^2(E; X) \cong X \otimes L^2(E; \mathbb{R}).$$

Equivalently, we can redefine the projections as

$$P_V u \stackrel{\text{def}}{=} P_{(X\otimes V)}u = (\text{Id}_X \otimes P_V)u, \quad \forall u \in L^2(E; X \otimes H) \cong X \otimes L^2(E; H),$$

and

$$P_{\delta(V)}F \stackrel{\text{def}}{=} P_{(X\otimes\delta(V))}F = (\text{Id}_X \otimes P_{\delta(V)})F, \quad \forall F \in L^2(E; X) \cong X \otimes L^2(E; \mathbb{R}),$$

so equation (1.15) can be written as

$$P_{X\otimes\delta(V)}F = \delta(P_{X\otimes V}\nabla F) = (\text{Id}_X \otimes \delta P_V)\nabla F, \quad \forall F \in L^2(E; X). \tag{4.53}$$

In a similar fashion, a subspace of $L^2(E; X \otimes H)$ of isometries under this extended

δ can be defined as in Section 1.4, using the extended isometry condition

$$\mathbb{E}|\delta u|_X^2 = \mathbb{E}|u|_{X\otimes H}^2. \tag{4.54}$$

Lemma 4.4.1. *$P_{(X\otimes V)}$ is an orthogonal projection on $L^2(E; X \otimes H)$. Its image $X \otimes V$ is an isometry subspace of $L^2(E; X\otimes H)$ in the sense of (4.54).*

Proof. That $P_{(X\otimes V)}$ is a projection is obvious, since $(\mathrm{Id}_X \otimes P_V)^2 = \mathrm{Id}_X \otimes P_V$ with P_V being an orthogonal projection on $L^2(E; H)$. It is self-adjoint since

$$(\mathrm{Id}_X \otimes P_V)^* = \mathrm{Id}_X \otimes P_V^* = \mathrm{Id}_X \otimes P_V.$$

The rest is clear, and we have, for all $u, v \in X \otimes V$,

$$\mathbb{E} < \delta u, \delta v >_X = \mathbb{E} < u, v >_{X\otimes H} . \qquad \square$$

With the above reinterpretation, we restate the commutation formula in Lemma 3.1.1 as (cf. Remark 3.1.5)

$$\nabla(\delta u) = (\delta \otimes \mathrm{Id}_H)\nabla u + u, \quad \forall u \in L^2(E; H) \cap \mathrm{Dom}(\delta) \text{ s.t. } \delta u \in \mathbb{D}^{2,1}. \tag{4.55}$$

Denote by $L_0^2(E; X)$ the set of elements in $L^2(E; X)$ with zero expectation. Condition (1.17) can be expressed as

$$\delta(X \otimes V) = L_0^2(E; X). \tag{4.56}$$

Lemma 4.4.2 (Wu [63]: Vector-valued Clark-Ocone Formula)**.** *If $V \in \Phi$ satisfies (4.56), we have*

$$F = \mathbb{E}F + \delta[P_{(X\otimes V)}\nabla]F = \mathbb{E}F + [\mathrm{Id}_X \otimes (\delta P_V)]\nabla F, \quad \forall F \in L^2(E; X). \tag{4.57}$$

Remark 4.4.3. As usual, the composed operator $[P_{(X\otimes V)}\nabla]$ extends to a bounded linear operator on $L^2(E; X)$.

Proof. This follows directly from (4.53) and (4.56) (as in [63], or see Section 1.4). Alternatively, we can apply the usual proof for the Clark-Ocone formula: without loss of generality we assume $\mathbb{E}F = 0$, and taking any $u \in L^2(E; X\otimes H)$, we calculate

$$\begin{aligned}
\mathbb{E} < F, \delta P_{(X\otimes V)}u >_X &= \mathbb{E} < [P_{(X\otimes V)}\nabla]F, P_{(X\otimes V)}u >_{X\otimes H} \\
&= \mathbb{E} < \delta[P_{(X\otimes V)}\nabla]F, \delta P_{(X\otimes V)}u >_X,
\end{aligned}$$

so the condition (4.56) implies the conclusion. $\square$

To establish a result similar to (4.23), we need to study the relationship between ∇, δ and the projection operators to obtain something like (4.28), i.e.,

$$\nabla P_V u = [P_{V\otimes H}\nabla]u, \quad u \in L^2(E;H). \tag{4.58}$$

or equivalently,

$$< \nabla P_V u, h >= P_V < \nabla u, h >, \quad \forall h \in H.$$

To prove an analogue of Proposition 4.2.1, we only need

$$(\delta \otimes \mathrm{Id}_H)\nabla P_V u = [(\delta P_V) \otimes \mathrm{Id}_H]\nabla u, \tag{4.59}$$

or even less, in fact,

$$P_V^{\perp}(\delta \otimes \mathrm{Id}_H)\nabla P_V u = P_V^{\perp}[(\delta P_V) \otimes \mathrm{Id}_H]\nabla u. \tag{4.60}$$

All the above conditions (4.58), (4.59) and (4.60) are equivalent on the classical Wiener space. We will assume (4.60) in what follows.

Lemma 4.4.4. *Given $V \in \Phi$, we have, for all $u \in L^2(E;H)$,*

$$u - \nabla\delta P_V u \in \mathrm{Image}(P_V^{\perp}), \tag{4.61}$$

and

$$(\delta \otimes \mathrm{Id}_H)\nabla P_V u \in \mathrm{Image}(P_V^{\perp}). \tag{4.62}$$

As a result, we have

$$u = [P_V\nabla]\delta u, \quad \forall u \in V. \tag{4.63}$$

Proof. By the definition of Φ, we have, for all $u, v \in L^2(E;H)$,

$$\mathbb{E} < P_V u, v >_H = \mathbb{E} < P_V u, P_V v >_H = \mathbb{E}(\delta P_V u)(\delta P_V v) = \mathbb{E} < [P_V\nabla]\delta P_V u, v >_H,$$

where the last equality is justified by the duality between $P_V\nabla$ and δP_V in $L^2(E;H)$, since the composed operator $[P_V\nabla]$ has as its domain $L^2(E;H)$, rather than $\mathbb{D}^{2,1}(H)$, by virtue of (1.18). Hence

$$P_V u = [P_V\nabla]\delta P_V u,$$

and (4.61) and (4.63) follow.

Again by the definition of the subspace of isometries, we derive from Equation (3.6) that, for all $u, v \in L^2(E; H)$,

$$\begin{aligned} 0 &= \mathbb{E} < \nabla P_V u, \tau \nabla P_V v >_{H\otimes H} \\ &= \mathbb{E} < (\delta \otimes \mathrm{Id}_H)\nabla P_V u, P_V v >_H \\ &= \mathbb{E} < P_V(\delta \otimes \mathrm{Id}_H)\nabla P_V u, v >_H, \end{aligned}$$

so

$$P_V(\delta \otimes \mathrm{Id}_H)\nabla P_V u = 0$$

and (4.62) follows. □

Remark 4.4.5. From the commutation formula (4.55), we see that (4.61) and (4.62) are related and, in fact, equivalent, since

$$u - \nabla\delta P_V u = u - [(\delta \otimes \mathrm{Id}_H)\nabla P_V u + P_V u] = P_V^{\perp} u - (\delta \otimes \mathrm{Id}_H)\nabla P_V u.$$

Remark 4.4.6. We have applied the following form of the integration by parts formula in the above proof

$$\mathbb{E} < \nabla u, \tau\nabla v >_{H\otimes H} = \mathbb{E} < (\delta \otimes \mathrm{Id}_H)\nabla u, v >_H, \quad \forall u, v \in L^2(E; H),$$

cf. (3.2). Similarly, we have

$$\mathbb{E} < \nabla u, \nabla v >_{H\otimes H} = \mathbb{E} < (\mathrm{Id}_H \otimes \delta)\nabla u, v >_H, \quad \forall u, v \in L^2(E; H).$$

Proposition 4.4.7. *Suppose $V \in \Phi$ satisfies (4.56) and $\phi \in L^2\Gamma H^*$ is in* $\mathrm{Dom}(d^1)$. *Assume also*

$$P_V^{\perp}(\delta \otimes \mathrm{Id}_H)\nabla P_V \phi^{\sharp} = P_V^{\perp}[(\delta P_V) \otimes \mathrm{Id}_H]\nabla\phi^{\sharp}. \tag{4.64}$$

Then we have $CO(\phi) \in \mathbb{D}^{2,1}$, and

$$\nabla CO(\phi) - \phi^{\sharp} = 2P_V^{\perp}[\mathrm{Id}_H \otimes (\delta P_V)](d^1\phi)^{\sharp}.$$

Therefore, we have $\|\nabla CO(\phi) - \phi^{\sharp}\|_{L^2(E;H)} \leq 2\,\|(d^1\phi)^{\sharp}\|_{L^2(E;\wedge^2 H)}$.

Proof. Apply the Clark-Ocone formula (4.57) to write

$$\phi^{\sharp} = \mathbb{E}\phi^{\sharp} + [\mathrm{Id}_H \otimes (\delta P_V)]\nabla\phi^{\sharp}, \tag{4.65}$$

so we have

$$P_V\phi^\sharp = \mathbb{E}\phi^\sharp + P_V[\mathrm{Id}_H \otimes (\delta P_V)]\nabla\phi^\sharp, \tag{4.66}$$

and

$$P_V^\perp\phi^\sharp = \phi^\sharp - P_V\phi^\sharp = P_V^\perp[\mathrm{Id}_H \otimes (\delta P_V)]\nabla\phi^\sharp. \tag{4.67}$$

By the commutation rule (4.55) and the hypothesis (4.64)

$$\nabla CO(\phi) = \nabla\delta P_V\phi^\sharp = (\delta \otimes \mathrm{Id}_H)\nabla P_V\phi^\sharp + P_V\phi^\sharp = [(\delta P_V) \otimes \mathrm{Id}_H]\nabla\phi^\sharp + P_V\phi^\sharp,$$

therefore,

$$P_V^\perp\nabla CO(\phi) = P_V^\perp\nabla\delta P_V\phi^\sharp = P_V^\perp[(\delta P_V) \otimes \mathrm{Id}_H]\nabla\phi^\sharp. \tag{4.68}$$

So

$$\begin{aligned}
\nabla CO(\phi) - \phi^\sharp &= \nabla\delta P_V\phi^\sharp - \phi^\sharp \\
&= P_V^\perp(\nabla\delta P_V\phi^\sharp - \phi^\sharp) \\
&= P_V^\perp[(\delta P_V) \otimes \mathrm{Id}_H]\nabla\phi^\sharp - P_V^\perp[\mathrm{Id}_H \otimes (\delta P_V)]\nabla\phi^\sharp \\
&= 2P_V^\perp[\mathrm{Id}_H \otimes (\delta P_V)](d^1\phi)^\sharp,
\end{aligned}$$

where the second equality follows from Lemma 4.4.4, the third from (4.68) and (4.67), and the last from (4.52). Now by the isometry (4.54)

$$\begin{aligned}
\|\nabla CO(\phi) - \phi^\sharp\|^2_{L^2(E;H)} &= 4\|P_V^\perp[\mathrm{Id}_H \otimes (\delta P_V)](d^1\phi)^\sharp\|^2_{L^2(E;H)} \\
&\leq 4\|[\mathrm{Id}_H \otimes (\delta P_V)](d^1\phi)^\sharp\|^2_{L^2(E;H)} \\
&= 4\|[\mathrm{Id}_H \otimes P_V](d^1\phi)^\sharp\|^2_{H\otimes V} \\
&\leq 4\|(d^1\phi)^\sharp\|^2_{L^2(E;\wedge^2 H)}.
\end{aligned}$$

□

Here we have a slightly weaker result for the constant.

Remark 4.4.8. Just as in Remark 4.2.5, we comment that we have shown here

$$P_{H\otimes V}(d^1\phi) = 0 \iff \phi = \nabla f \iff d^1\phi = 0,$$

just as the Clark-Ocone formula implies

$$P_V\nabla f = 0 \iff f = \text{constant} \iff \nabla f = 0.$$

Chapter 5

The Representation-Theoretic Approach to the Hodge Theory on Abstract Wiener Spaces

In this chapter, we give another proof of Shigekawa's result that any abstract Wiener space (E, H, μ) is cohomologically trivial.

Our motivation starts with the well-known isomorphism between the symmetric Fock space and the L^2 Gaussian space (see, e.g., [45; 35]): here for an abstract Wiener space (E, H, μ),

$$\Psi : \mathbf{F}_s(H) \cong L^2(E, \mu; \mathbb{R}), \tag{5.1}$$

where $\mathbf{F}_s(H) = \bigoplus_{k=0}^{\infty} H^{\odot k}$ is the symmetric (or Boson) Fock space over H, and $H^{\odot k}$ denotes the k-fold symmetric tensor product of H. On the classical Wiener space $(E = C_0, H = L_0^{2,1}, \gamma)$, the Itô-Wiener chaos expansion [36] expresses this isomorphism by expanding any square integrable functional into a sum of multiple stochastic integrals, so for any $F \in L^2(E, \mu; \mathbb{R})$, we have

$$F = \sum_{k=0}^{\infty} I_k(f_k), \tag{5.2}$$

where $I_0 = \mathbb{E}F$, and $I_k(f_k)$ is the multiple Itô integral. The non-random n-parameter kernel f_k can be taken as symmetric, i.e., $f_k \in H^{\odot k}$, hence uniquely determined by F. We have the same expansion (5.10) on a general abstract Wiener space, where I_k still maps the k-particle symmetric subspace isometrically onto the k-th Wiener chaos, albeit without the appealing interpretation as an Itô integral.

The chaos expansion is a powerful tool in stochastic analysis, and can be used to prove the classical Clark-Ocone formula (see, e.g., [48]), as well as the Clark-Ocone formulae for L^2 H-one-forms (Propositions 4.2.1 and 4.2.9). This method, however, becomes untractable for the study of higher-order differential forms, due to the complication introduced by higher degrees of skew-symmetry. Nualart and Zakai [51] (Proposition 2.7) gave a decomposition, analogous to their earlier result for H-valued vector fields in [50] (Theorem 4.4), for the subspace of completely symmetric higher-order vector fields. They did not study exterior derivatives or differential forms, and considered only iterated H-derivatives ∇^k and their adjoints δ^k, which do not involve any skew-symmetry.

For our purpose, observe that the isomorphism in (5.1) extends to

$$\Psi^q : \mathbf{F}_s(H) \otimes H^{\wedge q} \cong L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge q}, \tag{5.3}$$

where $H^{\wedge q}$ is the q-fold skew-symmetric tensor product. It is not difficult to see that Ψ^q intertwines the exterior derivatives d^q and their adjoints d^{q*} with symmetrisation and skew-symmetrisation operators, giving the following commutative diagram

$$\begin{array}{ccc} \mathbf{F}_s(H) \otimes H^{\wedge q} & \xrightarrow{\Psi^q} & L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge q} \\ {\scriptstyle \breve{d}^q} \big\downarrow & & \big\downarrow {\scriptstyle d^{q\sharp}} \\ \mathbf{F}_s(H) \otimes H^{\wedge q+1} & \xrightarrow{\Psi^{q+1}} & L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge q+1}. \end{array}$$

We describe the corresponding operators $\breve{d}^q$ and their adjoints $\breve{d}^{q*}$ on the extended Fock spaces in Section 5.2. This interplay of symmetry and skew-symmetry is best understood through the representation theory of symmetric groups, from which we borrow the machinery of the Littlewood-Richardson rule to decompose certain special S_n-invariant subspaces of $H^{\otimes n}$. This decomposition, proved in Section 5.1, leads immediately to the Hodge decomposition on our abstract Wiener space, which we explain in the last section. Our representation-theoretic notation follows that of James and Kerber [39], and we refer the reader to standard textbooks on representation theory for a more detailed description.

5.1 Some Representation Theory

Fix $k, n \in \mathbb{N}$, with $k < n$. We consider $H^{\otimes n}$, the completed n-th tensor powers of the Hilbert space H, and its two subspaces $H^{\odot n}$ and $H^{\wedge n}$, of the symmetric and skew-symmetric tensor powers, respectively, both completed using the Hilbert space cross norm inherited from $H^{\otimes n}$. Let $q = n - k$. We also have $H^{\odot k} \otimes H^{\wedge q}$, the subspace of elements of $H^{\otimes n}$ symmetric in the *first* k components and skew-symmetric in the last q components.

More generally, denote by $H^{\odot[k],\wedge[q]}$ the subspace spanned by elements of $H^{\otimes n}$ symmetric in *any* k components and skew-symmetric in the remaining q components; note that we are not fixing the order of the symmetric and skew-symmetric parts with respect to each other inside the tensor product. The symmetric group of degree n, S_n, acts naturally on $H^{\otimes n}$ by permuting the n components. The vector subspace $H^{\odot k} \otimes H^{\wedge q}$ is not an S_n-invariant subspaces of $H^{\otimes n}$, but $H^{\odot[k],\wedge[q]}$ is.

To give a more specific description of $H^{\odot[k],\wedge[q]}$, let $\mathbf{n} = \{1, 2, \cdots, n\}$, and define the set of the k-subsets of $\mathbf{n}$

$$\mathbf{n}^{[k]} = \{\mathbf{a} \subseteq \mathbf{n} || \mathbf{a}| = k\}.$$

For each $\mathbf{a} \in \mathbf{n}^{[k]}$, denote by $\mathbf{a}^c$ its complement in $\mathbf{n}$, and by $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$ the subspace of elements of $H^{\otimes n}$ symmetric in the chosen components, specified by $\mathbf{a}$, and skew-symmetric in the remaining components, specified by $\mathbf{a}^c$. Similarly we have $H^{\odot\mathbf{a},\otimes\mathbf{a}^c}$, the subspace of elements of $H^{\otimes n}$ only restricted to be symmetric in the chosen components specified by $\mathbf{a}$, and $H^{\wedge\mathbf{a},\otimes\mathbf{a}^c}$, the subspace of elements of $H^{\otimes n}$ skew-symmetric in the chosen components specified by $\mathbf{a}$. For example, $H^{\odot k} \otimes H^{\wedge q}$ correspond to the choice of $\mathbf{a} = \{1, \cdots, k\}$, and hence $\mathbf{a}^c = \{k+1, \cdots, n\}$, so in this notation can be written as $H^{\odot\{1,\cdots,k\},\wedge\{k+1,\cdots,n\}}$.

The subspace $H^{\odot[k],\wedge[q]}$ is the span of all $C(n, k)$ subspaces $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$, corresponding to the $C(n, k)$ possible choices of $\mathbf{a}$ in $\mathbf{n}^{[k]}$, of elements invariant under a permutation of a specific set of k variables, and anti-invariant under a permutation of the rest. For each choice, say, $\mathbf{a}$ and hence $\mathbf{a}^c$, of the k and q variables, the action of the corresponding $S_k \times S_q$ stabilises $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$, while the elements of $S_n/(S_k \times S_q)$ permute the spaces $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$ with different choices of $\mathbf{a}$'s. The structure of $H^{\odot[k],\wedge[q]}$ is similar to that of the representation induced from $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$, but the spaces $H^{\odot\mathbf{a},\wedge\mathbf{a}^c}$ with different choices of $\mathbf{a}$'s in general can intersect non-trivially, and therefore may not form a direct sum.

Recall the symmetrisation and skew-symmetrisation operators S^n and A^n, which project elements of $H^{\otimes n}$ onto the closed subspaces $H^{\odot n}$ and $H^{\wedge n}$, respectively. Corresponding to an element $\mathbf{a} \in \mathbf{n}^{[k]}$, we define the operator $S^{\mathbf{a}} : H^{\otimes n} \to H^{\odot\mathbf{a},\otimes\mathbf{a}^c}$, which symmetrises any n-tensor in its k components specified by $\mathbf{a}$, and the operator $A^{\mathbf{a}} : H^{\otimes n} \to H^{\wedge\mathbf{a},\otimes\mathbf{a}^c}$, which skew-symmetrises any n-tensor in its k components specified by $\mathbf{a}$. To be more precise, for $h \in H^{\otimes n}$ and $\rho \in S_k$, denote by $\rho^{\mathbf{a}} h$ the element of $H^{\otimes n}$ that has its $\mathbf{a}$ components permuted by ρ and the remaining components fixed, so $S^{\mathbf{a}}$ is defined by

$$S^{\mathbf{a}}h = \frac{1}{k!}\sum_{\rho \in S_k} \rho^{\mathbf{a}} h,$$

and $A^{\mathbf{a}}$ is defined, similarly to the alternating map A^n in (1.3), by

$$A^{\mathbf{a}}h = \frac{1}{k!}\sum_{\rho \in S_k} \operatorname{sgn}(\rho)\rho^{\mathbf{a}} h.$$

From our earlier discussion, it is clear that $H^{\odot \mathbf{a}, \otimes \mathbf{a}^c}$ is the image of $H^{\otimes n}$ under $S^{\mathbf{a}}$, $H^{\wedge \mathbf{a}, \otimes \mathbf{a}^c}$ is the image under $A^{\mathbf{a}}$, and $H^{\odot \mathbf{a}, \wedge \mathbf{a}^c}$ is the image under $S^{\mathbf{a}}A^{\mathbf{a}^c}$.

If $\{e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal basis of H, $\{e_{i_1}\otimes\cdots\otimes e_{i_n}\}_{i_1,\cdots,i_n=1}^{\infty}$ is a complete orthonormal basis of $H^{\otimes n}$. We can choose a basis of $H^{\odot n}$ with elements of the form

$$\frac{1}{n!}\sum_{p\in S_n} e_{i_{p(1)}}\otimes\cdots\otimes e_{i_{p(n)}},$$

with $i_1, \cdots, i_n$ all integers. Similarly, we also have a basis of $H^{\wedge n}$ with elements of the form

$$\frac{1}{n!}\sum_{p\in S_n} \operatorname{sgn}(p) e_{j_{p(1)}}\otimes\cdots\otimes e_{j_{p(n)}},$$

with $j_1, \cdots, j_q$ all *distinct* integers. For $H^{\odot k}\otimes H^{\wedge q}$, we can similarly take basis elements of the form

$$\frac{1}{k!}\left(\sum_{\rho\in S_k} e_{i_{\rho(1)}}\otimes\cdots\otimes e_{i_{\rho(k)}}\right)\otimes\frac{1}{q!}\left(\sum_{\pi\in S_q}\operatorname{sgn}(\pi) e_{j_{\pi(1)}}\otimes\cdots\otimes e_{j_{\pi(q)}}\right), \qquad (5.4)$$

where each of the indices $i_1, \cdots, i_k$, $j_1, \cdots$ and j_q run from 1 to ∞, and the j's have to be all distinct.

For $H^{\odot[k],\wedge[q]}$, its typical basis elements look almost like (5.4), but the positions of the components which are symmetric and those which are skew-symmetric depend on one of the $C(n,k)$ choices from $\mathbf{n}^{[k]}$. A given basis element of $H^{\odot k}\otimes H^{\wedge q}$ of the form (5.4), say, b, corresponds to two specific collections of basis elements of H, counted with multiplicity:

$$E_{b_i} = \{e_{i_1}, \cdots, e_{i_k}\}, \quad \text{and } E_{b_j} = \{e_{j_1}, \cdots, e_{j_q}\}.$$

The vector b also corresponds to the element $\mathbf{a}_b = \{1, \cdots, k\}$ in $\mathbf{n}^{[k]}$. Similarly, a basis element of $H^{\odot[k],\wedge[q]}$ involves first the choice of two collections of basis elements of H, counted with multiplicity, where one set (of k elements specified by the i-indices) forms the symmetric part of the basis element, and the other set (of q distinct elements specified by the j-indices) forms the skew-symmetric part; and secondly the choice of an element in $\mathbf{n}^{[k]}$, for the positioning of the k symmetric components. For $b \in H^{\odot k} \otimes H^{\wedge q}$ as in (5.4), since the action of S_n permutes the n components of b, the orbit O_b of b under the action of S_n covers all the $C(n,k)$ possibilities of the positioning. We can therefore enumerate all our basis elements of $H^{\odot[k],\wedge[q]}$ by going through the basis elements of $H^{\odot k} \otimes H^{\wedge q}$ of the form (5.4) (for our purpose, we don't need to worry about the possible repetitions). Denote by V_{O_b} the span of the vectors in O_b, which is a subspace of $H^{\odot[k],\wedge[q]}$. We have thus proved the following

Lemma 5.1.1. *Given any $k, q \in \mathbb{N}$, we have*

$$H^{\odot[k],\wedge[q]} = \mathrm{Span}\left(\bigcup_b V_{O_b}\right), \tag{5.5}$$

where the union is taken over all basis elements of a complete orthonormal basis of $H^{\odot k} \otimes H^{\wedge q}$.

In the sequel, we will often study the basis elements of $H^{\odot[k],\wedge[q]}$ through those of $H^{\odot k} \otimes H^{\wedge q}$, which give easier notation for explicit expressions.

For a fixed basis element b of $H^{\odot[k],\wedge[q]}$, there is a subgroup of S_n isomorphic to $S_k \times S_q$ whose representation on the one-dimensional space spanned by b is $[k]\sharp[1^q]$, the outer tensor product of $[k]$ and $[1^q]$, an irreducible representation of $S_k \times S_q$ (e.g., see Section 2.3 of [39]). The irreducible representation $[k]\sharp[1^q]$ on Span(b) induces into S_n the representation $[k][1^q]$ on V_{O_b}. A simple application of the Littlewood-Richardson rule (Theorem 2.8.13, or more directly, Corollary 2.8.14, of [39]) yields the following decomposition into irreducible constituents:

$$[k][1^q] = [k+1, 1^{q-1}] \oplus [k, 1^q]. \tag{5.6}$$

Hence, every subspace V_{O_b} splits into a direct sum of two irreducible components

$$V_{O_b} = V_{O_b}^+ \oplus V_{O_b}^-, \tag{5.7}$$

where $V_{O_b}^+$ and $V_{O_b}^-$ correspond to $[k+1, 1^{q-1}]$ and $[k, 1^q]$, respectively.

Suppose we have another basis element b' of $H^{\odot[k],\wedge[q]}$, with a corresponding orbit $O_{b'}$ and an associated subspace $V_{O_{b'}}$. As in the discussion earlier, in terms of the basis elements of H appearing in the expression of b and b', we have two sets

$E_{b_i} = \{e_{i_1}, \cdots, e_{i_k}\}$ and $E_{b_j} = \{e_{j_1}, \cdots, e_{j_q}\}$, where the i's and j's $\in \mathbb{N}$ and the j's are all distinct, and similarly $E_{b'_i} = \{e_{i'_1}, \cdots, e_{i'_k}\}$ and $E_{b'_j} = \{e_{j'_1}, \cdots, e_{j'_q}\}$, where the i''s and j''s $\in \mathbb{N}$ and the j''s are all distinct.

Observe that the orbits O_b and $O_{b'}$ are disjoint and the spaces V_{O_b} and $V_{O_{b'}}$ have a trivial intersection, as long as the sequences (E_{b_i}, E_{b_j}) and $(E_{b'_i}, E_{b'_j})$ differ. Therefore, each V_{O_b} intersects at most finitely many other subspaces $V_{O_{b'}}$. If we have a non-trivial element $v \in V_{O_b} \cap V_{O_{b'}}$, the orbit of v under the action of S_n spans an invariant subspace of both V_{O_b} and $V_{O_{b'}}$. Our earlier discussion shows that, either these two spaces coincide, i.e.,

$$V_{O_b} = V_{O_{b'}} (= V_{O_b} \cap V_{O_{b'}}),$$

or their intersection corresponds to one of the two components in (5.6), $[k+1, 1^{q-1}]$ and $[k, 1^q]$, i.e., in terms of (5.7),

$$V_{O_b} \cap V_{O_{b'}} = V^+_{O_b} \text{ or } V^-_{O_b}.$$

In summary, the action of S_n splits the collection of our basis elements of $H^{\odot[k],\wedge[q]}$ into disjoint subsets, each of which spans a vector subspace of $H^{\odot[k],\wedge[q]}$, which is a copy of the representation $[k][1^q]$ of S_n. Any non-trivial intersection of these vector subspaces, when they do not coincide, is limited to be one of the two irreducible components, as shown above. Therefore, the space $H^{\odot[k],\wedge[q]}$ is made of infinitely many finite-dimensional isomorphic representations of S_n, each isomorphic to $[k][1^q]$, mostly disjoint from the rest, but possibly intersecting a few along its irreducible components.

This discussion enables us to state the following decomposition of $H^{\odot[k],\wedge[q]}$ inside $H^{\otimes n}$.

Lemma 5.1.2. *Given any $k, q \in \mathbb{N}$, let $n = k + q$. Then we have*

$$H^{\odot[k],\wedge[q]} = (H^{\odot[k],\wedge[q]} \cap H^{\odot[k+1],\wedge[q-1]}) \oplus (H^{\odot[k],\wedge[q]} \cap H^{\odot[k-1],\wedge[q+1]}), \qquad (5.8)$$

where the equality is understood to take place inside $H^{\otimes n}$.

Proof. As mentioned above, S_n acts naturally on $H^{\otimes n}$ by permuting the n components. The vector subspaces in question, i.e., $H^{\odot[k],\wedge[q]}$, $H^{\odot[k+1],\wedge[q-1]}$, and $H^{\odot[k-1],\wedge[q+1]}$, as well as their intersections appearing in (5.8), are S_n-invariant subspaces of $H^{\otimes n}$.

Lemma 5.1.1 and the discussion afterwards show that $H^{\odot[k],\wedge[q]}$ consists of subspaces isomorphic to the representation $[k][1^q]$ of S_n, each intersecting finitely many others, with the non-trivial intersection being one of the two irreducible components of $[k][1^q]$. The same statements can be made for $H^{\odot[k+1],\wedge[q-1]}$ and $H^{\odot[k-1],\wedge[q+1]}$, but replacing $[k][1^q]$ with $[k+1][1^{q-1}]$ and $[k-1][1^{q+1}]$, respectively.

Similar to (5.6), we also have the Littlewood-Richardson decompositions for $[k+1][1^{q-1}]$ and $[k-1][1^{q+1}]$, i.e.,

$$[k+1][1^{q-1}] = [k+2, 1^{q-2}] \oplus [k+1, 1^{q-1}]$$

and

$$[k-1][1^{q+1}] = [k, 1^q] \oplus [k-1, 1^{q+1}],$$

respectively. Observe that $[k][1^q]$ has exactly one irreducible component in common with $[k+1][1^{q-1}]$, which is $[k+1, 1^{q-1}]$, and exactly one with $[k-1][1^{q+1}]$, which is $[k-1, 1^{q+1}]$, and no other ones.

Now Lemma 5.1.1 implies that $H^{\odot[k+1],\wedge[q-1]}$ and $H^{\odot[k-1],\wedge[q+1]}$ have only a trivial intersection; indeed, the intersection would have to be S_n-invariant, but (5.5) shows that it has to be trivial. Therefore, we only need to show that $H^{\odot[k],\wedge[q]}$ does intersect $H^{\odot[k+1],\wedge[q-1]}$ and $H^{\odot[k-1],\wedge[q+1]}$ separately, in a manner corresponding to the way $[k][1^q]$ intersects with $[k+1][1^{q-1}]$ and $[k-1][1^{q+1}]$, which then gives us the direct sum as in (5.8).

Again we can look at an arbitrary basis element b of the form (5.4), i.e.,

$$b = \frac{1}{k!}\left(\sum_{\rho\in S_k} e_{i_{\rho(1)}} \otimes \cdots \otimes e_{i_{\rho(k)}}\right) \otimes \frac{1}{q!}\left(\sum_{\pi\in S_q} \operatorname{sgn}(\pi) e_{j_{\pi(1)}} \otimes \cdots \otimes e_{j_{\pi(q)}}\right),$$

and its associated vector subspace $V_{O_b} \subset H^{\odot[k],\wedge[q]}$. All we need is to find two vectors,

$$v^+ \in V_{O_b} \cap H^{\odot[k+1],\wedge[q-1]}, \qquad \text{and } v^- \in V_{O_b} \cap H^{\odot[k-1],\wedge[q+1]},$$

since the two disjoint invariant subspaces, $V_{O_b} \cap H^{\odot[k+1],\wedge[q-1]}$ and $V_{O_b} \cap H^{\odot[k-1],\wedge[q+1]}$, of V_{O_b} have to correspond to the $[k+1, 1^{q-1}]$ and $[k, 1^q]$ components, respectively.

Using the operator τ defined earlier in (4.5), we can express the result of swapping the l-th and $(k+1)$-th components of b as $\tau_{l,k+1}b$, which is an element of O_b and of $H^{\odot\mathbf{a}(l),\wedge\mathbf{a}(l)^c}$, where $\mathbf{a}(l) = \{1, \cdots, \hat{l}, \cdots, k, k+1\} \in \mathbf{n}^{[k]}$, and l ranges from 1 to $k+1$. Similarly, for each $m = k+1, \cdots, n$, we have $\tau_{k,m}b$, an element of O_b and of $H^{\odot\mathbf{a}<m>,\wedge\mathbf{a}<m>^c}$, with $\mathbf{a}<m>= \{1, \cdots, k-1, m\} \in \mathbf{n}^{[k]}$. We conclude the proof by setting

$$v^+ = \frac{1}{k+1}\sum_{l=1}^{k+1}\tau_{l,k+1}b$$

and

$$v^- = \frac{1}{q+1}(b - \sum_{m=k+1}^{n}\tau_{k,m}b). \qquad \square$$

Corollary 5.1.3. *For each* $\mathbf{a} \in \mathbf{n}^{[k]}$, *we have*

$$H^{\odot\mathbf{a},\wedge\mathbf{a}^c} = (H^{\odot\mathbf{a},\wedge\mathbf{a}^c} \cap H^{\odot[k+1],\wedge[q-1]}) \oplus (H^{\odot\mathbf{a},\wedge\mathbf{a}^c} \cap H^{\odot[k-1],\wedge[q+1]}). \tag{5.9}$$

Proof. For any $g \in H^{\odot\mathbf{a},\wedge\mathbf{a}^c} \subset H^{\odot[k],\wedge[q]}$, we have

$$g = A^{\mathbf{a}^c}S^{\mathbf{a}}g.$$

Lemma 5.1.2 gives a direct-sum decomposition for g

$$g = \tilde{g} + \hat{g},$$

with $\tilde{g} \in H^{\odot[k],\wedge[q]} \cap H^{\odot[k+1],\wedge[q-1]}$, and $\hat{g} \in H^{\odot[k],\wedge[q]} \cap H^{\odot[k-1],\wedge[q+1]}$. So we have

$$g = A^{\mathbf{a}^c}S^{\mathbf{a}}g = A^{\mathbf{a}^c}S^{\mathbf{a}}\tilde{g} + A^{\mathbf{a}^c}S^{\mathbf{a}}\hat{g},$$

with $A^{\mathbf{a}^c}S^{\mathbf{a}}\tilde{g} \in (H^{\odot[k],\wedge[q]} \cap H^{\odot[k+1],\wedge[q-1]}) \cap H^{\odot\mathbf{a},\wedge\mathbf{a}^c} = H^{\odot\mathbf{a},\wedge\mathbf{a}^c} \cap H^{\odot[k+1],\wedge[q-1]}$, and $A^{\mathbf{a}^c}S^{\mathbf{a}}\hat{g} \in (H^{\odot[k],\wedge[q]} \cap H^{\odot[k-1],\wedge[q+1]}) \cap H^{\odot\mathbf{a},\wedge\mathbf{a}^c} = H^{\odot\mathbf{a},\wedge\mathbf{a}^c} \cap H^{\odot[k-1],\wedge[q+1]}$. By the uniqueness of the direct-sum decomposition, we have $\tilde{g} = A^{\mathbf{a}^c}S^{\mathbf{a}}\tilde{g}$ and $\hat{g} = A^{\mathbf{a}^c}S^{\mathbf{a}}\hat{g}$, and the proof is complete. $\square$

5.2 Higher Order Forms on Abstract Wiener Spaces

To apply the above general result to an abstract Wiener space (E, H, μ), we first recall the Itô-Wiener chaos expansion [36], which allows us to express $F \in L^2(E, \mu; \mathbb{R})$ as

$$F = \sum_{k=0}^{\infty} I_k(f_k), \quad f_k \in H^{\odot k}. \tag{5.10}$$

Similarly, any H-q-form $\phi \in L^2\Gamma(H^{\wedge q})^*$, or equivalently, $u = \phi^\sharp \in L^2\Gamma H^{\wedge q}$, can be expressed as

$$u = \sum_{k=0}^{\infty} I_k(f_k), \quad f_k \in H^{\odot k} \otimes H^{\wedge q}. \tag{5.11}$$

Applying Corollary 5.1.3 to the subspace $H^{\odot k} \otimes H^{\wedge q}$, we have the following direct-sum decomposition inside $H^{\otimes(k+q)}$:

$$H^{\odot k} \otimes H^{\wedge q} = \Big[(H^{\odot k} \otimes H^{\wedge q}) \cap H^{\odot[k+1],\wedge[q-1]}\Big] \oplus \Big[(H^{\odot k} \otimes H^{\wedge q}) \cap H^{\odot[k-1],\wedge[q+1]}\Big].$$

In the sequel, we write $H_{k,q} = H^{\odot k} \otimes H^{\wedge q}$, $H^{+}_{k,q} = H_{k,q} \cap H^{\odot[k+1],\wedge[q-1]}$, and $H^{-}_{k,q} = H_{k,q} \cap H^{\odot[k-1],\wedge[q+1]}$.

The isomorphism $\Psi : \mathbf{F}_s(H) \cong L^2(E, \mu; \mathbb{R})$, which sends elements of the symmetric Fock space to elements of the L^2 space on E, is defined by (see [45])

$$\Psi(\exp \odot h) = \exp(I(h) - \frac{1}{2}\|h\|^2), \quad h \in H,$$

where

$$\exp \odot h = \sum_{k \geq 0} \frac{1}{k!} h^{\odot k} \in \bigoplus_{k=0}^{\infty} H^{\odot k} = \mathbf{F}_s(H).$$

We extend Ψ to isomorphisms $\Psi^q : \mathbf{F}_s(H) \otimes H^{\wedge q} \cong L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge q}$, by setting

$$\Psi^q = \Psi \otimes \mathrm{Id}_{H^{\wedge q}}.$$

Recall the correspondence between L^2 differential forms and skew-symmetric L^2 vector fields via the Riesz representation; see Section 4.1. Thus we have, corresponding to each H-q-form $\phi \in L^2\Gamma(H^{\wedge q})^*$, a skew-symmetric H-vector field $u = \phi^\sharp \in L^2\Gamma H^{\wedge q} \cong L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge q}$, and an element $\breve{u} = (\Psi^q)^{-1} u \in \mathbf{F}_s(H) \otimes H^{\wedge q}$, which can be expressed as

$$\breve{u} = \sum_{k=0}^{\infty} f_k, \quad f_k \in H^{\odot k} \otimes H^{\wedge q}. \tag{5.12}$$

The following commutative diagram

$$\begin{array}{ccccc}
\bigoplus_{k=0}^{\infty} H^{\odot k} \otimes H^{\wedge q} & \xrightarrow{\Psi^q} & L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge q} & \xrightarrow{\cong} & L^2(E; (H^{\wedge q})^*) \\
\downarrow \breve{d}^q & & \downarrow d^{q\sharp} & & \downarrow d^q \\
\bigoplus_{k=1}^{\infty} H^{\odot(k-1)} \otimes H^{\wedge(q+1)} & \xrightarrow{\Psi^{q+1}} & L^2(E, \mu; \mathbb{R}) \otimes H^{\wedge(q+1)} & \xrightarrow{\cong} & L^2(E; (H^{\wedge(q+1)})^*)
\end{array}$$

shows that, to study the Hodge theory on our abstract Wiener space, it suffices to study the counterparts of d^q and d^{q*} on the Fock spaces, i.e., the operator $\breve{d}^q$ on $\mathbf{F}_s(H) \otimes H^{\wedge q}$, and its adjoint operator $\breve{d}^{q*}$. For $h \in H$ and $x \in H^{\wedge q}$, we have

$\Psi^q[(\exp \odot h) \otimes x] = \exp(I(h) - \frac{1}{2}\|h\|^2) \otimes x$, and

$$\begin{aligned} d^{q\sharp}\Psi^q[(\exp \odot h) \otimes x] &= d^{q\sharp}[\exp(I(h) - \frac{1}{2}\|h\|^2) \otimes x] \\ &= A^{q+1}[\exp(I(h) - \frac{1}{2}\|h\|^2) \otimes h \otimes x] \\ &= \exp(I(h) - \frac{1}{2}\|h\|^2) \otimes (h \wedge x) \\ &= \Psi^{q+1}[(\exp \odot h) \otimes (h \wedge x)]. \end{aligned}$$

Since Ψ^q is an isomorphism for each q, we can deduce, from $d^{q\sharp}\Psi^q = \Psi^{q+1}\breve{d}^q$, that

$$\breve{d}^q[(\exp \odot h) \otimes x] = (\exp \odot h) \otimes (h \wedge x).$$

We can understand $\breve{d}^q$ more clearly by looking at its action on each chaos. For the k-th chaos, any element $v \in H^{\odot k} \otimes H^{\wedge q}$ can be written as a sum of terms of the form

$$(\sum_{\rho \in S_k} h_{\rho(1)} \otimes \cdots \otimes h_{\rho(k)}) \otimes x,$$

for some $h_i \in H$ for all $i = 1$ to k, and $x \in H^{\wedge q}$. The above calculation gives

$$\begin{aligned} &\breve{d}^q \left(\sum_{\rho \in S_k} h_{\rho(1)} \otimes \cdots \otimes h_{\rho(k)} \otimes x \right) \\ = \quad &(\mathrm{Id}_{H^{\odot(k-1)}} \otimes A^{q+1}) \left[\sum_{\rho \in S_k} (h_{\rho(1)} \otimes \cdots \otimes h_{\rho(k-1)}) \otimes (h_{\rho(k)} \otimes x) \right], \end{aligned}$$

so the restriction of $\breve{d}^q$ to the k-th chaos is given by

$$\breve{d}^q v = \mathrm{Id}_{H^{\odot(k-1)}} \otimes A^{q+1} v, \quad v \in H^{\odot k} \otimes H^{\wedge q}. \tag{5.13}$$

Lemma 5.2.1. $\mathrm{Ker}(\breve{d}^q|_{H_{k,q}}) = H^+_{k,q}$.

Proof. With (5.13), it suffices to show $\mathrm{Ker}(A^{q+1}) = H^+_{1,q}$. We note $H_{1,q} = H^+_{1,q} \oplus H^-_{1,q}$ with $H^+_{1,q} = (H \otimes H^{\wedge q}) \cap H^{\odot[2],\wedge[q-1]}$, and $H^-_{1,q} = (H \otimes H^{\wedge q}) \cap H^{\wedge(q+1)}$. It is clear that $H^+_{1,q} \subset \mathrm{Ker}(A^{q+1})$.

Conversely, we show $\mathrm{Ker}(A^{q+1}) \subset H^+_{1,q}$. Since $A^{q+1}|_{H^{\wedge(q+1)}} = \mathrm{Id}_{H^{\wedge(q+1)}}$, we see $A^{q+1}|_{H^-_{1,q}} = \mathrm{Id}_{H^-_{1,q}}$. Now any element $h \in H_{1,q}$ can be written as $h = h^+ + h^-$, with $h^+ \in H^+_{1,q} \subset \mathrm{Ker}(A^{q+1})$, and $h^- \in H^-_{1,q}$. If $h \in \mathrm{Ker}(A^{q+1})$, we have

$$0 = A^{q+1}h = A^{q+1}h^+ + A^{q+1}h^- = A^{q+1}h^- = h^-,$$

therefore, $\mathrm{Ker}(A^{q+1}) = H^+_{1,q}$, and we are done. $\square$

Similarly, we can study the adjoint operator $\breve{d}^{q*}$, whose restriction to each chaos, $\breve{d}^{q*}|_{H^{\odot(k-1)}\otimes H^{\wedge(q+1)}} : H^{\odot(k-1)} \otimes H^{\wedge(q+1)} \subset H^{\otimes n} \to H^{\odot k} \otimes H^{\wedge q}$, is given by

$$\breve{d}^{q*}v = S^k \otimes \mathrm{Id}_{H^{\wedge q}} v, \quad v \in H^{\odot(k-1)} \otimes H^{\wedge(q+1)},$$

since any $v \in H^{\odot(k-1)} \otimes H^{\wedge(q+1)}$ can be written as a sum of terms of the form

$$\sum_{\rho \in S_{q+1}} \mathrm{sgn}(\rho) y \otimes h_{\rho(1)} \otimes \cdots \otimes h_{\rho(q+1)},$$

with $h_i \in H$ for all $i = 1$ to $q+1$, and $y \in H^{\odot(k-1)}$, and we can derive

$$\begin{aligned} & \breve{d}^{q*}\left(\sum_{\rho \in S_{q+1}} \mathrm{sgn}(\rho) y \otimes h_{\rho(1)} \otimes \cdots \otimes h_{\rho(q+1)}\right) \\ = \; & (S^k \otimes \mathrm{Id}_{H^{\wedge q}}) \left[\sum_{\rho \in S_k} \mathrm{sgn}(\rho)(y \otimes h_{\rho(1)}) \otimes (h_{\rho(2)} \otimes \cdots \otimes h_{\rho(q+1)})\right]. \end{aligned} \tag{5.14}$$

A similar reasoning as in Lemma 5.2.1 shows that

$$\mathrm{Ker}(\breve{d}^{q*}|_{H_{k-1,q+1}}) = H^-_{k-1,q+1}.$$

Summarising the above discussion, we have the following counterpart on the Fock spaces to Shigekawa's Hodge decomposition on abstract Wiener spaces [55].

Corollary 5.2.2. *For k, $q \in \mathbb{N}$, we have*

1. $\mathrm{Ker}(\breve{d}^q|_{H^{\odot k}\otimes H^{\wedge q}}) = H^+_{k,q}$.

2. $\mathrm{Ker}(\breve{d}^{(q-1)*}|_{H^{\odot k}\otimes H^{\wedge q}}) = H^-_{k,q}$.

The direct sum decomposition $H^{\odot k} \otimes H^{\wedge q} = H^+_{k,q} \oplus H^-_{k,q}$ shows that

$$\mathrm{Ker}(\breve{d}^{(q-1)*}|_{H^{\odot k}\otimes H^{\wedge q}}) \cap \mathrm{Ker}(\breve{d}^q|_{H^{\odot k}\otimes H^{\wedge q}}) = \emptyset.$$

Thus, the kernels of $\breve{d}^q$ and $\breve{d}^{(q-1)*}$ do not intersect inside any chaos. Correspondingly, the kernels of the operators d^q and $d^{(q-1)*}$ also intersect trivially; that is, there is no L^2 harmonic q-form on abstract Wiener spaces for $q > 0$.

Professor J. Rawnsley showed us that the following exact sequences

$$0 \to H^{\odot n} \xrightarrow{\breve{d}^0} \cdots \xrightarrow{\breve{d}^{q-1}} H_{k,q} \xrightarrow{\breve{d}^q} H_{k-1,q+1} \xrightarrow{\breve{d}^{q+1}} \cdots \xrightarrow{\breve{d}^n} H^{\wedge n} \to 0$$

and

$$0 \to H^{\wedge n} \xrightarrow{\breve{d}^{n*}} \cdots \xrightarrow{\breve{d}^{(q+1)*}} H_{k-1,q+1} \xrightarrow{\breve{d}^{q*}} H_{k,q} \xrightarrow{\breve{d}^{(q-1)*}} \cdots \xrightarrow{\breve{d}^{0*}} H^{\odot n} \to 0$$

can be proved using the following identity on $H_{k,q} = H^{\odot k} \otimes H^{\wedge q}$:

$$\breve{d}^{q*} \circ \breve{d}^{q} + \breve{d}^{q-1} \circ \breve{d}^{(q-1)*} = (k+q)\mathrm{Id}_{H_{k,q}}, \tag{5.15}$$

since $\breve{d}$ and $\breve{d}^*$ are invertible on the kernel of each other. Our Corollary 5.2.2 above shows how the exact sequences split. Equation (5.15) follows directly from (5.13) and (5.14), and can be seen as the analogue of Shigekawa's Weitzenböck identity in Lemma 4.3.7, since ∇, δ and $\delta\nabla$ correspond, respectively, to the annihilation, creation and number operators on the Fock space. This gives a simple proof of the Hodge decomposition in [55].

It would be interesting to extend this approach to other settings where analogues of the chaos expansion have been established, for example, over compact Lie groups with respect to the heat kernel measure [34; 33], or for compensated Poisson processes [62; 37; 38], normal martingales, and more general Levy processes [49].

We do not expect this approach to work in general for curved path spaces, nor the loop spaces where obstructions to log-Sobolev inequality and spectral gap inequality exist [17]. It is the path and loop spaces on compact Lie groups that we wish to generalise our idea to, and perhaps give a meaningful comparison with the two existing vanishing results, one by Fang and Franchi [31] for the path groups, using the flat left-invariant connection and the Itô map, and the other by Aida [2] for the loop groups, applying tools from the rough path theory. We also hope to relate this approach to the work on configuration spaces [53; 3; 13].

Chapter 6

Differential Forms and L^2 de Rham Cohomology on Riemannian Path Spaces

We would like to establish analogues of Theorems 4.3.8 and 4.3.14 on the based path space $C_{x_0}M$ of a smooth compact Riemannian manifold M, with $x_0 \in M$ fixed. The natural objects to study are the $\mathcal{H}$-vector-fields and $\mathcal{H}$-forms, since the 'admissible directions' for differentiation are subspaces of the tangent spaces, i.e., the 'Bismut tangent spaces' (see Section 2.1). However, the brackets of $\mathcal{H}$-vector-fields, which appear in the invariant formula of exterior derivatives (4.3.8), are not in general sections of $\mathcal{H}$ (see [12]), so such formula ceases to make sense for, say, an $\mathcal{H}$-q-form ϕ_σ defined only on $\wedge^q \mathcal{H}_\sigma$, for each $\sigma \in C_{x_0}M$.

There have been many efforts to circumvent the problem; see [26] or [44] for a survey. Elworthy and Li [23] proposed to replace the Hilbert spaces $\wedge^q \mathcal{H}_\sigma$ by a family of Hilbert spaces $\mathcal{H}_\sigma^{(q)}$ for $q \geq 2$, which are continuously included in $\wedge^q T_\sigma C_{x_0}M$, while keeping the exterior derivative a closure of the classical exterior derivative on smooth cylindrical forms. A detailed description of the case $q = 2$ is given in [26], where $\mathcal{H}^{(2)}$ is shown to be a deformation inside $\wedge^2 T_\sigma C_{x_0}M$ of the exterior product of the Bismut tangent bundle $\wedge^2 \mathcal{H}$, by the curvature of the damped Markovian connection. The analysis in [23; 26] proves the closability of exterior differentiation on the corresponding L^2 $\mathcal{H}$-one-forms, defines a self-adjoint Hodge-Kodaira Laplacian on such L^2 $\mathcal{H}$-one-forms, and establishes the resultant Hodge decomposition, i.e.,

$$L^2\Gamma(\mathcal{H}^*) = \mathrm{Ker}(\Delta^1) \oplus \mathrm{Image}(D) \oplus \overline{\mathrm{Image}(\bar{d}^{1*})}, \tag{6.1}$$

where Δ^1 is the Hodge-Kodaira operator defined by $\bar{d}^{1*}\bar{d}^1 + DD^*$, and $\bar{d}^1$ is the closure of the exterior derivative d^1. It holds that $\bar{d}^1 D = 0$, and every cohomology class in

$$L^2H^1(C_{x_0}M) = \frac{\mathrm{Ker}(\bar{d}^1)}{\mathrm{Image}(D)}$$

has a unique representative in $\mathrm{Ker}(\Delta^1)$. For a similar result established using a different approach, see Kusuoka [42].

After introducing some more concepts and notation on the path spaces and a few preliminary results, we prove the Clark-Ocone formula for L^2 $\mathcal{H}$-one-forms, following the steps in Section 4.2. This implies immediately that $\mathrm{Ker}(\Delta^1) = \{0\}$, the image of $\bar{d}^1$ is closed, the Hodge Laplacian for one-forms has a spectral gap, and we have an improved decomposition

$$L^2\Gamma(\mathcal{H}^*) = \mathrm{Image}(D) \oplus \mathrm{Image}(\bar{d}^{1*}). \tag{6.2}$$

The formulae for higher-order forms have not been developed completely, and we only give some partial results here.

We end this chapter by presenting an earlier and different approach, similar to that of Fang and Franchi [31], where the Itô map is used to pull back to the classical Wiener space. It proves the vanishing result for weak derivatives in general path spaces, while giving the result for strong derivatives in the case of symmetric spaces. We hope to study in future how the two different approaches can complement each other and perhaps work together to resolve the difficulty with higher-order forms.

6.1 Notions and Notation

We start by picking up where we left off in Section 2.1 to continue the setup on the manifold, and then proceed to state a few preliminary results.

First we review the definitions of $\mathcal{H}^{(q)}$ and $\mathcal{H}$-q-forms, following [26]. Recall that the Bismut tangent space $\mathcal{H}_\sigma$, defined for almost all $\sigma \in C_{x_0}M$, is a Hilbert space equipped with the inner product induced by the damped parallel translation W. The H-derivative of the Itô map $\mathcal{I}$ of our SDE (2.7) is a continuous linear map

$$T_w\mathcal{I} : H = L_0^{2,1} \to T_{\mathcal{I}(w)}C_{x_0}M, \qquad \text{a.e. } w \in C_0.$$

Theorem 3.3 of [26] shows that, for almost all $w \in C_0$, the map of q-vectors

$$\wedge^q(T_w\mathcal{I}) : \wedge^q H \to \wedge^q(T_{\mathcal{I}(w)}C_{x_0}M)$$

is a continuous linear operator, where $\wedge^q H$ is the Hilbert space completion of the q-th exterior power of H, and $\wedge^q(T_{\mathcal{I}(w)}C_{x_0}M)$ is the projective exterior power of the tangent space $T_{\mathcal{I}(w)}C_{x_0}M$, i.e., the completion of the algebraic tensor products using the projective cross norm. The conditional expectation of $\wedge^q(T_w\mathcal{I})$,

$$\overline{\wedge^q T\mathcal{I}}_\sigma : \wedge^q H \to \wedge^q(T_\sigma C_{x_0}M), \qquad \text{a.e. } \sigma \in C_{x_0}M,$$

is defined in the same way as $\overline{T\mathcal{I}}$ in Section 2.1 and gives a continuous linear map. We define $\mathcal{H}_\sigma^{(q)}$, for almost all $\sigma \in C_{x_0}M$, to be the image of $\overline{\wedge^q T\mathcal{I}}_\sigma$ in $\wedge^q(T_\sigma C_{x_0}M)$, together with the inner product induced by the linear bijection

$$\overline{\wedge^q T\mathcal{I}}_\sigma|_{[\mathrm{Ker}(\overline{\wedge^q T\mathcal{I}}_\sigma)]^\perp} : [\mathrm{Ker}(\overline{\wedge^q T\mathcal{I}}_\sigma)]^\perp \to \mathcal{H}_\sigma^{(q)},$$

so these are Hilbert spaces with natural continuous linear inclusions into $\wedge^q(T_\sigma C_{x_0}M)$.

Recall that $\mathcal{H} = \coprod \mathcal{H}_\sigma$ inherits a vector bundle structure, over a subset of full measure in $C_{x_0}M$, from the Hilbert bundle $L^2TC_{x_0}M$ via the map $\frac{\mathbb{D}}{dt}$, and so does the dual $\mathcal{H}^* = \coprod \mathcal{H}_\sigma^*$. Similarly here, we have $\mathcal{H}^{(q)} = \coprod \mathcal{H}_\sigma^{(q)}$, the vector bundle over $C_{x_0}M$ with fibres $\mathcal{H}_\sigma^{(q)}$, and $(\mathcal{H}^{(q)})^*$, the dual bundle; see [25; 26] for a detailed description of their vector bundle structure. Here $\mathcal{H} = \mathcal{H}^{(1)}$. Sections of $\mathcal{H}^{(q)}$ or $(\mathcal{H}^{(q)})^*$ are called $\mathcal{H}$-q-vector fields and $\mathcal{H}$-q-forms, respectively.

The case of $q = 2$ has been extensively studied in [26]. In particular, Theorem 4.3 of [26] gives an alternative expression for the space $\mathcal{H}_\sigma^{(2)} = \{\overline{\wedge^2 T\mathcal{I}}(h), h \in H^{\wedge 2}\}$ by proving that

$$\overline{\wedge^2 T\mathcal{I}}(h^1 \wedge h^2) = (1 + Q)[\wedge^2 \overline{T\mathcal{I}}(h^1 \wedge h^2)], \quad h^1, h^2 \in H. \tag{6.3}$$

Here the linear map $Q_\sigma : \wedge^2\mathcal{H} \to \wedge^2 T_\sigma C_{x_0}M$, which gives the difference between the two objects $\overline{\wedge^2 T\mathcal{I}}(h^1 \wedge h^2)$ and $\wedge^2\overline{T\mathcal{I}}(h^1 \wedge h^2)$, is defined by

$$Q_\sigma(G)_{s,t} = (W_s \otimes W_t)\left(\wedge^2(W_\cdot^{-1})W_\cdot^2 \int_0^\cdot (W_r^2)^{-1}[\mathcal{R}_{\sigma(r)}(G_{r,r})]dr\right)_{s\wedge t}.$$

We identify elements of $\wedge^2 T_\sigma C_{x_0}M$ with elements $G(s,t) \in T_{\sigma(s)}M \otimes T_{\sigma(t)}M$, continuous in (s,t), so $Q(G)$ is determined by the values $Q(G)(s,t)$ for $s < t$. The map Q is related to the curvature operator, $\boldsymbol{R} : \wedge^2 T_\sigma C_{x_0}M \to \mathcal{L}_{skew}(\mathcal{H}_\sigma; \mathcal{H}_\sigma)$, of the damped Markovian connection on $\mathcal{H}$, in that $1 + Q$ and $1 - \boldsymbol{R}$ are inverse of each other ([26] Lemma 4.2) and give the isometry between $\mathcal{H}^{(2)}$ and $\wedge^2\mathcal{H}$. Thus $u \in \mathcal{H}^{(2)}$ iff $u - \boldsymbol{R}(u) \in \wedge^2\mathcal{H}$, and the Hilbert space $\mathcal{H}_\sigma^{(2)}$, as a a deformation of $\wedge^2\mathcal{H}_\sigma$ inside $\wedge^2 T_\sigma C_{x_0}M$, is isomorphic to $\wedge^2\mathcal{H}_\sigma$ and determined only by the Riemannian structure of M, independent of the initial embedding used to obtain the Itô map.

By definition, a geometric differential one-form $\phi \in (TC_{x_0}M)^*$ on $C_{x_0}M$ gives a continuous linear map $\phi_\sigma : T_\sigma C_{x_0}M \to \mathbb{R}$, for each $\sigma \in C_{x_0}M$. Similarly, any $\mathcal{H}$-one-form $\phi \in \mathcal{H}^*$ gives a continuous linear map $\phi_\sigma : \mathcal{H}_\sigma \to \mathbb{R}$. Given a geometric differentiable one-form ϕ, the exterior derivative d^1 is defined by the usual formula

$$2d^1\phi(V_1, V_2) = \mathcal{L}_{V_1}\phi(V_2) - \mathcal{L}_{V_2}\phi(V_1) - \phi([V_1, V_2]), \quad \forall V_1, V_2 \in \mathbb{D}^{2,1}(TC_{x_0}M), \tag{6.4}$$

which can be restricted to give $\mathcal{H}$-two-forms. Again we pick as our initial domain the smooth cylindrical forms, and from Theorem 4.2 of [23] we know that the restricted exterior derivative considered as an operator $d^1 : \mathrm{Dom}(d^1) \subset L^2\Gamma\mathcal{H}^* \to L^2\Gamma(\mathcal{H}^{(2)})^*$ is closable. We denote its closure by $\bar{d}^1$.

Define $\tilde{\nabla}^\sharp : \mathbb{D}^{2,1}(L^2TC_{x_0}M) \to L^2\Gamma(L^2TC_{x_0}M \otimes \mathcal{H})$ by

$$_{(2)}< \tilde{\nabla}^\sharp U, V >_{\mathcal{H}} = \tilde{\nabla}_V U, \quad U \in \mathbb{D}^{2,1}(L^2TC_{x_0}M), V \in \mathcal{H},$$

and $\boldsymbol{\nabla}^\sharp : \mathbb{D}^{2,1}(\mathcal{H}) \to L^2\Gamma(\mathcal{H}^{\otimes 2})$ by

$$_{(2)}< \boldsymbol{\nabla}^\sharp U, V >_{\mathcal{H}} = \boldsymbol{\nabla}_V U, \quad U \in \mathbb{D}^{2,1}(\mathcal{H}), V \in \mathcal{H},$$

so equation (2.12) implies

$$\boldsymbol{\nabla}^\sharp = [(\frac{\mathbb{D}}{d.})^{-1} \otimes \mathrm{Id}]\tilde{\nabla}^\sharp \frac{\mathbb{D}}{d.} \tag{6.5}$$

Equivalently, since

$$\tilde{\nabla}_V U = \tilde{X} D(\tilde{Y}U)(V) = _{(2)}< (\tilde{X} \otimes \mathrm{Id})\nabla_{\mathcal{H}}(\tilde{Y}U), V >_{\mathcal{H}},$$

we have $\tilde{\nabla}^\sharp U = (\tilde{X} \otimes \mathrm{Id})\nabla_{\mathcal{H}}(\tilde{Y}U)$, and

$$\boldsymbol{\nabla}^\sharp U = [(\frac{\mathbb{D}}{d.})^{-1} \otimes \mathrm{Id}](\tilde{X} \otimes \mathrm{Id})\nabla_{\mathcal{H}}(\tilde{Y}\frac{\mathbb{D}}{d.}U) = (\boldsymbol{X} \otimes \mathrm{Id})\nabla_{\mathcal{H}}(\boldsymbol{Y}U). \tag{6.6}$$

From (6.4), we have, for a smooth geometric differential form ϕ and vector fields $V_1, V_2 \in \mathbb{D}^{2,1}(\mathcal{H})$,

$$\begin{aligned}
& 2\, d^1\phi(V_1, V_2) \\
= \;& <\boldsymbol{\nabla}_{V_1}\phi^\sharp, V_2>_{\mathcal{H}} - <\boldsymbol{\nabla}_{V_2}\phi^\sharp, V_1>_{\mathcal{H}} + \phi(\boldsymbol{\nabla}_{V_1}V_2) - \phi(\boldsymbol{\nabla}_{V_2}V_1) - \phi([V_1, V_2]) \\
= \;& < \tau(\boldsymbol{\nabla}^\sharp\phi^\sharp) - \boldsymbol{\nabla}^\sharp\phi^\sharp, V_1 \otimes V_2 >_{\mathcal{H}^{\otimes 2}} + \phi[\boldsymbol{T}(V_1, V_2)] \\
= \;& < 2\,\mathfrak{D}^1\phi^\sharp + (\phi \circ \boldsymbol{T})^\sharp, V_1 \otimes V_2 >_{\mathcal{H}^{\otimes 2}},
\end{aligned} \tag{6.7}$$

where $\mathfrak{D}^1 : \text{Dom}(\mathfrak{D}^1) \subset L^2\Gamma\mathcal{H} \to L^2\Gamma(\mathcal{H}^{\wedge 2})$ is defined by

$$\mathfrak{D}^1(V) = \frac{1}{2}[\tau(\boldsymbol{\nabla}^\sharp V) - \boldsymbol{\nabla}^\sharp V],$$

with initial domain $\mathbb{D}^{2,1}(\mathcal{H})$ (we show later that it is a closable operator and take its closure), the operator $\boldsymbol{T}$ is the torsion for the damped Markovian connection given by

$$\boldsymbol{T}(V_1, V_2) = \boldsymbol{\nabla}_{V_1}V_2 - \boldsymbol{\nabla}_{V_2}V_1 - [V_1, V_2], \quad V_1, V_2 \in \mathbb{D}^{2,1}(\mathcal{H}),$$

and $(\phi \circ \boldsymbol{T})^\sharp \in \Gamma(\mathcal{H}^{\otimes 2})$ is defined by $< (\phi \circ \boldsymbol{T})^\sharp, V_1 \otimes V_2 >_{\mathcal{H}^{\otimes 2}} = \phi[\boldsymbol{T}(V_1, V_2)]$. Observe that, since $\mathfrak{D}^1$ arises from the anti-symmetrisation of the covariant derivative, it corresponds to the exterior derivative only for the Levi-Civita connection, i.e., when the torsion term vanishes; this is noted by Cruzeiro and Fang [9] in their study of the anti-symmetrisation of general connections.

Let $\boldsymbol{\nabla}^* : L^2\Gamma(\mathcal{H}^{\otimes 2}) \to L^2\Gamma\mathcal{H}$ be the L^2-adjoint of $\boldsymbol{\nabla}^\sharp$. When restricted to act on $\mathbb{D}^{2,1}(\wedge^2\mathcal{H})$, it gives

$$\boldsymbol{\nabla}^*(V_1 \wedge V_2) = -\frac{1}{2}[\tau(\boldsymbol{\nabla}^\sharp) - \boldsymbol{\nabla}^\sharp]^*(V_1 \wedge V_2) = -(\mathfrak{D}^1)^*(V_1 \wedge V_2),$$

Since $(\mathfrak{D}^1)^* = -\boldsymbol{\nabla}^*$ is closed and densely defined in $L^2\Gamma(\mathcal{H}^{\wedge 2})$, the operator $\mathfrak{D}^1$ has a closed extension $\mathfrak{D}^{1**}$ (see, e.g., [65], page 196). Therefore,

Lemma 6.1.1. *The operator $\mathfrak{D}^1$ is closable.*

A q-vector-field V on $C_{x_0}M$ is said to have a divergence if there exists $\text{Div}(V) \in L^1\Gamma \wedge^{q-1} TC_{x_0}M$ such that, for any smooth cylindrical $(q-1)$-form ϕ,

$$\mathbb{E}\, d^1\phi(V) = -\mathbb{E}\, \phi(\text{Div}(V)).$$

This is a more general concept than being in the domain of div, the operator adjoint to ∇ or $d^{q\sharp}$. For examples of q-vector fields that have a divergence, see [24]. Note that if a q-vector field V has a divergence $\text{Div}(V)$, for $q > 1$, $\text{Div}(V)$ has a vanishing divergence. Theorem 9.3 of [26] proves that, for two $\mathcal{H}$-vector fields V_1 and V_2 adapted to $\mathcal{F}^{x_0}_{\cdot}$,

$$\boldsymbol{T}(V_1, V_2) = 2\,\text{Div}(Q(V_1 \wedge V_2)), \tag{6.8}$$

that is, $\boldsymbol{T}(V_1, V_2)$ is actually a divergence, when V_1 and V_2 are both adapted. This explains the earlier result of Cruzeiro and Fang [11] (Theorem 2.6) that $\boldsymbol{T}$ acting on two smooth adapted vector fields has a vanishing divergence, that is, for all f

cylindrical, V_1 and V_2 in L^2 and adapted,

$$\mathbb{E}Df[\boldsymbol{T}(V_1, V_2)] = 0. \tag{6.9}$$

We extend (6.8) and (6.9) to the following larger subspace of two-tensor fields:

$$\mathcal{V}^{(2)} = \{U \in L^2\Gamma(\otimes^2\mathcal{H}) : \frac{\mathbb{D}}{ds} \otimes \frac{\mathbb{D}}{dt} U_{s,t} \in \mathcal{F}^{x_0}_{s\vee t}, \text{ a.e. } s,t \in [0,T]\}.$$

Recall the shorthand notation $s \vee t = \max(s,t)$, and our identification of elements of $\otimes^2 T_\sigma C_{x_0}M$ with elements $u_{s,t} \in T_{\sigma(s)}M \otimes T_{\sigma(t)}M$, continuous in (s,t). Thus, as in Section 4.1, a vector $u \in \otimes^2 T_\sigma C_{x_0}M$ is in $\otimes^2\mathcal{H}$ iff we can write

$$u_{s,t} = (\boldsymbol{W}_s \otimes \boldsymbol{W}_t)(\frac{\mathbb{D}}{d.} \otimes \frac{\mathbb{D}}{d.})u,$$

with $\frac{\mathbb{D}}{ds} \otimes \frac{\mathbb{D}}{dt} u_{s,t} \in \otimes^2 L^2 T_\sigma C_{x_0}M$. Denote the set of tensor products of adapted vector fields by

$$\mathcal{S}^{(2)} = \{V_1 \otimes V_2 : V_1, V_2 \in L^2\Gamma\mathcal{H}, \text{ both adapted to } \{\mathcal{F}^{x_0}_t\}_{t\in[0,T]}\}.$$

Lemma 6.1.2. *The set of adapted primitive two-tensor-fields $\mathcal{S}^{(2)}$ is total in $\mathcal{V}^{(2)}$. Similarly, the set of adapted primitive skew-symmetric two-tensor-fields is total in the skew-symmetric subspace of $\mathcal{V}^{(2)}$, i.e., $\mathcal{S}^{(2)} \cap L^2\Gamma(\mathcal{H}^{\wedge 2})$ is total in $\mathcal{V}^{(2)} \cap L^2\Gamma(\mathcal{H}^{\wedge 2})$.*

Remark 6.1.3. Equivalently, we could state the L^2 version of the lemma, replacing $\mathcal{V}^{(2)}$ and $\mathcal{S}^{(2)}$, respectively, with

$$\mathcal{V}^{(2)\prime} = \{U \in L^2\Gamma(\otimes^2 L^2 T_\sigma C_{x_0}M) : U_{s,t} \in \mathcal{F}^{x_0}_{s\vee t}, \text{ a.e. } s,t \in [0,T]\}.$$

and

$$\mathcal{S}^{(2)\prime} = \{V_1 \otimes V_2 : V_1, V_2 \in L^2\Gamma(L^2 T_\sigma C_{x_0}M), \text{ both adapted to } \{\mathcal{F}^{x_0}_t\}_{t\in[0,T]}\}.$$

Proof. Take an orthonormal basis $\{h^{x_0,j}\}_{j\in\mathbb{N}}$ of $L^2([0,T]; T_{x_0}M)$, and set

$$h^j_t(\sigma) = /\!/^\sigma_t h^{x_0,j}_t, \quad j \in \mathbb{N}, t \in [0,T], \sigma \in C_{x_0}M.$$

Given any $U \in \mathcal{V}^{(2)}$, we can approximate $\frac{\mathbb{D}}{ds} \otimes \frac{\mathbb{D}}{dt} U_{s,t} \in \mathcal{F}^{x_0}_{s\vee t}$ by finite sums

$$\sum_{j,k=1}^{n} \lambda^{j,k}_{s\vee t}\, h^j_s \otimes h^k_t,$$

where $\lambda^{j,k} : [0,T] \times C_{x_0}M \to \mathbb{R}$ are bounded and adapted, i.e., $\lambda_r^{j,k} \in \mathcal{F}_r^{x_0}$ for all $r \in [0,T]$. Therefore, we only need to show that each term in the finite sums above can be approximated in L^2 by sums of terms of the form $(V_1)_s \otimes (V_2)_t$, with V_1, $V_2 \in L^2\Gamma L^2 TC_{x_0}M$, both adapted.

Since $\lambda_r^{j,k}$ is the L^2 limit of sums of elementary processes of the form

$$(r, \sigma) \mapsto f(\sigma)\mathbf{1}_{(a,b]}(r), \quad \sigma \in C_{x_0}M, a, b \in [0,T], f \in \mathcal{F}_a^{x_0},$$

we can write $\lambda_{s\vee t}^{j,k}$ as the limit of finite sums of functions of the form $f(\sigma)\mathbf{1}_{(a,b]}(s\vee t)$. Observe that

$$\begin{aligned} \mathbf{1}_{(a,b]}(s \vee t) &= \mathbf{1}_{[0,a]}(s)\mathbf{1}_{(a,b]}(t) + \mathbf{1}_{[0,a]}(t)\mathbf{1}_{(a,b]}(s) + \mathbf{1}_{(a,b]}(t)\mathbf{1}_{(a,b]}(s) \\ &= \mathbf{1}_{[0,b]}(s)\mathbf{1}_{(a,b]}(t) + \mathbf{1}_{[0,a]}(t)\mathbf{1}_{(a,b]}(s), \end{aligned}$$

so we write

$$f(\sigma)\mathbf{1}_{(a,b]}(s \vee t)h_s^j \otimes h_t^k = \mathbf{1}_{[0,b]}(s)h_s^j \otimes f(\sigma)\mathbf{1}_{(a,b]}(t)h_t^k + f(\sigma)\mathbf{1}_{(a,b]}(s)h_s^j \otimes \mathbf{1}_{[0,a]}(t)h_t^k.$$

This shows that $\lambda_{s\vee t}^{j,k} h_s^j \otimes h_t^k$ is indeed a limit of sums of $(V_1)_s \otimes (V_2)_t$ with V_1 and V_2 adapted, so the conclusion holds. The skew-symmetric part is clear. □

We collect below a few consequences of the above approximation result.

Corollary 6.1.4. *For all* $U \in \mathcal{V}^{(2)} \cap L^2\Gamma(\mathcal{H}^{\wedge 2})$*, we have*

$$\boldsymbol{T}(U) = 2\,\mathrm{Div}(Q(U)),$$

and therefore $\mathrm{Div}(\boldsymbol{T}(U)) = 0$*, i.e., given any smooth cylindrical* $f : C_{x_0}M \to \mathbb{R}$*,*

$$\mathbb{E}[Df\boldsymbol{T}(U)] = 0.$$

Proof. Take an approximating sequence $V_{1j} \wedge V_{2j}$ for $U \in \mathcal{V}^{(2)} \cap L^2\Gamma(\mathcal{H}^{\wedge 2})$ as in Lemma 6.1.2, that is, adapted vector fields $V_{1j}, V_{2j} \in L^2\Gamma\mathcal{H}$, such that $V_{1j} \wedge V_{2j} \to U$ in L^2 as $j \to \infty$. Then we also have $Q(V_{1j} \wedge V_{2j}) \to Q(U)$ and $T(V_{1j} \wedge V_{2j}) \to T(U)$.

Given any smooth cylindrical one-form $\phi \in \mathrm{Dom}(d^1)$, we apply (6.8) to see

$$\mathbb{E}\,\phi(\boldsymbol{T}(U)) = \lim_{j\to\infty} \mathbb{E}\,\phi(\boldsymbol{T}(V_{1j}\wedge V_{2j})) = \lim_{j\to\infty} -2\mathbb{E}\,d^1\phi(Q(V_{1j}\wedge V_{2j})) = -2\mathbb{E}\,d^1\phi(Q(U)),$$

so the result holds. □

Lemma 6.1.2 also shows, for $U \in \mathcal{V}^{(2)} \cap \mathrm{Dom}(\boldsymbol{\nabla}^*)$,

$$-\boldsymbol{\nabla}^*(U) = (\mathfrak{D}^1)^*(U) = d^{1*\sharp}(1+Q)(U).$$

Or, we can extend directly the statements in Proposition 9.6 and Corollary 9.7 of [26] to state

Corollary 6.1.5. *If $U \in \mathcal{V}^{(2)}$ has a divergence, we have $U \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, and*

$$\boldsymbol{\nabla}^* U = \mathrm{div}(1+Q)(U) = \mathrm{Div}(U) + \frac{1}{2}\boldsymbol{T}(U). \tag{6.10}$$

It is convenient to define an operator $\overline{\mathfrak{D}^1} : \mathrm{Dom}(\overline{\mathfrak{D}^1}) \subset L^2\Gamma\mathcal{H}^* \to L^2\Gamma(\mathcal{H}^{\otimes 2})$ by

$$(\frac{\mathbb{D}}{ds} \otimes \frac{\mathbb{D}}{dt})(\overline{\mathfrak{D}^1}\phi)_{s,t} = \mathbb{E}[(\frac{\mathbb{D}}{ds} \otimes \frac{\mathbb{D}}{dt})(\mathfrak{D}^1\phi^\sharp)_{s,t} | \mathcal{F}^{x_0}_{s\vee t}],$$

for $\phi \in L^2\Gamma\mathcal{H}^*$ such that $\phi^\sharp \in \mathrm{Dom}(\mathfrak{D}^1)$. Clearly $\mathrm{Dom}(\overline{\mathfrak{D}^1}) \supset \mathbb{D}^{2,1}(\mathcal{H}^*)$. If we denote by $P_{\mathcal{V}^{(2)}}$ the projection of $L^2\Gamma(\mathcal{H}^{\otimes 2})$ onto $\mathcal{V}^{(2)}$, it is clear that

$$\overline{\mathfrak{D}^1}\phi = P_{\mathcal{V}^{(2)}}(\mathfrak{D}^1\phi^\sharp). \tag{6.11}$$

For any smooth cylindrical one-form ϕ and vector fields V_1, V_2,

$$\begin{aligned}
\mathbb{E} < \overline{\mathfrak{D}^1}\phi, V_1 \wedge V_2 >_{\wedge^2\mathcal{H}} &= \mathbb{E} < \mathfrak{D}^1\phi^\sharp, P_{\mathcal{V}^{(2)}}(V_1 \wedge V_2) >_{\wedge^2\mathcal{H}} \\
&= \mathbb{E} < (d^1\phi)^\sharp - \frac{1}{2}(\phi \circ \boldsymbol{T})^\sharp, P_{\mathcal{V}^{(2)}}(V_1 \wedge V_2) >_{\wedge^2\mathcal{H}} \\
&= \mathbb{E}\, d^1\phi(P_{\mathcal{V}^{(2)}}(V_1 \wedge V_2)) - \mathbb{E}\,\phi[\mathrm{Div}(Q(P_{\mathcal{V}^{(2)}}(V_1 \wedge V_2)))] \\
&= \mathbb{E}\, d^1\phi[(1+Q)P_{\mathcal{V}^{(2)}}(V_1 \wedge V_2)] \\
&= \mathbb{E} < P_{\mathcal{V}^{(2)}}(1+Q)^*(d^1\phi)^\sharp, V_1 \wedge V_2 >_{\wedge^2\mathcal{H}} .
\end{aligned}$$

Since any $\phi \in \mathbb{D}^{2,1}$ can be approximated by a sequence of smooth cylindrical one-forms, we have actually shown

Lemma 6.1.6.

$$\overline{\mathfrak{D}^1}\phi = P_{\mathcal{V}^{(2)}}(1+Q)^*(\bar{d}^1\phi)^\sharp, \quad \forall \phi \in \mathbb{D}^{2,1}. \tag{6.12}$$

We prove the closability of $\overline{\mathfrak{D}^1}$ in the next section.

6.2 One-Forms on Path Spaces

Recall the Clark-Ocone formula for Riemannian path spaces given in Subsection 2.2.4, for any $\mathbb{D}^{2,1}$ function $F : C_{x_0}M \to \mathbb{R}^m$,

$$F(\sigma) = \mathbb{E}F + \int_0^T < \mathbb{E}[\frac{\mathbb{D}}{dt}(\nabla_{\mathcal{H}}F)_t|\mathcal{F}_t^{x_0}], d\{\sigma\}_t >_{\sigma_t}, \quad \mu_{x_0}\text{-a.e. } \sigma \in C_{x_0}M, \quad (2.29)$$

where $d\{\sigma\}_t$ is the martingale part of $\circ d\sigma_t$ defined in Section 2.1.

We follow the ideas in Section 4.2 to develop the Clark-Ocone formula for square-integrable $\mathcal{H}$-one-forms. Analogously to (4.16), we define, for $\phi \in L^2\Gamma\mathcal{H}^*$,

$$CO(\phi) = \int_0^T < \mathbb{E}(\frac{\mathbb{D}}{dt}\phi_t^\sharp|\mathcal{F}_t^{x_0}), d\{\sigma\}_t >_{\sigma_t}, \tag{6.13}$$

where $\phi^\sharp \in L^2\Gamma\mathcal{H}$ is given by $\phi_\sigma(h) =< \phi_\sigma^\sharp, h >_{\mathcal{H}}$ for $h \in \mathcal{H}$. As before, we note that CO is continuous as a map from $L^2\Gamma\mathcal{H}^*$ to $L^2(C_{x_0}M, \mathbb{R})$.

Proposition 6.2.1 (Clark-Ocone Formula for One-Forms). *If $\phi \in L^2\Gamma\mathcal{H}^*$ is in* $\mathrm{Dom}(\bar{d}^1)$*, we have $CO(\phi) \in \mathbb{D}^{2,1}$ and*

$$\begin{aligned} & \frac{\mathbb{D}}{dt}[\nabla_{\mathcal{H}}CO(\phi) - \phi^\sharp]_t \\ = \;& 2\int_t^T {}_{(2)}< \mathbb{E}[(\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})(\mathfrak{D}^1\phi^\sharp)_{t,s}|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s} & (6.14) \\ = \;& 2\int_t^T {}_{(2)}< (\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})(\overline{\mathfrak{D}^1}\phi)_{t,s}, d\{\sigma\}_s >_{\sigma_s} & (6.15) \\ = \;& 2\int_t^T {}_{(2)}< \mathbb{E}[(\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})(1+Q)^*(\bar{d}^1\phi)^\sharp)_{t,s}|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s} . & (6.16) \end{aligned}$$

Proof. Suppose first $\phi \in \mathbb{D}^{2,1}$, so we can apply the Clark-Ocone formula (2.29) to write, for almost all $t \in [0, T]$,

$$Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp(\sigma) = \mathbb{E}Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp(\sigma) + \int_0^T {}_{(2)}< \mathbb{E}[\frac{\mathbb{D}}{ds}\nabla_{\mathcal{H}}(Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp)_s|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s} . \tag{6.17}$$

Applying $X_{\sigma_t} = X(\sigma_t)$ to both sides, we have

$$\frac{\mathbb{D}}{dt}\phi_t^\sharp(\sigma) = X_{\sigma_t}\mathbb{E}Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp(\sigma) + X_{\sigma_t}\int_0^T {}_{(2)}< \mathbb{E}[\frac{\mathbb{D}}{ds}\nabla_{\mathcal{H}}(Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp)_s|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s} .$$

Taking conditional expectation with respect to $\mathcal{F}_t^{x_0}$, we obtain

$$\mathbb{E}[\frac{\mathbb{D}}{dt}\phi_t^\sharp|\mathcal{F}_t^{x_0}] = X_{\sigma_t}\mathbb{E}Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp(\sigma) + X_{\sigma_t}\int_0^t {}_{(2)}\!< \mathbb{E}[\frac{\mathbb{D}}{ds}\nabla_{\mathcal{H}}(Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp)_s|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s},$$

hence

$$\frac{\mathbb{D}}{dt}\phi_t^\sharp(\sigma) - \mathbb{E}[\frac{\mathbb{D}}{dt}\phi_t^\sharp(\sigma)|\mathcal{F}_t^{x_0}] = X_{\sigma_t}\int_t^T {}_{(2)}\!< \mathbb{E}[\frac{\mathbb{D}}{ds}\nabla_{\mathcal{H}}(Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp)_s|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s} . \tag{6.18}$$

Note that the conditional expectation of a vector field $U \in L^2\Gamma\mathcal{H}$, written above as $\mathbb{E}(U_\cdot|\mathcal{F}_\cdot^{x_0})$, is interpreted as

$$\mathbb{E}(U_t|\mathcal{F}_t^{x_0}) = /\!\!/_t^\sigma \mathbb{E}[(/\!\!/_t^\sigma)^{-1}U_t|\mathcal{F}_t^{x_0}],$$

which is independent of the choice of the connection used, since the difference between any two parallel translations is a linear operation from $T_{x_t}M$ into itself, and is measurable with respect to $\mathcal{F}_t^{x_0}$.

Recall that in Subsection 2.2.4, we have written (6.13) in the form of

$$CO(\phi) = -\text{div}(P_{\mathcal{V}}\phi^\sharp) = -\text{div}[(\frac{\mathbb{D}}{d.})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_\cdot^\sharp|\mathcal{F}_\cdot^{x_0})],$$

where $P_{\mathcal{V}}$ is the projection onto adapted processes defined by (2.30). Since $\phi \in \mathbb{D}^{2,1}$, Lemma 2.2.6 shows the adapted process $P_{\mathcal{V}}\phi \in \mathbb{D}^{2,1}$. By the communtation Lemma 3.2.1, we have $CO(\phi) \in \mathbb{D}^{2,1}$, and for $V = \overline{T\mathcal{I}}(h)$ with $h \in H$,

$$\begin{aligned} DCO(\phi)(V) &= -D\text{div}[(\frac{\mathbb{D}}{d.})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_\cdot^\sharp|\mathcal{F}_\cdot^{x_0})](V) \\ &= -\text{div}\boldsymbol{\nabla}_-[(\frac{\mathbb{D}}{d.})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_\cdot^\sharp|\mathcal{F}_\cdot^{x_0})](V) + < (\frac{\mathbb{D}}{d.})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_\cdot^\sharp|\mathcal{F}_\cdot^{x_0}), V >_{\mathcal{H}} \\ &= \int_0^T < \frac{\mathbb{D}}{ds}\boldsymbol{\nabla}_-(\frac{\mathbb{D}}{ds})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_\cdot^\sharp|\mathcal{F}_\cdot^{x_0}), d\{\sigma\}_s >_{\sigma_s} (V) \\ &\quad + < (\frac{\mathbb{D}}{d.})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_\cdot^\sharp|\mathcal{F}_\cdot^{x_0}), V >_{\mathcal{H}} . \end{aligned}$$

Therefore

$$\begin{aligned} DCO(\phi)(V) - < \phi^\sharp, V >_{\mathcal{H}} &= \int_0^T < \frac{\mathbb{D}}{ds}\boldsymbol{\nabla}_-(\frac{\mathbb{D}}{ds})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_\cdot^\sharp|\mathcal{F}_\cdot^{x_0}), d\{\sigma\}_s >_{\sigma_s} (V) \\ &\quad - < \phi^\sharp - (\frac{\mathbb{D}}{d.})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_\cdot^\sharp|\mathcal{F}_\cdot^{x_0}), V >_{\mathcal{H}} . \end{aligned} \tag{6.19}$$

For the first term on the right-hand side of (6.19), we apply Lemma 2.2.6 to write

$$\frac{\mathbb{D}}{ds}\nabla_{\mathcal{H}}\mathbb{E}(Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp|\mathcal{F}_t^{x_0}) = \mathbb{E}(\frac{\mathbb{D}}{ds}\nabla_{\mathcal{H}}Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp|\mathcal{F}_t^{x_0})\mathbf{1}_{[0,t]}(s), \quad \text{a.e.} \tag{6.20}$$

and then use (6.6) to calculate

$$\begin{aligned}
& \int_0^T < \frac{\mathbb{D}}{ds}\boldsymbol{\nabla}_-(\frac{\mathbb{D}}{ds})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_.^\sharp|\mathcal{F}_.^{x_0}), d\{\sigma\}_s >_{\sigma_s} (V) \\
= & \int_0^T < X_{\sigma_s}D[Y_{\sigma_s}\mathbb{E}(\frac{\mathbb{D}}{ds}\phi_s^\sharp|\mathcal{F}_s^{x_0})], d\{\sigma\}_s >_{\sigma_s} (V) \\
= & < \int_0^T X_{\sigma_s}\ {}_{(1)}\!< \nabla_{\mathcal{H}}[Y_{\sigma_s}\mathbb{E}(\frac{\mathbb{D}}{ds}\phi_s^\sharp|\mathcal{F}_s^{x_0})], d\{\sigma\}_s >_{\sigma_s}, V >_{\mathcal{H}} \\
= & \int_0^T < \int_0^T {}_{(1)}\!< (X_{\sigma_s}\otimes\frac{\mathbb{D}}{dt})\nabla_{\mathcal{H}}[\mathbb{E}(Y_{\sigma_s}\frac{\mathbb{D}}{ds}\phi_s^\sharp|\mathcal{F}_s^{x_0})]_t, d\{\sigma\}_s >_{\sigma_s}, \frac{\mathbb{D}}{dt}V_t >_{\sigma_t} dt \\
= & \int_0^T < \int_t^T {}_{(1)}\!< \mathbb{E}[(X_{\sigma_s}\otimes\frac{\mathbb{D}}{dt})\nabla_{\mathcal{H}}(Y_{\sigma_s}\frac{\mathbb{D}}{ds}\phi_s^\sharp)_t|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s}, \frac{\mathbb{D}}{dt}V_t >_{\sigma_t} dt \\
= & \int_0^T < \int_t^T {}_{(1)}\!< \mathbb{E}[(\frac{\mathbb{D}}{ds}\otimes\frac{\mathbb{D}}{dt})(\boldsymbol{\nabla}^\sharp\phi^\sharp)_{s,t}|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s}, \frac{\mathbb{D}}{dt}V_t >_{\sigma_t} dt \\
= & \int_0^T < \int_t^T {}_{(2)}\!< \mathbb{E}[(\frac{\mathbb{D}}{dt}\otimes\frac{\mathbb{D}}{ds})\tau(\boldsymbol{\nabla}^\sharp\phi^\sharp)_{t,s}|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s}, \frac{\mathbb{D}}{dt}V_t >_{\sigma_t} dt.
\end{aligned}$$

The second term on the right-hand side of (6.19) gives

$$\begin{aligned}
& < \phi^\sharp - (\frac{\mathbb{D}}{d.})^{-1}\mathbb{E}(\frac{\mathbb{D}}{d.}\phi_.^\sharp|\mathcal{F}_.^{x_0}), V >_{\mathcal{H}} \\
= & \int_0^T < \frac{\mathbb{D}}{dt}\phi_t^\sharp - \mathbb{E}(\frac{\mathbb{D}}{dt}\phi_t^\sharp|\mathcal{F}_t^{x_0}), \frac{\mathbb{D}}{dt}V_t >_{\sigma_t} dt \\
= & \int_0^T < X_{\sigma_t}\int_t^T {}_{(2)}\!< \mathbb{E}[\frac{\mathbb{D}}{ds}\nabla_{\mathcal{H}}(Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp)_s|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s}, \frac{\mathbb{D}}{dt}V_t >_{\sigma_t} dt \\
= & \int_0^T < \int_t^T {}_{(2)}\!< (X_{\sigma_t}\otimes \mathrm{Id})\mathbb{E}[\frac{\mathbb{D}}{ds}\nabla_{\mathcal{H}}(Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp)_s|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s}, \frac{\mathbb{D}}{dt}V_t >_{\sigma_t} dt \\
= & \int_0^T < \int_t^T {}_{(2)}\!< \mathbb{E}[(X_{\sigma_t}\otimes\frac{\mathbb{D}}{ds})\nabla_{\mathcal{H}}(Y_{\sigma_t}\frac{\mathbb{D}}{dt}\phi_t^\sharp)_s|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s}, \frac{\mathbb{D}}{dt}V_t >_{\sigma_t} dt \\
= & \int_0^T < \int_t^T {}_{(2)}\!< \mathbb{E}[(\frac{\mathbb{D}}{dt}\otimes\frac{\mathbb{D}}{ds})(\boldsymbol{\nabla}^\sharp\phi^\sharp)_{t,s}|\mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s}, \frac{\mathbb{D}}{dt}V_t >_{\sigma_t} dt,
\end{aligned}$$

where the second line follows from equation (6.18), and the last line from (6.6). The above identities combined with (6.12) lead to

$$
\begin{aligned}
& DCO(\phi)(V) - < \phi^\sharp, V >_{\mathcal{H}} \\
= & \int_0^T < \int_t^T {}_{(2)}< \mathbb{E}\{(\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})[\tau(\boldsymbol{\nabla}^\sharp \phi^\sharp) - \boldsymbol{\nabla}^\sharp \phi^\sharp]_{t,s} | \mathcal{F}_s^{x_0}\}, d\{\sigma\}_s >_{\sigma_s}, \frac{\mathbb{D}}{dt} V_t >_{\sigma_t} dt \\
= & < 2\,(\frac{\mathbb{D}}{dt})^{-1} \int_t^T {}_{(2)}< \mathbb{E}[(\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})(\mathfrak{D}^1 \phi^\sharp)_{t,s} | \mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s}, V >_{\mathcal{H}} \\
= & < 2\,(\frac{\mathbb{D}}{dt})^{-1} \int_t^T {}_{(2)}< (\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})(\overline{\mathfrak{D}^1} \phi)_{t,s}, d\{\sigma\}_s >_{\sigma_s}, V >_{\mathcal{H}} \\
= & < 2\,(\frac{\mathbb{D}}{dt})^{-1} \int_t^T {}_{(2)}< \mathbb{E}[(\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})(1+Q)^*(\bar{d}^1 \phi)^\sharp)_{t,s} | \mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s}, V >_{\mathcal{H}},
\end{aligned}
$$

which proves the results for $\phi \in \mathbb{D}^{2,1}$.

A general L^2 $\mathcal{H}$-one-from $\phi \in \mathrm{Dom}(\bar{d}^1)$ can be approximated by a sequence of cylindrical one-forms $\phi_j \in \mathbb{D}^{2,1}$ such that $\phi_j \to \phi$ and $\bar{d}^1 \phi_j \to \bar{d}^1 \phi$ in L^2, so the above computation shows that $DCO(\phi_j)$ converges in L^2, as

$$
\begin{aligned}
& \frac{\mathbb{D}}{dt}[\nabla_{\mathcal{H}} CO(\phi_j)]_t \\
= & \frac{\mathbb{D}}{dt}(\phi_j^\sharp)_t + 2 \int_t^T {}_{(2)}< \mathbb{E}[(\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})(1+Q)^*(\bar{d}^1 \phi_j)^\sharp)_{t,s} | \mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s} \\
\to & \frac{\mathbb{D}}{dt} \phi_t^\sharp + 2 \int_t^T {}_{(2)}< \mathbb{E}[(\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})(1+Q)^*(\bar{d}^1 \phi)^\sharp)_{t,s} | \mathcal{F}_s^{x_0}], d\{\sigma\}_s >_{\sigma_s} .
\end{aligned}
$$

Since CO is continuous, we have $CO(\phi_j) \to CO(\phi)$, and by the closedness of D we get $CO(\phi) \in \mathbb{D}^{2,1}$, with $\frac{\mathbb{D}}{dt}[\nabla_{\mathcal{H}} CO(\phi)]_t$ given by the limit above, so we are done. □

Corollary 6.2.2. *If $\phi \in L^2\Gamma\mathcal{H}^*$ is in* $\mathrm{Dom}(\bar{d}^1)$*, we have*

$$
\|DCO(\phi) - \phi\|_{L^2\Gamma\mathcal{H}^*} = \sqrt{2}\|\overline{\mathfrak{D}^1}\phi\|_{L^2\Gamma(\mathcal{H}^{\otimes 2})} \leq \sqrt{2}\|\bar{d}^1\phi\|_{L^2\Gamma(\mathcal{H}^{(2)})^*}
$$

and similarly $\|DCO(\phi) - \phi\|_{L^2\Gamma\mathcal{H}^*} \leq \|\mathfrak{D}^1\phi^\sharp\|_{L^2\Gamma(\mathcal{H}^{\wedge 2})}$.

Proof. The inequalities are clear from equations (6.11) and (6.12). The equality is a simple consequence of the Itô isometry, since

$$\begin{aligned}\int_0^T \|\frac{\mathbb{D}}{dt}[\nabla CO(\phi)-\phi^\sharp]_t\|_{L^2}^2 dt &= 4\int_0^T \mathbb{E}\int_t^T [(\frac{\mathbb{D}}{dt}\otimes\frac{\mathbb{D}}{ds})(\overline{\mathfrak{D}^1}\phi)_{t,s}]^2 ds dt \\ &= 4\int_0^T\int_0^T \mathbf{1}_{[t,T]}(s)\mathbb{E}|(\frac{\mathbb{D}}{dt}\otimes\frac{\mathbb{D}}{ds})(\overline{\mathfrak{D}^1}\phi)_{t,s}|^2 dt ds \\ &= 2\int_0^T\int_0^T \mathbb{E}|(\frac{\mathbb{D}}{dt}\otimes\frac{\mathbb{D}}{ds})(\overline{\mathfrak{D}^1}\phi)_{t,s}|^2 dt ds \\ &= 2\|\overline{\mathfrak{D}^1}\phi\|_{L^2\Gamma(\mathcal{H}\otimes\mathcal{H})}^2. \quad \square\end{aligned}$$

Corollary 6.2.3. *The operator $\overline{\mathfrak{D}^1}$ is closable on $\mathbb{D}^{2,1}$.*

Proof. Given any sequence of $\phi_j \in \mathbb{D}^{2,1}$, $j \in \mathbb{N}$, such that $\phi_j \to 0$ and $\overline{\mathfrak{D}^1}\phi_j \to U$ in L^2, Proposition 6.2.1 implies

$$\begin{aligned}\frac{\mathbb{D}}{dt}[\nabla_{\mathcal{H}} CO(\phi_j)]_t &= \frac{\mathbb{D}}{dt}(\phi_j^\sharp)_t + 2\int_t^T {}_{(2)}< (\frac{\mathbb{D}}{dt}\otimes\frac{\mathbb{D}}{ds})(\overline{\mathfrak{D}^1}\phi_j)_{t,s}, d\{\sigma\}_s >_{\sigma_s} \\ &\to 2\int_t^T {}_{(2)}< (\frac{\mathbb{D}}{dt}\otimes\frac{\mathbb{D}}{ds})U_{t,s}, d\{\sigma\}_s >_{\sigma_s}\end{aligned}$$

in L^2. Since $CO(\phi_j) \to 0$, and $\nabla_{\mathcal{H}}$ is closed, we get $\frac{\mathbb{D}}{dt}[\nabla_{\mathcal{H}} CO(\phi_j)]_t \to 0$, hence the Itô integral

$$2\int_t^T {}_{(2)}< (\frac{\mathbb{D}}{dt}\otimes\frac{\mathbb{D}}{ds})U_{t,s}, d\{\sigma\}_s >_{\sigma_s} = 0, \qquad \text{a.e. } t \in [0,T].$$

Therefore $U = 0$, that is, $\overline{\mathfrak{D}^1}\phi_j \to 0$. □

Corollary 6.2.4. *If $\phi \in L^2\Gamma\mathcal{H}^*$ satisfies any of the following conditions*

1. $\overline{\mathfrak{D}^1}\phi = 0$,
2. $\mathfrak{D}^1\phi^\sharp = 0$, *or*
3. $\bar{d}^1\phi = 0$,

we have $\phi = Df$ with $f = CO(\phi) \in \mathbb{D}^{2,1}$. Moreover, either of the conditions 2. and 3. implies 1. As a result, the first L^2 de Rham cohomology vanishes and $\mathrm{Ker}(\Delta^1) = \{0\}$.

Remark 6.2.5. As in Remark 4.2.5, we have actually shown, for $\phi \in \mathrm{Dom}(\bar{d}^1)$,

$$
\begin{aligned}
& \mathbb{E}[(\frac{\mathbb{D}}{ds} \otimes \frac{\mathbb{D}}{dt})(\mathfrak{D}^1 \phi^\sharp)_{s,t} | \mathcal{F}^{x_0}_{s \vee t}] = 0, \text{ a.e. } s, t \in [0, T] \\
\iff \quad & \mathbb{E}[[(\frac{\mathbb{D}}{dt} \otimes \frac{\mathbb{D}}{ds})(1+Q)^*(\bar{d}^1 \phi)^\sharp)_{t,s} | \mathcal{F}^{x_0}_{s \vee t}] = 0, \text{ a.e. } s, t \in [0, T] \\
\iff \quad & DCO(\phi) = \phi \iff \overline{\mathfrak{D}^1} \phi = 0 \iff \bar{d}^1 \phi = 0.
\end{aligned}
$$

Theorem 6.2.6 (Spectral Gap of Δ^1)**.** *The operator $\bar{d}^1$ has a closed range, so does $\bar{d}^{1*}$, hence Δ^1 has a spectral gap. Moreover, the Hodge decomposition (6.1) given in [23] does improve to (6.2).*

Proof. That $\bar{d}^1$ has a closed range follows from the bound given in Corollary 6.2.2 and the fact that it is a closed operator, i.e., since $\|\phi\|_{L^2\Gamma\mathcal{H}^*} \leq \sqrt{2}\|\bar{d}^1\phi\|_{L^2\Gamma(\mathcal{H}^{(2)})^*}$ for all $\phi \in \mathrm{Dom}(\bar{d}^1|_{\mathrm{Ker}(\bar{d}^1)^\perp})$, and $\bar{d}^1|_{\mathrm{Ker}(\bar{d}^1)^\perp}$ is closed, the injective operator $\bar{d}^1|_{\mathrm{Ker}(\bar{d}^1)^\perp}$ has a closed range (see, e.g, page 205 of [65], or Page 312 of [54]), and so does $\bar{d}^1$.

Since D also has closed range (Fang [30]), so does $\Delta^1 = \bar{d}^{1*}\bar{d}^1 + DD^*$ (see Zucker [66], page 446). By the result of Donnelly [14], this is equivalent to the existence of a spectral gap for Δ^1. The last statement is clear. □

6.3 Higher Order Forms on Path Spaces

Recall the invariant formula for d^q on a geometric q-form ϕ, i.e., $\phi_\sigma \in \wedge^q(T_\sigma C_{x_0}M)^*$,

$$
\begin{aligned}
(q+1)d^q\phi_\sigma(V_0, V_1, \cdots, V_q) \; = \; & \sum_{j=0}^{q}(-1)^j \mathcal{L}_{V_j}[\phi(V_0, \cdots, \hat{V}_j, \cdots, V_q)](\sigma) \\
& + \sum_{i<j}(-1)^{i+j}\phi_\sigma([V_i, V_j], V_0, \cdots, \hat{V}_i, \cdots, \hat{V}_j, \cdots, V_q),
\end{aligned}
\tag{6.21}
$$

where $V_0, V_1, \cdots, V_q$ are smooth $\mathcal{H}$-vector fields. Extending the closed covariant derivative operator $\boldsymbol{\nabla} : \mathbb{D}^{2,1}(\mathcal{H}) \subset L^2\Gamma\mathcal{H} \to \mathcal{L}_2(\mathcal{H}; \mathcal{H})$ in a natural way to an operator on higher-order tensor products, i.e, $\boldsymbol{\nabla} : \mathbb{D}^{2,1}(\mathcal{H}^{\otimes q}) \subset L^2\Gamma(\mathcal{H}^{\otimes q}) \to \mathcal{L}_2(\mathcal{H}; \mathcal{H}^{\otimes q})$, we can express the above formula as

$$\begin{aligned}
& (q+1)d^q\phi_\sigma(V_0, V_1, \cdots, V_q) \\
= & \sum_{j=0}^{q}(-1)^j < \boldsymbol{\nabla}_{V_j}\phi_\sigma^\sharp, V_0\wedge\cdots\wedge\hat{V}_j\wedge\cdots\wedge V_q >_{\mathcal{H}^{\wedge q}} \\
& + \sum_{j=0}^{q}(-1)^j\phi_\sigma[\boldsymbol{\nabla}_{V_j}(V_0\wedge\cdots\wedge\hat{V}_j\wedge\cdots\wedge V_q)] \\
& + \sum_{i<j}(-1)^{i+j}\phi_\sigma([V_i, V_j]\wedge V_0\wedge\cdots\wedge\hat{V}_i\wedge\cdots\wedge\hat{V}_j\wedge\cdots\wedge V_q). \qquad (6.22)
\end{aligned}$$

We can formulate (6.22) in terms of $\boldsymbol{\nabla}^\sharp : \mathbb{D}^{2,1}(\mathcal{H}^{\otimes q}) \subset L^2\Gamma(\mathcal{H}^{\otimes q}) \to L^2\Gamma(\mathcal{H}^{\otimes q+1})$, given by

$$_{(q+1)}< \boldsymbol{\nabla}^\sharp U, V >_{\mathcal{H}} = \boldsymbol{\nabla}_V U, \quad \forall U \in \mathbb{D}^{2,1}(\mathcal{H}^{\otimes q}), V \in \mathcal{H},$$

and corresponding to equations (6.5) and (6.6), we have here

$$\begin{aligned}
\boldsymbol{\nabla}^\sharp U &= (\underbrace{\boldsymbol{W}\otimes\cdots\otimes\boldsymbol{W}}_{q \text{ times}}\otimes\mathrm{Id})\tilde{\nabla}^\sharp(\underbrace{\frac{\mathbb{D}}{d.}\otimes\cdots\otimes\frac{\mathbb{D}}{d.}}_{q \text{ times}}U) \\
&= (\underbrace{\boldsymbol{X}\otimes\cdots\otimes\boldsymbol{X}}_{q \text{ times}}\otimes\mathrm{Id})\nabla_{\mathcal{H}}(\underbrace{\boldsymbol{Y}\otimes\cdots\otimes\boldsymbol{Y}}_{q \text{ times}}U).
\end{aligned}$$

As in Section 6.1, we can define a skew-symmetrised covariant derivative operator $\mathfrak{D}^q$:Dom$(\mathfrak{D}^q) \subset L^2\Gamma(\mathcal{H}^{\wedge q}) \to L^2\Gamma(\mathcal{H}^{\wedge q+1})$ by

$$< (q+1)\mathfrak{D}^q U, V_0\wedge\cdots\wedge V_q >_{\mathcal{H}^{\wedge q+1}} = (-1)^q < (q+1)\boldsymbol{\nabla}^\sharp U, V_0\wedge\cdots\wedge V_q >_{\mathcal{H}^{\wedge q+1}},$$

which corresponds to the first term in (6.22),

$$\begin{aligned}
& < (q+1)\mathfrak{D}^q U, V_0\wedge\cdots\wedge V_q >_{\mathcal{H}^{\wedge q+1}} \\
= & (-1)^q\sum_{j=0}^{q}(-1)^{q-j} < \boldsymbol{\nabla}_{V_j}U, V_0\wedge\cdots\wedge\hat{V}_j\wedge\cdots\wedge V_q >_{\mathcal{H}^{\wedge q}} \\
= & \sum_{j=0}^{q}(-1)^j < \boldsymbol{\nabla}_{V_j}U, V_0\wedge\cdots\wedge\hat{V}_j\wedge\cdots\wedge V_q >_{\mathcal{H}^{\wedge q}} .
\end{aligned}$$

cf. equation (4.8) for the flat spaces. If we define $\mathfrak{T}^q : \wedge^{q+1}\mathcal{H}_\sigma \to \wedge^q T_\sigma C_{x_0}M$ by mixing the torsion terms, that is,

$$
\begin{aligned}
& \mathfrak{T}^q(V_0, V_1, \cdots, V_q) \\
= & \sum_{i<j} (-1)^{i+j-1} \boldsymbol{T}(V_i, V_j) \wedge V_0 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q \\
= & \sum_{i<j} (-1)^{i+j-1} (\boldsymbol{\nabla}_{V_i} V_j - \boldsymbol{\nabla}_{V_j} V_i - [V_i, V_j]) \wedge V_0 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q, \\
= & \sum_{j=0}^{q} (-1)^j \boldsymbol{\nabla}_{V_j} (V_0 \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q) \\
& + \sum_{i<j} (-1)^{i+j} [V_i, V_j] \wedge V_0 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q,
\end{aligned}
$$

the invariant formula for $d^q\phi$ acting on a smooth cylindrical q-form ϕ simplifies into

$$
\begin{aligned}
& (q+1) d^q \phi_\sigma(V_0, V_1, \cdots, V_q) \\
= & < (q+1)\mathfrak{D}^q \phi^\sharp, V_0 \wedge \cdots \wedge V_q >_{\mathcal{H}^{\wedge q+1}} + \phi \circ \mathfrak{T}^q(V_0, V_1, \cdots, V_q) \\
= & < (q+1)\mathfrak{D}^q \phi^\sharp + (\mathfrak{T}^{q*}\phi)^\sharp, V_0 \wedge \cdots \wedge V_q >_{\mathcal{H}^{\wedge q+1}},
\end{aligned}
$$

where the operator $\mathfrak{T}^{q*}_\sigma : (\wedge^q T_\sigma C_{x_0} M)^* \to (\wedge^{q+1} \mathcal{H}_\sigma)^*$ is given by

$$
\mathfrak{T}^{q*}_\sigma \phi = \phi \circ \mathfrak{T}^q_\sigma.
$$

For example, with $q = 1$, $\mathfrak{T}^1 = \boldsymbol{T}$, and we get (6.7), which is our motivation. For $q = 2$, given a geometric two-form ϕ and three vector fields $V_1, V_2, V_3 \in \mathbb{D}^{2,1}(\mathcal{H})$, we have

$$
\begin{aligned}
& 3\, d^2\phi(V_1, V_2, V_3) \\
= & \mathcal{L}_{V_1}\phi(V_2, V_3) - \mathcal{L}_{V_2}\phi(V_1, V_3) + \mathcal{L}_{V_3}\phi(V_1, V_2) \\
& -\phi([V_1, V_2], V_3) + \phi([V_1, V_3], V_2) + \phi([V_2, V_3], V_1) \\
= & < \boldsymbol{\nabla}_{V_1}\phi^\sharp, V_2 \wedge V_3 >_{\wedge^2\mathcal{H}} + < \boldsymbol{\nabla}_{V_2}\phi^\sharp, V_3 \wedge V_1 >_{\wedge^2\mathcal{H}} + < \boldsymbol{\nabla}_{V_3}\phi^\sharp, V_1 \wedge V_2 >_{\wedge^2\mathcal{H}} \\
& + < \phi^\sharp, \boldsymbol{\nabla}_{V_1}(V_2 \wedge V_3) + \boldsymbol{\nabla}_{V_2}(V_3 \wedge V_1) + \boldsymbol{\nabla}_{V_3}(V_1 \wedge V_2) >_{\wedge^2\mathcal{H}} \\
& + \phi(-[V_1, V_2] \wedge V_3 + [V_1, V_3] \wedge V_2 + [V_2, V_3] \wedge V_1) \\
= & < \boldsymbol{\nabla}^\sharp \phi^\sharp, V_2 \wedge V_3 \wedge V_1 + V_3 \wedge V_1 \wedge V_2 + V_1 \wedge V_2 \wedge V_3 >_{\wedge^3\mathcal{H}} \\
& + \phi[\boldsymbol{T}(V_1, V_2) \wedge V_3 + \boldsymbol{T}(V_2, V_3) \wedge V_1 + \boldsymbol{T}(V_3, V_1) \wedge V_2] \\
= & < 3\mathfrak{D}^2\phi^\sharp + \mathfrak{T}^{2*}\phi^\sharp, V_1 \wedge V_2 \wedge V_3 >_{\wedge^3\mathcal{H}},
\end{aligned}
$$

with the operators $\mathfrak{D}^2$ and $\mathfrak{T}^2$ defined as above, e.g.,

$$\mathfrak{T}^2(V_1, V_2, V_3) = \boldsymbol{T}(V_1, V_2) \wedge V_3 + \boldsymbol{T}(V_2, V_3) \wedge V_1 + \boldsymbol{T}(V_3, V_1) \wedge V_2.$$

The L^2-adjoint of $\boldsymbol{\nabla}^\sharp$ is $\boldsymbol{\nabla}^* : \mathrm{Dom}(\boldsymbol{\nabla}^*) \subset L^2\Gamma(\mathcal{H}^{\otimes q+1}) \to L^2\Gamma(\mathcal{H}^{\otimes q})$, given by

$$\mathbb{E} < \boldsymbol{\nabla}^*(U \otimes V), W >_{\mathcal{H}^{\otimes q}} = \mathbb{E} < U \otimes V, \boldsymbol{\nabla}^\sharp W >_{\mathcal{H}^{\otimes q+1}} = \mathbb{E} < U, \boldsymbol{\nabla}_V W >_{\mathcal{H}^{\otimes q}},$$

for $U \in \mathcal{H}^{\otimes q}$, $V \in \mathcal{H}$, such that $U \otimes V \in \mathrm{Dom}(\boldsymbol{\nabla}^\sharp)$, and $W \in \mathbb{D}^{2,1}(\mathcal{H}^{\otimes q})$. It has been shown in [25] that $\mathrm{Dom}(\boldsymbol{\nabla}^*)$ includes $\mathbb{D}^{2,1}$.

Since the divergence operator is the L^2-adjoint of the exterior derivative, i.e., to say that U has a divergence means that, for any cylindrical ϕ,

$$\mathbb{E}\, d^q\phi(U) = -\mathbb{E}\, \phi\, (\mathrm{Div} U),$$

we can relate the divergence to $\boldsymbol{\nabla}^*$. Indeed, for $q = 0$, $\nabla_{\mathcal{H}}$ is the natural analogue for $\boldsymbol{\nabla}$, so if V has a divergence, we have

$$\mathrm{Div} V = -\nabla^*_{\mathcal{H}} V.$$

In general, for $V_1, \cdots, V_q \in \mathrm{Dom}(\mathrm{div}) \subset L^2\Gamma(\mathcal{H})$, we have, from (6.21),

$$\begin{aligned} q\, \mathrm{Div}(V_1 \wedge \cdots \wedge V_q) &= \sum_{j=1}^{q} (-1)^{j-1} \mathrm{Div}(V_j)(V_1 \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q) \\ &\quad - \sum_{i<j} (-1)^{i+j} [V_i, V_j] \wedge V_1 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q. \end{aligned} \tag{6.23}$$

The following lemma continues the comparison of div and $\boldsymbol{\nabla}^*$ for $q > 0$, and extends Proposition 9.6 of [26].

Lemma 6.3.1. *Suppose $U \in L^\infty\Gamma(\mathcal{H}^{\otimes q}) \cap \mathbb{D}^{2,1}(\mathcal{H}^{\otimes q})$, and $V \in L^\infty\Gamma\mathcal{H} \cap \mathrm{Dom}(\nabla^*_{\mathcal{H}})$, where $q \geq 1$. Then we have $U \otimes V \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, and*

$$\boldsymbol{\nabla}^*(U \otimes V)(\sigma) = U(\sigma)(\nabla^*_{\mathcal{H}} V)(\sigma) - \boldsymbol{\nabla}_{V(\sigma)} U. \tag{6.24}$$

In particular, if $V \in L^\infty\Gamma(\mathcal{H}^{\wedge q}) \cap \mathbb{D}^{2,1}(\mathcal{H}^{\wedge q})$, we have $V \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, and

$$\boldsymbol{\nabla}^* V = (-1)^q\, \mathrm{div}(V) + \frac{1}{q}(-1)^q\, \mathfrak{T}^{q-1}(V). \tag{6.25}$$

Remark 6.3.2. For $q = 1$, we see from Proposition 9.6 in [26] that equation (6.24) takes the same form, while (6.25) with $q = 2$ reduces to

$$\boldsymbol{\nabla}^*(V_1 \wedge V_2) = \operatorname{div}(V_1 \wedge V_2) + \frac{1}{2}\boldsymbol{T}(V_1, V_2), \quad \forall V_1, V_2 \in \mathbb{D}^{2,1}(\mathcal{H}).$$

Proof. The proof for the first part mirrors that of Proposition 9.6 in [26]. For any $Z \in \mathbb{D}^{2,1}(\mathcal{H}^{\otimes q})$,

$$\begin{aligned}
\mathbb{E} < (\boldsymbol{\nabla} Z)_\sigma, U \otimes V(\sigma) >_{\mathcal{H}^{\otimes q+1}} &= \mathbb{E} < \boldsymbol{\nabla}_{V_\sigma} Z, U(\sigma) >_{\mathcal{H}^{\otimes q}} \\
&= \mathbb{E}[D(< Z, U >_{\mathcal{H}^{\otimes q}})(V) - < Z, \boldsymbol{\nabla}_{V_\sigma} U >_{\mathcal{H}^{\otimes q}}] \\
&= \mathbb{E}[< Z, U >_{\mathcal{H}^{\otimes q}} (\nabla^*_{\mathcal{H}} V) - < Z, \boldsymbol{\nabla}_{V_\sigma} U >_{\mathcal{H}^{\otimes q}}],
\end{aligned}$$

since $\boldsymbol{\nabla}$ is a metric connection.

The second part follows immediately if V is primitive, i.e., $V = V_1 \wedge \cdots \wedge V_q$, where $V_1, \cdots, V_q \in L^\infty\Gamma\mathcal{H} \cap \mathbb{D}^{2,1}(\mathcal{H})$: indeed, since $V_1 \wedge \cdots \wedge V_q \in \operatorname{Dom}(\boldsymbol{\nabla}^*)$, we can apply (6.23) to calculate

$$\begin{aligned}
& q\boldsymbol{\nabla}^*(V_1 \wedge \cdots \wedge V_q) \\
= & \sum_{j=1}^{q} (-1)^{q-j} \left[(\nabla^*_{\mathcal{H}} V_j)(V_1 \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q) - \boldsymbol{\nabla}_{V_j}(V_1 \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q) \right] \\
= & (-1)^q \sum_{j=1}^{q} (-1)^{j-1} \operatorname{div}(V_j)(V_1 \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q) \\
& + (-1)^q \sum_{i<j} (-1)^{i+j-1} (\boldsymbol{\nabla}_{V_i} V_j - \boldsymbol{\nabla}_{V_j} V_i)(V_1 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q) \\
= & (-1)^q \Bigg[q \operatorname{div}(V_1 \wedge \cdots \wedge V_q) \\
& + \sum_{i<j} (-1)^{i+j-1} (\boldsymbol{\nabla}_{V_i} V_j - \boldsymbol{\nabla}_{V_j} V_i - [V_i, V_j]) \wedge V_1 \wedge \cdots \wedge \hat{V}_i \wedge \cdots \wedge \hat{V}_j \wedge \cdots \wedge V_q \Bigg] \\
= & (-1)^q [q \operatorname{div}(V_1 \wedge \cdots \wedge V_q) + \mathfrak{T}^{q-1}(V_1, \cdots, V_q)].
\end{aligned}$$

A general $V \in L^\infty\Gamma(\mathcal{H}^{\wedge q}) \cap \mathbb{D}^{2,1}(\mathcal{H}^{\wedge q})$ can be approximated by primitive elements of the form above, so we are done. □

Motivated by these results, we introduce a new notation. Given $U \in \mathbb{D}^{2,1}(\mathcal{H}^{\otimes q})$, and $V = V_1 \otimes \cdots \otimes V_p$, where $V_i \in \mathcal{H}$ for all $i = 1$ to p, define ${}_{(p)}\boldsymbol{\nabla}_V U \in \otimes^{p+q-1}\mathcal{H}$ by

$$_{(p)}\boldsymbol{\nabla}_V U = {}_{(q+1)}\!< (\boldsymbol{\nabla}^\sharp U) \otimes V_1 \otimes \cdots \otimes V_{p-1}, V_p >_{\mathcal{H}} . \tag{6.26}$$

If we have $U \in L^\infty\Gamma(\mathcal{H}^{\otimes q}) \cap \mathbb{D}^{2,1}(\mathcal{H}^{\otimes q})$, and $V = V_1 \otimes \cdots \otimes V_p \in L^\infty\Gamma\mathcal{H}^{\otimes p}$, with $V_p \in L^\infty\Gamma\mathcal{H} \cap \mathrm{Dom}(\nabla^*_{\mathcal{H}})$, equation (6.24) shows

$$\boldsymbol{\nabla}^*(V) = (V_1 \otimes \cdots \otimes V_{p-1})(\nabla^*_{\mathcal{H}} V_p) - \boldsymbol{\nabla}_{V_p}(V_1 \otimes \cdots \otimes V_{p-1}),$$

and

$$\begin{aligned}\boldsymbol{\nabla}^*(U \otimes V) &= (U \otimes V_1 \otimes \cdots \otimes V_{p-1})(\nabla^*_{\mathcal{H}} V_p) - \boldsymbol{\nabla}_{V_p}(U \otimes V_1 \otimes \cdots \otimes V_{p-1}) \\ &= U \otimes \boldsymbol{\nabla}^* V - {}_{(p)}\boldsymbol{\nabla}_V U.\end{aligned}$$

Since we can approximate any element of $\mathcal{H}^{\otimes p}$ by the primitive tensor products, the notation ${}_{(p)}\boldsymbol{\nabla}_V U$ defined in (6.26) extends to $V \in \mathcal{H}^{\otimes p}$. Therefore, approximating elements in $L^\infty\Gamma(\mathcal{H}^{\otimes p}) \cap \mathrm{Dom}(\boldsymbol{\nabla}^*)$ by primitive elements enables us to extend (6.24) to the following

Corollary 6.3.3. *If $U \in L^\infty\Gamma(\mathcal{H}^{\otimes q}) \cap \mathbb{D}^{2,1}(\mathcal{H}^{\otimes q})$ and $V \in L^\infty\Gamma(\mathcal{H}^{\otimes p}) \cap \mathrm{Dom}(\boldsymbol{\nabla}^*)$, we have $U \otimes V \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, and*

$$\boldsymbol{\nabla}^*(U \otimes V) = U \otimes \boldsymbol{\nabla}^* V - {}_{(p)}\boldsymbol{\nabla}_V U.$$

In light of the above, we restate our commutation formula in Lemma 3.2.1.

Lemma 6.3.4. *Given a vector field $U \in \mathbb{D}^{2,1}(\mathcal{H})$ adapted to $\{\mathcal{F}_t^{x_0}\}_{t\in[0,T]}$, we have $\tau\boldsymbol{\nabla}^\sharp U \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, $\nabla^*_{\mathcal{H}} U \in \mathbb{D}^{2,1}$, and*

$$\nabla_{\mathcal{H}} \nabla^*_{\mathcal{H}} U = \boldsymbol{\nabla}^* \tau \boldsymbol{\nabla}^\sharp U + U. \tag{6.27}$$

In other words, if $v \in \mathcal{H}_\sigma$, we have

$$\nabla_v \nabla^* U = < \boldsymbol{\nabla}^* \tau \boldsymbol{\nabla}^\sharp U, v >_{\mathcal{H}_\sigma} + < U, v >_{\mathcal{H}_\sigma} .$$

Remark 6.3.5. The τ above is $\tau_{1,2}$, defined in (4.5). For higher-order tensors, we will write the subscripts explicitly for clarity.

The Riemann curvature operator on the manifold, $\mathcal{R} : \wedge^2 TM \to \wedge^2 TM$, is defined by

$$\mathcal{R}(u \wedge v)w = \nabla^2 w(u, v) - \nabla^2 w(v, u), \quad \forall u, v, w \in TM.$$

We denote by $\tilde{R}$ the Riemann curvature operator for the pointwise connection on the L^2 tangent bundle, i.e., $\tilde{R} : \wedge^2 TC_{x_0}M \to \mathcal{L}_{skew}(L^2TC_{x_0}M; L^2TC_{x_0}M)$, and by $\boldsymbol{R}$ the curvature operator for the damped Markovian connection on $\Gamma\mathcal{H}$, i.e., $\boldsymbol{R} : \wedge^2 T_\sigma C_{x_0}M \to \mathcal{L}_{skew}(\mathcal{H}_\sigma;\mathcal{H}_\sigma)$. They are conjugate to each other via the map $\frac{\mathbb{D}}{d.}$: as in Section 9 of [26],

$$[\boldsymbol{R}(U)v]_t = (\frac{\mathbb{D}}{dt})^{-1}\left(\tilde{R}_\sigma(U_\sigma)\frac{\mathbb{D}}{d.}v\right)_t = W_t\int_0^t W_s^{-1}\mathcal{R}_{\sigma_s}(U_{s,s})(\frac{\mathbb{D}}{ds}v_s)ds.$$

We can now state the following

Corollary 6.3.6 (Commutation Formula for Two-Tensors)**.** *Given a vector field $U_1 \in L^\infty\Gamma\mathcal{H}\cap\mathbb{D}^{2,2}(\mathcal{H})$, and an adapted vector field $U_2 \in L^\infty\Gamma\mathcal{H}\cap\mathbb{D}^{2,1}(\mathcal{H})$, we have $U_1\otimes U_2 \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, $\boldsymbol{\nabla}^*(U_1\otimes U_2)\in\mathbb{D}^{2,1}$, $\tau_{2,3}\boldsymbol{\nabla}^\sharp(U_1\otimes U_2)\in\mathrm{Dom}(\boldsymbol{\nabla}^*)$, and*

$$\boldsymbol{\nabla}^\sharp\boldsymbol{\nabla}^*(U_1\otimes U_2) = \boldsymbol{\nabla}^*\tau_{2,3}\boldsymbol{\nabla}^\sharp(U_1\otimes U_2) + U_1\otimes U_2 - \boldsymbol{R}((-)\wedge U_2)U_1.$$

In other words, for any $v\in\mathcal{H}_\sigma$,

$$\boldsymbol{\nabla}_v\boldsymbol{\nabla}^*(U_1\otimes U_2) = {}_{(2)}\!< \boldsymbol{\nabla}^*\tau_{2,3}\boldsymbol{\nabla}^\sharp(U_1\otimes U_2), v>_{\mathcal{H}_\sigma} + {}_{(2)}\!< U_1\otimes U_2, v>_{\mathcal{H}_\sigma} - \boldsymbol{R}(v\wedge U_2)U_1.$$

Proof. That $U_1\otimes U_2\in\mathrm{Dom}(\boldsymbol{\nabla}^*)$ follows from (6.24), or Proposition 9.6 in [26], which says

$$\boldsymbol{\nabla}^*(U_1\otimes U_2)(\sigma) = U_1(\sigma)(\nabla^*_{\mathcal{H}}U_2)(\sigma) - \boldsymbol{\nabla}_{U_2(\sigma)}U_1.$$

Now we apply Lemma 6.3.4 to see that $\boldsymbol{\nabla}^*(U_1\otimes U_2)\in\mathbb{D}^{2,1}$, with

$$\begin{aligned}\boldsymbol{\nabla}_v\boldsymbol{\nabla}^*(U_1\otimes U_2) &= \boldsymbol{\nabla}_v[U_1(\nabla^*_{\mathcal{H}}U_2) - \boldsymbol{\nabla}_{U_2}U_1]\\ &= (\boldsymbol{\nabla}_vU_1)\nabla^*_{\mathcal{H}}U_2 + U_1 < \nabla_{\mathcal{H}}\nabla^*_{\mathcal{H}}U_2, v>_{\mathcal{H}_\sigma} - \boldsymbol{\nabla}_v\boldsymbol{\nabla}_{U_2}U_1\\ &= (\boldsymbol{\nabla}_vU_1)\nabla^*_{\mathcal{H}}U_2 + U_1 < \boldsymbol{\nabla}^*\tau\boldsymbol{\nabla}^\sharp U_2, v>_{\mathcal{H}_\sigma} + U_1 < U_2, v>_{\mathcal{H}_\sigma} - \boldsymbol{\nabla}_v\boldsymbol{\nabla}_{U_2}U_1.\end{aligned}$$

Lemma 6.3.1 and Corollary 6.3.3 show that $\tau_{2,3}\boldsymbol{\nabla}^\sharp(U_1\otimes U_2)\in\mathrm{Dom}(\boldsymbol{\nabla}^*)$, and

$$\begin{aligned}&{}_{(2)}\!< \boldsymbol{\nabla}^*\tau_{2,3}\boldsymbol{\nabla}^\sharp(U_1\otimes U_2), v>_{\mathcal{H}_\sigma}\\ =\ &{}_{(2)}\!< \boldsymbol{\nabla}^*(\boldsymbol{\nabla}^\sharp U_1\otimes U_2), v>_{\mathcal{H}_\sigma} + {}_{(2)}\!< \boldsymbol{\nabla}^*\tau_{2,3}(U_1\otimes\boldsymbol{\nabla}^\sharp U_2), v>_{\mathcal{H}_\sigma}\\ =\ &(\boldsymbol{\nabla}_vU_1)\nabla^*_{\mathcal{H}}U_2 - {}_{(2)}\!< \boldsymbol{\nabla}_{U_2}\boldsymbol{\nabla}^\sharp U_1, v>_{\mathcal{H}_\sigma} + U_1 < \boldsymbol{\nabla}^*\tau\boldsymbol{\nabla}^\sharp U_2, v>_{\mathcal{H}_\sigma}\\ &- {}_{(2)}\!< \boldsymbol{\nabla}_{\tau\boldsymbol{\nabla}^\sharp U_2}U_1, v>_{\mathcal{H}_\sigma}.\end{aligned} \tag{6.28}$$

Comparing the above expressions, we see that the conclusion will follow if we can show

$$\boldsymbol{\nabla}_v\boldsymbol{\nabla}_{U_2}U_1 - {}_{(2)}\!< \boldsymbol{\nabla}_{U_2}\boldsymbol{\nabla}^\sharp U_1, v>_{\mathcal{H}_\sigma} - < \boldsymbol{\nabla}_{\tau\boldsymbol{\nabla}^\sharp U_2}U_1, v>_{\mathcal{H}_\sigma} = \boldsymbol{R}(v\wedge U_2)U_1. \tag{6.29}$$

We prove (6.29) for the case where we can write $\boldsymbol{\nabla}^\sharp U_2$ in the form of $V_1 \otimes V_2$, with $V_1, V_2 \in L^\infty\Gamma\mathcal{H}\cap\mathbb{D}^{2,1}(\mathcal{H})$, and leave the rest to a simple approximation argument as in [25]: first take any approximating sequence of $\boldsymbol{Y}U_2$ that consists of finite sums of elements of the form $f(\sigma)_j h_j$, with $h_j \in H$ and $f_j : C_{x_0}M \to \mathbb{R}$ smooth cylindrical, the apply equation (6.6) to see that $\boldsymbol{\nabla}^\sharp U_2$ can be approximated by a sequence of finite sums of elements of the form $\boldsymbol{X}(h_j) \otimes \nabla_{\mathcal{H}} f_j$.

In this case, the last term in (6.28) becomes

$$\begin{aligned} & < \boldsymbol{\nabla}_{\tau\boldsymbol{\nabla}^\sharp U_2} U_1, v >_{\mathcal{H}_\sigma} = {}_{(2)}< \boldsymbol{\nabla}_{V_1} U_1 \otimes V_2, v >_{\mathcal{H}_\sigma} \\ = & {}_{(2)}< \boldsymbol{\nabla}^\sharp U_1, V_1 >_{\mathcal{H}_\sigma} < V_2, v >_{\mathcal{H}_\sigma} \\ = & {}_{(2)}< \boldsymbol{\nabla}^\sharp U_1, {}_{(2)}< V_1 \otimes V_2, v >_{\mathcal{H}_\sigma} >_{\mathcal{H}_\sigma} \\ = & {}_{(2)}< \boldsymbol{\nabla}^\sharp U_1, \boldsymbol{\nabla}_v U_2 >_{\mathcal{H}_\sigma} = \boldsymbol{\nabla}_{\boldsymbol{\nabla}_v U_2} U_1. \end{aligned}$$

Since

$${}_{(2)}< \boldsymbol{\nabla}_{U_2} \boldsymbol{\nabla}^\sharp U_1, v >_{\mathcal{H}_\sigma} = -\boldsymbol{\nabla}^2 U_1(U_2, v),$$

and

$$\boldsymbol{\nabla}_v \boldsymbol{\nabla}_{U_2} U_1 = \boldsymbol{\nabla}^2 U_1(v, U_2) + \boldsymbol{\nabla}_{\boldsymbol{\nabla}_v U_2} U_1,$$

the three terms on the left-hand side of (6.29) do combine to produce $\boldsymbol{R}(v \wedge U_2)U_1$, so we are done. □

Corollary 6.3.7 (Commutation Formula for Skew-Symmetric Two-Tensors)**.** *Given adapted vector fields* $U_1, U_2 \in L^\infty\Gamma\mathcal{H} \cap \mathbb{D}^{2,2}(\mathcal{H})$, *we have* $U_1 \wedge U_2 \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, $\boldsymbol{\nabla}^*(U_1 \wedge U_2) \in \mathbb{D}^{2,1}$, $\tau_{2,3}\boldsymbol{\nabla}^\sharp(U_1 \wedge U_2) \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, *and*

$$\boldsymbol{\nabla}^\sharp\boldsymbol{\nabla}^*(U_1 \wedge U_2) = \boldsymbol{\nabla}^*\tau_{2,3}\boldsymbol{\nabla}^\sharp(U_1 \wedge U_2) + U_1 \wedge U_2 - \frac{1}{2}\boldsymbol{R}(U_1 \wedge U_2).$$

In other words,for any $v \in \mathcal{H}_\sigma$,

$$\boldsymbol{\nabla}_v\boldsymbol{\nabla}^*(U_1 \wedge U_2) = {}_{(2)}< \boldsymbol{\nabla}^*\tau_{2,3}\boldsymbol{\nabla}^\sharp(U_1 \wedge U_2), v >_{\mathcal{H}_\sigma} + {}_{(2)}< U_1 \wedge U_2, v >_{\mathcal{H}} - \frac{1}{2}\boldsymbol{R}(U_1 \wedge U_2)v. \tag{6.30}$$

More generally, if $U \in L^\infty\Gamma(\mathcal{H}^{\wedge 2}) \cap \mathbb{D}^{2,2}(\mathcal{H}^{\wedge 2}) \cap \mathcal{V}^{(2)}$, *we have*

$$\boldsymbol{\nabla}^\sharp\boldsymbol{\nabla}^* U = \boldsymbol{\nabla}^*\tau_{2,3}\boldsymbol{\nabla}^\sharp U + U - \frac{1}{2}\boldsymbol{R}(U).$$

Proof. This follows from Corollary 6.3.6 and the first Bianchi identity

$$\boldsymbol{R}(U_1, U_2)V + \boldsymbol{R}(U_2, V)U_1 + \boldsymbol{R}(V, U_1)U_2 = 0.$$

The more general statement for non-primitive U follows from the usual approximation argument, as primitive adapted two-tensors are dense in $\mathcal{V}^{(2)}$ (Lemma 6.1.2), and we can approximate primitive adapted two-tensors by smooth adapted ones, so we are done. $\square$

Define $\mathfrak{R}^q : \otimes^{q+2}\mathcal{H}_\sigma \to \otimes^{q+1}\mathcal{H}_\sigma$ by mixing the curvature terms as follows:

$$\mathfrak{R}^q(U_1, \cdots, U_q, V, W) = \sum_{i=1}^{q-1} U_1 \otimes \cdots \otimes [\boldsymbol{R}(V \wedge W)U_i] \otimes \cdots \otimes U_q.$$

We can state now

Corollary 6.3.8 (Commutation Formula for Primitive q-Tensors)**.** *Given vector fields $U_1, \cdots, U_{q-1} \in L^\infty\Gamma\mathcal{H} \cap \mathbb{D}^{2,2}(\mathcal{H})$, and adapted $U_q \in L^\infty\Gamma\mathcal{H} \cap \mathbb{D}^{2,1}(\mathcal{H})$, we have $U_1 \otimes \cdots \otimes U_q \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, $\boldsymbol{\nabla}^*(U_1 \otimes \cdots \otimes U_q) \in \mathbb{D}^{2,1}$, $\tau_{q,q+1}\boldsymbol{\nabla}^\sharp(U_1 \otimes \cdots \otimes U_q) \in \mathrm{Dom}(\boldsymbol{\nabla}^*)$, and*

$$\begin{aligned} \boldsymbol{\nabla}^\sharp\boldsymbol{\nabla}^*(U_1 \otimes \cdots \otimes U_q) \quad = \quad & \boldsymbol{\nabla}^*\tau_{q,q+1}\boldsymbol{\nabla}^\sharp(U_1 \otimes \cdots \otimes U_q) + U_1 \otimes \cdots \otimes U_q \\ & - \mathfrak{R}^{q-1}(U_1, \cdots, U_{q-1}, -, U_q). \end{aligned}$$

More generally, if $U \in L^\infty\Gamma(\mathcal{H}^{\otimes q-1}) \cap \mathbb{D}^{2,2}(\mathcal{H}^{\otimes q-1})$, with U_q as above, we have

$$\boldsymbol{\nabla}^\sharp\boldsymbol{\nabla}^*(U \otimes U_q) = \boldsymbol{\nabla}^*\tau_{q,q+1}\boldsymbol{\nabla}^\sharp(U \otimes U_q) + U \otimes U_q - \mathfrak{R}^{q-1}(U, -, U_q).$$

Proof. The proof is essentially the same as that of Corollary 6.3.6, as we can apply Lemmas 6.3.1 and 6.3.4 to calculate

$$\begin{aligned} & \boldsymbol{\nabla}^\sharp\boldsymbol{\nabla}^*(U_1 \otimes \cdots \otimes U_q) \\ = \quad & \boldsymbol{\nabla}^\sharp[(U_1 \otimes \cdots \otimes U_{q-1})(\nabla^* U_q) - \boldsymbol{\nabla}_{U_q}(U_1 \otimes \cdots \otimes U_{q-1})] \\ = \quad & \boldsymbol{\nabla}^\sharp(U_1 \otimes \cdots \otimes U_{q-1})(\nabla^* U_q) + (U_1 \otimes \cdots \otimes U_{q-1}) \otimes (\boldsymbol{\nabla}^*\tau\boldsymbol{\nabla}^\sharp U_q) \\ & + U_1 \otimes \cdots \otimes U_q - \boldsymbol{\nabla}^\sharp\boldsymbol{\nabla}_{U_q}(U_1 \otimes \cdots \otimes U_{q-1}), \end{aligned}$$

and similarly

$$\begin{aligned} & \boldsymbol{\nabla}^*\tau_{q,q+1}\boldsymbol{\nabla}^\sharp(U_1 \otimes \cdots \otimes U_q) \\ = \quad & \boldsymbol{\nabla}^\sharp(U_1 \otimes \cdots \otimes U_{q-1})(\nabla^* U_q) - \boldsymbol{\nabla}_{U_q}\boldsymbol{\nabla}^\sharp(U_1 \otimes \cdots \otimes U_{q-1}) \\ & + (U_1 \otimes \cdots \otimes U_{q-1}) \otimes (\boldsymbol{\nabla}^*\tau\boldsymbol{\nabla}^\sharp U_q) - \boldsymbol{\nabla}_{\tau\boldsymbol{\nabla}^\sharp U_q}(U_1 \otimes \cdots \otimes U_{q-1}), \end{aligned}$$

so the conclusion follows from the observation that

$$\begin{aligned}
& -\boldsymbol{\nabla}^\sharp \boldsymbol{\nabla}_{U_q}(U_1 \otimes \cdots \otimes U_{q-1}) + \boldsymbol{\nabla}_{U_q}\boldsymbol{\nabla}^\sharp(U_1 \otimes \cdots \otimes U_{q-1}) + \boldsymbol{\nabla}_{\tau \boldsymbol{\nabla}^\sharp U_q}(U_1 \otimes \cdots \otimes U_{q-1}) \\
= & \sum_{i=1}^{q-1} U_1 \otimes \cdots \otimes [\boldsymbol{\nabla}^2 U_i(U_q, -) - \boldsymbol{\nabla}^2 U_i(-, U_q)] \otimes \cdots \otimes U_{q-1} \\
= & -\mathfrak{R}^{q-1}(U_1, \cdots, U_{q-1}, -, U_q). \quad \square
\end{aligned}$$

We can also derive the version of the commutation formula for higher-order skew-symmetric tensor products, similar to Corollary 6.3.7. Such formulae are necessary for the development of Clark-Ocone-type formulae for differential forms on the path space, as in the classical Wiener space. Extra complication arises here due to appearance of the curvature and torsion terms, see, e.g., equations (6.25) and (6.30). We also need a good analogue of (6.8) to continue the programme.

6.4 The Pullback Method

In this section, we use the Itô map $\mathcal{I} : C_0 \to C_{x_0}M$ of the SDE (2.7) to pull back to the classical Wiener space.

Lemma 6.4.1. *If $\alpha_\cdot(x_\cdot) \in \mathbb{L}(T_{x_\cdot}M; \mathbb{R})$ is adapted and in L^2_{loc},*

$$(\int_0^T \alpha_t d\{x\}_t) \circ \mathcal{I} = \int_0^T (\alpha_t \circ \mathcal{I}) X(x_t \circ \mathcal{I}) dB_t.$$

Proof. From the SDE (2.7),

$$\begin{aligned}
\int_0^T \alpha_t \circ dx_t &= \int_0^T \alpha_t X(x_t) \circ dB_t \\
&= \int_0^T \alpha_t X(x_t) dB_t + \frac{1}{2}\mathrm{Tr} \int_0^t \alpha_t \nabla X[X(x_t)(-)](-)dt \\
&= \int_0^T \alpha_t X(x_t) dB_t,
\end{aligned}$$

where the last equation follows from Assumption 2.1.1 and equation (2.8). Consequently, the martingale part $\int_0^t \alpha_t d\{x\}_t$ of the Stratonovich integral $\int_0^T \alpha_t \circ dx_t$ is given by $\int_0^t \alpha_t X(x_t) dB_t$, so

$$(\int_0^T \alpha_t \circ dx_t) \circ \mathcal{I} = \int_0^T (\alpha_t \circ \mathcal{I}) X(x_t \circ \mathcal{I}) \circ dB_t + \text{drift term},$$

where the martingale part is $\int_0^T (\alpha_t \circ \mathcal{I}) X(x_t \circ \mathcal{I}) dB_t$. □

Proposition 6.4.2. *$\forall \phi \in L^2\Gamma\mathcal{H}^*$, $CO(\phi) \circ \mathcal{I} = CO(\mathcal{I}^*\phi)$, i.e,.*

$$[\int_0^T < \mathbb{E}(\frac{\mathbb{D}}{dt}\phi_t^\sharp | \mathcal{F}_t^{x_0}), d\{x\}_t >_{x_t}] \circ \mathcal{I} = \int_0^T < \mathbb{E}[\frac{d}{dt}(\mathcal{I}^*\phi)_t^\sharp | \mathcal{F}_t], dB_t >_{\mathbb{R}^m}$$

Proof. As both sides have zero expectation, it suffices to test against all functions of the form $\int_0^T < \dot{a}_t, dB_t >_{\mathbb{R}^m}$ for $\{a_t\}_{t\in[0,T]}$ adapted, sample-continuous and bounded on $[0,1] \times C_0$. From the left hand side, we get

$$\begin{aligned}
& \int_{C_0} \int_0^T < \dot{a}_t, dB_t >_{\mathbb{R}^m} [\int_0^T < \mathbb{E}(\frac{\mathbb{D}}{dt}\phi_t^\sharp | \mathcal{F}_t^{x_0}), d\{x\}_t >_{x_t}] \circ \mathcal{I}\, d\gamma \\
= & \int_{C_0} \mathbb{E}[\int_0^T < \dot{a}_t, dB_t >_{\mathbb{R}^m} | \mathcal{F}_t^{\mathcal{I}}] \int_0^T < \mathbb{E}(\frac{\mathbb{D}}{dt}\phi_t^\sharp \circ \mathcal{I} | \mathcal{F}_t^{\mathcal{I}}), X(x_t^{\mathcal{I}}) dB_t >_{\mathbb{R}^m} d\gamma \\
= & \int_{C_0} \int_0^T < \mathbb{E}(\dot{a}_t | \mathcal{F}_t^{\mathcal{I}}), dB_t >_{\mathbb{R}^m} \int_0^T < Y_{x_t^{\mathcal{I}}} \mathbb{E}(\frac{\mathbb{D}}{dt}\phi_t^\sharp \circ \mathcal{I} | \mathcal{F}_t^{\mathcal{I}}), dB_t >_{\mathbb{R}^m} d\gamma \\
= & \int_{C_0} \int_0^T < \mathbb{E}(\dot{a}_t | \mathcal{F}_t^{\mathcal{I}}), Y_{x_t^{\mathcal{I}}} \mathbb{E}(\frac{\mathbb{D}}{dt}\phi_t^\sharp \circ \mathcal{I} | \mathcal{F}_t^{\mathcal{I}}) >_{\mathbb{R}^m} dt\, d\gamma \\
= & \int_{C_{x_0}M} \int_0^T < X(x_t)\mathbb{E}(\dot{a}_t | \mathcal{I}_t = x_t), \frac{\mathbb{D}}{dt}\phi_t^\sharp(x) >_{\mathbb{R}^m} dt\, d\mu_{x_0}(x) \\
= & \int_{C_{x_0}M} \phi_x(\overline{T\mathcal{I}_x}[\mathbb{E}(a_\cdot | \mathcal{I} = x)]) d\mu_{x_0}(x),
\end{aligned}$$

where we have used Lemma 2.2.4 and Lemma 6.4.1 in the second line, the Itô isometry in the fourth line, and equation (2.10) in the last. Applying the Itô isometry again, we obtain from the right hand side

$$\begin{aligned}
& \int_{C_0} \int_0^T < \dot{a}_t, dB_t >_{\mathbb{R}^m} \int_0^T < \mathbb{E}[\frac{d}{dt}(\mathcal{I}^*\phi)_t^\sharp | \mathcal{F}_t], dB_t >_{\mathbb{R}^m} d\gamma \\
= & \int_{C_0} \int_0^T < \dot{a}_t, \mathbb{E}[\frac{d}{dt}(\mathcal{I}^*\phi)_t^\sharp | \mathcal{F}_t] >_{\mathbb{R}^m} dt\, d\gamma \\
= & \int_{C_0} < a, (\mathcal{I}^*\phi)^\sharp >_H d\gamma \\
= & \int_{C_0} \phi[\overline{T\mathcal{I}(a)}] \circ \mathcal{I}\, d\gamma \\
= & \int_{C_{x_0}M} \phi_x[\overline{T\mathcal{I}(a)}_x] d\mu_{x_0}(x) \\
= & \int_{C_{x_0}M} \phi_x\{\overline{T\mathcal{I}_x}[\mathbb{E}(a_\cdot | \mathcal{I} = x)]\} d\mu_{x_0}(x),
\end{aligned}$$

where the fifth line follows from Corollary 3.7 of [25], and the last from equation (2.11) and the adaptedness of $\{a_t\}_{t\in[0,T]}$. $\square$

Proposition 6.4.3. *$\forall\phi \in L^2\Gamma\mathcal{H}^*$, $\bar{d}^1\phi = 0 \implies d^1\mathcal{I}^*\phi = 0$.*

Proof. Approximate ϕ by smooth cylindrical one-forms $\{\phi_j\}_{j\in\mathbb{N}}$ such that $\phi_j \to \phi$ in $L^2\Gamma\mathcal{H}^*$ and $d^1\phi_j \to 0$ in $L^2\Gamma(\mathcal{H}^{(2)})^*$. For any h_1, $h_2 \in H$ and smooth $\lambda : C_0 \to \mathbb{R}$ in finite chaos, we have $\overline{\wedge^2 T\mathcal{I}(\lambda h_1 \wedge h_2)} \in L^2\Gamma\mathcal{H}^{(2)}$ (see [27]). Therefore by the continuity of $\mathcal{I}^*$ on $\mathcal{H}$-one-forms ([25] Theorem 3.4), we obtain

$$\begin{aligned}
\int_{C_0} \lambda d^1\mathcal{I}^*\phi(h_1\wedge h_2)d\gamma &= -\int_{C_0} \mathcal{I}^*\phi \operatorname{div}(\lambda h_1\wedge h_2)d\gamma \\
&= -\lim_{j\to\infty}\int_{C_0} \mathcal{I}^*\phi_j \operatorname{div}(\lambda h_1\wedge h_2)d\gamma \\
&= \lim_{j\to\infty}\int_{C_0} d^1\mathcal{I}^*\phi_j(\lambda h_1\wedge h_2)d\gamma \\
&= \lim_{j\to\infty}\int_{C_0} \mathcal{I}^*(d^1\phi_j)(\lambda h_1\wedge h_2)d\gamma \\
&= \lim_{j\to\infty}\int_{C_{x_0}M} d^1\phi_j(x)[\wedge^2 T\mathcal{I}((\lambda h_1\wedge h_2))]d\mu_{x_0}(x) \\
&= \lim_{j\to\infty}\int_{C_{x_0}M} d^1\phi_j(x)[\overline{\wedge^2 T\mathcal{I}(\lambda h_1\wedge h_2)}]d\mu_{x_0}(x) \\
&= 0.
\end{aligned}$$

Since the collection of such λ is total in L^2, this shows that $d^1\mathcal{I}^*\phi(h_1\wedge h_2) = 0$, a.s. for each $h_1 \wedge h_2$, hence $d^1\mathcal{I}^*\phi = 0$, a.s. $\square$

With the above propositions, we obtain a weaker result for the path spaces than for the Wiener space, where D is replaced by the weak derivative defined by $\tilde{D} = [(D)^*|_{\mathbb{D}^{2,1}\mathcal{H}^*})]^*$ (see [25] for a detailed discussion). What we obtain here is different from the results in Section 6.2, stated in the usual (strong) derivatives. This reflects the general phenomenon that when we use the Itô map to pull back, we usually obtain results in terms of weak derivatives. These coincide with the usual derivatives on the flat space, but not necessarily so on the general path spaces; see, again, [25] for a detailed discussion.

Proposition 6.4.4. *$\forall\phi \in L^2\Gamma\mathcal{H}^*$, $\bar{d}^1\phi = 0 \implies \phi = \tilde{D}f$, for some $f \in \operatorname{Dom}(\tilde{D})$.*

Proof. We apply the flat space result in Section 4.2 to write $\mathcal{I}^*\phi = D\,g$, with $g = CO(\mathcal{I}^*\phi) \in \mathbb{D}^{2,1}$. By Proposition 6.4.2, $g = CO(\phi)\circ\mathcal{I}$, hence $CO(\phi) \in \mathbb{W}^{2,1}$

using Theorem 6.1 of [25], therefore $\mathcal{I}^*[\tilde{D}CO(\phi)] = D\,g = \mathcal{I}^*\phi$. The injectivity of $\mathcal{I}^*$ proved in Theorem 3.4 of [25] shows in turn that $\phi = \tilde{D}[CO(\phi)]$. □

Remark 6.4.5. This proof will also work for ϕ such that $\tilde{d}^1\phi = 0$, with a suitable definition for the weak exterior derivative $\tilde{d}^1$, such that $\tilde{d}^1\tilde{D} = 0$. This would give a Hodge decomposition in terms of the weak derivatives.

In the proof of Lemmas 2.2.6 and 3.2.1, we get around the problem of weak derivatives by taking an approximating sequence and making use of the fact that our derivative operators are closed. On the other hand, such a technique does not help us to show that $\tilde{d}^1\phi = 0$ implies $\phi = Df$ here, since we do not know that the Itô map is continuous on two-forms.

If we assume $\mathcal{I}$ to be continuous on two-forms, the argument would go as follows. We approximate ϕ by smooth cylindrical one-forms $\{\phi_j\}_{j\in\mathbb{N}}$ such that $d^1\phi_j \to 0$. Since $\mathcal{I}$ is a continuous linear map on one-forms by Theorem 3.4 of [25], we know $\mathcal{I}^*\phi_j \to \mathcal{I}^*\phi$. The continuity of the Itô map on two-forms would imply

$$d^1(\mathcal{I}^*\phi_j) = \mathcal{I}^*(d^1\phi_j) \to 0, \tag{6.31}$$

so we would be able to apply Corollary 4.2.3 to obtain

$$\|DCO(\mathcal{I}^*\phi_j) - \mathcal{I}^*\phi_j\|_{L^2} \le \sqrt{2}\|d^1\mathcal{I}^*\phi_j\|_{L^2},$$

and as a result,

$$\|DCO(\mathcal{I}^*\phi_j) - \mathcal{I}^*\phi\|_{L^2} \le \|DCO(\mathcal{I}^*\phi_j) - \mathcal{I}^*\phi_j\|_{L^2} + \|\mathcal{I}^*\phi_j - \mathcal{I}^*\phi\|_{L^2} \to 0, \quad \text{as } j \to \infty.$$

By Proposition 6.4.2, $CO(\mathcal{I}^*\phi_j) = CO(\phi_j)\circ\mathcal{I} \to CO(\phi)\circ\mathcal{I}$ in $\mathbb{D}^{2,1}$. Since we know $CO(\phi_j) \in \mathbb{D}^{2,1}$, and the set $\{f\circ\mathcal{I} | f \in \mathbb{D}^{2,1}\}$ is closed in $\mathbb{D}^{2,1}$ (Corollary 4.3 of [25]), we would conclude that $CO(\phi) \in \mathbb{D}^{2,1}$ and $\phi = DCO(\phi)$, as in Proposition 6.2.1.

As has been announced in [21], the above argument carries through on a symmetric space, where the pullback map $\mathcal{I}^* : L^2\Gamma\mathcal{H}^{(2)} \to L^2(C_0; \wedge^2 H^*)$ is indeed a continuous linear map, so (6.31) holds, and we obtain the same vanishing result, as in Proposition 6.2.1, for symmetric spaces. For the general Riemannian path spaces, we hope to combine the different approaches and work towards an analogue of Shigekawa's result on the flat space.

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