THE UNIVERSITY OF WARWICK

University of Warwick institutional repository: http://go.warwick.ac.uk/wrap

**A Thesis Submitted for the Degree of PhD at the University of Warwick**

http://go.warwick.ac.uk/wrap/55920

This thesis is made available online and is protected by original copyright.

Please scroll down to view the document itself.

Please refer to the repository record for this item for information to help you to cite it. Our policy information is available from the repository home page.

THE UNIVERSITY OF WARWICK

AUTHOR: **Paul Cadman** DEGREE: **Ph.D.**

TITLE: **Deformations of Plane Curve Singularities and the $\delta$-Constant Stratum**

DATE OF DEPOSIT: ................................

I agree that this thesis shall be available in accordance with the regulations governing the University of Warwick theses.

I agree that the summary of this thesis may be submitted for publication.

I **agree** that the thesis may be photocopied (single copies for study purposes only).

Theses with no restriction on photocopying will also be made available to the British Library for microfilming. The British Library may supply copies to individuals or libraries. subject to a statement from them that the copy is supplied for non-publishing purposes. All copies supplied by the British Library will carry the following statement:

> "Attention is drawn to the fact that the copyright of this thesis rests with its author. This copy of the thesis has been supplied on the condition that anyone who consults it is understood to recognise that its copyright rests with its author and that no quotation from the thesis and no information derived from it may be published without the author's written consent."

AUTHOR'S SIGNATURE: ........................................................

---

USER'S DECLARATION

1. I undertake not to quote or make use of any information from this thesis without making acknowledgement to the author.

2. I further undertake to allow no-one else to use this thesis while it is in my care.

DATE SIGNATURE ADDRESS

.........................................................................

.........................................................................

.........................................................................

.........................................................................

.........................................................................

# Deformations of Plane Curve Singularities and the $\delta$-Constant Stratum

by

**Paul Cadman**

**Thesis**

Submitted to the University of Warwick

for the degree of Doctor of Philosophy

**Doctor of Philosophy**

**Mathematics**

June 2011

To Peter

# Contents

# List of Figures

# Acknowledgments

I would like to thank my supervisor, David Mond, for his encouragement, kindness and patience. I would also like to thank Duco van Straten, Christian Sevenheck, Bradford Hovinen and Ragnar-Olaf Buchweitz for useful discussions and Diane Maclagan for her helpful comments on my final draft.

I am very grateful to my housemates Christine Granger and Ben Sharp for their support as I was writing up my thesis and to the whole Warwick maths community for their friendship throughout my time at Warwick.

# Declarations

I declare that, to the best of my knowledge and unless otherwise stated, all the work in this thesis is original. I confirm that this thesis has not been submitted for a degree at another university.

# Abstract

Consider the germ of a plane curve $(C_0, 0) := V(f) \subset \mathbb{C}^2$ with an isolated singularity at 0 where $f \in \mathcal{O}_{\mathbb{C}^2,0}$. The $\delta$-invariant of $(C_0, 0)$ can be interpreted as the maximum number of singularities that can *pile up* on the zero level set of a deformation of $f$. Let $F \in \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^\mu,0}$ be a miniversal deformation of $f$ then the $\delta$-constant stratum $D(\delta)$ in the discriminant of $F$ is the set of parameters where the $\delta$-invariant of the deformed curve is equal to the $\delta$-invariant of the original curve.

Givental and Varchenko showed that when $(C_0, 0)$ is irreducible, then $D(\delta)$ is an example of a Lagrangian singularity with respect to a symplectic form arising from the intersection pairing on the deformed curves. More recently van Straten and Sevenheck have developed a theory of deformations of Lagrangian singularities and conjecture that $D(\delta)$ is a rigid Lagrangian singularity when $(C_0, 0)$ is an irreducible plane curve singularity.

In this thesis we will show how to compute the symplectic form explicitly in the case of an irreducible simple singularity. Using this symplectic form we construct a maximal Cohen-Macaulay module on the discriminant that can be used to find equations for $D(\delta)$ for the $A_{2k}, E_6$ and $E_8$ singularities. We will add weight to the conjecture of van Straten and Sevenheck by showing that $D(\delta)$ is Cohen-Macaulay for $E_6$ and $E_8$.

# Introduction

This thesis studies deformations of plane curves, 1-dimensional complex germs $(C_0, 0) \subset (\mathbb{C}^2, 0)$, with isolated singularities. The underlying idea is to take an isolated plane curve singularity and deform it into an object that is more easily studied.

For plane curve singularity the space of all possible deformations (up to a notion of equivalence) can be parametrised using a finite dimensional variety and this information can be encoded in a miniversal deformation. The geometry of the parametrising variety is interesting in its own right. The variety of parameters such that the deformed curve is singular is called the discriminant and this has been studied extensively (see for example the work of Tessier in [Tes76], and Diaz and Harris in [DH88]).

We can define a plane curve singularity by an equation $(C_0, 0) = V(f)$ for $f \in \mathcal{O}_{\mathbb{C}^2,0}$ or as the image of a parametrisation $p : \prod_{i=1}^{r}(\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ (where $r$ is the number of branches of $(C_0, 0)$). A miniversal deformation of $(C_0, 0)$ can be obtained by deforming its equation and we can use a miniversal deformation $F \in \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^\mu}$ (where $\mu$ is the Milnor number of the curve) to parametrise the deformations of $C_0$.

Only those deformations arising from a deformation of the normalisation $n : (\overline{C_0}, \overline{0}) \to (C_0, 0)$ of the curve can be obtained by deforming the parametrisation. Moreover, this occurs when the deformation $C_s$ has the same $\delta$-invariant as the original curve where the $\delta$-invariant is defined to be the dimension $\dim_{\mathbb{C}} \mathcal{O}_{\overline{C_0},0}/\mathcal{O}_{C_0,0}$. The $\delta$-invariant measures how many singularities can *pile up* on a deformation of the curve.

We call the set of parameters where the deformed curve has the same $\delta$-invariant as the original curve the $\delta$-constant stratum, which we will denote by $D(\delta)$. This will be the main object of investigation in the thesis.

The Milnor fibre $C_s$ of an irreducible curve is a genus $\mu/2$ surface with one boundary component. The intersection of 1-cycles on this surface induces an irreducible skew-symmetric bilinear pairing on $H^1(C_s)$. Givental and Varchenko prove in [GV82] that this pairing can be pulled back by a nondegenerate period map to a symplectic structure called the intersection form $\Phi$ on the parameter space $(\Lambda, 0)$. (A nondegenerate period map is an isomorphism from the tangent space at each point of the parameter space into the cohomology of the deformed curve at that point). They also show that this structure identifies $(D(\delta), 0)$ as a Lagrangian subvariety.

We will compute the symplectic structure $\Phi$ explicitly for simple singularities using a method discussed in chapter 15 of [AGZV88] and it is this computation that will be the basis of investigation of $D(\delta)$ in the thesis.

The coefficients of $\Phi = \sum_{1 \leq i < j \leq \mu} g_{ij} \mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j$ can be placed in a skew-symmetric matrix $\Omega = (g_{ij})$ and using this we will define the intersection module $M_\Omega$ on the discriminant $D$ with the following presentation:

$$\mathcal{O}_{D,0} \xrightarrow{\chi^t \Omega \chi} \mathcal{O}_{D,0} \longrightarrow M_\Omega \longrightarrow 0$$

where $\chi$ is the matrix, known as the Saito matrix, with columns equal to the vector of coefficients of vector fields that are tangent to the discriminant and $D$ is the discriminant of the miniversal deformation.

We prove that the entries of the skew-symmetric matrix $\chi^t \Omega \chi$ generate an ideal defining $D(\delta)$ and moreover that (in theorem 4.22, p.76):

$$V\left(\mathrm{pf}_{2(m+1)}(\chi^t \Omega \chi)\right) = D\left(\delta(C_0) - m\right)$$

where $\mathrm{pf}_k(S)$ is the ideal of principal Pfaffians of a skew-symmetric matrix $S$ and $D(k)$ is the stratum of parameters $\lambda$ in the discriminant where $\delta(C_\lambda) \geq k$. The varieties also correspond to strata where $M_\Omega$ requires more than $2k$ generators.

We prove that $M_\Omega$ is a maximal Cohen-Macaulay $\mathcal{O}_{D,0}$-module and consequently can be viewed as an infinitesimal deformation of the module $\mathcal{O}_{\widetilde{D},0} := \pi_* \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^\mu,0}/J(F)$ but we do not yet understand the geometric significance of this fact. We can however use this knowledge to say something about $D(\delta)$ in particular cases.

Recently van Straten and Sevenheck have studied the deformation theory of Lagrangian singularities (see [vSS03]) and have conjectured that for irreducible singularities $(D(\delta), 0)$ is a rigid Lagrangian singularity. They prove that the conjecture is true if $(D(\delta), 0)$ is Cohen-Macaulay and show this in the case of the $A_{2k}$ singularities using a result of Givental (see [Giv90]). We use the result above to show the conjecture is true for $E_6$ and $E_8$ singularities by proving that $(D(\delta), 0)$ is Cohen-Macaulay.

In the following paragraphs we give an outline of the thesis. In chapter 1 we present the deformation theory of plane curve singularities, the $\delta$-invariant and the Milnor fibration with its associated cohomology bundle. The cohomology bundle admits a canonical flat connection called the Gauss-Manin connection. Using results from Kulikov's book (see [Kul98]) we show how to calculate the covariant connection with respect to the Gauss-Manin connection of a section given by a holomorphic form.

In chapter 2 we discuss the construction of the intersection pairing on the Milnor fibre of an irreducible simple singularity and give a clear account of how this can be computed as a residue at the unique point at infinity on the affine Milnor fibre.

We also recall the construction of the symplectic form $\Phi$ and we show that in the case of quasihomogeneous singularities we can make some deductions about the coefficients of $\Phi$. For example we prove that the form is quasihomogeneous and use this to calculate some entries in the matrix of coefficients. For the $A_{2k}$ singularities we prove a formula which computes explicitly the coefficients that appear along the diagonal of the matrix.

The two remaining chapters contain the main results of the thesis. In chapter 3 we introduce the module $M_\Omega$ described above and show that it can be realised as an infinitesimal deformation of the module $\mathcal{O}_{\widetilde{D},0}$. This investigation raises the possibility that the intersection form could be defined purely in terms of the deformation theory of $\mathcal{O}_{\widetilde{D},0}$ but as yet we have found no such description.

In chapter 4 we prove that the Pfaffian ideals of the presentation matrix for $M_\Omega$ define the $\delta$-constant strata $D(k)$ as described above. We use these calculations to prove that $D(\delta)$ is Cohen-Macaulay for the $E_6$ and $E_8$ singularities. In the final section we present some experimental evidence to show that the intersection form defined by a degenerate period map could be used to find ideals defining the strata $D(k)$.

Following this there is an appendix containing the *Maple* and *Macaulay2* code used to make computations in the thesis together with some of the results of the computations. Finally there is an appendix concerning a result of Heymanns (see [Hey69]) about the expansion of a minor of a matrix in terms of its Pfaffians.

# Chapter 1

# Deformations of Plane Curve Singularities

This chapter is an exposition of the basic objects used in this the thesis. We will define deformations of curves and deformations of the parametrisation curve and also the Milnor fibration. We will describe the Gauss-Manin connection, a canonical flat connection on the vector bundle of cohomology associated to the Milnor fibration. We will show how to compute the Gauss-Manin connection explicitly using the Leray residue theorem. Finally we will introduce the Saito matrix of a miniversal deformation.

The main reference for deformations of curves is Greuel, Lossen and Shustin [GM07]. A good reference for the Gauss-Manin connection is [Kul98] and for the the Saito matrix is [Bru84].

## 1.1 Deformations of Curves

Let $\mathbf{f} : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated singularity at the origin. By a germ we mean an equivalence class of maps $f_i : U_i \to \mathbb{C}$ defined on neighbourhoods $U_i$ of 0 in $\mathbb{C}^2$ such that two maps $f_i, f_j$ are equivalent if they coincide on some neighbourhood $U \subset U_1 \cap U_2$ of 0.

In the case of holomorphic functions, the two functions $f_i, f_j$ are equivalent if and only if their power series expansions at 0 coincide. So we can choose a representative of the class in the local ring of convergent power series at 0, denoted $\mathbb{C}\{x,y\}$, which is equal to $\mathcal{O}_{\mathbb{C}^2,0}$.

Let $f \in \mathcal{O}_{\mathbb{C}^2,0}$ be a representative of the germ $\mathbf{f}$ above and define $C_0 = V(f)$ to be the corresponding affine curve. We will assume the curve is reduced (as this will be useful when we consider the normalisation of the curve).

**Definition 1.1.** A *deformation of the curve* $(C_0,0)$ over $(\mathbb{C}^k,0)$ consists of a flat map $p : (\mathcal{A},0) \to (\mathbb{C}^k,0)$ of complex germs together with an isomorphism from $(C_0,0)$ to the central fibre $(\mathcal{A}_0,0)$ of $(\mathcal{A},0)$,

$$(C_0,0) \xrightarrow{\simeq} (p^{-1}(0),0) = (\mathcal{A}_0,0)\ .$$

**Remark 1.2.** The flatness condition is imposed because we want to study only geometrically meaningful deformations of the curve $C_0$. In particular the flatness condition implies that all of the fibres of the deformation have the same dimension if we take small enough representatives of $\mathcal{A}$ and $\mathbb{C}^k$ (see [GM07] page 223).

**Example 1.3.** As an example of this, consider the map $p : (V(xy),0) \to (\mathbb{C},0)$ induced from the projection $\pi : (\mathbb{C}^2,0) \to (\mathbb{C},0)$. The special fibre $p^{-1}(0)$ is 1-dimensional whilst other fibres are 0–dimensional and the map $p$ is not flat. Indeed, if $p^* : \mathbb{C}\{x\} \to \mathbb{C}\{x,y\}/(xy)$ was flat then the image of a generating set of the maximal ideal of $\mathbb{C}\{x\}$ would be a regular sequence of $\mathbb{C}\{x,y\}/(xy)$ (see [GM07] Theorem B.8.11 page 419) however $p^*(x)$ is a zero divisor of $\mathbb{C}\{x,y\}/(xy)$.

There are two natural ways we can deform $C_0$. The first is to deform deform $f$, the equation of the curve. Secondly, in the case of plane curves we can parametrise $C_0$ and deform this parametrisation. We will describe both methods.

### 1.1.1 Deformations of the equation

**Definition 1.4.** A *deformation of the equation* $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ is a germ $F : (\mathbb{C}^2 \times \mathbb{C}^k, 0) \to (\mathbb{C}, 0)$ with representative $F \in \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^k, 0}$ defined by:

$$F = f + \sum_{i=1}^{k} g_i \lambda_i$$

where $g_i \in \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^k, 0}$.

A deformation $F$ of $f$ is said to be *versal* if for any other deformation $F' : (\mathbb{C}^2 \times \mathbb{C}^l, 0) \to (\mathbb{C}, 0)$ there exists a family of germs of diffeomorphisms $g : (\mathbb{C}^2 \times \mathbb{C}^l) \to (\mathbb{C}^2, 0)$ and a smooth germ $\theta : (\mathbb{C}^l, 0) \to (\mathbb{C}^k, 0)$ such that:

$$F'(x, \lambda') = F(g(x, \lambda'), \theta(\lambda'))$$

where $g(x, 0) = x$ and $\theta(0) = 0$. A versal deformation is said to be *miniversal* if the dimension of its parameter space is minimal.

**Theorem 1.5.** *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ with isolated singularity of multiplicity $\mu$, then a miniversal deformation of the germ is a map $F : (\mathbb{C}^2 \times \mathbb{C}^\mu, 0) \to (\mathbb{C}, 0)$ defined by:*

$$F(x, \lambda) = f(x) + \sum_{i=1}^{\mu} \lambda_i \phi_i(x)$$

*where $\phi_1, \ldots, \phi_\mu$ forms a basis of the $\mathbb{C}$–vector space $M_f = \mathcal{O}_{\mathbb{C}^2,0}/J_f$ where $J_f$ is the Jacobian ideal of $f$.*

We will often perform calculations on germs $f$ that have quasihomogeneous representatives. We recall the definitions here.

**Definition 1.6.** The map

$$f : \mathbb{C}^n \to \mathbb{C}^m : x_1, \ldots, x_n \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

is said to be *quasihomogeneous* with weights $w = (w_1, \ldots, w_n)$ and degrees $d =$

$(d_1, \ldots, d_m)$ if for $a \in \mathbb{C}^*$ the following equality holds:

$$f(a^{w_1}x_1, \ldots, a^{w_n}x_n) = (a^{d_1}f_1(x_1, \ldots, x_n), \ldots, a^{d_m}f_m(x_1, \ldots, x_m))$$

A hypersurface $V(f)$ is said to be quasihomogeneous if the map $f$ is quasihomogeneous. For a quasihomogeneous map $g$ we will denote its quasihomogeneous degree (with respect to the weights and degrees of the map) as $\operatorname{qdeg} g$.

A consequence of quasihomogeneity of a hypersurface is the following:

**Proposition 1.7.** *The hypersurface $V(f) \subset \mathbb{C}^n$ is invariant under the $\mathbb{C}^*$–action defined for $a \in \mathbb{C}^*$ by:*

$$h_a : \mathbb{C}^n \to \mathbb{C}^n : z \mapsto h_a(z) = a \cdot z = (a^{w_1}z_1, \ldots, a^{w_n}z_n)$$

*Proof.* If $(z_1, \ldots, z_n) \in V(f)$ then $f(a^{w_1}z_1, \ldots, a^{w_n}z_n) = a^d f(z_1, \ldots, z_n) = 0$ and hence $(a^{w_1}z_1, \ldots, a^{w_n}z_n) \in V(f)$. $\square$

**Remark 1.8.** A holomorphic $k$–form $\omega \in \Omega^k_{\mathbb{C}^n}$ is said to be quasihomogeneous with respect to weights $(w_1, \ldots, w_n)$ and degree $d$ if

$$h_a^*\omega = a^d\omega$$

where $h_a$ is the $\mathbb{C}^*$–action on $\mathbb{C}^n$ defined above. We will denote the quasihomogeneous degree of $\omega$ as $\operatorname{qdeg} \omega$.

**Definition 1.9.** A germ $f \in \mathcal{O}_{\mathbb{C}^2,0}$ of an isolated singularity is called a *simple singularity* if there is a biholomorphic change of coordinates that takes $f$ to one of the following germs at the origin:

| Singularity | | Weights $(x, y)$ |
|---|---|---|
| $A_n$ : | $y^2 + x^{n+1}$ (for $n \geq 1$) | $(2, n+1)$ |
| $D_n$ : | $x^{n-1} + xy^2$ (for $n \geq 4$) | $(2, n-2)$ |
| $E_6$ : | $x^4 + y^3$ | $(3, 4)$ |
| $E_7$ : | $x^3y + y^3$ | $(2, 3)$ |
| $E_8$ : | $x^5 + y^3$ | $(3, 5)$ |

Each of the germs in the table is quasihomogeneous. The final column in the table gives a possible choice for weights of the variables.

Before further considering the deformation of quasihomogeneous singularities we will need the following result of K. Saito that a miniversal deformation of a simple singularity has a quasihomogeneous representative.

**Theorem 1.10** (see [Sai74], Theorem 0.8, p.291)**.** *Let $f$ be a simple singularity then $f$ has a miniversal deformation $F_\lambda$ in which all the parameters $\lambda_i$ have positive weights.*

*Proof.* The proof follows by a result of K. Saito ([Sai74], Theorem 0.8, p.291) where it is shown that $f$ is a simple singularity if and only if $\operatorname{qdeg} \phi < \operatorname{qdeg} f$ where $\phi \in \mathcal{O}_{\mathbb{C}^2}/J_f$. We deduce from this, considering the representative of $F_\lambda$ given in theorem 1.5, that the weights of $\lambda_i$ must be positive because $\operatorname{qdeg} F_\lambda = \operatorname{qdeg} f$. □

We now return to considering how a deformation of the equation of $f$ relates to a deformation of the curve $C_0$.

**Remark 1.11.** The deformation $F$ of $f$ induces a deformation of $(C_0, 0)$ if we define $(\mathcal{A}, 0) = (F^{-1}(0), 0)$:

$$\begin{array}{ccc} (C_0, 0) & \overset{i}{\hookrightarrow} & (\mathcal{A}, 0) \\ \downarrow & & \downarrow p \\ \{0\} & \hookrightarrow & (\mathbb{C}^k, 0) \end{array}$$

where $p$ is the restriction of the map $\pi : (\mathbb{C}^2 \times \mathbb{C}^k, 0) \to (\mathbb{C}^k, 0)$.

The converse is also true, a deformation $p : (\mathcal{A}, 0) \to (\mathbb{C}^k, 0)$ of $(C_0, 0)$ can be described using a deformation of $f$ (see [GM07] Proposition 1.5 page 228). The total space of the deformation can be given as $(\mathcal{A}, 0) = (V(F), 0) \subset (\mathbb{C}^2 \times \mathbb{C}^k, 0)$ for some deformation $F \in \mathcal{O}_{\mathbb{C}^2\times\mathbb{C}^k,0}$ of $f$. The map $p$ is given by the restriction of the projection to the second component of $(\mathbb{C}^2 \times \mathbb{C}^k, 0)$.

A versal deformation of $(C_0, 0)$ is a deformation that contains information about all possible deformations of the curve. We now recall how to construct a deformation of $f$ that corresponds to a versal deformation of $C_0$ in the isolated singularity case.

**Definition 1.12.** A deformation $p : (\mathcal{A}, 0) \to (\mathbb{C}^k, 0)$ of $(C_0, 0)$ is said to be *versal* if for any other deformation $q : (\mathcal{B}, 0) \to (\mathbb{C}^l, 0)$ of $(C_0, 0)$ there exists a base change map $b : (\mathbb{C}^l, 0) \to (\mathbb{C}^k, 0)$ such that $q$ is isomorphic to $b^*p$:

$$\begin{array}{ccccc} (\mathcal{B},0) & \xrightarrow{\simeq} & b^*(\mathcal{A},0) & \xrightarrow{\mathrm{pr}_1} & (\mathcal{A},0) \\ & \searrow^{q} & \downarrow & & \downarrow p \\ & & (\mathbb{C}^l,0) & \xrightarrow[b]{} & (\mathbb{C}^k,0)\,. \end{array}$$

A versal deformation $p$ is called *miniversal* if the base space $(\mathbb{C}^k, 0)$ has the smallest possible dimension.

In the case of $C_0$, a plane curve with isolated singularity, a miniversal deformation exists and can be expressed using a deformation of its equation.

**Theorem 1.13** (see [GM07] Corollary 1.17 page 239)**.** *Let $(C_0, 0)$ be a plane curve with isolated singularity defined by $f \in \mathcal{O}_{\mathbb{C}^2,0}$. Choose generators $\phi_1, \ldots, \phi_\tau$ ( as a $\mathbb{C}$–vector space) of the Tjurina algebra:*

$$T^1_{(C_0,0)} = \mathcal{O}_{\mathbb{C}^2}/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$$

*and let $F$ be a deformation of $f$ defined by:*

$$F(x, y, \lambda_1, \ldots, \lambda_\tau) = f(x, y) + \sum_{i=1}^{\tau} \lambda_i \phi_i(x, y)$$

*then $F$ induces a miniversal deformation of $(C_0, 0)$. We call $\tau$ the Tjurina number of $(C_0, 0)$.*

### 1.1.2 Deformations of the parametrisation

In this section we recall how a parametrisation of the curve $(C_0, 0)$ can be obtained from its normalisation. We also recall that a deformation of $(C_0, 0)$ can be obtained by deforming its parametrisation if and only if the $\delta$–invariant of the deformed fibres remains constant. We will describe parametrisations of $(C_0, 0)$ using the normalisation of the curve. Consider the commutative diagram:

$$\begin{array}{ccc} (\overline{C_0}, \overline{0}) & & \\ {\scriptstyle n}\downarrow & \overset{\phi}{\searrow} & \\ (C_0, 0) & \xrightarrow{\ i\ } & (\mathbb{C}^2, 0) \end{array}$$

where $n$ is the normalisation, $\overline{0} = n^{-1}(0)$ so that $(\overline{C_0}, \overline{0})$ is a multigerm which can be identified with a multigerm $(\mathbb{C}, B)$ where $B$ is a set of cardinality equal to the number of branches of $(C_0, 0)$. If we write $(C_0, 0) = (C_0^1, 0) \cup \cdots \cup (C_0^r, 0)$ as the decomposition of $(C_0, 0)$ into irreducible factors then $(\overline{C_0}, \overline{0}) = (\overline{C_0^1}, \overline{0}^1) \cup \cdots \cup (\overline{C_0^r}, \overline{0}^r)$ is a multigerm with $(\overline{C_0^i}, \overline{0}^i) \simeq (\mathbb{C}, 0)$ for which $n_|(\overline{C_0^i}, \overline{0}^i) = (C_0^i, 0)$. In terms of local rings this becomes:

$$\begin{array}{ccc} \mathcal{O}_{\overline{C_0}, \overline{0}} \simeq \oplus_{i=1}^r \mathcal{O}_{\overline{C_0^i}, \overline{0}^i} \simeq \oplus_{i=1}^r \mathbb{C}\{t_i\} & & \\ {\scriptstyle n^\sharp}\uparrow & \overset{\phi^\sharp}{\nwarrow} & \\ \mathbb{C}\{x, y\}/(f) \simeq \mathcal{O}_{C_0, 0} & \longleftarrow & \mathcal{O}_{\mathbb{C}^2, 0} \simeq \mathbb{C}\{x, y\}\ . \end{array}$$

Consider the map

$$\phi^\sharp = (\phi_i^\sharp)_{i=1}^r : \mathbb{C}\{x, y\} \to \oplus_{i=1}^r \mathbb{C}\{t_i\}$$

and let $x_i = \phi_i^\sharp(x)$ and $y_i = \phi_i^\sharp(y)$ then $\phi$ defines a parametrisation $\phi_i : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0) : t_i \mapsto (x_i(t_i), y_i(t_i))$ of each branch $(C_0^i, 0)$ of $(C_0, 0)$.

**Definition 1.14.** We will define a *deformation of the parametrisation* by making the following observations.

Since both $(\overline{C_0}, \overline{0})$ and $(\mathbb{C}^2, 0)$ are smooth the Tjurina number of $\overline{C_0}$ and hence the dimension of the base of its versal deformation is 0 so any deformation of these germs is isomorphic to a product deformation. This allows us to write a deformation of the parametrisation $\phi : (\overline{C_0}, \overline{0}) \to (\mathbb{C}^2, 0)$ over $(\mathbb{C}^k, 0)$ using the following commutative diagram:

$$\begin{array}{ccccc}
(\overline{C_0}, \overline{0}) & \hookrightarrow & (\overline{\mathcal{B}}, \overline{0}) & \xrightarrow{\simeq} & (\overline{C_0} \times \mathbb{C}^k, \overline{0}) \\
\downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\
(\mathbb{C}^2, 0) & \hookrightarrow & (\mathcal{M}, 0) & \xrightarrow{\simeq} & (\mathbb{C}^2 \times \mathbb{C}^k, 0) \\
\downarrow & & \downarrow{\scriptstyle \psi_0} & \swarrow{\scriptstyle \mathrm{pr}_2} & \\
\{0\} & \hookrightarrow & (\mathbb{C}^k, 0) & &
\end{array}$$

where $(\overline{\mathcal{B}}, \overline{0}) = \cup_{i=1}^{r} (\overline{\mathcal{B}^i}, \overline{0}^i)$ and $(\overline{\mathcal{B}^i}, \overline{0}^i) \simeq (\overline{C_0^i} \times \mathbb{C}^k, \overline{0}^i)$.

We call the map $\psi : (\overline{C_0}, \overline{0}) \to (\mathbb{C}^2 \times \mathbb{C}^k, \overline{0})$ a *deformation of* $\phi$ over $(\mathbb{C}^k, 0)$.

**Remark 1.15.** Define $(\mathcal{B}, 0) = \psi(\overline{\mathcal{B}}, \overline{0})$ then the restriction $\psi_0| : (\mathcal{B}, 0) \to (\mathbb{C}^k, 0)$ is a deformation of $(C_0, 0)$ (see [GM07] p.300) for which the ideal in $\mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^k, 0}$ defining $(\mathcal{A}, 0)$ as the kernel of $\psi^\sharp : \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^k, 0} \to \mathcal{O}_{\overline{\mathcal{A}}, \overline{0}}$.

The deformation of the parametrisation can be described with the $r$ components of $\psi : (\overline{C_0} \times \mathbb{C}^k, \overline{0}) \to (\mathbb{C}^2 \times \mathbb{C}^k, 0)$ which we write as $\{\psi_i = (X_i(t_i, s), Y_i(t_i, s))\}_{i=1}^{r}$

$$X_i(t_i, s) = x_i(t_i) + A_i(t_i, s)$$
$$Y_i(t_i, s) = y_i(t_i) + B_i(t_i, s)$$

where $X_i, Y_i \in \mathcal{O}_{\overline{C_0}^i \times \mathbb{C}^k, \overline{0}^i}$, $A_i(t_i, 0) = B_i(t_i, 0) = 0$ and $s \in (T, 0)$.

**Proposition 1.16** ([GM07] Proposition 2.27 p.305)**.** *We have the following equality of vector spaces:*

$$T^1_\phi = \left(\mathcal{O}_{\overline{C_0}, \overline{0}} \oplus \mathcal{O}_{\overline{C_0}, \overline{0}}\right) / \left(D\phi \cdot \mathcal{O}_{\overline{C_0}, \overline{0}} + \mathcal{O}_{C_0, 0} \oplus \mathcal{O}_{C_0, 0}\right)$$

*with basis* $(v_1, w_1), \ldots, (v_k, w_k)$ *where* $v_j = (v_j^1, \ldots, v_j^r)$ *and* $w_j = (w_j^1, \ldots w_j^r)$ *are elements of* $\mathfrak{m}_{\overline{C_0}, \overline{0}} = \oplus_{i=1}^{r} t_i \mathbb{C}\{t_i\}$*. The deformation of the parametrisation defined* *by:*

$$X_i(t_i, s_1, \ldots, s_k) = x_i(t_i) + \sum_{j=1}^{k} s_j v_j^i$$

$$Y_i(t_i, s_1, \ldots, s_k) = y_i(t_i) + \sum_{j=1}^{k} s_j w_j^i$$

*is a miniversal deformation of the parametrisation $\phi$ over $(\mathbb{C}^k, 0)$. $T^1_\phi$ has dimension $\tau - \delta$ where $\delta$ is defined in the next section.*

**Remark 1.17.** Let $\psi : (\overline{C_0} \times \mathbb{C}^k, \overline{0}) \to (\mathbb{C}^2 \times \mathbb{C}^k, 0)$ be a deformation of the parametrisation; we have seen that it induces a deformation of $(C_0, 0)$. Let $G \in \mathcal{O}_{\mathbb{C}^2 \times \mathbb{C}^k, 0}$ be such that $\psi^\sharp(G) = 0$ then $G$ is a deformation of the equation which induces the same deformation of $(C_0, 0)$ as $\psi$.

Let $p : (\mathcal{A}, 0) \to (\mathbb{C}^\mu, 0)$ be a miniversal deformation of $(C_0, 0)$. By definition of versality there is a map $b : (\mathbb{C}^k, 0) \to (\mathbb{C}^\mu, 0)$ of the base spaces so that the deformation induced by $G$ is isomorphic to the pullback of $p$:

$$\begin{array}{ccc} (G^{-1}(0), 0) & \xrightarrow{\simeq} & b^*(\mathcal{A}, 0) \\ {\scriptstyle \mathrm{pr}_2} \downarrow & \swarrow {\scriptstyle b^*p} & \\ (T, 0) & & \end{array}$$

### 1.1.3 The $\delta$–invariant

The quotient $n_* \mathcal{O}_{\overline{C_0}, 0} / \mathcal{O}_{C_0, 0}$ is concentrated on the set of singular points of $C_0$ because $C_0$ is normal outside of this set. Hence it is a finite dimensional vector space (see[GM07] Corollary 3.29 p.200).

**Definition 1.18.** The *$\delta$–invariant* of $(C_0, 0)$ is defined as follows:

$$\delta(C_0, 0) = \dim_{\mathbb{C}} \left( \mathcal{O}_{\overline{C_0}, 0} / \mathcal{O}_{C_0, 0} \right) .$$

Let $p : \mathcal{A} \to \mathbb{C}^k$ be a representative of a deformation of $(C_0, 0)$ then the $\delta$–invariant of the fibres of the deformation is defined as follows:

$$\delta(\mathcal{A}_s) = \sum_{x \in \mathcal{A}_s} \delta(\mathcal{A}_s, x)$$

which is finite since $\mathcal{A}_s$ has only finitely many singularities by assumption.

The deformation is called *$\delta$–constant* if there is a representative such that the function $s \mapsto \delta(\mathcal{A}_s)$ is constant on $S$.

We will show that a deformation of $(C_0, 0)$ can be induced from a deformation of its parametrisation if and only if the deformation is $\delta$–constant. This result was proved by Tessier (see [Tes76] §3.2 p.607.).

**Definition 1.19.** Let $p : \mathcal{A} \to \mathbb{C}^k$ be a representative of a deformation of $(C_0, 0)$ then a *simultaneous normalisation* of $p$ is a finite map $\nu : Z \to \mathcal{A}$ such that $\overline{p} = p \circ \nu : Z \to S$ is normal and such that the map $\nu_s : Z_s \to \mathcal{A}_s$ is the normalisation of $\mathcal{A}_s$.

The map $p$ is called *equinormalisable* if the normalisation of $\mathcal{A}$ $n : \overline{\mathcal{A}} \to \mathcal{A}$ is a simultaneous normalisation of $p$. The map $p$ is called *equinormalisable at* $0 \in \mathcal{A}$ if $\overline{p} = p \circ n : \overline{\mathcal{A}} \to S$ is flat at each point of $n^{-1}(0)$ and if for $s = p(x)$ the induced map of fibres $\overline{\mathcal{A}_s} \to \mathcal{A}_s$ is the normalisation.

**Remark 1.20.** Equinormalisability is an open property (see [GM07] proposition 2.55 page 347) and so if we find a representative of a deformation that is equinormalisable at 0 then we can modify the representative so that it is a simultaneous normalisation.

**Theorem 1.21** (see [GM07] Theorem 2.56 page 348)**.** *Let $p : \mathcal{A} \to S$ be a representative of a deformation of $(C_0, 0)$ then $p$ is equinormalisable if and only $p$ is locally $\delta$–constant.*

**Corollary 1.22.** *A deformation $p : (\mathcal{A}, 0) \to (S, 0)$ of $(C_0, 0)$ is $\delta$–constant if and only if it can be induced from a deformation of the parametrisation of $(C_0, 0)$.*

*Proof.* By the previous theorem a $\delta$-constant deformation is equinormalisable and so the map $n : (\overline{\mathcal{A}, \overline{0}}) \to (\mathcal{A}, 0)$ satisfies the conditions to be a deformation of the parametrisation of $(C_0, 0)$. Conversely, a representative of a deformation of the parametrisation of $(C_0, 0)$ defines an equinormalisation of $(\mathcal{A}, 0)$ and so is $\delta$–constant and consequently the induced deformation of $(C_0, 0)$ is $\delta$–constant. □

We now turn our attention to studying the strata in the base of a deformation where the restriction of the deformation is equinormalisable.

**Definition 1.23.** For a representative $p : \mathcal{A} \to \mathbb{C}^k$ of a deformation of $(C_0, 0)$ we define the *discriminant* $D$ of $p$ as the critical values of $p$:

$$D = p(\operatorname{Sing} p) \subset \mathbb{C}^k$$

and define:

$$D(n) = \{s \in \mathbb{C}^k : \delta(\mathcal{A}_s) \geq n) \ .$$

We will call $D(\delta(C_0, 0))$ the $\delta$–constant stratum and use the notation $D(\delta)$ for this set.

**Remark 1.24.** If the fibre $\mathcal{A}_s$ a has positive $\delta$–invariant then it must have at least one singular point and so $D(1) = D$.

**Example 1.25.** Figure 1.1 shows a real picture of the discriminant of a miniversal deformation of the $A_3$ singularity. It is the set of parameters $(\lambda_1, \lambda_2, \lambda_3)$ such that the zero level set of $F(x, y, \lambda) = -y^2 + x^4 + \lambda_1 x^2 + \lambda_2 x + \lambda_3$ has a singular point. The $\delta$-constant stratum is the closure of points such that $F(x, y, \lambda) = (x - a)^2(x + a)^2$

**Theorem 1.26** (see [GM07] Theorem 2.30 p.355, [DH88] p.435)**.** *Let $p : \mathcal{A} \to \mathbb{C}^k$ and $\psi : (\overline{C_0} \times \mathbb{C}^l, \overline{0}) \to (C_0 \times \mathbb{C}^l, 0)$ be sufficiently small representatives of a miniversal deformation of $(C_0, 0)$ and a miniversal deformation of the parametrisation of $(C_0, 0)$ respectively. Then $D(\delta)$ is irreducible of dimension $\tau - \delta$.*

By remark 1.17 a map of the base spaces $b : \mathbb{C}^l \to \mathbb{C}^k$ is induced by versality of the deformation $p$. Moreover, the deformations of the parametrisation are precisely those which are $\delta$–constant (by corollary 1.22) and so the image of $b$ is $D(\delta)$ in $\mathbb{C}^k$.

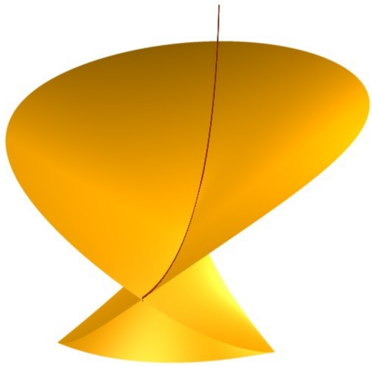


Figure 1.1: The real part of the discriminant of the $A_3$ singularity with $\delta$–constant stratum marked (the curve on self-intersection of two sheets of the discriminant in general position).

**Proposition 1.27.** *$D(\delta)$ is the closure of the space in the parameter space $\Lambda \subset \mathbb{C}^\mu$ of the deformation for which fibres over points in this space have precisely $\delta$ nondegenerate singular points.*

**Example 1.28.** Consider the singularity $A_{2k}$, $f = y^2 - x^{2k+1}$, the corresponding curve $C_{A_{2k}} = V(f)$ has parametrisation $t \mapsto (t^{2k+1}, t^2)$. We compute the $\delta$–invariant:

$$\delta(C_{A_{2k}}, 0) = \dim_{\mathbb{C}} \left( \mathbb{C}\{t\}/\mathbb{C}\{t^2, t^{2k+1}\} \right).$$

The vector space $\mathbb{C}\{t\}/\mathbb{C}\{t^2, t^{2k+1}\}$ has basis $t, t^3, t^5, \ldots, t^{2k-1}$ and so $\delta(C_{A_{2k}}, 0) = k$.

**Remark 1.29.** An interesting comparison between $\delta$ and $\mu$ is noted by Tessier (see [Tes76] remark 5.6.3 page 658). The maximum number of critical points that can *pile up* on a level set of a deformation of $f$ is measured by $\delta$ while $\mu$ measures the maximum number of critical points that a deformation of $f$ can *spread out*.

For instance consider a deformation $F : \mathbb{C}^2 \times \mathbb{C}^k \to \mathbb{C}$ of the $A_{2k}$–singularity defined as follows:

$$F(x, y, a_1, \ldots, a_k) = y^2 + (x + 2\sum_{i=1}^{k} a_i) \prod_{i=1}^{k} (x - a_i)^2 .$$

Let $F_a = F(x, y, a_1, \ldots, a_k)$ then the induced deformation of $C_{A_{2k}}$ over $\mathbb{C}^k$ is a $\delta$–constant deformation. When the $a_i$ are pairwise distinct the singular set of the fibre $F_a^{-1}(0)$ consists of $k$ $A_1$–singularities (i.e. $k$ nondegenerate critical points). Indeed, $(x - a_i)$ is a factor of $\frac{\partial F}{\partial x}$ for each $k = 1, \ldots, k$ and so at $(a_i, 0) \in F_a^{-1}(0)$ the fibre has a nondegenerate critical point. This is the maximum number of $A_1$ singularities that can *pile up* on the fibre and is equal to the the $\delta$–invariant of the singularity.

Let $\epsilon$ be a complex number and define a deformation $F_\epsilon(x, y) = y^2 + x^{2k+1} + \epsilon$ of the $A_{2k}$ singularity. The critical point at the origin of the map $F_0$ *spreads out* to $2k$ critical points of the map $F_\epsilon$ for nonzero $\epsilon$. This is the maximum number of critical points that any deformation of $A_{2k}$ can have and is equal to the Milnor number of the singularity.

A further comparison between and $\delta$ and $\mu$ is provided by Milnor:

**Theorem 1.30** (see [Mil68], Theorem 10.5, p.85)**.** *Suppose $r$ is the number of branches of the curve $C_0$ (that is, removing the assumption of irreducibility of $C_0$ for which $r = 1$) passing through the origin then $\delta, \mu$ satisfy:*

$$2\delta = \mu + r - 1 \ .$$

**Theorem 1.31** (see [Tes76] Theorem 4.8.2 p. 640)**.** *Let $D \subset \mathbb{C}^\mu$ be a representative of the discriminant in the base space of a miniversal deformation of a singularity $C_0$. Define $\mathrm{Cr}_D(k)$ as the set of points $\lambda \in D$ for which $D$ is locally isomorphic to $k$ transversely intersecting nonsingular hypersurfaces in general position:*

$$\delta(C_0, 0) = \max\{k : 0 \in \overline{\mathrm{Cr}_D(k)}\} \, .$$

This result is a consequence of Tessier's "product decomposition theorem" and gives a decomposition of the $\delta$–constant stratum as a transversal intersection of $\delta$ nonsingular leaves of the discriminant in general position (see figure 1.2).

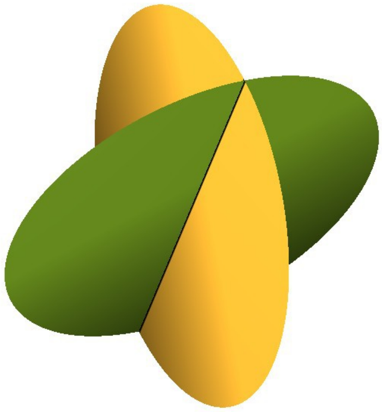


Figure 1.2: An illustration of the $\delta$–constant stratum as a transversal intersection of nonsingular leaves of the discriminant.

### 1.1.4 How the genus of the normalisation relates to the $\delta$–invariant

In this section we consider a deformation $p : (\mathcal{A}, 0) \to (\mathbb{C}, 0)$ of the plane curve $(C_0, 0)$ where the fibre $(C_0, 0)$ is irreducible with Milnor number $\mu$. We will show that the difference in $\delta$-invariants $\delta(C_0) - \delta(C_t)$ is equal to the genus of the normalization of $C_t$.

For $t \neq 0$ the singular point on $C_0$ will split into multiple singular points on $C_t$. Label these singular points as $a_1 \dots a_p$ and for each singularity let $\mu_i$ be its Milnor number and let $r_i$ be the number of branches of $C_t$ at $a_i$. Denote the genus of the smooth curve $\overline{C_t}$ by $g$.

**Lemma 1.32** (see [BKS86],p.618)**.** *The first Betti numbers $b_1(C_0)$ and $b_1(C_t)$ are related as follows:*

$$b_1(C_t) = 2g + \sum_{i=1}^{p} (r_i - 1).$$

*Proof.* Each singularity $a_i$ on $C_t$ has $r_i$ preimages under the normalization map. We can recover $C_t$ from its normalization $\overline{C_t}$ by identifying these preimages to a single point. Each of these identifications increases the first Betti number by $r_i - 1$. Moreover, the Betti number of $\overline{C_t}$ is equal to $2g$ and so we get the desired equality. □

**Theorem 1.33** (see [BG80], Theorem 4.1.4, p. 257)**.**

$$\delta(C_0) - \delta(C_t) = g$$

*Proof.* As the singularity on $C_0$ deforms, it splits into singularities of smaller Milnor numbers on $C_t$. The Milnor number of $C_0$ and the Milnor numbers of singularities on $C_t$ are related by the Betti number of $C_t$:

$$\mu = b_1(C_t) + \sum_{i=1}^{p} \mu_i$$

and since $C_0$ is irreducible we know that $\mu = 2\delta(C_0)$ (by theorem 1.30). From this and by lemma 1.32 we deduce:

$$2\delta(C_0) = 2g + \sum_{i=1}^{p} (r_i - 1 + \mu_i).$$

The $\delta$-invariant at each of the singular points $a_i$ is $\frac{1}{2}(r_i - 1 + \mu_i)$ and so the previous equation becomes the equality we were trying to prove. □

## 1.2 Milnor Fibration

The Milnor fibration is a type of deformation of $f$ where we take care to choose a representative of $p : (\mathcal{A}, 0) \to (S, 0)$ such that the fibres have good topological properties and the deformation becomes a locally trivial fibre bundle over the complement to the discriminant.

We recall the definition of the Milnor fibration found in [AGZV88] pages 285–287. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with isolated singularity at the origin with Milnor number $\mu$. Let $F : (\mathbb{C}^2 \times \mathbb{C}^\mu, 0) \to (\mathbb{C}, 0)$ be a miniversal deformation of $f$ and $G : (\mathbb{C}^2 \times \mathbb{C}^\mu, 0) \to (\mathbb{C} \times \mathbb{C}^\mu)$ be the corresponding unfolding given by $G(z, \lambda) = (F(z, \lambda), \lambda)$.

In what follows we choose sufficiently small representatives of $f, F$ and $G$. We use the notation $F_\lambda$ for the substitution $F(\cdot, \lambda) : \mathbb{C}^2 \to \mathbb{C}$.

**Definition 1.34** (Milnor Fibration)**.** For positive real numbers $\epsilon, \eta$ and $\delta$ define the following balls:

$$\begin{aligned} B^2_\epsilon &= \{z \in \mathbb{C}^2 : |z| < \epsilon\} \\ B^1_\eta &= \{t \in \mathbb{C} : |t| < \eta\} \\ B^\mu_\delta &= \{\lambda \in \mathbb{C}^\mu : |\lambda| < \delta\} \end{aligned}$$

and set:

$$\begin{aligned} S &= B^1_\eta \times B^\mu_\delta \\ Y &= (B^2_\epsilon \times B^\mu_\delta) \cap G^{-1}(S) \\ Y_s &= Y \cap G^{-1}(s) \text{ for } s \in S. \end{aligned}$$

Choose $\epsilon$ small enough so that for all $r$ satisfying $0 < r \leq \epsilon$ we have that $\partial B^2_r$ is transverse to $f^{-1}(0)$. Then choose $\eta$ and $\delta$ small enough so that for all $(t, z) \in S$ we have that $\partial B^2_\epsilon$ is transverse to $F^{-1}_\lambda(t)$.

Define the discriminant $D$ to be the set of critical values of the restriction $G_| : X \to S$ that is:

$$D = \{s \in S : X_s \text{ is singular}\}.$$

If we choose $\epsilon, \nu$ and $\delta$ with the above properties and let $Y' = Y \backslash G^{-1}(D), S; = S \backslash D$ then define the bundle:

$$\mathcal{Y} : \; Y' \xrightarrow{G_|} S' \; .$$

The fibre of the bundle is known as the *Milnor fibre.* The topology of the fibre is independent of $\epsilon, \eta, \delta$.

If we consider the sub-bundle consisting of fibres of the Milnor fibration over $\{0\} \times B^\mu_\delta \subset S$ we get a new bundle called the central Milnor fibration.

**Definition 1.35** (Central Milnor Fibration)**.** Define the following sets:

$$\begin{aligned} \Lambda &= B^{\mu}_{\delta} \\ X_\lambda &= F_\lambda^{-1}(0) \cap B^2_\epsilon \text{ for } \lambda \in \Lambda \\ X &= F^{-1}(0) = \bigcup_{\lambda \in \Lambda} X_\lambda \subset \mathbb{C}^2 \times \mathbb{C}^\mu. \end{aligned}$$

Furthermore define $\Sigma^{rel}$ as the critical space:

$$\Sigma^{rel} = \{(x, \lambda) \in B^2_\epsilon \times B^\mu_\delta \, : \, \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = F = 0\}$$

and $D = \pi_|(\Sigma^{rel})$ as the discriminant where $\pi_|$ is the restriction of the projection map:

$$\pi : B^2_\epsilon \times B^\mu_\delta \to B^\mu_\delta \, .$$

Let $X' = X \setminus \Sigma^{rel}$ and $\Lambda' = \Lambda \setminus \Delta$. We define a fibre bundle called the *central Milnor fibration* as:

$$\mathcal{X} : \; X' \xrightarrow{\pi_|} \Lambda' \; .$$

**Remark 1.36.** The central Milnor fibration is a sub-bundle of the Milnor fibration:

$$\begin{CD} X' @>{i_1}>> Y' \\ @VVV @VVV \\ \Lambda' @>>{i_2}> S' \end{CD}$$

where $i_1$ identifies the fibre $X_\lambda \subset X'$ with $X_\lambda \times \{0\} \subset Y'$ and $i_2$ identifies the point $\lambda \in \Lambda'$ with $(\lambda, 0) \in S'$.

**Remark 1.37.** For each $\lambda \in \Lambda'$ there is a Milnor fibration (perhaps after choosing smaller $\epsilon, \eta$), let:

$$
\begin{aligned}
T &= B^1_\eta \\
{}^\lambda Y &= B^2_\epsilon \cap F_\lambda^{-1}(T) \\
{}^\lambda Y_t &= {}^\lambda Y \cap F_\lambda^{-1}(t) \text{ for } t \in T
\end{aligned}
$$

with critical space ${}^\lambda\Sigma = \{t \in t : {}^\lambda Y_t \text{ is singular}\}$. Define ${}^\lambda Y' = {}^\lambda Y \setminus F_\lambda^{-1}({}^\lambda\Sigma)$, $T' = T \setminus {}^\lambda\Sigma$ then we have the Milnor fibration for $F_\lambda$:

$$ {}^\lambda\mathcal{Y} : {}^\lambda Y' \xrightarrow{(F_\lambda)_|} T' \;. $$

This is a sub-bundle of $Y'$ in a similar way to the central Milnor fibration:

$$
\begin{array}{ccc}
{}^\lambda Y' & \xrightarrow{j_1} & Y' \\
\downarrow & & \downarrow \\
T' & \xrightarrow[j_2]{} & S'
\end{array}
$$

where $j_1$ identifies the fibre ${}^\lambda Y_t$ with $\{0\} \times {}^\lambda Y_t \subset Y'$ and $j_2$ identifies the point $t \in T'$ with $(0, t) \in S'$.

**Theorem 1.38** (see [Mil68])**.** *The Milnor and central Milnor fibrations are locally trivial fibre bundles. The fibre of each bundle is homotopy equivalent to a wedge of spheres, with the number of spheres equal to the Milnor number of $f$.*

**Remark 1.39.** When $f$ is a quasihomogeneous polynomial, we can extend the domain of $G$ to a global Milnor fibration:

$$ G : \mathbb{C}^2 \times \mathbb{C}^\mu \setminus G^{-1}(D) \to \mathbb{C} \times \mathbb{C}^\mu \setminus D \,. $$

The fibre of this bundle is an affine algebraic curve diffeomorphic to the corresponding Milnor fibre. A similar global version of the central Milnor fibration and the Milnor fibration at a point $\lambda$ exists in this case also (see [Mil68] chapter 9 page 76, [Dim92] §3.1.12 page 72).

The theorem tells us that the rank of the first homology and cohomology groups of the fibres of these bundles is $\mu$. Consequently by taking the first cohomology and homology groups of the fibres of the Milnor and deformation bundles we obtain vector bundles.

**Definition 1.40.** We define a vector bundle $\underline{\mathrm{H}} \to S'$ where the total space is

$$\underline{\mathrm{H}} = \coprod_{(t,\lambda)\in S'} H_1(X_{(t,\lambda)}; \mathbb{C})$$

and define the dual vector bundle $\underline{\mathrm{H}}^* \to S'$ where the total space is

$$\underline{\mathrm{H}}^* = \mathrm{Hom}(\underline{\mathrm{H}}, \mathbb{C}_{S'}) = \coprod_{(t,\lambda)\in S'} H^1(X_{(t,\lambda)}; \mathbb{C})\,.$$

We call $\underline{\mathrm{H}}$ and $\underline{\mathrm{H}}^*$ the *homology* and *cohomology bundles* respectively.

There is a pairing between these vector bundles which extends to the sheaf of holomorphic sections:

$$(,) : \underline{\mathrm{H}}^* \times \underline{\mathrm{H}} \to \mathbb{C}_{S'}$$
$$\langle , \rangle : \mathcal{H}^* \times \mathcal{H} \to \mathcal{O}_{S'}$$

where $\mathcal{H}^*$ is the sheaf of holomorphic sections of the cohomology bundle and $\mathcal{H}$ is the sheaf of holomorphic sections of the homology bundle.

Let $\mathcal{H}^*_{DR}$ be the sheaf of relatively closed holomorphic 1-forms on the total space of the Milnor bundle. We will need the following result:

**Proposition 1.41** ([Kul98] page 15)**.** *The pairing given by integrating the de Rham form representing the class in $\mathcal{H}^*_{DR}$ over a section of $\mathcal{H}$:*

$$\mathcal{H}^*_{DR} \times \mathcal{H} \to \mathcal{O}_{S'}$$

*is a perfect pairing.*

**Remark 1.42.** In fact, any holomorphic 1-form defined on the total space of the

Milnor bundle is relatively closed and so $\mathcal{H}^*_{DR} = \Omega^1(\mathcal{X})/\mathrm{d}\mathcal{O}_{\mathcal{X}}$. Indeed let $\omega \in \Omega^1(\mathcal{X})$ be such a form then $\mathrm{d}\omega|_{X_{(t,s)}} = 0$ because $\mathrm{d}\omega$ is a holomorphic 2-form that is restricted to a complex 1-dimensional curve $X_{(t,s)}$.

**Remark 1.43.** This proposition identifies $\underline{\mathrm{H}}^*_{DR}$ with the dual of $\underline{\mathrm{H}}$ (i.e $\underline{\mathrm{H}}^*$). We can represent a section $s \in \mathcal{H}^*$ evaluated on $\sigma \in \mathcal{H}$ by integrating a holomorphic 1-form $\omega$ over $\sigma$.

$$s(\sigma) = \int_\sigma \omega$$

Any holomorphic 1-form $\omega$ defines a section of $\underline{\mathrm{H}}$ by restricting the form to each fibre. We have seen in the previous remark that $\omega|_{X_{(t,s)}}$ is closed and therefore defines a cohomology class in $H^1(X_{(t,s)};\mathbb{C})$.

### 1.2.1 Trivialisations of the cohomology bundle

In this section we will show that the transition maps for the cohomology bundle are locally constant. This is significant because we can define a canonical flat connection on a vector bundle which has locally constant transition maps. From now on we will use cohomology groups with complex coefficients.

Over a contractible subset $U$ in the base space $S$ a deformation retraction of $U$ to a point $\lambda$ in $U$ lifts to a homotopy of fibres $X_U$ of $X_\lambda$ in the Milnor fibration. This homotopy becomes an isomorphism in cohomology:

$$i^*_\lambda : H^1(X_U) \longrightarrow H^1(X_\lambda) \ .$$

The cohomology $H^1(X_U)$ is a complex vector space of rank $\mu$ and so these maps can be used to define a trivialisation for the bundle over $U$.

To define trivialisation maps of the bundle over $U$ note that the preimage under the projection map of the bundle $\pi^{-1}(U)$ is a collection of cohomology classes $c_\lambda \in H^1(X_\lambda)$. The maps $(i^*_\lambda)^{-1}$ map each of these classes to a cohomology class in

$H^1(X_U)$:

$$\begin{array}{ccc} \coprod_{\lambda \in U} H^1(X_\lambda) & \longrightarrow & U \times H^1(U) \\ c_\lambda & \longmapsto & (\lambda, (i_\lambda^*)^{-1} c_\lambda) \end{array}$$

**Proposition 1.44.** *The transition maps for $\underline{H}$ are locally constant.*

*Proof.* Let $U_1, U_2$ be contractible subsets in the base space of the vector bundle $\Lambda \setminus D$ such that $U_1 \cap U_2$ is connected. Then we can find the transition map between respective trivialisations of $U_1, U_2$ over $U_1 \cap U_2$. Let $i_\lambda^* : H^1(X_\lambda) \to H^1(X_{U_1})$ and $j_\lambda^* : H^1(X_\lambda) \to H^1(X_{U_2})$ be the isomorphisms used in the construction of the trivialisation maps. The transition map is indicated in the diagram below:

$$\begin{array}{ccc} & & U_1 \cap U_2 \times H^1(X_{U_1}) \\ & \nearrow & \big\downarrow {\scriptstyle (\lambda,b) \mapsto (\lambda, g(\lambda,b))} \\ \coprod_{\lambda \in U_1 \cap U_2} H^1(X_\lambda) & & \\ & \searrow & \\ & & U_1 \cap U_2 \times H^1(X_{U_2}) \end{array}$$

where $g : U_1 \cap U_2 \times H^1(X_{U_1}) \to H^1(X_{U_2})$ is defined as follows:

$$g(\lambda, b) = (j_\lambda^*)^{-1}(i_\lambda^* b) .$$

We will show that this map is independent of $\lambda$. Indeed, we can construct isomorphisms $I^*, J^*$ in a similar way to $i^*, j^*$:

$$I^* : H^1(X_{U_1}) \to H^1(X_{U_1 \cap U_2})$$
$$J^* : H^1(X_{U_2}) \to H^1(X_{U_1 \cap U_2}).$$

Using the maps indicated in the diagram above it is possible to compute $g(p, b)$ and $g(q, b)$ for $p, q \in U_1 \cap U_2$ via the maps $I^*$ and $J^*$ which are independent of $p, q$:

$$g(p, b) = (j_p^*)^{-1}(i_p^* b) = (J^*)^{-1}(I^*(b)) = (j_q^*)^{-1}(i_q^* b) = g(q, b)$$

$$
\begin{array}{ccccc}
 & & H^1(X_{U_1}) & & \\
 & \swarrow^{i_p^*} & \downarrow{\scriptstyle I^*} & \searrow^{i_q^*} & \\
H^1(X_p) & \longleftarrow & H^1(X_{U_1 \cap U_2}) & \longrightarrow & H^1(X_q) \\
 & \nwarrow_{j_p^*} & \uparrow{\scriptstyle J^*} & \nearrow_{j_q^*} & \\
 & & H^1(X_{U_2}) & &
\end{array} .
$$

In summary the second component of the transition map in the bundle is independent of the first component and can be considered as a linear map $g : H^1(X_{U_1}) \to H^1(X_{U_2})$. □

## 1.3 Gauss-Manin connection

In this section we show how to construct a flat connection on a vector bundle where the transition maps are locally constant.

Let $E$ be a locally trivial vector bundle of rank $k$ with locally constant transition maps and let $\mathcal{E}$ be the sheaf of holomorphic sections of the bundle.

Let us describe a connection locally. Choose a point $b \in B$ and let $s_1 \dots s_k$ be $k$ linearly independent sections defined in some neighbourhood $U$ of $b$ coming from a trivialisation of $E|U$. We define a local flat connection on $E|U$ by setting $\nabla(s_i) = 0$ for all $i$. We can propagate this definition to obtain a global flat connection on the bundle due to the fact that the transition maps are locally constant. Indeed, let $V \subset B$ be another contractible set containing $b$ with trivialising sections $t_1, \dots, t_k$ then on $U \cap V$

$$t_i = \sum_{j=1}^{k} g_{ij} s_j$$

where $(g_{ij}) = G \in \mathrm{Gl}_k(\mathbb{C})$ is the matrix of the locally constant transition function between the two trivialisations. Then by the Leibniz rule for a connection:

$$\nabla(t_i) = \sum_{j=1}^{k} \mathrm{d}g_{ij} s_j + g_{ij} \nabla(s_j) = 0$$

so the definition of $\nabla$ is independent of a trivialisation and therefore $\nabla$ is well defined. We call such a connection the *Gauss-Manin* connection on $E$.

The holomorphic sections of the bundle are a $\mathcal{O}_B$–module generated by the horizontal sections:

$$\mathcal{E} = \ker \nabla \otimes_{\mathbb{C}_B} \mathcal{O}_B \simeq E \otimes_{\mathbb{C}} \mathcal{O}_B.$$

Indeed, we associate to each $e \in E$ a unique horizontal section $s \in \ker \nabla$ such that $s(\pi(e)) = e$ and conversely to each horizontal section $(\ker \nabla)_b$ at $b \in B$ we can associate the point $s(b) \in E$.

Turning our attention to the case of a locally trivial vector bundle $\underline{\mathrm{H}}$ with locally constant transition maps we deduce that $\underline{\mathrm{H}}$ has a Gauss-Manin connection on $\mathcal{H} = \mathcal{O}_{S'} \otimes_{\mathbb{C}'_S s} \underline{\mathrm{H}}^*$. To construct horizontal sections of this bundle over $U \subset S'$ we use the isomorphism given at the beginning of section 1.2.1:

$$i^* : H^1(X_U; \mathbb{C}) \to H^1(X_\lambda).$$

We choose $\sigma \in H^1(X_U; \mathbb{C})$ and define a section $h_\sigma(s) = i_s^*(\sigma)$.

**Theorem 1.45** (see [Kul98] page 9)**.** *Let $\omega$ be a holomorphic 1-form representing a section of $\underline{H}^*$ and $\sigma$ a horizontal section of $\underline{H}$ then we have the following equality:*

$$\frac{\partial}{\partial \lambda_i} \langle \omega, \sigma \rangle = \langle \nabla_{\frac{\partial}{\partial \lambda_i}} \omega, \sigma \rangle .$$

*Proof.* The space of sections of the bundle $\underline{\mathrm{H}}$ is generated by the horizontal sections as an $\mathcal{O}_{S'}$–module. The result will follow once we have checked equality on the horizontal sections because the pairing $\langle\, , \rangle$ is $\mathcal{O}_{S'}$–linear in both components. Indeed, for horizontal sections $\langle \omega, \sigma \rangle$ is constant and $\nabla_{\frac{\partial}{\partial \lambda_i}} \omega$ is zero. □

To compute the Gauss-Manin connection on an explicit form we will need to make use of the Gelfand–Leray form and Leray residues, which we outline now.

**Proposition 1.46** (see [AGZV88] page 215)**.** *Let $\omega$ be a holomorphic 1–form on the total space $Y'$ of the Milnor fibration. There exists a neighbourhood of $x_0 \in Y'$*

*and a 1–form $\phi$ on this neighbourhood for which:*

$$d\omega = dF_\lambda \wedge \phi.$$

*The form $\phi$ is thus unique when restricted to a fibre.*

*Proof.* Existence of $\phi$ is established by the following construction. Restrict $F_\lambda$ to a neighbourhood of $x_0$ on which it is a submersion then on this neighbourhood let $x := F_\lambda$ and $y$ can be coordinates on $Y'$. Using these coordinates we find that $\mathrm{d}\omega = h(x, y)\mathrm{d}F_\lambda \wedge \mathrm{d}y$ and so we define $\phi$ as:

$$\phi = h(x, y)\mathrm{d}y$$

which satisfies the equation of the proposition.

For the uniqueness of $\phi$ the set of forms which satisfy the equation of the proposition after restriction to a fibre differ by $\alpha \wedge \mathrm{d}x$ but $\mathrm{d}x$ vanishes when restricted to a fibre. □

**Definition 1.47.** The form $\phi$ restricted to a fibre is called the *Gelfand–Leray form* of $\mathrm{d}\omega$ and will be denoted $\frac{\mathrm{d}\omega}{\mathrm{d}F_\lambda}$

For a fixed parameter $\lambda_0 \in \Lambda'$ consider the sub-bundle ${}^{\lambda_0}\mathcal{Y}$ of the Milnor fibration $\mathcal{Y}$ (see remark 1.37). The projection map of the new bundle is the holomorphic function $F_{\lambda_0}$.

Applying the methods in [Kul98, page 16], take a 1-form $\omega_{\lambda_0}$ on the total space of the bundle ${}^{\lambda_0}Y$. The restriction $\omega_{|{}^{\lambda_0}Y_{t_0}}$ is represented by the Poincaré residue of the form $\frac{\mathrm{d}F_{\lambda_0} \wedge \omega}{F_{\lambda_0} - t}$:

$$\omega_{|{}^{\lambda_0}Y_{t_0}} = \mathrm{Res}_{{}^{\lambda_0}Y_{t_0}} \frac{\mathrm{d}F_{\lambda_0} \wedge \omega_{\lambda_0}}{F_{\lambda_0} - t}.$$

As in the definition of the Gelfand–Leray form the function $F_{\lambda_0}$ can be viewed as a local parameter for ${}^{\lambda_0}Y$, the Poincaré residue is then an extension to a complex manifold of the residue at a pole in complex analysis.

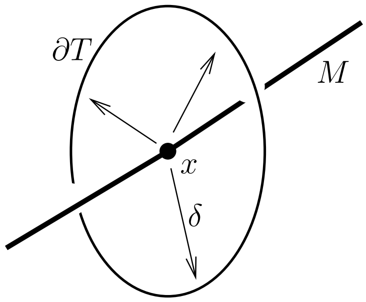



Figure 1.3: Illustrating the construction of the Leray coboundary operator.

**Definition 1.48.** Let $N$ be a complex manifold of dimension $n$ and $M$ a complex submanifold of $N$ of codimension 1. We define the *Leray coboundary operator* as a map:

$$\delta : H_{n-1}(M) \to H_n(N \setminus M)$$

by the following construction (illustrated in figure 1.3).

Let $T$ be a tubular neighbourhood of $M$ in $N$ considered as a locally trivial fibre bundle with base $M$ and fibre $T_x$ homeomorphic to a disc. We associate to each $(n-1)$-chain $\sigma_{n-1}$ in $M$ an $n$-chain

$$\delta\sigma_{n-1} = \bigcup_{x \in \mathrm{supp}(\sigma_{n-1})} \delta x$$

in $T \setminus M$ where $\delta x = \partial T_x$.

Alternatively we can define the Leray coboundary $\delta$ using the following commutative diagram (see [BKS86], p.641 ):

$$\begin{array}{ccc} H_{n-1}(M) & \xrightarrow{\delta} & H_n(N \setminus M) \\ D^P \uparrow \simeq & & \simeq \uparrow D^A \\ H^{n-1}(M) & \xrightarrow{\partial} & H^n(N, M) \end{array}$$

where $D^P$ and $D^A$ are the Poincaré and Alexander duality isomorphisms respectively and the map $\partial$ is the boundary homomorphism in the long exact sequence of cohomology of the pair $(N, M)$.

**Theorem 1.49** (Leray Residue Theorem, see [Kul98] page 16). *Let $\sigma(t,\lambda)$ be a horizontal section of $\underline{H}$ which is the homology vector bundle of the Milnor fibration $\mathcal{Y}'$ (see definition 1.40). Furthermore let $\omega$ be a holomorphic 1-form on the total space of $\mathcal{Y}'$ then:*

$$2\pi \mathrm{i} \int_{\sigma(\mathrm{t},\lambda)} \omega = \int_{\delta\sigma(\mathrm{t},\lambda)} \frac{\mathrm{dF}_\lambda \wedge \omega}{\mathrm{F}_\lambda - \mathrm{t}}.$$

**Theorem 1.50.** *Let $\nabla_{\frac{\partial}{\partial \lambda_i}} : \mathcal{H}^* \to \mathcal{H}^*$ be the covariant derivative of the Gauss-Manin connection with respect to the $\lambda_i$ coordinate direction and let $\omega$ be a holomorphic 1-form on the total space of the Milnor bundle (which by remark 1.43 defines a section of the cohomology bundle by restriction to each fibre) then:*

$$\nabla_{\frac{\partial}{\partial \lambda_i}} \omega = \frac{\partial \omega}{\partial \lambda_i} - \frac{\partial F_\lambda}{\partial \lambda_i} \frac{\mathrm{d}\omega}{\mathrm{d}F_\lambda}.$$

*Proof.* This result can be found in [AGZV88] (p.285) but we give a proof here with full details. By theorem 1.45 we can regard $\langle \nabla_{\frac{\partial}{\partial \lambda_i}} \omega, \sigma \rangle$ as the partial derivative $\frac{\partial}{\partial \lambda_i} \langle \omega, \sigma \rangle$. We find:

$$2\pi \mathrm{i} \frac{\partial}{\partial \lambda_i} \int_{\sigma(t,\lambda)} \omega = \frac{\partial}{\partial \lambda_i} \int_{\delta\sigma(t,\lambda)} \frac{\mathrm{d}F_\lambda \wedge \omega}{F_\lambda - t}$$

(by Leray residue theorem)

$$= \int_{\delta\sigma(t,\lambda)} \frac{\mathrm{d}F_\lambda \wedge \frac{\partial \omega}{\partial \lambda_i}}{F_\lambda - t} + \frac{\frac{\partial}{\partial \lambda_i}(\mathrm{d}F_\lambda) \wedge \omega}{F_\lambda - t} - \frac{\partial F_\lambda}{\partial \lambda_i} \frac{\mathrm{d}F_\lambda \wedge \omega}{(F_\lambda - t)^2}$$

(differentiating the integrand)

$$= \int_{\delta\sigma(t,\lambda)} \frac{\mathrm{d}F_\lambda \wedge \frac{\partial \omega}{\partial \lambda_i}}{F_\lambda - t} - \frac{\partial F_\lambda}{\partial \lambda_i} \frac{\mathrm{d}\omega}{F_\lambda - t}$$

(since $\mathrm{d}\left(\frac{\frac{\partial F_\lambda}{\partial \lambda_i} \omega}{F_\lambda - t}\right) = \frac{\mathrm{d}(\frac{\partial F_\lambda}{\partial \lambda_i}) \wedge \omega}{F_\lambda - t} + \frac{\partial F_\lambda}{\partial \lambda_i} \frac{\mathrm{d}\omega}{F_\lambda - t} - \frac{\partial F_\lambda}{\partial \lambda_i} \frac{\mathrm{d}F_\lambda \wedge \omega}{(F_\lambda - t)^2}$)

$$= \int_{\sigma(t,\lambda)} \frac{\partial \omega}{\partial \lambda_i} - \frac{\partial F_\lambda}{\partial \lambda_i} \frac{\mathrm{d}\omega}{\mathrm{d}F_\lambda}$$

(by the Leray residue theorem).

□

### 1.3.1 An example computation of the Gauss-Manin connection

Let $\underline{\mathrm{H}}^*_{A_4}$ be the cohomology bundle of the $A_4$ singularity with miniversal deformation:

$$F(x, y, \lambda_3, \lambda_2, \lambda_1, \lambda_0) = y^2 + x^5 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0.$$

Let $\omega$ be the section defined by the 1–form $y\mathrm{d}x$. We will compute $\nabla_{\frac{\partial}{\partial\lambda_2}}\omega$. By the formula above this is equal to the the Gelfand–Leray form $\frac{\mathrm{d}x\wedge\mathrm{d}y}{\mathrm{d}F}$. We need to find a 1-form $\nu = a\mathrm{d}x + b\mathrm{d}y$ satisfying:

$$\mathrm{d}x \wedge \mathrm{d}y = (a\mathrm{d}x + b\mathrm{d}y) \wedge (2y\mathrm{d}y + (5x^4 + 3\lambda_3 x^2 + 2\lambda_2 x + \lambda_1)\mathrm{d}x).$$

Such a 1-form is $\frac{1}{2y}\mathrm{d}x$ and therefore the covariant derivative of $\omega$ is:

$$\nabla_{\frac{\partial}{\partial\lambda_2}}\omega = \frac{x^2}{2y}\mathrm{d}x \wedge \mathrm{d}y\,.$$

### 1.3.2 The Saito Matrix

Using the notation from subsection 1.2 let $\mathcal{X} : X' \xrightarrow{\pi} \Lambda'$ be the central Milnor fibration of a miniversal deformation $F$, which we assume to be quasihomogeneous, of an isolated singularity $f$ with Milnor number $\mu$. The Milnor algebra $\mathcal{O}_{X_0,0}/J(f)$ is a $\mu$–dimensional $\mathbb{C}$–vector space with basis $\phi_1, \ldots, \phi_\mu$. The Malgrange preparation theorem shows that $\pi_*\left(\mathcal{O}_{X,0}/J(F)\right)$ is freely generated by $\phi_1, \ldots, \phi_\mu$ over $\mathcal{O}_{\Lambda,0}$. It follows that we can find functions $a_{ij} \in \mathcal{O}_{\Lambda,0}$ satisfying:

$$F\phi_j = \sum_{i=1}^{\mu} a_{ij}\phi_i \mod J(F). \tag{1.3.1}$$

By a theorem of K. Saito (see [Sai81] pages 777–778) the vector fields

$$\theta_j = \sum_{i=1}^{\mu} a_{ij}\frac{\partial}{\partial\lambda_i}$$

are a free basis for the $\mathcal{O}_{\Lambda,0}$–module $\mathrm{Der}(-\log D)$ of logarithmic vector fields:

$$\mathrm{Der}(-\log D) = \{\theta \in \Theta_0(\Lambda) : \theta \cdot F \in \mathrm{I}(h)\}$$

and these are the vector fields tangent to the discriminant. For explanations and proofs of these assertions see [Bru84] pages 562–566.

**Definition 1.51** (Saito Matrix)**.** Define $\chi$ to be the $\mu \times \mu$ matrix with entries $\chi_{ij} = a_{ij}$. This is known as the *Saito matrix* of $F$.

**Proposition 1.52.** *There exists a presentation for the $\mathcal{O}_{\Lambda,0}$–module $\pi_*\mathcal{O}_{\Sigma,0} := \pi_*\mathcal{O}_{X,0}/J(F)$ of the form:*

$$\mathcal{O}^{\mu}_{\Lambda,0} \xrightarrow{\chi} \mathcal{O}^{\mu}_{\Lambda,0} \xrightarrow{p} \pi_*\mathcal{O}_{\Sigma,0}$$

*where $p$ maps the $i$th basis vector $e_i$ of $\mathcal{O}^{\mu}_{\Lambda,0}$ to the generator $\phi_i$.*

*Proof.* The fact that the first syzygies of the module $\pi_*\mathcal{O}_{\Sigma,0}$ are given by the columns of the matrix $\chi$ can be deduced from equation 1.3.1. Indeed, let $\chi_i = (a_{1,i}, \ldots, a_{\mu,i})^t$ be a column of $\chi$. Since $F$ is quasihomogeneous we have $F \in J(F)$ and therefore that $p(\chi_i) = F\phi_i = 0$ in $\pi_*\mathcal{O}_{X,0}/J(F)$ (by equation 1.3.1). □

**Remark 1.53.** If the miniversal deformation is quasihomogeneous with respect to weights $(\mathrm{wt}\,\lambda_1, \ldots, \mathrm{wt}\,\lambda_\mu)$ then the Euler vector field:

$$\chi_E = \sum_{i=1}^{\mu} \mathrm{wt}\,\lambda_i \cdot \lambda_i \frac{\partial}{\partial \lambda_i}$$

is tangent to the discriminant and so the column of weights $E = (\mathrm{wt}\,\lambda_1, \ldots, \mathrm{wt}\,\lambda_\mu)^t$ is a syzygy of the module $\pi_*\mathcal{O}_{X,0}/J(F)$ (i.e $p(E) = 0$). We can use this information to find the other syzygies of the module. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
\mathcal{O}^{\mu}_{\Lambda,0} & \xrightarrow{\chi} & \mathcal{O}^{\mu}_{\Lambda,0} & \xrightarrow{p} & \pi_*\mathcal{O}_{\Sigma,0} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle L_{\phi_i}} & & \downarrow{\scriptstyle \cdot\phi_i} & & \\
\mathcal{O}^{\mu}_{\Lambda,0} & \xrightarrow{\chi} & \mathcal{O}^{\mu}_{\Lambda,0} & \xrightarrow{p} & \pi_*\mathcal{O}_{\Sigma,0} & \longrightarrow & 0.
\end{array}$$

The map defined by multiplication $\cdot\phi_i$ on $\pi_*\mathcal{O}_{\Sigma,0}$ induces a map $L_{\phi_i} : \mathcal{O}^{\mu}_{\Lambda,0} \to \mathcal{O}^{\mu}_{\Lambda,0}$. We get such a map for each basis element $\phi_i$ and they can be used to find other syzygies:

$$p(L_{\phi_i}(E)) = p(E) \cdot \phi_i = 0$$

and so we can construct the Saito matrix where the columns are given by $L_{\phi_i}E$

$$\chi = (L_{\phi_1}E = E, L_{\phi_2}E, \ldots, L_{\phi_\mu}E).$$

This method for computing the Saito matrix is implemented as a *Macaulay2* script for a miniversal deformation of the $E_8$ singularity in appendix C.3.2.

# Chapter 2

# The Intersection Form

The aim of this chapter is to describe a result of Givental and Varchenko in [VG82] which constructs a symplectic structure called the intersection form on the base space of a miniversal deformation of a plane curve singularity. The intersection form is induced from the intersection pairing of homology cycles on the smooth Milnor fibre.

We will show how the intersection form can be computed explicitly in the case of $A_{2k}$, $E_6$ and $E_8$. We will also prove a result about the form of the matrix of coefficients of the intersection form for quasihomogeneous singularities.

## 2.1 The intersection pairing on the Milnor fibre

In this section we will introduce the intersection pairing on an irreducible affine curve and show how it can be computed as a residue at its unique point at infinity.

Following [Lam86, §11, p.154–158] we define an intersection pairing in the middle cohomology of the Milnor fibre.

Let $M$ be an oriented, compact $n$ dimensional complex manifold with boundary $\partial M$ then for each $k$ there is a Poincaré duality isomorphism (see [Gre67], p.188)

$$D_k : H_k(M; \partial M; \mathbb{Z}) \to H^{2n-k}(M; \mathbb{Z})$$

and this can be used to define a pairing in the middle homology of $M$.

**Definition 2.1.** The *intersection pairing* on $M$ is a map

$$\mathrm{I} : H_n(M;\mathbb{Z}) \times H_n(M;\mathbb{Z}) \to \mathbb{Z}$$

which is $(-1)^n$-symmetric, defined by

$$(\sigma_1, \sigma_2) \mapsto \langle D_n(j_*\sigma_1), \sigma_2 \rangle$$

where the map $j_* : H_n(M;\mathbb{Z}) \to H_n(M, \partial M;\mathbb{Z})$ comes from the inclusion $j : M \to (M, \partial M)$ in the long exact sequence of homology of the pair $(M, \partial M)$. The pairing $\langle\ ,\ \rangle$ is the evaluation of the cohomology class $D_n(\sigma_1)$ on the homology class $\sigma_2$ (if we regard $H^n(M)$ as the dual $H_n(M)^*$).

**Remark 2.2** (see [Lam86], p.157)**.** Let us now consider $M$ as a $2n$ dimensional real manifold and let $a, b$ be oriented compact submanifolds of dimension $n$ that intersect transversely. The oriented intersection number $a{\cdot}b$ is equal to the intersection pairing $I([a],[b])$ where $[a],[b]$ are homology classes in $H_n(M)$ determined by their inclusion in $M$ and their orientations.

The intersection pairing defines a map

$$p : H_n(M;\mathbb{Z}) \to H^n(M;\mathbb{Z}) : \sigma \mapsto I(\sigma,\ ).$$

We now define a dual intersection pairing on the image of this map.

**Definition 2.3.** The *intersection pairing* in the middle cohomology of $M$ is defined on the image of $p$ as follows:

$$\mathrm{I}^\vee : \operatorname{Im} p \times \operatorname{Im} p \to \mathbb{Z} : (\omega_1, \omega_2) \mapsto \mathrm{I}\left(p^{-1}(\omega_1), p^{-1}(\omega_2)\right).$$

**Remark 2.4.** Suppose $M$ is the Milnor fibre of an irreducible plane curve singularity then the singularity has even Milnor number by Milnor's formula (see Theorem 1.30, page 17). The map $j_* : H_n(M) \to H_n(M, \partial M)$ is an isomorphism and so we deduce that I is nondegenerate. Since I is nondegenerate the map $p : H_1(M; \mathbb{Z}) \to H^1(M)$ is an isomorphism and so $\mathrm{I}^\vee$ is defined on the whole cohomology group $H^1(M; \mathbb{Z})$.

**Remark 2.5.** Let $\sigma_1, \ldots \sigma_k$ be generators of $H^n(M; \mathbb{Z})$ then define

$$\mathrm{I}^\vee_{\mathbb{C}}(a\sigma_i, b\sigma_j) = a\,\mathrm{I}^\vee(\sigma_i, b\sigma_j)$$

for $a, b \in \mathbb{C}$ and $0 \le i, j \le k$. In this way we extend the definition of $\mathrm{I}^\vee$ to a map:

$$\mathrm{I}^\vee_{\mathbb{C}} : H^n(M; \mathbb{C}) \times H^n(M; \mathbb{C}) \to \mathbb{C}\,.$$

By these remarks we can define an intersection pairing on each smooth fibre of the central Milnor fibration.

**Proposition 2.6.** *Let $\mathcal{X}$ be the central Milnor fibration of an irreducible singularity $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ (see definition 1.35) with Milnor fibre $X_\lambda$. Then the Milnor number is even and there exists an intersection pairing (constructed as $I^\vee$ on each fibre)*

$$\mathrm{I}^\vee_\lambda : H^1(\overline{X}_\lambda; \mathbb{C}) \times H^1(\overline{X}_\lambda; \mathbb{C}) \to \mathbb{C}$$

*on the closed Milnor fibre $\overline{X}_\lambda$ which is skew symmetric and nondegenerate.*

### 2.1.1 The intersection pairing on an affine curve

When the central Milnor fibration is defined by an irreducible quasihomogeneous singularity then the Milnor fibre is an affine curve (see remark 1.39, p. 22). In this section we will see how to compute the intersection pairing on such a curve at its unique point at infinity.

In what follows let $Y$ be an affine irreducible curve in $\mathbb{C}^2$ and $\overline{Y}$ be the corresponding projective curve in $\mathbb{P}^2$.

The affine curve $Y$ is a 1-dimensional complex manifold so it is natural to use holomorphic forms on $Y$. The cohomology group $H^1(Y; \mathbb{C})$ is isomorphic to the vector space of holomorphic 1-forms $\Omega^1_Y$ because $Y$ is a Stein manifold. In fact any manifold that can be embedded as a closed submanifold of $\mathbb{C}^N$ is a Stein manifold.

Indeed, $Y$ is Stein implies that $H^k(Y, F) = 0$ for any coherent analytic sheaf $F$ and for any $k > 0$. The sheaf $\Omega^1_Y$ of holomorphic $k$-forms on $Y$ is coherent so we deduce that the the complex:

$$\Omega^{\cdot}_Y : \Omega^0_Y \xrightarrow{d} \Omega^1_Y \xrightarrow{d} \Omega^2_Y = 0$$

is an acyclic resolution (note that $\Omega^2_Y = 0$ because $Y$ is 1-dimensional). Moreover $\Omega^{\cdot}_Y$ is a resolution of the constant sheaf $\mathbb{C}_Y$ so we deduce that:

$$H^1(Y, \mathbb{C}) = H^1(Y, \mathbb{C}_Y) = H^1(\Gamma(Y, \Omega^{\cdot}_Y)).$$

We conclude that we can represent a cohomology class $c \in H^1(Y, \mathbb{C})$ by a holomorphic 1-form $\omega$ which is necessarily closed because $Y$ is 1-dimensional.

**Remark 2.7.** The affine curve $Y$ is noncompact, so our definition of the intersection pairing needs to be modified. For noncompact manifolds the Poincaré–Lefschetz duality isomorphism is

$$D_k : H_n(X_\lambda; \mathbb{C}) \to H^{2n-k}_c(X_\lambda; \mathbb{C})$$

where $H^{2n-k}_c(X_\lambda; \mathbb{C})$ is the compact cohomology (see [Gre67],§26.6,p.164). Since $Y$ is a Stein manifold any cohomology class on $Y$ has a holomorphic representative that we will consider to be a holomorphic differential form (by the de Rham theorem).

Any cohomology class defined by a homomorphic form $\omega \in \Omega^1_{X_\lambda}$ has a representative $\omega_c$ which has compact support on $X_\lambda$ (such a representative is explicitly constructed in lemma 2.10 (p. 40) ). By Poincaré–Lefschetz duality $\omega_c$ is dual to an absolute homology class.

In this interpretation the intersection pairing becomes:

$$I^\vee(\omega, \eta) = I^\vee(\omega_c, \eta) = \int_{X_\lambda} \omega_c \wedge \eta$$

where $\omega, \eta \in \Omega_1(X_\lambda)$ (see [Ste],§2.3,p.520).

**Proposition 2.8.** *Let $Y := V(f)$ be 1–dimensional complex irreducible curve defined by a quasihomogeneous polynomial $f \in \mathbb{C}[x, y]$ then the projective curve $\bar{Y} \subset \mathbb{P}^2$ has a unique point at infinity.*

*Proof.* Assume $f$ has the form:

$$f = a_1 x^p y^q + \cdots + a_{n-1} y^t + a_n x^s$$

where $a_1, \ldots a_n \in \mathbb{C}$, $a_1 \neq 0$. Since we assume that $f$ is irreducible it follows that $t > s$. Indeed if $t = s$ then $f$ is homogeneous and as such factorises as a product of linear forms and so is reducible. We assign weight $t$ to $x$ and $s$ to $y$ so that $f$ has quasihomogeneous degree $s + t$.

The lowest degree terms in $x$ and $y$ must be unmixed because $Y$ is irreducible.

The proposition is equivalent to showing that the highest degree monomial in $f$ is unmixed in $x$ and $y$. Indeed, with this assumption, after homogenizing $f$ has the form (with relabeled coefficients):

$$\overline{f} = b_1 y^t + \cdots + b_n z^{t-s} x^s$$

and then curve $\overline{Y} = V(\overline{f})$ has the single point $[1 : 0 : 0]$ at infinity.

Assume for a contradiction that the highest degree monomial is $x^p y^q$ with $p, q > 0$. Since $f$ is quasihomogeneous we deduce from the highest order monomial that $st = tp + sq$. We assume that $p + q \geq t$ else $x^p y^q$ is not the highest degree monomial in $f$. If $p + q = t$ then we get a contradiction since we deduce that:

$$st = tp + qs$$

(by quasihomogeneity)

$$> s(p + q)$$

(since $t > s$)

$$= st$$

(by assumption $p + q = t$).

Now we consider the case where $p + q > t$, then as $s(p + q) < tp + sq$ we have a contradiction since:

$$s(p + q) < st$$

($sp + tq = st$ and $s < t$)

$$< s(p + q)$$

(by assumption $p + q > t$).

□

**Proposition 2.9.** *Let $Y$ and $f$ be as in the previous proposition, with the additional assumption that $f$ is a simple singularity. Now let $F_\lambda$ be a quasihomogeneous miniversal deformation of $f$, then $Y_\lambda$ has a singularity at infinity whenever $Y$ does.*

*Proof.* By the proof of the previous proposition we may assume (perhaps exchanging $x$ and $y$) that $f$ has the form:

$$f = a_1 y^t + \cdots + a_m x^a y^b + \cdots + a_n x^s$$

where $a_1 \neq 0$, and that $x$ and $y$ have weights such that the quasihomogeneous degree of $f$ is $st$. If $\bar{Y}$ has a singularity at infinity then $f$ does not contain a monomial $x^{t-1}$. Indeed, the homogenous form of $x^{t-1}$ is $zx^{t-1}$ and the partial derivative with respect to $z$ at the point at infinity $(1 : 0 : 0)$ does not vanish. We deduce therefore that $s \leq (t - 2)$.

The polynomial $F_\lambda(x, y) = f(x, y) + \sum_{i=1}^{\mu} \lambda_i \phi_i(x, y)$ is quasihomogeneous in $x, y$ and $\lambda$ so none of the monomials $\phi_i$ have the form $x^{t-1}$. Indeed, the quasihomogeneous degree of $x^{t-1}$ is $t(t - 1)$ and $t(t - 1) < st$ since $F_\lambda$ is quasihomogeneous with respect to the weights of $\lambda_i$ and these weights are positive (by a result of K. Saito stated in proposition 1.10). From the first part of the proof we know that $s < t-2$ and hence we deduce that

$$t(t-1) < st \leq t(t-2),$$

which is a contradiction. We conclude that $\bar{Y_\lambda}$ also has a singularity at infinity. □

Let $\omega$ be a closed holomorphic 1-form representing a cohomology class $[\omega] \in H^1(Y)$.

**Lemma 2.10.** *The cohomology class $[\omega]$ may be represented by a form with compact support on $Y$.*

*Proof.* Let $V$ be a contractible neighbourhood of the unique point at infinity on the curve $\bar{Y}$. Then since closed forms are exact on a contractible neighbourhood by the Poincaré lemma, the restriction $\omega_{|V}$ can be expressed as $\mathrm{d}\alpha$ for some holomorphic function $\alpha$ on $V$. By means of a bump function we will extend the domain of definition of $\alpha$ to the whole of $Y$. Let $U$ be a neighbourhood of infinity contained in $V$ and define the function:

$$b(z) = \begin{cases} 1 & \text{when } z \in U \\ 0 < b(z) < 1 & \text{when } z \in V \setminus U \\ 0 & \text{when } z \notin V \end{cases}.$$

Then $\omega - d(b\alpha)$ is a form with compact support belonging to the same cohomology class as $\omega$. □

### 2.1.2 Computation of the intersection pairing

We may now compute the intersection pairing of two cohomology classes $[\omega_1], [\omega_2]$ on the curve $Y$. Again, since $Y$ is a Stein manifold we may represent the classes $[\omega_1]$ and $[\omega_2]$ by closed holomorphic 1-forms. By lemma 2.10 we may choose a representative of $[\omega_1]$ with compact support on $Y$ and compute the intersection pairing (defined in this case in remark 2.7).

$$I([\omega_1],[\omega_2]) = \int_Y (\omega_1 - d(b\alpha_1)) \wedge \omega_2$$

(by lemma 2.10)

$$= -\int_Y d(b\alpha_1) \wedge \omega_2$$

($\omega_1$ and $\omega_2$ are holomorphic forms so $\omega_1 \wedge \omega_2 = 0$ on the complex 1-manifold Y)

$$= -\int_{V \setminus U} d(b\alpha_1) \wedge \omega_2$$

(integrand is supported on $V \setminus U$ (where $U, V$ are defined in the definition of the bump function $b$) because $b = 1$ on $U$ therefore $\mathrm{d}(b\alpha_1)$ is holomorphic, hence $\omega_2 \wedge \mathrm{d}(b\alpha_1) = 0$)

$$= -\int_{V \setminus U} d(b\alpha_1 \omega_2)$$

($\omega_2$ is closed)

$$= -\int_{\partial(V \setminus U)} b\alpha_1 \omega_2$$

(Stokes's theorem)

$$= \int_{\partial U} \alpha_1 \omega_2$$

($b$ is 0 on $\partial V$, sign change due to orientation of $\partial$U)

$$= 2\pi i \, \mathrm{Res}_\infty(\alpha_1 \omega_2)$$

(Cauchy integral theorem).

The form $\alpha\omega_2$ is a holomorphic 1–form on the curve $Y$ but in general it is a meromorphic form on the projective curve $\bar{Y_\lambda}$, defined by the polynomial $F_\lambda$. The computation above shows that the residue of this meromorphic form is equal to the intersection pairing of the corresponding classes.

In section 2.3.1 we will see how to compute these residues explicitly using a parametrisation of the curve at its unique point at infinity.

## 2.2 The period map and the intersection form

We have seen that the Milnor fibre of an isolated singularity $f : \mathbb{C}^2 \to \mathbb{C}$ carries an intersection pairing. The aim of this section is to show that, when nondegenerate, this pairing can be pulled back, using a suitable section of the cohomology bundle, to a symplectic form on the base space $\mathbb{C}^\mu$. The symplectic form contains information about the strata in the discriminant over which the fibres have particular types of singular points. In particular we will show that the intersection form can be used to find equations for the $\delta$–constant strata $D(k)$ (see definition 1.23, p. 15).

We begin by describing the type of sections of the cohomology bundle we will require. We can construct global sections of the cohomology bundle by restricting a de Rham cohomology class on the total space of the Milnor bundle to each fibre of the bundle (see proposition 1.41). Since the fibres are Stein manifolds we can choose holomorphic representatives (see section 2.1.1).

Let us consider the case where we consider the Milnor fibre of a germ of a holomorphic function $f : \mathbb{C}^2 \to \mathbb{C}$ which we assume to be irreducible with isolated singularity at the origin (and consequently even Milnor number $\mu$). Then both the tangent space at a point $\lambda$ in the base and the corresponding fibre in the cohomology bundle $\underline{\mathrm{H}}^*$ over $\lambda$ are vector spaces of rank $\mu$. On each fibre of $\underline{\mathrm{H}}^*$ there is a nondegenerate intersection pairing (by proposition 2.6).

The reference for this section is [VG82].

**Definition 2.11** (Period Map)**.** Consider a holomorphic 1–form, $\omega$ on the total space of the Milnor bundle, we call the section of $\underline{\mathrm{H}}^*$ defined by this form the *period map* of $\omega$:

$$P_\omega(\lambda) = [\omega_{|X_\lambda}].$$

The covariant Gauss-Manin derivative of a period map at a point $\lambda$ in the base is a map:

$$\nabla P_\omega(\lambda) : T_\lambda B \to H^1(X_\lambda; \mathbb{C}) \simeq \mathbb{C}^\mu\,.$$

We will be interested in sections of the cohomology bundle where this map is an isomorphism.

**Definition 2.12** (Nondegenerate Period Map)**.** A period map $P_\omega$ is called *nondegenerate* at $\lambda \in \Lambda$ if $\nabla P_\omega(\lambda)$ is an isomorphism.

**Remark 2.13.** Let $\gamma_1, \dots, \gamma_\mu$ be flat sections of the homology bundle that give a basis of each fibre. The period map $P_\omega$ is nondegenerate if the determinant of the matrix

$$J = \begin{pmatrix} \nabla_{\frac{\partial}{\partial \lambda_1}} P_\omega(\gamma_1) & \cdots & \nabla_{\frac{\partial}{\partial \lambda_1}} P_\omega(\gamma_\mu) \\ \vdots & & \vdots \\ \nabla_{\frac{\partial}{\partial \lambda_\mu}} P_\omega(\gamma_1) & \cdots & \nabla_{\frac{\partial}{\partial \lambda_\mu}} P_\omega(\gamma_\mu) \end{pmatrix}$$

does not vanish outside the discriminant $D$ sufficiently near the origin in $\Lambda$. The columns of the matrix form a basis of the fibre under the identification of the fibre $H^1(X_\lambda; \mathbb{C})$ with $\mathbb{C}^\mu$ given by integration:

$$\sigma \mapsto \left( \int_{\gamma_1(\lambda)} \sigma, \dots, \int_{\gamma_\mu} \sigma \right) \in \mathbb{C}^\mu.$$

**Definition 2.14.** The map $P_\omega$ is called *infinitesimally nondegenerate* if when restricted to the $\lambda_\mu$-axis through the origin in $\Lambda$ (the "free" parameter of the miniversal deformation) the determinant of the matrix $J$ is nonzero as $\lambda_\mu \to 0$.

**Remark 2.15.** It is shown in [VG82, Theorem 1], that if $P_\omega$ is infinitesimally nondegenerate then it is nondegenerate.

**Proposition 2.16.** *If $f$ is quasihomogeneous then the period map $P_\omega$ defined by $\omega = y\,dx$ is nondegenerate.*

*Proof.* Recall that a miniversal deformation of $f$ is given by:

$$F(x, \lambda) = f(x) + \sum_{i=1}^{\mu} \phi_i \lambda_i$$

where $\phi_i$ are monomials forming a basis of the Milnor algebra $\mathcal{O}_{\mathbb{C}^2,0}/J_f$.

The section $\nabla_{\frac{\partial}{\partial \lambda_i}} P_\omega$ is represented by the class $\phi_i \frac{\mathrm{d}x \wedge \mathrm{d}y}{\mathrm{d}F_\lambda}$ (by theorem 1.50) which is equal to $\phi_i \frac{\mathrm{d}x \wedge \mathrm{d}y}{\mathrm{d}f}$ after restriction to the $\lambda_\mu$ axis.

The form can be decomposed in a series:

$$\phi_i \frac{\mathrm{d}x \wedge \mathrm{d}y}{\mathrm{d}f} = \sum_{\alpha} \lambda_\mu^\alpha \log(\lambda_\mu) A_\alpha(\lambda_\mu) \tag{2.2.1}$$

where $\alpha$ are nonzero rational numbers and $A_\alpha$ are flat sections of the cohomology bundle. The minimal $\alpha$ such that $A_\alpha$ is nonzero in the expansion of the form $\phi_i \frac{\mathrm{d}x\wedge\mathrm{d}y}{\mathrm{d}f}$ is called the order and is denoted by $\alpha(\phi_i \frac{\mathrm{d}x\wedge\mathrm{d}y}{\mathrm{d}f})$.

In the case of a quasihomogeneous singularity $f$ with weights $(u, v)$ so that $\operatorname{qdeg} f = 1$ we have that

$$\alpha\left(\frac{x^{m_1} y^{m_2} \mathrm{d}x \wedge \mathrm{d}y}{\mathrm{d}f}\right) = (m_1 + 1)u + (m_2 + 1)v - 1$$

and the orders $\alpha\left(\phi_i \frac{\mathrm{d}x\wedge\mathrm{d}y}{\mathrm{d}f}\right)$ are equal to the spectral numbers of $f$ (see [AGZV88], §13.3, p.380). It is known that for a germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with isolated singularity the sum of the spectral numbers is $\mu(n/2 - 1)$. In our case $n$ is equal to 2 and so the sum is equal to 0.

Each monomial of the determinant of $J$ contains a factor from each row of the matrix, and so the order of the matrix along the $\lambda_\mu$ axis is equal to the sum of the spectral numbers. Hence the order of $J$ along the $\lambda_\mu$ axis is zero and we conclude that $P_\omega$ is infinitesimally nondegenerate and hence nondegenerate. □

We will now use a nondegenerate period map to pull back the intersection pairing $\mathrm{I}_\lambda^\vee$ to the base of the cohomology bundle.

**Definition 2.17** (Intersection form)**.** Let $\mathrm{I}_\lambda^\vee$ be the intersection pairing on the cohomology of the Milnor fibre $X_\lambda$ then we define the *intersection form*, $\Phi$, a 2–form on $\Lambda \setminus D$:

$$\Phi = P_\omega^* \, \mathrm{I}_\lambda^\vee$$

which we can compute on a basis of the tangent space to the parameter space:

$$\Phi_\lambda\left(\frac{\partial}{\partial\lambda_i}, \frac{\partial}{\partial\lambda_j}\right) = \mathrm{I}_\lambda^\vee\left(\nabla_{\frac{\partial}{\partial\lambda_i}} P_\omega(\lambda), \nabla_{\frac{\partial}{\partial\lambda_j}} P_\omega(\lambda)\right).$$

The intersection form in the case where $f$ is quasihomogeneous is unique up to a diffeomorphism of the base space $\Lambda$ that preserves the discriminant $D$ (see [VG82] Theorem 5).

**Theorem 2.18** (see [VG82] Theorem 4, page 85)**.** *If $P_\omega$ is nondegenerate then the intersection form $\Phi$ extends holomorphically to $T^*\Lambda$ to a symplectic form on $\Lambda$.*

**Remark 2.19.** We will provide an alternative proof of this for the $A_{2k}$ singularities in theorem 2.36 by explicitly calculating the form $\Phi$.

Consider the stratum $D_0 \subset D$ of $\lambda$ such that the fibre $X_\lambda$ has exactly $\frac{\mu}{2}$ nondegenerate critical points.

**Theorem 2.20** (see [VG82],Theorem 9,p.87)**.** *The stratum $D_0$ is a Lagrangian subvariety of the symplectic space* $(\Lambda, \Phi)$*.*

**Remark 2.21.** In the case where $f$ is an irreducible simple singularity $2\delta = \mu$ (by Milnor's formula see Theorem 1.30, page 17) and therefore the stratum $D_0$ is equal to the $\delta$–constant stratum $D(\delta)$.

## 2.3 Computation of the intersection form

We can compute coefficients of the intersection form on the Milnor fibre using the formulas for the intersection pairing in 2.1.2 (see [VA93], p.101).

In the coordinates $\lambda_1, \dots, \lambda_\mu$ of the base space $\mathbb{C}^\mu \setminus D$ the intersection form has the following form:

$$\Phi(\lambda) = \sum_{1 \le i < j \le \mu} g_{ij} \, \mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j$$

where $g_{ij} = \mathrm{I}^\vee_\lambda \left( \nabla_{\frac{\partial}{\partial \lambda_i}} P_\omega(\lambda), \nabla_{\frac{\partial}{\partial \lambda_j}} P_\omega(\lambda) \right)$ and where $P_\omega$ is a nondegenerate period map. The form $\omega = x \, \mathrm{d}y$ will produce a nondegenerate period map (by proposition 2.16).

### 2.3.1 Calculations for quasihomogeneous simple singularities

Using the results above we will perform some calculations for the case where the germ $f$ is quasihomogeneous. Let $F$ be a quasihomogeneous miniversal deformation of an irreducible simple singularity with Milnor number $\mu$ (see theorem 1.5, p. 7). We know from proposition 2.9, p. 39 that $F$ has the following form:

$$F = x^\alpha + \sum_{\substack{0 \le n < \alpha \\ 0 \le m < \beta \\ n\beta + m\alpha = \alpha\beta \\ n+m<\alpha}} (c_{nm} x^n y^m) - y^\beta + \sum_{i=1}^{\mu} \phi_i(x,y)\lambda_i$$

where $\alpha > \beta$, $c_{nm} \in \mathbb{C}$ and $F$ is quasihomogeneous with weights $(\beta, \alpha, w_1, \ldots, w_\mu)$ and degree $\alpha\beta$. We know also that $\operatorname{qdeg} \phi_i < \alpha\beta$ because the weights of $\lambda_i$ are positive (a result of K. Saito see Proposition 1.10, p9).

To perform calculations of the intersection form in this case we need to find a parametrisation of each fibre of the relative Milnor bundle $X_\lambda$ in a neighbourhood of its unique point at infinity.

**Proposition 2.22** (see [GM07], Corollary 3.8, Page 171)**.** *Let $X_\lambda$ be a smooth fibre of the relative Milnor bundle associated to $F_\lambda$. There exists a neighbourhood of infinity $U_\lambda$ on $X_\lambda$ such that there is a parametrisation $n_\lambda : \mathbb{C}^* \to U_\lambda$ of the form:*

$$n_\lambda(t) = (x(t,\lambda), t^{-\alpha})$$

*where $x(t)$ is a power series $x(t) = \sum_{n=-\beta}^{\infty} a_n(\lambda) t^n$.*

*Proof.* The map $n_\lambda$ can be found in the reference with the modification that $n_\lambda$ parametrises a neighbourhood of infinity on the curve.

It remains to show that the lowest power of $t$ that appears in $x(t)$ is $t^{-\beta}$.

Assume that $x(t) = \sum_{n=-k}^{\infty} a_n(\lambda) t^n$ for $k > \beta$ then $n^*(F) \neq 0$ since $n^*(F)$ contains the monomial $t^{k\alpha}$ (from $n^*(x^\alpha)$). This monomial, however, cannot be found in the pullback of the other terms of $F$ and so is not cancelled out. Indeed, $n^*(y^\beta) = t^{-\alpha\beta} \neq t^{-k\alpha}$ because $\alpha\beta < k\alpha$. Also $n^*(c_{nm}x^n y^m)$ does not contain the monomial $t^{-k\alpha}$ because we assume that the order of $F$ is $\alpha$ and therefore $n + m < \alpha$

which implies $kn + \beta m < \alpha k$. Thus the lowest order monomial in $n^*(c_{nm}x^n y^m)$ is $t^{-kn-\beta m} \neq t^{-k\alpha}$.

A similar argument applies to $n^*(\phi_i(x,y)\lambda_i)$ as $\phi_i(x,y)$ has degree less than $\alpha$. $\square$

Let $n : \mathbb{C}^* \times \mathbb{C}^\mu \to \mathbb{C}^2 \times \mathbb{C}^\mu$ be the map defined by:

$$n(t, \lambda) = (n_\lambda(t), \lambda)$$

then the following is true:

**Lemma 2.23.** *The map $n$ is quasihomogeneous with respect to weights $(-1, w_1, \ldots, w_\mu)$ and degrees $(\beta, \alpha, w_1, \ldots, w_\mu)$.*

*Proof.* Apply the $\mathbb{C}^*$–action to $n(t,\lambda) \in V(F)$ for some $a \in \mathbb{C}^*$ (writing $a^w\lambda$ for $(a^{w_1}\lambda_1, \ldots a^{w_\mu}\lambda_\mu)$). We also know from proposition 1.7 above that $a \cdot n(t,\lambda) \in V(F)$. In particular, for $a$ sufficiently near 1, $a \cdot n(t,\lambda)$ is in the image of $n$, and so there exists $(T,l) \in \mathbb{C}^* \times \mathbb{C}^\mu$ such that $n(T,l) = a \cdot n(t,\lambda) = (a^\beta x(t,\lambda), a^\alpha t^{-\alpha}, a^w\lambda)$. By considering the second and third components on the right hand side of this equation we conclude that $T = a^{-1}t$ and $l = a^w\lambda$ and so $x(a^{-1}t, a^w\lambda) = a^\beta x(t,\lambda)$.

We have shown that

$$n(a^{-1}t, a^w) = (a^\beta x(t,\lambda), a^\alpha t^{-\alpha}, a^w\lambda)$$

and thus that $n$ is quasihomogeneous with respect to the specified weights and degrees. $\square$

**Corollary 2.24.** *Let $x(t,\lambda) = \sum_{n=-\beta}^{\infty} a_n(\lambda)t^n$ be the power series above, then* $\operatorname{qdeg} a_n(\lambda) \geq 0$.

*Proof.* Since $\operatorname{qdeg} x(t,\lambda) = \beta$ we deduce that $\operatorname{qdeg} a_n(\lambda) = \beta - \operatorname{qdeg}(t^n) = \beta + n$ and so $\operatorname{qdeg} a_n(\lambda) \geq 0$ since $n \geq -\beta$. $\square$

**Remark 2.25.** A way of expressing this result is to note that $\operatorname{qdeg} a_n(\lambda)$ lies in the span $\mathbb{Z}_{>0}\langle w_1, \ldots, w_\mu\rangle$. In particular, since the weights of $\lambda_i$ are positive this means that $a_n(\lambda)$ contains only non-negative powers of $\lambda_i$.

We can now compute the weight of the Gauss-Manin derivative of a section of the cohomology bundle. Suppose $\omega \in \Omega^1_{\mathbb{C}^2\times\mathbb{C}^\mu}$ defines a section $\lambda \mapsto \omega_{|X_\lambda}$ of the cohomology bundle (see section 1.3) then by theorem 1.50:

$$\nabla_{\frac{\partial}{\partial\lambda_i}}\omega_{|X_\lambda} = \phi_i \frac{\mathrm{d}\omega_{|X_\lambda}}{\mathrm{d}F_\lambda}.$$

From the definition of the Gelfand–Leray form (see theorem 1.49, p. 30) we can compute $\operatorname{qdeg}\frac{\mathrm{d}\omega_{|X_\lambda}}{\mathrm{d}F_\lambda} = \operatorname{qdeg}\omega_{|X_\lambda} - \operatorname{qdeg}F$, and since $\operatorname{qdeg}\phi_i = \operatorname{qdeg}F - \operatorname{qdeg}\lambda_i$ we have:

**Lemma 2.26.**

$$\operatorname{qdeg}\nabla_{\frac{\partial}{\partial\lambda_i}}\omega_{|X_\lambda} = \operatorname{qdeg}\omega_{|X_\lambda} - \operatorname{qdeg}\lambda_i\,.$$

□

### 2.3.2 Calculation of the intersection form for singularities of type $A_{2k}$

We will now turn our attention to the specific case of a germ $f$ with critical point of type $A_{2k}$. The miniversal deformation of $F$ has the form:

$$F(x, y, \lambda_1, \dots, \lambda_{2k}) = x^{2k+1} - y^2 + \sum_{i=1}^{2k} x^{2k-i}\lambda_i$$

which is quasihomogeneous with weights $(2, 2k+1, 4, \dots, 2+2i, \dots, 4k+2)$ and degree $4k+2$. The normalisation map is given by:

$$n(t, \lambda) = (x(t, \lambda), t^{-2k-1}, \lambda_1, \dots, \lambda_{2k})$$

where $x(t, \lambda) = t^{-2} + \sum_{n=-1}^{\infty} a_n t^n$.

**Remark 2.27.** Observe that for nonzero $a_n(\lambda)$, $\operatorname{qdeg} a_n(\lambda)$ lies in the span $\mathbb{Z}_{>0}\langle 4, \dots, 2+2i, \dots, 4k+2\rangle$ (by the remark after corollary 2.24). From this we can deduce that the coefficients $a_{2m-1}(\lambda)$ are equal to 0 for $m \geq 0$ because $\operatorname{qdeg} n^*x = 2$ (by lemma

2.23) which implies that $\operatorname{qdeg} a_{2m-1}(\lambda) = 1 + 2m$ which is odd and so does not lie in $\mathbb{Z}_{>0}\langle 4, \ldots, 2+2i, \ldots, 4k+2\rangle$.

Furthermore, $a_0(\lambda) = 0$ because if it was not then $\operatorname{qdeg} a_0(\lambda) = 2$ which does not lie in $\mathbb{Z}_{>0}\langle 4, \ldots, 2+2i, \ldots, 4k+2\rangle$.

We will now describe how to compute the intersection form using the method in section 2.1.2.

Let $\omega = x\mathrm{d}y$ and recall (see section 2.2) that the period map induced from $\omega$ can be used to pull back the intersection pairing on the cohomology bundle to its base. The result is a 2–form on the base which we can write as:

$$\Phi(\lambda) = \sum_{1 \le i < j \le \mu} g_{ij}(\lambda)\mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j$$

where $\lambda \in \mathbb{C}^\mu \setminus \Sigma$ and the coefficients are computed using the intersection pairing on the Milnor fibre

$$g_{ij}(\lambda) = \langle \nabla_{\frac{\partial}{\partial\lambda_i}} \omega_{|X_\lambda}, \nabla_{\frac{\partial}{\partial\lambda_j}} \omega_{|X_\lambda} \rangle \,.$$

It will also be convenient to write the coefficients of $\Phi$ as a $2k \times 2k$ skew-symmetric matrix $G$, with entries $g_{ij}$.

Recall the procedure in section 2.1.2 (p. 40) to compute the coefficients $g_{ij}$. We use the Poincaré lemma to find a holomorphic function $\alpha$ such that $\mathrm{d}\alpha = \nabla_{\frac{\partial}{\partial\lambda_i}} \omega_{|X_\lambda}$ on a sufficiently small neighbourhood of the unique point at infinity on $X_\lambda$. $g_{ij}$ can be computed as follows:

$$g_{ij} = \operatorname{Res}_\infty \left( \alpha \nabla_{\frac{\partial}{\partial\lambda_j}} \omega_{|X_\lambda} \right) .$$

We will construct $\alpha$ explicitly using the parametrisation $n_\lambda$ we have found for $X_\lambda$ at infinity. Denote by $\omega_i$ the Gauss–Manin derivative $\nabla_{\frac{\partial}{\partial\lambda_i}} \omega_{|X_\lambda}$ which by theorem 1.50 (p. 30) is equal to:

$$\omega_i = \frac{x^{2k-i}}{2y}\mathrm{d}x \,.$$

**Lemma 2.28.** *The coefficient $b_{-1}$ in the expansion of $n_\lambda^*(\omega_i) = (\sum_{n=-k}^{\infty} b_n t^n)dt$ is* *equal to zero.*

*Proof.* Since $n_\lambda^*(\omega_i) = \frac{1}{2}t^{2k+1}n_\lambda^*(x^{2k-i}\mathrm{d}x)$ the coefficient $b_{-1}$ is equal to the coefficient of $t^{-2k-2}$ in $n_\lambda^*(x^{2k-i}\mathrm{d}x)$.

Recall that the coefficients of odd powers of $t$ in $n_\lambda^* x$ are all zero (by remark 2.27) which implies that the coefficients of even powers of $t$ in $n_\lambda^*(\mathrm{d}x)$ are zero. From this we deduce that the coefficients of even powers of $t$ in $n_\lambda^*(x^{2k-i}\mathrm{d}x)$ are zero and so in particular $b_{-1}$ is equal to zero. □

**Definition 2.29.** Define a holomorphic function $\alpha_i$ with domain equal to the image of the parametrisation $n_\lambda$ as follows:

$$n_\lambda^*(\alpha_i) = \sum_{\substack{n=-k \\ n\neq -1}}^{\infty} \frac{b_n}{n+1} t^{n+1}\,.$$

This function has the property that $\mathrm{d}n_\lambda^*(\alpha_i) = n_\lambda^*(\omega_i)$. This can be verified by taking the exterior derivative of the expansion of $n_\lambda^*(\alpha_i)$ while noting that $b_{-1} = 0$ by the previous lemma. Thus the function can be used in the computation of the intersection pairing. Using the formula in section 2.1.2 and the fact that the residue is unchanged when computed using the parametrisation we have:

$$g_{ij} = \langle \omega_i, \omega_j \rangle = \mathrm{Res}_\infty\, n_\lambda^*(\alpha_i \omega_j) \tag{2.3.1}$$

and

$$\mathrm{qdeg}\, n^*\omega_i = 2k + 1 - 2i \tag{2.3.2}$$

$$\mathrm{qdeg}(n^*(\alpha_i\omega_j) = 2\,\mathrm{qdeg}(\omega) - \mathrm{qdeg}\,\lambda_i - \mathrm{qdeg}\,\lambda_j \tag{2.3.3}$$

by lemma 2.26.

**Lemma 2.30.** *Let $\omega$ be the holomorphic 1-form defining the section of the cohomology bundle defining the intersection form $\Phi$. Then $\Phi$ satisfies $\mathrm{qdeg}\,\Phi = 2\,\mathrm{qdeg}(\omega)$.*

*Proof.* From the calculations above, each coefficient of $\Phi$ satisfies $\mathrm{qdeg}\, g_{ij} = 2\,\mathrm{qdeg}(\omega) -$

$\operatorname{qdeg}\lambda_i - \operatorname{qdeg}\lambda_j$ and therefore $\operatorname{qdeg}(g_{ij}\mathrm{d}\lambda_i \wedge \mathrm{d}\lambda_j) = 2\operatorname{qdeg}(\omega)$. □

**Lemma 2.31.** *If* $\operatorname{qdeg}\lambda_i + \operatorname{qdeg}\lambda_j > 2\operatorname{qdeg}(\omega)$ *then*

$$\operatorname{Res}_\infty(\alpha_i\omega_j) = 0$$

*Proof.* Consider the expansion:

$$n^*(\alpha_i\omega_j) = \cdots + \operatorname{Res}_\infty(\alpha_i\omega_j)t^{-1}\mathrm{d}t + \ldots;$$

since $\operatorname{qdeg} t^{-1}\mathrm{d}t = 0$ we infer that if $\operatorname{Res}_\infty(\alpha_i\omega_j)$ is nonzero then $\operatorname{qdeg}\operatorname{Res}_\infty(\alpha_i\omega_j) < 0$ (by the formula for $\operatorname{qdeg}(n^*(\alpha_i\omega_j))$.

Now $\operatorname{Res}_\infty(\alpha_i\omega_j)$ is some power series of the coefficients $a_n$ of $n^*(x)$. All the powers of the $a_n$ are positive because the formula for $\omega_i$ contains only positive powers of $x$. From remark 2.27 we know that $a_n$ has only positive weights, so $\operatorname{qdeg}\operatorname{Res}_\infty(\alpha_i\omega_j) > 0$. We conclude that $\operatorname{Res}_\infty(\alpha_i\omega_j) = 0$. □

**Remark 2.32.** We can now deduce that the entries below the anti–diagonal in the matrix $G$ are zero. Indeed, when $i + j > 2k + 1$, $\operatorname{qdeg}\lambda_i + \operatorname{qdeg}\lambda_j = 4 + 2(i + j) > 4k + 6 = 2\operatorname{qdeg}(\omega)$ and so the entries $g_{ij}$ are equal to 0 when $i + j > 2k + 1$ by the previous lemma.

**Remark 2.33.** By a similar argument the entries $g_{ij}$ such that $i + j = 2k$ are also zero. Indeed, if $i + j = 2k$ then if $g_{ij}$ is nonzero it must satisfy $\operatorname{qdeg} g_{ij} = 2(2k + 3) - 2(2k) - 4 = 2$. However the lowest weight of the parameters $\lambda_i$ is 4, so we conclude that $g_{ij} = 0$.

We can also calculate the entries on the anti–diagonal of $G$ ($g_{ij} : i + j = 2k + 1$) explicitly using the following lemma.

**Lemma 2.34.**

$$g_{i,2k+1-i} = \frac{1}{2k + 1 - 2i}$$

*Proof.* Since $\operatorname{qdeg}\lambda_i + \operatorname{qdeg}\lambda_{2k+1-i} = 2\operatorname{qdeg}(\omega)$ we have that $\operatorname{qdeg} n^*(\omega_i\alpha_{2k+1-i}) = 0$ (by equation 2.3.3 above). Consider the expansion:

$$n^*(\omega_i\alpha_{2k+1-i}) = \cdots + \operatorname{Res}_\infty(\omega_i\alpha_{2k+1-i})t^{-1}\mathrm{d}t + \ldots;$$

then $\operatorname{qdeg}\operatorname{Res}_\infty(n^*(\omega_i\alpha_{2k+1-i})) = 0$ because $\operatorname{qdeg}(t^{-1}\mathrm{d}t) = 0$. From this we deduce that $\operatorname{Res}_\infty(n^*(\omega_i\alpha_{2k+1-i}))$ is equal to the product of the degree 0 coefficients of $n^*(\omega_i)$ and $n^*(\alpha_{2k+1-i})$. Consider the expansions:

$$n^*(\omega_j) = \frac{1}{2}t^{2k+1}n^*(x^{2k-i}\mathrm{d}x) = (-t^{-2k-2+2j} + \ldots)\mathrm{d}t$$
$$n^*(\alpha_j) = -\frac{1}{-2k-1+2j}t^{-2k-1+2j} + \ldots\ .$$

By equation 2.3.2, $\operatorname{qdeg} n^*(\omega_j) = \operatorname{qdeg} n^*(\alpha_j) = 2k+1-2j$ and since the weight of $t$ is $-1$ the degree 0 coefficients are $-1$ and $-\frac{1}{-2k-1+2j}$ respectively. We conclude that:

$$g_{i,2k+1-i} = \operatorname{Res}_\infty(\omega_i\alpha_{2k+1-i}) = \frac{1}{2i-2k-1}.$$

□

**Example 2.35.** In the case of $A_6$ the matrix $G$ has the following form:

$$\begin{pmatrix} 0 & g_{12} & g_{13} & g_{14} & 0 & -\frac{1}{5} \\ -g_{12} & 0 & g_{23} & 0 & -\frac{1}{3} & 0 \\ -g_{13} & -g_{23} & 0 & -1 & 0 & 0 \\ -g_{14} & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using a Maple procedure (see appendix C.1, p. 97) we can compute the remaining residues directly. The procedure uses equation 2.3.1, (p.50) to compute the entries

$g_{ij}$:

$$\begin{pmatrix} 0 & -\frac{1}{5}\lambda_1^2 - \frac{1}{15}\lambda_3 & -\frac{2}{5}\lambda_2 & \frac{3}{5}\lambda_1 & 0 & -\frac{1}{5} \\ \frac{1}{5}\lambda_1^2 + \frac{1}{15}\lambda_3 & 0 & -\frac{1}{3}\lambda_1 & 0 & -\frac{1}{3} & 0 \\ \frac{2}{5}\lambda_2 & \frac{1}{3}\lambda_1 & 0 & -1 & 0 & 0 \\ -\frac{3}{5}\lambda_1 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using these calculations we can prove the following theorem about the extension of the intersection form over the discriminant. This theorem can be found in more generality in a paper by Givental and Varchenko [VG82]. The proof given here elucidates their result in the case of a simple singularity.

**Theorem 2.36.** *The intersection form $\Phi$ for $A_{2k}$ is a nondegenerate 2–form on $\mathbb{C}^\mu \setminus D$ which extends over the discriminant to define a nondegenerate 2–form on $\mathbb{C}^\mu$.*

*Proof.* $\Phi$ is nondegenerate at $\lambda$ if $\bigwedge^{\frac{\mu}{2}} \Phi(\lambda) \neq 0$ and by [Bou58] (chap. II, §5, no. 2, page 83) we have that

$$\bigwedge^{\frac{\mu}{2}} \Phi = \frac{\mu}{2}!\,\mathrm{Pf}(G)\lambda_1 \wedge \cdots \wedge \lambda_\mu .$$

If we reverse the order of columns in the matrix $G$ (computed in lemmas 2.31 and 2.34) it becomes upper triangular with diagonal entries $g_{i,2k+1-i}$ equal to $\frac{1}{2k+1-2i}$ which implies that the Pfaffian of $G$ (given by $\mathrm{Pf}(G) = \pm\Pi_{i=1}^k \frac{1}{2k+1-2i}$) is nonzero and so we deduce that $\Phi$ is a nondegenerate 2–form on $\mathbb{C}^\mu \setminus D$.

We can extend $\Phi$ over the discriminant by extending the domain of the coefficients $g_{ij}$ over the discriminant, the Pfaffian of the corresponding matrix is still nonzero everywhere and so the extended form is nondegenerate. □

Using equation 2.3.1 we can compute the matrix of the intersection form for miniversal deformations of the $E_6$ and $E_8$ singularities. The matrices can be found in appendix A.2 (p.86).

**Theorem 2.37.** *The intersection form* $\Phi$ *in the case of the* $E_6$ *and* $E_8$ *singularities is a nondegenerate* 2*–form on* $\mathbb{C}^\mu \setminus D$ *which extends over the discriminant to define a nondegenerate* 2*–form on* $\mathbb{C}^\mu$*.*

*Proof.* We observe that the matrices (in appendix A.2) of the intersection forms in these cases are upper triangular with nonzero constants on the diagonal after reversing the order of the columns of the matrix. The intersection forms are therefore nondegenerate by the argument in the proof of the previous theorem. □

# Chapter 3

# Deformations of Modules

In this chapter we will recall the theory of deformations of maximal Cohen-Macaulay modules over hypersurface rings. We will then use the intersection form constructed in previous chapters to define a maximal Cohen-Macaulay module $M_\Omega$ on the discriminant of a miniversal deformation of a plane curve singularity.

We go on to prove that $M_\Omega$ can be realised as an infinitesimal deformation of the module $\mathcal{O}_{\widetilde{D},0} := \pi_*\mathcal{O}_{\Sigma,0}$ which was discussed in section 1.3.2 (p.31).

For an overview of maximal Cohen-Macaulay modules on hypersurface rings we refer the reader to the paper [BGS87] and for the deformation theory of maximal Cohen-Macaulay modules to the papers [PP96] and [HP97]. Eisenbud developed the theory of matrix factorisations for maximal Cohen-Macaulay modules in [Eis80].

## 3.1 Maximal Cohen-Macaulay modules over hypersurface rings

**Definition 3.1.** Let $R$ be a commutative Noetherian regular local ring. For an irreducible $h \in R$ define the hypersurface ring $S = R/(h)$. An $S$-module $M$ is called a *Cohen-Macaulay module* if its depth is equal to its dimension. The module $M$ is called a *maximal Cohen-Macaulay module (MCM)* if in addition its depth is equal to the dimension of the ring $S$ (the largest possible value for the depth).

**Remark 3.2.** Over a regular local ring Cohen-Macaulay modules are free.

Over a hypersurface ring, the set of MCM modules is equivalent to the set of modules that have 2–periodic free resolutions.

**Definition 3.3.** A pair of square matrices of the same size $(A, B)$ with entries in $R$ is called a *matrix factorisation* of $h$ if they satisfy $AB = hI_{R^k}$ and $BA = hI_{R^k}$ where $I_{R^k}$ is the identity matrix.

**Theorem 3.4** (see [Eis80],Theorem 6.1,(p.52) or [YM90],Proposition 7.2,(p.56))**.** *Let $M$ be a $S$–module, then $M$ is a MCM module if and only if $M$ has a 2-periodic $S$–free resolution coming from a matrix factorisation $(A, B)$ of $h$.*

*A 2-periodic resolution of $M$ is a free resolution of the form:*

$$\cdots \xrightarrow{A} S^k \xrightarrow{B} S^k \xrightarrow{A} S^k \longrightarrow M \longrightarrow 0 \, .$$

**Remark 3.5.** Suppose we promote the MCM $S$-module $M$ to a $R$-module then according to the Auslander-Buchsbaum formula:

$$\operatorname{proj.dim}_R(M) = \operatorname{depth}_R(R) - \operatorname{depth}_R(M) = 1$$

therefore, $M$ has a free resolution over $R$ of the form:

$$0 \longrightarrow R^k \xrightarrow{A} R^k \longrightarrow M \longrightarrow 0 \, .$$

**Definition 3.6.** $R$ is a regular local ring so any irreducible $h \in R$ is a prime nonzero divisor and hence $S = R/(h)$ is an integral domain. Let $K$ be the quotient field of $S$. For an $S$-module $M$ we define the *rank of* $M$, denoted $\operatorname{rank}_S(M)$, to be the dimension of the $K$–vector space $K \otimes_R M$.

**Proposition 3.7** ([Eis80],Proposition 5.6, p. 51)**.** *Let $(A, B)$ be a matrix factorisation such that* $\det A = h^k u$ *where $u \notin (h)$ then* $\operatorname{rank}_S(\operatorname{coker} A)$ *is equal to $k$.*

**Remark 3.8.** When $M$ is a rank 1 MCM $S$-module we can construct the corresponding matrix factorisation using a presentation matrix of $M$ and its adjugate matrix. Indeed let $A$ be the presentation matrix of $M$ and denote $\operatorname{Adj}(A)$ the adjugate matrix of $A$ (the matrix of cofactors, or the classical adjoint of $A$). This matrix satisfies:

$$A \operatorname{Adj}(A) = hI_{S^n}$$

so $(A, \operatorname{Adj}(A))$ is a matrix factorisation of $M$.

When $M$ is a MCM $S$-module of rank 2 over a hypersurface ring and the presentation matrix $A$ is skew-symmetric then we can define the Pfaffian of $A$ denoted $\operatorname{pf}(A)$ (see appendix B.1).

Using a theorem of Heymans ([Hey69]) on expansion of Pfaffians (described in theorem B.8, p. 94) the submaximal minors of $A$ are divisible by $\operatorname{pf}(A)$ and hence the entries of $\operatorname{Adj}(A)$ are divisible by $\operatorname{pf}(A)$. We conclude from this that the matrix $\frac{1}{\operatorname{pf}(A)} \operatorname{Adj}(A)$ has entries in $S$ and we can construct a matrix factorisation of $M$ in the following way.

**Lemma 3.9.** *Let $A$ be a presentation matrix for a MCM rank 2 $S$–module $M$. The pair $(A, \frac{1}{\operatorname{pf} A} \operatorname{Adj}(A))$ is a matrix factorization for $M$.*

*Proof.* We need to check that the pair satisfies the definition of a matrix factorisation, indeed:

$$A \cdot (\frac{1}{\operatorname{pf} A} \operatorname{Adj} A) = \frac{\det(A)}{\operatorname{pf}(A)} I_{R^k} = h\, I_{R^k}.$$

□

**Example 3.10.** $R = \mathbb{C}[[a, b, c, d]]$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $h = ad - bc$ and $M = \operatorname{coker} A$.

$$0 \longrightarrow R^2 \xrightarrow{A} R^2 \longrightarrow M \longrightarrow 0\ .$$

So $M$ is a module over $R$ defined with generators $X_1, X_2$ and relations $aX_1 + bX_2 = 0, cX_1 + dX_2 = 0$. This means $hX_1 = 0, hX_2 = 0$ and so $hM = 0$ thus $M$ is an

$R/(h) =: S$–module. Let $(x\,y) \in \operatorname{coker} A$ then

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} xa + yc & xb + yd \end{pmatrix} = 0\,.$$

Then as a $R$–module, $M$ has rank 0 and as a $S$–module, $M$ has rank 1 (since $f = 0$ in $S$). Then $\operatorname{Adj} A = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and $(A, \operatorname{Adj} A)$ is a matrix factorisation with 2–periodic $S$–free resolution.

$$\cdots \longrightarrow S^2 \xrightarrow{A} S^2 \xrightarrow{\operatorname{Adj} A} S^2 \xrightarrow{A} S^2 \longrightarrow M \longrightarrow 0$$

**Example 3.11.** Let $\Sigma$ be the critical space of the miniversal deformation of the $A_2$ singularity then the $\mathbb{C}\{\lambda_1, \lambda_2\}$–module $\pi_* \mathcal{O}_{\Sigma,0}$ has a presentation

$$0 \longrightarrow \mathcal{O}_{\mathbb{C}^2} \xrightarrow{A} \mathcal{O}_{\mathbb{C}^2} \longrightarrow \pi_* \mathcal{O}_\Sigma \longrightarrow 0$$

where $A = \begin{pmatrix} -6\lambda_1 & -9\lambda_2 \\ -9\lambda_2 & 2\lambda_1^2 \end{pmatrix}$ is the Saito matrix (see Proposition 1.52, p. 32).

In a similar way to the previous example the module $\pi_* \mathcal{O}_{\Sigma,0}$ is a MCM when considered as a $\mathcal{O}_{D,0} = \mathbb{C}\{\lambda_1, \lambda_2\}/(h)$–module where $h = 4\lambda_1^3 + 27\lambda_2^2$ is the equation of the discriminant $D$ of the $A_2$ singularity. It has a 2–periodic $\mathcal{O}_{D,0}$–resolution with matrix factorization $(A, B)$ where $A$ is as before and

$$B = \begin{pmatrix} 2\lambda_1^2 & 9\lambda_2 \\ 9\lambda_2 & -6\lambda_1 \end{pmatrix}$$

the adjugate matrix of $A$.

**Example 3.12.** We will give an example of lemma 3.9. Let the matrix $A$ be as the previous example and let $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then

$$A^t \Omega A = \begin{pmatrix} 0 & 12\lambda_1^3 + 81\lambda_2^2 \\ -12\lambda_1^3 - 81\lambda_2^2 & 0 \end{pmatrix}.$$

The Pfaffian of $A\Omega A$ is equal to $12\lambda_1^3 + 81\lambda_2^2$ and we find

$$\frac{1}{\mathrm{pf}(A\Omega A)} \operatorname{Adj}(A\Omega A) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and so indeed

$$A \cdot \frac{1}{\mathrm{pf}(A)} \operatorname{Adj}(A) = \det(A) I_{S^2}.$$

## 3.2 Deformations of Modules

Let $M$ be a module over the hypersurface ring $S$. We will introduce infinitesimal deformations of $M$ as described in [HP97], §1, p. 678.

**Definition 3.13.** Let $S[\epsilon]$ be a polynomial ring over $S$ and $N$ a module over $\tilde{S} := S[\epsilon]/(\epsilon^2)$. Then $N \xrightarrow{\epsilon} N \xrightarrow{\epsilon} N$ is a complex. The module $N$ is called an *infinitesimal deformation* of $M$ if

1. $M \simeq N/\epsilon N$
2. $N \xrightarrow{\epsilon} N \xrightarrow{\epsilon} N$ is exact.

**Remark 3.14.** The first property defines a short exact sequence

$$0 \longrightarrow \epsilon N \longrightarrow N \longrightarrow M \longrightarrow 0$$

and the second property implies that $M \simeq \epsilon N = 0 :_N \epsilon$ which in turn implies the following short exact sequence

$$0 \longrightarrow M \xrightarrow{i_N} N \xrightarrow{p_N} M \longrightarrow 0\,.$$

This induces a long exact sequence of Hom

$$\cdots \longrightarrow \operatorname{Hom}_S(N,M) \longrightarrow \operatorname{Hom}_S(M,M) \xrightarrow{\delta} \operatorname{Ext}^1_S(M,M) \longrightarrow \cdots .$$

The map $N \mapsto \delta(\mathrm{id}_M)$ defines a bijection between the set of infinitesimal deformations of $M$ and $\operatorname{Ext}^1_S(M,M)$ (see [HP97], Theorem 1.1, p.680).

### 3.2.1 Deformations of maximal Cohen-Macaulay modules

It is possible to construct an infinitesimal deformation of a maximal Cohen-Macaulay module $M$ on a hypersurface ring by deforming the matrices $A$ and $B$ in its 2-periodic resolution (see [War03],p.86). To do this we find matrices $A_1$ and $B_1$ such that the $\tilde{S}$-module $N = \operatorname{coker}(A + \epsilon A_1)$ fits into the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
M & \longleftarrow & S^a & \xleftarrow{A} & S^a & \xleftarrow{B} & S^a & \xleftarrow{A} & \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \cdot\epsilon} & & \downarrow & & \downarrow & & \\
N & \longleftarrow & \tilde{S}^a & \xleftarrow{A+\epsilon A_1} & \tilde{S}^a & \xleftarrow{B+\epsilon B_1} & \tilde{S}^a & \xleftarrow{A+\epsilon A_1} & \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \mathrm{pr}} & & \downarrow & & \downarrow & & \\
M & \longleftarrow & S^a & \xleftarrow{A} & S^a & \xleftarrow{B} & S^a & \xleftarrow{A} & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & &
\end{array}$$

We define the maps $i$ and $j$ in the diagram by lifting the maps $\cdot\epsilon$ and pr respectively. The maps $i$ and $j$ fit into a short exact sequence

$$0 \longrightarrow M \xrightarrow{i} N \xrightarrow{j} M \longrightarrow 0$$

which by remark 3.14 defines an element of $\operatorname{Ext}^1_S(M,M)$ and consequently a deformation of $M$.

**Proposition 3.15.** *The module* $N = \operatorname{coker}(A+\epsilon A_1)$ *is an infinitesimal deformation* *of the module $M$ whenever the matrices $A, B, A_1, B_1$ satisfy:*

$$AB_1 + A_1 B = 0.$$

*Proof.* We need to prove that the middle complex is exact in the commutative diagram above. We first check that $\mathrm{Im}(B + \epsilon B_1) \subset \ker(A + \epsilon A_1)$, indeed:

$$(A + \epsilon A)(B + \epsilon B_1) = AB + \epsilon(AB_1 + A_1 B) = 0\,.$$

Now we check that $\ker(A + \epsilon A_1) \subset \mathrm{Im}(B + \epsilon B_1)$, indeed let $v_0 + \epsilon v_1 \in \ker(A + \epsilon A_1)$ then $Av_0 + \epsilon(A_1 v_0 + Av_1) = 0$ and so $Av_0 = 0$ and

$$A_1 v_0 + Av_1 = A_1 B w_0 + Av_1 = A(-B_1 w_0 + v_1) = 0\,.$$

Because $\ker A \subset \mathrm{Im}\, B$ there exists $w_0$ and $w_1$ such that $Bw_0 = v_0$ and $Bw_1 = -B_1 w_0 + v_1$ and therefore:

$$(B + \epsilon B_1)(w_0 + \epsilon w_1) = Bw_0 + \epsilon(B_1 w_0 + Bw_1) = v_0 + \epsilon v_1\,.$$

□

**Example 3.16.** Let $(D, 0)$ be the discriminant of a miniversal deformation of the $A_2$ singularity given by

$$F(x, y, \lambda_1, \lambda_2) = -y^2 + x^3 + \lambda_1 x + \lambda_2\,.$$

Recall from the construction of the Saito matrix in section 1.3.2 (p.31) that we defined the module $\mathcal{O}_{\widetilde{D},0} := \pi_* \mathcal{O}_{\Sigma,0}$ as an $\mathcal{O}_{\mathbb{C}^\mu,0}$-module. If we consider $\pi_* \mathcal{O}_{\Sigma,0}$ as a module over the hypersurface ring $\mathcal{O}_{D,0} = \mathcal{O}^{\mu}_{\mathbb{C}^\mu,0}/(h)$ (where $h = 4\lambda_1^3 + 27\lambda_2^2$ is the equation of the discriminant) then we have the following 2-periodic free resolution:

$$\cdots \xrightarrow{A} \mathcal{O}_{D,0} \xrightarrow{B} \mathcal{O}_{D,0} \xrightarrow{A} \mathcal{O}_{D,0} \longrightarrow \mathcal{O}_{\widetilde{D},0} \longrightarrow 0$$

where $A$ is the Saito matrix

$$A = \begin{pmatrix} -6\lambda_1 & -9\lambda_2 \\ -9\lambda_2 & 2\lambda_1^2 \end{pmatrix}$$

and $B$ is given by the adjugate matrix

$$B = \begin{pmatrix} 2\lambda_1^2 & 9\lambda_2 \\ 9\lambda_2 & -6\lambda_1 \end{pmatrix}.$$

We define infinitesimal deformations of the module $\mathcal{O}_{\widetilde{D},0}$ by deforming the Saito matrix in the way described above. For instance we can use the following matrices:

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3}\lambda_1 \end{pmatrix} \quad , \; B_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3}\lambda_1 \end{pmatrix}$$

or

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \; B_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Observe that the second of these deformations is of the form $\operatorname{coker}(\chi + \epsilon\Omega)$ where $\Omega$ is the matrix of coefficients of the intersection form for $A_2$.

## 3.3 The module $M_\Omega$ associated to the intersection form

Let $\Omega$ be the matrix of coefficients of the the intersection form $\Phi$ on the base space of the cohomology bundle $\Lambda$ (see definition 2.17). We will construct a rank 2 MCM $\mathcal{O}_{D,0}$-module that we will use in chapter 4 to extract information contained in $\Phi$ about strata in the discriminant $D \subset \Lambda$. Let $\chi$ be the Saito matrix associated to a deformation $F$ of $f$ and let $h \in \mathcal{O}_{\mathbb{C}^\mu,0}$ be the equation of the discriminant hypersurface $D \subset \Lambda$.

Recall from 1.3.2 that the module $\mathcal{O}_{\Sigma,0} := \mathcal{O}_{\mathbb{C}^2\times\mathbb{C}^\mu}/J(F)$ can be considered as an

$\mathcal{O}_{\mathbb{C}^\mu,0}$-module with presentation matrix:

$$0 \longrightarrow \mathcal{O}^\mu_{\mathbb{C}^\mu,0} \xrightarrow{\chi} \mathcal{O}^\mu_{\mathbb{C}^\mu,0} \longrightarrow \pi_*\mathcal{O}_{\Sigma,0} \longrightarrow 0\,.$$

Let $\mathcal{O}_{D,0}$ be the hypersurface ring $\mathcal{O}^\mu_{\mathbb{C}^\mu,0}/(h)$ then because $\chi \operatorname{Adj}\chi = \det\chi \cdot I_{\mathcal{O}^\mu_{\mathbb{C}^\mu,0}} = h \cdot I_{\mathcal{O}^\mu_{\mathbb{C}^\mu,0}}$ the module $\pi_*\mathcal{O}_{\Sigma,0}$ has a 2-periodic resolution of the form:

$$\longrightarrow \mathcal{O}^\mu_{D,0} \xrightarrow{\operatorname{Adj}\chi} \mathcal{O}^\mu_{D,0} \xrightarrow{\chi} \mathcal{O}^\mu_{D,0} \longrightarrow \pi_*\mathcal{O}_{\Sigma,0} \longrightarrow 0\;.$$

**Definition 3.17.** We define the *intersection module* $M_\Omega$ associated to $\Omega$ as an $\mathcal{O}_{D,0}$-module using the presentation matrix:

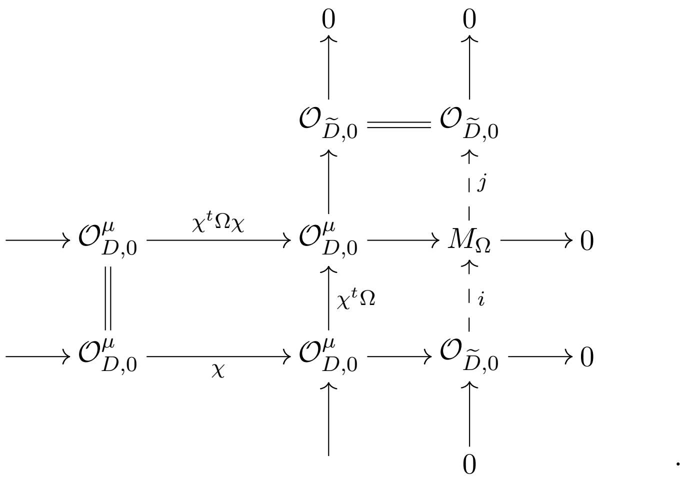
$$\mathcal{O}^\mu_{D,0} \xrightarrow{\chi^t\Omega\chi} \mathcal{O}^\mu_{D,0} \longrightarrow M_\Omega \longrightarrow 0\;.$$

**Proposition 3.18.** *The module $M_\Omega$ is a rank 2 maximal Cohen-Macaulay $\mathcal{O}_{D,0}$-module.*

*Proof.* The presentation matrix $\chi^t\Omega\chi$ is skew symmetric and so by lemma 3.9 we have that $\left(\chi^{-1}\Omega\chi, \frac{1}{\operatorname{pf}(\chi^t\Omega\chi)}\operatorname{Adj}(\chi^t\Omega\chi)\right)$ is a matrix factorisation and consequently is a 2-periodic $\mathcal{O}_{D,0}$-free resolution for $M_\Omega$. Thus $M_\Omega$ is a MCM $\mathcal{O}_{D,0}$-module by Eisenbud's theorem (theorem 3.4).

According to proposition 3.7 $M_\Omega$ is rank 2 because $\det(\Omega) \in \mathbb{C}^*$ and $\det(\chi^t\Omega\chi) = f^2\det(\Omega)$ since $\det(\chi) = f$. □

The matrix $\Omega$ is invertible and so $\operatorname{coker}\chi^t\Omega = \operatorname{coker}\chi = \mathcal{O}_{\tilde{D},0}$. Using this fact we construct the following commutative diagram:




The various resolutions in the diagram define maps $i$,$j$ that fit into the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\widetilde{D},0} \xrightarrow{\ i\ } M_\Omega \xrightarrow{\ j\ } \mathcal{O}_{\widetilde{D},0} \longrightarrow 0$$

and we conclude that $M_\Omega$ defines an element of $\text{Ext}^1_{\mathcal{O}_{D,0}}(\mathcal{O}_{\widetilde{D},0}, \mathcal{O}_{\widetilde{D},0})$. Consequently $M_\Omega$ can be viewed as an infinitesimal deformation of the module $\mathcal{O}_{\widetilde{D},0}$.

We will now show how to realise $M_\Omega$ as a $\mathcal{O}_{D,0}[\epsilon]/\epsilon^2$-module. Let $e_k$ be the $k$th generator of $\mathcal{O}^{\mu}_{D,0}$ and suppose it gets mapped to $m_k$, the $k$th generator of the module $\mathcal{O}_{\widetilde{D},0}$. Then if we consider $M_\Omega$ as a $\mathcal{O}_{D,0}[\epsilon]/\epsilon^2$-module, $i(m_k)$ is the image of $\epsilon e_k$ in $M_\Omega$. But by the construction of $i$ we have that $\epsilon e_k = \chi^t\Omega e_k$. Hence the vector $\chi^t\Omega e_k - \epsilon e_k$ is a relation between the generators of $M_\Omega$ as a $\mathcal{O}_{D,0}[\epsilon]/\epsilon^2$-module. So we deduce that:

$$M_\Omega = \text{coker}(\chi^t\Omega - \epsilon I)$$

as a $\mathcal{O}_{D,0}[\epsilon]/\epsilon^2$-module where $I$ is the identity matrix.

We can write $M_\Omega$ as a deformation of the presentation matrix $\chi$ of $\mathcal{O}_{\widetilde{D},0}$ in the manner of the previous section i.e. $M_\Omega = \text{coker}(\chi + \epsilon A_1)$ for some matrix $A_1$. We construct $A_1$ as follows:

$$\text{coker}(\chi^t\Omega - \epsilon I) = \text{coker}(-\chi\Omega - \epsilon I)$$

($\Omega$ is skew-symmetric)

$$= \operatorname{coker}(\chi + \epsilon\Omega^{-1})$$

($\Omega$ is invertible)

and we conclude that $A_1 = \Omega^{-1}$ for this deformation.

To construct the $B_1$ matrix we use a result of Buchweitz and Leuschke in ([BL07], theorem 2.8) which shows that there exists a skew-symmetric matrix $B_\Omega$ such that:

$$\chi^t B_\Omega = -\Omega^{-1} \operatorname{Adj} \chi.$$

Since the Saito matrix can be chosen to be symmetric (this is due to the fact that $\mathcal{O}_{\Sigma,0}$ is Gorenstein, see [MS10]):

$$\chi B_\Omega + \Omega^{-1} \operatorname{Adj} \chi = \chi^t B_\Omega + \Omega^{-1} \operatorname{Adj} \chi = 0$$

and so the matrices $\chi + \epsilon\Omega^{-1}$ and $\operatorname{Adj} \chi + \epsilon B_\Omega$ define an infinitesimal deformation by proposition 3.15.

**Theorem 3.19.** *Let $\Omega$ be the matrix of coefficients of the intersection form for a miniversal deformation of a singularity. We can consider the intersection module $M_\Omega$ as an infinitesimal deformation of the module $\mathcal{O}_{\widetilde{D},0} := \pi_* \mathcal{O}_{\Sigma,0}$ with the following presentation:*

$$\longrightarrow \mathcal{O}^\mu_{D,0}[\epsilon]/\epsilon^2 \xrightarrow{\operatorname{Adj}\chi + \epsilon B_\Omega} \mathcal{O}^\mu_{D,0}[\epsilon]/\epsilon^2 \xrightarrow{\chi + \epsilon\Omega^{-1}} \mathcal{O}^\mu_{D,0}[\epsilon]/\epsilon^2 \longrightarrow M_\Omega \longrightarrow 0\,.$$

□

**Question 3.20.** We have shown that $M_\Omega$ is an infinitesimal deformation of $\mathcal{O}_{\widetilde{D},0}$ but what is the geometric significance of this? Can we use the classification theory of maximal Cohen-Macaulay modules over hypersurface rings to identify $M_\Omega$? Perhaps this would be a better definition for $M_\Omega$ than the presentation matrix we are using.

# Chapter 4

# Finding Strata in the Discriminant

In this chapter we will discuss various methods for finding an ideal defining the $\delta$-constant stratum. We prove that the ideals of principal Pfaffians of the skew-symmetric presentation matrix for the intersection module $M_\Omega$ (defined in chapter 3) define the varieties $D^\delta(k)$, the set of parameters where the deformed curve has $\delta$-invariant greater than or equal to $k$.

Because we are able to compute the intersection form explicitly for $A_{2k}, E_6$ and $E_8$ singularities we can compute ideals defining these strata. We are able to check that $D(\delta)$ is Cohen-Macaulay for $E_6$ and $E_8$ which proves, by a result of van Straten and Sevenheck in [vSS03], that $D(\delta)$ is a rigid Lagrangian singularity for these singularities.

We will begin be describing a stratification of the discriminant by Milnor number and showing that we can define the intersection form on each of these strata.

## 4.1 The intersection form on strata of the discriminant

**Definition 4.1.** Let $(\mu_1, \ldots, \mu_v)$ be a collection of positive integers and define the $(\mu_1, \ldots, \mu_v)$-*stratum* as the set of all $\lambda \in \Lambda$ for which the fibre $X_\lambda$ has exactly $v$ singular points with Milnor numbers $\mu_1, \ldots, \mu_v$.

Let $P_\omega$ be a nondegenerate period map on $\Lambda$ and $X_\lambda$ a singular fibre of the central Milnor fibration. As the fibre is singular the intersection pairing $\mathrm{I}_\lambda^\vee$ could be degenerate. In what follows we will need to know that the pull back of $\mathrm{I}_\lambda^\vee$ by $P_\omega$ coincides with the intersection form $\Phi$ at $\lambda \in D$.

Let $\Pi$ be the regular part of the $(\mu_1, \ldots, \mu_v)$-stratum, then the fibres $X_\lambda$ over $\Pi$ are homeomorphic so we can define a locally trivial cohomology bundle $\coprod_{\lambda \in \Pi} H^1(X_\lambda; \mathbb{C}) \to \Pi$. In a similar way to the constructions for the cohomology bundle over $\Lambda \setminus D$ there is a canonical flat Gauss-Manin connection in this bundle.

The stratum $\Pi$ has codimension equal to $\sum_{i=1}^{v} \mu_i$ in $\Lambda$ and $H^1(X_\lambda; \mathbb{C})$ has rank $\mu - \sum_{i=1}^{v} \mu_i$ and so the nondegenerate period map $P_\omega$ for the bundle over $\Lambda \setminus D$ is also nondegenerate when restricted to the bundle over $\Pi$. We can therefore pull back the intersection pairing $\mathrm{I}_\lambda^\vee$ on fibres over $\Pi$ via the nondegenerate period map to an intersection form $\Phi_\Pi$ on $\Pi$.

**Theorem 4.2** (see[Var89], Theorem 7, p.72)**.** *Let $\Phi$ be the intersection form on the base space $\Lambda$ of a miniversal deformation of a plane curve singularity. Let $\Phi_\Pi$ be the intersection form constructed over a stratum $\Pi$ of the discriminant as above then:*

$$\Phi|_\Pi = \Phi_\Pi.$$

## 4.2 An example of the $\delta$–constant stratum

In this section we will give an example of finding equations for the $\delta$-constant stratum in $\Lambda$ using the well known fact that that the $\delta$-constant deformations of a plane curve singularity $(C_0, 0)$ are those which are induced from a deformation of its parametrisation (this result was discussed in 1.1.3).

**Example 4.3.** A miniversal deformation of the equation of the $A_4$ singularity in $\mathbb{C}^2$ is given by proposition 1.13, p.10:

$$F(x, y, \lambda) = -y^2 + x^5 + \sum_{i=1}^{4} \lambda_i x^{4-i}$$

and a miniversal deformation of its parametrisation $t \mapsto (t^2, t^5)$ is given by proposition 1.16, p.12:

$$\psi(t, u) = (t^2, t^5 + u_1 t^3 + u_2 t)\,.$$

We first describe $\psi$ as a deformation (using remark 1.15, p. 12) of the equation by finding $G \in \mathbb{C}\{x, y, u_1, u_2\}$ such that $\psi^\sharp(G) = 0$. To do this, let

$$G = y^2 + \sum_{i=1}^{5} a_i x^{5-i}$$

and find the coefficients $a_i$ for which $\psi^\sharp(G) = 0$, thus:

$$\begin{aligned}
a_1 &= 2u_1 \\
a_2 &= 2u_2 + u_1^2 \\
a_3 &= 2u_2 + u_1 \\
a_4 &= u_2^2 \\
a_5 &= 0
\end{aligned}$$

which defines a deformation of the equation over $\mathbb{C}^2$.

By versality, this deformation is isomorphic to a pullback of the miniversal deformation $F$ under some map $b : \mathbb{C}^2 \to \mathbb{C}^4$. We find $b$ by noting that applying the coordinate transformation $(y \mapsto y, x \mapsto x - \frac{2u_1}{5})$ to $G$ removes the $x^4$ term. We find the map $\mathbb{C}^2 \to \mathbb{C}^4 : (u_1, u_2) \mapsto (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ where

$$\begin{aligned}
\lambda_1 &= 2u_2 - \frac{3}{5}u_1^2 \\
\lambda_2 &= \frac{2}{25}u_1^3 - \frac{12}{5}u_2 u_1 + 2u_2 + u_1 \\
\lambda_3 &= \frac{12}{125}\,{u_1}^4 + \frac{24}{25}\,u_2 {u_1}^2 - \frac{8}{5}\,u_2 u_1 - \frac{4}{5}\,{u_1}^2 + {u_2}^2 \\
\lambda_4 &= -\frac{72}{3125}\,{u_1}^5 - \frac{16}{125}\,u_2 {u_1}^3 + \frac{8}{25}\,u_2 {u_1}^2 + \frac{4}{25}\,{u_1}^3 - \frac{2}{5}\,{u_2}^2 u_1
\end{aligned}$$

so that $G_{u_1,u_2} = F_{b(u_1,u_2)}$.

We can therefore find an ideal $I \subset \mathbb{C}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ defining $D(\delta)$ by eliminating

$u_1, u_2$ from the equations above.

## 4.3 The intersection form identifies the $\delta$–constant stratum in the discriminant

The aim of this section is to establish the link between the $\delta$-invariant of fibres over a stratum of the discriminant and the number of generators that the module $M_\Omega$ requires on the stratum. We will then use this fact to find equations for the $\delta$-constant stratum from the ideal of Pfaffians from the skew-symmetric presentation matrix for $M_\Omega$.

### 4.3.1 Examples

In this section we will use the techniques described previously to prove that the ideals defining the strata $D(k)$ can be found as the principal Pfaffians of the presentation matrix of $M_\Omega$ where $\Omega$ is the matrix of the intersection form (see definition 2.17, p.44).

Recall the definition of the $\mathcal{O}_{\mathbb{C}^\mu,0}$–module $M_\Omega$ defined with the presentation matrix $S := \chi^t \Omega \chi$

$$\mathcal{O}^\mu_{\mathbb{C}^\mu,0} \xrightarrow{S} \mathcal{O}^\mu_{\mathbb{C}^\mu,0} \longrightarrow M_\Omega \longrightarrow 0$$

where $\chi$ is the Saito matrix whose columns are coefficients of a basis of $\mathrm{Der}(-\log D)$ (see section 1.3.2).

$M_\Omega$ can also be considered as an $\mathcal{O}_D$–module because $\det S \in (h^2)$ where $h$ is the equation of $D$.

**Example 4.4.** In the case of the $A_2$ singularity the presentation matrix for $M_\Omega$ (as an $\mathcal{O}_{\mathbb{C}^\mu,0}$–module) has the form (see example 3.11, p.58):

$$S = \begin{pmatrix} 0 & 12\lambda_1^3 + 81\lambda_2^2 \\ -12\lambda_1^3 - 81\lambda_2^2 & 0 \end{pmatrix}$$

where $12\lambda_1^3 + 81\lambda_2^2$ is the equation of the discriminant of a miniversal deformation of $A_2$. At a point on the discriminant, $S$ is equal to the zero matrix and therefore $M_\Omega$ requires 2 generators.

We will show that $M_\Omega$ requires at least 2 generators everywhere because the presentation matrix for the module is skew-symmetric.

**Lemma 4.5.** *Let $M_\Omega$ be the $\mathcal{O}_{\mathbb{C}^\mu,0}$-module defined above, then $M_\Omega$ requires at least 2 generators at any point on the discriminant.*

*Proof.* For $\lambda \in D$ we have $\operatorname{rank} S(\lambda) \leq \mu - 1$ which implies that $\operatorname{rank} S(\lambda) \leq \mu - 2$ because $S$ is skew-symmetric (by the remark after proposition B.10) and so $M_\Omega = \operatorname{coker} S$ requires at least 2 generators. □

**Example 4.6.** For $A_4$ the presentation matrix for $M_\Omega$ can be found in appendix A.1 (p.84) and here we make some observations about the number of generators that the module requires on different strata in the discriminant.

1. At a smooth point on $D$ (the (1)-stratum) the module requires 2 generators.
2. At a point on the (2)-stratum the module requires 2 generators.
3. At a point on the $(1, 1)$-stratum the module requires 4 generators.
4. At the origin (the (4)-stratum) the module requires 4 generators.

Indeed, at a point on the $(1, 1)$-stratum the presentation matrix is the zero matrix.

Notice that there is a link between the $\delta$-invariant of fibres of a stratum and the number of generators that $M_\Omega$ requires. That link is the topic of the next section.

### 4.3.2 Fitting ideals of $M_\Omega$

A natural way to define strata where $M_\Omega$ requires a number of generators is to use Fitting ideals.

**Definition 4.7.** Let $M$ be an $\mathcal{O}_{\mathbb{C}^\mu,0}$–module with presentation matrix

$$\mathcal{O}^n_{\mathbb{C}^\mu,0} \xrightarrow{A} \mathcal{O}^n_{\mathbb{C}^\mu,0} \longrightarrow 0$$

then the $k$th *Fitting ideal* of $M$, denoted $\mathcal{F}_k(M)$ is an ideal with generators equal to the $n-k$ minors of the matrix $A$, denoted $\mathrm{Min}_{n-k}(A)$.

**Proposition 4.8** (see [Eis95] Proposition 20.6, p.498)**.** *The $k$th Fitting ideal $\mathcal{F}_k(M)$ defines a variety consisting of points $p \in \mathrm{supp}(M)$ where the module $M(p)$ needs at least $k$ generators, or equivalently where the matrix $A(p)$ has rank less than $n-k$.*

$\square$

The fact that the presentation matrix for $M_\Omega$ is skew-symmetric is important, it allows us to stratify the discriminant using varieties defined by the even indexed Fitting ideals of $M_\Omega$.

**Proposition 4.9.**

$$D = \coprod_{k=1}^{\frac{\mu}{2}} V(\mathcal{F}_{\mu-2k}(M_\Omega)) \setminus V(\mathcal{F}_{\mu-2(k+1)}(M_\Omega))$$

*Proof.* We prove this proposition using the following lemma which establishes the equality of the varieties defined by the Fitting ideals indexed by $\mu - 2k - 1$ and $\mu - 2k$ for $k$ from 1 to $\frac{\mu}{2}$. $\square$

**Lemma 4.10.**

$$V(\mathcal{F}_{\mu-2k}(M_\Omega)) = V(\mathcal{F}_{\mu-(2k-1)}(M_\Omega))$$

*Proof.* This follows from proposition B.10 and its following remark (p.95). There it is shown that $V(\mathrm{Min}_{2k}(S)) = V(\mathrm{Min}_{2k-1}(S))$ for a skew–symmetric matrix $S$. $\square$

### 4.3.3 The intersection pairing on the normalisation of a singular curve

Let $n : \overline{X}_\lambda \to X_\lambda$ be the normalisation of the curve $X_\lambda$. Since $X_\lambda$ has isolated singularities we can recover it as a quotient of its normalisation $X_\lambda \simeq \overline{X}_\lambda / S$ which glues together on $\overline{X}_\lambda$ the preimage under $n$ of singular points of $X_\lambda$ (see [BKS86], p.618).

**Example 4.11.** Let $X_{\lambda_0}$ be a deformation of an $A_4$ singularity over the (1)-stratum. We can write $X_{\lambda_0}$ as the following variety:

$$X_\lambda = V\left(-y^2 + (x-a)^2(x^3 + 2ax^2 + bx + c)\right)$$

for $a, b, c \in \mathbb{C}$. The normalisation of this curve under the map

$$n^\sharp : \mathcal{O}_{X_{\lambda_0}} \to \mathcal{O}_{\overline{X}_{\lambda_0}} : (x, y) \mapsto (x, \frac{y}{x-a}) = (u, v)$$

is equal to the nonsingular curve

$$\overline{X}_\lambda = V(-v^2 + u^3 + 2au + bu + c).$$

The singularity at $(a, 0)$ on $X_\lambda$ has two preimages under the normalisation map. The nonsingular curve has genus 1 and the rank of the intersection form is 2.

The cokernel of the natural homomorphism $H_1(\overline{X}_\lambda) \to H_1(X_\lambda)$ is equal to the kernel of the intersection pairing on the curve $X_\lambda$.

**Proposition 4.12** (see [Var89],p.70)**.** *The kernel of the homomorphism* $n^* : H^1(X_\lambda) \to H^1(\overline{X}_\lambda)$ *is equal to the kernel of the intersection form* $\mathrm{I}_\lambda^\vee$.

**Lemma 4.13.** *The intersection pairing* $\mathrm{I}_\lambda^\vee$ *on* $H^1(X_\lambda; \mathbb{C})$ *passes to the normalisation, i.e.:*

$$\mathrm{I}_\lambda^\vee(a, b) = \mathrm{I}_{\overline{X}_\lambda}^\vee(n^*a, n^*b)$$

*for* $a, b \in H^1(X_\lambda; \mathbb{C})$.

*Proof.* We prove this result by showing that

$$\int_{[X_\lambda]} a_c \wedge b = \int_{[\overline{X}_\lambda]} n^*(a_c \wedge b)$$

where $a_c$ is a representative of $a$ with compact support (see the definition of the intersection pairing in remark 2.7, p. 37).

The normalisation $n$ is an isomorphism outside a finite set of points (and so $\overline{X}_\lambda$ and $X_\lambda$ have the same set of 1-chains and 2-chains). We deduce from this that the

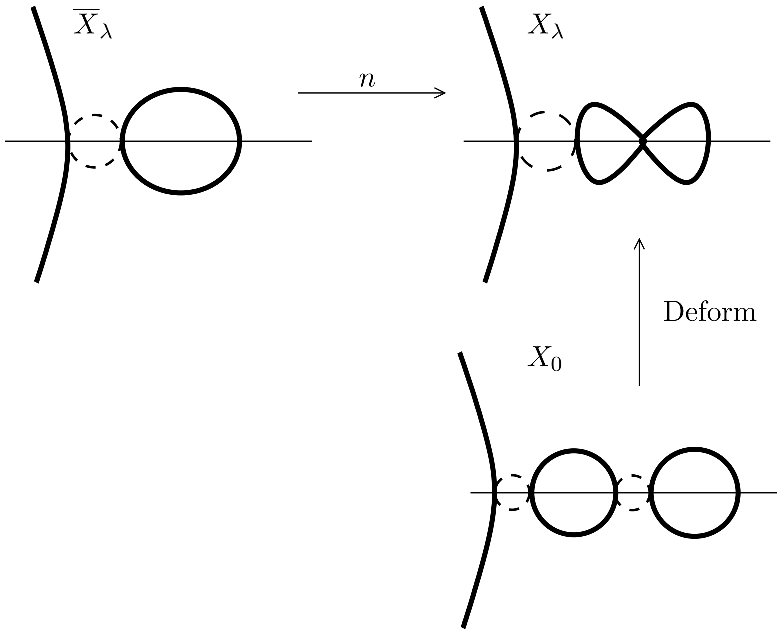



Figure 4.1: A diagram illustrating the deformation and normalisation of the $A_4$ singularity in example 4.11. The curves shown are real with additional cycles marked with a dotted line to show the full topology of the complex curve.

homomorphism

$$n_* : H_2(\overline{X}_\lambda) \to H_2(X_\lambda)$$

is the identity map and therefore that $n_*[\overline{X}_\lambda] = [X_\lambda]$. Then we have

$$\int_{[\overline{X}_\lambda]} n^*(a_c \wedge b) = \int_{n_*[\overline{X}_\lambda]} a_c \wedge b = \int_{[X_\lambda]} a_c \wedge b.$$

□

### 4.3.4 The relationship between the rank of the intersection pairing and the genus of the normalisation

**Definition 4.14.** Recall that $H^1(X_\lambda)$ is a $\mu$-dimensional vector space. We define the *rank* of the intersection pairing $\mathrm{I}_\lambda^\vee$, denoted $R(\mathrm{I}_\lambda^\vee)$, as the following dimension:

$$R(\mathrm{I}_\lambda^\vee) = \dim_{\mathbb{C}}\{a \in H^1(X_\lambda) : \exists b \in H^1(X_\lambda) \text{ such that } \mathrm{I}_\lambda^\vee(a, b) \neq 0\}.$$

**Remark 4.15.** The rank of the intersection pairing is also equal to the following dimensions:

$$\begin{aligned} R(\mathrm{I}^\vee_\lambda) &= \dim_\mathbb{C} H^1(X_\lambda) - \dim_\mathbb{C}\{a \in H^1(X_\lambda) : \mathrm{I}^\vee_\lambda(a,b) = 0 \ \ \forall b \in H^1(X_\lambda)\} \\ &= \dim_\mathbb{C} H^1(X_\lambda) - \dim_\mathbb{C} \ker \mathrm{I}^\vee_\lambda \\ &= \dim_\mathbb{C} H^1(X_\lambda) - \dim_\mathbb{C} \ker n^* \end{aligned}$$

where the last equality is true by proposition 4.12.

**Lemma 4.16.** *The genus of the normalisation and the rank of the intersection pairing are related as follows:*

$$g(\overline{X}_\lambda) = \frac{1}{2} R(\mathrm{I}^\vee_\lambda).$$

*Proof.* Recall that kernel of $\mathrm{I}^\vee_\lambda$ is equal to the kernel of $n^*$ and $n^*$ is a surjection onto $H^1(\overline{X}_\lambda)$. The rank of $\mathrm{I}^\vee_\lambda$ is therefore equal to the rank of $\mathrm{I}^\vee_{\overline{X}_\lambda}$ by lemma 4.13. Furthermore the rank of $\mathrm{I}^\vee_{\overline{X}_\lambda}$ is equal to $2g(\overline{X}_\lambda)$ because is is a smooth algebraic curve of genus $g$ and so $g(\overline{X}_\lambda) = \frac{1}{2}R(\mathrm{I}^\vee_\lambda)$. □

**Corollary 4.17.**

$$\delta(X_0) - \delta(X_\lambda) = \frac{1}{2} R(\mathrm{I}^\vee_\lambda)$$

*Proof.* This follows by the formula relating the genus of the normalisation and the $\delta$-invariant given in theorem 1.33 (p.19). □

Using the results proved so far we obtain a simple proof that the $\delta$-constant stratum is Lagrangian.

**Theorem 4.18.** *The $\delta$-constant stratum is Lagrangian with respect to the symplectic structure provided by the intersection form.*

*Proof.* To show that the $\delta$-constant stratum is Lagrangian we must show that its dimension is half that of the parameter space and that the symplectic form vanishes on it.

Proposition 1.16 (p.12) gives the dimension of the $\delta$-constant stratum as $\tau - \delta$ which is equal to $\frac{\mu}{2}$ by Milnor's formula (see Theorem 1.30, p.17) because $\mu$ is equal to $\tau$ in this case. This proves the first part.

By corollary 4.17 we have that the rank of the intersection pairing is 0 for any fibre $X_\lambda$ over the $\delta$-constant stratum. The intersection form is the pullback via a nondegenerate period map $P_\omega$ of the intersection pairing:

$$\Phi(\lambda) = P_\omega^* \mathrm{I}_\lambda^\vee$$

which for $\lambda \in D(\delta)$ is therefore equal to the zero form. This proves the second part. □

We will now show that a stratification of the discriminant in terms of the $\delta$-invariant is the same as a stratification of the discriminant in terms of the rank of the intersection pairing. Indeed, we use the rank of the intersection form to define the following varieties in the discriminant.

**Definition 4.19.**

$$R_{2m}(M_\Omega) = \{\lambda \in D : R(\mathrm{I}_\lambda^\vee) \leq 2m\}$$

for $m$ from 0 to $\delta(X_0) - 1$.

Then there is the following relation between these varieties and the varieties defined by the $\delta$-invariant.

**Theorem 4.20.**

$$R_{2m}(M_\Omega) = D(\delta - m)$$

*for* $m = 0, \ldots, \delta - 1$.

*Proof.* Corollary 4.17 shows that $R_{2m}$ is equal to the set of $\lambda \in \Lambda$ where

$$\delta(X_\lambda) \geq \delta(X_0) - m$$

which is precisely the definition of $D(\delta - m)$. □

### 4.3.5 Finding equations for $D(k)$ using a presentation matrix for $M_\Omega$

In this section we will use theorem 4.20 to find equations for the varieties $D(k)$ using ideals generated by Pfaffians of the skew-symmetric presentation matrix $S$ for $M_\Omega$.

**Lemma 4.21.**

$$R_{2m}(M_\Omega) = V(\mathrm{Min}_{2m+1}(S))$$

*Proof.* It is sufficient to prove that the rank of the matrix $S(\lambda)$ is equal to the rank of the intersection pairing $\mathrm{I}_\lambda^\vee$.

The $ij$th entry in the matrix $S(\lambda)$ is equal to

$$\Omega(\chi_i, \chi_j)(\lambda) = \mathrm{I}_\lambda^\vee \left(\nabla_{\chi_i} P_\omega, \nabla_{\chi_j} P_\omega\right) .$$

Since the vector fields $\chi_i$ are a basis for vector fields tangent to the discriminant the forms $\nabla_{\chi_i} P_\omega$ are a basis for $H^1(X_\lambda)$. $\square$

**Theorem 4.22.**

$$D(\delta - m) = V\left(\mathrm{pf}_{2(m+1)}(S)\right)$$

*for $m$ from $0$ to $\delta - 1$.*

*Proof.* We prove the theorem by establishing a chain of equalities starting with the variety $D(\delta - m)$ and ending with $V\left(\mathrm{pf}_{2(m+1)}(S)\right)$.

$$D(\delta - m) = R_{2m}(M_\Omega)$$

(by theorem 4.20)

$$= V(\mathrm{Min}_{2m+1}(S))$$

(by lemma 4.21)

$$= V\left(\mathcal{F}_{\mu-(2m+1)}(M_\Omega)\right)$$

(by definition)

$$= V\left(\mathcal{F}_{\mu-2(m+1)}(M_\Omega)\right)$$

(by lemma 4.10)

$$= V\left(\mathrm{pf}_{2(m+1)}(S)\right)$$

(by proposition B.10 (p.95) ).

□

**Remark 4.23.** Whenever we know the intersection form and the Saito matrix for an irreducible plane curve singularity we can use this theorem to find equations for the $\delta$-constant stratum and the other strata $D(k)$.

## 4.4 $D(\delta)$ is Cohen–Macaulay for $A_{2k}, E_6$ and $E_8$ singularities

The proof that $D(\delta)$ is Cohen–Macaulay for $A_{2k}$ is proved by Sevenheck in [Sev03], (Theorem 1.15.) using a theorem of Givental in [Giv90] concerning open-swallowtails. We will outline these results in this section.

We will go on to prove that $D(\delta)$ is Cohen-Macaulay in the case of the $E_6$ and $E_8$ singularities using the structure for $D(\delta)$ we found in theorem 4.22.

### 4.4.1 The open swallowtail

In this section we will introduce the open-swallowtail for the $A_{2k}$ singularities. We shall follow [Giv90], §7, p. 3261 and [Sev03], §1.2, p.19, in which the open swallowtail is shown to be a Lagrangian Cohen-Macaulay singularity.

**Definition 4.24** (see [Giv90], §7, p. 3261)**.** Define $\mathcal{P}_{2k+1}$ to be the space of polynomials of odd degree with sum of roots equal to 0, which is equal to the miniversal deformation of the equation of the $A_{2k}$ singularity in $\mathbb{C}$.

$$\mathcal{P}_{2k+1} = \left\{ x^{2k+1} + \frac{a_2}{(2n+1)!} x^{2k-1} + \cdots + a_{2n+1} \,:\, a_i \in \mathbb{C} \right\} \simeq \mathbb{C}^{2n-1}.$$

According to Givental ([Giv90], §5, p. 3256), this is a symplectic space with symplectic structure:

$$\omega = \sum_{i=2}^{n+1} (-1)^i \mathrm{d}a_i \wedge \mathrm{d}a_{2k+3-i}$$

and the open swallowtail $\Sigma_n \subset \mathcal{P}_{2n+1}$, (the space of polynomials with a root of multiplicity $\geq n$) is a Lagrangian subvariety of $\mathcal{P}_{2n+1}$.

**Theorem 4.25** (see [Sev03], Theorem 1.15, p. 22)**.** *The open swallowtail $\Sigma_n$ is a Cohen–Macaulay singularity.*

*Proof.* We sketch the proof given in [Sev03], Theorem 1.15.

Consider the map

$$\Sigma_n \to \mathcal{P}_{n+1}$$

defined by $n$–fold differentiation of polynomials in $\Sigma_n$. This map is of degree $n+1$ because a generic polynomial $p \in \mathcal{P}_{n+1}$ has $n+1$ preimages. Indeed if $p$ has roots $t_1, \ldots, t_{n+1}$ then $p$ has $n+1$ preimages $(x - t_j)^{n+1} \prod_{\substack{i=1 \\ i \neq j}}^{n+1} (x - t_i)$ under this map. This implies that $\mathcal{O}_{\Sigma_n,0}$ is a finitely generated module over $\mathcal{O}_{\mathcal{P}_{n+1},0}$ of rank $n+1$.

A difficult theorem of Givental ([Giv90], Theorem 10, p. 3261) shows that $\mathcal{O}_{\Sigma_n,0}$ is generated by $1, a_2, \ldots, a_{2n+1}$ over $\mathcal{O}_{\mathcal{P}_{n+1},0}$ and so is free. Since $\mathcal{P}_{n+1} \simeq \mathbb{C}^{n-1}$ is smooth then any finitely generated module over $\mathcal{P}_{n+1}$ is Cohen–Macaulay if and only if it is free (see for instance [GM07] Corollary B.8.12, page 419). We therefore conclude that $\mathcal{O}_{\Sigma_n,0}$ is Cohen–Macaulay. □

Givental shows ([Giv90], Theorem 7, p. 3257) that there is a symplectomorphism between $(\mathbb{C}^{2n-1}, \omega)$ and $(\mathbb{C}^{2n-1}, \Phi)$ where $\Phi$ is the intersection form, the symplectic structure defined in section 2.2. This symplectomorphism carries $\Sigma_n$ to the closure of the space of polynomials with $n$ nondegenerate singularities, this is the same as $D^\delta$ by proposition 1.27. From this we deduce the following theorem.

**Theorem 4.26.** *The $\delta$–constant stratum is a Lagrangian Cohen–Macaulay singularity for $A_{2k}$ singularities*

□

### 4.4.2 $D(\delta)$ is Cohen-Macaulay for $E_6$ and $E_8$

The aim of this section is to prove that the structure given for $D^\delta$ in theorem 4.22:

$$\mathcal{O}_{D(\delta),0} = \mathcal{O}_{\mathbb{C}^\mu,0}/\operatorname{pf}_2(\chi^t\Omega\chi)$$

(where $\chi$ is the Saito matrix and $\Omega$ is the matrix of coefficients of the intersection form for the corresponding singularity) is reduced and a Cohen-Macaulay $\mathcal{O}_{\mathbb{C}^\mu,0}$-module for the $E_6$ and $E_8$ singularities.

We will use this fact together with a result of Sevenheck and van Straten in [vSS03] to prove that $D(\delta)$ is an example of a rigid Lagrangian singularity for these singularities.

**Theorem 4.27.** *For the $E_6$ and $E_8$ plane curve singularities, $\mathcal{O}_{D(\delta),0}$ is Cohen-Macaulay and is a reduced structure for $D(\delta)$.*

*Proof.* To show that $\mathcal{O}_{D(\delta),0}$ is Cohen-Macaulay it is sufficient to find regular sequences on $\mathcal{O}_{D(\delta),0}$ of length 3 and 4 for $E_6$ and $E_8$ respectively (indeed, the dimension of $D(\delta)$ is 3 in the case of $E_6$ and 4 in the case of $E_8$ (see theorem 1.26) so the existence of such regular sequences implies that $\mathcal{O}_{D(\delta),0}$ is Cohen-Macaulay).

We can compute both the Saito matrix $\chi$ (using the *Macaulay2* scripts given in appendix C.3 for $E_6$ and $E_8$) and the matrix of the intersection form $\Omega$ for $E_6$ and $E_8$ (both computed using the methods of section 2.3.1 and given in appendix A.2).

The matrix $\chi^t\Omega\chi$ is therefore known explicitly and so the *Depth* package of *Macaulay2* ([GS]) can be used to check (for instance) that $(\lambda_1, \lambda_2, \lambda_4)$ is a regular sequence in the case of $E_6$ and $(\lambda_1, \lambda_2, \lambda_3, \lambda_8)$ is a regular sequence in the case of $E_8$.

In [FGVS99] it is shown that the geometric degree of $D(\delta)$ (defined as the number of points in which a generic complementary dimensional hyperplane intersects $D(\delta)$

near 0) for a deformation of a plane curve singularity $C_0$, is equal to the Euler characteristic of the compactified Jacobian of $C_0$. This Euler characteristic is calculated for quasihomogeneous singularities in [Pio07] and is equal to 5 in the case of $E_6$ and 7 in the case of $E_8$. Using *Singular* we can verify that the algebraic degree of $\mathcal{O}_{D(\delta),0}$ for $E_6$ and $E_8$ (given as a coefficient of its Hilbert polynomial) coincides with these values and we conclude that $\mathcal{O}_{D(\delta),0}$ gives a reduced structure.

Indeed, as $\mathcal{O}_{D(\delta),0}$ is Cohen-Macaulay it has no embedded components so we can relate the degree $\deg \mathcal{O}_{D(\delta),0}$ given by the Hilbert polynomial and the degree $\deg D(\delta)$ by the formula ( see [BM92] (p.26, following definition 3.4) or [EH00] (p.148 Bézout's theorem with multiplicities)):

$$\deg \mathcal{O}_{D(\delta),0} = \mathrm{mult}(\mathrm{pf}_2(S)) \deg D(\delta)$$

and by the above computations we see that $\mathrm{mult}(\mathrm{pf}_2(S)) = 1$ from which it follows that the ideal $\mathrm{pf}_2(S)$ is radical and consequently that $\mathcal{O}_{D^{\delta},0}$ is a reduced structure.

□

## 4.5 An application: $D(\delta)_{E_6}$ and $D(\delta)_{E_8}$ are rigid Lagrangian singularities

In [vSS03], van Straten and Sevenheck discuss the deformation theory of Lagrangian singularities. In particular they prove that for irreducible plane curve singularities if $D(\delta)$ is Cohen-Macaulay then $D(\delta)$ is an example of a rigid Lagrangian deformation. They also conjecture that this is true for irreducible plane curve singularities and we prove this for $E_6$ and $E_8$.

**Definition 4.28.** Let $M$ be a $2n$-dimensional complex symplectic manifold i.e. a $2n$ dimensional complex manifold endowed with a closed nondegenerate 2-form, $\omega$. Suppose $L \subset M$ is a reduced subspace of dimension $n$ such that $L$ is a Lagrangian submanifold in a neighbourhood of each of its smooth points.

A *Lagrangian deformation* of $L$ over $S$ is a flat family $\mathcal{L} \to S$ with the condition that $\mathcal{L}_s$ is a Lagrangian submanifold of $M$ in a neighbourhood of each of its smooth points.

It is shown in [vSS03] and [vS06] that if $D(\delta)$ is Cohen-Macaulay then $D(\delta)$ is a rigid Lagrangian singularity. Van Straten and Sevenheck conjecture that for an irreducible plane curve singularity this is always true.

**Theorem 4.29.** *The $\delta$-constant stratum, $D^\delta$ in the base space of a miniversal deformation of the $E_6$ and $E_8$ singularities is an example of a rigid Lagrangian singularity.*

*Proof.* By conjecture 14 in [vSS03] the theorem is true if $D^\delta$ is Cohen-Macaulay. We proved this in section 4.4. □

Van Straten and Sevenheck prove this conjecture for the $A_{2n}$ singularities but they note that the only missing piece of the proof for other cases is to show that $D^\delta$ is Cohen-Macaulay. We have shown that $D(\delta)$ is Cohen-Macaulay for $E_6$ and so we deduce that it is an example of a rigid Lagrangian singularity.

## 4.6 The intersection form for a degenerate period map

In this section we will study intersection forms on the base space of a miniversal deformation constructed as the pullback of the intersection pairing via period maps which are degenerate. As before such a construction produces a closed 2-form on the base space but now the form is degenerate. On the basis of computer algebra experiments with these forms we are able to make some conjectures.

First we will consider the meaning of a degenerate period map. Recall that the period map, (discussed in section 2.2 p. 42) is a section of the cohomology bundle associated to an irreducible plane curve singularity.

To define $P_\omega$ we take a holomorphic 1–form $\omega$ on the total space of the central Milnor bundle $\mathcal{Y}'$ and restrict this form to each fibre to get a section of the cohomology bundle:

$$P_\omega(\lambda) = [\omega|_{X_\lambda}].$$

This section is called nondegenerate if the map

$$\nabla P_\omega(\lambda) : T_\lambda \Lambda' \to H^1(X_\lambda) \simeq \mathbb{C}^\mu$$

is an isomorphism for each $\lambda \in \Lambda \setminus D$. We use this map to pull back the intersection pairing $\mathrm{I}^\vee_\lambda$ on the fibres to a symplectic form on the base space $\Lambda$.

In proposition 2.16 it was shown that the period map of the form $y\mathrm{d}x$ is non-degenerate for quasihomogeneous singularities. In what follows we will consider degenerate period maps in the case of the $A_{2k}$ singularities.

Let $\omega = y^{2k}\mathrm{d}x$ then the period map of the form $\omega$ at a point $\lambda \in \Lambda \setminus D$ is:

$$P_\omega(\lambda) = y^{2k}\mathrm{d}x|_{X_\lambda} = \left( \sum_{i=0}^{2k-1} x^i \right)^k \mathrm{d}x|_{X_\lambda}$$

which is an exact form and so $P_\omega$ is the zero section of the cohomology bundle. This period map is therefore degenerate.

Let $\omega = y^{2k+1}\mathrm{d}x$ then the period map of the form $\omega$ at a point $\lambda \in \Lambda \setminus D$ is not exact but the period map is degenerate and we can use the *Maple* procedure given in appendix C.2 to compute a degenerate intersection form.

**Definition 4.30.** Let $P_\omega$ be a degenerate period map we call the pullback of the intersection pairing on fibres of the cohomology bundle to the base space a *degenerate intersection form* associated to the form $\omega$.

**Example 4.31.** In the case of the $A_2$ singularity we can compute the degenerate intersection form associated to the form $y^{2k+1}\mathrm{d}x$ for small $k$ using the *Maple* script in appendix C.2. The matrix of the resulting degenerate intersection forms is

$$\begin{pmatrix} 0 & h^k \\ -h^k & 0 \end{pmatrix}$$

where $h$ is the equation defining the discriminant of $A_2$ in $\Lambda$.

**Question 4.32.** Is this the matrix of the degenerate intersection form for all $k$?

I believe this to be true and perhaps this can be shown using a similar computation to those in section 2.3.1 using the quasihomogeneity of the miniversal deformation of $A_2$.

For $A_4$ the entries in the matrix of the degenerate intersection form for $y^3\mathrm{d}x$ are given in appendix A.3. We can check that the entries of this matrix generate an ideal defining the $\delta$-constant stratum. We can also check that the ideals of principal Pfaffians of the matrix define the varieties $D(k)$.

**Conjecture 4.33.** *Let $B$ be be matrix of coefficients of the degenerate intersection form associated to the form $y^3dx$ for the $A_{2k}$ singularity, then:*

$$D(\delta - m) = V\left(\mathrm{pf}_{2(m+1)}(B)\right)$$

*for $m$ from $0$ to $\delta - 1$, where $\delta$ is equal to $k$ the $\delta$-invariant of the $A_{2k}$ singularity.*

The varieties defined by the ideals of principal Pfaffians for $B$ coincide with the same varieties for the presentation matrix of the intersection module $M_\Omega$ where $\Omega$ is the nondegenerate intersection form for the same singularity. We pose the following question:

**Question 4.34.** Define a new module $M_B$ with presentation:

$$\mathcal{O}^{\mu}_{\mathbb{C}^\mu,0} \xrightarrow{B} \mathcal{O}^{\mu}_{\mathbb{C}^\mu,0} \longrightarrow M_B \longrightarrow 0$$

then what is the relationship between the module $M_B$ and the intersection module $M_\Omega$? Is $M_B$ a maximal Cohen-Macaulay $\mathcal{O}_{D,0}$ module?

# Appendix A

# Results of Computations

## A.1 A presentation matrix for the module $M_\Omega$ in the case of $A_4$

The matrix $\chi$ whose columns consist of coefficients of a basis of vector fields tangent to the discriminant of $A_4$ is:

$$\chi_{A_4} = \begin{pmatrix} -10\lambda_1 & -15\lambda_2 & 6\lambda_1^2 - 20\lambda_3 & 4\lambda_1\lambda_2 - 25\lambda_4 \\ -15\lambda_2 & 6\lambda_1^2 - 20\lambda_3 & 13\lambda_1\lambda_2 - 25\lambda_4 & 6\lambda_2^2 + 2\lambda_1\lambda_3 \\ -20\lambda_3 & 4\lambda_1\lambda_2 - 25\lambda_4 & 6\lambda_2^2 + 2\lambda_1\lambda_3 & 11\lambda_2\lambda_3 - 15\lambda_1\lambda_4 \\ -25\lambda_4 & 2\lambda_1\lambda_3 & 3\lambda_2\lambda_3 & 4\lambda_3^2 \end{pmatrix}$$

The matrix $\Omega$ of coefficients of the intersection form for $A_4$ is:

$$\Omega = \begin{pmatrix} 0 & -\frac{1}{3}\lambda_1 & 0 & -\frac{1}{3} \\ \frac{1}{3}\lambda_1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 \end{pmatrix}$$

The presentation matrix $S = \chi^t \Omega \chi$ of the module $M_\Omega$ appears over the page.

$$
\begin{pmatrix}
0 & 20\lambda_1^4+135\lambda_1\lambda_2^2-180\lambda_1^2\lambda_3+400\lambda_3^2-250\lambda_2\lambda_4 & \frac{40}{3}\lambda_1^3\lambda_2+90\lambda_2^3-120\lambda_1\lambda_2\lambda_3-\frac{400}{3}\lambda_1^2\lambda_4+\frac{2000}{3}\lambda_3\lambda_4 & \frac{20}{3}\lambda_1^3\lambda_3+45\lambda_2^2\lambda_3-\frac{80}{3}\lambda_1\lambda_3^2-\frac{400}{3}\lambda_1\lambda_2\lambda_4+\frac{625}{3}\lambda_4^2 \\
 & 0 & 12\lambda_1^5+81\lambda_1^2\lambda_2^2-88\lambda_1^3\lambda_3+135\lambda_2^2\lambda_3+160\lambda_1\lambda_3^2-550\lambda_1\lambda_2\lambda_4+625\lambda_4^2 & 8\lambda_1^4\lambda_2+54\lambda_1\lambda_2^3-72\lambda_1^2\lambda_2\lambda_3+240\lambda_2\lambda_3^2+40\lambda_1^3\lambda_4-150\lambda_2^2\lambda_4-200\lambda_1\lambda_3\lambda_4 \\
 & & 0 & \frac{16}{3}\lambda_1^3\lambda_2^2+36\lambda_2^4-4\lambda_1^4\lambda_3-75\lambda_1\lambda_2^2\lambda_3+\frac{28}{3}\lambda_1^2\lambda_3^2+\frac{80}{3}\lambda_3^3+\frac{160}{3}\lambda_1^2\lambda_2\lambda_4+250\lambda_2\lambda_3\lambda_4-\frac{500}{3}\lambda_1\lambda_4^2 \\
 & & & 0
\end{pmatrix}
$$

The presentation matrix for the module $M_\Omega$ for the $A_4$ singularity.

The matrix is skew-symmetric so the entries below the antidiagonal can be deduced from those given here.

## A.2 The matrix of coefficients of the intersection form for $E_6$ and $E_8$

The matrix of coefficients of the intersection form corresponding to a miniversal deformation of the $E_6$ singularity (the parameters $\lambda_1, \ldots, \lambda_6$ are ordered by ascending weight)

$$F_\lambda(x,y) = x^3 + y^4 + \lambda_1 y^2 x + \lambda_2 xy + \lambda_3 x + \lambda_4 y^2 + \lambda_5 y + \lambda_6$$

is given by:

$$\begin{pmatrix} 0 & \frac{1}{15}\lambda_1\lambda_2 & -\frac{2}{15}\lambda_1^2 & -\frac{1}{5}\lambda_4 & 0 & -\frac{1}{5} \\ -\frac{1}{15}\lambda_1\lambda_2 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\ \frac{2}{15}\lambda_1^2 & 0 & 0 & 1 & 0 & 0 \\ \frac{1}{5}\lambda_4 & 0 & -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix of coefficients of the intersection form corresponding to a miniversal deformation of the $E_8$ singularity (the parameters $\lambda_1, \ldots, \lambda_8$ are ordered by ascending weight):

$$F_\lambda(x,y) = x^3 + y^5 + \lambda_1 y^3 x + \lambda_2 y^2 x + \lambda_3 y^3 + \lambda_4 xy + \lambda_5 y^2 + \lambda_6 x + \lambda_7 y + \lambda_8$$

is given over the page.

$$
\begin{pmatrix}
0 & -\frac{1}{35}\lambda_1^3\lambda_2^2-\frac{1}{105}\lambda_1^5\lambda_3-\frac{1}{14}\lambda_1\lambda_2\lambda_3+\frac{2}{105}\lambda_1^4\lambda_4-\frac{1}{42}\lambda_2\lambda_4+\frac{13}{420}\lambda_1^2\lambda_5-\frac{1}{42}\lambda_1\lambda_6 & -\frac{1}{105}\lambda_1^5\lambda_2+\frac{1}{21}\lambda_1\lambda_2^2-\frac{2}{35}\lambda_1^3\lambda_3+\frac{4}{35}\lambda_1^2\lambda_4+\frac{1}{14}\lambda_5 & \frac{2}{105}\lambda_1^4\lambda_2-\frac{2}{21}\lambda_2^2+\frac{4}{35}\lambda_1^2\lambda_3-\frac{4}{21}\lambda_1\lambda_4 & \frac{1}{105}\lambda_1^6-\frac{23}{105}\lambda_1^2\lambda_2-\frac{3}{7}\lambda_3 & -\frac{1}{105}\lambda_1^5+\frac{1}{7}\lambda_1\lambda_2 & -\frac{1}{35}\lambda_1^3 & \frac{1}{7} \\
 & 0 & \frac{1}{12}\lambda_1^2\lambda_2 & -\frac{1}{6}\lambda_1\lambda_2 & -\frac{1}{12}\lambda_1^3 & \frac{1}{12}\lambda_1^2 & \frac{1}{4} & 0 \\
 & & 0 & 0 & -\frac{1}{2}\lambda_1 & \frac{1}{2} & 0 & 0 \\
 & & & 0 & 1 & 0 & 0 & 0 \\
 & & & & 0 & 0 & 0 & 0 \\
 & & & & & 0 & 0 & 0 \\
 & & & & & & 0 & 0 \\
 & & & & & & & 0
\end{pmatrix}
$$

The matrix of coefficients of the intersection form for the $E_8$ singularity.

The matrix is skew–symmetric so the entries below the antidiagonal can be deduced from those given here.

## A.3 The matrix of coefficients of a degenerate intersection form of $A_4$

The matrix of coefficients of a degenerate intersection form (where the period map is defined by $y^3\mathrm{d}x$) corresponding to a miniversal deformation of the $A_4$ singularity

$$F(x, y, \lambda) = -y^2 + x^5 + \lambda_1 x^3 + \lambda_2 x^2 + \lambda_3 x + \lambda_4$$

is given over the page.

$$\Omega_{1,2} = -7\lambda_1^6 - \frac{1041}{20}\lambda_1^3\lambda_2^2 + \frac{281}{5}\lambda_1^4\lambda_3 - \frac{162}{5}\lambda_2^4 - \frac{27}{10}\lambda_1\lambda_2^2\lambda_3 - \frac{543}{5}\lambda_1^2\lambda_3^2 + \frac{2547}{10}\lambda_1^2\lambda_2\lambda_4 - \frac{84}{5}\lambda_3^3 - \frac{459}{2}\lambda_2\lambda_3\lambda_4 - \frac{745}{4}\lambda_1\lambda_4^2$$

$$\Omega_{1,3} = -\frac{44}{5}\lambda_1^4\lambda_2 - \frac{297}{5}\lambda_1\lambda_2^3 + \frac{396}{5}\lambda_1^2\lambda_2\lambda_3 - \frac{154}{5}\lambda_1^3\lambda_4 - \frac{1188}{5}\lambda_2\lambda_3^2 + \frac{297}{2}\lambda_2^2\lambda_4 + 154\lambda_1\lambda_3\lambda_4$$

$$\Omega_{1,4} = \frac{77}{5}\lambda_1^5 + \frac{2079}{20}\lambda_1^2\lambda_2^2 - \frac{726}{5}\lambda_1^3\lambda_3 - \frac{891}{20}\lambda_2^2\lambda_3 + \frac{1672}{5}\lambda_1\lambda_3^2 - \frac{121}{2}\lambda_1\lambda_2\lambda_4 - \frac{825}{4}\lambda_4^2$$

$$\Omega_{2,3} = -13\lambda_1^5 - \frac{351}{4}\lambda_1^2\lambda_2^2 + \frac{494}{5}\lambda_1^3\lambda_3 - \frac{2457}{20}\lambda_2^2\lambda_3 - \frac{936}{5}\lambda_1\lambda_3^2 + \frac{1053}{2}\lambda_1\lambda_2\lambda_4 - \frac{2275}{4}\lambda_4^2$$

$$\Omega_{2,4} = -\frac{78}{5}\lambda_1^3\lambda_2 - \frac{1053}{10}\lambda_2^3 + \frac{702}{5}\lambda_1\lambda_2\lambda_3 + 156\lambda_1^2\lambda_4 - 780\lambda_3\lambda_4$$

$$\Omega_{3,4} = -\frac{143}{5}\lambda_1^4 - \frac{3861}{20}\lambda_1\lambda_2^2 + \frac{1287}{5}\lambda_1^2\lambda_3 - 572\lambda_3^2 + \frac{715}{2}\lambda_2$$

The entries of the matrix $\Omega$ of coefficients of the intersection form for $A_4$ singularity where the period map is degenerate (defined by the form $y^3\mathrm{d}x$).

The matrix is skew–symmetric so the entries below the antidiagonal can be deduced from those given here.

# Appendix B

# On The Expansion of Pfaffians

The aim of this appendix is to state a result of Heymans [Hey69] in which it is shown how a minor of a skew-symmetric matrix can be expanded in Pfaffians of the matrix. We then show a consequence of these expansions for determinantal varieties defined by such matrices.

## B.1 The Pfaffian of a skew-symmetric matrix

We will begin by defining the Pfaffian and establishing our notation.

Let $M$ be a $2n \times 2n$ skew-symmetric matrix defined over a commutative ring $R$, that is a matrix $M = (x_{ij})$ satisfying $x_{ij} = -x_{ji}$. The Pfaffian is a multilinear form of degree $n$ in the entries of the matrix.

**Definition B.1** (Pfaffian)**.** Let $\Pi$ be the set of unordered partitions of $\{1, \ldots, 2n\}$ into $n$ unordered pairs. A typical element of $\Pi$ can be written as

$$\{\{i_1, j_1\}, \ldots, \{i_n, j_n\}\}$$

For each partition $p \in \Pi$ let $o(p)$ be any particular ordering:

$$o(p) = ((i_1, j_1), \ldots, (i_n, j_n))$$

and define a permutation and a monomial associated to $o(p)$:

$$\pi_{o(p)} = \begin{bmatrix} 1 & 2 & \cdots & 2n-1 & 2n \\ i_1 & j_1 & \cdots & i_n & j_n \end{bmatrix} \qquad X_{o(p)} = \prod_{k=1}^{n} x_{i_k j_k}$$

We then define the *Pfaffian* of $M$ as follows:

$$\mathrm{pf}_{2n}(M) = \sum_{p \in \Pi} \mathrm{sgn}(\pi_{o(p)}) X_{o(p)}$$

For this to be well defined we must check that the Pfaffian is independent of choice of $o(p)$.

**Proposition B.2.** $\mathrm{sgn}(\pi_{o(p)})X_{o(p)}$ *is independent of the choice of ordering.*

*Proof.* Firstly, if we make a different choice for $o(p)$ where the entries in the first pair, say, are transposed:

$$o(p)' = ((j_1, i_1), (i_2, j_2), \ldots, (i_n, j_n))$$

then

$$X_{o(p)'} = x_{j_1 i_1} \prod_{k=2}^{n} x_{i_k, j_k} = -x_{i_1, j_1} \prod_{k=2}^{n} x_{i_k, j_k} = -X_{o(p)}$$

Furthermore, since $\pi_{o(p)'}$ is obtained from $\pi_{o(p)}$ by composing with a transposition $(i_1, j_1)$ we see that the sign of $\pi_{o(p)'}$ is equal to minus the sign of $\pi_{o(p)}$:

$$\mathrm{sgn}((i_1, j_1) \circ \pi_{o(p)'}) = -\,\mathrm{sgn}(\pi_{o(p)})$$

and so:

$$\mathrm{sgn}(\pi_{o(p)})X_{o(p)} = \mathrm{sgn}(\pi_{o(p)'})X_{o(p)'}$$

Secondly, if we choose $o(p)''$ where the first two pairs, say, have exchanged positions:

$$o(p)'' = ((i_2, j_2), (i_1, j_1), (i_3, j_3), \ldots, (i_n, j_n))$$

Then $\pi_{o(p)''}$ is obtained from $\pi_{o(p)}$ by composing with two transpositions and so the sign remains unchanged:

$$\mathrm{sgn}((i_1, i_2) \circ (j_1, j_2) \circ \pi_{o(p)''}) = \mathrm{sgn}(\pi_{o(p)})$$

Furthermore for the monomials:

$$X_{o(p)''} = x_{i_2 j_2} x_{i_1 j_1} \prod_{k=3}^{n} x_{i_k j_k} = x_{i_1 j_1} x_{i_2 j_2} \prod_{k=3}^{n} x_{i_k j_k} = X_{o(p)}$$

and so choosing $o(p)''$ does not change the value of the Pfaffian.

Since all choices of orderings for $p$ differ by operations of these two types the value of the Pfaffian does not depend on the choice of ordering. □

We can simplify the definition of the Pfaffian further by observing that for each partition $p$ we can make a choice of ordering $o_+(p)$ such that the associated permutation $\pi_{o_+(p)}$ has positive sign. Then using the definition of the Pfaffian we see:

$$\mathrm{pf}(M) = \sum_{p \in \Pi} X_{o_+(p)}$$

This is the form of the Pfaffian we will use subsequently.

## B.2 Quadratic forms in Pfaffians

First we will establish a notation for minors and Pfaffians of the matrix $M \in \mathrm{SkewMat}(2n, \mathbb{C})$.

**Definition B.3.** Let $I = \{i_1, \ldots, i_s\}$ and $J = \{j_1, \ldots, j_s\}$ be subsets of $L = \{1, \ldots, 2n\}$.

Define $\min(I; J)$ as the minor of the submatrix consisting of the rows and columns of $M$ indexed by the sets $I$ and $J$ respectively.

**Remark B.4.** Let $K = I \cap J$ then $\min(K; K)$ is the largest symmetrically placed or 'principal' minor of the matrix $M$.

Similarly we define $\mathrm{pf}(I)$ as the Pfaffian of the submatrix formed by the rows and columns of $M$ indexed by the set $I$.

We will need to state a result of Heymann's on the expansions of minors of $M$ as a quadratic form in Pfaffians.

**Definition B.5.** For a minor $\min_s(I;J)$ let $K = I \cap J$ and define $\mathrm{pf}_{n,m}(I;J)$ to be the following sum:

$$\sum_{\substack{I_1 \subset I \setminus K,\, |I_1|=n \\ J_1 \subset J \setminus K,\, |J_1|=m}} \mathrm{pf}(I_1 \cup J_1 \cup K)\,\mathrm{pf}(I_2 \cup J_2 \cup K)$$

where $I_2 = (I \setminus K) \setminus I_1$ and $J_2 = (J \setminus K) \setminus J_1$

**Example B.6.** Let $M$ be a $8 \times 8$ skew-symmetric matrix. We will compute the quadratic form $\mathrm{pf}_{1,0}$ for the minor $\min(\{1,2,3,4\},\{4,5,6,7\})$.

By the definition of the quadratic form there are 3 terms, since there are 3 possible choices for the sets $I_1$ and $I_2$ namely

$$(I_1, I_2) = (\{1\},\{2,3\}), (\{2\},\{1,3\}) \text{ or } (\{3\},\{1,3\})$$

there is only 1 choice for the sets $J_1$ and $J_2$ namely

$$(J_1, J_2) = (\emptyset, \{5,6,7\})$$

We conclude that the quadratic form is equal to:

$$\begin{aligned}\mathrm{pf}_{1,0} = {} & \mathrm{pf}_2(\{1,4\})\,\mathrm{pf}_6(\{2,3,5,6,7,4\}) + \mathrm{pf}_2(\{2,4\})\,\mathrm{pf}_6(\{1,3,5,6,7,4\}) + \\ & + \mathrm{pf}_2(\{3,4\})\,\mathrm{pf}_6(\{1,2,5,6,7,4\})\end{aligned}$$

Heymans describes many expansions of minors in terms of Pfaffians, we will need the 'canonical expansion'. The following theorem shows how to expand an arbitrary minor $\min(I;J)$ in terms of Pfaffians. The form of the expansion depends on the parity of $|I|$ and $K$.

**Theorem B.7** (see [Hey69] page 743, equations 3.42-3.45)**.** *Let $I, J$ be subsets of $L = \{1, \dots, n\}$ of cardinality $s$. Let $K = I \cap J$, $k = |K|$ and $t = s - k$. Then* $\min(I; J) =$

$$\begin{cases} \sum_{p=0}^{a} \lambda_a(p)\,\mathrm{pf}_{a+p,a-p}, & t = 2a, k = 2b \quad \textit{(Case 1)} \\ 2\sum_{p=0}^{a-1} \lambda_a(p)\,\mathrm{pf}_{a-1+p,a-p}, & t = 2a-1, k = 2b \quad \textit{(Case 2)} \\ 2\sum_{p=0}^{a-1} \nu_a(p)\,\mathrm{pf}_{a+p,a-p}, & t = 2a, k = 2b+1 \quad \textit{(Case 3)} \\ \sum_{p=0}^{a} \nu_a(p)\,\mathrm{pf}_{a+p+1,a-p}, & t = 2a+1, k = 2b+1 \quad \textit{(Case 4)} \end{cases}$$

*where the coefficients $\lambda_a(p)$ for $p = 0, \dots, s$ are rational numbers defined by the following recursive formulas*

$$\begin{cases} \binom{2a}{a}\lambda_a(0) = 1 \\ \sum_{p=0}^{r} \sum_{u=0}^{r} \binom{r}{u}\binom{r}{u-p}\binom{2a-2r-1}{a+p-2u-1}\lambda_a(p) = 0 \quad (r = 1, \dots, a) \end{cases}$$

*and the coefficients $\nu_a(p)$ are integers defined by:*

$$\nu_a(0) = 2\lambda_{a+1}(0) - \lambda_{a+1}(1)$$
$$\nu_a(p) = \lambda_{a+1}(p) - \lambda_{a+1}(p+1)$$

We summarise this theorem with the following theorem.

**Theorem B.8.** *Using the formulas for expansion of Pfaffians in the previous theorem we reach the following conclusions:*

1. *Minors of a skew-symmetric matrix $M$ of even order $2s$ can be expanded in terms of products of pairs Pfaffians of order $2s$. (Cases 1 and 4)*

$$\mathrm{Min}_{2s}(M) \subseteq (\mathrm{Pf}_{2s}(M))^2$$

2. *Minors of a skew-symmetric matrix of odd order $2s-1$ can be expanded in terms of products of Pfaffians of order $2s-2$ and $2s$. (Cases 2 and 3)*

$$\mathrm{Min}_{2s-1}(M) \subseteq \mathrm{Pf}_{2s}(M)\,\mathrm{Pf}_{2s-2}(M)$$

*where* $\mathrm{Min}_k(M)$ *and* $\mathrm{Pf}_k(M)$ *are the ideals of order* $k$ *minors and principal Pfaffians of the matrix* $M$ *respectively.*

**Example B.9.** Let $M = (x_{ij})_{i,j=1}^{6}$ be a $6 \times 6$ skew-symmetric matrix. We will expand the minor $\mathrm{min}_3(\{1,2,3\};\{1,2,4\})$ in terms of Pfaffians using the previous theorem. We have $I \setminus K = \{3\}$, $J \setminus K = \{4\}$ and $K = \{1,2\}$ then we are in case 2 of the theorem with $t = 1$ and $k = 1$. So the required expansion is:

$$2\lambda_1(0)\,\mathrm{pf}_{0,0}(I;J)$$

To compute the coefficients in the expansion use that $\binom{2}{1}\lambda_1(0) = 1$ so $\lambda_1(0) = \frac{1}{2}$ and the quadratic form $\mathrm{pf}_{0,1}(I,J)$ has only one term meaning the expansion becomes:

$$\mathrm{pf}_2(1,2)\,\mathrm{pf}_4(1,2,3,4) = x_{12}(x_{14}x_{23} + x_{31}x_{24} + x_{12}x_{34})$$

which is indeed equal to the minor.

## B.3 An application to rank loci

For a $2n \times 2n$ skew-symmetric matrix $M$ the ideal generated by order $2k$ minors defines the "rank$< 2k$ locus" of the matrix. We will use Heymans' theorem to show that this locus can be defined using the Pfaffians of $M$ instead. Working in the ring $\mathbb{C}[x_{ij}]$, $(1 \le i < j \le 2n)$ let $\mathrm{Min}_{2k}(M)$ and $\mathrm{pf}_{2k}(M)$ denote the ideals generated by the order $k$ minors and order $k$ Pfaffians of the matrix $M$ respectively.

**Proposition B.10.** *Using the notation in the previous paragraph we have the following equality of varieties:*

$$V(\mathrm{Min}_{2k-1}) = V(\mathrm{pf}_{2k})$$

*Proof.* $\mathrm{Min}_{2k-1}$ is contained in $\mathrm{pf}_{2k}$ since a minor $m \in \mathrm{Min}_{2k-1}$ can be expanded in terms of products of pairs of Pfaffians of order $2k$ and $2k-2$ (by theorem B.7), therefore $m \in \mathrm{pf}_{2k}$. We deduce that $V(\mathrm{pf}_{2k}) \subset V(\mathrm{min}_{2k-1})$.

$\mathrm{pf}_{2k}$ is contained in the radical of $\mathrm{Min}_{2k}$ since the square of a Pfaffian $p \in \mathrm{pf}_{2k}$ is equal to a principal order $2k$ minor of the matrix. Furthermore, since every order $2k$ minor can be expanded in terms of order $2k-1$ minors, $\mathrm{Min}_{2k}$ is contained in $\mathrm{Min}_{2k-1}$ and hence $\mathrm{pf}_{2k}$ is contained in the radical of $\mathrm{Min}_{2k-1}$. By Hilbert's Nullstellensatz we deduce that $V(\mathrm{Min}_{2k-1}) \subset V(\mathrm{pf}_{2k})$ □

**Remark B.11.** By the previous proposition we see that the "rank$< 2k$ locus" is equal to the "rank$< (2k-1)$ locus" of $M$. Indeed, if all order $2k$ minors vanish then in particular all order $2k$ Pfaffians vanish and by the proposition so then do all the order $2k-1$ minors. In this case the matrix $M$ must have rank less than or equal to $2k-2$.

# Appendix C

# Maple and Macaulay2 scripts

## C.1 A Maple procedure to compute the coefficients of the intersection form associated to $A_{2n}$

```
#maple procedure to calculate the intersection form coefficients
#for a miniversal deformation of the A_{2N} singularity.
#
#Input: The integer N, corresponding to A_{2N}
#Output: intform(N) = Coefficients g_{ij} of the intersection 2-form
#where 1<=i<=n , i+1<=j<=n+1-i
#other coefficents can be obtained from these since
#the 2-form is skew symmetric and g_{ij}=0 whenever i+j > n+1
#
#Limitation: Can be used to calculate form upto A_{20} on a 2010 desktop machine
#

intform := proc(N::integer)
local t,c,a,n,f,i,co,B,h,k,g,G,A,m,X,x,y,Y,w,Temp,j,C,F:
global R,P:
n := 2*N:

#Define the A_n singularity in C^2

f := -y^2 +x^(n+1):

#Define its miniversal deformation with deformation parameters c[i]

for i from 1 to n do
```

```
    f := f + c[i] *x^(n-i):
od:

#co[i] stores the basis of the Milnor algebra to use later.

for i from 1 to n do
    co[i] := coeff(f,c[i]);
od:

B := diff(f,y):

#Find a parametrisation for the deformed curve.
#We know that the y component of the parametrisation is of this form:

h := subs(y=t^(-(n+1)),f):

#Finding the x component of the parametrisation:

k := 1:

for i from 1 to n do
    k := k + a[2*i]*t^(2*i); #only even coefficients are nonzero
od:

g := subs(x=t^(-2)*k,h): #t^(-2) is the lowest power of t that occurs

G[2] := g;

#G[2] := convert(series(g,t=0,2*(n+1)),polynom):

#In the pullback of the equation of the curve,
#recursively the coefficient of t^p gives an equation involving a[p]
#and the c[i]. We can solve this equation to find a[p].

for i from 1 to n do
    A[2*i] := solve(coeff(G[2*i],t,2*i-2*(n+1))=0,a[2*i]);
    G[2*i+2] := subs(a[2*i]=A[2*i],G[2*i]);
od:


m := nops([op(convert(A,list))]): #finds the number of entries in A

#Initialisation of the pullback of x,y,co[i]
```

```
X := 1:

for i from 1 to m do
    X := X + A[2*i]*t^(2*i);
od:

X := t^(-2)*X:
Y := t^(-(n+1)):

B := subs(y=Y,B):

for i from 1 to n do
    co[i] := subs(x=X,co[i]):
od:

#Call the difforms package

with(difforms):
defform(t=scalar,u=scalar,v=scalar):
for i from 1 to n do
    defform(c[i]=scalar): #c[i] is a fixed parameter
od:

for i from 1 to n do
#Calculate the pull back of the Gauss-Manin derivative
#with respect to each coordinate direction in the base:
#
#w[i] := (1/(B))*co[i]*coeff(d(X),d(t)):
#
#Truncate the resulting polynomial for speedup.

    w[i] := convert(series((1/(B))*co[i]*coeff(d(X),d(t)),t=0,2*n),polynom):

    Temp := expand(w[i],t):

    #find the antiderivative of w[i] in the image of the parametrisation
    for j from -2*n to -1 do
        C[j] := coeff(Temp,t,j-1)/(j):
    od:

    C[0] := 0:
```

```
    for j from 1 to 2*n do
        C[j] := coeff(Temp,t,j-1)/(j):
    od:

    F[i] := 0:

    for j from -2*n to 2*n do
        F[i] := F[i] + C[j]*t^j:
    od:

od:

#Residues

for i from 1 to n do
    for j from i+1 to n do
        if i+j <= n+1 then #entries g_ij in the matrix where i+j > n+1 are zero
            R[i][j] := expand(coeff(F[i]*w[j],t^(-1))):
            print(R[i][j]):
        end if:
    od:
od:


#check resulting form is closed (if not then there is a problem with the calculation)

P := 0:

for i from 1 to n do
    for j from i+1 to n+1-i do
        P := P +R[i][j]*(d(c[j])&^d(c[i])):
    od:
od:

if (simpform(d(P)) <> 0) then
    RETURN(NOTCLOSED):
end if:

if (simpform(d(P)) = 0) then
    RETURN():
end if:

end proc:
```

## C.2 Maple procedure for computing the degenerate intersection form for $A_{2n}$

Similar maple code to compute the coefficients of the degenerate intersection form for $A_{2n}$ computed using the period map associated to $y^3 \mathrm{d}x$.

```
#Maple procedure to calculate the coefficients of the degenerate intersection form for the
#A_{2N} singularity (using period map defined my y^3 dx
#
#Input: The integer N, corresponding to A_{2N}
#Output: intform(N) = Coefficients g_{ij} of the degenerate intersection 2-form
#where 1<=i<=n , i+1<=j<=n
#other coefficents can be obtained from these since
#the 2-form is skew symmetric
#
#Limitation: Can be used to calculate form upto A_{6} on a 2010 desktop machine
#

intform := proc(N::integer)
local t,c,a,f,i,co,B,h,k,g,G,A,m,x,y,Y,Temp,j,C,P:
global R,n,X,w,F:
n := 2*N:
f := -y^2 +x^(n+1):

for i from 1 to n do
    f := f + c[i] *x^(n-i):
od:

for i from 1 to n do
    co[i] := coeff(f,c[i]);
od:

B := diff(f,y):

h := subs(y=t^(-(n+1)),f):

k := 1:

for i from 1 to 10*n do
    k := k + a[2*i]*t^(2*i);
od:
```

```
g := subs(x=t^(-2)*k,h):

G[2] := convert(series(g,t=0,20*(n+1)),polynom):

for i from 1 to 4*n do
    A[2*i] := solve(coeff(G[2*i],t,2*i-2*(n+1))=0,a[2*i]);
    G[2*i+2] := subs(a[2*i]=A[2*i],G[2*i]);
od:

m := nops([op(convert(A,list))]):
X := 1:

for i from 1 to m do
    X := X + A[2*i]*t^(2*i);
od:

X := t^(-2)*X:
Y := t^(-(n+1)):

B := subs(y=Y,B):

for i from 1 to n do
    co[i] := subs(x=X,co[i]):
od:

with(difforms):
defform(t=scalar,u=scalar,v=scalar):
for i from 1 to n do
    defform(c[i]=scalar):
od:

for i from 1 to n do
    w[i] := convert(series(Y*co[i]*coeff(d(X),d(t)),t=0,10*n),polynom):

    Temp := expand(w[i],t):

    for j from -10*n to -1 do
        C[j] := coeff(Temp,t,j-1)/(j):
    od:

    C[0] := 0:

    for j from 1 to 10*n do
```

```
        C[j] := coeff(Temp,t,j-1)/(j):
    od:

    F[i] := 0:

    for j from -10*n to 10*n do
        F[i] := F[i] + C[j]*t^j:
    od:

od:

#Residues

for i from 1 to n do
    for j from i+1 to n do
        R[i][j] := expand(coeff(F[i]*w[j],t^(-1))):
        print(R[i][j]):
    od:
od:


#check closed

P := 0:

for i from 1 to n do
    for j from i+1 to n do
        P := P + R[i][j]*(d(c[j])&^d(c[i])):
    od:
od:

if (simpform(d(P)) <> 0) then
        RETURN(NOTCLOSED):
end if:

if (simpform(d(P)) = 0) then
        RETURN():
end if:

end proc:
```

## C.3 Macaulay2 scripts to compute the Saito matrix for $E_6$ and $E_8$

### C.3.1 $E_6$

```
--Script to compute the delta-constant stratum for E6 and check that its depth is 3.
--First define the ring defining the critical space O_{\sigma}
loadPackage "Depth";

--First deine the Tjurina algebra for E_6
R = QQ[x,y,a,b,c,d,e,f,Degrees=>{3,4,2,5,8,6,9,12},MonomialSize=>32];
f = y^3+x^4+a*x^2*y+b*x*y+c*y+d*x^2+e*x+f;
p = matrix{{f,diff(x,f),diff(y,f)}};
M = cokernel(p);

--use the pushFoward function to write the Tjurina algebra as an \OO_{\Cm^{\mu}}--module.
S = QQ[A,B,C,D,E,F,Degrees=>{2,5,8,6,9,12},MonomialSize=>32];
fn = map(R,S,matrix{{a,b,c,d,e,f}})
en = pushForward(fn,M);
crit = trim(en);

--dis is ideal defining the discriminant.
dis = fittingIdeal(0,crit);

--the calculation of a matrix whose columns are coefficients of a basis of \der(-log \D)
jdisc = gens(ideal(jacobian(disc)));
gdisc = gens disc;
der = modulo(jdisc,gdisc);

--the matrix of coefficients of the intersection form
ME6 = matrix{{0,1/15*A*B,-2/15*A^2,-1/5*D,0,-1/5},{-1/15*A*B,0,0,0,-1/2,0},
{2/15*A^2,0,0,1,0,0},{1/5*D,0,-1,0,0,0},{0,1/2,0,0,0,0},{1/5,0,0,0,0,0}};

--the presentation matrix for the module M_{\Omega}, the entries of which form an ideal
--defining the \delta-constant stratum
A = transpose(der)*ME6*der;
qIdelta = ideal(A);
Idelta = trim(A);
```

### C.3.2 $E_8$

Using the procedure in remark 1.53,(p.32) we compute the Saito matrix for a miniversal deformation of $E_8$.

```
R = QQ[x,y,a_1..a_8,Degrees=>{5,3,1,4,6,7,9,10,12,15}];
F = x^3+y^5+a_1*x*y^3+a_2*x*y^2+a_3*y^3+a_4*x*y+a_5*y^2+a_6*x+a_7*y+a_8;
Fx = diff(x,F);
Fy = diff(y,F);
S = R/(Fx,Fy);

--a function which takes f and returns the coefficients of f in a basis of S.
cof1 = f -> transpose((coefficients(f,Variables=>{x,y},
        Monomials=>{x*y^3,x*y^2,y^3,x*y,y^2,x,y,1}))_1)

--MY is the matrix of the action of multiplying by y on S.
--first write y(x*y^3) and y(y^3) in terms of the basis:

XY4 = promote(x*y^4,S);
Y4 = promote(y^4,S);

MY = transpose(matrix{{cof1(XY4)},{cof1(x*y^3)},{cof1(Y4)},{cof1(x*y^2)},{cof1(y^3)},
        {cof1(x*y)},{cof1(y^2)},{cof1(y)}});

--MX is the matrix of the action of multiplying by x on S.
--first write x(xy^3), x(xy^2), x(xy), and x(x) in terms of a basis of S.

X2Y3 = promote(x^2*y^3,S);
X2Y2 = promote(x^2*y^2,S);
X2Y = promote(x^2*y,S);
X2 = promote(x^2,S);

MX = transpose(matrix{{cof1(X2Y3)},{cof1(X2Y2)},{cof1(x*y^3)},{cof1(X2Y)},{cof1(x*y^2)},
        {cof1(X2)},{cof1(x*y)},{cof1(x)}});

--the column of coefficients of the Euler vector field
E = transpose(matrix{{a_1,4*a_2,6*a_3,7*a_4,9*a_5,10*a_6,12*a_7,15*a_8}});

--the following columns are syzergies of S as an QQ[a_1..a_8]-module.

XE = MX*E;
YE = MY*E;
XYE = MX*YE;
```

```
Y2E = MY*YE;
Y3E = MY*Y2E;
XY2E = MX*Y2E;
XY3E = MX*Y3E;

--the Saito matrix
der = E | YE | XE | Y2E | XYE | Y3E | XY2E | XY3E;
```

# Bibliography


[AGZV88] V.I Arnold, S.M. Gusein-Zade, and A.N. Varchenko. *Singularities of Differentiable Maps*, volume 2. Birkhauser, 1988.

[BG80] R.O. Buchweitz and G.M. Greuel. The Milnor number and deformations of complex curve singularities. *Inventiones Mathematicae*, 58(3):241–281, 1980.

[BGS87] R.O. Buchweitz, G.M. Greuel, and F.O. Schreyer. Cohen-Macaulay modules on hypersurface singularities II. *Inventiones mathematicae*, 88(1):165–182, 1987.

[BKS86] E. Brieskorn, H. Knörrer, and J. Stillwell. *Plane algebraic curves.* Birkhäuser, 1986.

[BL07] Ragnar-Olaf Buchweitz and Graham J. Leuschke. Factoring the adjoint and maximal Cohen–Macaulay modules over the generic determinant. *Amer. J. Math.*, 129(4):943–981, 2007.

[BM92] Dave Bayer and David Mumford. What can be computed in algebraic geometry? In *in Computational Algebraic Geometry and Commutative Algebra*, pages 1–48. University Press, 1992.

[Bou58] N. Bourbaki. *Algèbre, livre II.* Paris (Hermann and Co), 1958.

[Bru84] JW Bruce. Functions on discriminants. *Journal of the London Mathematical Society*, 2(3):551, 1984.

[DH88] S. Diaz and J. Harris. Ideals associated to deformations of singular plane curves. *Transactions of the American Mathematical Society*, 309(2):433–468, 1988.

[Dim92] A. Dimca. *Singularities and topology of hypersurfaces*. Springer, 1992.

[EH00] D. Eisenbud and J. Harris. *The geometry of schemes*. Springer Verlag, 2000.

[Eis80] D. Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Transactions of the American Mathematical Society*, 260(1):35–64, July 1980.

[Eis95] D. Eisenbud. *Commutative algebra with a view toward algebraic geometry*. Springer, 1995.

[FGVS99] B. Fantechi, L. Göttsche, and D. Van Straten. Euler number of the compactified Jacobian and multiplicity of rational curves. *Journal of Algebraic Geometry*, 8:115–133, 1999.

[Giv90] A.B. Givental. Singular lagrangian manifolds and their lagrangian mappings. *J. Soviet Math,*, 52(4):3246–3278, 1990.

[GM07] E.Shustin G.-M.Greuel, C.Lossen. *Introduction to Singularities and Deformations*. Springer, 2007.

[Gre67] M. J. Greenberg. *Lectures on Algebraic Topology*. W.A.Benjamin, 1967.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[GV82] A.B. Givental and A.N. Varchenko. Period mapping and intersection form. *Funct. Anal. Appl.*, 16(2):pp. 83–93, 1982.

[Hey69] P Heymans. Pfaffians and skew-symmetric matrices. *Proc. London Math. Soc*, 1969.

[HP97] J. Herzog and D. Popescu. Thom-Sebastiani problems for maximal Cohen-Macaulay modules. *Mathematische Annalen*, 309(4):677–700, 1997.

[Kul98] V. S. Kulikov. *Mixed Hodge structures and singularities.* CUP, 1998.

[Lam86] K. Lamotke. Regular Solids and Isolated Singularities, Advanced Lectures in Mathematics, 1986.

[Mil68] J. Milnor. *Singular Points of Complex Hypersurfaces.* Princeton University Press, 1968.

[MS10] D. Mond and M. Schulze. Adjoint divisors and free divisors. *Arxiv preprint arXiv:1001.1095*, 2010.

[Pio07] J. Piontkowski. Topology of the compactified Jacobians of singular curves. *Mathematische Zeitschrift*, 255(1):195–226, 2007.

[PP96] G. Pfister and D. Popescu. Deformations of maximal Cohen-Macaulay modules. *Mathematische Zeitschrift*, 223(1):309–332, 1996.

[Sai74] K. Saito. Einfach-elliptische Singularitäten. *Inventiones Mathematicae*, 23(3):289–325, 1974.

[Sai81] K. Saito. Primitive forms for a universal unfolding of a function with an isolated critical point. *J. Fac. Sci. Univ. Tokyo Sect. IA Math*, 28(3):775–792, 1981.

[Sev03] C. Sevenheck. *Lagrange-Singularitäten.* PhD thesis, 2003.

[Ste] JHM Steenbrink. Mixed hodge structures associated with isolated singularities. In *Proc. Sumpos. Pure Math*, volume 40, pages 513–536.

[Tes76] B. Tessier. The hunting of invariants in the geometry of discriminants. *Real and Complex Singularities, Oslo*, pages 565–677, 1976.

[VA93] O.V.Lyashko V.A.Vasil'ev V.I Arnold, V.V.Goryunov. *Singularity Theory I.* Springer, 1993.

[Var89] A.N. Varchenko. The period map and the discriminant. *Sbornik: Mathematics*, 62(1):65–81, 1989.

[VG82] AN Varchenko and AB Givental'. Mapping of periods and intersection form. *Functional Analysis and Its Applications*, 16(2):83–93, 1982.

[vS06] D. van Straten. Some Problems on Lagrangian Singularities. In *Singularities and computer algebra*, London Math. Soc. Lecture Notes Series, No. 324, pages 333–349. Cambridge Univ Press, 2006.

[vSS03] D. van Straten and C. Sevenheck. Rigid and complete intersection lagrangian singularities. *Mathematische Annalen*, 327(1):79–102, 2003.

[War03] T. Warmt. *Gorenstein-Dualität und topologische Invarianten von Singularitäten*. PhD thesis, Mainz, 2003.

[YM90] Y. Yoshino and H. Matsumura. *Cohen-Macaulay modules over Cohen-Macaulay rings*. Cambridge Univ Pr, 1990.