On the Optimal Stopping Problem driven by Spectrally Negative Lévy Processes

by

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Contents

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Declarations

The work contained in this thesis is original, except as acknowledged, and has not been submitted previously for a degree at any university. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person, except where due reference is made.

Chapter 1

Introduction and preliminaries

1.1 Lévy Processes

The theory of Lévy processes goes back to the 1920s, when the key stones of modern probability were laid. In the early literature, Lévy processes can be found under a number of different names. Not until the 1990s the term "Lévy process" became standard for referring to processes with stationary and independent increments. It is named after the French mathematician Paul Pierre Lévy (1886-1971), honouring his pioneering work in understanding processes with this property. Surprisingly, this simple property leads to a large class of processes, including most of the commonly used stochastic processes, like Brownian motion (whose increments are of normal distribution and sample pathes are $\mathbb{P}$-a.s. continuous) and compound Poisson processes (whose sample paths are constant between jumps).

In this section we give a brief introduction to Lévy processes and fluctuation theory. For a detailed account, we refer, among others, to Applebaum [3], Bertoin [7], Kyprianou [49], Protter [69], and Sato [72].

Let $\Omega = D([0,\infty),\mathbb{R})$ be the set of paths $\omega : [0,\infty) \to \mathbb{R}$, which are right continuous on $[0,\infty)$ with left limits denoted by $\omega(s-)$ for any $s \in (0,\infty)$. Let $X = \{X_t, t \geq 0\}$ be the coordinate process, where for all $t \geq 0$

$$X_t = X_t(\omega) = \omega(t).$$

Denote by X_{s-} and ΔX_s the left limit and the jump at time $s \in (0,\infty)$ of the process X, that is,

$$X_{s-} = X_{s-}(\omega) = \omega(s-), \qquad \Delta X_s = X_s - X_{s-} \quad \text{for all } s \in (0,\infty).$$

Define for all $t \geq 0$ the shift operator $\theta_t : \Omega \to \Omega$ by setting

$$\theta_t\omega(s) = \omega(t+s) \quad \text{for all } s \geq 0.$$

Let $\mathcal{F}$ be the Borel sigma algebra of the set Ω, and $\mathcal{F}_t$ be the sigma algebra generated by the process X_s for all $0 \leq s \leq t$. Denote by $\mathbb{P}_x$ the probability measure of X when it is started at $X_0 = x$, and by $\mathbb{E}_x$ the associated expectation operator. For convenience we write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

Then we can formally define a Lévy process as follows.

Definition 1.1 (Lévy Process). *The coordinate process $\{X_t, t \geq 0\}$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be a Lévy process if it has stationary and independent increments with paths that are $\mathbb{P}$-a.s. right continuous with left limits(càdlàg).*

One of the most common tools to characterize the probability distribution of a Lévy process is the Lévy Khintchine exponent Ψ, as the distribution of a Lévy process is uniquely determined by its Lévy Khinchine exponent, which is explained in the next theorem.

Theorem 1.2 (Lévy Khintchine Formula). *[Theorem 1 in Chapter 1 Bertoin [7]] Consider $\mu \in \mathbb{R}$, $\sigma \geq 0$, and a measure Π on $\mathbb{R} \setminus \{0\}$ such that $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge |y|^2)\Pi(dy) < \infty$. For this triple (μ, σ, Π), define for each $\beta \in \mathbb{R}$ a continuous function $\Psi : \mathbb{R} \to \mathbb{C}$ by setting*

$$\Psi(\beta) = i\mu\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{i\beta y} + i\beta y \mathbb{1}_{\{|y|<1\}})\Pi(dy). \tag{1.1}$$

Then, there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ under which X is a Lévy process with characteristic exponent Ψ, i.e.

$$\mathbb{E}_0(e^{i\beta X_t}) = e^{-t\Psi(\beta)} \tag{1.2}$$

for all $\beta \in \mathbb{R}$ and $t \geq 0$.

Thus, the triple (μ, σ, Π) giving the characteristic exponent Ψ, uniquely determines the Lévy process X. This triple is normally referred to as the characteristic triple, the measure Π is referred to as Lévy measure, the constant σ is referred to as the Gaussian coefficient, and μ is referred to as drift coefficient in the literature.

There is a simpler expression than equation (1.1) for the Lévy-Khintchine formula in the case when the sample paths of the Lévy process have bounded variation on every compact time interval $\mathbb{P}$-a.s. (in this case, we simply say that the Lévy process has bounded variation). Specifically, a Lévy process has bounded variation if and only if $\sigma = 0$ and the Lévy measure Π satisfies the following condition

$$\int_{\mathbb{R}\setminus\{0\}} (1 \wedge |y|)\Pi(dy) < \infty.$$

Then the integral $\int_{\mathbb{R}\setminus\{0\}}(\beta y \mathbb{1}_{\{|y|<1\}})\Pi(dy)$ is well defined for all $\beta \in \mathbb{R}$, and the equation (1.1) can be reduced to the following simpler form,

$$\Psi(\beta) = ib\beta + \int_{\mathbb{R}\setminus\{0\}} (1 - e^{i\beta x})\Pi(dx),$$

for some $b \in \mathbb{R}$.

We say that a Lévy process X is a subordinator if X has $\mathbb{P}$-a.s. non decreasing paths. Thus, it must be of bounded variation, and $\Pi((-\infty, 0]) = 0$ and $b \geq 0$.

Also for any Lévy process X, the probability that X jumps at a fixed time $t \geq 0$ is equal to 0, that is,

$$\mathbb{P}(\Delta X_t = 0) = 0 \text{ for all } t \in [0, \infty). \tag{1.3}$$

Any Lévy process X admits the following Lévy-Itô decomposition,

$$X_t = X_t^1 + X_t^2 + X_t^3, \qquad t \geq 0, \tag{1.4}$$

where $X_t^1 = -\mu t + \sigma B_t$, $X_t^2 = \sum_{0\leq s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|\geq 1\}}$, σ and μ are as in the characteristic triple, B is a standard Brownian motion, and the process X^3 is obtained as the uniform $L^2(\mathbb{P})$-limit on compact time intervals, $\epsilon \downarrow 0$, of the family $(X^{3,\epsilon}, \epsilon > 0)$ given by

$$X_t^{3,\epsilon} = \sum_{0\leq s\leq t} \Delta X_s \mathbb{1}_{\{\epsilon<|\Delta X_s|<1\}} - t\int_{\mathbb{R}} y\mathbb{1}_{\{\epsilon<|y|<1\}}\Pi(dy), \qquad t \geq 0. \tag{1.5}$$

To finish this section, let us introduce the definition of regularity of a point for an open or closed set $\mathcal{O}$ for a Lévy process.

Definition 1.3. *For a Lévy process X, the point $x \in \mathbb{R}$ is said to be regular for an open or closed set $\mathcal{O}$ if*

$$\mathbb{P}_x(\tau_{\mathcal{O}} = 0) = 1,$$

where $\tau_{\mathcal{O}} = \inf\{t \geq 0 : X_t \in \mathcal{O}\}$.

Intuitively speaking, x is regular for $\mathcal{O}$ if the Lévy process X hits $\mathcal{O}$ immediately after starting from x.

1.1.1 Spectrally negative Lévy processes

We say that a Lévy process $X = \{X_t : t \geq 0\}$ is spectrally negative if $\Pi((0,\infty)) = 0$ and X is not a negative subordinator.

For spectrally negative Lévy processes, $\mathbb{E}_x\left(e^{\beta X_t}\right)$ is well defined for all $x \in \mathbb{R}$ and $\beta \geq 0$. Hence, we can define the Laplace exponent $\psi(\beta) : [0,\infty) \to \mathbb{R}$ by the relationship for all $t \geq 0$

$$\mathbb{E}(e^{\beta X_t}) = e^{\psi(\beta)t}. \tag{1.6}$$

Then, from (1.1) it follows that for all $\beta \geq 0$,

$$\psi(\beta) = -\Psi(-i\beta) = -\mu\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(-\infty,0)} \left(e^{\beta y} - 1 - \beta y \mathbb{1}_{\{y>-1\}}\right) \Pi(dx).$$

If X has bounded variation, the Laplace exponent can be reduced to:

$$\psi(\beta) = b\beta - \int_{(-\infty,0)} (1 - e^{\beta y})\Pi(dy), \tag{1.7}$$

where we require $b = -\mu - \int_{-1}^{0} y\Pi(dy) > 0$ to in order to exclude negative subordinators.

The Laplace exponent $\psi : [0,\infty) \to \mathbb{R}$ is infinitely differentiable and convex, and satisfies $\psi(0) = 0$ as well as $\lim_{\beta\uparrow\infty} \psi(\beta) = \infty$. As a result of the shape of the graph of ψ, we can define its right inverse for all $q \geq 0$ by:

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}. \tag{1.8}$$

Thanks to the existence of the Laplace exponent ψ, the process $\{\mathcal{E}_t, t \geq 0\}$, defined for all $t \geq 0$ by

$$\mathcal{E}_t = e^{\beta X_t - \psi(\beta)t}, \tag{1.9}$$

is a martingale for all $\beta \geq 0$. Since $\mathbb{E}\left(\mathcal{E}_t\right) = 1$ for all $t \geq 0$, we can transform measures via

$$d\mathbb{P}^\beta|_{\mathcal{F}_t} = \mathcal{E}_t d\mathbb{P}|_{\mathcal{F}_t}. \tag{1.10}$$

This measure transform is known as the Esscher transform, which is a natural generalization of the Cameron Martin Girsanov transformation. We denote by $\mathbb{E}^\beta$ the expectation operator associated with the probability measure $\mathbb{P}^\beta$.

Under the new probability measure $\mathbb{P}^\beta$, X is still a spectrally negative Lévy process with Laplace exponent $\psi^\beta(\lambda) = \psi(\lambda+\beta) - \psi(\beta)$ and $\Phi^\beta(q) = \Phi(q+\psi(\beta)) - \beta$.

We remark here that for all $\{\mathcal{F}_t\}$-stopping time τ, the transformation given by (1.10) can be extended to τ on the set $\{\tau < \infty\}$.

1.1.2 Scale functions for spectrally negative Lévy processes

Throughout this subsection, we assume X is a spectrally negative Lévy process. For fixed real numbers a and b, let τ_b^+ be the first time the process X_t goes above the level b, τ_a^- be the first time X goes below the level a and $\tau_{a,b}$ be the first time that X goes outside the interval $[a, b]$. That is

$$\tau_b^+ = \inf\{t \geq 0 : X_t > b\}, \tag{1.11}$$
$$\tau_a^- = \inf\{t \geq 0 : X_t < a\}, \tag{1.12}$$
$$\tau_{a,b} = \min\{\tau_b^+, \tau_a^-\} \tag{1.13}$$

Thanks to the absence of positive jumps, $X_{\tau_b^+} = b$ on the set $\{\tau_b^+ < \infty\}$ $\mathbb{P}$-a.s..

The Laplace transforms of τ_b^+, τ_a^- and $\tau_{a,b}$ have been calculated by many authors using the q scale functions W^q and Z^q, which we shall now define.

Definition 1.4 **(q scale functions).** *For a given spectrally negative Lévy process X with Laplace exponent ψ, for every $q \geq 0$ there exists a function $W^q : \mathbb{R} \to [0, \infty)$ such that $W^q(x) = 0$ for all $x < 0$, which is strictly increasing and continuous on $(0, \infty)$, and has the Laplace transform*

$$\int_0^\infty e^{-\beta x} W^q(x)dx = \frac{1}{\psi(\beta) - q}, \tag{1.14}$$

for all $\beta \geq \Phi(q)$. We also define $Z^q : \mathbb{R} \to \mathbb{R}$ as

$$Z^q(x) = 1 + q\int_{-\infty}^x W^q(y)dy, \quad \text{for all } x \in \mathbb{R}. \tag{1.15}$$

We shall write for short $W^0 = W$ and $Z^0 = Z$, and call W^q and Z^q the q scale functions.

The Laplace transforms of τ_b^+, τ_a^- and $\tau_{a,b}$ are evaluated in the next Theorem.

Theorem 1.5. **(One and two sided exit formulae)***[Theorem 8.1 in [49]] For each fixed $q \geq 0$, we have the following assertions.*

(i) for $x \leq b$, we have

$$\mathbb{E}_x\left(e^{-q\tau_b^+}\mathbb{1}_{\{\tau_b^+ < \infty\}}\right) = e^{-\Phi(q)(b-x)} \tag{1.16}$$

(ii) For $x \in \mathbb{R}$, we have

$$\mathbb{E}_x\left(e^{-q\tau_a^-}\mathbb{1}_{\{\tau_a^-<\infty\}}\right) = Z^q(x-a) - \frac{q}{\Phi(q)}W^q(x-a), \tag{1.17}$$

where $\frac{q}{\Phi(q)}$ is understood in the limiting sense for $q = 0$, so that

$$\mathbb{P}_x\left(\tau_a^- < \infty\right) = \begin{cases} 1 - \psi'(0+)W(x-a) & \text{if } \psi'(0+) > 0 \\ 1 & \text{if } \psi'(0+) \leq 0 \end{cases} \tag{1.18}$$

(iii) For $x \leq b$, we have

$$\mathbb{E}_x\left(e^{-q\tau_b^+}\mathbb{1}_{\{\tau_a^->\tau_b^+\}}\right) = \frac{W^q(x-a)}{W^q(b-a)} \tag{1.19}$$

and

$$\mathbb{E}_x\left(e^{-q\tau_a^-}\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right) = Z^q(x-a) - Z^q(b-a)\frac{W^q(x-a)}{W^q(b-a)} \tag{1.20}$$

Because of the role played by W^q and Z^q in obtaining the Laplace transforms of the exit times, in comparison with q scale functions for linear continuous diffusions in Itô and McKean [40], they are referred as q scale functions in the literature. The functions W^q and Z^q have appeared in a number of one and two sided exit problems for spectrally negative Lévy processes. See for example [5], [14], [26], [70] and [68].

Due to the definition of the scale functions, it is not straightforward to see the analytical properties of W^q and Z^q. However, in recent literature some authors found closed formulas for a certain class of spectrally negative Lévy processes and conditions under which W^q and Z^q are smooth and convex. Hubalek and Kyprianou [37] described a parametric family of scale functions explicitly. They constructed a spectrally negative Lévy process having a particular pre-determined Wiener-Hopf factorization. Kyprianou and Rivero [46] employed the approach proposed in [37] and combined it with methods of the potential analysis for subordinators. The smoothness and convexity properties of the scale functions were studied in [12] and [50]. Surya [79] developed a robust numerical method to compute the scale function of a general spectrally negative Lévy process. The method is based on the Esscher transform of measures under which the scale function is determined.

Here we summarize some of the properties of the scale functions W^q and Z^q, which we will frequently use in latter chapters.

(i) If X has unbounded variation, i.e. $\sigma \neq 0$ or $\int_{-\infty}^{0}(1 \wedge |x|)\Pi(dx) = \infty$, then

$W^q \in C^1(\mathbb{R} \setminus \{0\})$ and $W^q(0) = 0$. If $\sigma \neq 0$, then $W^q \in C^2(\mathbb{R} \setminus \{0\})$.

(ii) If X has bounded variation, i.e. $\sigma = 0$ and $\int_{-\infty}^{0}(1 \wedge |x|)\Pi(dx) < \infty$, W^q is continuous and has right and left derivatives on $(0, \infty)$, and they agree if and only if the Lévy measure Π has no atoms, and $W^q(0) = 1/b$, where b is as defined in (1.7).

(iii) For fixed $q \geq 0$, and $n \geq 0$ such that $q \geq \psi(n)$, let $W_n^{q-\psi(n)}$ denote the scale functions of the spectrally negative Lévy process X under the measure $\mathbb{P}^n$ as defined in (1.10). Then

$$W^q(x) = e^{nx} W_n^{q-\psi(n)}(x) \qquad \text{for all } x \in \mathbb{R}. \tag{1.21}$$

(iv) For any fixed real values a and x, $\mathbb{E}_x\left(e^{-q\tau_a^-}\mathbb{1}_{\{\tau_a^- < \tau_b^+\}}\right)$ strictly increases to $\mathbb{E}_x\left(e^{-q\tau_a^-}\mathbb{1}_{\{\tau_a^- < \infty\}}\right)$ as b increases to infinity. Thus, based on the right hand sides of (1.17) and (1.20), we can deduce that $\frac{Z^q(x)}{W^q(x)}$ is strictly decreasing to $\frac{q}{\Phi(q)}$ as $x \uparrow \infty$. The same result applies to $\frac{Z_n^{q-\psi(n)}(x)}{W_n^{q-\psi(n)}(x)}$ for $q - \psi(n) \geq 0$.

(v) If $q > 0$, then for x large enough, $W^q(x)$ behaves asymptotically as $\frac{e^{\Phi(q)x}}{\psi'(\Phi(q))}$.

(vi) $\{e^{-qt\wedge\tau_0^-} W^q(X_{t\wedge\tau_0^-}), t \geq 0\}$ is a $\mathbb{P}_x$ martingale, and $\{e^{-qt\wedge\tau_{0,b}} Z^q(X_{t\wedge\tau_{0,b}}), t \geq 0\}$ is also a $\mathbb{P}_x$ martingale for all real value $b > 0$.

We finish this section presenting the distributions of $\underline{X}_{e_q}$ and $\overline{X}_{e_q}$, which are defined by

$$\underline{X}_{e_q} = \inf\{X_s : 0 \leq s \leq e_q\}, \tag{1.22}$$

$$\overline{X}_{e_q} = \sup\{X_s : 0 \leq s \leq e_q\}, \tag{1.23}$$

where e_q is exponential random variable with parameter $q > 0$ which is independent of X.

It is well known for spectrally negative Lévy processes that $\overline{X}_{e_q}$ has the exponential distribution with parameter $\Phi(q)$, and X_{e_q} can be decomposed into the sum of the following two parts:

$$X_{e_q} = \underline{X}_{e_q} + (X_{e_q} - \underline{X}_{e_q}),$$

where $\underline{X}_{e_q}$ and $X_{e_q} - \underline{X}_{e_q}$ are independent, and $X_{e_q} - \underline{X}_{e_q}$ has the same distribution as $\overline{X}_{e_q}$. See Greenwood [34] for details. The probability distribution function of the random variable $\underline{X}_{e_q}$ can be obtained from the Laplace transform of τ_y^-. Indeed for all $y \leq 0$, we have

$$\mathbb{P}\left(\underline{X}_{e_q} < y\right) = \mathbb{E}\left(\mathbb{1}_{\{e_q > \tau_y^-\}}\right) = \mathbb{E}\left(e^{-q\tau_y^-}\right) = Z^q(-y) - \frac{q}{\Phi(q)} W^q(-y).$$

In the case where X has unbounded variation, as a result of the differentiability of W^q, the probability density function of $\underline{X}_{e_q}$ is found to be

$$f_{\underline{X}_{e_q}}(y) = -qW^q(-y) + \frac{q}{\Phi(q)}\left(W^q\right)'(-y),$$

for all $y < 0$. Further, in the case where X has bounded variation, thanks to the existence of the left and right derivatives of W^q on $(0, \infty)$, we derive for all $y < 0$

$$\frac{d}{dy}\mathbb{P}\left(\underline{X}_{e_q} < y+\right) = -qW^q(-y) + \frac{q}{\Phi(q)}\left(W^q\right)'((-y)-) \tag{1.24}$$

$$\frac{d}{dy}\mathbb{P}\left(\underline{X}_{e_q} < y-\right) = -qW^q(-y) + \frac{q}{\Phi(q)}\left(W^q\right)'((-y)+). \tag{1.25}$$

Thus, the distribution of $\underline{X}_{e_q}$ only has atoms if X has bounded variation, and this only happens at $y = 0$. Moreover, we have

$$\begin{aligned}
\mathbb{P}\left(\underline{X}_{e_q} = 0\right) &= \mathbb{P}\left(\underline{X}_{e_q} \leq 0\right) - \mathbb{P}\left(\underline{X}_{e_q} < 0\right) \\
&= 1 - \left(Z^q(0) - \frac{q}{\Phi(q)} W^q(0)\right) \\
&= \frac{q}{\Phi(q)} W^q(0).
\end{aligned}$$

1.2 Stochastic calculus for Lévy processes

In this Section, we present some existing result on stochastic calculus for Lévy processes. For a detailed account, we refer to Applebaum [3], Jacod and Shiryaev [42] and Sato [72].

1.2.1 Poisson stochastic integrals

To begin this Section, let us introduce the definition of Poisson random measure.

Definition 1.6. *For any Lévy process X on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, denote by N the unique measure on $([0, \infty) \times \mathbb{R}\backslash\{0\}, \mathcal{B}([0, \infty)) \times \mathcal{B}(\mathbb{R}\backslash\{0\}))$, which corresponds* *to*

$$N(t,B) = \#\{s \le t : \Delta X_s \in B\} = \sum_{s \le t} \mathbb{1}_{\{\Delta X_s \in B\}}$$

for all $t \ge 0$ *and* $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$. *Then* N *is called the Poisson random measure on* $([0,\infty) \times \mathbb{R} \setminus \{0\}, \mathcal{B}([0,\infty)) \times \mathcal{B}(\mathbb{R} \setminus \{0\}))$ *with respect to* X.

Below we collect some properties for the Poisson random measure N.

(i) For each $\omega \in \Omega$, $t \ge 0$, the set function $B \mapsto N(t, B)(\omega)$ is a counting measure on $\mathcal{B}(\mathbb{R} \setminus \{0\})$.

(ii) For all $t > 0$ and $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, $N(t, B)$ is an Poisson random variable with parameter $t \int_B \Pi(dy)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(iii) If A_1 and A_2 are mutually disjoint sets in $\mathcal{B}([0,\infty)) \times \mathcal{B}(\mathbb{R} \setminus \{0\})$, then $N(A_1)$ and $N(A_2)$ are independent.

(iv) If all elements from the sequence $\{A_i, i \in \mathbb{N}\}$ are mutually disjoint sets in $\mathcal{B}([0,\infty)) \times \mathcal{B}(\mathbb{R} \setminus \{0\})$, then

$$N(\bigcup_{i \in \mathbb{N}} A_i) = \sum_{i \in \mathbb{N}} N(A_i).$$

For any set $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ which is bounded below (0 is not in the closure of B), we define the compensated Poisson random measure by

$$\widetilde{N}(t, B) = N(t, B) - t\Pi(B).$$

Therefore, $\{\widetilde{N}(t, B), t \ge 0\}$ is a martingale. Furthermore, $\widetilde{N}$ extends to a martingale valued measure with forbidden set $\{0\}$.

Definition 1.7. *For all* $t > 0$, *define* $\mathcal{H}_2(t, \mathbb{R})$ *to be the linear space of all equivalent class of mappings* $f : [0,t] \times \mathbb{R} \times \Omega \to \mathbb{R}$ *which coincide almost everywhere with respect to* $dt \times \Pi(dy) \times \mathbb{P}(d\omega)$. *and which satisfy the following conditions:*

(i) f *is predictable,*

(ii) $\int_0^t \int_{\mathbb{R}} \mathbb{E}\left(|f(s,y)|^2\right) dt\Pi(dy) < \infty$.

We define the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_2}$ *on* $\mathcal{H}_2(t, \mathbb{R})$ *by setting*

$$\langle f_1, f_2 \rangle_{\mathcal{H}_2} = \int_0^t \int_{\mathbb{R}} \mathbb{E}\left(f_1(s,y) f_2(s,y)\right) dt\Pi(dy).$$

Then $\mathcal{H}_2(t,\mathbb{R})$ is a real Hilbert space. For all $t \geq 0$ and sets $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ such that $\Pi(B) < \infty$, define

$$\int_0^t \int_B f(s,y)N(ds,dy) = \sum_{0 \leq s \leq t} f(s,\Delta X_s)\mathbb{1}_{\{\Delta X_s \in B\}}.$$

Then $\int_0^t \int_B f(s,y)N(ds,dy)$ is well defined as it is only a finite random sum. Furthermore, the stochastic integral

$$\int_0^t \int_B f(s,y)\widetilde{N}(ds,dy) = \int_0^t \int_B f(s,y)N(ds,dy) - \int_0^t \int_B f(s,y)ds\Pi(dy)$$

is well defined as well for all $t \geq 0$ and $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ such that $\Pi(B) < \infty$. And we have the following Theorem

Theorem 1.8 (Theorem 4.3.4 in [3]). *For fixed $t > 0$, for all $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$ and $f \in \mathcal{H}_2(t,\mathbb{R})$, then*

(i) there exists a sequence of sets $\{B_i, i \in \mathbb{N}\}$ in $\mathcal{B}(\mathbb{R} \setminus \{0\})$ with $\Pi(B_i) < \infty$ for all $i \in \mathbb{N}$, and $B_i \uparrow B$ as $i \to \infty$ for which

$$\lim_{i\to\infty} \int_0^t \int_{B_i} f(s,y)\widetilde{N}(ds,dy) = \int_0^t \int_B f(s,y)\widetilde{N}(ds,dy) \quad \mathbb{P}\text{-a.s.}$$

and the convergence is $L^2(\mathbb{P})$ uniform on compact intervals of $[0,t]$.

(ii) if

$$\mathbb{E}\left(\int_0^t \int_B |f(s,y)|ds\Pi(dy)\right) < \infty,$$

then

$$\int_0^t \int_B f(s,y)\widetilde{N}(ds,dy) = \int_0^t \int_B f(s,y)N(ds,dy) - \int_0^t \int_B f(s,y)ds\Pi(dy).$$

Moreover, we have the following Theorem.

Theorem 1.9 (Theorem 4.2.3 in [3]). *For all fixed $t \geq 0$, for all $f \in \mathcal{H}_2(t,\mathbb{R})$ and $B \in \mathcal{B}(\mathbb{R} \setminus \{0\})$, then*

(i)

$$
\begin{aligned}
\mathbb{E}\left(\int_0^t \int_B f(s,y)\widetilde{N}(ds,dy)\right) &= 0 \\
\mathbb{E}\left(\left(\int_0^t \int_B f(s,y)\widetilde{N}(ds,dy)\right)^2\right) &= \int_0^t \int_B \mathbb{E}\left((f(s,y))^2\right) ds\Pi(dy)
\end{aligned}
$$

(ii) $\{\int_0^t \int_B f(s,y)\widetilde{N}(ds,dy), t \geq 0\}$ *is* $\mathcal{F}_t$ *adapted.*

(iii) $\{\int_0^t \int_B f(s,y)\widetilde{N}(ds,dy), t \geq 0\}$ *is a square integrable martingale.*

Thus, the mapping

$$
f \mapsto \int_0^t \int_{\mathbb{R}} f(s,y)\widetilde{N}(ds,dy) \tag{1.26}
$$

is an isometry from $\mathcal{H}_2(t,\mathbb{R})$ to $L^2(\mathbb{P})$.

We remark here that as a result of the Theorems above, X^2 and X^3 from the Lévy Itô decomposition (1.4) can be rewritten as follows,

$$
\begin{aligned}
X_t^2 &= \int_{|y|\geq 1} yN(t,dy) \qquad t \geq 0, \\
X_t^3 &= \int_{|y|<1} y\widetilde{N}(t,dy) \qquad t \geq 0.
\end{aligned}
$$

Therefore, for all $t \geq 0$

$$
\begin{aligned}
dX_t &= dX_t^1 + dX_t^2 + dX_t^3 \\
&= \mu dt + \sigma dB_t + \int_{|y|\geq 1} yN(dt,dy) + \int_{|y|<1} y\widetilde{N}(dt,dy).
\end{aligned}
$$

1.2.2 Generalized Itô's formula

Itô's formula has become a cornerstone of stochastic calculus. It is the extension of the chain rule to the stochastic integral. Itô's formula is firstly established by Itô [39] for a standard Brownian motion and twice continuously differentiable functions F. Since then there are many extensions of the Itô formula both on the underlying processes X and the function F. The best known extensions is the Itô-Tanaka formula, that is the Itô's formula for the function $F(x) = |x|$ where the underlying process is Brownian motion, which is introduced by Tanaka [80]. Later on, many authors extended this result to continuous semi martingales and absolutely continuous functions F which have locally bounded first derivative F', see for example, Applebaum [3], Bouleau and Yor [8]. Elworthy et al [25], Kunita and Watanabe [35], Lévy [51], Meyer [55], Peskir [62] and Peskir [64], Wang [82], etc.

Recently, there is renewed interest in the Itô's formula for Lévy processes and for functions F whose derivatives only exist as Random Nikodym distribution sense. Various authors have been studying to find the minimum conditions on the Lévy measure and the function F such that the Itô's formula holds true, see for example [24], [27], [28] [48]. We mention here that Eisenbaum [22] studied this class of functions for standard Brownian motions, and expressed the correction term as an area integral with respect to both the time variable s and the space variable x of the local time by combining the results in [8] and [29] with the time-reversal property of standard Brownian motion. Later on, this result is generalized to Lévy processes by Eisenbaum in [23] on the condition that $\sum_{0\leq s\leq t}|\Delta X_s| < \infty$ for all fixed $t > 0$.

Eisenbaum and Kyprianou [21] completed the recent accumulation of results concerning extended Itô's formula for a one dimensional Lévy process with the following result.

Let $\mathbf{F}$ be the class of functions defined on $\mathbb{R} \times \mathbb{R}^+$ to $\mathbb{R}$. Then we introduce the following subclass of functions of $\mathbf{F}$.

$\mathcal{D}_{1,1}$ consists of functions $F \in \mathbf{F}$ such that $\partial F/\partial x$ and $\partial F/\partial t$ exist as Radon Nikodym derivatives with respect to the Lebesgue measure, and are locally bounded,

$\mathcal{D}_{2,1}$ consists of functions $F \in \mathbf{F}$ such that $\partial^2 F/\partial x^2$ and $\partial F/\partial t$ exist as Radon Nikodym derivatives with respect to the Lebesgue measure, and are locally bounded,

$\mathcal{I}_1$ consists of functions $F \in \mathbf{F}$ such that

$$\int_{\mathbb{R}} \left|F(x+y,t) - F(x,t)\right|\Pi(dy)$$

is well defined and locally bounded in (x,t),

$\mathcal{I}_2$ consists of functions $F \in \mathbf{F}$ such that $\partial F/\partial x$ exists as a Random Nikodym derivative with respect to Lebesgue measure and

$$\int_{\mathbb{R}} \left|F(x+y,t) - F(x,t) - \frac{\partial F}{\partial x}(x,t)y\mathbb{1}_{\{|y|<1\}}\right|\Pi(dy)$$

is well defined and locally bounded in (x,t), Let $\mathbb{L}_X$ be the integro-differential operator for the Lévy process X such that

$$\begin{aligned}
\mathbb{L}_X F(x,t) &= \mu\frac{\partial F}{\partial x}(x,t) + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(x,t) \\
&\quad + \int_{\mathbb{R}} \left[F(x+y,t) - F(x,t) - \frac{\partial F}{\partial x}(x,t) y \mathbb{1}_{\{|y|<1\}}\right]\Pi(dy)
\end{aligned} \tag{1.27}$$

whenever $\frac{\partial F}{\partial t}$, $\frac{\partial F}{\partial x}$ and $\frac{\partial^2 F}{\partial x^2}$ exist as Radon Nokodym derivatives and the integral is well defined. Then the following theorem hold true.

Theorem 1.10 (Theorem 2 and Theorem 3 in Eisenbaum and Kyprianou [21]). *Suppose that X is a Lévy process with the triple (μ, σ, Π), and $\alpha_1 = \mathbb{1}_{\{\sigma \neq 0\}}$ and $\alpha_2 = \mathbb{1}_{\{\sigma=0,\ \int_{-1}^{1}|x|\Pi(dx)=\infty\}}$. Then for all $F \in \mathcal{D}_{1+\alpha_1,1} \cap \mathcal{I}_{1+\alpha_2}$.*

(i) $(\frac{\partial}{\partial t} + \mathbb{L}_X)F$ is well defined for all x and t.

(ii) For all $t \geq 0$, we have $\mathbb{P}$-a.s.

$$F(t, X_t) = F(0, X_0) + \int_0^t (\frac{\partial}{\partial t} + \mathbb{L}_X)F(s, X_s)ds + M_t^F \tag{1.28}$$

where

$$\begin{aligned}
M_t^F &= \int_0^t \sigma\frac{\partial F}{\partial x}(X_s, s)dB_s + \int_0^t \int_{(-1,1)\setminus\{0\}} y\frac{\partial F}{\partial x}(X_{s-}, s)\widetilde{N}(ds, dy) \\
&\quad + \int_0^t \int_{\mathbb{R}\setminus\{0\}} (F(X_{s-} + y, s) - F(X_{s-}, s) - y\frac{\partial F}{\partial x}(X_{s-}, s)\mathbb{1}_{\{|y|<1\}})\widetilde{N}(ds, dy))
\end{aligned} \tag{1.29}$$

for all $t \geq 0$, is a local martingale.

As a result of the role played by $\mathbb{L}_X$ in the generalized Itô's formula (1.28), from now on we will refer to $\mathbb{L}_X$ as the infinitesimal generator of the Lévy processes X.

1.3 Optimal stopping problems

Optimal stopping theory has a long history in probability theory. It was first formulated to solve the problem of sequential analysis in mathematical statistics for discrete time stochastic processes. See Wald [81]. Then a general stopping rule was found for optimal stopping problem in discrete time by Snell [77]. The first general results for continuous time optimal stopping problems were derived by Dynkin [20], where the fundamental concepts of excessive and superharmonic functions were introduced. For a complete survey on optimal stopping problems we refer to Shiryaev

[76], and the reference therein, where also the existence and uniqueness of value functions were found for general gain functions and Markov processes.

Optimal stopping problems have applications in many areas. For example, in mathematical statistics, where the Bayesian approach to the sequential analysis on testing two statistical hypotheses can be solved by reducing the initial problems to optimal stopping problems, see Peskir and Shiryaev [63]. Optimal stopping problems are also used to derive sharp inequalities, for example Doob's inequality for Brownian motion and Bessel processes, see [17], [19], [32], [33] and [41]. In mathematical finance, it is used to study the pricing of options, the most famous example is the pricing of an American options done by Mckean [54] in the Black-Scholes model and the pricing of Russian options in Larry and Shiryaev [74].

Recently, there is a renewed interest in the optimal stopping problem where the underlying uncertainty is modeled by a Lévy process. This is partly due to the applications to financial modelling. In the traditional Black-Scholes model, stock prices are modeled by geometric Brownian motions. However, it has been observed by many authors that the tails of Brownian motions are too light compared with the true market observations ([31], [38]). In order to overcome this problem, many authors began to use the exponentials of a Lévy process in financial modelling. See [11], [15], [73] and [18] for a more detailed accounts of application of Lévy process in option pricing.

The purpose of this section is to review the formulation and methodologies used in the existing literature for the optimal stopping problems. For the general theory of optimal stopping problems and many more examples, we refer to the books Peskir and Shiryaev [66] and Shiryaev [76].

In the meanwhile, we shall also define the optimal stopping problem that we are interested in and study in this thesis.

Let $X = (X_t, t \geq 0)$ be a real valued spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ satisfying the usual condition (that is, $\{\mathcal{F}_t\}$ is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$ null sets in $\mathcal{F}$). Assume we are given a continuous function $g : \mathbb{R} \to \mathbb{R}$, satisfying the following conditions:

$$\mathbb{P}_x\left(\lim_{t\to\infty} e^{-qt} g(X_t) = 0\right) = 1, \tag{1.30}$$

$$\mathbb{E}_x\left(\sup_{0\leq t\leq\infty} e^{-qt}|g(X_t)|\right) < \infty, \tag{1.31}$$

for all $x \in \mathbb{R}$. Then let us consider the following optimal stopping problem:

$$V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} g(X_\tau)) \tag{1.32}$$

for all $x \in \mathbb{R}$, where $q > 0$, $\mathcal{T}_{[0,\infty]}$ is the family of stopping time with respect to $\{\mathcal{F}_t\}$ taking values in $[0, \infty]$. We will refer to V and g as the value function and the gain function. We say that a stopping time $\tau^\star$ is optimal if

$$V(x) = \mathbb{E}_x(e^{-q\tau^\star}(X_{\tau^\star})) \qquad \text{for all } x \in \mathbb{R}. \tag{1.33}$$

An optimal stopping problem consists of finding the pair $(V, \tau^\star)$, where V is the value function, and $\tau^\star$ is the optimal stopping time. Note that the optimal stopping time $\tau^\star$ may be not unique.

By taking the stopping time $\tau = 0$, it follows immediately that

$$\mathbb{E}_x(e^{-q\tau} g(X_\tau)\big|_{\tau=0}) = g(x) \qquad \text{for all } x \in \mathbb{R}.$$

Therefore, by definition of the value function we must have $V(x) \geq g(x)$ for all $x \in \mathbb{R}$. It turns out to be helpful when the domain of $V(x)$ is split into the following two regions

$$\text{continuation region} \qquad \mathcal{C} = \{x \in \mathbb{R} : V(x) > g(x)\}, \tag{1.34}$$

$$\text{stopping region} \qquad \mathcal{D} = \{x \in \mathbb{R} : V(x) = g(x)\}. \tag{1.35}$$

It has been shown that, under very week conditions (g is continuous and X is a Feller process), V is lower semicontinuous (see page 49 in [66]). Furthermore, under these conditions, $\mathcal{C}$ is open and $\mathcal{D}$ is closed. Thus, we obtain that the random time

$$\tau_{\mathcal{C}} = \inf\{t \geq 0 : X_t \notin \mathcal{C}\} \tag{1.36}$$

is a $\{\mathcal{F}_t\}$-stopping time since $\mathcal{C}$ is open and both X_t and $\{\mathcal{F}_t\}_{t\geq 0}$ are right continuous.

The following well-known theorems give the necessary and sufficient conditions for the function V to be the value function of the optimal stopping problem (1.32) and $\tau_{\mathcal{C}}$ to be the smallest optimal stopping time.

Theorem 1.11 (Necessary condition for optimal stopping). *[Theorem 2.4 in [66]]. Suppose that τ_1 is an optimal stopping time for the optimal stopping problem* (1.32). *that is*

$$V(x) = \mathbb{E}_x(e^{-q\tau_1} g(X_{\tau_1})) \qquad \textit{for all } x \in \mathbb{R}. \tag{1.37}$$

Then $\{e^{-qt}V(X_t), t \geq 0\}$ is the smallest right continuous supermartingale such that $V(x) \geq g(x)$ on $\mathbb{R}$. We also have

1. *the stopping time τ_C given in* (1.36) *is optimal, and $\tau_C \leq \tau_1$ $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}$.*

2. *the process $\{e^{-qt\wedge\tau_C}V_{t\wedge\tau_C}, t \geq 0\}$ is a right continuous martingale under $\mathbb{P}_x$ for every $x \in \mathbb{R}$.*

Theorem 1.12 (Sufficient condition for optimal stopping). *[Theorem 2.7 in [66]] Consider the optimal stopping problem* (1.32) *upon assuming that the condition* (1.30) *and* (1.31) *are satisfied. Assume that $\{e^{-qt}\hat{V}(X_t), t \geq 0\}$ is the smallest right continuous supermartingale such that $\hat{V}(x) \geq g(x)$ on $\mathbb{R}$. Set $\hat{C} = \{x \in \mathbb{R} : \hat{V}(x) > g(x)\}$ and $\tau_{\hat{C}} = \inf\{t \geq 0 : X_t \notin \hat{C}\}$. then we have*

1. *If $\mathbb{P}_x(\tau_{\hat{C}} < \infty) = 1$ for all $x \in \mathbb{R}$, then $\hat{V} = V$ and $\tau_{\hat{C}}$ is optimal in* (1.32).

2. *If $\mathbb{P}_x(\tau_{\hat{C}} < \infty) < 1$ for some $x \in \mathbb{R}$, then there is no optimal stopping time (with probability 1) for* (1.32).

As a result of Theorem 1.11 and Theorem 1.12, the optimal problem (1.32) is equivalent to finding the smallest right continuous martingale $e^{-qt}\hat{V}(X_t)$ such that $\hat{V}(x) \geq g(x)$ for all $x \in \mathbb{R}$. Then $V(x) = \hat{V}(x)$ for all $x \in \mathbb{R}$, and the optimal stopping time is $\tau_{\hat{C}}$ given that $\mathbb{P}_x(\tau_{\hat{C}} < \infty) = 1$ for all $x \in \mathbb{R}$.

1.3.1 Free boundary approach

Generally, it is very difficult to apply Theorems 1.11 and 1.12 to solve any specific optimal stopping problem, as the solution can only be found as the smallest supermartingale that dominates the gain function. However, it is also stated in the Theorem 1.11 and 1.12 that the value function is such that $\{e^{-qt}V(X_t), t \geq 0\}$ is a global supermartingale and a martingale in C. So if V is smooth enough, it should solve the following free boundary problem.

$$\begin{aligned}
\mathbb{L}_X V(x) - qV(x) &= 0 \qquad \text{for all } x \in C, &(1.38)\\
\mathbb{L}_X V(x) - qV(x) &\leq 0 \qquad \text{for all } x \in \overset{\circ}{D}, &(1.39)\\
V(x) - g(x) &\geq 0 \qquad \text{for all } x \in \mathbb{R}, &(1.40)\\
V(x) - g(x) &= 0 \qquad \text{for all } x \in \overset{\circ}{D}, &(1.41)
\end{aligned}$$

where $\mathbb{L}_X$ is the infinitesimal generator of X as defined in (1.27).

Normally continuous pasting is used as one of the boundary conditions in the free boundary approach. That is, we assume that the value function V is continuous at the boundary of the continuation region $\mathcal{C}$,

$$V(x) \quad = \quad g(x) \qquad \text{for all } x \in \partial\mathcal{D}. \tag{1.42}$$

The idea that continuous pasting happens as a principle was first noticed in [63] and in [67]. The application of continuous pasting has been seen in [1], [4], [6], [47], [58], [59], and [78] .

Furthermore, for a smooth gain function g and a nice Lévy process X, the smooth pasting condition can be also used to get one of the boundary conditions. That is,

$$V'(x) \quad = \quad g'(x) \qquad \text{for all } x \in \partial\mathcal{D}, \tag{1.43}$$

The smooth pasting condition was first applied by Mikhalevich [56] for sequential analysis, and later on by Chernoff [13] and Lindley [52]. It is worth noting that Mckean [54] obtained the pricing formula for perpetual American put options in the Black-Schole model by using the smooth pasting property. It has been observed that the smooth pasting condition holds for a much wider class of processes than just a Brownian motion, see for example [66], [71] and [76] (and references therein). However, the failure of smooth pasting has also been observed in various examples, see [1], [4], [6], [58], [59], [47], and [78]. Amongst the aforementioned list of articles, it has also been observed that unlike the case when X is purely Gaussian, the value function V is only differentiable at the stopping boundary if the underlying process X is regular for the interior of $\mathcal{D}$ at the boundary of $\mathcal{D}$. In [65] it was also shown that a sufficient condition for smooth pasting to hold for a diffusion is that the diffusion leaves a symmetric interval upwards with probability $1/2$ in the limit when the length of this interval goes to zero.

There are several difficulties when applying the free boundary approach to solve an optimal stopping problem driven by a Lévy process. Unlike continuous Itô diffusions, the infinitesimal generator of a Lévy process is not a local operator, in fact it is actually an integro-differentiable operator, as stated in (1.27). Therefore, it is more challenging to obtain or guess any candidate solution for the free boundary problem when the underling process is a Lévy process. In addition to this, we still do not know whether the true value function V is in the domain of the infinitesimal generator $\mathbb{L}_X$ of X.

1.3.2 Guess and verify approach

In the recent literature, many authors use the so-called guess and verify approach to solve optimal stopping problems, see [1], [47], [49], [30], [57], [10] [75] and [78]. This approach is based on the following Lemma.

Lemma 1.13 (Guess and Verify Lemma). *(Lemma 9.1 in [49]) Consider the optimal stopping problem* (1.32) *where $g(x)$ is a non-negative measurable function satisfying* (1.30). *Let τ be a $\{\mathcal{F}_t\}$-stopping time and V_τ be*

$$V_\tau(x) := \mathbb{E}_x\left(e^{-q\tau}g(X_\tau)\right) \qquad \textit{for all } x \in \mathbb{R}, \tag{1.44}$$

then the pair $(V_\tau(x), \tau)$ is a solution if

(i) $V_\tau(x) \geq g(x)$ for all $x \in \mathbb{R}$,

(ii) the process $\{e^{-qt}V_\tau(X_t) : t \geq 0\}$ is a right continuous supermartingale.

The function V_τ is normally referred to as the candidate value function. There are several advantages of using the Guess and Verification Lemma to solve the optimal stopping problem. Firstly, instead of looking for the smallest supermartingale of the form $\{e^{-qt}\hat{V}(X_t), t \geq 0\}$ as suggested by Theorem 1.12, we only needs to concentrate on all candidate value functions V_τ, which are obtained by stopping at the stopping time τ. By dropping the idea of the "smallest" supermartingale, we can restrict ourselves to a much smaller class of functions to work with. Secondly, we can guess the form of the optimal stopping time and the pasting conditions at the boundary without proving them. This can be done as once the candidate value function V_τ is proved to be the true value function, then by the uniqueness of the value functions, all the conditions guessed (the candidate stopping time and pasting conditions at the boundary) become true without the need of proof. Thirdly, there is no requirement for the candidate value function V_τ to be in the domain of the infinitesimal generator. For some gain functions the supermartingale property for $\{e^{-qt}V_\tau(X_t), t \geq 0\}$ can be shown without using Itô's formula, see for example [1], [47] and [78].

There are also some drawbacks when applying the Guess and Verification Lemma with Lévy processes. Firstly, we need to be able to have a correct guess on the shape of the optimal stopping time. Normally this is not clear for a general gain function. However, for some gain functions, a reasonable guess can be made from the existing literature where the optimal stopping problem is solved for Brownian motions. In addition to this, we have to be able to calculate the candidate value function V_τ for this class of stopping times guessed. However, in the theory of Lévy processes, because of the jumps, it is generally difficult to obtain the candidate functions even for simple stopping time.

1.3.3 Averaging function approach

By using averaging functions, Surya [78] proposed a method to evaluate the candidate value functions V_τ where τ are the first passage times. An averaging function $A_g : \mathbb{R} \to \mathbb{R}$ with respect to g and $\underline{X}_{e_q}$ is a function such that

$$\mathbb{E}_x(A_g(\underline{X}_{e_q})) = g(x) \qquad \text{for all } x \in \mathbb{R}, \tag{1.45}$$

where $\underline{X}_{e_q} = \inf\{X_s : s \in [0, e_q]\}$, and e_q has exponential distribution with parameter q and is independent of X. It was shown that,

$$\mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^- < \infty\}}\right) = \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q} < a\}}\right),$$

for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$. And the following result is obtained in [78] for the optimal stoping problem.

Theorem 1.14. *Consider the optimal stopping problem* (1.32) *where the gain function g satisfies* (1.30) *and* (1.31). *If there exists a continuous averaging function $A_g : \mathbb{R} \to \mathbb{R}$ with respect to g and $\underline{X}_{e_q}$, and there exists $\hat{a} \in \mathbb{R}$ such that $A_g(\hat{a}) = 0$, A_g is non-increasing for $x < \hat{a}$ and $A_g(x) \leq 0$ for $x > \hat{a}$, then letting $a_\infty^- \leq \hat{a}$ be the smallest root of the equation $A_g(x) = 0$, we have*

$$V(x) = \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q} < a_\infty^-\}}\right) \qquad \text{for all } x \in \mathbb{R}. \tag{1.46}$$

And the corresponding optimal stopping time is $\tau^\star = \inf\{t \geq 0 : X_t < a_\infty^-\}$.

Using this approach, Surya reproduced the results of those discussed, among others, in [16], [59], [9], [1], [60] and [47], and showed that the smooth pasting condition at the boundary only depends on the regularity of the Lévy process at the boundary for the interior of the stopping region.

It is clear that in order to apply this approach, it is crucial that we find an averaging function for the gain function g, and this choice of averaging function satisfies the conditions in the theorem above. Furthermore, the averaging function approach proposed in [78] only treats the optimal stopping problem where the optimal stopping time is the first passage time.

1.4 Outline of this thesis

As discussed above, all methods presented in the previous section for solving the optimal stopping problems have certain disadvantages when the underlying uncertainty is modeled by Lévy processes. The aim of this thesis is to develop an effective approach to solve the optimal stopping problem (1.32) for complicated optimal stopping times and general gain functions, that does not require the existence of the pasting conditions at the boundaries, or the shape of the averaging functions.

The content of the chapters in this thesis is outlined in what follows.

Chapter 2 In this chapter we study the pricing of American Strangle options when the underlying uncertainty is modeled by spectrally negative Lévy processes, that is the optimal stopping problem (1.32) for the following gain function,

$$g(x) = (K_1 - e^x)^+ + (e^x - K_2)^+,$$

where K_1 and K_2 are reals satisfying $K_2 \geq K_1 > 0$. By using the convexity of $g \circ \log$, we first study the convexity of $V \circ \log$, then conclude that the continuation region must be of the form (a, b) with $-\infty < a < \log(K_1) \leq \log(K_2) < b < \infty$. Next we calculate the candidate value function $V_{\tau_{a,b}}$ for all a and b satisfying the condition above. Then by guessing that the pasting conditions at both boundaries, we obtain a system of two equations, to which we show there is a unique pair of solutions $(a^\star, b^\star)$. Finally we prove that the candidate pair $(V_{\tau_{a^\star,b^\star}}, \tau_{a^\star,b^\star})$ satisfies the conditions in the guess and verification Lemma, thus, is a solution. The pasting conditions we guessed hold true without the need of proving as discussed in Section 1.3.2. We also show the limitation of this approach when it is applied to other gain functions.

Chapter 3 In this chapter we study the optimal stopping problem (1.32) for general smooth gain functions which satisfy the Assumptions 3.3 and 3.24 and the underlying process is a spectrally negative Lévy process. Our approach is inspired by Surya [78]. However, our method does not have the requirement on the shape of the averaging function. Moveover, neither of any knowledge of the continuation region in advance or the pasting conditions at the boundary are needed. Instead, we work with the left semi-solution $\bar{V}$ up to the point b of the optimal stopping problem (1.32), which will be defined in Chapter 3. Also a sufficient condition is given for any function to be the left semi value function. This approach is totally based on finding the function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where

$$h(x, a) = \mathbb{E}_x \left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^- < \infty\}}\right) + \frac{q}{\Phi(q)} A_g(a) W^q(x - a)$$

for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$ where A_g is the averaging function with respect to g and $\underline{X}_{e_q}$. Then by choosing $a_1^\star$ to be such that $h_{g_1}(x, a_1^\star)$ is the smallest function dominating the gain function g and choosing $b_1^\star$ to be the largest value such that $h(x, a_1^\star) = g(x)$, we find that $\bar{V}$ is the left semi value function up to the point $b_1^\star$ for the optimal stopping problem (1.32), where $\bar{V}$ is defined to be $h(\cdot, a_1^\star)$ on $(-\infty, b_1^\star]$ and g otherwise. And the corresponding optimal stopping time for $x \le b_1^\star$ is that $\tau_{a^\star,b^\star}$. We also give the condition under which the pair $(\bar{V}, \tau_{a^\star,b^\star})$ is the global solution pair. In the case when $(\bar{V}, \tau_{a^\star,b^\star})$ is only a left semi-solution, we prove further that under suitable conditions, the above method can be repeated to study the value function for $x > b_1^\star$. Using our approach, we are able to reproduce the result in [78]. Thus, our approach shows no contradiction with the existing literature.

Chapter 4 The main purpose of this chapter is twofold. Firstly, we extend the approach in Chapter 3 to gain functions g which are non differentiable at a fixed number of points. This is done by introducing the class of extended functions, such that for all g_1 in the extended class the function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is well defined for g_1. However, $a_1^\star$ is chosen such that $h(x, a_1^\star)$ is the smallest function dominating the gain function g instead of g_1, and $b_1^\star$ is chosen to be the largest value such that $h(x, a_1^\star) = g(x)$. We prove that the value of $a_1^\star$ does not depend on choice of g_1. By the same argument as in Chapter 3, a left semi value function up to point $b_1^\star$ is obtained. Furthermore, we show that this procedure can be repeated to study value functions for $x > b_1^\star$. Secondly, we treat the class of gain functions such that $\lim_{x\to-\infty} g(x) \le 0$. We present an approach to find V_0 and $b_0^\star$ such that $(V_0, \tau_{b_0^\star}^+)$ is the left semi-solution pair up to the point $b_0^\star$. Finally, we show that the method suggested in the first part in this chapter can be applied to V_0 to study the value function function for $x > b_0^\star$.

Chapter 5 In this chapter we solve various optimal stopping problems using the approach suggested in Chapter 3 and Chapter 4. The examples include American call option, American put option and Novikov Shiryaev problem, etc. Out results show no contradiction with the existing literature, as established in Mordecki [59], Boyarchenko and Levendorskii [9], Alili and Kyprianou [1], Novikov and Shiryaev [60], and Kyprianou and Surya [47].

Chapter 2

On the Perpetual American Strangles driven by Spectrally Negative Lévy Processes

2.1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with characteristic triple (μ, σ, Π), where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge x^2)\Pi(dx) < \infty$. For any $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the law of X starting from x. We write simply $\mathbb{P}_0 = \mathbb{P}$, and denote by $\mathbb{E}_x$ and $\mathbb{E}$ the corresponding expectation operators.

Let Ψ be the Lévy Khinchine exponent and ψ be the Laplace exponent for X. The Laplace exponent $\psi(\beta)$ is well defined for all $\beta \geq 0$, and takes the following form,

$$\psi(\beta) = -\Psi(-i\beta) = -\mu\beta + \frac{1}{2}\sigma^2\beta^2 + \int_{(-\infty,0)}(e^{\beta x} - 1 - \beta x \mathbb{1}_{\{x>-1\}})\Pi(dx).$$

And we write $\Phi : [0, \infty) \to \mathbb{R}$ for the right inverse of ψ, i.e.

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}. \tag{2.1}$$

We refer to [7] and [49] for a detailed account on Lévy process.

The optimal stopping problem we consider in this chapter is of the following form,

$$\begin{aligned} V(x) &= \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau}g(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) \\ &= \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau}((K_1-e^{X_\tau})^+ + (e^{X_\tau}-K_2)^+)\mathbb{1}_{\{\tau<\infty\}}), \end{aligned} \tag{2.2}$$

where $q > \max\{\psi(1), 0\}$, K_1 and K_2 are reals satisfying $K_2 \geq K_1 > 0$, and the supremum is taken over the class $\mathcal{T}_{[0,\infty]}$ of Markov stopping times taking values in $[0,\infty]$ with respect to $\{\mathcal{F}_t\}$. We say a pair $(V^\star, \tau^\star)$ is a solution to the optimal stopping problem (2.2) if

$$V^\star(x) = \mathbb{E}_x(e^{-q\tau^\star}g(X_{\tau^\star})\mathbb{1}_{\{\tau<\infty\}}) = \sup_{\tau\in\mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau}g(X_\tau)\mathbb{1}_{\{\tau<\infty\}})$$

for all $x \in \mathbb{R}$. Note that the optimal stopping time $\tau^\star$ may not unique. And the stopping time

$$\tau_C = \inf\{t \geq 0 : X_t \notin C\}. \tag{2.3}$$

is the smallest stopping time $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}$, where C is the continuity region, i.e.

$$C = \{x \in \mathbb{R} : V(x) > g(x)\}.$$

We refer to [76] and [66] for more details for general theory of optimal stopping problems.

The value function (2.2) is normally referred to as the perpetual American Strangle option. Strangles are classical ways to build volatility strategies, and are formed by holding simultaneously a long position in an American put option and a long position in an American call option. Both options of this portfolio have the same underlying asset, but strike prices are most often different (that of the put option being the smaller one). Unfortunately, models based on Brownian motion can not reproduce neither heavy tails and skewness of the return distributions nor the volatility smile. Hence, many authors are using Lévy processes to model the underlying asset, which allows us to overcome the 'heavy tail' problem.

In literature, various authors have established optimal stopping strategies for the perpetual American Strangles for continuous diffusions or a certain class of Lévy processes. In all cases, it has been shown that an optimal strategy is the first exit time from a bounded interval by the underlying processes, that is:

$$\tau^\star = \inf\{t \geq 0 : X_t \notin [a^\star, b^\star]\} \tag{2.4}$$

for some specific finite values of $a^\star$ and $b^\star$. See [57] for the pricing of American Strangles in the Black-Sholes model, [30] where free boundary approach was applied for continuous diffusion, and [10] where price of American Strangles were found for a class of Lévy processes with the Lévy measure Π satisfying a certain form.

In this chapter, we apply the guess and verify approach to solve the optimal stopping problem (2.2). The biggest advantage of this method is that we can assume any condition for the candidate value function $V_\tau(x) := \mathbb{E}_x(e^{-q\tau}g(X_\tau))$ and the candidate optimal stopping time τ. For example, the shape of the continuation region and the pasting conditions at the boundaries. Under these assumptions, if the corresponding candidate pair $(V_\tau(x), \tau)$ is a solution to the optimal stopping problem (by guess and verification lemma), then, by the uniqueness of the value function, all conditions assumed become true without proving them. However, in order to apply this method, we need to be able to calculate the candidate value function V_τ. When the underlying uncertainty is modeled by Lévy processes, because of the jumps, it is generally difficult to do this. We shall show that with the help of scale functions W^q and Z^q, a closed form formula for the candidate value function $V_{\tau_{a,b}}$ can be obtained, where

$$\tau_{a,b} = \inf\{t \geq 0 : X_t \notin [a, b]\},$$

for all choices of real values $-\infty < a < \log(K_1) \leq \log(K_2) < b < \infty$. By guessing the continuous pasting and the smooth pasting at a and b, we are able to find a unique pair $(a^\star, b^\star)$ such that $V_{\tau_{a^\star,b^\star}}$ is the value function of the optimal stopping problem (2.2).

This chapter is organized as follows: In Section 2, we study the convexity of $V \circ \log$, which allows us to conclude on the shape of the continuation region. In section 3, we calculate the candidate value function $V_{\tau_{a,b}}$ by using the scale functions W^q and Z^q. In Section 4, we show that $(V_{\tau_{a^\star,b^\star}}, \tau_{a^\star,b^\star})$ is a solution to the optimal stopping problem (2.2), where $a^\star$ and $b^\star$ are the unique pair obtained from guessing the pasting conditions at the boundaries. In Section 5, we show the limitation of this method, and explain the difficulties when it is applied for other gain functions. The last section contains some of proofs of results from the previous sections.

2.2 The shape of continuation region $\mathcal{C}$

The following Theorem studies the shape of the continuation region $\mathcal{C}$ for the optimal stopping problem (2.2).

Theorem 2.1. *Consider the optimal stopping problem* (2.2), *then*

(i) the function $V \circ \log : (0, \infty) \to \mathbb{R}$ is convex on $(0, \infty)$. Therefore, V is continuous in $\mathbb{R}$.

(ii) the continuation region $\mathcal{C}$ is of the form (a, b), where a and b are two real numbers satisfying $-\infty < a < \log(K_1) \leq \log(K_2) < b < \infty$.

Proof for Theorem 2.1. *(i)*

Define $Y_t = e^{X_t}$ for all $t \geq 0$. Then we can rewrite V given in (2.2) as,

$$\bar{V}(y) := V(\log(y)) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_y \left(e^{-q\tau} \bar{g}(Y_\tau) \mathbb{1}_{\{\tau < \infty\}} \right), \tag{2.5}$$

for all $y = e^x$ and $x \in \mathbb{R}$, where $\bar{g}(x) = g \circ \log(x)$ for all $x \in \mathbb{R}$. Note that for all $0 < y_1 \leq y_2 < \infty$ and $c \in [0, 1]$,

$$\begin{aligned}
& c\bar{V}(y_1) + (1-c)\bar{V}(y_2) \\
&= c \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_{y_1}(e^{-q\tau}\bar{g}(Y_\tau)\mathbb{1}_{\{\tau<\infty\}}) + (1-c) \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_{y_2}(e^{-q\tau}\bar{g}(Y_\tau)\mathbb{1}_{\{\tau<\infty\}}) \\
&= \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_1(ce^{-q\tau}\bar{g}(y_1 Y_\tau)\mathbb{1}_{\{\tau<\infty\}}) + \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_1((1-c)e^{-q\tau}\bar{g}(y_2 Y_\tau)\mathbb{1}_{\{\tau<\infty\}}) \\
&\geq \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \left(\mathbb{E}_1(ce^{-q\tau}\bar{g}(y_1 Y_\tau)\mathbb{1}_{\{\tau<\infty\}}) + \mathbb{E}_1((1-c)e^{-q\tau}\bar{g}(y_2 Y_\tau)\mathbb{1}_{\{\tau<\infty\}})\right) \\
&= \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_1\left(e^{-q\tau}\left(c\bar{g}(y_1 Y_\tau) + (1-c)\bar{g}(y_2 Y_\tau)\right)\mathbb{1}_{\{\tau<\infty\}}\right).
\end{aligned}$$

Then, it follows from the convexity of $\bar{g}$ that for all $0 < y_1 \leq y_2 < \infty$ and $c \in [0, 1]$,

$$\begin{aligned}
c\bar{V}(y_1) + (1-c)\bar{V}(y_2) &\geq \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_1\left(e^{-q\tau}\left(\bar{g}(cy_1 Y_\tau + (1-c)y_2 Y_\tau)\mathbb{1}_{\{\tau<\infty\}}\right)\right) \\
&\geq \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_{cy_1+(1-c)y_2}\left(e^{-q\tau}\left(\bar{g}(Y_\tau)\mathbb{1}_{\{\tau<\infty\}}\right)\right) \\
&= \bar{V}(cy_1 + (1-c)y_2),
\end{aligned} \tag{2.6}$$

as required.

(ii)

Since $q > \max(0, \psi(1))$, it is clear that the optimal stopping time is a finite stopping time $\mathbb{P}$-a.s.. Thus, the continuation region $\mathcal{C}$ is a open and strict subset of

$\mathbb{R}$, then $\mathcal{C}$ must be of the following form,

$$\mathcal{C} = (-\infty, a_0) \cup (\bigcup_{i=1}^{\infty}(a_i, b_i)) \cup (b_0, +\infty) \tag{2.7}$$

where $a_0 \in \mathbb{R} \cup \{-\infty\}$, $b_0 \in \mathbb{R} \cup \{\infty\}$, and a_i and b_i are real numbers for all $i \in \mathbb{N}$ and $i \geq 1$, and $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for all $i \neq j$ where $i, j \in \mathbb{N}$, and

$$a_0 \leq \inf\{a_i : i \in \mathbb{N}, i \geq 1\} \leq \sup\{b_i : i \in \mathbb{N}, i \geq 1\} \leq b_0.$$

Next, we will show the non existence of $(-\infty, a_0)$ and (b_0, ∞) in $\mathcal{C}$ for all $a_0 \in \mathbb{R}$ and $b_0 \in \mathbb{R}$.

Suppose there exists a disjoint interval (∞, a_0) in $\mathcal{C}$ with $a_0 \in \mathbb{R}$. Because of the absence of positive jumps for X, we have $\tau_{a_0}^+ = \tau_{\mathcal{C}}$ $\mathbb{P}_x$-a.s. for all $x < a_0$, where $\tau_{a_0}^+ = \{t \geq 0 : X_t > a_0\}$. Then, from the result in Theorem 1.5, it follows that for all $x < a_0$:

$$\begin{aligned} V(x) &= \mathbb{E}_x\left(e^{-q\tau_{\mathcal{C}}} g(X_{\tau^\star}) \mathbb{1}_{\{\tau_{\mathcal{C}}<\infty\}}\right) \\ &= \mathbb{E}_x\left(e^{-q\tau_{a_0}^+} g(X_{\tau_{a_0}^+}) \mathbb{1}_{\{\tau_{a_0}^+<\infty\}}\right) \\ &= g(a_0)e^{-\Phi(q)(a_0-x)}. \end{aligned}$$

Therefore, $V(x)$ decreases to 0 as $x \to -\infty$, which clearly contradicts the fact $V(x) \geq g(x)$ on $\mathbb{R}$. Thus, it is impossible to have a disjoint interval of the form $(-\infty, a_0)$ for any $a_0 \in \mathbb{R}$ in the continuation region $\mathcal{C}$.

Now let us assume that $\mathcal{C}$ contains a disjoint interval of the form (b_0, ∞) for some real number b_0. Then, on the set $\{\tau_{\mathcal{C}} < \infty\}$, $X_{\tau_{\mathcal{C}}} \leq b_0$ $\mathbb{P}_x$-a.s. for all $x > b_0$. As g is bounded on $(-\infty, b_0]$, we obtain that for all $x > b_0$,

$$\begin{aligned} V(x) &= \mathbb{E}_x\left(e^{-q\tau_{\mathcal{C}}} g(X_{\tau_{\mathcal{C}}}) \mathbb{1}_{\{\tau_{\mathcal{C}}<\infty\}}\right) \\ &\leq c(b_0)\mathbb{E}_x\left(e^{-q\tau_{\mathcal{C}}} \mathbb{1}_{\{\tau_{\mathcal{C}}<\infty\}}\right) \\ &\leq c(b_0)\mathbb{E}_x\left(e^{-q\tau_{b_0}^-} \mathbb{1}_{\{\tau_{b_0}^-<\infty\}}\right), \end{aligned} \tag{2.8}$$

where $c(b_0)$ is the upper bound for g on $(-\infty, b_0]$, and the last inequality is due to $\tau_{b_0}^- \leq \tau_{\mathcal{C}}$ $\mathbb{P}_x$-a.s. for all $x > b_0$ on the set $\{\tau_{\mathcal{C}} < \infty\}$.

Clearly the last term in (2.8) goes to 0 as $x \to \infty$. Therefore, $V(x) < g(x)$ for all x large enough, which clearly contradicts the definition of V. Hence, we can not have a disjoint interval of the form (b_0, ∞) for any $b_0 \in \mathbb{R}$ in the continuation region $\mathcal{C}$.

Now suppose that there exists a disjoint interval (a, b) in the continuation region $\mathcal{C}$ such that $a < b \leq \log(K_1)$. Note that as $g \circ \log$ is linear on the interval $[e^a, e^b]$, then $g \circ \log$ is the largest convex function on $[e^a, e^b]$ among all the convex functions $f : [e^a, e^b] \to \mathbb{R}$ with $f(e^a) = g(a)$ and $f(e^b) = g(b)$. However, by part (i), $V \circ \log$ is convex on $[e^a, e^b]$ with $V \circ \log(e^a) = g(a)$, $V \circ \log(e^b) = g(b)$. Also by definition of value function, $V \circ \log(x) > g \circ \log(x)$ on (e^a, e^b). Clearly, the above two statements contradict each other. Therefore, we can not have a disjoint interval of the form (a, b) in $\mathcal{C}$ with $a < b \leq \log(K_1)$

A similar argument can be applied to show the non existence of (a, b) where $\log(K_2) \leq a < b$.

Finally, we consider for all $x \in [\log(K_1), \log(K_2)]$. Note that for all $b > \log(K_2)$ and $x \in [\log(K_1), \log(K_2)]$,

$$V(x) \geq \mathbb{E}_x \left(e^{-q\tau_b^+} g(X_{\tau_b^+}) \mathbb{1}_{\{\tau_b^+ < \infty\}} \right) > 0 = g(x).$$

Therefore, we must have $[\log(K_1), \log(K_2)] \subset \mathcal{C}$. Hence, we can conclude that $\mathcal{C} = (a, b)$ for some a and b such that $a < \log(K_1) \leq \log(K_2) < b$. □

2.3 Evaluating the candidate value function

As a result of the previous calculation, we know that the continuation region is of the form (a, b) where $-\infty < a < \log(K_1) \leq \log(K_2) < b < \infty$. The aim of this section is to evaluate the candidate value function $V_{\tau_{a,b}}$ for these choices of a and b. Throughout this section, we assume that $-\infty < a < \log(K_1) \leq \log(K_2) < b < \infty$.

Let τ_a^- and τ_b^+ be such that

$$\tau_b^+ = \inf\{t \geq 0 : X_t > b\}, \quad (2.9)$$

$$\tau_a^- = \inf\{t \geq 0 : X_t < a\}. \quad (2.10)$$

Then, $\tau_{a,b} = \min\{\tau_b^+, \tau_a^-\}$. And for all $x < b$ we have:

$$\begin{aligned}
V_{\tau_{a,b}}(x) &= \mathbb{E}_x\left(e^{-q\tau_{a,b}} g(X_{\tau_{a,b}})\mathbb{1}_{\{\tau_{a,b}<\infty\}}\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right) + \mathbb{E}_x\left(e^{-q\tau_b^+} g(X_{\tau_b^+})\mathbb{1}_{\{\tau_a^->\tau_b^+\}}\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_a^-} (K_1 - e^{X_{\tau_a^-}})\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right) + \mathbb{E}_x\left(e^{-q\tau_b^+} g(b)\mathbb{1}_{\{\tau_a^->\tau_b^+\}}\right) \\
&= K_1\mathbb{E}_x\left(e^{-q\tau_a^-}\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right) - \mathbb{E}_x\left(e^{-q\tau_a^-} e^{X_{\tau_a^-}}\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right) \\
&\quad + g(b)\mathbb{E}_x\left(e^{-q\tau_b^+}\mathbb{1}_{\{\tau_a^->\tau_b^+\}}\right). \qquad (2.11)
\end{aligned}$$

Define a change of measure via

$$d\mathbb{P}^1|_{\mathcal{F}_{\tau_a^-}} = \mathcal{E}_{\tau_a^-} d\mathbb{P}|_{\mathcal{F}_{\tau_a^-}}$$

where $\mathcal{E}_t = e^{X_t - \psi(1)t}$ for all $t \geq 0$. By using $\mathbb{P}^1$, we can rewrite the second term in (2.11) for all $x < b$,

$$\mathbb{E}_x\left(e^{-q\tau_a^-} e^{X_{\tau_a^-}}\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right) = e^x\mathbb{E}_x^1\left(e^{-(q-\psi(1))\tau_a^-}\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right).$$

Therefore, thanks to Theorem 1.5, we can evaluate each term in (2.11) by the scale functions W^q and Z^q, and obtain for all $x < b$,

$$\begin{aligned}
V_{\tau_{a,b}}(x) &= \left(e^b - K_2\right)\frac{W^q(x-a)}{W^q(b-a)} + K_1\left(Z^q(x-a) - Z^q(b-a)\frac{W^q(x-a)}{W^q(b-a)}\right) \\
&\quad - e^x\left(Z_1^{q-\psi(1)}(x-a) - Z_1^{q-\psi(1)}(b-a)\frac{W_1^{q-\psi(1)}(x-a)}{W_1^{q-\psi(1)}(b-a)}\right) \\
&= K_1 Z^q(x-a) - e^x Z_1^{q-\psi(1)}(x-a) \\
&\quad + \frac{W^q(x-a)}{W^q(b-a)}\left(e^b Z_1^{q-\psi(1)}(b-a) + e^b - K_2 - K_1 Z^q(b-a)\right) \\
&= h(x,a) + \frac{W^q(x-a)}{W^q(b-a)}\left(g(b) - h(b,a)\right), \qquad (2.12)
\end{aligned}$$

where

$$h(x,a) = K_1 Z^q(x-a) - e^x Z_1^{q-\psi(1)}(x-a), \qquad (2.13)$$

and the second equality is due to $W_1^{q-\psi(1)}(x) = e^{-x}W^q(x)$ on $\mathbb{R}$. So that, overall, we derive,

$$
\begin{aligned}
V_{\tau_{a,b}}(x) &= \mathbb{E}_x\left(e^{-q\tau_{a,b}} g(X_{\tau_{a,b}})\right) \\
&= \begin{cases} h(x,a) + \frac{W^q(x-a)}{W^q(b-a)}\left(g(b) - h(b,a)\right) & \text{if } x < b \\ e^x - K_2 & \text{if } x \geq b. \end{cases}
\end{aligned}
\tag{2.14}
$$

Remark 2.2. *Recall that* $-\infty < a < \log(K_1) \leq \log(K_2) < b < \infty$. *Clearly,* $V_{\tau_{a,b}}$ *given in* (2.14) *is continuous at* b. *However,* $V_{\tau_{a,b}}$ *is continuous at* $x = a$ *if and only if*

$$\frac{W^q(0+)}{W^q(b-a)}\left(g(b) - h(b,a)\right) = 0. \tag{2.15}$$

Thus, $V_{\tau_{a,b}}$ *is continuous at* $x = a$, *if and only if that* $W^q(0+) = 0$ *or* $g(b) = h(b,a)$. *As* $W^q(0+) = 0$ *if and only if* X *has unbounded variation, we can conclude that* $V_{\tau_{a,b}}$ *is continuous at* $x = a$ *if and only if that* X *has unbounded variation or* $g(b) = h(b,a)$.

Also, as a result of the existence of left and right derivatives of W^q *on* $(0,\infty)$, *we obtain for all* $x \in (a,b)$,

$$V'_{\tau_{a,b}}(x+) = \frac{\partial h}{\partial x}(x,a) + \frac{(W^q)'((x-a)+)}{W^q(b-a)}\left(g(b) - h(b,a)\right), \tag{2.16}$$

and

$$V'_{\tau_{a,b}}(x-) = \frac{\partial h}{\partial x}(x,a) + \frac{(W^q)'((x-a)-)}{W^q(b-a)}\left(g(b) - h(b,a)\right). \tag{2.17}$$

Therefore, $V_{\tau_{a,b}}$ *is differentiable at* $x \in (a,b)$ *if and only if*

$$\frac{(W^q)'((x-a)+)}{W^q(b-a)}\left(g(b) - h(b,a)\right) = \frac{(W^q)'((x-a)-)}{W^q(b-a)}\left(g(b) - h(b,a)\right).$$

Moreover,

$$
\begin{aligned}
V'_{\tau_{a,b}}(a+) &= qK_1 W^q(0) - e^a Z_1^{q-\psi(1)}(0) - (q - \psi(1))e^a W_1^{q-\psi(1)}(0) \\
&\quad + \frac{(W^q)'(0+)}{W^q(b-a)}\left(g(b) - h(b,a)\right) \\
&= -e^a + qK_1 W^q(0) - (q - \psi(1))e^a W_1^{q-\psi(1)}(0) \\
&\quad + \frac{(W^q)'(0+)}{W^q(b-a)}\left(g(b) - h(b,a)\right).
\end{aligned}
$$

So, a sufficient condition for $V_{\tau_{a,b}}$ *to be differentiable at* $x = a$ *would be that* X *has* *unbounded variation and* $h(b,a) = g(b)$. *And from equation* (2.17), *it follows that a sufficient condition for* $V_{\tau_{a,b}}$ *to be differentiable at* $x = b$ *is that* $g(b) = h(b,a)$ *and* $\frac{\partial h}{\partial x}(b,a) = g'(b)$.

2.4 Main results

We have the following Theorem for a solution to the optimal stopping problem (2.2).

Theorem 2.3. *Consider the optimal stopping problem* (2.2). *Then* $(V_{\tau_{a^\star,b^\star}}, \tau_{a^\star,b^\star})$ *is a solution to the optimal stopping problem* (2.2), *where* $a^\star < b^\star$. *and* $(a^\star, b^\star)$ *is the unique pair of solution to the following system of equations:*

$$\begin{cases} h(b,a) = e^b - K_2, \\ \frac{\partial h}{\partial x}(b,a) = e^b, \end{cases} \tag{S1}$$

where h *is as defined in equation* (2.13).

The following Lemmas and Propositions are needed for proving Theorem 2.3.

Proposition 2.4.

(i) There exists at least one solution pair $(a^\star, b^\star)$ *with* $a^\star < b^\star$ *for the system of equations (S1).*

(ii) If $(a^\star, b^\star)$ *is a solution pair with* $a^\star < b^\star$ *to the system of equations (S1), then*

- *(a)* $a^\star < a_p < \log(K_1)$ *where* $a_p = \log\left(\frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}K_1\right)$.
- *(b)* $b^\star > \log(K_2)$ *and* $h(x,a^\star) > g(x)$ *on* $(a^\star, \log(K_2))$.
- *(c)* $\frac{\partial h}{\partial x}(x,a^\star) < e^x$ *on* $(-\infty, a^\star) \cup (a^\star, b^\star)$ *and* $\frac{\partial h}{\partial x}(x,a^\star) > e^x$ *on* $(b^\star, \infty)$.
- *(d)* $h(x,a^\star) > g(x)$ *on* $(a^\star, b^\star) \cup (b^\star, \infty)$, $h(b^\star, a^\star) = e^{b^\star} - K_2$.
- *(e)* $(a^\star, b^\star)$ *is the unique solution such that* $b^\star > a^\star$.

Lemma 2.5. *Suppose* $f : \mathbb{R} \to \mathbb{R}$ *is a decreasing function, then*

$$x \mapsto \int_{-\infty}^{0} f(x+y)\Pi(dy)$$

is decreasing in x *in the domain where* $\int_{-\infty}^{0} f(x+y)\Pi(dy)$ *is finite.*

Proposition 2.6. *The stochastic process* $\{e^{-qt\wedge\tau_0^-} Z^q(X_{t\wedge\tau_0^-}), t \geq 0\}$ *is a martingale. And* $\mathbb{L}_X Z^q(x) - qZ^q(x) = 0$ *for all* $x > 0$, *where* $\mathbb{L}_X$ *is infinitesimal generator for* X.

Proposition 2.7. *The stochastic process* $\{e^{-qt}V_{\tau_{a^\star,b^\star}}(X_t), t \geq 0\}$ *is a supermartingale.*

The proofs for Proposition 2.4, Lemma 2.5, Proposition 2.6 and Proposition 2.7 can be found respectively on page 33, 37, 37 and 39. Hence, Theorem 2.3 can be obtained as a direct result of the guess and verification lemma.

Remark 2.8. *Thanks to the system of equations (S1), it follows from Remark 2.2 that* $V_{\tau_{a^\star,b^\star}}$ *is continuous at* $x = a^\star$*, and continuously differentiable at* $x = b^\star$*. And* $V_{\tau_{a^\star,b^\star}}$ *is differentiable at* $x = a^\star$ *if* X *has unbounded variation.*

2.5 Limitation

The method applied in this chapter totally depends on the existence of a unique solution to the system of equations (S1), which is obtained by guessing the smooth and continuous pasting properties at the boundaries. In the case of American Strangle, the two equations from (S1) are nice enough for us to find a unique solution. However, for other gain functions, it may be very difficult or impossible to solve the system.

We illustrate the difficulties with another example. Consider the following optimal stopping problem,

$$V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau}g(X_\tau)\mathbb{1}_{\{\tau<\infty\}}) \tag{2.18}$$

where $\{X_t, t \geq 0\}$ is a spectrally negative Lévy process, $g(x) = e^{2x} + \alpha e^x + \beta$ and $q > \max\{0, \psi(2)\}$, $\beta > 0$, $\alpha < -\sqrt{\frac{4q(q-\psi(2))\beta}{(q-\psi(1))^2}}$, and the supremum is taken over the class $\mathcal{T}_{[0,\infty]}$ of Markov stopping times with respect to $\{\mathcal{F}_t\}$.

By checking the infinitesimal generator, we have for all $x \in \mathbb{R}$

$$\mathbb{L}_X g(x) - qg(x) = (\psi(2) - q)e^{2x} + \alpha(\psi(1) - q)e^x - q\beta. \tag{2.19}$$

So $\mathbb{L}_X g(x) - qg(x) < 0$ for all $x \in \mathbb{R} \setminus (a_L^-, a_L^+)$, and $\mathbb{L}_X g(x) - qg(x) > 0$ for all $x \in (a_L^-, a_L^+)$, where a_L^- and a_L^+ with $a_L^- < a_L^+$ are the solutions to $\mathbb{L}_X g(x) - qg(x) = 0$. Together with the shape of $g(x)$, it is reasonable to guess that the continuation region $\mathcal{C}$ is of the form of a bounded interval (a, b).

By applying Esscher transform, see equation (1.9) in Chapter 1, we can derive a closed formula for the candidate value function $V_{\tau_{a,b}}$ for all $a < b$ and $x \leq b$.

$$\begin{aligned} V_{\tau_{a,b}}(x) &= \mathbb{E}_x\left(e^{-q\tau_{a,b}}g(X_{\tau_{a,b}})\mathbb{1}_{\{\tau_{a,b}<\infty\}}\right) \\ &= \mathbb{E}_x\left(e^{-q\tau_{a,b}}e^{2X_{\tau_{a,b}}}\right) + \alpha\mathbb{E}_x\left(e^{-q\tau_{a,b}}e^{X_{\tau_{a,b}}}\right) + \beta\mathbb{E}_x\left(e^{-q\tau_{a,b}}\right) \\ &= e^{2x}\mathbb{E}_x^2\left(e^{-(q-\psi(2))\tau_{a,b}}\right) + \alpha e^x\mathbb{E}_x^1\left(e^{-(q-\psi(1))\tau_{a,b}}\right) + \beta\mathbb{E}_x\left(e^{-q\tau_{a,b}}\right) \\ &= e^{2x}\mathbb{E}_x^2\left(e^{-(q-\psi(2))\tau_{a,b}}\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right) + e^{2x}\mathbb{E}_x^2\left(e^{-(q-\psi(2))\tau_{a,b}}\mathbb{1}_{\{\tau_a^->\tau_b^+\}}\right) \\ &\quad +\alpha e^x\mathbb{E}_x^1\left(e^{-(q-\psi(1))\tau_{a,b}}\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right) + \alpha e^x\mathbb{E}_x^1\left(e^{-(q-\psi(1))\tau_{a,b}}\mathbb{1}_{\{\tau_a^->\tau_b^+\}}\right) \\ &\quad +\beta\mathbb{E}_x\left(e^{-q\tau_{a,b}}\mathbb{1}_{\{\tau_a^->\tau_b^+\}}\right) + \beta\mathbb{E}_x\left(e^{-q\tau_{a,b}}\mathbb{1}_{\{\tau_a^-<\tau_b^+\}}\right). \end{aligned}$$

Then, from Theorem 1.5, it follows that for all $a < b$ and $x \leq b$

$$\mathbb{E}_x\left(e^{-q\tau_{a,b}}g(X_{\tau_{a,b}})\right) = h_1(x,a) + \frac{W^q(x-a)}{W^q(b-a)}\left(g(b) - h_1(b,a)\right), \tag{2.20}$$

where $h_1(x,a) = e^{2x}Z_2^{q-\psi(2)}(x-a) + \alpha e^x Z_1^{q-\psi(1)}(x-a) + \beta Z^q(x-a)$. As before, we can guess that the optimal stopping boundaries $a^\star$ and $b^\star$ satisfy the following system of equations,

$$\begin{cases} h_1(b,a) = g(b), \\ \frac{\partial h_1}{\partial x}(b,a) = 2e^{2b} + \alpha e^b. \end{cases} \tag{2.21}$$

And by a similar argument as in the American strangle case, we see that the system of equations (2.21) guarantees that the continuous pasting happens at both $a^\star$ and $b^\star$, and the smooth pasting happens at $b^\star$. Moreover, the smooth pasting happens at $a^\star$ if X has unbounded variation. Let $v = b - a$, then the system of equations (2.21) can be rewritten as follows:

$$e^{2b} + e^b + \beta = e^{2b}Z_2^{q-\psi(2)}(v) + e^b Z_1^{q-\psi(1)}(v) + \beta Z^q(v) \tag{2.22}$$

$$e^{2b} + \alpha e^b = \begin{cases} 2e^{2b}Z_2^{q-\psi(2)}(v) + (q-\psi(2))e^{2b}W_2^{q-\psi(2)}(v) \\ +\alpha e^b Z_1^{q-\psi(1)}(v) + \alpha(q-\psi(1))e^b W_1^{q-\psi(1)}(v) \\ +\beta q W^q(v) \end{cases} \tag{2.23}$$

We see that each equation above gives two solutions for e^b in terms of v, and it is very hard to eliminate any of them. Furthermore, unlike the American strangle case, it is extremely difficult to obtain any solutions for e^b from the system of equations (2.21).

2.6 Proof

Proof for Proposition 2.4.

Proof of (i)

By rewriting (S1), we get

$$e^a = e^{-v}\frac{K_2 + K_1 Z^q(v)}{1 + Z_1^{q-\psi(1)}(v)} \tag{2.24}$$

$$e^a = \frac{qe^{-v}W^q(v)K_1}{1 + Z_1^{q-\psi(1)}(v) + (q-\psi(1))W_1^{q-\psi(1)}(v)} \tag{2.25}$$

where $v = b - a$. From now on, we denote by $f_1(v)$ and $f_2(v)$ the RHS of (2.24) and (2.25). Note that finding a solution pair (a, b) for the system of equations (S1) with $a < b$ is equivalent to finding a solution v to $f_1(v) = f_2(v)$ with $v > 0$.

By differentiating f_1, we get for all $v > 0$

$$\begin{aligned}
f_1'(v) &= -\frac{e^{-v}(K_2 + K_1 Z^q(v))(1 + Z_1^{q-\psi(1)}(v))}{(1 + Z_1^{q-\psi(1)}(v))^2} \\
&\quad -\frac{e^{-v}(K_2 + K_1 Z^q(v))(q-\psi(1))W_1^{q-\psi(1)}(v)}{(1 + Z_1^{q-\psi(1)}(v))^2} \\
&\quad +\frac{e^{-v}qK_1W^q(v)(1 + Z_1^{q-\psi(1)}(v))}{(1 + Z_1^{q-\psi(1)}(v))^2} \\
&= -\frac{e^{-v}(K_2 + K_1 Z^q(v))\left(1 + Z_1^{q-\psi(1)}(v) + (q-\psi(1))W_1^{q-\psi(1)}(v)\right)}{(1 + Z_1^{q-\psi(1)}(v))^2} \\
&\quad +\frac{e^{-v}qK_1W^q(v)(1 + Z_1^{q-\psi(1)}(v))}{(1 + Z_1^{q-\psi(1)}(v))^2} \\
&= \frac{\left(1 + Z_1^{q-\psi(1)}(v) + (q-\psi(1))W_1^{q-\psi(1)}(v)\right)}{1 + Z_1^{q-\psi(1)}(v)}\left(-\frac{e^{-v}(K_2 + K_1 Z^q(v))}{1 + Z_1^{q-\psi(1)}(v)}\right. \\
&\quad \left.+\frac{e^{-v}qK_1W^q(v)}{1 + Z_1^{q-\psi(1)}(v) + (q-\psi(1))W_1^{q-\psi(1)}(v)}\right) \\
&= \frac{\left(1 + Z_1^{q-\psi(1)}(v) + (q-\psi(1))W_1^{q-\psi(1)}(v)\right)}{1 + Z_1^{q-\psi(1)}(v)}(-f_1(v) + f_2(v)).
\end{aligned} \tag{2.26}$$

Thus, $v^\star > 0$ is a solution to $f_1(v) = f_2(v)$ if and only if $f_1'(v^\star) = 0$. So, the problem is reduced to show the existence of a local stationary point for f_1 on $(0, \infty)$.

Now, we will show that f_1 converges as $v \to \infty$, and $f_1(0) > \lim_{v\to\infty} f_1(v)$ and $f_1(v) < \lim_{v\to\infty} f_1(v)$ for all v large enough. This will guarantee the existence of at least one local minimum point for f_1 on $(0,\infty)$, and therefore, will guarantee the existence of at least one solution pair $(a^\star, b^\star)$ to (S1) with $a^\star < b^\star$.

First we show the existence of $\lim_{v\to\infty} f_1(v)$. For all $v > 0$, we have

$$f_1(v) = e^{-v}\frac{K_2 + K_1 Z^q(v)}{1 + Z_1^{q-\psi(1)}(v)} = e^{-v}\frac{\frac{K_2}{W_1^{q-\psi(1)}(v)} + \frac{K_1 Z^q(v)}{W_1^{q-\psi(1)}(v)}}{\frac{1}{W_1^{q-\psi(1)}(v)} + \frac{Z_1^{q-\psi(1)}(v)}{W_1^{q-\psi(1)}(v)}}$$

$$= \frac{\frac{K_2}{W^q(v)} + \frac{K_1 Z^q(v)}{W^q(v)}}{\frac{1}{W_1^{q-\psi(1)}(v)} + \frac{Z_1^{q-\psi(1)}(v)}{W_1^{q-\psi(1)}(v)}}.$$

Note that $\frac{W^q(v)}{Z^q(v)}$ decreases to $\frac{q}{\Phi(q)}$, $\frac{W_1^{q-\psi(1)}(v)}{Z_1^{q-\psi(1)}(v)}$ decreases to $\frac{q-\psi(1)}{\Phi(q)-1}$ and $W_1^{q-\psi(1)}(v)$ increases to ∞ as $v \uparrow \infty$. As a consequence, $f_1(v)$ converges as $v \uparrow \infty$, and

$$\lim_{v\uparrow\infty} f_1(v) = \frac{q(\Phi(q)-1)}{\Phi(q)(q-\psi(1))}K_1 < K_1.$$

The last inequality happens as a result of the convexity of Laplace exponent ψ and $q = \psi(\Phi(q))$.

Next, note that $f_1(0) > \lim_{v\uparrow\infty} f_1(v)$, is true by a direct calculation

$$f_1(0) = \frac{K_1 + K_2}{2} \geq K_1 > \lim_{v\uparrow\infty} f_1(v).$$

Finally, we show that for v large enough

$$f_1(v) < \lim_{v\uparrow\infty} f_1(v) = \frac{q(\Phi(q)-1)}{\Phi(q)(q-\psi(1))}K_1.$$

Define for all $v > 0$,

$$\begin{aligned}
\epsilon_v^q &= \frac{Z^q(v)}{W^q(v)} - \frac{q}{\Phi(q)}, \\
\epsilon_v^{q-\psi(1)} &= \frac{Z_1^{q-\psi(1)}(v)}{W_1^{q-\psi(1)}(v)} - \frac{q-\psi(1)}{\Phi(q)-1}, \\
v_v^q &= \frac{1}{W^q(v)}.
\end{aligned}$$

So ϵ_v^q, $\epsilon_v^{q-\psi(1)}$ and v_v^q all decrease to 0 as $v \to \infty$. Also for all $v > 0$, define c_v^q to be such that

$$c_v^q := \frac{\epsilon_v^q}{v_v^q} = \frac{\frac{Z^q(v)}{W^q(v)} - \frac{q}{\Phi(q)}}{\frac{1}{W^q(v)}} = Z^q(v) - \frac{q}{\Phi(q)} W^q(v) = \mathbb{E}_v \left(e^{-q\tau_0^-} \mathbb{1}_{\{\tau_0^- < \infty\}} \right).$$

Clearly, $\mathbb{E}_v \left(e^{-q\tau_0^-} \mathbb{1}_{\{\tau_0^- < \infty\}} \right)$ decreases to 0 as $v \uparrow \infty$. So $c_v^q \downarrow 0$ as $v \uparrow \infty$. Now by using ϵ_v^q, $\epsilon_v^{q-\psi(1)}$, v_v^q and c_v^q, we can rewrite f_1 for all $v > 0$ and get

$$f_1(v) = \frac{\frac{K_2}{W^q(v)} + \frac{K_1 Z^q(v)}{W^q(v)}}{\frac{1}{W_1^{q-\psi(1)}(v)} + \frac{Z_1^{q-\psi(1)}(v)}{W_1^{q-\psi(1)}(v)}} = \frac{\frac{K_2}{W^q(v)} + \frac{K_1 Z^q(v)}{W^q(v)}}{\frac{e^v}{W^q(v)} + \frac{Z_1^{q-\psi(1)}(v)}{W_1^{q-\psi(1)}(v)}}$$
$$= \frac{v_v^q K_2 + \epsilon_v^q K_1 + \frac{q}{\Phi(q)} K_1}{e^v v_v^q + \epsilon_v^{q-\psi(1)} + \frac{q-\psi(1)}{\Phi(q)-1}}.$$

Note that for all strictly positive numbers a_1, a_2, a_3 and a_4, if $\frac{a_1}{a_2} < \frac{a_3}{a_4}$, then $\frac{a_1+a_3}{a_2+a_4} < \frac{a_3}{a_4}$. So if we can show that

$$\frac{v_v^q K_2 + \epsilon_v^q K_1}{e^v v_v^q + \epsilon_v^{q-\psi(1)}} < \frac{\frac{q}{\Phi(q)}}{\frac{q-\psi(1)}{\Phi(q)-1}} K_1$$

for all v large enough, then $f_1(v) < \frac{q(\Phi(q)-1)}{\Phi(q)(q-\psi(1))} K_1$ for all v large enough. This is indeed true because for all $v > 0$

$$\frac{v_v^q K_2 + \epsilon_v^q K_1}{e^v v_v^q + \epsilon_v^{q-\psi(1)}} = \frac{v_v^q K_2 + v_v^q c_v^q K_1}{e^v v_v^q + \epsilon_v^{q-\psi(1)}} < \frac{v_v^q K_2 + v_v^q c_v^q K_1}{e^v v_v^q} = \frac{K_2 + c_v^q K_1}{e^v}.$$

Clearly, the last term decreases to 0 as v goes to ∞. So there exists a number v_0 such that for all $v > v_0$, $f_1(v) < \frac{q(\Phi(q)-1)}{\Phi(q)(q-\psi(1))} K_1$. Therefore, there exists at least one strictly positive solution to the problem $f_1(v) = f_2(v)$.

Proof of (a)

Let $a^\star$ and $b^\star$ be a pair of solution with $a^\star < b^\star$. From the definition of h, we have for all $x > a^\star$:

$$\frac{\partial h}{\partial x}(x, a^\star) = qK_1 W^q(x - a^\star) - e^x Z_1^{q-\psi(1)}(x - a^\star) - (q - \psi(1)) e^x W_1^{q-\psi(1)}(x - a^\star).$$

By using $W^q(x) = e^x W_1^{q-\psi(1)}(x)$ for all $x \in \mathbb{R}$, we can rewrite the above formula for all $x > a^\star$ as

$$\frac{\partial h}{\partial x}(x,a^\star) = e^x W_1^{q-\psi(1)}(x-a^\star)\left(qK_1e^{-a^\star} - (q-\psi(1)) - \frac{Z_1^{q-\psi(1)}(x-a^\star)}{W_1^{q-\psi(1)}(x-a^\star)}\right). \tag{2.27}$$

Note that $\frac{Z_1^{q-\psi(1)}(x-a^\star)}{W_1^{q-\psi(1)}(x-a^\star)}$ decreases to $\frac{q-\psi(1)}{\Phi(q)-1}$ as x goes to ∞. So if $e^{a^\star} \geq e^{a_p}$, we get $qK_1e^{-a^\star} - (q-\psi(1)) - \frac{Z_1^{q-\psi(1)}(x-a^\star)}{W_1^{q-\psi(1)}(x-a^\star)} \leq 0$ for all $x \in \mathbb{R}$. Thus, $\frac{\partial h}{\partial x}(x,a^\star) \leq 0$ for all $x > a^\star$. Therefore, there doesn't exist any $b > a^\star$ such that $\frac{\partial h}{\partial x}(b,a^\star) = e^b > 0$. Hence, we must have $e^{a^\star} < e^{a_p} < K_1$, where the last inequality is due to the convexity of the Laplace exponent ψ.

<u>*Proof of (b)*</u>

Note that $h(x,a_p)$ is the American put value function for the gain function $(K_1 - e^x)^+$ (see [49]), where a_p is as defined in part (a). Furthermore, $h(x,a_p) > (K_1 - e^x)^+$ for all $x \in (a_p,\infty)$, and $h(x,a_p)$ goes to 0 as $x \to \infty$. Thus, there exists $b_0 > \log(K_2)$ such that $h(b_0,a_p) = e^{b_0} - K_2$, and $h(x,a_p) > g(x)$ on $x \in (a^\star, b_0)$.

Next, for all $a < a_p$ we have for all $x \in \mathbb{R}\setminus\{a\}$

$$\begin{aligned}\frac{\partial h}{\partial a}(x,a) &= -qK_1W^q(x-a) + e^x(q-\psi(1))W_1^{q-\psi(1)}(x-a)\\ &= W^q(x-a)\left(e^a(q-\psi(1)) - qK_1\right).\end{aligned} \tag{2.28}$$

As $e^a < e^{a_p} < \frac{qK_1}{q-\psi(1)}$, from the fundamental theorem of calculus and equation (2.28), we can conclude that $h(x,a^\star) > h(x,a_p) \geq g(x)$ for all $x \in (a^\star, b_0)$. Finally, from (S1) it follows that $b^\star > b_0 > \log(K_2)$.

<u>*Proof of (c)*</u>

As $\frac{Z_1^{q-\psi(1)}(x-a^\star)}{W_1^{q-\psi(1)}(x-a^\star)}$ is strictly decreasing in x for all $x > a^\star$,

$$qK_1e^{-a^\star} - (q-\psi(1)) - \frac{Z_1^{q-\psi(1)}(x-a^\star)}{W_1^{q-\psi(1)}(x-a^\star)}$$

is strictly increasing in x on the set $(a^\star,\infty)$. It follows from $a^\star < a_p$ in part (a), that there is a unique point $a_s < b^\star$ such that

$$\begin{aligned}W_1^{q-\psi(1)}(x-a^\star)\left(qK_1e^{-a^\star} - (q-\psi(1)) - \frac{Z_1^{q-\psi(1)}(x-a^\star)}{W_1^{q-\psi(1)}(x-a^\star)}\right) &< 0 \text{ on } (a^\star, a_s),\\ W_1^{q-\psi(1)}(x-a^\star)\left(qK_1e^{-a^\star} - (q-\psi(1)) - \frac{Z_1^{q-\psi(1)}(x-a^\star)}{W_1^{q-\psi(1)}(x-a^\star)}\right) &> 0 \text{ on } (a_s, \infty).\end{aligned}$$

Furthermore, $W_1^{q-\psi(1)}(x-a^\star)\left(qK_1e^{-a^\star}-(q-\psi(1))-\frac{Z_1^{q-\psi(1)}(x-a^\star)}{W_1^{q-\psi(1)}(x-a^\star)}\right)$ is strictly increasing in x on (a_s,∞). Thus, by comparing to equation equation (2.27), it follows from the system of equations (S1), that $\frac{\partial h}{\partial x}(x,a^\star)<e^x$ on $(-\infty,a^\star)\cup(a^\star,b^\star)$, and $\frac{\partial h}{\partial x}(x,a^\star)>e^x$ on $(b^\star,\infty)$.

Proof of (d)

As a result of part (c), we get from the fundamental theorem that for all $x<b^\star$

$$h(b^\star,a^\star)<h(x,a^\star)+\int_x^{b^\star}e^y dy=h(x,a^\star)+e^{b^\star}-e^x.$$

As $h(b^\star,a^\star)=e^{b^\star}-K_2$, we have for all $x<b^\star$

$$h(x,a^\star)>e^x-K_2.$$

Together with part (iii), we can conclude that $h(x,a^\star)>g(x)$ for all $x\in(a^\star,b^\star)$. A similar argument can be applied for $x>b^\star$ to get the rest of the statement.

Proof of (e)

First note that as a result of part (d), there is a unique $b^\star$ for each $a^\star$. Suppose there exists another pair of solution $(a_1^\star,b_1^\star)$, without lost of generality let us assume that $a_1^\star<a^\star<a_p$. Then, thanks to part (d) and equation (2.28), it follows from the fundamental theorem of calculus that $h(x,a_1^\star)>h(x,a^\star)\geq g(x)$ for all $x>a_1^\star$. So such $b_1^\star$ does not exists. Therefore, $(a^\star,b^\star)$ is unique.

□

Proof for Lemma 2.5. Suppose $x_1<x_2$, then

$$\begin{aligned}&\int_{-\infty}^0 f(x_1+y)\Pi(dy)-\int_{-\infty}^0 f(x_2+y)\Pi(dy)\\ =&\int_{-\infty}^0(f(x_1+y)-f(x_2+y))\Pi(dy)\\ \leq&\ 0\end{aligned}$$

as required. □

Proof for Proposition 2.6. By rewriting (1.17), we have for all $x\in\mathbb{R}$

$$\mathbb{E}_x\left(e^{-q\tau_0^-}\mathbb{1}_{\{\tau_0^-<\infty\}}\right)=Z^q(x)-\frac{q}{\Phi(q)}W^q(x). \tag{2.29}$$

Note that $Z^q(X_{\tau_0^-})-\frac{q}{\Phi(q)}W^q(X_{\tau_0^-})=\mathbb{1}_{\{\tau_0^-<\infty\}}$ $\mathbb{P}$-a.s.. This is true as: in the event

$\{\tau_0^- < \infty\}$, if X has unbounded variation, $X_{\tau_0^-} \leq 0$ $\mathbb{P}$-a.s., so by the definition of Z and W, we have $Z^q(X_{\tau_0^-}) + \frac{q}{\Phi(q)}W^q(X_{\tau_0^-}) = 1$ $\mathbb{P}$-a.s.. If X has bounded variation, $X_{\tau_0^-} < 0$ $\mathbb{P}$-a.s., again $Z^q(X_{\tau_0^-}) + \frac{q}{\Phi(q)}W^q(X_{\tau_0^-}) = 1$. In the event $\{\tau_0^- = \infty\}$, we have $X_{\tau_0^-} = \infty$, thus

$$\lim_{x\uparrow\infty}\left(Z^q(x) - \frac{q}{\Phi(q)}W^q(x)\right) = \lim_{x\uparrow\infty}\mathbb{E}_x\left(e^{-q\tau_0^-}\mathbb{1}_{\{\tau_0^-<\infty\}}\right) = 0.$$

By using equation (2.29) we have for all $x \in \mathbb{R}$

$$\begin{aligned} Z^q(x) - \frac{q}{\Phi(q)}W^q(x) &= \mathbb{E}_x\left(e^{-q\tau_0^-}\mathbb{1}_{\{\tau_0^-<\infty\}}\right) \\ &= \mathbb{E}_x\left(e^{-q\tau_0^-}\left(Z^q(X_{\tau_0^-}) - \frac{q}{\Phi(q)}W^q(X_{\tau_0^-})\right)\right). \end{aligned}$$

By tower property and the strong Markov property, we get for all $x \in \mathbb{R}$

$$\begin{aligned} &Z^q(x) - \frac{q}{\Phi(q)}W^q(x) \\ &= \mathbb{E}_x\left(\mathbb{E}_x\left(e^{-q\tau_0^-}\left(Z^q(X_{\tau_0^-}) - \frac{q}{\Phi(q)}W^q(X_{\tau_0^-})\right)\Big|\mathcal{F}_t\right)\right) \\ &= \mathbb{E}_x\left(\mathbb{1}_{\{t\geq\tau_0^-\}}e^{-q\tau_0^-}\left(Z^q(X_{\tau_0^-}) - \frac{q}{\Phi(q)}W^q(X_{\tau_0^-})\right)\right) \\ &\quad+\mathbb{E}_x\left(\mathbb{1}_{\{t<\tau_0^-\}}\mathbb{E}_x\left(e^{-q\tau_0^-}\left(Z^q(X_{\tau_0^-}) - \frac{q}{\Phi(q)}W^q(X_{\tau_0^-})\right)\Big|\mathcal{F}_t\right)\right) \\ &= \mathbb{E}_x\left(\mathbb{1}_{\{t\geq\tau_0^-\}}e^{-q\tau_0^-}\left(Z^q(X_{\tau_0^-}) - \frac{q}{\Phi(q)}W^q(X_{\tau_0^-})\right)\right) \\ &\quad+\mathbb{E}_x\left(\mathbb{1}_{\{t<\tau_0^-\}}e^{-qt}\mathbb{E}_{X_t}\left(e^{-q\tilde{\tau}_0^-}\left(Z^q(\widetilde{X}_{\tilde{\tau}_0^-}) - \frac{q}{\Phi(q)}W^q(\widetilde{X}_{\tilde{\tau}_0^-})\right)\right)\right), \end{aligned}$$

where $\widetilde{X}$ is an independent copy of X with respect to $\mathbb{P}_{X_t}$, and $\tilde{\tau}_0^-$ is the first time

$\widetilde{X}$ goes below 0. Thus, by equation (2.29) we have for all $x \in \mathbb{R}$

$$
\begin{aligned}
Z^q(x) - \frac{q}{\Phi(q)} W^q(x) &= \mathbb{E}_x\left(\mathbb{1}_{\{t \geq \tau_0^-\}} e^{-q\tau_0^-} \left(Z^q(X_{\tau_0^-}) - \frac{q}{\Phi(q)} W^q(X_{\tau_0^-}) \right) \right) \\
&\quad + \mathbb{E}_x\left(\mathbb{1}_{\{t < \tau_0^-\}} e^{-qt} \mathbb{E}_{X_t}\left(e^{-q\tau_0^-} \mathbb{1}_{\{\tau_0^- < \infty\}} \right) \right) \\
&= \mathbb{E}_x\left(\mathbb{1}_{\{t \geq \tau_0^-\}} e^{-q\tau_0^-} \left(Z^q(X_{\tau_0^-}) - \frac{q}{\Phi(q)} W^q(X_{\tau_0^-}) \right) \right) \\
&\quad + \mathbb{E}_x\left(\mathbb{1}_{\{t < \tau_0^-\}} e^{-qt} \left(Z^q(X_t) - \frac{q}{\Phi(q)} W^q(X_t) \right) \right) \\
&= \mathbb{E}_x\left(e^{-q\tau_0^- \wedge t} \left(Z^q(X_{\tau_0^- \wedge t}) - \frac{q}{\Phi(q)} W^q(X_{\tau_0^- \wedge t}) \right) \right).
\end{aligned}
$$

So by independent and stationary increments of Lévy processes X and the strong Markov property of X, $\{e^{-q\tau_0^- \wedge t} \left(Z^q(X_{\tau_0^- \wedge t}) - \frac{q}{\Phi(q)} W^q(X_{\tau_0^- \wedge t}) \right), t \geq 0\}$ is a martingale for all $x \in \mathbb{R}$. Since $\{e^{-q\tau_0^- \wedge t} W^q(X_{\tau_0^- \wedge t}), t \geq 0\}$ is a martingale, for example see [7], so by linearity we have $\{e^{-q\tau_0^- \wedge t} Z^q(X_{\tau_0^- \wedge t}), t \geq 0\}$ is a martingale.

Next, denote by $\mathbb{L}_X$ the infinitesimal generator of X. As a result of the differentiability of W^q, $Z^q \in C^{\delta_1} \cap C^{1+\delta_1}(\mathbb{R} \setminus \{0\})$ where $\delta_1 = \mathbb{1}_{\{X \text{ has unbounded variation}\}}$. Then $\mathbb{L}_X Z^q(x)$ is well defined for all $x \in \mathbb{R} \setminus \{0\}$. Then, it follows from Itô's formula and the Doob Meyer decomposition that

$$
\int_0^{t \wedge \tau_0^-} e^{-qs} \left(\mathbb{L}_X Z^q(X_s) - qZ^q(X_s) \right) \mathbb{1}_{\{X_s \neq 0\}} ds = 0 \tag{2.30}
$$

$\mathbb{P}_x$-a.s. for all $t \geq 0$ and $x \in \mathbb{R}$. By dividing both sides of equation (2.30) by $t > 0$ and letting $t \downarrow 0$, we obtain from the right continuity of Lévy processes that

$$
\mathbb{L}_X Z^q(x) - qZ^q(x) = 0,
$$

for all $x > 0$

□

Proof for Proposition 2.7. First, we shall show that

$$
\mathbb{L}_X V_{\tau_{a^\star, b^\star}}(x) - qV_{\tau_{a^\star, b^\star}}(x) \leq 0
$$

for all $x \in \mathbb{R} \setminus \{a^\star, b^\star\}$.

For all $x < a^\star$, by definition of h we have

$$\mathbb{L}_X h(x, a^\star) - qh(x, a^\star) = \mathbb{L}_X g(x) - qg(x).$$

Then, it follows from part part (a) in Proposition 2.4 that

$$\mathbb{L}_X h(x, a^\star) - qh(x, a^\star) = \mathbb{L}_X g(x) - qg(x) < 0$$

for all $x < a^\star$. Next, we consider $x \in (a^\star, b^\star)$. Note that by Esscher transform and Proposition 2.6, we obtain that $\{e^{-qt} e^{X_{t\wedge\tau_0^-}} Z_1^{q-\psi(1)}(X_{t\wedge\tau_0^-}), t \geq 0\}$ is a $\mathbb{P}_x$ martingale as well. Furthermore, by a similar argument as in Proposition 2.6,

$$\mathbb{L}_X e^{x-a^\star} Z_1^{q-\psi(1)}(x - a^\star) - qe^{x-a^\star} Z_1^{q-\psi(1)}(x - a^\star) = 0$$

for all $x > a^\star$. Therefore, we obtain from equation (2.14) and (S1) that

$$\mathbb{L}_X V_{\tau_{a^\star,b^\star}}(x) - qV_{\tau_{a^\star,b^\star}}(x) = \mathbb{L}_X h(x, a^\star) - qh(x, a^\star) = 0$$

for all $x \in (a^\star, b^\star)$.

Now, we consider $x > b^\star$. Let $g_1(x) = e^x - K_2$ for all $x \in \mathbb{R}$, then $V_{\tau_{a^\star,b^\star}}(x) = g_1(x)$ for all $x > b^\star$. So for all $x > b^\star$

$$\begin{aligned}
&\mathbb{L}_X V_{\tau_{a^\star,b^\star}}(x) - qV_{\tau_{a^\star,b^\star}}(x) \\
&= -qV_{\tau_{a^\star,b^\star}}(x) + \mu V'_{\tau_{a^\star,b^\star}}(x) + \frac{1}{2}\sigma^2 V''_{\tau_{a^\star,b^\star}}(x) \\
&\quad + \int_{(-\infty,0)} \left(V_{\tau_{a^\star,b^\star}}(x+y) - V_{\tau_{a^\star,b^\star}}(x) - yV'_{\tau_{a^\star,b^\star}}(x)\mathbb{1}_{\{|y|<1\}}\right) \Pi(dy) \\
&= \mathbb{L}_X g_1(x) - qg_1(x) + \int_{(-\infty,0)} \left(V_{\tau_{a^\star,b^\star}}(x+y) - g_1(x+y)\right) \Pi(dy) \\
&= -(q - \psi(1))e^x + qK_2 + \int_{(-\infty,0)} \left(V_{\tau_{a^\star,b^\star}}(x+y) - g_1(x+y)\right) \Pi(dy).
\end{aligned} \tag{2.31}$$

Then, thanks to part (c) in Proposition 2.4, $V_{\tau_{a^\star,b^\star}}(x) - g_1(x)$ is a decreasing function in x for all $x > b^\star$. Therefore, it follows from Lemma 2.5, that $\mathbb{L}_X V_{\tau_{a^\star,b^\star}}(x) - qV_{\tau_{a^\star,b^\star}}(x)$ is decreasing in x on the set $(b^\star, \infty)$. Then, once we have shown that

$$\mathbb{L}_X V_{\tau_{a^\star,b^\star}}(b^\star+) - qV_{\tau_{a^\star,b^\star}}(b^\star+) \leq 0,$$

we can conclude that $\mathbb{L}_X V_{\tau_{a^\star,b^\star}}(x) - qV_{\tau_{a^\star,b^\star}}(x) \leq 0$ on $\mathbb{R} \setminus \{a^\star, b^\star\}$.

First suppose that X has unbounded variation. Note that, by part (c) in Proposition 2.4, there exists $d > 0$ such that $|V_{\tau_{a^\star,b^\star}}(x)| < e^x + d$ and $|V'_{\tau_{a^\star,b^\star}}(x)| < e^x + d$ for all $x \in \mathbb{R}$. Furthermore, $|V''_{\tau_{a^\star,b^\star}}(x)| < e^x + d$ for all $x \in \mathbb{R} \setminus \{a^\star, b^\star\}$. Then, from the fundamental theorem of calculus, for all $x > b^\star$ and $y \in (-1, 0)$ there exists $x_y \in (x + y, x)$ such that

$$\begin{aligned} |V_{\tau_{a^\star,b^\star}}(x+y) - V_{\tau_{a^\star,b^\star}}(x) - yV'_{\tau_{a^\star,b^\star}}(x)| &\leq y^2|V''_{\tau_{a^\star,b^\star}}(x_y)| \\ &\leq y^2(e^{x_y} + d) \\ &\leq y^2(e^x + d). \end{aligned}$$

For all $x > b^\star$, $y \leq -1$, we have

$$\begin{aligned} |V_{\tau_{a^\star,b^\star}}(x+y) - V_{\tau_{a^\star,b^\star}}(x)| &\leq |V_{\tau_{a^\star,b^\star}}(x+y)| + |V_{\tau_{a^\star,b^\star}}(x)| \\ &\leq e^{x+y} + d + e^x + d \\ &\leq 2(e^x + d). \end{aligned}$$

By combining the above two equations together, we obtain that for all $x > b^\star$ and $y < 0$,

$$|V_{\tau_{a^\star,b^\star}}(x+y) - V_{\tau_{a^\star,b^\star}}(x) - yV'_{\tau_{a^\star,b^\star}}(x)\mathbb{1}_{\{y\in(-1,0)\}}| \leq \tilde{c}(1 \wedge y^2)(e^x + d),$$

where $\tilde{c}$ is some strictly positive constant. Then, from the dominated convergence theorem, it follows that

$$\begin{aligned} &\lim_{\epsilon\downarrow 0}\left(\mathbb{L}_X V_{\tau_{a^\star,b^\star}}(b^\star+\epsilon) - qV_{\tau_{a^\star,b^\star}}(b^\star+\epsilon)\right) \\ &\quad= \lim_{\epsilon\downarrow 0}\left(\mu V'_{\tau_{a^\star,b^\star}}(b^\star+\epsilon) + \frac{\sigma^2}{2}V''_{\tau_{a^\star,b^\star}}(b^\star+\epsilon) - qV_{\tau_{a^\star,b^\star}}(b^\star+\epsilon)\right) \\ &\qquad+ \lim_{\epsilon\downarrow 0}\int_{-\infty}^{0}(V_{\tau_{a^\star,b^\star}}(b^\star+y+\epsilon) - V_{\tau_{a^\star,b^\star}}(b^\star+\epsilon) - yV'_{\tau_{a^\star,b^\star}}(b^\star+\epsilon)\mathbb{1}_{\{|y|<1\}})\Pi(dy) \\ &\quad= \mu V'_{\tau_{a^\star,b^\star}}(b^\star+) + \frac{\sigma^2}{2}V''_{\tau_{a^\star,b^\star}}(b^\star+) - qV_{\tau_{a^\star,b^\star}}(b^\star+) \\ &\qquad+ \int_{-\infty}^{0}(V_{\tau_{a^\star,b^\star}}((b^\star+y)+) - V_{\tau_{a^\star,b^\star}}(b^\star+) - yV'_{\tau_{a^\star,b^\star}}(b^\star+)\mathbb{1}_{\{|y|<1\}})\Pi(dy). \end{aligned}$$

By Remark 2.2 and (S1), we see that $V_{\tau_{a^\star,b^\star}} \in C^1(\mathbb{R})$ in the case when X has unbounded variation. Therefore, by equation (2.14), we can rewrite the above the equation as

$$
\begin{aligned}
&\mathbb{L}_X V_{\tau_{a^\star,b^\star}}(b^\star+) - qV_{\tau_{a^\star,b^\star}}(b^\star+)\\
&= \mu V'_{\tau_{a^\star,b^\star}}(b^\star) + \frac{\sigma^2}{2}V''_{\tau_{a^\star,b^\star}}(b^\star+) - qV_{\tau_{a^\star,b^\star}}(b^\star)\\
&\quad + \int_{-\infty}^{0}(V_{\tau_{a^\star,b^\star}}(b^\star+y) - V_{\tau_{a^\star,b^\star}}(b^\star) - yV'_{\tau_{a^\star,b^\star}}(b^\star)\mathbb{1}_{\{|y|<1\}})\Pi(dy)\\
&= \mu\frac{\partial h}{\partial x}(b^\star,a^\star) + \frac{\sigma^2}{2}V''_{\tau_{a^\star,b^\star}}(b^\star+) - qh(b^\star,a^\star)\\
&\quad + \int_{-\infty}^{0}(h(b^\star+y,a^\star) - h(b^\star,a^\star) - y\frac{\partial h}{\partial x}(b^\star,a^\star)\mathbb{1}_{\{|y|<1\}})\Pi(dy)\\
&= \mathbb{L}_X h(b^\star,a^\star) - qh(b^\star,a^\star) + \frac{\sigma^2}{2}(V''_{\tau_{a^\star,b^\star}}(b^\star+) - \frac{\partial^2 h}{\partial x^2}(b^\star,a^\star))\\
&\le 0,
\end{aligned}
$$

where the last equality is due to

$$\mathbb{L}_X h(b^\star,a^\star) - qh(b^\star,a^\star) = 0,$$

and $h(x,a^\star) > V_{\tau_{a^\star,b^\star}}(x)$ for all $x > a^\star$ which can be seen from part (c) in Proposition 2.4. The case where X has bounded variation can be done by a similar argument as above. Summing up, we derive that for all $x \in \mathbb{R}\setminus\{a^\star,b^\star\}$

$$\mathbb{L}_X V_{\tau_{a^\star,b^\star}}(x) - qV_{\tau_{a^\star,b^\star}}(x) \le 0. \tag{2.32}$$

Now, we return to prove the supermartingale property of $\{e^{-qt}V_{\tau_{a^\star,b^\star}}(X_t), t \ge 0\}$. Note that for spectrally negative Lévy processes, the occupancy time in the set $\{a^\star,b^\star\}$ is 0 $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}$. Then by Itô's formula, we have

$$
\begin{aligned}
&e^{-qt}V_{\tau_{a^\star,b^\star}}(X_t)\\
&= V_{\tau_{a^\star,b^\star}}(X_0) + \int_0^t e^{-qs}\left(\mathbb{L}_X V_{\tau_{a^\star,b^\star}}(X_s) - qV_{\tau_{a^\star,b^\star}}(X_s)\right)\mathbb{1}_{\{X_s\notin\{a^\star,b^\star\}\}}ds + M_t^V\\
&\le V_{\tau_{a^\star,b^\star}}(X_0) + M_t^V
\end{aligned}
$$

$\mathbb{P}$-a.s. for all $t \ge 0$, where M_t^V is a local martingale. As local martingales which are bounded below are true martingales, we can conclude that for all $x \in \mathbb{R}$

$$\mathbb{E}_x\left(e^{-qt}V_{\tau_{a^\star,b^\star}}(X_t)\right) \le V_{\tau_{a^\star,b^\star}}(x).$$

Finally, using stationary and independent increments properties of Lévy processes, we conclude that $\{e^{-qt}V_{\tau_{a^\star,b^\star}}(X_t), t \geq 0\}$ is a $\mathbb{P}_x$ supermartingale.

$\square$

Chapter 3

On the Left Semi-Solution of the Optimal Stopping Problem for Smooth Gain Functions

3.1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with characteristic triple (μ, σ, Π), where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge x^2)\Pi(dx) < \infty$. For any $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the law of X starting from x, and we write simply $\mathbb{P}_0 = \mathbb{P}$. And denote by $\mathbb{E}_x$ and $\mathbb{E}$ the corresponding expectation operators. Throughout this chapter we assume the spectrally negative Lévy process X satisfying the following condition:

X has bounded variation, or X has unbounded variation and the q scale function $W^q \in C^2(\mathbb{R} \setminus \{0\})$. (3.1)

One example of a spectrally negative Lévy process in this class would be any spectrally negative Lévy process with the Brownian exponent, which guarantees X has unbounded variation and $W^q \in C^2(\mathbb{R} \setminus \{0\})$. We refer to [12], [50] and [46] for more conditions under which the above condition is satisfied.

In this chapter we consider the following optimal stopping problem

$$V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} g(X_\tau)), \tag{3.2}$$

where $q > 0$, the gain function g is sufficiently smooth (in the class $D^2(I)$ and satisfying Assumption 3.24), and the supremum is taken over the class $\mathcal{T}_{[0,\infty]}$ of Markov stopping times taking values in $[0,\infty]$ with respect to $\{\mathcal{F}_t\}$.

A stopping time $\tau^\star$ is optimal if for all $x \in \mathbb{R}$

$$V(x) = \mathbb{E}_x(e^{-q\tau^\star} g(X_{\tau^\star})). \tag{3.3}$$

Recall that under very general condition of the gain function g, the stopping time

$$\tau_C = \inf\{t \geq 0 : X_t \notin C\}$$

is the smallest optimal stopping time, where C is the continuation region, i.e.

$$C \quad = \quad \{x \in \mathbb{R} : V(x) > g(x)\}.$$

Furthermore, $\{e^{-qt}V(X_t), t \geq 0\}$ is the smallest supermartingale such that $V \geq g$ on $\mathbb{R}$, and $\{e^{-qt}V(X_t), t \geq 0\}$ is a martingale inside the continuation region C, that is, $\{e^{-qt\wedge\tau_C}V(X_{t\wedge\tau_C}), t \geq 0\}$ is a martingale. We refer to [66] and [76] for a detailed account on the theory of optimal stopping problems.

Many approaches have been established in the literature to solve the optimal stopping problems, for example, the Stefan's free boundary approach and guess and verify approach, etc. However, they all have certain disadvantages when the underlying uncertainty is modeled by Lévy processes. For instance, in the guess and verify approach, because of the jumps it is not easy to obtain a formula for the candidate value function V_τ, where $V_\tau(x) = \mathbb{E}_x\left(e^{-q\tau}g(X_\tau)\right)$ for all $x \in \mathbb{R}$ and τ is a $\{\mathcal{F}_t\}$-stopping time. We point out here that, by using the averaging functions, Surya [78] found a closed formula for the candidate value function V_τ in the case where τ is a first passage time (from above or below), and showed that V_τ is the value function under the condition that the averaging function exists and satisfies a particular shape.

In this chapter we propose an approach to solve the optimal stopping problem in a very general setting. Our approach is inspired by Surya [78]. However, unlike [78], our approach requires neither the shape of the averaging function, nor any knowledge of the continuation region in advance, nor the pasting conditions at the boundary. Instead, our method relies on obtaining the left semi-solutions for the optimal stopping problem, which is defined as follows.

Definition 3.1. *A pair $(\bar{V}, \bar{\tau})$ is called a closed (open) left semi-solution of the optimal stopping problem* (3.2) *up to the point $b \in \mathbb{R}$ if the following statements hold true,* *(i)* $\bar{V}(x) = V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} g(X_\tau))$ *for all* $x \leq (<)\, b$. *and* $\bar{V}(x) = g(x)$ *otherwise,*

(ii) $\bar{V}(x) = \mathbb{E}_x\left(e^{-q\bar{\tau}} g(X_{\bar{\tau}})\right)$ *for all* $x \leq (<)\, b$.

We say that $\bar{V}$ *is the closed (open) left semi value function up to the point* b. *and* $\bar{\tau}$ *is a closed (open) left semi optimal stopping time up to the point* b. *and* C_b *is the closed (open) left semi continuation region up to the point* b. *where*

$$C_b = \{x \in C : x \leq (<)\, b\}.$$

A sufficient condition for a pair $(\bar{V}, \bar{\tau})$ to be the closed (open) semi-solution up to the point b is given below.

Lemma 3.2 (Sufficient Lemma for Left Semi-solution). *Consider the optimal stopping problem* (3.2). *and suppose there exists a function* $T : \mathbb{R} \to \mathbb{R}$ *such that* $\{e^{-qt}T(X_t), t \geq 0\}$ *is a supermartingale, and* $T(x) \geq g(x)$ *for all* $x \in \mathbb{R}$. *If there exist* $b \in \mathbb{R}$ *and a stopping time* $\bar{\tau}$ *such that*

$$T(x) = \mathbb{E}_x\left(e^{-q\bar{\tau}} g(X_{\bar{\tau}})\right)$$

for all $x \leq (<)\, b$, *then the pair* $(\bar{V}, \bar{\tau})$ *is a closed (open) left semi-solution up to the point* b, *where*

$$\bar{V}(x) = \begin{cases} T(x) & x \leq (<)\, b \\ g(x) & x > (\geq)\, b. \end{cases}$$

The proof for Lemma 3.2 can be found on page 67. From Lemma 3.2, we see that finding a left semi-solution of the optimal stopping problem is closely related to finding the function T. The main aim of this chapter is to present an effective approach to construct a function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, such that $h(\cdot, a)$ satisfies all conditions in Lemma 3.2 for some suitable choices of $a \in \mathbb{R}$. This function h is constructed based on the averaging function and q scale function W^q. Moreover,

$$\mathbb{E}_x\left(e^{-q\tau_{a,b}} g(X_{\tau_{a,b}})\right) = h(x, a).$$

for some $a < b$ and for all $x \leq b$, where

$$\begin{aligned} \tau_a^- &= \inf\{t \geq 0 : X_t < a\} \\ \tau_b^+ &= \inf\{t \geq 0 : X_t > b\} \\ \tau_{a,b} &= \tau_a^- \wedge \tau_b^+ = \inf\{t \geq 0 : X_t \notin [a,b]\}. \end{aligned}$$

Then under Assumption 3.24, the optimal stopping boundary $a^\star$ is chosen to be such that $h(x, a^\star)$ is the smallest function dominating the gain function g, and $b^\star$ is chosen to be the last point such that $h(x, a^\star) = g(x)$. Then, the pair $(\bar{V}, \tau_{a^\star, b^\star})$ is a closed left semi-solution up to $b^\star$ for the optimal stopping problem, where $\bar{V}$ is defined to be $h(\cdot, a^\star)$ on $(-\infty, b^\star] \cap \mathbb{R}$ and g otherwise. With the knowledge on the closed left semi value function $\bar{V}$, we can work out the closed left semi continuation region $\mathcal{C}_{b^\star}$. And all pasting conditions can be seen directly from the path properties of $h(\cdot, a)$. We also show that, under some suitable conditions, this method can be repeated to study the value function V for $x > b^\star$.

This chapter is organized as follows. In Section 2, we state the class of the gain functions we are working with, and also gather some useful properties for g in this class. In Section 3, we first introduce a closed form formula for an averaging function using infinitesimal generator of Lévy processes, and then construct the function h based on the averaging function and q scale function. We finish this section with some probabilistic and path properties of h. In Section 4, we find $a^\star$ and $b^\star$ such that the pair $(\bar{V}, \tau_{a^\star, b^\star})$ is a closed left semi-solution of the optimal stopping problem up to the point $b^\star$. We also give a sufficient condition under which $(\bar{V}, \tau_{a^\star, b^\star})$ is a solution. Furthermore, under suitable conditions, we show that the above construction (the averaging function, the function h and the pair $(a^\star, b^\star)$) can be repeated to study the value function $V(x)$ for $x > b$. We reproduce the results from [78] in Section 5 by using the approach suggested in this chapter. Section 6 consists of conclusion and discussion. Finally, in Section 7, we presents details of derivation of the results of Sections 2-5.

3.2 Properties on the gain function

Let ψ be the Laplace exponent for X, then ψ is well defined for all $\beta \geq 0$, and,

$$\mathbb{E}(e^{\beta X_t}) = e^{\psi(\beta) t} \qquad \text{for all } t \geq 0 \text{ and } \beta \geq 0.$$

As introduced in Chapter I, ψ is infinitely differentiable, convex, $\psi(0) = 0$ and $\lim_{\beta\uparrow\infty}\psi(\beta) = \infty$. As before denote by,

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) = q\} \tag{3.4}$$

its right inverse.

First let's specify the class of gain functions we are considering in this chapter. Let

$$\delta_1 = \mathbb{1}_{\{X \text{ has unbounded variation}\}}, \tag{3.5}$$
$$\delta_2 = \mathbb{1}_{\{\int_{-1}^{0} y\Pi(dy)=\infty\}}. \tag{3.6}$$

Definition 3.3. *Let $D^2(I)$ be the set consisting functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the following conditions, where I is a subset of $\mathbb{R}$ with only finite number of points.*

(i) $f \in C^{\delta_1}(\mathbb{R}) \cap C^{1+\delta_1}(\mathbb{R}\setminus I)$. f converges as $x \to -\infty$, and f' converges to 0 as $x \to -\infty$.

(ii) If X has bounded variation, there exists $c \in (0, \Phi(q))$ and $d > 0$ such that $|f(x)| < e^{cx} + d$ and $|\max\{f'(x+), f'(x-)\}| < e^{cx} + d$ for all $x \in \mathbb{R}$.

(iii) If X has unbounded variation, there exists $c \in (0, \Phi(q))$ and $d > 0$ such that $|f(x)| < e^{cx} + d$, $|f'(x)| < e^{cx} + d$ and $|\max\{f''(x+), f''(x-)\}| < e^{cx} + d$ for all $x \in \mathbb{R}$, and $\lim_{x\to-\infty} f''(x) = 0$.

It can be verified easily that any function $g \in D^2(I)$ is in the class $\mathcal{D}_{1+\alpha_1,1} \cap \mathcal{I}_{1+\alpha_2}$ (see Section 1.2.2 in Chapter 1 for the definition). Thus, by Theorem 1 in [21], $\mathbb{L}_X g(x)$ is well defined for all $x \in \mathbb{R}$, where

$$\mathbb{L}_X g(x) = \mu g'(x) + \frac{1}{2}\sigma^2 g''(x) + \int_{\mathbb{R}} \left(g(x+y) - g(x) - yg'(x)\mathbb{1}_{\{|y|<1\}}\right)\Pi(dy), \tag{3.7}$$

for all $x \in \mathbb{R}$. Furthermore, as the occupancy time in I is 0 $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}$, then by Itô's formula (see [21]) we have that $\mathbb{P}$-a.s. for all $t \geq 0$

$$\begin{aligned} e^{-qt}g(X_t) &= g(X_0) + \int_0^t e^{-qs}\left(\mathbb{L}_X - q\right)g(X_s)ds + M_t^g \\ &= g(X_0) + \int_0^t e^{-qs}\left(\mathbb{L}_X - q\right)g(X_s)\mathbb{1}_{\{X_s\notin I\}}ds + M_t^g, \end{aligned} \tag{3.8}$$

where

$$M_t^g = \int_0^t \sigma e^{-qs} g'(X_{s-}) dB_s + \int_0^t \int_{-1}^0 y e^{-qs} g'(X_{s-}) \widetilde{N}(ds, dy) \tag{3.9}$$
$$+ \int_0^t \int_{-\infty}^0 e^{-qs} (g(X_{s-} + y) - g(X_{s-}) - y g'(X_{s-}) \mathbb{1}_{\{|y|<1\}}) \widetilde{N}(ds, dy).$$

for all $t \geq 0$, is a local martingale, and $\widetilde{N}$ is the compensated Poisson random measure.

Then we have the following Theorem on the local martingale M^g.

Theorem 3.4. *For all $g \in D^2(I)$, the local martingale M^g defined in (3.9) is a martingale.*

The proof for Theorem 3.4 can be found on page 69. The following Lemma is needed for proving Theorem 3.4.

Lemma 3.5. *For all $g \in D^2(I)$,*

(i) There exists $c_1 > 0$ such that for all $x \in \mathbb{R}$ and $y < 0$

$$|g(x+y) - g(x) - \delta_2 y g'(x) \mathbb{1}_{\{y \in (-1,0)\}}| \leq c_1 (1 \wedge |y|^{1+\delta_2})(e^{cx} + d),$$

where c and d are as defined in the definition of $D^2(I)$ (see Definition 3.3).

(ii) $\mathbb{L}_X g$ is continuous on $\mathbb{R} \setminus I$, and the left and right limits of $\mathbb{L}_X g(x)$ exist for all $x \in I$.

(iii) There exists $c_2 > 0$ such that

$$\max\{|\mathbb{L}_X g(x+) - q g(x+)|, |\mathbb{L}_X g(x-) - q g(x-)|\} \leq c_2 (e^{cx} + d)$$

for all $x \in \mathbb{R}$, where c and d are as defined in the definition of $D^2(I)$.

(iv) $\lim_{x \to -\infty} (\mathbb{L}_X g(x) - q g(x)) = -q g(-\infty)$.

The proof for Lemma 3.5 is on page 67. Furthermore, we have the following Lemma for the gain function g.

Lemma 3.6. *For all functions $g : \mathbb{R} \to \mathbb{R}$ such that there exist $c \in (0, \Phi(q))$ and $d > 0$ for which $|g(x)| < e^{cx} + d$ for all $x \in \mathbb{R}$, then*

$$\mathbb{E}_x \left(e^{-q\tau} g(X_\tau) \mathbb{1}_{\{\tau = \infty\}} \right) = 0$$

for all $\{\mathcal{F}_t\}$-stopping time τ.

The proof for Lemma 3.6 is on page 71. And we finish this section with the following Remark.

Remark 3.7. *For all $a < b$, then for any $t \in \mathbb{R}^+$, $t \wedge \tau_{a,b}$ is a bounded stopping time. By optional sampling theorem we have $\mathbb{E}\left(M^g_{t\wedge\tau_{a,b}}\right) = 0$ for all $t \geq 0$.*

3.3 Construction of the function h

The goal of this section is to derive the a closed form formula for an averaging function $A_g : \mathbb{R} \to \mathbb{R}$ and the function $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, such that h satisfies the conditions in Lemma 3.2 for some suitable choices of a.

3.3.1 Averaging Functions

An averaging function $A_g : \mathbb{R} \to \mathbb{R}$ with respect to the random variable $\underline{X}_{e_q}$ and the gain function g is a function satsifying

$$\mathbb{E}_x\left(A_g(\underline{X}_{e_q})\right) = g(x)$$

for all $x \in \mathbb{R}$, where $\underline{X}_{e_q}$ is the infimum of the spectrally negative Lévy process X up to an exponentially distributed time e_q with parameter q, which is independent of X, i.e.

$$\underline{X}_{e_q} = \inf\{X_s : 0 \leq s \leq e_q\}. \tag{3.10}$$

Similarly $\overline{X}_{e_q}$ is defined by

$$\overline{X}_{e_q} = \sup\{X_s : 0 \leq s \leq e_q\}. \tag{3.11}$$

It is well known from the literature on Lévy processes that X_{e_q} can be decomposed into the sum of the following two parts:

$$X_{e_q} = \underline{X}_{e_q} + (X_{e_q} - \underline{X}_{e_q}).$$

Observe that $\underline{X}_{e_q}$ and $X_{e_q} - \underline{X}_{e_q}$ are independent. and that $X_{e_q} - \underline{X}_{e_q}$ has the same distribution as $\overline{X}_{e_q}$ (see [34] for details). In the case of spectrally negative Lévy processes, $\overline{X}_{e_q}$ has an exponential distribution with parameter $\Phi(q)$. The probability distribution function of the random variable $\underline{X}_{e_q}$ can be obtained from the Laplace transform of τ_y^-. Indeed, for all $y \le 0$, we have that

$$\mathbb{P}\left(\underline{X}_{e_q} < y\right) = \mathbb{E}\left(\mathbb{1}_{\{e_q > \tau_y^-\}}\right) = \mathbb{E}\left(e^{-q\tau_y^-}\right) = Z^q(-y) - \frac{q}{\Phi(q)} W^q(-y)$$

where Z^q and W^q are the q scale functions for the spectrally negative Lévy process X. If X has unbounded variation, $\underline{X}_{e_q}$ has the pdf,

$$f_{\underline{X}_{e_q}}(y) = -qW^q(-y) + \frac{q}{\Phi(q)} \left(W^q\right)'(-y), \qquad y < 0.$$

In the case where X has bounded variation, both of the left and right derivatives exist for W^q on $(0, \infty)$. So we have for all $y < 0$:

$$\frac{d}{dy}\mathbb{P}\left(\underline{X}_{e_q} < y+\right) = -qW^q(-y) + \frac{q}{\Phi(q)} \left(W^q\right)'((-y)-), \tag{3.12}$$

$$\frac{d}{dy}\mathbb{P}\left(\underline{X}_{e_q} < y-\right) = -qW^q(-y) + \frac{q}{\Phi(q)} \left(W^q\right)'((-y)+). \tag{3.13}$$

However, it follows from Theorem 1.5 that, even in this case, the distribution of $\underline{X}_{e_q}$ has has maximally one atoms, which happens at $y = 0$.

$$\begin{aligned}
\mathbb{P}\left(\underline{X}_{e_q} = 0\right) &= \mathbb{P}\left(\underline{X}_{e_q} \le 0\right) - \mathbb{P}\left(\underline{X}_{e_q} < 0\right) \\
&= 1 - \left(Z^q(0) - \frac{q}{\Phi(q)} W^q(0)\right) \\
&= \frac{q}{\Phi(q)} W^q(0).
\end{aligned} \tag{3.14}$$

With the help of the distribution functions of $\underline{X}_{e_q}$ and $\overline{X}_{e_q}$, we can find a closed formula for the averaging function based on the infinitesimal generator $\mathbb{L}_X$.

Proposition 3.8. *For all $g \in D^2(I)$,*

$$\mathbb{E}_x\left(\left(\mathbb{L}_X g(X_{e_q}) - qg(X_{e_q})\right)\mathbb{1}_{\{X_{e_q} \notin I\}}\right) = -qg(x)$$

for all $x \in \mathbb{R}$.

The proof for Proposition 3.8 is on page 71.

Proposition 3.9. *For all $g \in D^2(I)$, the function $A_g : \mathbb{R} \to \mathbb{R}$ defined by,*

$$A_g(x) = \frac{\Phi(q)}{q} e^{\Phi(q)x} \int_x^\infty e^{-\Phi(q)y} \left(qg(y) - \mathbb{L}_X g(y)\right) \mathbb{1}_{\{y \notin I\}} dy \tag{3.15}$$

for all $x \in \mathbb{R}$, is an averaging function with respect to $\underline{X}_{e_q}$ and g.

The proof for Proposition 3.9 will be on page 72. We have the following remarks on this choice of averaging function A_g (3.15).

Remark 3.10. *Note that the averaging function A_g defined in* (3.15) *is continuous for all $x \in \mathbb{R}$. But the differentiability of A_g depends on the continuity of $\mathbb{L}_X g$. Moreover on the set $\mathbb{R} \setminus I$, it follows directly from equation* (3.15) *that*

$$\frac{q}{\Phi(q)} A_g'(x) - qA_g(x) = \mathbb{L}_X g(x) - qg(x). \tag{3.16}$$

And on the set I, the right and left derivative of A_g exist and

$$\begin{aligned} \frac{q}{\Phi(q)} A_g'(x-) - qA_g(x) &= \mathbb{L}_X g(x-) - qg(x), & (3.17)\\ \frac{q}{\Phi(q)} A_g'(x+) - qA_g(x) &= \mathbb{L}_X g(x+) - qg(x). & (3.18) \end{aligned}$$

Remark 3.11. *As I is a finite subset of $\mathbb{R}$, it follows from* (3.15) *that $A_g(x)$ is continuously differentiable for all $x < \inf\{I\}$ with convention that $\inf\{\emptyset\} = \infty$. Therefore by applying L'Hopital's rule together with part (iv) in Lemma 3.5 we obtain*

$$\begin{aligned} \lim_{x \to -\infty} A_g(x) &= \lim_{x \to -\infty} \frac{\Phi(q) \int_x^\infty e^{-\Phi(q)y}(qg(y) - \mathbb{L}_X g(y))\mathbb{1}_{\{y \notin I\}} dy}{qe^{-\Phi(q)x}} \\ &= \lim_{x \to -\infty} \frac{-\Phi(q)e^{-\Phi(q)x}(qg(x) - \mathbb{L}_X g(x))}{-\Phi(q)qe^{-\Phi(q)x}} \\ &= g(-\infty). \end{aligned}$$

Therefore, for all gain functions $g \in D^2(I)$, the averaging function $A_g(x)$ is bounded on the interval $(-\infty, x_0]$ for any $x_0 \in \mathbb{R}$.

Furthermore, from part (iv) in Lemma 3.5 and equations (3.17) *and* (3.18), *we see that both $A_g'(x+)$ and $A_g'(x-)$ are bounded on the interval $(-\infty, x_0]$ for any $x_0 \in \mathbb{R}$.*

Remark 3.12. *It is possible to derive an averaging function with respect to $\underline{X}_{e_q}$ and g from other methods, and the results from Remark 3.10 and Remark 3.11 still hold true for this choice of averaging function. For example, consider the function $g(x) = K - e^x$ where K is some strictly positive real value. By using the fluctuation theory for spectrally negative Lévy processes, see [78] for example, we have for all $x \in \mathbb{R}$*

$$A_g(x) = K - \frac{e^x}{\mathbb{E}(e^{\underline{X}_{e_q}})} = K - \frac{q - \psi(1)}{\Phi(q) - 1} \frac{\Phi(q)}{q} e^x, \tag{3.19}$$

and

$$\mathbb{L}_X g(x) - qg(x) = (q - \psi(1))e^x - qK.$$

It can be checked easily that all results from Remark 3.10 and Remark 3.11 hold true for this choice of averaging function A_g (3.19).

Finally, note that if we assume that $\psi(1) > 0$ *and* $q \in (0, \psi(1))$, *then clearly* $g \notin D^2(\emptyset)$, *and the formula in Proposition 3.9 is not well defined. However, the function* A_g (3.19) *is still well defined, and holds true as an averaging function.*

Clearly, from Remark 3.12, we see that the results from Remarks 3.10 and 3.11 hold true for a bigger class of averaging functions than equation (3.15). We finish this section with the definition of the following class of averaging functions.

Definition 3.13. *For all functions* $f : \mathbb{R} \to \mathbb{R}$ *such that the left and right limits of* $\mathbb{L}_X f(x)$ *exist for all* $x \in \mathbb{R}$, *let* A_f *be an averaging function w.r.t.* f *and* $\underline{X}_{e_q}$. *Then we say that this averaging function* A_f *is of the type (L) if the following conditions hold true.*

1. A_f *is continuous, and the left and right limits of* A'_f *exist and are bounded on* $(-\infty, x_0]$ *for all* $x_0 \in \mathbb{R}$.

2.

$$\begin{aligned}
\lim_{x \to -\infty} A_f(x) &= \lim_{x \to -\infty} f(x) \\
\frac{q}{\Phi(q)} A'_f(x-) - qA_f(x) &= \mathbb{L}_X f(x-) - qf(x) \qquad \textit{for all } x \in \mathbb{R}, \\
\frac{q}{\Phi(q)} A'_f(x+) - qA_f(x) &= \mathbb{L}_X f(x+) - qf(x) \qquad \textit{for all } x \in \mathbb{R}.
\end{aligned}$$

3.3.2 The function $h(x, a)$

In this section, instead of the class of $D^2(I)$, we are working with the following class of functions.

Definition 3.14. $D_h^2(I)$ *is the set consisting functions* $f \in C^{\delta_1}(\mathbb{R}) \cap C^{\delta_1+1}(\mathbb{R} \setminus I)$ *where* I *is a subset of* $\mathbb{R}$ *with only finite number points. Furthermore,* $f(x)$ *converges to a constant and* $f'(x)$ *converges to* 0 *as* x *goes to* $-\infty$. *The left and right limits of the* $(1 + \delta_1)^{th}$ *derivative of* f *exist for all* $x \in I$. *If* X *has unbounded variation,* $\lim_{x \to -\infty} f''(x) = 0$. *There exists a type (L) averaging function* A_f *with respect to* f *and* $\underline{X}_{e_q}$.

Remark 3.15. *If the function $g \in D^2(I)$, then by Proposition 3.9 an averaging function A_g with respect to $\underline{X}_{e_q}$ and g can be found, and is of the type (L). Therefore, $D^2(I) \subseteq D^2_h(I)$.*

For all $g \in D^2_h(I)$, let A_g be a type (L) averaging function. We define $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by setting

$$h(x,a) \quad = \quad \mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^- < \infty\}}\right) + \frac{q}{\Phi(q)} A_g(a) W^q(x-a), \tag{3.20}$$

for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$. Note that by Theorem 3.1 in [78], we can rewrite the first term in (3.20) using averaging functions. Thus,

$$h(x,a) = \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q} < a\}}\right) + \frac{q}{\Phi(q)} A_g(a) W^q(x-a), \tag{3.21}$$

for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$. It is clearly that from the representation above that the function h is uniquely determined by the choice of averaging function A_g. For this reason, we say that the function h is constructed from this averaging function A_g.

Then we have the following Theorem for the exist time for the gain function g.

Theorem 3.16. *Suppose that $g \in D^2_h(I)$. Let h be as defined in* (3.21) *for a type (L) averaging function A_g (see Definition 3.13). And suppose further that there exist $a \in \mathbb{R}$ and $b \in \mathbb{R} \cup \{\infty\}$ such that one of the following holds true,*

(i) $b \in \mathbb{R}$, $b > a$ and $h(b,a) = g(b)$,

(ii) $A_g(a) = 0$.

Then for this particular pair (a,b),

$$\mathbb{E}_x\left(e^{-q\tau_{a,b}} g(X_{\tau_{a,b}})\mathbb{1}_{\{\tau_{a,b} < \infty\}}\right) = h(x,a), \qquad \text{for all } x \in (-\infty, b] \cap \mathbb{R}. \tag{3.22}$$

where b is understood to be ∞ in the second case, and $\tau_{a,b}$ is understood to be τ_a^-.

The proof for Theorem 3.16 can be found on page 81. The existence of such pair (a,b) will be established in the next section. The following Lemma and Proposition are needed for proving Theorem 3.16.

Lemma 3.17. *Suppose that $g \in D^2_h(I)$. Let h be as defined in* (3.21) *for a type (L) averaging function A_g. For each fixed $a \in \mathbb{R}$, let $I_a = \{x \in I : x < a\}$. Then $h(\cdot, a)$*

is in the class $C^{1+\delta_1}(\mathbb{R} \setminus (I_a \cup \{a\})) \cap C^{\delta_1}(\mathbb{R})$, *and for all* $x > a$

$$\begin{aligned}\frac{\partial}{\partial x} h(x,a) &= \int_{-\infty}^{a-x} A_g'(y+x)\mathbb{1}_{\{x+y \notin I_a\}}\mathbb{P}(\underline{X}_{e_q} \in dy) + A_g(a)qW^q(x-a) \\ \frac{\partial h}{\partial x}(a+,a) &= g'(a+) - W^q(0)(\mathbb{L}_X g(a+) - qg(a)).\end{aligned}$$

The proof for Lemma 3.17 is on page 73.

Proposition 3.18. *Suppose that* $g \in D_h^2(I)$. *Let* h *be as defined in* (3.21) *for a type (L) averaging function* A_g. *Then for all* $a \in \mathbb{R}$, *the stochastic process* $\{e^{-qt\wedge\tau_a^-} h(X_{t\wedge\tau_a^-}, a), t \geq 0\}$ *is a* $\mathbb{P}_x$ *martingale. And* $\mathbb{L}_X h(x,a) - qh(x,a) = 0$ *for all* $x \in (a, \infty)$.

The proof for Proposition 3.18 can be found on page 79.

Below we collect some further properties of $h(x,a)$, which are needed for finding the pair (a,b) in Theorem 3.16.

Lemma 3.19. *Suppose that* $g \in D_h^2(I)$. *Let* h *be as defined in* (3.21) *for a type (L) averaging function* A_g. *For each fixed* $x \in \mathbb{R}$, *let* $I_x = \{y \in I : y < x\}$. *Then the function* $h(x,\cdot)$ *is in the class* $C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus (I_x \cup \{x\}))$, *and for all* $a \in \mathbb{R} \setminus (I_x \cup \{x\})$

$$\frac{\partial}{\partial a} h(x,a) = W^q(x-a)(\mathbb{L}_X g(a) - qg(a)). \tag{3.23}$$

The proof for Lemma 3.19 is on page 81.

Lemma 3.20. *Suppose that* $g \in D_h^2(I)$. *Let* h *be as defined in* (3.21) *for a type (L) averaging function* A_g. *Then it holds true that,*

(i) If $a \in \{x \in \mathbb{R} \setminus I : \mathbb{L}_X g(x) - qg(x) < 0\}$, *then there exists* $\epsilon > 0$ *such that* $h(x,a) > g(x)$ *for all* $x \in (a, a+\epsilon)$.

(ii) If $a \in \{x \in \mathbb{R} \setminus I : \mathbb{L}_X g(x) - qg(x) > 0\}$, *then there exists* $\epsilon > 0$ *such that* $h(x,a) < g(x)$ *for all* $x \in (a, a+\epsilon)$.

The proof for Lemma 3.20 is on page 83.

Proposition 3.21. *Suppose that* $g \in D_h^2(I)$. *Let* h *be as defined in* (3.21) *for a type (L) averaging function* A_g. *If there exists* $a \in \mathbb{R}$ *such that* $A_g(a) \geq 0$, *and both the left and right limits of* $\mathbb{L}_X g(x) - qg(x)$ *are non positive on* $(-\infty, a)$. *Then* $\{e^{-qt} h(X_t, a), t \geq 0\}$ *is a supermartingale.*

The proof for Proposition 3.21 can be found on page 84.

Proposition 3.22. *Suppose that $g \in D^2_h(I)$, and let A_g be a type (L) averaging function with respect to g and $\underline{X}_{e_q}$. Then A_g is the unique.*

The proof for Proposition 3.22 can be found on page 85.

We finish this section with the following Remark.

Remark 3.23. *Suppose that $g \in D^2(I)$, then g is bounded on $(-\infty, a]$ for all fixed $a \in \mathbb{R}$. Therefore, for all fixed $a \in \mathbb{R}$,*

$$\lim_{x\to\infty} \mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^-<\infty\}}\right) = 0.$$

As W^q behave asymptotically as $e^{\Phi(q)x}/\psi'(\Phi(q))$ when x is large, we can conclude that $h(\cdot, a)$ behaves asymptotically similar to $qA_g(a)e^{\Phi(q)x}/(\Phi(q)\psi'(\Phi(q)))$ when x is large. Therefore, it follows from the definition of $D^2(I)$ that

1. *if $A_g(a) > 0$, $h(x,a) > g(x)$ for all x large enough,*
2. *if $A_g(a) < 0$, $h(x,a) < g(x)$ for all x large enough,*
3. *if $A_g(a) = 0$, $h(\cdot, a)$ converges to 0 as $x \to \infty$,*
4. *if $A_g(a) \geq 0$, $h(\cdot, a)$ is bounded below in $\mathbb{R}$.*

3.4 Left semi-solution for the optimal stopping problem

In this section we return to the optimal stopping problem (3.2), and study the left semi-solutions by using the function h. In addition, we also assume that the gain function g satisfies the following assumption.

Assumption 3.24. *The function $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions: $\lim_{x\to-\infty} f(x) > 0$, and $f \in C^{\delta_1}(\mathbb{R}) \cap C^{1+\delta_1}(\mathbb{R}\setminus I)$ where I is a subset of $\mathbb{R}$ with only finite elements, and the left and right limits of the $(1+\delta_1)^{th}$ derivative of f exist for all $x \in I$. The constant a_L is well defined, where*

$$\begin{aligned} a_L &= \sup\{a \in \mathbb{R} : \textit{the left and right limits of } \mathbb{L}_X f(x) - qf(x) \textit{ exist and} \\ &\qquad \textit{are non positive for all } x \leq a\}. \end{aligned}$$

Furthermore, the type (L) averaging function A_f with respect to $\underline{X}_{e_q}$ and f exists. Finally, there exists $x > a_L$ such that $h(x, a_L) < f(x)$, where the function h is as defined in equation (3.21) for this choice of averaging function A_f.

Remark 3.25. *A sufficient condition for the existence of $x > a_L$ such that*

$$h(x, a_L) < f(x),$$

is that there exists $\epsilon > 0$ such that $\mathbb{L}_X f(x) - qf(x) \geq 0$ for all $x \in (a_L, a_L + \epsilon)$.

If the latter statement holds true, then it follows from the definition of a_L that there exists $a_1 \in (a_L, a_L+\epsilon)$ such that $\mathbb{L}_X f(a_1) - qf(a_1) > 0$. Thus, by Lemma 3.20, there exist $x_0 > a_1$ such that $h(x_0, a_1) < f(x_0)$. So, by Lemma 3.19, we have,

$$\begin{aligned} h(x_0, a_1) &= h(x_0, a_L) + \int_{a_L}^{a_1} \frac{\partial}{\partial a} h(x_0, a) da \\ &= h(x_0, a_L) + \int_{a_L}^{a_1} W^q(x_0 - a)\big(\mathbb{L}_X f(a) - qf(a)\big) da \end{aligned}$$

As $\mathbb{L}_X f(x) - qf(x) \geq 0$ on (a_L, a_1), we have $h(x_0, a_L) \leq h(x_0, a_1) < f(x_0)$ as required.

Let I_g be the empty set. From now on, we will assume that $g \in D^2(I_g)$ and satisfies Assumption 3.24 with the set I_g. That is, g is continuously twice differentiable. We denote by $a_{L,1}$ the constant a_L in Assumption 3.24 for the function g, and by A_1 this choice of averaging function with respect to $\underline{X}_{e_q}$ and g as defined in equation (3.15), then A_1 is the unique averaging function of type (L). Finally, we write h as the function defined in (3.21) for this averaging function A_1. Then we have the following theorem for the left semi-solution of the optimal stopping problem (3.2).

Theorem 3.26. *Consider the optimal stopping problem* (3.2) *with respect to a gain function $g \in D^2(I_g)$ which satisfies the Assumption 3.24 with the set $I_g = \emptyset$. Define $a_1^\star$ and $b_1^\star$ to be*

$$a_1^\star = \sup\{a < a_{L,1} : h(x,a) \geq g(x) \text{ for all } x \in \mathbb{R}\}, \tag{3.24}$$

$$b_1^\star = \begin{cases} \sup \mathcal{N}_1 & \text{if } A_1(a_1^\star) > 0 \text{ and } \mathcal{N}_1 \neq \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{3.25}$$

where $\mathcal{N}_1 = \{b > a_1^\star : \ h(b, a_1^\star) = g(b)\}$. Then $a_1^\star \in \mathbb{R}$, and

(i) If $b_1^\star < \infty$, then the pair $(V_1, \tau_1^\star)$ is a closed left semi-solution up to the point

$b_1^\star$, *where*

$$
\begin{aligned}
V_1(x) &= \begin{cases} h(x, a_1^\star) & \textit{if } x \in (-\infty, b_1^\star] \cap \mathbb{R} \\ g(x) & \textit{otherwise}, \end{cases} && (3.26) \\
\tau_1^\star &= \inf\{t \geq 0 : X_t \notin [a_1^\star, b_1^\star] \cap \mathbb{R}\} && (3.27)
\end{aligned}
$$

(ii) The pair $(V_1, \tau_1^\star)$ *is a solution for the optimal stopping problem* (3.2) *if one of the following statements hold true,*

(a) $b_1^\star = \infty$,

(b) $\mathbb{L}_X V_1(x) - qV_1(x) \leq 0$ *for all* $x \in (b_1^\star, \infty)$.

The proof for Theorem 3.26 can be found on page 94. Using the closed left semi value function $V_1(x)$, we can work out $\mathcal{C}_{b_1^\star}$, the closed left semi continuation region up to $b_1^\star$. Indeed,

$$
\begin{aligned}
\mathcal{C}_{b_1^\star} &= \{x \in \mathcal{C} : x \leq b_1^\star\} \\
&= \{x \in \mathbb{R} : V_1(x) > g(x)\},
\end{aligned}
$$

where $\mathcal{C} = \{x \in \mathbb{R} : V(x) > g(x)\}$ is the global continuation region. Let $\partial\mathcal{C}_{b_1^\star}$ denote the boundaries of $\mathcal{C}_{b_1^\star}$. Note that as a result of Lemma 3.30 below, $a_1^\star$ is always in the boundary set $\partial\mathcal{C}_{b_1^\star}$, while $b_1^\star$ may not be. Furthermore, $\partial\mathcal{C}_{b_1^\star}$ may contain more points than just $a_1^\star$ and $b_1^\star$. Then we have the following theorem for the pasting conditions for all $x \in \partial\mathcal{C}_{b_1^\star}$.

Theorem 3.27. *Suppose that all conditions in Theorem 3.26 hold true. Then* $V_1(x) = g(x)$ *for all* $x \in \partial\mathcal{C}_{b_1^\star}$, *and* $V_1'(x) = g'(x)$ *for all* $x \in \partial\mathcal{C}_{b_1^\star} \setminus \{a_1^\star\}$. *Furthermore,* $V_1'(a_1^\star+) = g'(a_1^\star)$ *if the underlying spectrally negative Lévy process* X *has unbounded variation.*

The pasting condition at $x = a_1^\star$ can be seen from Lemma 3.17, and the pasting conditions for $x \in \partial\mathcal{C}_{b_1^\star} \setminus \{a_1^\star\}$ can be proved by using a similar argument as in part (vi) in Lemma 3.30 below. Hence the proof for Theorem 3.27 is omitted.

Remark 3.28. *The purpose of requiring Assumption 3.24 holding true for* g *is twofold. Firstly, the requirement of* $a_{L,1}$ *being well defined is to eliminate the case when* $\{e^{-qt}g(X_t), t \geq 0\}$ *is a supermartingale. Otherwise, the optimal stopping problem is solved for* $V(x) = g(x)$ *for all* $x \in \mathbb{R}$ *without any calculation. Secondly, the requirement for the existence of* $x > a_{L,1}$ *with* $h(x, a_{L,1}) < g(x)$ *guarantees that the set* $\mathcal{N}_1$ *is non empty in the case when* $A_1(a_1^\star) > 0$, *or equivalently,* $b_1^\star = \infty$ *if* *and only if $A_1(a_1^\star) = 0$. This claim is proved in Lemma 3.30. If this requirement is dropped, it is possible to have $\mathcal{N}_1 = \emptyset$ and $A_1(a_1^\star) > 0$. In this case by definition, $b_1^\star$ is equal to infinity. Therefore, the conditions in Theorem 3.16 break down, and equation* (3.22) *does not hold true for $x < b_1^\star$ anymore.*

Remark 3.29. *Note that if we drop the condition* (3.1)*, all results in Section 3.3.2 would still hold true apart from the twice differentiability of $h(\cdot, a)$ for all $x > a$. If we replace the condition* (3.1) *with an extra condition on g, i.e. $g \in C^3(\mathbb{R})$, then by following a similar argument as in the proof for the differentiability of $h(\cdot, a)$. we have $h(\cdot, a) \in C^{\delta_1}(\mathbb{R}) \cap C^{1+\delta_1}(\mathbb{R} \setminus \{a\})$ as well. Furthermore, by a similar argument, Theorem 3.26 and Theorem 3.27 would still hold true.*

3.4.1 Preliminary results for V_1

In this section we provide some Lemmas which are needed for the proof of Theorem 3.26.

It is clear that $b_1^\star$ is well defined as long as a finite $a_1^\star$ exists. The following Lemma establishes the latter statement.

Lemma 3.30. *Suppose that all conditions in Theorem 3.26 hold true. Then*

(i) $-\infty < a_1^\star < a_{L,1}$.

(ii) $h(x, a_1^\star) \geq g(x)$ *for all* $x \in \mathbb{R}$, *and for all* $\epsilon > 0$ *there exists* $x \in (a_1^\star, a_1^\star + \epsilon)$ *such that* $h(x, a_1^\star) > g(x)$.

(iii) $A_1(a_1^\star) \geq 0$.

(iv) For all $\epsilon \in (0, a_{L,1} - a_1^\star)$, *there exists* $x \in (a_1^\star, a_1^\star + \epsilon)$ *such that*

$$\mathbb{L}_X g(x) - qg(x) < 0.$$

(v) $b_1^\star = \infty$ *if and only if* $A_1(a_1^\star) = 0$.

(vi) If $b_1^\star < \infty$, *then* $g(b_1^\star) = V_1(b_1^\star)$ *and* $g'(b_1^\star) = V_1'(b_1^\star -)$.

The proof for Lemma 3.30 is on page 86.

Lemma 3.31. *Suppose that all conditions in Theorem 3.26 hold true. Then* $V_1 \in D^2(I_{V_1})$, *where*

$$I_{V_1} = \Big\{ x \in \{a_1^\star, b_1^\star\} \cap \mathbb{R} : \textit{the left and right } (1+\delta_1)^{th} \textit{ derivatives of } V_1 \textit{ do not agree at } x. \Big\}$$

Furthermore, if $b_1^\star \neq \infty$, *then* $\mathbb{L}_X V_1(b_1^\star+) - qV_1(b_1^\star+)$ *is well defined and non positive.*

The proof for Lemma 3.31 is on page 91. We can also obtain the following Corollary for the function $h(\cdot, a_1^\star)$.

Corollary 3.32. *Suppose that all conditions in Theorem 3.26 hold true. Then* $h(x, a_1^\star) > 0$ *for all* $x \in \mathbb{R}$, *and there exists* $x_0 \in (a_1^\star, b_1^\star)$ *such that* $\mathbb{L}_X g(x_0) - qg(x_0) > 0$.

The proof for Corollary 3.32 is on page 92

3.5 Value function for $x > b_1^\star$

In this section we show that a similar construction can be applied to the closed left semi value function V_1 to calculate the value function for $x \in (b_1^\star, \infty)$.

In order to do this, we require that $V_1 \in D^2(I_{V_1})$ and satisfies the Assumption 3.24. It has been shown in Lemma 3.31 that $V_1 \in D^2(I_{V_1})$. However, it is not clear that whether V_1 would satisfy Assumption 3.24 at all. Therefore, from now on we will assume that V_1 satisfies Assumption 3.24 with I_{V_1}. And we denote by $a_{L,2}$ the constant a_L in Assumption 3.24 for the function V_1. We also point out here that if V_1 satisfies Assumption 3.24, then $b_1^\star < \infty$. Otherwise, $a_{L,2}$ will not be well defined as required by Assumption 3.24.

Let $A_2 : \mathbb{R} \to \mathbb{R}$ be the function as defined in equation (3.15) for V_1, then A_2 is the unique type (L) averaging function w.r.t. V_1 and $\underline{X}_{e_q}$. And we write $h_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as the function (3.21) for this particular choice of averaging function A_2. Therefore, all results from Lemma 3.17, Lemma 3.19, Lemma 3.20, Proposition 3.18, Proposition 3.21 and Proposition 3.22 hold for $h_2(x, a)$. Then we have the following Theorem.

Theorem 3.33. *Consider the optimal stopping problem* (3.2) *where the gain function* $g \in D^2(I_g)$ *and satisfies Assumption 3.24 with the set* $I_g = \emptyset$, *and suppose that* V_1 *satisfies Assumption 3.24 with* I_{V_1}. *Define* $a_2^\star$ *and* $b_2^\star$ *to be*

$$
\begin{aligned}
a_2^\star &= \sup\{a < a_{L,2} : h_2(x, a) \geq V_1(x) \text{ for all } x \in \mathbb{R}\}, \\
b_2^\star &= \begin{cases} \sup \mathcal{N}_2 & \text{if } A_2(a_2^\star) > 0 \text{ and } \mathcal{N}_2 \neq \emptyset \\ \infty & \text{otherwise,} \end{cases}
\end{aligned}
$$

$\mathcal{N}_2 = \{b > a_2^\star : h_2(b, a_2^\star) = V_1(b)\}$. *Then* $a_2^\star \in \mathbb{R}$, *and*

(i) *If $b_2^\star < \infty$, then the pair $(V_2, \tau_2^\star)$ is a closed left semi-solution up to the point $b_2^\star$, where*

$$\begin{aligned} V_2(x) &= \begin{cases} h_2(x, a_2^\star) & \text{if } x \in (-\infty, b_2^\star] \cap \mathbb{R} \\ V_1(x) & \text{otherwise,} \end{cases} && (3.28) \\ \tau_2^\star &= \inf\{t \geq 0 : X_t \notin \{[a_1^\star, b_1^\star] \cup [a_2^\star, b_2^\star]\} \cap \mathbb{R}\}. && (3.29) \end{aligned}$$

(ii) *The pair $(V_2, \tau_2^\star)$ is a solution for the optimal stopping problem if one of the following statements hold true,*

(a) $b_2^\star = \infty$,

(b) $\mathbb{L}_X V_2(x) - qV_2(x) \leq 0$ *for all* $x \in (b_2^\star, \infty)$.

By using Theorem 3.38 below, Theorem 3.33 can be proved by a similar argument as in Theorem 3.26. Hence, it is omitted. Note that because of the absence of the positive jumps, there is no contradiction between Theorem 3.26 and Theorem 3.33 for $x \leq b_1^\star$.

Again by using the closed left semi value function $V_2(x)$, we can work out $C_{b_2^\star}$, the closed left semi continuation region up to $b_2^\star$. Thus,

$$C_{b_2^\star} = \{x \in \mathbb{R} : V_2(x) > g(x)\}.$$

Let $\partial C_{b_2^\star}$ denote the boundary of $C_{b_2^\star}$. Again we observe (by Lemma 3.36 below) that $b_2^\star$ may not be in the set $\partial C_{b_2^\star}$, and $\partial C_{b_2^\star}$ may contain more points than just $\{a_1^\star, b_1^\star, a_2^\star, b_2^\star\}$. Then we have the following theorem for the pasting conditions for all $x \in \partial C_{b_2^\star}$.

Theorem 3.34. *Suppose that all conditions in Theorem 3.33 hold true. Then $V_2(x) = g(x)$ for all $x \in \partial C_{b_2^\star}$, and $V_2'(x) = g'(x)$ for all $x \in \partial C_{b_2^\star} \setminus \{a_1^\star, a_2^\star\}$. Furthermore, $V_2'(x+) = g'(x)$ for all $x \in \{a_1^\star, a_2^\star\}$ if the underlying spectrally negative Lévy process X has unbounded variation.*

The proof for Theorem 3.34 can be done using a similar argument as in Theorem 3.27. Hence, it is omitted.

Suppose that $b_2^\star < \infty$ and V_2 is only a closed left semi value function up to $b_2^\star$. Then by a similar argument as Lemma 3.31, we can show that $V_2 \in D^2(I_{V_2})$ for some finite set I_{V_2}. Thus, under the condition that V_2 satisfies the Assumption 3.24, we can repeat the above procedure to study the value function for $x > b_2^\star$. In fact, we can repeat this procedure for the $(i+1)^{th}$ time, $i \in \mathbb{N}$, as long as V_i satisfies the

Assumption 3.24 with the set I_{V_i}, where

$$I_{V_i} = \Big\{x \in \{a_j^\star, b_j^\star : 1 \le j \le i\} \cap \mathbb{R} : \text{the left and right } (1+\delta_1)^{th} \text{ derivatives of } V_i \text{ do not agree at } x.\Big\}$$

Let $n^\star$ be the first time the above condition breaks down. That is

$$n^\star = \sup\{n \in \mathbb{N} : V_i \text{ satisfies Assumption 3.24 for all } i \le n.\}$$

Then we obtain three sequences, $a_i^\star$, $b_i^\star$ and V_i. We point out here $n^\star \in \mathbb{N} \cup \{\infty\}$. That is, it is possible to have countable number of elements for $a_i^\star$, $b_i^\star$ and V_i. Furthermore, in the case $n^\star = \infty$, the strictly increasing sequences $a_i^\star$ and $b_i^\star$ may or may not converge. Examples where $a_i^\star$ and $b_i^\star$ diverge and converge are illustrated in Chapter 5. To sum up, we have the following theorem.

Theorem 3.35. *Consider the optimal stopping problem* (3.2) *with respect to the gain function $g \in D^2(I_g)$ which satisfies the Assumption 3.24 with the set $I_g = \emptyset$. Then,*

(i) If $n^\star < \infty$, then the pair $(V_{n^\star}, \tau_{n^\star}^\star)$ is a closed left semi-solution up to the point $b_{n^\star}^\star$, where $b_{n^\star}^\star = \lim_{i\to\infty} b_{i\wedge n^\star}^\star$, and $V_{n^\star}(x) = \lim_{i\to\infty} V_{i\wedge n^\star}(x)$ for all $x \in \mathbb{R}$, and

$$\tau_{n^\star}^\star = \inf\{t \ge 0 : X_t \notin \Big\{\bigcup_{i=1}^{n^\star}[a_i^\star, b_i^\star]\Big\} \cap \mathbb{R}\}.$$

(ii) If $n^\star = \infty$ and $b_{n^\star}^\star < \infty$, then the pair $(V_{n^\star}, \tau_{n^\star}^\star)$ is an open left semi-solution pair up to the point $b_{n^\star}^\star$.

(iii) The pair $(V_{n^\star}, \tau_{n^\star}^\star)$ is a solution to the optimal stopping problem (3.2) *if one of the following statements hold true.*

(a) $b_{n^\star}^\star < \infty$, and $\mathbb{L}_X V_{n^\star}(x) - qV_{n^\star}(x) \le 0$ for all $x > b_{n^\star}^\star$.

(b) $b_{n^\star}^\star = \infty$.

The proof for Theorem 3.35 can be found on page 99. We remark here that, in the case when $\{a_i^\star, 1 \le i \le n^\star, i \in \mathbb{N}\}$ and $\{b_i^\star, 1 \le i \le n^\star, i \in \mathbb{N}\}$ converge, it is not clear what happens after the limit point.

3.5.1 Preliminary results for V_2

The following results are needed for the proof of Theorem 3.33.

Lemma 3.36. *Suppose that all conditions in Theorem 3.33 hold true. Then*

(i) $h(x, a_1^\star) = h_2(x, b_1^\star)$ *for all* $x \in \mathbb{R}$.

(ii) $b_1^\star \leq a_2^\star < a_{L,2}$.

(iii) $h_2(x, a_2^\star) \geq V_1(x)$ *for all* $x \in \mathbb{R}$, *and for all* $\epsilon > 0$ *there exists* $x \in (a_2^\star, a_2^\star + \epsilon)$ *such that* $h_2(x, a_2^\star) > V_1(x)$.

(iv) $A_2(a_2^\star) \geq 0$.

(v) For all $\epsilon \in (0, a_{L,2} - a_2^\star)$, *there exists* $x \in (a_2^\star, a_2^\star + \epsilon)$ *such that* $\mathbb{L}_X V_1(x) - qV_1(x) < 0$.

(vi) $b_2^\star = \infty$ *if and only if* $A_2(a_2^\star) = 0$.

(vii) If $b_2^\star < \infty$, $V_2(b_2^\star) = g(b_2^\star)$ *and* $V_2'(b_2^\star-) = g'(b_2^\star)$.

(viii) $b_2^\star > a_2^\star > b_1^\star$.

The proof for Lemma 3.36 is on page 95

Lemma 3.37. *Suppose that all conditions in Theorem 3.33 hold true. Then* $V_2 \in D^2(I_{V_2})$, *where*

$$I_{V_2} = \Big\{x \in \{a_1^\star, b_1^\star, a_2^\star, b_2^\star\} \cap \mathbb{R} : \text{the left and right } (1+\delta_1)^{th} \text{ derivatives of } V_2 \text{ do not agree at } x.\Big\}$$

Furthermore, if $b_2^\star < \infty$, *then* $\mathbb{L}_X V_2(b_2^\star+) - qV_2(b_2^\star+)$ *is well defined and non positive.*

The proof for Lemma 3.37 can be done by a similar argument as in proof for Lemma 3.31.

Theorem 3.38.

(i) Let $\{a_i, i \in \mathbb{N}, i \leq n\}$ *and* $\{b_i, i \in \mathbb{N}, i \leq n\}$, $n \in \mathbb{N}$, *be two sequences of finite elements, such that* $-\infty \leq a_1 < b_1 < a_2 < b_2 < ... < b_n < \infty$. *Suppose there exist two continuous functions* $f : \mathbb{R} \to \mathbb{R}$ *and* $f_1 : \mathbb{R} \to \mathbb{R}$ *with the following properties,*

(a) both $\lim_{x \to -\infty} f(x)$ *and* $\lim_{x \to -\infty} f_1(x)$ *exist in* $\mathbb{R}$,

(b) $\{e^{-qt\wedge\tau_{a_i,b_i}} f_1(X_{t\wedge\tau_{a_i,b_i}}), t \geq 0\}$ *is a* $\mathbb{P}_x$ *martingale for all* $1 \leq i \leq n$, *where* τ_{a_1,b_1} *is understood to be* $\tau_{b_1}^+$ *in the case* $a_1 = -\infty$,

(c) $f_1(x) = f(x)$ *for all* $x \in B_n^c$ *where* $B_n = \left\{\bigcup_{i=1}^n [a_i, b_i]\right\} \cap \mathbb{R}$.

Then

$$\mathbb{E}_x\left(e^{-q\tau_{B_n}} f(X_{\tau_{B_n}})\right) = f_1(x), \tag{3.30}$$

for all $x \in \mathbb{R}$, where

$$\tau_{B_n} \quad = \quad \inf\{t \geq 0 : X_t \notin B_n\}. \tag{3.31}$$

(ii) *Suppose further that there exists $c \in (0, \Phi(q))$ and $d > 0$ such that $|f(x)| < e^{cx} + d$ for all $x \in \mathbb{R}$, and $\lim_{x\to\infty} f_1(x)$ exists in $\mathbb{R}$, and $B_n \subset \mathbb{R}$. Then equation* (3.30) *holds true for $b_n = \infty$ for all $x \in \mathbb{R}$.*

By Theorem 3.38, it follows from the Proposition 3.18 and definition of V_2 that

$$\mathbb{E}_x\left(e^{-q\tau_2^\star} g(X_{\tau_2^\star})\right) = V_2(x)$$

for all $x \in \mathbb{R}$, and furthermore,

$$\mathbb{E}_x\left(e^{-q\tau_2^\star} g(X_{\tau_2^\star})\right) = h_2(x, a_2^\star)$$

for all $x \in (-\infty, b_2^\star] \cap \mathbb{R}$. Thus, the condition in Lemma 3.2 is satisfied.

3.6 Consistency with the existing literature

In [78] averaging functions are used to evaluate candidate value function V_τ, where τ is first passage time. In his paper Surya showed that if we can find an averaging function A_g for $\underline{X}_{e_q}$ and gain function g, and furthermore, this averaging function A_g is continuous and there exists $\hat{a}$ such that $A_g(\hat{a}) = 0$. A_g is non-increasing for $x < \hat{a}$ and $A_g(x) \leq 0$ for $x > \hat{a}$, then the value function V takes the following form.

$$V(x) = \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q} < a_\infty^-\}}\right) \text{ for all } x \in \mathbb{R}, \tag{3.32}$$

where a_∞^- is the smallest root of the equation

$$A_g(x) = 0.$$

And the corresponding optimal stopping time is $\tau^\star = \inf\{t \geq 0 : X_t < a_\infty^-\}$.

Here we show that we can recover Surya's result by the approach proposed in this chapter.

Lemma 3.39. *Suppose that $g \in D^2(I_g)$ and satisfies the Assumption 3.24 with $I_g = \emptyset$. Let $A_g : \mathbb{R} \to \mathbb{R}$ be as defined in* (3.15). *If there exists $\hat{a}$ such that $A_g(\hat{a}) = 0$, A_g is non-increasing for $x < \hat{a}$ and $A_g(x) \leq 0$ for $x > \hat{a}$, then let a_∞^- be the smallest root of the equation $A_g(x) = 0$, and a_∞^+ be*

$$a_\infty^+ = \sup\{a \geq a_\infty^- : A_g(x) = 0 \textit{ for all } x \in (a_\infty^-, a)\}.$$

Then $a_1^\star = a_\infty^+$, and $h(x, a_1^\star) = \mathbb{E}_x\left(e^{-q\tau^-_{a_\infty^-}} g(X_{\tau^-_{a_\infty^-}})\right) = \mathbb{E}_x\left(e^{-q\tau^-_{a_\infty^+}} g(X_{\tau^-_{a_\infty^+}})\right)$ for all $x \in \mathbb{R}$.

As $A_g(a_1^\star) = A_g(a_\infty^+) = 0$, by definition of $b_1^\star$ we have $b_1^\star = \infty$. Then by Theorem 3.26 we have $h(x, a_\infty^+) = V(x)$ for all $x \in \mathbb{R}$.

Proof for Lemma 3.39. First we show that $a_1^\star = a_\infty^+$. As $A_g(x) \leq 0$ for all $x \geq a_\infty^+$, we have for all $x \in \mathbb{R}$,

$$\begin{aligned}
h(x, a_\infty^+) &= \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q} < a_\infty^+\}}\right) \\
&\geq \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q} < a_\infty^+\}}\right) + \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q} \in [a_\infty^+, \infty)\}}\right) \\
&= \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\right) \\
&= g(x).
\end{aligned}$$

Furthermore, for all $x \leq a_\infty^+$,

$$\mathbb{L}_X g(x) - qg(x) = \frac{q}{\Phi(q)} A_g'(x) - qA_g(x) \leq 0,$$

where the last inequality is due to A_g is non increasing and non negative on $(-\infty, a_\infty^+]$. Therefore, $a_\infty^+ \leq a_L$. Thus, by definition of $a_1^\star$, we derive that $a_1^\star \geq a_\infty^+$.

Next we show that $a_1^\star \leq a_\infty^+$. This is done by contradiction. Suppose $a_1^\star > a_\infty^+$. Note that by definition of a_∞^+, $A_g(a) \leq 0$ for all $a > a_\infty^+$. Then we either have $A_g(a_1^\star) < 0$ or $A_g(a_1^\star) = 0$. If $A_g(a_1^\star) < 0$, then by Remark 3.23, $h(x, a_1^\star) < g(x)$ for all x large enough, which contradicts the definition of $a_1^\star$. So we can't have $A_g(a_1^\star) < 0$.

If $A_g(a_1^\star) = 0$ and $a_1^\star > a_\infty^+$, then by definition of a_∞^+ and continuous differentiability of A_g on $\mathbb{R}$, there must exists $a_1 \in (a_\infty^+, a_1^\star)$ such that $A_g'(a_1) > 0$ and $A_g(a_1) < 0$. Thus, by Remark 3.10,

$$\mathbb{L}_X g(a_1) - qg(a_1) = \frac{q}{\Phi(q)} A_g'(a_1) - qA_g(a_1) > 0.$$

Then, $a_{L,1} < a_1^\star$, which clearly contradicts part (i) in Lemma 3.30. So $a_1^\star \leq a_\infty^+$, which allows us to conclude that $a_1^\star = a_\infty^+$.

As $A_g(a) = 0$ for all $a \in [a_\infty^-, a_\infty^+]$, then for all $x \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{E}\left(A_g(\underline{X}_{e_q} + x)\mathbb{1}_{\{\underline{X}_{e_q}+x<a_\infty^-\}}\right) \\
&= \mathbb{E}\left(A_g(\underline{X}_{e_q} + x)\mathbb{1}_{\{\underline{X}_{e_q}+x<a_\infty^-\}}\right) + \mathbb{E}\left(A_g(\underline{X}_{e_q} + x)\mathbb{1}_{\{\underline{X}_{e_q}+x\in[a_\infty^-,a_\infty^+)\}}\right) \\
&= \mathbb{E}\left(A_g(\underline{X}_{e_q} + x)\mathbb{1}_{\{\underline{X}_{e_q}+x<a_\infty^+\}}\right) \\
&= h(x, a_\infty^+).
\end{aligned}$$

This completes the proof. □

3.7 Conclusion and Discussion

In this chapter, we proposed an approach that solves the optimal stopping problem (3.2) in a general setting. Our approach does not require any knowledge on the continuation region in advance, or the shape of the averaging function, or the pasting condition at the boundaries. The method is based on finding the function $h(x, a)$ (3.21) by using averaging function and the scale functions for spectrally negative Lévy processes, and the optimal stopping boundaries $a_i^\star$ and $b_i^\star$ are selected such that h is the smallest function that dominates the gain function g. Then a close left semi-solution $(V_i, \tau_i^\star)$ is obtained up to the point $b_i^\star$. Under certain conditions $(V_i, \tau_i^\star)$ is even the global solution to the optimal stopping problem (3.2). And the pasting coditions on the boundary of the closed left semi continuation region can be observed directly from the path property of h. The left semi value function and the pasting conditions found in this chapter show no contradiction with the existing results the literature, for example in [1], [2], [9], [36], [53], [47] and [78], where a sufficient condition for the smooth pasting is found to be closely related to the regularity of the underlying process X at the stopping boundary. Moreover, our conclusion over the left semi value functions and the pasting conditions extend further the aforementioned recent work into more general payoff functions.

Finally we remark on the sequences $a_i^\star$ and $b_i^\star$. It is possible to obtain two sequences $a_i^\star$ and $b_i^\star$ which have countable number of elements. Examples are constructed in Chapter 5 where $a_i^\star$ and $b_i^\star$ are diverging and converging. We point out here that if $a_i^\star$ and $b_i^\star$ are converging, it is not clear on what happens after the limit point.

3.8 Proofs

Proof for Lemma 3.2. We will only prove for the case of closed left semi-solutions. The argument below can be repeated to show for the case of open left semi-solutions.

Consider the following optimal stopping problem,

$$V^T(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} T(X_\tau)),$$

where $q > 0$, X is a spectrally negative Lévy process, and the supremum is taken over the class $\mathcal{T}_{[0,\infty]}$ of Markov stopping times taking values in $[0,\infty]$ with respect to $\mathcal{F}$. By the sufficient theorem for optimal stopping problems, see Theorem 2.7 in [66], $T(x) = V^T(x)$ for all $x \in \mathbb{R}$. Thus,

$$\begin{aligned} T(x) &= \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} T(X_\tau)) \\ &\geq \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} g(X_\tau)) \\ &= V(x), \end{aligned}$$

where the inequality is due to $T(x) \geq g(x)$ for all $x \in \mathbb{R}$. On other hand, for all $x \leq b$,

$$\begin{aligned} V(x) &= \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} g(X_\tau)) \\ &\geq \mathbb{E}_x(e^{-q\bar{\tau}} g(X_{\bar{\tau}})) \\ &= T(x). \end{aligned}$$

Thus, by definition 3.1, the pair $(\bar{V}, \bar{\tau})$ is a closed left semi-solution for optimal stopping problems up to the point b. □

Proof for Lemma 3.5. Throughout this proof, we will assume that $\delta_2 = 1$. A similar argument can be done for the case $\{\delta_2 = 0\}$ to obtain the result as required.

(i)

Under the definition of $D^2(I_g)$, from the Mean Value Theorem, it follows that for all $x \in \mathbb{R}$ and $y \in (-1, 0)$, there exists $x_0 \in (x+y, x)$ such that

$$|g(x+y) - g(x) - yg'(x)| \leq \frac{1}{2}y^2 |\max\{g''(x_0+), g''(x_0-)\}| \leq \frac{1}{2}y^2(e^{cx_0} + d) \leq \frac{1}{2}y^2(e^{cx} + d$$

where c and d are as in the Definition 3.3, and the last equality is due to the fact that $c, d > 0$, and $e^{cx} + d$ is increasing in x.

For $y \le -1$, we have

$$|g(x+y) - g(x)| \le e^{c(x+y)} + d + e^{cx} + d \le 2e^{cx} + 2d, \qquad \text{for all } x \in \mathbb{R}.$$

By putting the above two equations together we get:

$$|g(x+y) - g(x) - yg'(x)\mathbb{1}_{\{y\in(-1,0)\}}| \le c_1(1 \wedge y^2)(e^{cx} + d),$$

for all $x \in \mathbb{R}$ and $y < 0$ where c_1 is some positive constant.

(ii)

Let $\{\epsilon_n, n \in \mathbb{N}\}$ be any sequence converging to 0 as n goes to ∞. So for this sequence, let $\epsilon > 0$ be such that $\epsilon > \epsilon_n$ for all $n \in \mathbb{N}$. Then thanks to part (i), we have for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$,

$$\begin{aligned}
&\int_{-\infty}^{0} \left|g(x+y+\epsilon_n) - g(x+\epsilon_n) - yg'(x+\epsilon_n)\mathbb{1}_{\{|y|<1\}}\right|\Pi(dy) \\
&< c_1(e^{cx+c\epsilon_n} + d)\int_{-\infty}^{0}(1 \wedge y^2)\Pi(dy) \\
&< c_1(e^{cx+c\epsilon} + d)\int_{-\infty}^{0}(1 \wedge y^2)\Pi(dy) \\
&< \infty.
\end{aligned}$$

Therefore, by letting n goes to ∞ and applying the dominated convergence theorem, we get

$$\begin{aligned}
&\lim_{n\to\infty}\int_{-\infty}^{0}\left(g(x+y+\epsilon_n) - g(x+\epsilon_n) - yg'(x+\epsilon_n)\mathbb{1}_{\{|y|<1\}}\right)\Pi(dy) \\
&= \int_{-\infty}^{0}\left(g(x+y) - g(x) - yg'(x)\mathbb{1}_{\{|y|<1\}}\right)\Pi(dy)
\end{aligned}$$

for all $x \in \mathbb{R}$ and $y < 0$. Thus, $\int_{-\infty}^{0}\left(g(x+y) - g(x) - yg'(x)\mathbb{1}_{\{|y|<1\}}\right)\Pi(dy)$ is continuous in x for all $x \in \mathbb{R}$. Finally, by the definition of $D^2(I)$, g' is continuous on $\mathbb{R}$, and g'' is continuous on $\mathbb{R}\setminus I$ with left and right limits on I. Therefore, from the equation (3.7), we can conclude that the statement in part (ii) holds true.

(iii)

Thanks to the definition of $D^2(I)$ and part (i), together with $\int_{-\infty}^{0}(1 \wedge$

$y^2)\Pi(dy) < \infty$, we have for all $x \in \mathbb{R} \setminus I$,

$$
\begin{aligned}
&|\mathbb{L}_X g(x) - qg(x)| \\
\leq\ & q|g(x)| + |\mu g'(x)| + \frac{1}{2}\sigma^2 |g''(x)| + \int_{-\infty}^{0} |g(x+y) - g(x) - yg'(x)\mathbb{1}_{\{|y|<1\}}|\Pi(dy) \\
\leq\ & q(e^{cx} + d) + |\mu|(e^{cx} + d) + \frac{1}{2}\sigma^2(e^{cx} + d) + c_1 \int_{-\infty}^{0} (1 \wedge y^2)\Pi(dy)(e^{cx} + d) \\
=\ & c_2(e^{cx} + d), \qquad (3.33)
\end{aligned}
$$

where c_1 is as identified in part (ii), and $c_2 = q + |\mu| + \sigma^2/2 + c_1 \int_{-\infty}^{0}(1 \wedge y^2)\Pi(dy)$. Finally, by Part (ii), we obtain the equation as required.

(iv)

It follows from the definition of $D^2(I)$ that,

$$
\begin{aligned}
&\lim_{x \to -\infty} (\mathbb{L}_X g(x) - qg(x)) \\
&= \lim_{x \to -\infty} \int_{-\infty}^{0} \left(g(x+y) - g(x) - yg'(x)\mathbb{1}_{\{y>-1\}}\right) \Pi(dy) - qg(-\infty).
\end{aligned}
$$

Then, as a result of part (i), we can apply the dominated convergence theorem and obtain

$$
\begin{aligned}
&\lim_{x \to -\infty} (\mathbb{L}_X g(x) - qg(x)) \\
&= \int_{-\infty}^{0} \lim_{x \to -\infty} \left(g(x+y) - g(x) - yg'(x)\mathbb{1}_{\{y>-1\}}\right) \Pi(dy) - qg(-\infty) \\
&= -qg(-\infty). \qquad (3.34)
\end{aligned}
$$

□

Proof for Theorem 3.4. Suppose that $\delta_2 = 1$. We will show that each term in (3.9) is a martingale.

First, consider the local martingale $\{M_t^{g,1}, t \geq 0\}$ where

$$M_t^{g,1} = \sigma \int_0^t e^{-qs} g'(X_{s-})dB_s, \qquad t \geq 0.$$

Under the definition of $D^2(I)$, it follows from the Fubini's theorem that for all $t \geq 0$

$$\begin{aligned}
\mathbb{E}\left(\langle M^{g,1}\rangle_t\right) &= \mathbb{E}\left(\sigma^2 \int_0^t e^{-2qs}(g'(X_{s-}))^2 ds\right) \\
&\leq \sigma^2 \int_0^t e^{-2qs}\mathbb{E}(e^{cX_{s-}} + d)^2 ds \\
&= \sigma^2 \int_0^t e^{-2qs}(e^{s\psi(2c)} + 2de^{s\psi(c)} + d^2)ds \\
&< \infty,
\end{aligned}$$

where the last equality is due to $\mathbb{E}(e^{cX_{s-}}) = \mathbb{E}(e^{cX_s}) = e^{s\psi(c)}$ for all $c \geq 0$. Hence, by the Burkholder-Davis-Gundy inequality (for example see [45] page 166), $\{M_t^{g,1}, t \geq 0\}$ is a martingale.

Next, consider the term $M^{g,2}$, where

$$M_t^{g,2} = \int_0^t \int_{-1}^0 ye^{-qs}g'(X_{s-})\widetilde{N}(ds, dy), \qquad t \geq 0.$$

For each fixed $t > 0$, thanks to the definition of $D^2(I)$, there exists $c \in (0, \Phi(q))$ and $d > 0$ such that

$$\begin{aligned}
&\int_0^t \int_{-1}^0 \mathbb{E}\left((ye^{-qs}g'(X_{s-}))^2\right) ds\Pi(dy) \\
&\leq \int_0^t \int_{-1}^0 \mathbb{E}\left((ye^{-qs}(e^{cX_{s-}} + d))^2\right) ds\Pi(dy) \\
&= \int_{-1}^0 y^2\Pi(dy) \int_0^t e^{-2qs}\mathbb{E}\left(e^{2cX_{s-}} + 2de^{cX_{s-}} + d^2\right) ds \\
&= \int_{-1}^0 y^2\Pi(dy) \int_0^t e^{-2qs}\left(e^{s\psi(2c)} + 2de^{s\psi(c)} + d^2\right) ds \\
&< \infty.
\end{aligned}$$

The condition $\int_{-1}^0 y^2\Pi(dy) < \infty$ is applied to obtain the last inequality. Thus, the function $ye^{-qs}g'(X_{s-})\mathbb{1}_{\{y\in(0,1)\}}$ is in the class of $\mathcal{H}_2(t, \mathbb{R})$ (see Section 1.2.1 in Chapter 1 for the definition). Then it follows from Theorem 4.2.3 that $\{M_t^{g,2}, t \geq 0\}$ is a martingale.

Finally, we consider the term, for all $t \geq 0$

$$M_t^{g,3} = \int_0^t \int_{-\infty}^0 e^{-qs}(g(X_{s-} + y) - g(X_{s-}) - yg'(X_{s-})\mathbb{1}_{\{|y|<1\}})\widetilde{N}(ds, dy).$$

By using part (i) in Lemma 3.5, it follows from a similar argument as for $M^{g,2}$ that

$\{M_t^{g,3}, t \geq 0\}$ is a martingale.

If $\delta_2 = 0$, then

$$M_t^g = \sigma \int_0^t e^{-qs} g'(X_{s-}) dB_s + \int_0^t \int_{-\infty}^0 e^{-qs}(g(X_{s-} + y) - g(X_{s-}))\tilde{N}(ds.dy).$$

Then by applying the a similar argument as above under the new condition $\int_{-\infty}^0 (1 \wedge y)\Pi(dy) < 0$, we can conclude that M^g is a martingale. And this completes the proof. □

Proof for Lemma 3.6. Clearly, for all $x \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}_x \left(\lim_{t\to\infty} e^{-qt} |g(X_t)| \right) &\leq \mathbb{E}_x \left(\lim_{t\to\infty} e^{-qt}(e^{cX_t} + d) \right) \\ &= \mathbb{E}_x \left(\lim_{t\to\infty} e^{-qt} e^{cX_t} \right) \end{aligned}$$

Also as $\{e^{-qt+cX_t}, t \geq 0\}$ is a $\mathbb{P}_x$ supermartingale with last element equals to 0 $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}$, we obtain that

$$\mathbb{E}_x \left(\lim_{t\to\infty} e^{-qt} |g(X_t)| \right) = 0 \tag{3.35}$$

for all $x \in \mathbb{R}$. Note that for any stopping time τ and $x \in \mathbb{R}$,

$$\mathbb{E}_x \left(\lim_{t\to\infty} e^{-qt} |g(X_t)| \right) \geq \mathbb{E}_x \left(e^{-q\tau} |g(X_\tau)| \mathbb{1}_{\{\tau=\infty\}} \right) \geq 0. \tag{3.36}$$

The first inequality holds true because the LHS is evaluated with respect to all paths of X, while the RHS is evaluated only with respect to paths of X on the set $\{\tau = \infty\}$. So by putting (3.35) and (3.36) together we have the equality as required. □

Proof of Proposition 3.8. Under the definition of $D^2(I)$, it follows from Theorem 3.4 that for all $x \in \mathbb{R}$ and $t \geq 0$

$$\mathbb{E}_x \left(e^{-qt} g(X_t) \right) - g(x) = \mathbb{E}_x \left(\int_0^t e^{-qs} (\mathbb{L}_X g(X_s) - qg(X_s)) \mathbb{1}_{\{X_s \notin I\}} ds \right).$$

By the Fubini's Theorem and Esscher transform, we get for all $x \in \mathbb{R}$ and $t \geq 0$

$$
\begin{aligned}
&\int_0^t \mathbb{E}_x \left(qe^{-qs} \left(\mathbb{L}_X g(X_s) - qg(X_s)\right) \mathbb{1}_{\{X_s \notin I\}}\right) ds \\
&= \mathbb{E}_x \left(\int_0^t qe^{-qs} \left(\mathbb{L}_X g(X_s) - qg(X_s)\right) \mathbb{1}_{\{X_s \notin I\}} ds \right) \\
&= \mathbb{E}_x \left(qe^{-qt} g(X_t) - qg(x)\right) \\
&= qe^{\Phi(q)x} \mathbb{E}_x^{\Phi(q)} \left(e^{-\Phi(q)X_t} g(X_t)\right) - qg(x).
\end{aligned}
$$

Note that under the measure $\mathbb{P}^{\Phi(q)}$, X remains as a spectrally negative Lévy process, and goes to ∞ $\mathbb{P}_x^{\Phi(q)}$-a.s. for all $x \in \mathbb{R}$. Then, thanks to the definition of $D^2(I)$, by letting t go to infinity, we have for all $x \in \mathbb{R}$

$$
\begin{aligned}
&\int_0^\infty \mathbb{E}_x \left(qe^{-qs} \left(\mathbb{L}_X g(X_s) - qg(X_s)\right) \mathbb{1}_{\{X_s \notin I\}}\right) ds \\
&= \lim_{t \to \infty} \int_0^t \mathbb{E}_x \left(qe^{-qs} \left(\mathbb{L}_X g(X_s) - qg(X_s)\right) \mathbb{1}_{\{X_s \notin I\}}\right) ds \\
&= \lim_{t \to \infty} \left(qe^{\Phi(q)x} \mathbb{E}_x^{\Phi(q)} \left(e^{-\Phi(q)X_t} g(X_t)\right) - qg(x)\right) \\
&= -qg(x).
\end{aligned} \tag{3.37}
$$

On the other hand from part (iii) in Lemma 3.5 we have for all $x \in \mathbb{R}$

$$
\begin{aligned}
&\int_0^\infty \mathbb{E}_x \left(\left|qe^{-qs} \left(\mathbb{L}_X g(X_s) - qg(X_s)\right) \mathbb{1}_{\{X_s \notin I\}}\right|\right) ds \\
&\leq \int_0^\infty \mathbb{E}_x \left(qe^{-qs} c_1 (e^{cX_s} + d)\right) ds \\
&= \int_0^\infty \left(qe^{-qs} c_1 (e^{s\psi(c)} + d)\right) ds \\
&< \infty.
\end{aligned}
$$

The last inequality is due to $c \in (0, \Phi(q))$. Thus by Fubini's theorem again we obtain for all $x \in \mathbb{R}$

$$
\begin{aligned}
&\int_0^\infty \mathbb{E}_x \left(qe^{-qs} \left(\mathbb{L}_X g(X_s) - qg(X_s)\right) \mathbb{1}_{\{X_s \notin I\}}\right) ds \\
&= \mathbb{E}_x \left(\int_0^\infty qe^{-qs} \left(\mathbb{L}_X g(X_s) - qg(X_s)\right) \mathbb{1}_{\{X_s \notin I\}} ds \right) \\
&= \mathbb{E}_x \left(\mathbb{L}_X g(X_{e_q}) - qg(X_{e_q}) \mathbb{1}_{\{X_{e_q} \notin I\}}\right).
\end{aligned} \tag{3.38}
$$

By combining (3.37) and (3.38), we obtain the result as required. $\square$

Proof for Proposition 3.9. As $X_{e_q} = \underline{X}_{e_q} + X_{e_q} - \underline{X}_{e_q}$, and $X_{e_q} - \underline{X}_{e_q}$ is independent of $\underline{X}_{e_q}$ and has exponential distribution with parameter $\Phi(q)$, it follows from Proposition 3.8 that for all $x \in \mathbb{R}$

$$\begin{aligned}
-qg(x) &= \mathbb{E}_x\left(\mathbb{L}_X g(X_{e_q}) - qg(X_{e_q})\mathbb{1}_{\{X_{e_q}\notin I\}}\right)\\
&= \mathbb{E}_x\left(\mathbb{L}_X g(\underline{X}_{e_q} + X_{e_q} - \underline{X}_{e_q}) - qg(\underline{X}_{e_q} + X_{e_q} - \underline{X}_{e_q})\mathbb{1}_{\{X_{e_q}\notin I\}}\right)\\
&= \mathbb{E}_x\left(\int_0^\infty \left(\mathbb{L}_X g(\underline{X}_{e_q} + y) - qg(\underline{X}_{e_q} + y)\mathbb{1}_{\{\underline{X}_{e_q}+y\notin I\}}\right)\Phi(q)e^{-\Phi(q)y}dy\right)\\
&= \Phi(q)\mathbb{E}_x\left(e^{\Phi(q)\underline{X}_{e_q}}\int_{\underline{X}_{e_q}}^\infty \left(\mathbb{L}_X g(y) - qg(y)\mathbb{1}_{\{y\notin I\}}\right)e^{-\Phi(q)y}dy\right)\\
&= \mathbb{E}_x\left(-qA_g(\underline{X}_{e_q})\right)
\end{aligned}$$

as required.

$\square$

Proof for Lemma 3.17. Fix $a \in \mathbb{R}$. For each $x \in (-\infty, a)$, by the definition of the averaging function A_g and scale function W^q, we have

$$\begin{aligned}
h(x,a) &= \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q}<a\}}\right) + \frac{q}{\Phi(q)}W^q(x-a)A_g(a)\\
&= g(x).
\end{aligned}$$

As $g(x) \in C^{\delta_1}(\mathbb{R})\cap C^{1+\delta_1}(\mathbb{R}\backslash I)$, we have $h(\cdot,a) \in C^{\delta_1}((-\infty,a))\cap C^{1+\delta_1}((-\infty,a)\backslash I_a)$.

Now we study the continuity and differentiability of $h(\cdot,a)$ for all $x > a$. For all $\epsilon > 0$,

$$\begin{aligned}
h(x+\epsilon,a) &= \mathbb{E}_{x+\epsilon}\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q}<a\}}\right) + \frac{q}{\Phi(q)}W^q(x+\epsilon-a)A_g(a)\\
&= \mathbb{E}\left(A_g(\underline{X}_{e_q}+x+\epsilon)\right) - \mathbb{E}\left(A_g(\underline{X}_{e_q}+x+\epsilon)\mathbb{1}_{\{\underline{X}_{e_q}\geq a-\epsilon-x\}}\right)\\
&\quad+\frac{q}{\Phi(q)}W^q(x+\epsilon-a)A_g(a)\\
&= g(x+\epsilon) + \frac{q}{\Phi(q)}W^q(x+\epsilon-a)A_g(a)\\
&\quad-\int_{y\in(-\infty,0]} A_g(y+x+\epsilon)\mathbb{1}_{\{y\geq a-x-\epsilon\}}\mathbb{P}(\underline{X}_{e_q}\in dy). \qquad (3.39)
\end{aligned}$$

where the definition of the averaging function is used in the last equality. Note that from Remark 3.11, $A_g(x)$ is bounded on $(-\infty, x_0]$ for all $x_0 \in \mathbb{R}$. So by letting ϵ go to 0 and applying the dominated convergence theorem, we get

$$h(x+,a) = g(x) - \int_{y\in[a-x,0]} A_g(y+x)\mathbb{P}(\underline{X}_{e_q} \in dy) + \frac{q}{\Phi(q)} W^q(x-a)A_g(a)$$

for all $x > a$. By a similar argument as above, we find the left limit of $h(\cdot, a)$ for all $x > a$ as

$$h(x-,a) = g(x) - \int_{y\in(a-x,0]} A_g(y+x)\mathbb{P}(\underline{X}_{e_q} \in dy) + \frac{q}{\Phi(q)} W^q(x-a)A_g(a)$$

As $\underline{X}_{e_q}$ has no atom apart from 0, we can conclude that $x \mapsto h(x,a)$ is continuous on (a, ∞).

At $x = a$, the right limit is

$$\begin{aligned} h(a+,a) &= g(a) - \int_{y\in[0,0]} A_g(y+a)\mathbb{P}(\underline{X}_{e_q} \in dy) + \frac{q}{\Phi(q)} W^q(0)A_g(a) \\ &= g(a) - A_g(a)\mathbb{P}(\underline{X}_{e_q} = 0) + \frac{q}{\Phi(q)} W^q(0)A_g(a) \\ &= g(a), \end{aligned}$$

which is also the left limit of $h(x,a)$ at $x = a$. This allows us to conclude that $h(\cdot, a)$ is continuous on $\mathbb{R}$.

Next we calculate the right and left derivative $h(\cdot, a)$ at $x > a$. For all $\epsilon \in (0,1)$ we have

$$\begin{aligned} & h(x+\epsilon,a) - h(x,a) \\ &= \mathbb{E}_{x+\epsilon}\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q}<a\}}\right) - \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q}<a\}}\right) \\ &\quad + \frac{q}{\Phi(q)}\left(W^q(x+\epsilon-a)A_g(a) - W^q(x-a)A_g(a)\right) \\ &= \left(\int_{y<a-x-\epsilon} A_g(y+x+\epsilon)\mathbb{P}(\underline{X}_{e_q} \in dy) - \int_{y<a-x} A_g(y+x)\mathbb{P}(\underline{X}_{e_q} \in dy)\right) \\ &\quad + \frac{q}{\Phi(q)}\left(W^q(x+\epsilon-a)A_g(a) - W^q(x-a)A_g(a)\right). \end{aligned} \tag{3.40}$$

The first term in the last equality can be rewritten as:

$$\begin{aligned}
&\int_{y<a-x-\epsilon} A_g(y+x+\epsilon)\mathbb{P}(\underline{X}_{e_q} \in dy) - \int_{y<a-x} A_g(y+x)\mathbb{P}(\underline{X}_{e_q} \in dy) \\
&= \left(\int_{y<a-x-\epsilon} A_g(y+x+\epsilon)\mathbb{P}(\underline{X}_{e_q} \in dy) - \int_{y<a-x-\epsilon} A_g(y+x)\mathbb{P}(\underline{X}_{e_q} \in dy)\right) \\
&\quad + \left(\int_{y<a-x-\epsilon} A_g(y+x)\mathbb{P}(\underline{X}_{e_q} \in dy) - \int_{y<a-x} A_g(y+x)\mathbb{P}(\underline{X}_{e_q} \in dy)\right) \\
&= \left(\int_{y<a-x-\epsilon} \left(A_g(y+x+\epsilon) - A_g(y+x)\right)\mathbb{P}(\underline{X}_{e_q} \in dy)\right) \\
&\quad - \int_{y\in[a-x-\epsilon,a-x)} A_g(y+x)\mathbb{P}(\underline{X}_{e_q} \in dy).
\end{aligned} \tag{3.41}$$

As $|A_g'(x+)|$ is bounded on $(-\infty, b]$ for all $b \in \mathbb{R}$, then for each fixed x and $\epsilon \in (0,1)$, by the Mean Value Theorem, the term $\left|\frac{1}{\epsilon}\left(A_g(y+x+\epsilon) - A_g(y+x)\right)\right|$ is bounded for all $y \leq 0$. So by dividing both sides of (3.41) by $\epsilon \in (0,1)$, letting $\epsilon \downarrow 0$, and finally applying the bounded convergence theorem, we obtain for all $x > a$

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\left(\int_{y<a-x-\epsilon} A_g(y+x+\epsilon)\mathbb{P}(\underline{X}_{e_q} \in dy) - \int_{y<a-x} A_g(y+x)\mathbb{P}(\underline{X}_{e_q} \in dy)\right) \\
&= \int_{y<a-x} \lim_{\epsilon\downarrow 0} \frac{A_g(y+x+\epsilon) - A_g(y+x)}{\epsilon}\mathbb{P}(\underline{X}_{e_q} \in dy) \\
&\quad - A_g(a)\frac{d}{dy}\mathbb{P}\left(\underline{X}_{e_q} < (a-x)-\right) \\
&= \int_{y<a-x} A_g'((y+x)+)\mathbb{P}(\underline{X}_{e_q} \in dy) \\
&\quad - A_g(a)\left(-qW^q(x-a) + \frac{q}{\Phi(q)}\left(W^q\right)'((x-a)+)\right),
\end{aligned} \tag{3.42}$$

where equation (3.13) is applied in the last equality. By plugging this into (3.40), we obtain for all $x > a$

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \frac{h(x+\epsilon, a) - h(x,a)}{\epsilon} \\
&= \int_{y<a-x} A_g'((y+x)+)\mathbb{P}(\underline{X}_{e_q} \in dy) + A_g(a)qW^q(x-a).
\end{aligned} \tag{3.43}$$

By performing a similar calculation, we observe that the left derivative of $h(\cdot, a)$ for

$x > a$ by

$$\lim_{\epsilon \uparrow 0} \frac{h(x+\epsilon, a) - h(x,a)}{\epsilon}$$
$$= \int_{y \leq a-x} A_g'((y+x)-)\mathbb{P}(\underline{X}_{e_q} \in dy) + A_g(a) q W^q(x-a). \qquad (3.44)$$

Note that $A_g'(x+) = A_g'(x-)$ for all $x \in (-\infty, a) \setminus I_a$. So by comparing equations (3.43) and (3.44), together with the fact that $\underline{X}_{e_q}$ has no atom apart from 0, we can conclude that $h(\cdot, a)$ is differentiable for all $x > a$ with the following derivative

$$\frac{\partial}{\partial x} h(x,a) = \int_{-\infty}^{a-x} A_g'(y+x) \mathbb{1}_{\{x+y \notin I_a\}} \mathbb{P}(\underline{X}_{e_q} \in dy) + A_g(a) q W^q(x-a). \qquad (3.45)$$

Next we calculate the right limit of $\frac{\partial h}{\partial x}(\cdot, a)$ at $x = a$. From equation (3.43), it follows that

$$\begin{aligned} \frac{\partial h}{\partial x}(a+, a) &= \int_{y<0} A_g'((y+a)+)\mathbb{P}(\underline{X}_{e_q} \in dy) + A_g(a) q W^q(0) \\ &= \int_{y \leq 0} \lim_{\epsilon \downarrow 0} \frac{A_g(y+a+\epsilon) - A_g(y+a)}{\epsilon} \mathbb{P}(\underline{X}_{e_q} \in dy) \\ &\quad - A_g'(a+)\mathbb{P}(\underline{X}_{e_q} = 0) + A_g(a) q W^q(0). \end{aligned} \qquad (3.46)$$

By the bounded convergence theorem, we can rewrite (3.46) as

$$\begin{aligned} &\frac{\partial h}{\partial x}(a+, a) \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \int_{y \leq 0} \Big(A_g(y+a+\epsilon) - A_g(y+a) \Big) \mathbb{P}(\underline{X}_{e_q} \in dy) \\ &\quad - W^q(0) \left(\frac{q}{\Phi(q)} A_g'(a+) - q A_g(a) \right) \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \Big(\mathbb{E}(A_g(\underline{X}_{e_q} + a + \epsilon)) - \mathbb{E}(A_g(\underline{X}_{e_q} + a)) \Big) - W^q(0) \left(\mathbb{L}_X g(a+) - q g(a) \right) \\ &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} (g(a+\epsilon) - g(a)) - W^q(0) \left(\mathbb{L}_X g(a+) - q g(a) \right) \\ &= g'(a+) - W^q(0) \left(\mathbb{L}_X g(a+) - q g(a) \right), \end{aligned} \qquad (3.47)$$

where the third equality is due to the definition of A_g and Remark 3.10. As in the case of unbounded variation, $W^q(0) = 0$, and g is differentiable for all $x \in \mathbb{R}$, therefore,

$$\frac{\partial h}{\partial x}(a, a) = g'(a).$$

Finally, and we show that $h(\cdot, a)$ is twice differentiable for all $x > a$ in the case that X has unbounded variation. Note that as X has unbounded variation, the distribution of $\underline{X}_{e_q}$ has a density $f_{\underline{X}_{e_q}}$,

$$f_{\underline{X}_{e_q}}(y) = -qW^q(-y) + \frac{q}{\Phi(q)} (W^q)'(-y) \tag{3.48}$$

for all $y < 0$. Under the condition (3.1), $f_{\underline{X}_{e_q}}$ is continuously differentiable on $\mathbb{R}\backslash\{0\}$. Furthermore, by using $W^q(x) = e^{\Phi(q)x}W_{\Phi(q)}(x)$ for all $x \in \mathbb{R}$, we can rewrite (3.48) for $y < 0$ as

$$\begin{aligned} f_{\underline{X}_{e_q}}(y) &= -qW^q(-y) + qW^q(-y) + \frac{q}{\Phi(q)} e^{-\Phi(q)y} W'_{\Phi(q)}(-y) \\ &= \frac{q}{\Phi(q)} e^{-\Phi(q)y} W'_{\Phi(q)}(-y). \end{aligned}$$

Also for all $y < 0$

$$\begin{aligned} f'_{\underline{X}_{e_q}}(y) &= -qe^{-\Phi(q)y} W'_{\Phi(q)}(-y) - \frac{q}{\Phi(q)} e^{-\Phi(q)y} W''_{\Phi(q)}(-y) \\ &= -\Phi(q) f_{\underline{X}_{e_q}}(y) - \frac{q}{\Phi(q)} e^{-\Phi(q)y} W''_{\Phi(q)}(-y). \end{aligned}$$

Note that $W_{\Phi(q)}$ increases to a constant as $x \uparrow \infty$, and under the measure $\mathbb{P}^{\Phi(q)}$, X drifts to ∞ as $t \to \infty$, and $W'_{\Phi(q)} > 0$ on $(0, \infty)$. By part (i) in Theorem 2.1 in [50], $W'_{\Phi(q)}$ is convex on $(0, \infty)$. Therefore, $W_{\Phi(q)}$ is concave on $(0, \infty)$, i.e. $W''_{\Phi(q)} \leq 0$ on $(0, \infty)$.

Let $\tilde{c} > 0$ be such that $|\max\{A'(x+), A'(x-)\}| < \tilde{c}$ on $(-\infty, a]$. Therefore, for all $x > a$, we have

$$\begin{aligned} &\int_{-\infty}^{a} \left| A'_g(z) \mathbb{1}_{\{z \notin I\}} f'_{\underline{X}_{e_q}}(z - x) \right| dz \\ \leq\ & \tilde{c} \int_{-\infty}^{a} \left| -\Phi(q) f_{\underline{X}_{e_q}}(z - x) - \frac{q}{\Phi(q)} e^{-\Phi(q)(x-z)} W''_{\Phi(q)}(x - z) \right| dz \\ \leq\ & \tilde{c} \int_{-\infty}^{a} \Phi(q) f_{\underline{X}_{e_q}}(z - x) dz - \tilde{c} \int_{-\infty}^{a} \frac{q}{\Phi(q)} e^{-\Phi(q)(x-z)} W''_{\Phi(q)}(x - z) dz \\ =\ & 2\tilde{c} \int_{-\infty}^{a} \Phi(q) f_{\underline{X}_{e_q}}(z - x) dz \\ & -\tilde{c} \int_{-\infty}^{a} \left(\Phi(q) f_{\underline{X}_{e_q}}(z - x) + \frac{q}{\Phi(q)} e^{-\Phi(q)(x-z)} W''_{\Phi(q)}(x - z) \right) dz \\ =\ & 2\tilde{c} \int_{-\infty}^{a} \Phi(q) f_{\underline{X}_{e_q}}(z - x) dz + \tilde{c} \int_{-\infty}^{a} f'_{\underline{X}_{e_q}}(z - x) dz \\ <\ & \textit{constant}. \end{aligned}$$

Next, we show the continuity of $x \mapsto \int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z \notin I\}} f'_{\underline{X}_{e_q}}(z-x)dz$ for all $x > a$. Let $\{\epsilon_n \in (0, x-a), i \in \mathbb{N}\}$ be a sequence decreasing to 0, and let $\epsilon_0 > \epsilon_n$ for all $n \in \mathbb{N}$. Then for any fixed $a \in \mathbb{R}$ and $x > a$, we have

$$\left|A_g'(y+x+\epsilon_n)\mathbb{1}_{\{y+x+\epsilon_n \notin I\}} f'_{\underline{X}_{e_q}}(y)\mathbb{1}_{\{y<a-x-\epsilon_n\}}\right| \leq \tilde{c}_{a+\epsilon_0}\left|f'_{\underline{X}_{e_q}}(y)\right|$$

for all $n \in \mathbb{N}$ and $y \leq a-x$, where $\tilde{c}_{a+\epsilon_0} = \sup_{x \leq a+\epsilon_0}\{|A'(x+)|, |A'(x-)|\}$. Note that by a similar argument as before, we have for all $a \in \mathbb{R}$ and $x > a$,

$$\int_{-\infty}^{a-x} \left|f'_{\underline{X}_{e_q}}(y)\right| dy < \infty.$$

So, by letting $n \to \infty$ and applying the dominated convergence theorem, we obtain for all $a \in \mathbb{R}$ and $x > a$,

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} f'_{\underline{X}_{e_q}}(z-x-\epsilon_n)dz \\
&= \lim_{n\to\infty} \int_{-\infty}^{a-x} A_g'(y+x+\epsilon_n)\mathbb{1}_{\{y+x+\epsilon_n \notin I\}} f'_{\underline{X}_{e_q}}(y)\mathbb{1}_{\{y<a-x-\epsilon_n\}}dy \\
&= \int_{y<a-x} A_g'((y+x)+)\mathbb{1}_{\{y+x\notin I\}} f'_{\underline{X}_{e_q}}(y)dy \\
&= \int_{y<a-x} A_g'(y+x)\mathbb{1}_{\{y+x\notin I\}} f'_{\underline{X}_{e_q}}(y)dy,
\end{aligned}$$

where the last equality is due to $A_g'(x+) = A_g'(x) = A_g'(x-)$ on the set $\mathbb{R} \setminus I$. By following a similar argument as above, we have for all $a \in \mathbb{R}$ and $x > a$,

$$\begin{aligned}
&\lim_{n\to\infty} \int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} f'_{\underline{X}_{e_q}}(z-x+\epsilon_n)dz \\
&\qquad = \int_{y\leq a-x} A_g'(y+x)\mathbb{1}_{\{y+x\notin I\}} f'_{\underline{X}_{e_q}}(y)dy,
\end{aligned}$$

which allows us to conclude that $x \mapsto \int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} f'_{\underline{X}_{e_q}}(z-x)dz$ is continuous for all $x > a$ and $a \in \mathbb{R}$.

Finally, it follows from the fundamental theorem of calculus and the Fubini's theorem that for all $x > a$ and $\tilde{c}_1 \in (a, x)$

$$\begin{aligned}
&\int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} f'_{\underline{X}_{e_q}}(z-x)dz \\
&= \frac{\partial}{\partial x}\int_{\tilde{c}_1}^{x}\int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} f'_{\underline{X}_{e_q}}(z-y)dzdy \\
&= \frac{\partial}{\partial x}\int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} \int_{\tilde{c}_1}^{x} f'_{\underline{X}_{e_q}}(z-y)dydz \\
&= \frac{\partial}{\partial x}\int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} \left(f_{\underline{X}_{e_q}}(z-\tilde{c}_1) - f_{\underline{X}_{e_q}}(z-x)\right)dz \\
&= -\frac{\partial}{\partial x}\int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} f_{\underline{X}_{e_q}}(z-x)dz,
\end{aligned}$$

where the last equality is due to the term $\int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} f_{\underline{X}_{e_q}}(z-\tilde{c}_1)dz$ is independent of x. Therefore, for all $x > a$, it follows from (3.45) that for all $x > a$

$$\frac{\partial^2}{\partial x^2}h(x,a) = -\int_{-\infty}^{a} A_g'(z)\mathbb{1}_{\{z\notin I\}} f'_{\underline{X}_{e_q}}(z-x)dz + A_g(a)qW^{q\prime}(x-a)$$

as required. □

Proof for Proposition 3.18. It has been proved in the literature that the process $\{e^{-qt\wedge\tau_a^-}W^q(X_{t\wedge\tau_a^-} - a), t \geq 0\}$ is a $\mathbb{P}_x$ martingale for all $x \in \mathbb{R}$ (see for example Chapter VII Lemma 11 in [7]), so the proof is completed once we show that $\{e^{-qt\wedge\tau_a^-}K(X_{t\wedge\tau_a^-})\}$ is a $\mathbb{P}_x$ martingale for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$, where $K(x) = \mathbb{E}_x\left(e^{-q\tau_a^-}g(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^-<\infty\}}\right)$.

Fix $a \in \mathbb{R}$, recall that θ is the shift operator. Then by the Markov property of Lévy processes, we have for all $x \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-q\tau_a^-}g(X_{\tau_a^-})\mathbb{1}_{\{t<\tau_a^-<\infty\}}\big|\mathcal{F}_t\right) \\
&= \mathbb{E}_x\left(e^{-q(t+\tau_a^-\circ\theta_t)}g(X_{t+\tau_a^-\circ\theta_t})\mathbb{1}_{\{t<\tau_a^-<\infty\}}\big|\mathcal{F}_t\right) \\
&= e^{-qt}\mathbb{E}_x\left(e^{-q\tau_a^-\circ\theta_t}g(X_{t+\tau_a^-\circ\theta_t})\mathbb{1}_{\{\tau_a^-\circ\theta_t<\infty\}}\mathbb{1}_{\{t<\tau_a^-<\infty\}}\big|\mathcal{F}_t\right) \\
&= e^{-qt}\mathbb{1}_{\{t<\tau_a^-<\infty\}}\mathbb{E}_{X_t}\left(e^{-q\tilde{\tau}_a^-}g(\tilde{X}_{\tilde{\tau}_a^-})\mathbb{1}_{\{\tilde{\tau}_a^-<\infty\}}\right) \\
&= e^{-qt}K(X_t)\mathbb{1}_{\{t<\tau_a^-<\infty\}},
\end{aligned} \tag{3.49}$$

where $\tilde{X}$ is a copy of the Lévy process with respect to $\mathbb{P}_{X_t}$, and $\tilde{\tau}_a^-$ is the first passage time below the level a for $\tilde{X}$.

Note also that on the set $\{\tau_a^- < \infty\}$, we have $K(X_{\tau_a^-}) = g(X_{\tau_a^-})$. This is true because in the case of bounded variation, $X_{\tau_a^-} < a$ $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}$.

Then, as $g = K$ on $(-\infty, a)$, it follows that $K(X_{\tau_a^-}) = g(X_{\tau_a^-})$ $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}$. In the case of X having unbounded variation, note that $\mathbb{P}_a(\tau_a^- = 0) = 1$. so $K(a) = \mathbb{E}_a\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\right) = g(a)$. Then, by the same argument as in the bounded variation case, we can conclude that $K(X_{\tau_a^-}) = g(X_{\tau_a^-})$ $\mathbb{P}_x$-a.s. for all $x \in \mathbb{R}$.

Thus,by applying equation (3.49) and the tower property, we obtain that for all $x \in \mathbb{R}$ and $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-qt\wedge\tau_a^-} K(X_{t\wedge\tau_a^-})\right) \\
&= \mathbb{E}_x\left(e^{-qt} K(X_t)\mathbb{1}_{\{t<\tau_a^-<\infty\}}\right) + \mathbb{E}_x\left(e^{-q\tau_a^-} K(X_{\tau_a^-})\mathbb{1}_{\{t\geq\tau_a^-\}}\right) \\
&= \mathbb{E}_x\left(\mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{t<\tau_a^-<\infty\}}\Big|\mathcal{F}_t\right)\right) + \mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{t\geq\tau_a^-\}}\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{t<\tau_a^-<\infty\}}\right) + \mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{t\geq\tau_a^-\}}\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^-<\infty\}}\right) \\
&= K(x).
\end{aligned} \tag{3.50}$$

Hence, for all $x \in \mathbb{R}$ and $0 \leq s \leq t$ we arrive at

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-qt\wedge\tau_a^-} K(X_{t\wedge\tau_a^-})\Big|_{\mathcal{F}_s}\right) \\
&= \mathbb{E}_x\left(e^{-qt\wedge\tau_a^-} K(X_{t\wedge\tau_a^-})\mathbb{1}_{\{s<\tau_a^-\}}\Big|_{\mathcal{F}_s}\right) + \mathbb{E}_x\left(e^{-qt\wedge\tau_a^-} K(X_{t\wedge\tau_a^-})\mathbb{1}_{\{s\geq\tau_a^-\}}\Big|_{\mathcal{F}_s}\right) \\
&= e^{-qs}\mathbb{E}_x\left(e^{-q(t-s)\wedge(\tau_a^- - s)} K(X_{s+(t-s)\wedge(\tau_a^- - s)})\mathbb{1}_{\{s<\tau_a^-\}}\Big|_{\mathcal{F}_s}\right) \\
&\quad + e^{-q\tau_a^-} K(X_{\tau_a^-})\mathbb{1}_{\{s\geq\tau_a^-\}} \\
&= e^{-qs}\mathbb{E}_x\left(e^{-q(t-s)\wedge(\tau_a^-\circ\theta_s)} K(X_{s+(t-s)\wedge(\tau_a^-\circ\theta_s)})\mathbb{1}_{\{s<\tau_a^-\}}\Big|_{\mathcal{F}_s}\right) \\
&\quad + e^{-q\tau_a^-} K(X_{\tau_a^-})\mathbb{1}_{\{s\geq\tau_a^-\}} \\
&= e^{-qs}\mathbb{1}_{\{s<\tau_a^-\}}\mathbb{E}_{X_s}\left(e^{-q(t-s)\wedge\tilde{\tau}_a^-} K(\widetilde{X}_{(t-s)\wedge\tilde{\tau}_a^-})\right) + e^{-q\tau_a^-} K(X_{\tau_a^-})\mathbb{1}_{\{s\geq\tau_a^-\}},
\end{aligned}$$

where $\tilde{\tau}_a^-$ and $\widetilde{X}$ are i.i.d. copies of τ_a^- and X with respect to $\mathbb{P}_{X_s}$. By applying (3.50), we get for all $x \in \mathbb{R}$ and $0 \leq s \leq t$

$$\begin{aligned}
\mathbb{E}_x\left(e^{-qt\wedge\tau_a^-} K(X_{t\wedge\tau_a^-})\Big|_{\mathcal{F}_s}\right) &= e^{-qs}\mathbb{1}_{\{s<\tau_a^-\}} K(X_s) + e^{-q\tau_a^-} K(X_{\tau_a^-})\mathbb{1}_{\{s\geq\tau_a^-\}} \\
&= e^{-qs\wedge\tau_a^-} K(X_{s\wedge\tau_a^-}).
\end{aligned}$$

Therefore, $\{e^{-qt\wedge\tau_a^-} K(X_{t\wedge\tau_a^-}), t \geq 0\}$ is a $\mathbb{P}_x$ martingale for all $x \in \mathbb{R}$.

From Lemma 3.17, $\mathbb{L}_X h(x,a) - qh(x,a)$ is well defined for all $x \in (a, \infty)$. Thus, by Itô's formula and the Doob's Meyer decomposition for supermartingle and submartingales, it follows that

$$\int_0^{t\wedge\tau_a^-} e^{-qs}\left(\mathbb{L}_X h(X_s,a) - qh(X_s,a)\right) ds = 0 \tag{3.51}$$

$\mathbb{P}_x$-a.s. for all $t \geq 0$ and $x \in \mathbb{R}$. Then for all $X_0 = x > a$, by dividing both sides of equation (3.51) by $t > 0$ and letting $t \downarrow 0$, it follows from the right continuity of the Lévy process that

$$\mathbb{L}_X h(x,a) - qh(x,a) = 0$$

for all $x > a$ and $a \in \mathbb{R}$ as required. □

Proof for Theorem 3.16. First suppose condition (i) in Theorem 3.16 is true. By Lemma 3.17, $h(x,a) = g(x)$ for all $x \leq a$. Also because $h(b,a) = g(b)$, we have

$$\mathbb{E}_x\left(e^{-q\tau_{a,b}} g(X_{\tau_{a,b}})\right) = \mathbb{E}_x\left(e^{-q\tau_{a,b}} h(X_{\tau_{a,b}}, a)\right) \qquad \text{for all } x \leq b. \tag{3.52}$$

Also from Proposition 3.18 and the optional sampling theorem, it follows that

$$\mathbb{E}_x\left(e^{-qt\wedge\tau_{a,b}} h(X_{t\wedge\tau_{a,b}}, a)\right) = h(x,a) \qquad \text{for all } x \in \mathbb{R}.$$

Note that $h(\cdot, a)$ is bounded on $(-\infty, b]$. So, by letting t go to ∞ and applying the bounded convergence theorem, we obtain

$$\mathbb{E}_x\left(e^{-q\tau_{a,b}} h(X_{\tau_{a,b}}, a)\right) = h(x,a) \qquad \text{for all } x \in \mathbb{R}. \tag{3.53}$$

By combining equations (3.52) and (3.53), it gives the result as required.

The case when $A_g(a) = 0$ can be seen from the definition (3.20), which completes the proof.

□

Proof for Lemma 3.19. Fix $x \in \mathbb{R}$. Note that $h(x,a) = g(x)$ for all $a > x$. So $a \mapsto h(x,a)$ is continuous on (x, ∞), and all partial derivatives of $h(x, \cdot)$ is equal to 0.

For $a < x$, from the definition (3.21) for $h(x,\cdot)$, it follows that

$$
\begin{aligned}
& h(x,a+\epsilon) - h(x,a) \\
= \; & \mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q}<a+\epsilon\}}\right) + \frac{q}{\Phi(q)}W^q(x-a-\epsilon)A_g(a+\epsilon) \\
& -\mathbb{E}_x\left(A_g(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q}<a\}}\right) - \frac{q}{\Phi(q)}W^q(x-a)A_g(a) \\
= \; & \int_{a-x\le y<a-x+\epsilon} A_g(y+x)\mathbb{P}(\underline{X}_{e_q}\in dy) \\
& +\frac{q}{\Phi(q)}\Big(W^q(x-a-\epsilon)A_g(a+\epsilon) - W^q(x-a)A_g(a)\Big).
\end{aligned} \tag{3.54}
$$

By letting ϵ go to 0, and using the continuity of A_g and W^q, together with the fact that the distribution of $\underline{X}_{e_q}$ has no atom apart from 0, we obtain that

$$\lim_{\epsilon\to 0}\left(h(x,a+\epsilon) - h(x,a)\right) = 0$$

Thus, $a \mapsto h(x,a)$ is continuous on $(-\infty, x)$. At $a = x$, by (3.54), the left limit would be

$$
\begin{aligned}
& \lim_{\epsilon\downarrow 0} h(x,x-\epsilon) \\
& = \lim_{\epsilon\downarrow 0}\left(\mathbb{E}\left(A_g(\underline{X}_{e_q}+x)\mathbb{1}_{\{\underline{X}_{e_q}<x-\epsilon-x\}}\right) + \frac{q}{\Phi(q)}W^q(x-x+\epsilon)A_g(x+\epsilon)\right) \\
& = g(x) - \lim_{\epsilon\downarrow 0}\mathbb{E}\left(A_g(\underline{X}_{e_q}+x)\mathbb{1}_{\{\underline{X}_{e_q}\ge-\epsilon\}}\right) + \frac{q}{\Phi(q)}W^q(0)A_g(x) \\
& = g(x) - \lim_{\epsilon\downarrow 0}\int_{y\ge-\epsilon} A_g(\underline{X}_{e_q}+x)\mathbb{P}(\underline{X}_{e_q}\in dy) + \frac{q}{\Phi(q)}W^q(0)A_g(x) \\
& = g(x) - \frac{q}{\Phi(q)}W^q(0)A_g(x) + \frac{q}{\Phi(q)}W^q(0)A_g(x) \\
& = g(x),
\end{aligned}
$$

which is the right limit of $h(x,\cdot)$ at $a = x$. Hence, we can conclude that $h(x,\cdot)$ is continuous in $\mathbb{R}$.

Next we turn to the differentiability of $h(x,\cdot)$ on $(-\infty, x)$. For $a < x$ and

$a \notin I_x$, by dividing both sides of (3.54) by $\epsilon > 0$ and letting $\epsilon \downarrow 0$, we have:

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \frac{h(x,a+\epsilon)-h(x,a)}{\epsilon} \\
&= \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\left(\int_{a-x\le y<a-x+\epsilon} A_g(y+x)\mathbb{P}(\underline{X}_{e_q}\in dy)\right) \\
&\quad + \lim_{\epsilon\downarrow 0} \frac{1}{\epsilon}\frac{q}{\Phi(q)}\Big(W^q(x-a-\epsilon)A_g(a+\epsilon) - W^q(x-a)A_g(a)\Big) \\
&= A_g(a)\frac{d}{dy}\mathbb{P}(\underline{X}_{e_q} < (a-x)+) - \frac{q}{\Phi(q)}\left(W^q\right)'((x-a)-)A_g(a) \\
&\quad + \frac{q}{\Phi(q)}W^q(x-a)A_g'(a).
\end{aligned}$$

Then it follows from equation (3.12) and the definition of $D_h^2(I)$ that

$$\begin{aligned}
\lim_{\epsilon\downarrow 0} \frac{h(x,a+\epsilon)-h(x,a)}{\epsilon} &= A_g(a)\left(-qW^q(x-a) + \frac{q}{\Phi(q)}\left(W^q\right)'((x-a)-)\right) \\
&\quad -\frac{q}{\Phi(q)}\left(W^q\right)'((x-a)-)A_g(a) + \frac{q}{\Phi(q)}W^q(x-a)A_g'(a) \\
&= W^q(x-a)\left(\frac{q}{\Phi(q)}A_g'(a) - qA_g(a)\right) \\
&= W^q(x-a)\left(\mathbb{L}_X g(a) - qg(a)\right).
\end{aligned}$$

By a similar argument as above, we can calculate the left derivative of $h(x,\cdot)$ for $a < x$ and $a \notin I_x$, which turns out to be equal to the right derivative. Thus, $h(x,\cdot)$ is differentiable for all $a < x$ and $a \notin I_x$. This completes the proof. □

Proof for Lemma 3.20. Proof of (i).

First, because $\mathbb{L}_X g(x) - qg(x)$ is continuous on $\mathbb{R}\setminus I$, there exist $\epsilon_0 > 0$ such that $\mathbb{L}_X g(x) - qg(x) < 0$ for all $x \in (a, a+\epsilon_0)$ and $(a, a+\epsilon_0)\cap I = \emptyset$.

If there does not exists $\epsilon > 0$ such that $h(x,a) > g(x)$ for all $x \in (a, a+\epsilon)$, then for all $\epsilon > 0$ there exists $x_\epsilon \in (a, a+\epsilon)$ such that $h(x_\epsilon, a) \le g(x_\epsilon)$. If the latter happens, a contradiction can be found. Without lose of generality, we can choose $\epsilon < \epsilon_0$. Then it follows from Lemma 3.17, Lemma 3.19 and the fundamental theorem of calculus that

$$\begin{aligned}
g(x_\epsilon) &= h(x_\epsilon, x_\epsilon) \\
&= h(x_\epsilon, a) + \int_a^{x_\epsilon} W^q(x_\epsilon - \tilde{a})(\mathbb{L}_X g(\tilde{a}) - qg(\tilde{a}))d\tilde{a} \\
&< h(x_\epsilon, a) \\
&\le g(x_\epsilon), \qquad (3.55)
\end{aligned}$$

where the first inequality is due to $W^q(x) > 0$ on $(0,\infty)$ and $\mathbb{L}_X g(\tilde{a}) - qg(\tilde{a}) < 0$ for all $\tilde{a} \in (a, x_\epsilon)$, and the last inequality are due to the construction of x_ϵ.

Clearly the equation (3.55) contradicts itself. Thus, x_ϵ does not exist, and the statement in the Lemma holds true.

(ii)

The proof of part (ii) can be done by a similar argument.

□

Proof for Proposition 3.21. First suppose that $\delta_1 = 1$, that is, X has unbounded variation. So, under the definition of $D^2_h(I)$, $h(\cdot, a)$ is in $C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus I_h)$, where

$$I_h \subseteq \{a\} \cup I,$$

and the left and right limits of the second derivative of $h(\cdot, a)$ exist on I_h.

So we can apply Itô's formula and get $\mathbb{P}$-a.s. for all $t \geq 0$,

$$e^{-qt}h(X_t, a) = h(X_0, a) + \int_0^t e^{-qs}(\mathbb{L}_X h(X_s, a) - qh(X_s, a))\mathbb{1}_{\{X_s \notin I_h\}} ds + M_t^h,$$

where M_t^h is a local martingale. Let τ_n be the localization sequence for it. Then we have for all $n \in \mathbb{N}$, $t \geq 0$ and $x \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-qt\wedge\tau_n} h(X_{t\wedge\tau_n}, a)\right) \\
&\quad = h(x, a) + \mathbb{E}_x\left(\int_0^{t\wedge\tau_n} e^{-qs}(\mathbb{L}_X h(X_s, a) - qh(X_s, a))\mathbb{1}_{\{X_s \notin I_h\}} ds\right).
\end{aligned}$$

As $h(x, a) = g(x)$ for all $x \in (-\infty, a]$, then for all $x \in (-\infty, a] \setminus I_h$

$$\mathbb{L}_X h(x, a) - qh(x, a) = \mathbb{L}_X g(x) - qg(x) \leq 0.$$

Together with Proposition 3.18, we have $\mathbb{L}_X h(x, a) - qh(x, a) \leq 0$ for all $x \in \mathbb{R} \setminus I_h$. Therefore,

$$\mathbb{E}_x\left(e^{-qt\wedge\tau_n} h(X_{t\wedge\tau_n}, a)\right) \leq h(x, a) \tag{3.56}$$

for all $x \in \mathbb{R}$, $t \geq 0$ and $n \in \mathbb{N}$.

As $A_g(a) \geq 0$, so $\frac{q}{\Phi(q)} A_g(a) W^q(x - a)$ goes to ∞ or 0 as $x \to \infty$. Together with the fact that,

$$\lim_{x\to\infty} \mathbb{E}_x\left(e^{-q\tau_a^-} g(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^- < \infty\}}\right) = 0,$$

we can conclude that $h(\cdot, a)$ is bounded below on $\mathbb{R}$. So, by Fatou's Lemma and equation (3.56), we get for all $t \geq 0$

$$\begin{aligned}\mathbb{E}_x\left(e^{-qt}h(X_t, a)\right) &= \mathbb{E}_x\left(\liminf_{n\to\infty} e^{-qt\wedge\tau_n}h(X_{t\wedge\tau_n}, a)\right) \\ &\leq \liminf_{n\to\infty} \mathbb{E}_x\left(e^{-qt\wedge\tau_n}h(X_{t\wedge\tau_n}, a)\right) \\ &\leq h(x, a).\end{aligned}$$

Finally by stationary and independent increments of Lévy processes, the stochastic process $\{e^{-qt}h(X_t, a), t \geq 0\}$ is a $\mathbb{P}_x$ supermartingale. □

Proof for Proposition 3.22. Suppose that there exists another averaging function $\tilde{A}_g$ of type (L) w.r.t. g and $\underline{X}_{e_q}$, and there exists $x_0 \in \mathbb{R}$ such that $A_g(x_0) \neq \tilde{A}_g(x_0)$. Let us denote by h and $\tilde{h}$ the functions constructed from A_g and $\tilde{A}_g$ as in equation (3.21), respectively. Then, from Lemma 3.17, it follows that

$$h(x, x_0) = g(x) = \tilde{h}(x, x_0)$$

for all $x \leq x_0$.

Now assume that $A_g(x_0) > \tilde{A}_g(x_0)$. Then from definition of h (3.20), we see that $h(x, x_0) > \tilde{h}(x, x_0)$ for all $x > x_0$. Let a and b be two real numbers such that $a < x_0 < b$. Then, thanks to Lemma 3.17, we can apply Itô's formula and get $\mathbb{P}$-a.s. for all $t \geq 0$,

$$\begin{aligned}e^{-qt}h(X_t, x_0) &= h(X_0, x_0) \\ &\quad + \int_0^t e^{-qs}(\mathbb{L}_X h(X_s, x_0) - qh(X_s, x_0))\mathbb{1}_{\{X_s \notin I_{x_0}\}} ds + M_t^h,\end{aligned}$$

where

$$I_{x_0} = \{x \in I : x \leq x_0\} \cup \{x_0\},$$

and M^h is the local martingale term. Let τ_n be the localization sequence for M^h. Then by the optional sampling theorem, we obtain for all $t \geq 0$,

$$\begin{aligned}&\mathbb{E}_{x_0}\left(e^{-qt\wedge\tau_n\wedge\tau_{a,b}}h(X_{t\wedge\tau_n\wedge\tau_{a,b}}, x_0)\right) \\ &= h(x_0, x_0) + \mathbb{E}_{x_0}\left(\int_0^{t\wedge\tau_n\wedge\tau_{a,b}} e^{-qs}(\mathbb{L}_X h(X_s, x_0) - qh(X_s, x_0))\mathbb{1}_{\{X_s \notin I_{x_0}\}} ds\right).\end{aligned} \tag{3.57}$$

Thanks to Proposition 3.18, $\mathbb{L}_X h(x, x_0) - qh(x, x_0) = 0$ for all $x > x_0$. Also because

$h(x,x_0)=g(x)$ for all $x\le x_0$, we can rewrite equation (3.57) for all $t\ge 0$ as follows,

$$\begin{aligned}
&\mathbb{E}_{x_0}\left(e^{-qt\wedge\tau_n\wedge\tau_{a,b}}h(X_{t\wedge\tau_n\wedge\tau_{a,b}},x_0)\right)\\
&=g(x_0)+\mathbb{E}_{x_0}\left(\int_0^{t\wedge\tau_n\wedge\tau_{a,b}}e^{-qs}(\mathbb{L}_X h(X_s,x_0)-qh(X_s,x_0))\mathbb{1}_{\{X_s\notin I_{x_0}\}}\mathbb{1}_{\{X_s<x_0\}}ds\right)\\
&=g(x_0)+\mathbb{E}_{x_0}\left(\int_0^{t\wedge\tau_n\wedge\tau_{a,b}}e^{-qs}(\mathbb{L}_X g(X_s)-qg(X_s))\mathbb{1}_{\{X_s\notin I_{x_0}\}}\mathbb{1}_{\{X_s<x_0\}}ds\right).
\end{aligned}$$

As $g\in D_h^2(I)$, both the left and right limits of $\mathbb{L}_X g(x)-qg(x)$ is bounded on $(-\infty,b]$, and $h(\cdot,x_0)$ is bounded on $(-\infty,b]$. By letting t and n go to ∞, and applying the dominated convergence theorem, we derive

$$\begin{aligned}
&\mathbb{E}_{x_0}\left(e^{-q\tau_{a,b}}h(X_{\tau_{a,b}},x_0)\right)\\
&\quad=g(x_0)+\mathbb{E}_{x_0}\left(\int_0^{\tau_{a,b}}e^{-qs}(\mathbb{L}_X g(X_s)-qg(X_s))\mathbb{1}_{\{X_s\notin I_{x_0}\}}\mathbb{1}_{\{X_s<x_0\}}ds\right).
\end{aligned}\tag{3.58}$$

By performing a similar calculation to $\tilde{h}(x,x_0)$, we also have

$$\begin{aligned}
&\mathbb{E}_{x_0}\left(e^{-q\tau_{a,b}}\tilde{h}(X_{\tau_{a,b}},x_0)\right)\\
&\quad=g(x_0)+\mathbb{E}_{x_0}\left(\int_0^{\tau_{a,b}}e^{-qs}(\mathbb{L}_X g(X_s)-qg(X_s))\mathbb{1}_{\{X_s\notin I_{x_0}\}}\mathbb{1}_{\{X_s<x_0\}}ds\right).
\end{aligned}\tag{3.59}$$

Thus, by comparing equations (3.58) and (3.59), we can conclude that

$$\mathbb{E}_{x_0}\left(e^{-q\tau_{a,b}}h(X_{\tau_{a,b}},x_0)\right)=\mathbb{E}_{x_0}\left(e^{-q\tau_{a,b}}\tilde{h}(X_{\tau_{a,b}},x_0)\right).\tag{3.60}$$

On the other hand, as $h(b,x_0)>\tilde{h}(b,x_0)$, and $h(x,x_0)=g(x)=\tilde{h}(x,x_0)$ for all $x\le x_0$, we derive

$$\mathbb{E}_{x_0}\left(e^{-q\tau_{a,b}}h(X_{\tau_{a,b}},x_0)\right)>\mathbb{E}_{x_0}\left(e^{-q\tau_{a,b}}\tilde{h}(X_{\tau_{a,b}},x_0)\right),$$

which clearly contradicts equation (3.60). Thus, such x_0 does not exist.

A similar argument as above can be applied to show that there does not exists any $x_0\in\mathbb{R}$ such that $A_g(x_0)<\tilde{A}_g(x_0)$. Then we can conclude that $A_g(x_0)=\tilde{A}_g(x_0)$ for all $x_0\in\mathbb{R}$ as required.

□

Proof for Lemma 3.30. Proof for (i).

From Assumption 3.24, there exists $x_0>a_{L,1}$ such that $h(x_0,a_{L,1})<g(x_0)$. From Lemma 3.19, $h(x_0,\cdot)$ is continuous. Thus, there exists $\epsilon>0$ such that

$h(x_0, a) < g(x_0)$ for all $a \in (a_{L,1} - \epsilon, a_{L,1}]$. Then, by the definition of $a_1^\star$, we can conclude that $a_1^\star < a_{L,1}$.

Next we show that $a_1^\star > -\infty$.

By part (iv) in Lemma 3.5, Remark 3.11 and Assumption 3.24, there exists $a_0 < a_{L,1}$ such that $A_1(a) > 0$ and $\mathbb{L}_X g(a) - qg(a) < -\epsilon_1$ for all $a \leq a_0$, where $\epsilon_1 \in (0, qg(-\infty))$. Then, as a result of Remark 3.23, the constant $d = \inf\{h(x, a_0) - g(x) : x \in \mathbb{R}\}$ is well defined in $\mathbb{R}$.

If $d \geq 0$, we have $h(x, a_0) \geq g(x)$ for all x, and by definition of $a_1^\star$ we have $a_1^\star \geq a_0 > -\infty$.

Now suppose that $d < 0$. Thanks to Lemma 3.20 and Remark 3.23, there exist x_1 and x_0 such that $a_0 < x_1 < x_0 < \infty$,

$$\{x \in \mathbb{R} : h(x, a_0) < g(x)\} \subseteq [x_1, x_0].$$

Then by the fundamental theorem of calculus and Lemma 3.19, we have for all $\tilde{a} < a_0$ and $x \in [x_1, x_0]$,

$$\begin{aligned} h(x, \tilde{a}) &= h(x, a_0) - \int_{\tilde{a}}^{a_0} \frac{\partial}{\partial a} h(x, a) da \\ &= h(x, a_0) - \int_{\tilde{a}}^{a_0} W^q(x - a)(\mathbb{L}_X g(a) - qg(a)) da \\ &\geq h(x, a_0) + \int_{\tilde{a}}^{a_0} W^q(x_1 - a_0)\epsilon_1 da \\ &\geq h(x, a_0) + W^q(x_1 - a_0)\epsilon_1(a_0 - \tilde{a}). \end{aligned}$$

We can choose $\tilde{a}$ such that $W^q(x_1 - a_0)\epsilon_1(a_0 - \tilde{a}) > -d$. Thus, we have for all $x \in [x_1, x_0]$

$$\begin{aligned} h(x, \tilde{a}) > h(x, a_0) - d &= g(x) + \big(h(x, a_0) - g(x)\big) - d \\ &\geq g(x) + \inf_x \big(h(x, a_0) - g(x)\big) - d \\ &= g(x). \end{aligned}$$

Also from Lemma 3.19, it follows that $h(x, \tilde{a}) \geq h(x, a_0) \geq g(x)$ for all $x \notin [x_1, x_0]$. Therefore, we can conclude that $h(x, \tilde{a}) \geq g(x)$ for all $x \in \mathbb{R}$. So $a_1^\star > \tilde{a} > -\infty$.

Proof of (ii).

If there exists $x_0 > a_1^\star$ such that $h(x_0, a_1^\star) < g(x_0)$, then, by the continuity of $h(x_0, \cdot)$ from Lemma 3.19, there exists $\epsilon > 0$ such that $h(x_0, a) < g(x_0)$ for all $a \in (a_1^\star - \epsilon, a_1^\star + \epsilon)$ which clearly contradicts the definition of $a_1^\star$. Therefore, we must have $h(x, a_1^\star) \geq g(x)$ for all $x \in \mathbb{R}$.

Next we prove by contradiction that for all $\epsilon > 0$ there exists $x \in (a_1^\star. a_1^\star + \epsilon)$ such that $h(x, a_1^\star) > g(x)$.

As $h(x, a_1^\star) \geq g(x)$ for all $x \in \mathbb{R}$, the contrary of the second statement in part (ii) is that, there exists $\epsilon > 0$ such that $h(x, a_1^\star) = g(x)$ for all $x \in (a_1^\star, a_1^\star + \epsilon)$. Without loss of generality we can choose $\epsilon \in (0, a_{L,1} - a_1^\star)$. Then, thanks to the absence of positive jumps and Proposition 3.18, we obtain for all $x \in (a_1^\star. a_1^\star + \epsilon)$

$$\mathbb{L}_X g(x) - qg(x) = \mathbb{L}_X h(x, a_1^\star) - qh(x, a_1^\star) = 0.$$

Hence, by Lemma 3.19 and the fundamental theorem of calculus, we can derive for all $x \in \mathbb{R}$

$$\begin{aligned} h(x, a_1^\star + \epsilon) &= h(x, a_1^\star) + \int_{a_1^\star}^{a_1^\star + \epsilon} W^q(x - a)\big(\mathbb{L}_X g(a) - qg(a)\big) da \\ &= h(x, a_1^\star) \\ &\geq g(x), \end{aligned} \tag{3.61}$$

where the last inequality is due to the first statement in part (ii). Clearly equation (3.61) contradicts the definition of $a_1^\star$, therefore, the second statement in part (ii) follows.

Proof of (iii).

If $A_1(a_1^\star) < 0$, from Remark 3.23 we obtain that $h(x, a_1^\star) < g(x)$ for all x large enough. This clearly contradicts to $h(x, a^\star) \geq g(x)$ in part (ii) of this Lemma. So we must have $A_1(a_1^\star) \geq 0$.

Proof of (iv).

Note that the contrary of the statement in part (iv) is that, there exists $\epsilon \in (0, a_{L,1} - a_1^\star)$, such $\mathbb{L}_X g(x) - qg(x) = 0$ for all $x \in (a_1^\star, a_1^\star + \epsilon)$. By Lemma 3.19 and the fundamental theorem of calculus we have for all $x \in (a_1^\star, a_1^\star + \epsilon)$

$$\begin{aligned} h(x, a_1^\star) &= h(x, x) - \int_{a_1^\star}^{x} W^q(x - \tilde{a})\big(\mathbb{L}_X g(\tilde{a}) - qg(\tilde{a})\big) d\tilde{a} \\ &= h(x, x) \\ &= g(x). \end{aligned}$$

Clearly this contradicts part (ii). Therefore, the statement in part (iv) holds true.

Proof of (v).

Suppose that $A_1(a_1^\star) = 0$. Then, by the definition of $b_1^\star$, we have $b_1^\star = \infty$. We will show that if $A_1(a_1^\star) > 0$, then $b_1^\star < \infty$. As a result of part (iii), we see that $A_1(a_1^\star) = 0$ becomes the necessary and sufficient condition for $b_1^\star = \infty$.

In order to do this, we prove that the set $\{b > a_1^\star : h(b, a_1^\star) = g(b)\}$ is non empty in the case when $A_1(a_1^\star) > 0$. Then, from Remark 3.23 it follows that the set $\{b > a_1^\star : h(b, a_1^\star) = g(b)\}$ is upper bounded. Thus,

$$b_1^\star = \sup\{b > a_1^\star : h(b, a_1^\star) = g(b)\} < \infty$$

as required.

The proof for the set $\{b > a_1^\star : h(b, a_1^\star) = g(b)\}$ being non empty in the case when $A_1(a_1^\star) > 0$ is done by contradiction. If the contrary is true, we have $h(x, a_1^\star) > g(x)$ for all $x > a_1^\star$, and the set $\{x \in \mathbb{R} : h(x, a) < g(x)\}$ is non empty for all $a \in (a_1^\star, a_{L,1})$.

Because of the continuity of A_1 and part (iv), we can choose $a_0 \in (a_1^\star, a_{L,1})$ such that $\mathbb{L}_X g(a_0) - qg(a_0) < 0$ and $A_1(a) > 0$ for all $a \in (a_1^\star, a_0)$. Then it follows from Remark 3.23 and Lemma 3.20 that, there exist a_1 and a_2 such that $a_0 < a_1 < a_2 < \infty$ and

$$\{x \in \mathbb{R} : h(x, a_0) < g(x)\} \subseteq [a_1, a_2].$$

Then, by Lemma 3.19 we obtain for all $x \in \mathbb{R}$ and $a \in (a_1^\star, a_0)$

$$\begin{aligned} h(x, a) &= h(x, a_0) - \int_a^{a_0} W^q(x - \tilde{a})(\mathbb{L}_X g(\tilde{a}) - qg(\tilde{a}))d\tilde{a} \\ &\geq h(x, a_0), \end{aligned}$$

where the inequality is due to $a_0 < a_{L,1}$. Therefore, we have for all $a \in (a_1^\star, a_0)$

$$\{x \in \mathbb{R} : h(x, a) < g(x)\} \subseteq \{x \in \mathbb{R} : h(x, a_0) < g(x)\} \subseteq [a_1, a_2], \tag{3.62}$$

and $\{x \in \mathbb{R} : h(x, a) < g(x)\}$ is not empty.

On the other hand, we know that $h(x, a_1^\star) > g(x)$ for all $x > a_1^\star$. Define d to be such that

$$d = \inf\{h(x, a_1^\star) - g(x) : x \in [a_1, a_2]\}.$$

And d is a strictly positive real number. By Lemma 3.19 and the fundamental theorem of calculus, we have for all $\epsilon \in (0, a_0 - a_1^\star)$ and $x \in [a_1. a_2]$

$$\begin{aligned} h(x, a_1^\star + \epsilon) &= h(x, a_1^\star) + \int_{a_1^\star}^{a_1^\star+\epsilon} W^q(x-a)\big(\mathbb{L}_X g(a) - qg(a)\big)da \\ &\geq h(x, a_1^\star) + \int_{a_1^\star}^{a_1^\star+\epsilon} W^q(a_2-a)\big(\mathbb{L}_X g(a) - qg(a)\big)da. \end{aligned}$$

Hence,

$$\begin{aligned} &\inf_{x\in[a_1,a_2]}\{h(x, a_1^\star+\epsilon) - g(x)\} \\ &\geq \inf_{x\in[a_1,a_2]}\{h(x, a_1^\star) - g(x)\} + \int_{a_1^\star}^{a_1^\star+\epsilon} W^q(a_2-a)\big(\mathbb{L}_X g(a) - qg(a)\big)da \\ &= d + \int_{a_1^\star}^{a_1^\star+\epsilon} W^q(a_2-a)\big(\mathbb{L}_X g(a) - qg(a)\big)da. \end{aligned}$$

As the second term in the above equation is continuous in ϵ, we can choose $\epsilon > 0$ such that

$$\int_{a_1^\star}^{a_1^\star+\epsilon} W^q(a_2-a)\big(\mathbb{L}_X g(a) - qg(a)\big)da > -d/2.$$

Therefore, we have

$$\inf_{x\in[a_1,a_2]}\{h(x, a_1^\star+\epsilon) - g(x)\} > 0.$$

This clearly contradicts equation (3.62). So we can not have that the set $\{b > a_1^\star : h(b, a_1^\star) = g(b)\}$ is empty in the case when $A_1(a_1^\star) > 0$.

Proof of (vi).

Suppose that $b_1^\star < \infty$ and $h(b_1^\star, a_1^\star) = V_1(b_1^\star) > g(b_1^\star)$. Note that by definition of $b_1^\star$, it follows from the continuity of both $h(\cdot, a_1^\star)$ and $g(\cdot)$ that, there exists $\epsilon > 0$ such that $h(x, a_1^\star) > g(x)$ for all $x \in (b_1^\star - \epsilon, \infty)$ with $b_1^\star - \epsilon > a_1^\star$, which clearly gives a contradiction to the definition of $b_1^\star$. Therefore together with part (ii) we can conclude that $g(b_1^\star) = V_1(b_1^\star)$ in the case $b_1^\star < \infty$.

Next, we turn to the differentiability of V_1 and g at $b_1^\star$. First suppose that $V_1'(b_1^\star-) > g'(b_1^\star)$. Note that $V_1'(b_1^\star-) = \frac{\partial h}{\partial x}(b_1^\star, a_1^\star)$. As $g(b_1^\star) = V_1(b_1^\star) = h(b_1^\star, a_1^\star)$, then there exists $\epsilon > 0$ such that $\frac{\partial h}{\partial x}(x, a_1^\star) > g'(x)$ for all $x \in (b_1^\star - \epsilon, b_1^\star)$. Then, by the fundamental theorem of calculus, we can conclude that $h(b_1^\star - \epsilon, a_1^\star) < g(b_1^\star - \epsilon)$, which contradicts the fact that $h(x, a_1^\star) \geq g(x)$ for all $x \in \mathbb{R}$.

Similar contradiction can be found for the case when $V_1'(b_1^\star-) < g'(b_1^\star)$. Therefore we can conclude that $V_1'(b_1^\star-) = g'(b_1^\star)$.

$\square$

Proof for Lemma 3.31. As $g \in D^2(I_g)$, it follows from Lemma 3.17 and Lemma 3.30 that $V_1 \in D^2(I_{V_1})$.

Next we calculate the right limit of $\mathbb{L}_X V_1(b_1^\star) - qV_1(b_1^\star)$ given that $b_1^\star < \infty$. First suppose that $\delta_2 = 1$, that is $\int_{-\infty}^0 (1 \wedge |y|)\Pi(dy) = \infty$. In this case for all $\epsilon \in (0,1)$,

$$\begin{aligned}
&\mathbb{L}_X V_1(b_1^\star + \epsilon) - qV_1(b_1^\star + \epsilon)\\
&= \mu V_1'(b_1^\star + \epsilon) + \frac{\sigma^2}{2} V_1''(b_1^\star + \epsilon) - qV_1(b_1^\star + \epsilon) \qquad (3.63)\\
&\quad + \int_{-\infty}^0 \Big(V_1(b_1^\star + y + \epsilon) - V_1(b_1^\star + \epsilon) - yV_1'(b_1^\star + \epsilon)\mathbb{1}_{\{y>-1\}} \Big)\Pi(dy).
\end{aligned}$$

As $V_1 \in D^2(I_{V_1})$, it follows from part (i) in Lemma 3.5 that for all $y < 0$ and $\epsilon \in (0,1)$

$$\begin{aligned}
\left| V_1(b_1^\star + y + \epsilon) - V_1(b_1^\star + \epsilon) - yV_1'(b_1^\star + \epsilon)\mathbb{1}_{\{y>-1\}} \right| &\leq c_1(1 \wedge y^2)(e^{c(b_1^\star + \epsilon)} + d)\\
&\leq c_1(1 \wedge y^2)(e^{c(b_1^\star + 1)} + d)
\end{aligned}$$

for some $c_1 > 0$, $c \in (0, \Phi(q))$ and $d > 0$. Therefore, as $\int_{-\infty}^0 (1 \wedge |y|^2)\Pi(dy) < \infty$, by letting ϵ goes to 0 and applying dominated convergence theorem, we obtain

$$\begin{aligned}
&\mathbb{L}_X V_1(b_1^\star +) - qV_1(b_1^\star +)\\
&= \mu V_1'(b_1^\star +) + \frac{\sigma^2}{2} V_1''(b_1^\star +) - qV_1(b_1^\star +) \qquad (3.64)\\
&\quad + \int_{-\infty}^0 \Big(V_1((b_1^\star + y)+) - V_1(b_1^\star +) - yV_1'(b_1^\star +)\mathbb{1}_{\{y>-1\}} \Big)\Pi(dy).
\end{aligned}$$

As $\delta_2 = 1$, then thanks to Lemma 3.17 and part (vi) in Lemma 3.30, V_1 is continuously differentiable on $\mathbb{R}$, and $V_1''(b_1^\star +)$ is well defined. Therefore, we can rewrite equation (3.64) as

$$\begin{aligned}
&\mathbb{L}_X V_1(b_1^\star +) - qV_1(b_1^\star)\\
&= \mu V_1'(b_1^\star -) + \frac{\sigma^2}{2} V_1''(b_1^\star +) - qV_1(b_1^\star)\\
&\quad + \int_{-\infty}^0 \Big(V_1((b_1^\star + y)-) - V_1(b_1^\star -) - yV_1'(b_1^\star -)\mathbb{1}_{\{y>-1\}} \Big)\Pi(dy)\\
&= \mathbb{L}_X V_1(b_1^\star -) - qV_1(b_1^\star) + \frac{\sigma^2}{2} \left(g''(b_1^\star) - \frac{\partial^2}{\partial x^2} h(b_1^\star, a_1^\star) \right)
\end{aligned}$$

As $V_1(x) = h(x, a_1^\star)$ on the set $(-\infty, b_1^\star]$, it follows from Proposition 3.18 that

$$\mathbb{L}_X V_1(b_1^\star -) - qV_1(b_1^\star -) = \mathbb{L}_X h(b_1^\star, a_1^\star) - qh(b_1^\star, a_1^\star) = 0.$$

Thus,

$$\mathbb{L}_X V_1(b_1^\star +) - qV_1(b_1^\star) \quad = \quad \frac{\sigma^2}{2}\left(g''(b_1^\star) - \frac{\partial^2}{\partial x^2} h(b_1^\star, a_1^\star)\right) \leq 0,$$

since $h(x, a^\star) > g(x)$ for all $x > b_1^\star$. So, the right limit exists and is non positive at $b^\star$.

The case when $\delta_2 = 0$ can be done using a similar argument. This completes the proof.

□

Proof for Corollary 3.32. The proof is done by contradiction.

Suppose there exists $x_0 \in \mathbb{R}$ such that $h(x_0, a_1^\star) \leq 0$. In the case where $b_1^\star = \infty$, we have $A_1(a_1^\star) = 0$. Then it follows from Remark 3.23 that $h(\cdot, a_1^\star)$ is bounded below and converges to 0 as $x \to \infty$. As $\lim_{x \to -\infty} g(x) > 0$, we can choose a such that $a < x_0$, and $g(x) > 0$ for all $x \leq a$. Therefore, we obtain the following equation

$$\mathbb{E}_{x_0}\left(e^{-q\tau_a^-} h(X_{\tau_a^-}, a_1^\star)\right) > 0. \tag{3.65}$$

On the other hand, thanks to Proposition 3.21 and Lemma 3.30, the process $\{e^{-qt}h(X_t, a_1^\star), t \geq 0\}$ is a supermartingale. So by optional sampling theorem, we have for all $t \geq 0$

$$\mathbb{E}_{x_0}\left(e^{-qt\wedge\tau_a^-} h(X_{t\wedge\tau_a^-}, a_1^\star)\right) \leq h(x_0, a_1^\star) \leq 0.$$

Then, it follows from the bounded convergence theorem that

$$\mathbb{E}_{x_0}\left(e^{-q\tau_a^-} h(X_{\tau_a^-}, a_1^\star)\right) \leq 0,$$

which clearly contradicts equation (3.65). Therefore, in the case $b_1^\star = \infty$ there does not exist $x_0 \in \mathbb{R}$ such that $h(x_0, a_1^\star) \leq 0$.

Now suppose that $b_1^\star < \infty$. As a result of part (v) in Lemma 3.30, $A_1(a_1^\star) > 0$. That is, $h(x, a_1^\star)$ goes to ∞ as x goes to ∞. So we can choose a_1 and a_2 such that $a_1 < x_0 < a_2$, and $h(x, a_1^\star) > 0$ for all $x \in (-\infty, a_1] \cup [a_2, \infty)$. So we have

$$\mathbb{E}_{x_0}\left(e^{-q\tau_{a_1,a_2}} h(X_{\tau_{a_1,a_2}}, a_1^\star)\right) > 0. \tag{3.66}$$

On the other hand, same as above, by Proposition 3.21 and Lemma 3.30, the stochastic process $\{e^{-qt}h(X_t, a_1^\star), t \geq 0\}$ is a supermartingale. Thus, by the optional sampling theorem, we have for all $t \geq 0$

$$\mathbb{E}_{x_0}\left(e^{-qt\wedge\tau_{a_1,a_2}}h(X_{t\wedge\tau_{a_1,a_2}}, a_1^\star)\right) \leq h(x_0, a_1^\star) \leq 0.$$

Thus, it follows from bounded convergence theorem that

$$\mathbb{E}_{x_0}\left(e^{-q\tau_{a_1,a_2}}h(X_{\tau_{a_1,a_2}}, a_1^\star)\right) \leq 0,$$

which clearly contradicts (3.66). So in the case when $b_1^\star < \infty$, there does not exist $x_0 \in \mathbb{R}$ such that $h(x_0, a_1^\star) \leq 0$ either.

Overall, we must have $h(x, a_1^\star) > 0$ for all $x \in \mathbb{R}$.

Next we show the existence of $x_0 \in (a_1^\star, b_1^\star)$ such that $\mathbb{L}_X g(x_0) - qg(x_0) > 0$. First suppose that $b_1^\star < \infty$ and $\mathbb{L}_X g(x) - qg(x) \leq 0$ for all $x \in (a_1^\star, b_1^\star)$. Then by Itô's formula and Theorem 3.4, we obtain for all $x \in \mathbb{R}$ and $t \geq 0$ that

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-qt\wedge\tau_{a_1^\star,b_1^\star}}g(X_{t\wedge\tau_{a_1^\star,b_1^\star}})\right) \\
=\ & g(x) + \mathbb{E}_x\left(\int_0^{t\wedge\tau_{a_1^\star,b_1^\star}} e^{-qs}\left(\mathbb{L}_X g(X_s) - qg(X_s)\right) ds\right) \\
\leq\ & g(x).
\end{aligned}$$

Thus, by the bounded convergence theorem we have for all $x \in \mathbb{R}$

$$\mathbb{E}_x\left(e^{-q\tau_{a_1^\star,b_1^\star}}g(X_{\tau_{a_1^\star,b_1^\star}})\right) = \lim_{t\to\infty}\mathbb{E}_x\left(e^{-qt\wedge\tau_{a_1^\star,b_1^\star}}g(X_{t\wedge\tau_{a_1^\star,b_1^\star}})\right) \leq g(x).$$

Since $h(x, a_1^\star) = g(x)$ on the set $(-\infty, a_1^\star] \cup \{b_1^\star\}$, then for all $x \leq b_1^\star$

$$\mathbb{E}_x\left(e^{-q\tau_{a_1^\star,b_1^\star}}h(X_{\tau_{a_1^\star,b_1^\star}}, a_1^\star)\right) = \mathbb{E}_x\left(e^{-q\tau_{a_1^\star,b_1^\star}}g(X_{\tau_{a_1^\star,b_1^\star}})\right) \leq g(x). \tag{3.67}$$

From the Proposition 3.18 and the bounded convergence theorem, we obtain that for all $x \in \mathbb{R}$ that

$$\mathbb{E}_x\left(e^{-q\tau_{a_1^\star,b_1^\star}}h(X_{\tau_{a_1^\star,b_1^\star}}, a_1^\star)\right) = \lim_{t\to\infty}\mathbb{E}_x\left(e^{-qt\wedge\tau_{a_1^\star,b_1^\star}}h(X_{t\wedge\tau_{a_1^\star,b_1^\star}}, a_1^\star)\right) = h(x, a_1^\star). \tag{3.68}$$

By combining equations (3.67) and (3.68), we arrive at

$$h(x, a_1^\star) \leq g(x) \qquad \text{for all } x \leq b_1^\star.$$

which clearly contradicts part (ii) in Lemma 3.30. Therefore, in the case $b_1^\star < \infty$, there must exist $x_0 \in (a_1^\star, b_1^\star)$ such that $\mathbb{L}_X g(x_0) - qg(x_0) > 0$.

Finally, we consider the case $b_1^\star = \infty$. If $\mathbb{L}_X g(x) - qg(x) \leq 0$ on the set $(a_1^\star, \infty)$, then if follows from part (i) in Lemma 3.30 that $a_{L,1}$ is not well defined. Therefore, in the case $b_1^\star = \infty$, there must exist $x_0 \in (a_1^\star, \infty)$ such that $\mathbb{L}_X g(x_0) - qg(x_0) > 0$. □

Proof for Theorem 3.26. By part (i) in Lemma 3.30, we have $a^\star \in \mathbb{R}$.

(i)

By using Lemma 3.30, Proposition 3.21 and Theorem 3.16, the result in part (i) follows from Lemma 3.2 .

(ii)

First suppose that $b_1^\star = \infty$. From Theorem 3.16 it follows that for all $x \in \mathbb{R}$

$$h(x, a_1^\star) = \mathbb{E}_x \left(e^{-q\tau^-_{a_1^\star}} g(X_{\tau^-_{a_1^\star}}) \right).$$

Finally, with the help from Lemma 3.30 and Proposition 3.21, we can apply guess and verification lemma and conclude that $(V_1, \tau_1^\star)$ is a solution.

Next we consider the case $b_1^\star < \infty$. From Theorem 3.16 and part (ii) in Lemma 3.30, it follows that

$$\mathbb{E}_x \left(e^{-q\tau_{a_1^\star, b_1^\star}} g(X_{\tau_{a_1^\star, b_1^\star}}) \right) = V_1(x)$$

for all $x \in \mathbb{R}$, and $V_1(x) \geq g(x)$ for all $x \in \mathbb{R}$. Therefore, we only need to prove that $\{e^{-qt} V_1(X_t), t \geq 0\}$ is a supermartingale, then by guess and verification lemma we can conclude that $(V_1, \tau_1^\star)$ is a solution.

Because $\mathbb{L}_X V_1(x) - qV_1(x) \leq 0$ for all $x > b_1^\star$, and thanks to part (i) in Lemma 3.30 and Proposition 3.18, we have $\mathbb{L}_X V_1(x) - qV_1(x) \leq 0$ for all $x \in \mathbb{R} \setminus \{a_1^\star, b_1^\star\}$. So we can apply Itô's formula and get $\mathbb{P}$-a.s. for all $t \geq 0$,

$$\begin{aligned} e^{-qt} V_1(X_t) &= V_1(X_0) + \int_0^t e^{-qs} (\mathbb{L}_X V_1(X_s) - qV_1(X_s)) \mathbb{1}_{\{X_s \notin \{a_1^\star, b_1^\star\}\}} ds + M_t^{V_1} \\ &\leq V_1(X_0) + M_t^{V_1}, \end{aligned}$$

where $M_t^{V_1}$ is a local martingale. Furthermore, by Lemma 3.31 and Theorem 3.4 it is a true martingale. Thus, we have for all $t \geq 0$ and $x \in \mathbb{R}$

$$\mathbb{E}_x \left(e^{-qt} V_1(X_t) \right) \leq V_1(x).$$

Finally by stationary and independent increments of Lévy processes, the stochastic process $\{e^{-qt}V_1(X_t), t \geq 0\}$ is a $\mathbb{P}_x$ supermartingale. This completes the proof. $\square$

Proof for Lemma 3.36.

(i)

By definition of h_2, $h_2(x, b_1^\star) = V_1(x) = h(x, a_1^\star)$ for all $x \leq b_1^\star$. So we only need to prove that $h_2(x, b_1^\star) = h(x, a_1^\star)$ for all $x > b_1^\star$. This is done by contradiction.

Now suppose there exists $b > b_1^\star$ such that $h(b, a_1^\star) > h_2(x, b_1^\star)$. By Proposition 3.18, we have for all $t \geq 0$

$$\mathbb{E}_{b_1^\star}\left(e^{-qt\wedge\tau_{a_1^\star,b}}h(X_{t\wedge\tau_{a_1^\star,b}}, a_1^\star)\right) = h(b_1^\star, a_1^\star).$$

As $h(\cdot, a_1^\star)$ is bounded on the set $(-\infty, b]$, by letting t go to ∞ and applying the bounded convergence theorem, we obtain

$$\mathbb{E}_{b_1^\star}\left(e^{-q\tau_{a_1^\star,b}}h(X_{\tau_{a_1^\star,b}}, a_1^\star)\right) = h(b_1^\star, a_1^\star) = g(b_1^\star), \tag{3.69}$$

where the last equality is due to $b_1^\star < \infty$ and Lemma 3.30.

Thanks to Proposition 3.18, $\mathbb{L}_X h(x, a_1^\star) - qh(x, a_1^\star) = 0$ for all $x > a_1^\star$, and $\mathbb{L}_X h_2(x, b_1^\star) - qh_2(x, b_1^\star) = 0$ for all $x > b_1^\star$. Thus, $\mathbb{L}_X h_2(x, b_1^\star) - qh_2(x, b_1^\star) = 0$ for all $x \in (a_1^\star, b_1^\star) \cup (b_1^\star, \infty)$. Also note that, by Lemma 3.17 and part (vi) in Lemma 3.30, $h_2(\cdot, b_1^\star)$ is continuously differentiable at $x = b_1^\star$. Therefore, we can apply Itô's formula and derive for all $t \geq 0$ $\mathbb{P}$-a.s.

$$\begin{aligned} &e^{-qt}h_2(X_t, b_1^\star) \\ &= h_2(X_0, b_1^\star) + \int_0^t e^{-qs}\left(\mathbb{L}_X h_2(X_s, b_1^\star) - qh_2(X_s, b_1^\star)\right)\mathbb{1}_{\{X_s \notin \{a_1^\star, b_1^\star\}\}}ds + M_t^{h_2} \end{aligned}$$

where M^{h_2} is the local martingale term. Let τ_n be the localization sequence for it. Then, as $\mathbb{L}_X h_2(x, b_1^\star) - qh_2(x, b_1^\star) = 0$ for all $x \in (a_1^\star, b_1^\star) \cup (b_1^\star, \infty)$, by the optional sampling Theorem we obtain for all $t \geq 0$

$$\mathbb{E}_{b_1^\star}\left(e^{-qt\wedge\tau_{a_1^\star,b}\wedge\tau_n}h_2(X_{t\wedge\tau_{a_1^\star,b}\wedge\tau_n}, b_1^\star)\right) = h_2(b_1^\star, b_1^\star).$$

Again, as $h_2(\cdot, b_1^\star)$ is bounded on $(-\infty, b]$, by letting $t \to \infty$ and $n \to \infty$ and applying the bounded convergence theorem, we can derive

$$\mathbb{E}_{b_1^\star}\left(e^{-q\tau_{a_1^\star,b}}h_2(X_{\tau_{a_1^\star,b}}, b_1^\star)\right) = h_2(b_1^\star, b_1^\star) = V_1(b_1^\star) = g(b_1^\star). \tag{3.70}$$

By combining equations (3.69) and (3.70) we obtain

$$\mathbb{E}_{b_1^\star}\left(e^{-q\tau_{a_1^\star,b}}h_2(X_{\tau_{a_1^\star,b}},b_1^\star)\right)=\mathbb{E}_{b_1^\star}\left(e^{-q\tau_{a_1^\star,b}}h(X_{\tau_{a_1^\star,b}},a_1^\star)\right). \tag{3.71}$$

However, as $h(b,a_1^\star) > h_2(x,b_1^\star)$, and $h(x,a_1^\star) = h_2(x,b_1^\star)$ for all $x \le b_1^\star$. we derive that

$$\mathbb{E}_{b_1^\star}\left(e^{-q\tau_{a_1^\star,b}}h_2(X_{\tau_{a_1^\star,b}},b_1^\star)\right)<\mathbb{E}_{b_1^\star}\left(e^{-q\tau_{a_1^\star,b}}h(X_{\tau_{a_1^\star,b}},a_1^\star)\right),$$

which clearly contradicts equation (3.71). Therefore, $h(x,a_1^\star) \le h_2(x,b_1^\star)$ for all $x > b_1^\star$. The argument for $h(x,a_1^\star) \ge h_2(x,b_1^\star)$ can be done in a similar way, which allows us to conclude that $h(x,a_1^\star) = h_2(x,b_1^\star)$ for all $x > b_1^\star$ as required.

(ii)

The fact $a_2^\star \ge b_1^\star$ can be seen directly from part (i). The proof for $a_2^\star < a_{L,2}$ can be done using a similar argument as in Lemma 3.30.

The proof for *(iii), (iv), (v), (vi) and (vii)* can be done using similar arguments as in the proof for Lemma 3.30.

(viii)

Suppose that $a_2^\star = b_1^\star$. Then, it follows from part (i) and the definition of $b_1^\star$ that $\mathcal{N}_2 = \emptyset$. Thus, by definition, $b_2^\star = \infty$. Also thanks to part (i), $h_2(\cdot, a_2^\star)$ diverges to ∞ as $x \to \infty$. So, from Remark 3.23, we can conclude that $A_2(a_2^\star) = 0$. Clearly, this contradicts part (vi). Therefore, $a_2^\star \ne b_1^\star$. Finally, it follows from part (ii) that $a_2^\star > b_1^\star$.

The statement of $b_2^\star > a_2^\star$ can be seen from the definition of $b_2^\star$. And this completes the proof.

□

Proof for Theorem 3.38.

(i)

Note that for all $x \notin B_n$, $\tau_{B_n} = 0$ $\mathbb{P}_x$-a.s.. Then,

$$\mathbb{E}_x\left(e^{-q\tau_{B_n}}f(X_{\tau_{B_n}})\right)=f(x)=f_1(x)$$

for all $x \notin B_n$. Thus, we only need to prove equation (3.30) for all $x \in B_n$. And this is done by induction for each interval $[a_i, b_i]$, $1 \le i \le n$.

First suppose that $a_1 > -\infty$. As $f = f_1$ on B_n^c, it follows from the continuity of f and f_1 that $f(x) = f_1(x)$ for all $x \in \{a_1, b_1, ..., a_n, b_n\}$. Therefore, because of the absence of positive jumps, we obtain that for all $x \in [a_1, b_1]$

$$\mathbb{E}_x\left(e^{-q\tau_{B_n}} f(X_{\tau_{B_n}})\right) = \mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}} f(X_{\tau_{a_1,b_1}})\right) = \mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}} f_1(X_{\tau_{a_1,b_1}})\right).$$

Also, as f_1 is bounded on $(-\infty, b_1]$, then from the martingale property of $e^{-qt} f_1(X_t)$ on $[a_1, b_1]$ and the bounded convergence theorem, it follows that for all $x \in [a_1, b_1]$

$$\begin{aligned}
\mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}} f_1(X_{\tau_{a_1,b_1}})\right) &= \mathbb{E}_x\left(\lim_{t\to\infty} e^{-qt\wedge\tau_{a_1,b_1}} f_1(X_{t\wedge\tau_{a_1,b_1}})\right)\\
&= \lim_{t\to\infty} \mathbb{E}_x\left(e^{-qt\wedge\tau_{a_1,b_1}} f_1(X_{t\wedge\tau_{a_1,b_1}})\right)\\
&= f_1(x).
\end{aligned}$$

By combining the above two equations, we derive that for all $x \in [a_1, b_1]$

$$\mathbb{E}_x\left(e^{-q\tau_{B_n}} f(X_{\tau_{B_n}})\right) = \mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}} f_1(X_{\tau_{a_1,b_1}})\right) = f_1(x).$$

Define $\tau_{B_i} = \inf\{t \geq 0 : X_t \notin B_i\}$ and $B_i = \left\{\bigcup_{j=1}^{i}[a_j, b_j]\right\} \cap \mathbb{R}$ for all $1 \leq i \leq n$. And suppose that

$$\mathbb{E}_x\left(e^{-q\tau_{B_n}} f(X_{\tau_{B_n}})\right) = \mathbb{E}_x\left(e^{-q\tau_{B_{m-1}}} f(X_{\tau_{B_{m-1}}})\right) = f_1(x) \tag{3.72}$$

for all $x \in \bigcup_{i=1}^{m-1}[a_i, b_i]$, $1 < m < n$, where the first equality is due to the absence of positive jumps.

Now we prove for the case where $x \in [a_m, b_m]$. Recall that θ is a shift operator. So, thanks to the strong Markov property and the absence of positive jumps, we have $\mathbb{P}_x$-a.s. for all $x \in [a_m, b_m]$,

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-q\tau_{B_m}} f(X_{\tau_{B_m}}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}} \Big| \mathcal{F}_{\tau_{a_m}^-}\right)\\
&= \mathbb{E}_x\left(e^{-q(\tau_{a_m}^- + \tau_{B_m}\circ\theta_{\tau_{a_m}^-})} f(X_{\tau_{a_m}^- + \tau_{B_m}\circ\theta_{\tau_{a_m}^-}}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}} \Big| \mathcal{F}_{\tau_{a_m}^-}\right)\\
&= e^{-q\tau_{a_m}^-} \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}} \mathbb{E}_x\left(e^{-q\tau_{B_m}\circ\theta_{\tau_{a_m}^-}} f(X_{\tau_{a_m}^- + \tau_{B_m}\circ\theta_{\tau_{a_m}^-}}) \Big| \mathcal{F}_{\tau_{a_m}^-}\right)\\
&= e^{-q\tau_{a_m}^-} \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}} \mathbb{E}_{X_{\tau_{a_m}^-}}\left(e^{-q\tilde{\tau}_{B_m}} f(\widetilde{X}_{\tilde{\tau}_{B_m}})\right)\\
&= e^{-q\tau_{a_m}^-} \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}} \mathbb{E}_{X_{\tau_{a_m}^-}}\left(e^{-q\tilde{\tau}_{B_{m-1}}} f(\widetilde{X}_{\tilde{\tau}_{B_{m-1}}})\right)\\
&= e^{-q\tau_{a_m}^-} \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}} f_1(X_{\tau_{a_m}^-})
\end{aligned} \tag{3.73}$$

where $\widetilde{X}$ is an independent copy of the Lévy process X, and $\tilde{\tau}_{B_{m-1}}$ is the first time that $\widetilde{X}$ exits from B_{m-1}, and the last equality is due to the induction step (3.72). Therefore, it follows from tower property and equation (3.73) that for all $x \in [a_m, b_m]$,

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-q\tau_{B_m}} f(X_{\tau_{B_m}}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}}\right) \\
&= \mathbb{E}_x\left(\mathbb{E}_x\left(e^{-q\tau_{B_m}} f(X_{\tau_{B_m}}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}} \Big| \mathcal{F}_{\tau_{a_m}^-}\right)\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_{a_m}^-} \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}} f_1(X_{\tau_{a_m}^-})\right). \qquad (3.74)
\end{aligned}$$

So by using the above equation and the fact $f = f_1$ on B_n^c, we have for all $x \in [a_m, b_m]$,

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-q\tau_{B_m}} f(X_{\tau_{B_m}}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}}\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_{B_m}} f(X_{\tau_{B_m}}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}}\right) \\
&\quad + \mathbb{E}_x\left(e^{-q\tau_{B_m}} f(X_{\tau_{B_m}}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \notin \bigcup_{i=1}^{m-1}[a_i,b_i]\}}\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_{a_m}^-} f_1(X_{\tau_{a_m}^-}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \in \bigcup_{i=1}^{m-1}[a_i,b_i]\}}\right) \\
&\quad + \mathbb{E}_x\left(e^{-q\tau_{a_m}^-} f_1(X_{\tau_{a_m}^-}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}} \mathbb{1}_{\{X_{\tau_{a_m}^-} \notin \bigcup_{i=1}^{m-1}[a_i,b_i]\}}\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_{a_m}^-} f_1(X_{\tau_{a_m}^-}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}}\right) \qquad (3.75)
\end{aligned}$$

Thus, we obtain that for all $x \in [a_m, b_m]$,

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-q\tau_{B_n}} f(X_{\tau_{B_n}})\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_{B_m}} f(X_{\tau_{B_m}})\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_{B_m}} f(X_{\tau_{B_m}}) \mathbb{1}_{\{\tau_{b_m}^+ < \tau_{a_m}^-\}}\right) + \mathbb{E}_x\left(e^{-q\tau_{B_m}} f(X_{\tau_{B_m}}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}}\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_{b_m}^+} f_1(X_{\tau_{b_m}^+}) \mathbb{1}_{\{\tau_{b_m}^+ < \tau_{a_m}^-\}}\right) + \mathbb{E}_x\left(e^{-q\tau_{a_m}^-} f_1(X_{\tau_{a_m}^-}) \mathbb{1}_{\{\tau_{b_m}^+ > \tau_{a_m}^-\}}\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_{a_m,b_m}} f_1(X_{\tau_{a_m,b_m}})\right), \qquad (3.76)
\end{aligned}$$

where the third equality is due to $f = f_1$ on $B_n^c \cup \{a_1, b_1, \dots, a_n, b_n\}$ and equation (3.75). Finally, by the martingale property of $\{e^{-qt\wedge\tau_{a_m,b_m}} f_1(X_{t\wedge\tau_{a_m,b_m}}), t \geq 0\}$ and the bounded convergence theorem, we have for all $x \in [a_m, b_m]$

$$\mathbb{E}_x \left(e^{-q\tau_{a_m,b_m}} f_1(X_{\tau_{a_m,b_m}}) \right) = f_1(x).$$

This completes the proof for part (i) in the case $a_1 > -\infty$.

Now we consider the case that $a_1 = -\infty$. As both $\lim_{x \to -\infty} f_1(x)$ and $\lim_{x \to -\infty} f(x)$ exist, we must have for all $x \leq b_1$

$$\mathbb{E}_x \left(e^{-q\tau_{b_1}^+} f(X_{\tau_{b_1}^+}) \mathbb{1}_{\{\tau_{b_1}^+ = \infty\}} \right) = \mathbb{E}_x \left(e^{-q\tau_{b_1}^+} f_1(X_{\tau_{b_1}^+}) \mathbb{1}_{\{\tau_{b_1}^+ = \infty\}} \right) = 0.$$

Together with $f_1(b_1) = f(b_1)$, we obtain

$$\mathbb{E}_x \left(e^{-q\tau_{b_1}^+} f(X_{\tau_{b_1}^+}) \right) = \mathbb{E}_x \left(e^{-q\tau_{b_1}^+} f_1(X_{\tau_{b_1}^+}) \right).$$

Therefore, a similar argument as above can be applied to derive equation (3.30) for the case $a_1 = \infty$.

(ii)

Suppose $b_n = \infty$. Then from part (i), it follows that equation (3.30) holds true for all $x < a_n$. For all $x \geq a_n$, by Lemma 3.6, we obtain that for all $x \in [a_n, \infty)$

$$\mathbb{E}_x \left(e^{-q\tau_{B_n}} f(X_{\tau_{B_n}}) \mathbb{1}_{\{\tau_{B_n} = \infty\}} \right) = \mathbb{E}_x \left(e^{-q\tau_{a_n}^-} f(X_{\tau_{a_n}^-}) \mathbb{1}_{\{\tau_{a_n}^- = \infty\}} \right) = 0.$$

So we can apply a similar argument as before and obtain equation (3.30).

□

Proof for Theorem 3.35. (i)

The proof for part (i) can be done by using the same argument as in part (i) in Theorem 3.33.

(ii)

First note that for each fixed $x < b^\star_{n^\star}$, there exists $i_x = \inf\{i \in \mathbb{N} : b^\star_i > x\}$. Thus, for each $x < b^\star_{n^\star}$, as $V_{i_x}(x) = V_{n^\star}(x)$, the problem is reduced to show that $V(x) = V_{i_x}(x)$, which can be done by using the same argument as in part (i) Theorem 3.33.

(iii)

The proof for part (iii) can be done by using the same argument as in part (ii) in Theorem 3.33.

□

Chapter 4

On the Left Semi-Solution of the Optimal Stopping Problem for non Smooth Gain function

4.1 Introduction

Let $X = \{X_t : t \geq 0\}$ be a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with characteristic triple (μ, σ, Π), where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and $\int_{\mathbb{R}\setminus\{0\}}(1 \wedge x^2)\Pi(dx) < \infty$. For any $x \in \mathbb{R}$, let $\mathbb{P}_x$ be the law of X starting from x, and we write simply $\mathbb{P}_0 = \mathbb{P}$. And denote $\mathbb{E}_x$ and $\mathbb{E}$ the corresponding expectation operators. Throughout this chapter we assume that the spectrally negative Lévy process X has unbounded variation, and the q scale function W^q is twice continuously differentiable for all $x > 0$.

Now, we consider the following optimal stopping problem,

$$V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} g(X_\tau)). \tag{4.1}$$

where $q > 0$, the supremum is taken over the class $\mathcal{T}_{[0,\infty]}$ of $\{\mathcal{F}_t\}$-stopping times taking values in $[0, \infty]$.

In Chapter 3, an approach has been suggested to derive the left semi-solutions for the optimal stopping problem. In order to apply this method, the gain function g has to be sufficiently smooth and $\lim_{x \to -\infty} g(x) > 0$, as required by the Assumptions 3.3 and 3.24. However, in the literature, there exists a large class of gain functions where the above conditions break down. For example, in the Novikov-Shiryaev problem, the gain function g takes the form $(x^+)^n$ for some $n \in \mathbb{N}$, so

$\lim_{x\to -\infty} g(x) = 0$. In the standard American put option, the gain function is given by $g(x) = (K - e^x)^+$ for some strictly positive real K, and has non differentiable point at $x = \log(K)$. Also in the American Strangle problem, the gain function takes the form $g(x) = (K_1 - e^x)^+ + (e^x - K_2)^+$ for some strictly positive reals K_1 and K_2 with $K_2 \geq K_1$, and is not differentiable at $x \in \{\log(K_1), \log(K_2)\}$.

The aim of this chapter is to provide an effective approach to find the left semi-solutions for the optimal stopping problem (4.1), where the gain functions have non differentiable points, or $\lim_{x\to -\infty} g(x) = 0$. First, we treat the case where the gain function has a non differentiable set I of finite elements and $\lim_{x\to -\infty} g(x) > 0$. In order to overcome the problem arisen from non differentiable points, we introduce the class of functions $\mathbf{G}_g$ (the extended class), which consists of functions that has the unique type (L) averaging function A, and are in the class $C^2([\inf\{I\}, \infty))$ and equal to g for all $x \leq \inf\{I\}$. Clearly, this class of functions $\mathbf{G}_g$ is non empty under suitable conditions for g. Hence, we can choose a function g_1 from this class, and construct h_{g_1} from the type (L) averaging function A w.r.t. g_1 and $\underline{X}_{e_q}$ (see Section 3.3.2 in Chapter 3 for the definition and properties of h_{g_1}). The crucial step that links h_{g_1} back to the gain function g is the definition of $a^\star$ and $b^\star$. Unlike Chapter 3, $a^\star$ is defined to the be the largest value such that h_{g_1} dominates the gain function g instead of g_1, and $b^\star$ is defined to be the last time that $h_{g_1}(x, a^\star) = g(x)$. Thus, by applying the same argument as in Chapter 3, a left semi-solution pair up the the point $b^\star$ can be found. We also prove that the choice of $g_1 \in \mathbf{G}_g$ does not affect $a^\star$, or $b^\star$, or the left semi value function at all. Furthermore, like in Chapter 3, we show that this construction can be repeated to study value function for $x > b^\star$.

The second part of this chapter is to study gain functions such that

$$\lim_{x\to -\infty} g(x) \leq 0.$$

For gain functions satisfying this property, we propose an approach for obtaining a closed left semi-solution $(V_0, \tau_0^\star)$ up to some point $b_0^\star$, where $\tau_0^\star = \inf\{t \geq 0 : X_t > b_0^\star\}$. Furthermore, we can perform a similar construction as in the previous case to study the value function for $x > b_0^\star$.

This chapter is organized as follows. In section 2, we specify the class of gain functions we are working with in this chapter, and define the extended class of gain functions. In section 3, by using the method discussed as above, we study the optimal stopping problem for the gain function with a non differentiable set and $\lim_{x\to\infty} g(x) > 0$. In section 4, we study gain function that has a non differentiable set and $\lim_{x\to\infty} g(x) \leq 0$. In section 5, by reproducing Surya's result [78], we show that there is no contradiction between our work and the existing literature. We give the concluding remarks in section 6. And section 7 consists of proofs for results in the previous sections.

4.2 Extended class of gain function

First, let us define the class of gain functions we are working with.

Definition 4.1. *Let $D^{1,2}(I^1, I^2)$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy the following conditions, where I^1 and I^2 are two disjoint subsets of $\mathbb{R}$ with only finite number of elements.*

(i) $f \in C(\mathbb{R}) \cap C^1(\mathbb{R} \setminus I^1) \cap C^2(\mathbb{R} \setminus \{I^1 \cup I^2\})$.

(ii) $\lim_{x \to -\infty} f(x)$ *exists, and also* $\lim_{x\to-\infty} f'(x) = \lim_{x\to-\infty} f''(x) = 0$. *Furthermore, the left and right first (second) derivative of f exist on I^1 (I^2), respectively.*

(iii) There exists $c \in (0, \Phi(q))$ and $d > 0$ such that $|f(x)| < e^{cx} + d$ for all $x \in \mathbb{R}$, and

$$\begin{aligned}
&|\max\{f'(x+), f'(x-)\}| < e^{cx} + d, \\
&\left|\max\{\lim_{\epsilon \downarrow 0} \frac{f'(x-\epsilon) - f'(x-)}{\epsilon}, \lim_{\epsilon \downarrow 0} \frac{f'(x+\epsilon) - f'(x+)}{\epsilon}\}\right| < e^{cx} + d,
\end{aligned}$$

for all $x \in \mathbb{R}$.

Next we define the extended class of functions, $\mathbf{G}_f$, for a function f.

Definition 4.2. *For all functions $f \in D^{1,2}(I^1, I^2)$, we say $\mathbf{G}$ is set of functions extended from the function f, in short we write $\mathbf{G}_f$, if*

(i) if $I^1 = \emptyset$, the set $\mathbf{G}_f$ only consists of the function f itself.

(ii) if $I^1 \neq \emptyset$, then $f_1 \in \mathbf{G}_f$ if the following conditions hold true,

(a) $f_1(x) = f(x)$ *for all* $x \leq \inf\{I^1\}$,

(b) $f_1 \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus I^2_{\inf\{I^1\}})$, *where*

$$I^2_{\inf\{I^1\}} = \{x \in I^2 : x < \inf\{I^1\}\}, \tag{4.2}$$

(c) the type (L) averaging function A_{f_1} w.r.t. f_1 and $\underline{X}_{e_q}$ exists (see Definition 3.13 in Chapter 3 for the type (L) averaging function).

If $f \in D^{1,2}(I^1, I^2)$, then the set $\mathbf{G}_f$ must be non empty. This is true as we can always choose functions $f_1 \in D^2(I^2_{\inf\{I^1\}})$ (see Definition 3.3 in Chapter 3 and equation (4.2)). Then by Proposition 3.9, the type (L) averaging function A_{f_1} exists.

Let f_1 be any choice in $\mathbf{G}_f$. Then $f_1 \in D^2_h(I^2_{\inf\{I_1\}})$ (see Definition 3.14 in Chapter 3). We denote by A_{f_1} the unique type (L) averaging function w.r.t. f_1 and $\underline{X}_{e_q}$ (see Proposition 3.22 for the uniqueness), and define $h_{f_1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} h_{f_1}(x,a) &= \mathbb{E}_x\left(e^{-q\tau_a^-} f_1(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^- < \infty\}}\right) + \frac{q}{\Phi(q)} A_{f_1}(a) W^q(x-a) \\ &= \mathbb{E}_x\left(A_{f_1}(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q} < a\}}\right) + \frac{q}{\Phi(q)} A_{f_1}(a) W^q(x-a), \end{aligned} \tag{4.3}$$

for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$, where W^q is the q scale function for the spectrally negative Lévy process X, and $\tau_a^- = \inf\{t \geq 0 : X_t < a\}$, and the second equality follows from Theorem 3.1 in [78]. Note that, as $f_1 \in D^2_h(I^2_{\inf\{I^1\}})$, all results in Section 3.3.2 in Chapter 3 hold true for h_{f_1}. Then we have the following Proposition.

Proposition 4.3. *Suppose that $f \in D^{1,2}(I^1, I^2)$. Let f_1 and f_2 be two functions in $\mathbf{G}_f$, and A_{f_1} and A_{f_2} be the corresponding type (L) averaging functions, and h_{f_1} and h_{f_2} be as defined in* (4.3) *for A_{f_1} and A_{f_2}, respectively. Then $h_{f_1}(x,a) = h_{f_2}(x,a)$ for all $x \in \mathbb{R}$ and $a \leq \inf\{I^1\}$.*

The proof for Proposition 4.3 can be done by following a similar argument as in the proof for Proposition 3.22 in Chapter 3. Hence, it is omitted.

Similar to Chapter 3, we also assume the following Assumption holds true.

Assumption 4.4. *The function $f \in D^{1,2}(I^1, I^2)$. And the constant*

$$a_L = \sup\Big\{a < \inf\{I^1\} : \textit{ the left and right limits of } \mathbb{L}_X f(x) - qf(x) \\ \textit{are non positive on } (-\infty, a)\Big\},$$

is well defined in $\mathbb{R}$. Finally, there exists $f_1 \in \mathbf{G}_f$ such that $h_{f_1}(x, a_L) < f(x)$ for some $x > a_L$, where h_{f_1} is as defined in (4.3) *for the type (L) averaging function A_{f_1}.*

As a result of Propsotion 4.3, we see that the choice of $f_1 \in \mathbf{G}_f$ does not affect the existence of $x > a_L$ such that $h_{f_1}(x, a_L) < f(x)$. Thus, it is either that there exists $x > a_L$ such that $h_{f_1}(x, a_L) < f(x)$ for all $f_1 \in \mathbf{G}_f$, or $h_{f_1}(x, a_L) \geq f(x)$ for all $x > a_L$ and for all $f_1 \in \mathbf{G}_f$.

Remark 4.5. *If one of the following conditions holds true, it is sufficient for the existence of $x > a_L$ such that $h_{f_1}(x, a_L) < f(x)$.*

(i) $a_L < \inf\{I^1\}$, and there exists $\epsilon > 0$ such that $\mathbb{L}_X f(x) - qf(x) \geq 0$ for all $x \in (a_L, a_L + \epsilon)$.

(ii) The set I^1 is not empty, $f'(\inf\{I^1\}+) > f'(\inf\{I^1\}-)$. and the left and right limits of $\mathbb{L}_X f(x) - qf(x)$ are both non positive on $(-\infty, \inf\{I^1\})$.

The proof for (i) can be done by a similar argument as in Remark 3.25. Note that in the second case, $a_L = \inf\{I^1\}$. So by Lemma 3.17,

$$\frac{\partial h_{f_1}}{\partial x}(\inf\{I^1\}+, \inf\{I^1\}) = f_1'(\inf\{I^1\}) = f'(\inf\{I^1\}-) < f'(\inf\{I^1\}+).$$

Therefore, by fundamental theorem of calculus, there must exist $x > \inf\{I^1\}$ such that $h_{f_1}(x, \inf\{I^1\}) < f(x)$.

4.3 Non differentiable gain functions

In this section we study the optimal stopping problem where the gain function $g \in D^{1,2}(I_g^1, I_g^2)$ and $\lim_{x \to -\infty} g(x) > 0$. And we denote by $a_{L,1}$ the constant a_L in Assumption 4.4 for the gain function g. Throughout this section, we fix $g_1 \in \mathbf{G}_g$.

4.3.1 Left semi-solution for the optimal stopping problem

We have the following Theorem for the closed left semi-solution of the optimal stopping problem (4.1).

Theorem 4.6. *Consider the optimal stopping problem* (4.1) *where the gain function $g \in D^{1,2}(I_g^1, I_g^2)$ and satisfies Assumption 4.4 with the sets I_g^1 and I_g^2. Furthermore, suppose that $\lim_{x \to -\infty} g(x) > 0$. Define $a_1^\star$ and $b_1^\star$ to be*

$$a_1^\star = \sup\{a < a_{L,1} : h_{g_1}(x, a) \geq g(x) \text{ for all } x \in \mathbb{R}\}, \tag{4.4}$$

$$b_1^\star = \begin{cases} \sup \mathcal{N}_1 & \text{if } A_{g_1}(a_1^\star) > 0 \text{ and } \mathcal{N}_1 \neq \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{4.5}$$

where $\mathcal{N}_1 = \{b > a_1^\star : h_{g_1}(b, a_1^\star) = g(b)\}$, and h_{g_1} is as defined in equation (4.3) *for the type (L) averaging function A_{g_1} w.r.t. g_1 and $\underline{X}_{e_q}$. Then, $a_1^\star \in \mathbb{R}$, and* *(i) If $b_1^\star < \infty$, then the pair $(V_1, \tau_1^\star)$ is a closed left semi-solution for the optimal stopping problem* (4.1) *up to the point $b_1^\star$. where*

$$V_1(x) = \begin{cases} h_{g_1}(x, a_1^\star) & \text{if } x \in (-\infty, b_1^\star] \cap \mathbb{R} \\ g(x) & \text{otherwise,} \end{cases} \tag{4.6}$$

$$\tau_1^\star = \inf\{t \geq 0 : X_t \notin [a_1^\star, b_1^\star] \cap \mathbb{R}\}. \tag{4.7}$$

(ii) The pair $(V_1, \tau_1^\star)$ is a solution for the optimal stopping problem (4.1) *if one of the following statements holds true.*

(a) $b_1^\star = \infty$,

(b) $V_1(x)$ is differentiable for all $x \geq b_1^\star$, and the left and right limits of $\mathbb{L}_X V_1(x) - qV_1(x)$ exist and are non positive on $(b_1^\star, \infty)$.

Theorem 4.6 can be proved by following a similar argument as for Theorem 3.26 in Chapter 3. Hence, it is omitted as well. Note that $a_1^\star$ is defined to be the largest value such that h_{g_1} dominates the gain function g instead of g_1. We also point out here that if $b_1^\star < \infty$, it is not clear about the relationship between $b_1^\star$, and $\inf\{I_g^1\}$ or $\sup\{I_g^1\}$. Furthermore, as a result of Proposition 4.3, neither of $a_1^\star$ (4.4), or $b_1^\star$ (4.5), or V_1 (4.6) are affected by different choices of $g_1 \in \mathbf{G}_g$.

Same as in Chapter 3, we can work out the closed left semi continuation region up to $b_1^\star$, that is

$$\mathcal{C}_{b_1^\star} = \{x \in \mathbb{R} : V_1(x) > g(x)\}.$$

Let $\partial\mathcal{C}_{b_1^\star}$ denote the set of all boundaries. Then we have the following theorem for the pasting condition on $\partial\mathcal{C}_{b_1^\star}$.

Theorem 4.7. *Under the same conditions as in Theorem 4.6. $V_1(x) = g(x)$ for all $x \in \partial\mathcal{C}_{b_1^\star}$. Furthermore, V_1 is differentiable for all $x < b_1^\star$. and $V_1'(x-) \geq g'(x+)$ for all $x \in \partial\mathcal{C}_{b_1^\star}$. And for all $x \in \partial\mathcal{C}_{b_1^\star}$. if g is differentiable at x, then $V_1'(x-) = g'(x)$.*

The proof for Theorem 4.7 can be done by a similar argument as in part (vi) in Lemma 4.8 below. Hence, it is omitted.

4.3.2 Preliminary results for V_1

We need some preliminary results given in the following series of lemmas.

Lemma 4.8. *Under the same conditions as in Theorem 4.6. the following properties holds true for $a_1^\star$ (4.4) and $b_1^\star$ (4.5).*

(i) $-\infty < a_1^\star < a_{L,1}$.

(ii) $h_{g_1}(x, a_1^\star) \geq g(x)$ *for all* $x \in \mathbb{R}$, *and for all* $\epsilon > 0$ *there exists* $x \in (a_1^\star, a_1^\star + \epsilon)$ *such that* $h_{g_1}(x, a_1^\star) > g(x)$.

(iii) $A_{g_1}(a_1^\star) \geq 0$.

(iv) *For all* $\epsilon \in (0, a_{L,1} - a_1^\star)$, *there exists* $x \in (a_1^\star, a_1^\star + \epsilon)$ *such that*

$$\mathbb{L}_X V_1(x) - qV_1(x) < 0.$$

(v) $b_1^\star = \infty$ *if and only if* $A_{g_1}(a_1^\star) = 0$.

(vi) *If* $b_1^\star < \infty$, *then* $g(b_1^\star) = V_1(b_1^\star)$ *and* $g'(b_1^\star+) \leq V_1'(b_1^\star-)$.

The proof of Lemma 4.8 can be found on page 116.

Lemma 4.9. *Under the same conditions as in Theorem 4.6,* $V_1 \in D^{1,2}(I^1_{V_1}, I^2_{V_1})$, *where*

$$I^1_{V_1} = \{x \in I^1_g : \ V_1'(x+) \neq V_1'(x-) \ \},$$

and

$$I^2_{V_1} = \Big\{x \in \{I^2_g \cup \{a_1^\star, b_1^\star\}\} \cap \mathbb{R} : V_1'(x+) = V_1'(x-) \text{ and } V_1''(x+) \neq V_1''(x-)\Big\}.$$

Furthermore, if $b_1^\star < \infty$ *and* $V_1'(b_1^\star-) = g'(b_1^\star+)$, *then* $\mathbb{L}_X V_1(b_1^\star+) - qV_1(b_1^\star)$ *is well defined and non positive.*

The proof for Lemma 4.9 can be done using a similar argument as for Lemma 3.31, hence, it is omitted.

4.3.3 Value function for $x > b_1^\star$

In the case when $b_1^\star < \infty$, the closed left semi-solution V_1 may not be the global solution to the optimal stopping problem (4.1). In this section we show that, like in Chapter 3, the approach proposed in the previous section can be repeated to study the value function V for $x > b_1^\star$.

As $V_1 \in D^{1,2}(I^1_{V_1}, I^2_{V_1})$, then the extended class of functions $\mathbf{G}_{V_1}$ is non empty. Throughout this section, we fix $g_2 \in \mathbf{G}_{V_1}$. Also from now on, we assume that V_1 satisfies Assumption 4.4 with the sets $I^1_{V_1}$ and $I^2_{V_1}$. And we denote by $a_{L,2}$ the constant a_L in Assumption 4.4 for V_1. We remark here that if V_1 satisfies Assumption 4.4, then $b_1^\star < \infty$. Otherwise, $a_{L,2}$ is not well defined.

Then we have the following result.

Theorem 4.10. *Suppose that all conditions in Theorem 4.6 hold true. Furthermore, suppose that V_1 satisfies Assumption 4.4 with the sets $I^1_{V_1}$ and $I^2_{V_1}$. Define $a^\star_2$ and $b^\star_2$ to be*

$$a^\star_2 = \sup\{a < a_{L,2} : h_{g_2}(x,a) \geq V_1(x) \text{ for all } x \in \mathbb{R}\} \tag{4.8}$$

$$b^\star_2 = \begin{cases} \sup \mathcal{N}_2 & \text{if } A_{g_2}(a^\star_2) > 0 \text{ and } \mathcal{N}_2 \neq \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{4.9}$$

where $\mathcal{N}_2 = \{b > a^\star_2 : h_{g_2}(b, a^\star_2) = V_1(b)\}$, and h_{g_2} is as defined in equation (4.3) for the type (L) averaging function A_{g_2} w.r.t. g_2 and $\underline{X}_{e_q}$. Then, $a^\star_2 \in \mathbb{R}$, and

(i) If $b^\star_2 < \infty$, then the pair $(V_2, \tau^\star_2)$ is a closed left semi-solution for the optimal stopping problem (4.1) up to the point $b^\star_2$, where

$$V_2(x) = \begin{cases} h_{g_2}(x, a^\star_2) & \text{if } x \in (-\infty, b^\star_2] \cap \mathbb{R} \\ g(x) & \text{otherwise.} \end{cases} \tag{4.10}$$

$$\tau^\star_2 = \inf\{t \geq 0 : X_t \notin \{[a^\star_1, b^\star_1] \cup [a^\star_2, b^\star_2]\} \cap \mathbb{R}\}. \tag{4.11}$$

(ii) The pair $(V_2, \tau^\star_2)$ is a solution for the optimal stopping problem (4.1) if one of the following statements holds true,

(a) $b^\star_2 = \infty$,

(b) $V_2(x)$ is differentiable for all $x \geq b^\star_2$, and the left and right limits of $\mathbb{L}_X V_2(x) - qV_2(x)$ exist and are non positive on $(b^\star_2, \infty)$.

The proof for Theorem 4.10 can be done using a similar argument as for Theorem 3.33 in Chapter 3. Hence, it is omitted.

Then the closed left semi continuation region $\mathcal{C}_{b^\star_2}$ up to the point $b^\star_2$ (4.9) can be derived as,

$$\mathcal{C}_{b^\star_2} = \{x \in \mathbb{R} : V_2(x) > g(x)\}.$$

And we have the following Theorem for the pasting conditions on the boundary set $\partial\mathcal{C}_{b^\star_2}$.

Theorem 4.11. *Under the same condition as in Theorem 4.10, $V_2(x) = g(x)$ for all $x \in \partial\mathcal{C}_{b^\star_2}$, and $V_2(x)$ is differentiable for all $x < b^\star_2$, and $V_2'(x-) \geq g'(x+)$ for all $x \in \partial\mathcal{C}_{b^\star_2}$. For all $x \in \partial\mathcal{C}_{b^\star_2}$, if g is differentiable at x, then $V_2'(x-) = g'(x)$.*

The proof for Theorem 4.11 can be done by using a similar argument as for Theorem 3.34 in Chapter 3. Hence, it is omitted.

Same as Chapter 3, we can keep repeating this procedure for the $(i+1)^{th}$ time, $i \in \mathbb{N}$, as long as $V_i \in D^{1,2}(I^1_{V_i}, I^2_{V_i})$, where

$$I^1_{V_i} = \{x \in I^1_g : V_i'(x+) \neq V_i'(x-)\},$$

and

$$\begin{aligned} I^2_{V_i} &= \Big\{x \in \{I^2_g \cup \{a^\star_j, b^\star_j : 1 \leq j \leq i\}\} \cap \mathbb{R} : \\ &\qquad V_i'(x+) = V_i'(x-) \text{ and } V_i''(x+) \neq V_i''(x-)\Big\}. \end{aligned}$$

Note that if V_i satisfies Assumption 4.4, then $b^\star_i < \infty$. Let $n^\star$ be the first time the above condition breaks down, that is

$$n^\star = \sup\{n \in \mathbb{N} : V_i \text{ satisfies Assumption 4.4 for all } i \leq n.\}$$

Note that $n^\star \in \mathbb{N} \cup \{\infty\}$. And we have the following Theorem.

Theorem 4.12. *Consider the optimal stopping problem* (4.1) *where the gain function $g \in D^{1,2}(I^1_g, I^2_g)$ and satisfies Assumption 4.4 with the sets I^1_g and I^2_g, and $\lim_{x \to -\infty} g(x) > 0$. Then,*

(i) If $n^\star < \infty$, then the pair $(V_{n^\star}, \tau^\star_{n^\star})$ is a closed left semi-solution for the optimal stopping problem (4.1) *up to the point $b^\star_{n^\star}$, where $b^\star_{n^\star} = \lim_{i\to\infty} b^\star_{i\wedge n^\star}$, and $V_{n^\star}(x) = \lim_{i\to\infty} V_{i\wedge n^\star}(x)$ for all $x \in \mathbb{R}$, and*

$$\tau^\star_{n^\star} = \inf\Big\{t \geq 0 : X_t \notin \{\bigcup_{i=1}^{n^\star}[a^\star_i, b^\star_i]\} \cap \mathbb{R}\Big\}.$$

(ii) If $n^\star = \infty$ and $b^\star_{n^\star} < \infty$, then the pair $(V_{n^\star}, \tau^\star_{n^\star})$ is an open left semi-solution for the optimal stopping problem (4.1) *up to the point $b^\star_{n^\star}$.*

(iii) The pair $(V_{n^\star}, \tau^\star_{n^\star})$ is a solution for the optimal stopping problem (4.1) *if one of the following statements holds true,*

(a) $b^\star_{n^\star} < \infty$, $V_{n^\star}$ is differentiable for all $x \geq b^\star_{n^\star}$, and the left and right limits of $\mathbb{L}_X V_{n^\star}(x) - qV_{n^\star}(x)$ exist and are non positive on $(b^\star_{n^\star}, \infty)$.

(b) $b^\star_{n^\star} = \infty$.

Theorem 4.12 can be done by using similar arguments as for Theorem 3.35 in Chapter 3. Hence, they are omitted.

4.3.4 Preliminary results for V_2

The following Lemmas are needed for the proof of Theorem 4.10, Theorem 4.11 and Theorem 4.12.

Lemma 4.13. *Under the same conditions as in Theorem 4.10, the following properties hold true for $a_2^\star$ (4.8) and $b_2^\star$ (4.9).*

(i) $h_{g_2}(x, b_1^\star) = h_{g_1}(x, a_1^\star)$ *for all* $x \in \mathbb{R}$.

(ii) $V_1'(b_1^\star-) = g'(b_1^\star+)$.

(iii) $b_1^\star \leq a_2^\star < a_{L,2}$.

(iv) $h_{g_2}(x, a_2^\star) \geq V_1(x)$ *for all* $x \in \mathbb{R}$, *and for all* $\epsilon > 0$ *there exists* $x \in (a_2^\star, a_2^\star + \epsilon)$ *such that* $h_{g_2}(x, a_2^\star) > V_1(x)$.

(v) $A_{g_2}(a_2^\star) \geq 0$.

(vi) *For all* $\epsilon \in (0, a_{L,2} - a_2^\star)$, *there exists* $x \in (a_2^\star, a_2^\star + \epsilon)$ *such that*

$$\mathbb{L}_X V_1(x) - qV_1(x) < 0.$$

(vii) $b_2^\star = \infty$ *if and only if* $A_{g_2}(a_2^\star) = 0$.

(viii) *If* $b_2^\star < \infty$, *then* $g(b_2^\star) = V_2(b_2^\star)$ *and* $g'(b_2^\star+) \leq V_2'(b_2^\star-)$.

(ix) $b_2^\star > a_2^\star > b_1^\star$.

The proof for Lemma 4.13 is on page 117.

Lemma 4.14. *Suppose that all conditions in Theorem 4.10 hold true, then* $V_2 \in D^{1,2}(I_{V_2}^1, I_{V_2}^2)$, *where*

$$I_{V_2}^1 = \{x \in I_g^1 : V_2'(x+) \neq V_2'(x-)\},$$

and

$$\begin{aligned} I_{V_2}^2 &= \Big\{x \in \{I_g^2 \cup \{a_1^\star, b_1^\star, a_2^\star, b_2^\star\}\} \cap \mathbb{R} : \\ &\qquad V_2'(x+) = V_2'(x-) \text{ and } V_2''(x+) \neq V_2''(x-)\Big\}. \end{aligned}$$

Furthermore, if $b_2^\star < \infty$ and $V_2'(b_2^\star -) = g'(b_2^\star +)$, then $\mathbb{L}_X V_2(b_2^\star +) - qV_2(b_2^\star)$ is well defined and non positive.

The proof for Lemma 4.14 can be done by using a similar argument as for Lemma 4.9.

4.4 The case $\lim_{x\to-\infty} g(x) \leq 0$

In this section we study the optimal stopping problem (4.1) for the following class of gain functions.

Definition 4.15. *$\widetilde{D}$ is the set consisting functions $f \in C(\mathbb{R})$. Furthermore, There exists $x \in \mathbb{R}$ such that $f(x) > 0$. There exists $c \in (0, \Phi(q))$ and $d > 0$ such that $|f(x)| < e^{cx} + d$ for all $x \in \mathbb{R}$. Finally, $\lim_{x\to-\infty} f(x)$ exists, and the function $S(x) := e^{-\Phi(q)x} f(x)$ converges to 0 or diverges to $-\infty$ as $x \to -\infty$.*

4.4.1 Left semi-solution for the optimal stopping problem

Theorem 4.16. *Consider the optimal stopping problem (4.1) for the gain function $g \in \widetilde{D}$. Then, the constant*

$$b_0^\star = \sup\{b \in \mathbb{R} : S(b) = \max\{S(x) : x \in \mathbb{R}\}\}, \tag{4.12}$$

is well defined in $\mathbb{R}$. Furthermore,

(i) the pair $(V_0, \tau_0^\star)$ is a closed left semi-solution for the optimal stopping problem (4.1) up to the point $b_0^\star$, where

$$V_0(x) = \begin{cases} h_0(x, b_0^\star) & \textit{for all } x \leq b_0^\star \\ g(x) & \textit{for all } x > b_0^\star, \end{cases} \tag{4.13}$$

$$\tau_0^\star = \inf\{t \geq 0 : X_t > b_0^\star\}, \tag{4.14}$$

$$h_0(x, b) = g(b)e^{-\Phi(q)(b-x)}. \tag{4.15}$$

(ii) if $\{e^{-qt}V_0(X_t), t \geq 0\}$ is a supermartingale, then $(V_0, \tau_0^\star)$ is a solution for the optimal stopping problem (4.1).

The proof of Theorem 4.16 can be found on page 117.

So, the closed left semi continuation region $C_{b_0^\star}$ can be found by

$$C_{b_0^\star} = \{x \in C : x \leq b_0^\star\} = \{x \in \mathbb{R} : V_0(x) > g(x)\}.$$

Let $\partial\mathcal{C}_{b_0^\star}$ denote the boundary set. Then we have the following theorem for the pasting conditions on the set $\partial\mathcal{C}_{b_0^\star}$.

Theorem 4.17. *Consider the optimal stopping problem* (4.1) *for the gain function* $g \in \widetilde{D}$. *Then* $V_0(x) = g(x)$ *for all* $x \in \partial\mathcal{C}_{b_0^\star}$. *Furthermore,* V_0 *is differentiable for all* $x < b_0^\star$, *and* $V_0'(x-) \geq g'(x+)$ *for all* $x \in \partial\mathcal{C}_{b_0^\star}$. *And for all* $x \in \partial\mathcal{C}_{b_0^\star}$, $V_0'(x) = g'(x)$ *if* g *is differentiable at* x.

Theorem 4.17 can be obtained as a direct application of Theorem 4.16. Hence, the proof is omitted. The following Lemma shows that $b_0^\star$ is well defined.

Lemma 4.18. *Suppose that all conditions in Theorem 4.16 hold true. Then the function* $S : \mathbb{R} \to \mathbb{R}$ *has at least one strictly positive global maximum point, and* $b_0^\star < \infty$.

The proof of Lemma 4.18 can be found on page 117.

To finish this Section, we present the following Lemma.

Lemma 4.19. *Suppose that all conditions in Theorem 4.16 hold true. If there exists* $\epsilon > 0$ *such that* $V_0 \in C^1((-\infty, b_0^\star + \epsilon]) \cap C^2((-\infty, b_0^\star + \epsilon] \setminus \{b_0^\star\})$, *and the left and right limits of the second derivatives of* V_0 *exist at* $b_0^\star$. *Then* $\mathbb{L}_X V_0(b_0^\star+) - qV_0(b_0^\star+)$ *is well defined and non positive.*

The proof for Lemma 4.19 can be done by using a similar argument as for Lemma 4.9.

4.4.2 Value function for $x > b_0^\star$

In this section we show that under the condition that V_0 satisfies the Assumption 4.4, then the approach suggested in Section 4.2 for non differentiable gain functions, can be applied to V_0 to study value function for $x > b_0^\star$.

Assume that the function V_0 satisfies the Assumption 4.4 with the sets $I_{V_0}^1$ and $I_{V_0}^2$, and denote by a_{L,V_0} the constant a_L in the Assumption 4.4 for V_0. Throughout this section, we fix $g_{0,1} \in \mathbf{G}_{V_0}$. Then we have the following Theorem.

Theorem 4.20. *Consider the optimal stopping problem* (4.1), *where the gain function* $g \in \widetilde{D}$, *and* V_0 *satisfies the Assumption 4.4 with the sets* $I_{V_0}^1$ *and* $I_{V_0}^2$. *Defined* $a_{0,1}^\star$ *and* $b_{0,1}^\star$ *by setting*

$$a_{0,1}^\star = \sup\{a < a_{L,V_0} : h_{g_{0,1}}(x,a) \geq V_0(x) \text{ for all } x \in \mathbb{R}\} \tag{4.16}$$

$$b_{0,1}^\star = \begin{cases} \sup \mathcal{N}_{V_0} & \text{if } A_{g_{0,1}}(a_{0,1}^\star) > 0 \text{ and } \mathcal{N}_{V_0} \neq \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{4.17}$$

where $\mathcal{N}_{V_0} = \{b > a^\star_{0,1} : h_{g_{0,1}}(b, a^\star_{0,1}) = V_0(b)\}$, *and* $h_{g_{0,1}}$ *is as defined in* (4.3) *for the type (L) averaging function* $A_{g_{0,1}}$ *with respect to* $g_{0,1}$ *and* $\underline{X}_{e_q}$. *Then* $a^\star_{0,1} < \infty$. *and*

(i) *If* $b^\star_{0,1} < \infty$, *the pair* $(V_{0,1}, \tau^\star_{0,1})$ *is a closed left semi-solution up to the point* $b^\star_{0,1}$, *where*

$$V_{0,1}(x) = \begin{cases} h_{g_{0,1}}(x, a^\star_{0,1}) & \text{if } x \in (-\infty, b^\star_{0,1}] \cap \mathbb{R} \\ g(x) & \text{otherwise,} \end{cases} \tag{4.18}$$

$$\tau^\star_{0,1} = \inf\left\{t \geq 0 : X_t \notin \{(-\infty, b^\star_0] \cup [a^\star_{0,1}, b^\star_{0,1}]\} \cap \mathbb{R}\right\}. \tag{4.19}$$

(ii) *The pair* $(V_{0,1}, \tau^\star_{0,1})$ *is a solution to the optimal stopping problem* (4.1) *if one of the following statements holds true,*

(a) $b^\star_{0,1} = \infty$,

(b) $b^\star_{0,1} < \infty$, $V_{0,1}(x)$ *is differentiable for all* $x \geq b^\star_{0,1}$, *and the left and right limits of* $\mathbb{L}_X V_{0,1}(x) - qV_{0,1}(x)$ *exist and is non positive on* $(b^\star_{0,1}, \infty)$.

Let $\mathcal{C}_{b^\star_{0,1}}$ be the closed left semi continuation region up to the point $b^\star_{0,1}$, then

$$\mathcal{C}_{b^\star_{0,1}} = \{x \in \mathbb{R} : V_{0,1}(x) > g(x)\}.$$

And we have the following theorem for the pasting conditions for all $x \in \partial\mathcal{C}_{b^\star_{0,1}}$.

Theorem 4.21. *Under the same condition as in Theorem 4.20,* $V_{0,1}(x) = g(x)$ *for all* $x \in \partial\mathcal{C}_{b^\star_{0,1}}$. *Furthermore,* $V_{0,1}$ *is differentiable for all* $x < b^\star_{0,1}$, *and* $V'_{0,1}(x-) \geq g'(x+)$ *for all* $x \in \partial\mathcal{C}_{b^\star_{0,1}}$. *And for all* $x \in \partial\mathcal{C}_{b^\star_{0,1}}$, *if* g *is differentiable at* x, *then* $g'(x) = V'_{0,1}(x-)$.

Same as before, we can keep repeating this procedure for the $(1+i)^{th}$ time, $i \in \mathbb{N}$, as long as $V_{0,i}$ satisfies the Assumption 4.4 with the sets $I^1_{V_{0,i}}$ and $I^2_{V_{0,i}}$, where

$$I^1_{V_{0,i}} = \{x \in I^1_{V_0} : V'_{0,i}(x+) \neq V'_{0,i}(x-)\ \},$$

and

$$\begin{aligned} I^2_{V_{0,i}} = \Big\{ & x \in \{I^2_{V_0} \cup \{a^\star_{0,j}, b^\star_{0,j} : 1 \leq j \leq i\}\} \cap \mathbb{R} : \\ & V'_{0,i}(x+) = V'_{0,i}(x-) \text{ and } V''_{0,i}(x+) = V''_{0,i}(x-)\Big\}. \end{aligned}$$

Let $n_0^\star$ be the first time the above condition breaks down, that is

$$n_0^\star = \sup\{n \in \mathbb{N} : V_{0,i} \text{ satisfies Assumption 4.4 for all } i \leq n.\}$$

Similar as before, $n_0^\star \in \mathbb{N} \cup \{\infty\}$. And we have the following Theorem.

Theorem 4.22. *Suppose that all conditions in Theorem 4.20 hold true. Then,*

(i) If $n_0^\star < \infty$, then the pair $(V_{0,n_0^\star}, \tau_{0,n_0^\star}^\star)$ is a closed left semi-solution for the optimal stopping problem (4.1) up to the point $b_{0,n_0^\star}^\star$. where $b_{0,n_0^\star}^\star = \lim_{i\to\infty} b_{0,i\wedge n_0^\star}^\star$, and $V_{0,n_0^\star}(x) = \lim_{i\to\infty} V_{0,i\wedge n_0^\star}(x)$ for all $x \in \mathbb{R}$, and

$$\tau_{0,n_0^\star}^\star = \inf\left\{t \geq 0 : X_t \notin \{(-\infty, b_0^\star] \cup \bigcup_{i=1}^{n_0^\star} [a_{0,i}^\star, b_{0,i}^\star]\} \cap \mathbb{R}\right\}.$$

(ii) If $n_0^\star = \infty$ and $b_{0,n^\star}^\star < \infty$, then the pair $(V_{0,n_0^\star}, \tau_{0,n_0^\star}^\star)$ is an open left semi-solution for the optimal stopping problem (4.1) up to the point $b_{0,n_0^\star}^\star$.

(iii) The pair $(V_{0,n_0^\star}, \tau_{0,n_0^\star}^\star)$ is a solution for the optimal stopping problem (4.1) if one of the following statements holds true,

(a) $b_{0,n_0^\star}^\star < \infty$, $V_{0,n_0^\star}$ is differentiable for all $x \geq b_{0,n_0^\star}^\star$, and the left and right limits of $\mathbb{L}_X V_{0,n_0^\star}(x) - qV_{0,n_0^\star}(x)$ exist and are non positive on $(b_{0,n_0^\star}^\star, \infty)$.

(b) $b_{0,n_0^\star}^\star = \infty$.

Theorem 4.20, Theorem 4.21 and Theorem 4.22 can be proved by following a similar argument as for Theorem 4.6, Theorem 4.11 and Theorem 4.12. Hence, they are omitted.

4.4.3 Preliminary results for $V_{0,1}$

The following Lemmas are needed in proving Theorem 4.20.

Lemma 4.23. *Under the same conditions as in Theorem 4.20, the following properties hold true for $a_{0,1}^\star$ (4.16) and $b_{0,1}^\star$ (4.17).*

(i) $h_0(x, b_0^\star) = h_{g_{0,1}}(x, b_0^\star)$ for all $x \in \mathbb{R}$.

(ii) $V_0'(b_0^\star-) = g'(b_0^\star+)$.

(iii) $b_0^\star \leq a_{0,1}^\star < a_{L,V_0}$.

(iv) $h_{g_{0,1}}(x, a_{0,1}^\star) \geq V_0(x)$ for all $x \in \mathbb{R}$, and for all $\epsilon > 0$ there exists $x \in (a_{0,1}^\star, a_{0,1}^\star + \epsilon)$ such that $h_{g_{0,1}}(x, a_{0,1}^\star) > V_0(x)$.

(v) $A_{g_{0,1}}(a^\star_{0,1}) \geq 0$.

(vi) *For all* $\epsilon \in (0, a_{L,V_0} - a^\star_{0,1})$, *there exists* $x \in (a^\star_{0,1}, a^\star_{0,1} + \epsilon)$ *such that*

$$\mathbb{L}_X V_0(x) - qV_0(x) < 0.$$

(vii) $b^\star_{0,1} = \infty$ *if and only if* $A_{g_{0,1}}(a^\star_{0,1}) = 0$.

(viii) *If* $b^\star_{0,1} < \infty$, *then* $g(b^\star_{0,1}) = V_{0,1}(b^\star_{0,1})$ *and* $g'(b^\star_{0,1}+) \leq V'_{0,1}(b^\star_{0,1}-)$.

(ix) $b^\star_{0,1} > a^\star_{0,1} > b^\star_0$.

Lemma 4.24. *Suppose that all conditions in Theorem 4.20 hold true, then* $V_{0,1} \in D^{1,2}(I^1_{V_{0,1}}, I^2_{V_{0,1}})$, *where*

$$I^1_{V_{0,1}} = \{x \in I^1_{V_0} : V'_{0,1}(x+) \neq V'_{0,1}(x-)\},$$

and

$$\begin{aligned} I^2_{V_{0,1}} \quad = \quad & \Big\{x \in \{I^2_{V_0} \cup \{b^\star_0, a^\star_{0,1}, b^\star_{0,1}\}\} \cap \mathbb{R} : \\ & \quad V'_{0,1}(x+) = V'_{0,1}(x-) \text{ and } V''_{0,1}(x+) \neq V''_{0,1}(x-)\Big\}. \end{aligned}$$

Furthermore, if $b^\star_{0,1} < \infty$ *and* $V'_{0,1}(b^\star_{0,1}-) = g'(b^\star_{0,1}+)$, *then the right limit of*

$$\mathbb{L}_X V_{0,1}(b^\star_{0,1}) - qV_{0,1}(b^\star_{0,1})$$

is well defined and non positive.

Corollary 4.25. *Suppose that all conditions in Theorem 4.20 hold true, then* $h_{g_{0,1}}(x, a^\star_{0,1}) > 0$ *for all* $x \in \mathbb{R}$. *Furthermore, suppose that* $V_0 \in C^2(a^\star_{0,1}, b^\star_{0,1})$, *then there exists* $x \in (a^\star_{0,1}, b^\star_{0,1})$ *such that* $\mathbb{L}_X V_0(x) - qV_0(x) > 0$.

The proof for Lemma 4.23, Lemma 4.24 and Corollary 4.25 can be done by following similar arguments as for Lemma 4.8, Lemma 4.9 and Corollary 3.32.

4.5 Consistency with the existing literature

Let $\overline{A} : \mathbb{R} \to \mathbb{R}$ be an averaging function with respect to $\overline{X}_{e_q}$ and g, that is $\mathbb{E}_x\left(\overline{A}(\overline{X}_{e_q})\right) = g(x)$ for all $x \in \mathbb{R}$. In [78], Surya proved that if there exists an averaging function $\overline{A}$, which satisfies certain conditions (there exists $x_0 \in \mathbb{R}$ such that $\overline{A}(x_0) = 0$, and $\overline{A}(x)$ is nondecreasing for $x \geq x_0$ and $\overline{A}(x) \leq 0$ for all $x < x_0$), then $(V_{\tau^+_{b_\infty}}, \tau^+_{b_\infty})$ is a solution to the optimal stopping problem (4.1), where b_∞ is found as the largest root of

$$\overline{A}(x) = 0, \tag{4.20}$$

and $V_{\tau^+_{b_\infty}}(x) = \mathbb{E}_x\left(e^{-q\tau^+_{b_\infty}} g(X_{\tau^+_{b_\infty}})\mathbb{1}_{\{\tau^+_{b_\infty}<\infty\}}\right)$ and $\tau^+_{b_\infty} = \inf\{t \geq 0 : X_t > b_\infty\}$. Here we show that under the same conditions on the averaging function $\overline{A}$, the constant $b_0^\star$ defined in (4.12), is the largest root of (4.20).

Lemma 4.26. *Suppose that $g \in \widetilde{D}$. Suppose further that there exists $b \in \mathbb{R}$ and a continuous averaging function $\overline{A}$ with respect to $\overline{X}_{e_q}$ and g, such that $\overline{A}(b) = 0$, $\overline{A}$ is nondecreasing on $(b, +\infty)$ and non positive on $(-\infty, b)$. Let b_∞ be the largest root of* (4.20), *then $b_0^\star = b_\infty$, where $b_0^\star$ is as defined in* (4.12).

As a result of Lemma above, V_0 (4.13) is equal to $V_{\tau^+_{b_\infty}}$ for all $x \in \mathbb{R}$, and $\tau_0^\star = \tau^+_{b_\infty}$. Finally, we point out here that by using the path properties of $\overline{A}$, Surya [78] prove that $\{e^{-qt}V_0(X_t), t \geq 0\}$ is a supermartingale. Hence, by Theorem 4.16, $(V_0, \tau_0^\star)$ is a solution.

Proof for Lemma 4.26. Note that $\overline{A}(x) > 0$ for all $x > b_\infty$. So, we have for all $b > b_\infty$,

$$\begin{aligned}
&\mathbb{E}\left(\overline{A}(\overline{X}_{e_q} + b_\infty)\mathbb{1}_{\{\overline{X}_{e_q}+b_\infty > b_\infty\}}\right) \\
&= \mathbb{E}\left(\overline{A}(\overline{X}_{e_q} + b_\infty)\mathbb{1}_{\{\overline{X}_{e_q}+b_\infty > b\}}\right) + \mathbb{E}\left(\overline{A}(\overline{X}_{e_q} + b_\infty)\mathbb{1}_{\{\overline{X}_{e_q}+b_\infty \in (b_\infty, b]\}}\right) \\
&> \mathbb{E}\left(\overline{A}(\overline{X}_{e_q} + b_\infty)\mathbb{1}_{\{\overline{X}_{e_q}+b_\infty > b\}}\right).
\end{aligned} \tag{4.21}$$

As for all $b \geq b_\infty$,

$$\begin{aligned}
\mathbb{E}\left(\overline{A}(\overline{X}_{e_q} + b_\infty)\mathbb{1}_{\{\overline{X}_{e_q}+b_\infty > b\}}\right) &= \mathbb{E}_{b_\infty}\left(e^{-q\tau_b^+} g(X_{\tau_b^+})\mathbb{1}_{\{\tau_b^+<\infty\}}\right) \\
&= g(b)e^{-\Phi(q)(b-b_\infty)},
\end{aligned} \tag{4.22}$$

where the first equality is due to Corollary 3.1 in [78]. By using equation (4.22), we can rewrite both sides of the inequality (4.21), and obtain that $S(b_\infty) > S(b)$ for all $b > b_\infty$, where $S(b) = g(b)e^{-\Phi(q)b}$.

Also for all $b < b_\infty$

$$\begin{aligned} g(b) &= \mathbb{E}\left(\overline{A}(\overline{X}_{e_q} + b)\right) \\ &\leq \mathbb{E}\left(\overline{A}(\overline{X}_{e_q} + b)\mathbb{1}_{\{\overline{X}_{e_q}+b>b_\infty\}}\right) \\ &= \mathbb{E}_b\left(e^{-q\tau^+_{b_\infty}} g(X_{\tau^+_{b_\infty}})\mathbb{1}_{\{\tau^+_{b_\infty}<\infty\}}\right) \\ &= g(b_\infty)e^{-\Phi(q)(b_\infty - b)}, \end{aligned}$$

where the inequality is due to $\overline{A}(x) \leq 0$ for all $x \leq b_\infty$. So $S(b_\infty) \geq S(b)$ for all $b \leq b_\infty$. Therefore, b_∞ is the largest constant that maximizes the function S. Then by definition of $b_0^\star$, we have $b_0^\star = b_\infty$.

□

4.6 Conclusions

In this chapter, we proposed a method that extends the approach in Chapter 3 to gain functions that has non differentiable points, or $\lim_{x\to-\infty} g(x) \leq 0$. This method is based on constructing the extended class of functions $\mathbf{G}$ for the gain function g. By reproducing Surya's result [78], we are able to conclude that there is no contradiction with our result and the existing literature, done by for example [1], [47] and [78]. By using the left semi-solution, we observe that the continuous pasting holds true at all boundaries of the left semi continuation region, and the smooth pasting holds true at both $a_i^\star$ and $a_{0,i}^\star$, $i \in \mathbb{N}$. For other points in the boundary set, it is interesting to observe that there is no fixed rule for the underlying process (for example, regularity at the boundary) under which the smooth pasting happens. A sufficient condition would be the differentiability of the gain function at the boundary. Finally, we remark here that, like in the smooth gain function case, the strictly increasing sequences, $\{a_i^\star, 1 \leq i \leq n^\star\}$, $\{b_i^\star, 1 \leq i \leq n^\star\}$, $\{a_{0,i}^\star, 1 \leq i \leq n_0^\star\}$ and $\{b_{0,i}^\star, 1 \leq i \leq n_0^\star\}$ may or may not converge. And it is not clearly what happens after the limit point.

4.7 Proofs

Proof for Lemma 4.8. Part (i), (ii), (iii), (iv), and (v) can be done using a similar argument as for Lemma 3.30.

(vi)

The proof for $g(b_1^\star) = V_1(b_1^\star, a_1^\star)$ can be done using a similar argument as for part (vi) in Lemma 3.30. If $g'(b_1^\star+) > V_1'(b_1^\star-) = \frac{\partial h_{g_1}}{\partial x}(b_1^\star, a_1^\star)$, then there exists

$\epsilon > 0$ such that $g(x) > h_{g_1}(x, a_1^\star)$ for all $x \in (b_1^\star, b_1^\star + \epsilon)$, which clearly contradicts part (ii) in this Lemma. Therefore, $g'(b_1^\star+) \leq \frac{\partial h_{g_1}}{\partial x}(b_1^\star, a_1^\star) = V_1'(b_1^\star-)$. $\square$

Proof for Lemma 4.13. *(i)*

Part (i) can be done by using a similar argument as for part (i) in Lemma 3.36.

(ii)

As a result of part (vi) in Lemma 4.8, we have $V_1'(b_1^\star-) \geq g'(b_1^\star+)$. If $V_1'(b_1^\star-) > g'(b_1^\star+)$, then by definition of $a_{L,2}$, we have $a_{L,2} = b_1^\star$. Note that under Assumption 4.4, we have $h_{g_2}(b, a_{L,2}) < g(b)$ for some $b > a_{L,2} = b_1^\star$. Then from part (i), it follows that $h_{g_1}(b, a_1^\star) = h_{g_2}(b, a_{L,2}) < g(b)$ for some $b > a_{L,2} = b_1^\star$, which clearly contradict part (ii) in Lemma 4.8. Therefore, $V_1'(b_1^\star-) = g'(b_1^\star+)$.

Part (iii) to (ix) can be proved by using a similar argument as in Lemma 3.36. $\square$

Proof for Lemma 4.18. Note that thanks to the definition of 4.15, there exists $c \in (0, \Phi(q))$ and $d > 0$ such that

$$\lim_{x\to\infty} e^{-\Phi(q)x}|g(x)| \leq \lim_{x\to\infty} e^{-\Phi(q)x}(e^{cx} + d) = 0.$$

Thus, $S(x)$ converges to 0 as x goes to ∞. Also from the definition 4.15, we have $S(x)$ converges to 0 or diverges to $-\infty$ as x goes to $-\infty$. Thus, by continuity of S we can conclude that the function $S(x)$ is bounded above, and $b_0^\star < \infty$. $\square$

Proof for Theorem 4.16. *(i)*

First note that $\{e^{-qt}h_0(X_t, b), t \geq 0\}$ is a $\mathbb{P}_x$ martingale for all $b \in \mathbb{R}$. Thus, it is a $\mathbb{P}_x$ supermartingale as well. Next, as $b_0^\star$ is the global maximum point for $S(x)$, so we obtain for all $x \in \mathbb{R}$

$$g(b_0^\star)e^{-\Phi(q)b_0^\star} = S(b_0^\star) \geq S(x) = g(x)e^{-\Phi(q)x}.$$

Thus, for all $x \in \mathbb{R}$

$$h_0(x, b_0^\star) = g(b_0^\star)e^{-\Phi(q)(b_0^\star - x)} \geq g(x). \tag{4.23}$$

Finally, we show that $h_0(x, b_0^\star)$ can be achieved by stopping at the stopping time $\tau_0^\star$ for all $x \leq b_0^\star$. Thanks to the definition 4.15, g is converging as x goes to $-\infty$, then for all $x \in \mathbb{R}$

$$\mathbb{E}_x\left(e^{-q\tau_0^\star} g(X_{\tau_0^\star})\mathbb{1}_{\{\tau_0^\star=\infty\}}\right) = 0.$$

Thus, we have for all $x \leq b_0^\star$

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-q\tau_0^\star} g(X_{\tau_0^\star})\right) \\
&= \mathbb{E}_x\left(e^{-q\tau_0^\star} g(X_{\tau_0^\star})\mathbb{1}_{\{\tau_0^\star=\infty\}}\right) + \mathbb{E}_x\left(e^{-q\tau_0^\star} g(X_{\tau_0^\star})\mathbb{1}_{\{\tau_0^\star<\infty\}}\right) \\
&= g(b_0^\star)\mathbb{E}_x\left(e^{-q\tau_0^\star}\mathbb{1}_{\{\tau_0^\star<\infty\}}\right) \\
&= g(b_0^\star)e^{-\Phi(q)(b_0^\star-x)} \\
&= h_0(x, b_0^\star).
\end{aligned}$$

Therefore, from Lemma 3.2, it follows that $(V_0, \tau_0^\star)$ is a closed left semi-solution up to the point $b_0^\star$.

(ii).

From the proof in part (i), we have $V_0(x) = \mathbb{E}_x\left(e^{-q\tau_0^\star} g(X_{\tau_0^\star})\right)$ for all $x \in \mathbb{R}$, and $V_0(x) \geq g(x)$. As $\{e^{-qt}V_0(X_t), t \geq 0\}$ is a supermartingale, then by Guess and Verification Lemma we can conclude that $V_0(x) = V(x)$ for all $x \in \mathbb{R}$.

□

Chapter 5

Examples

In this chapter, by applying the approach suggested in chapters 3 and 4, we study the following optimal stopping problem for various examples,

$$V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} G(X_\tau)), \tag{5.1}$$

where $q > 0$, the underlying uncertainty X is a spectrally negative Lévy process on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ with the triple (μ, σ, Π), $\sigma > 0$, and the supremum is taken over the class $\mathcal{T}_{[0,\infty]}$ of $\{\mathcal{F}_t\}$-stopping times taking values in $[0, \infty]$.

5.1 Perpetual American put option

In this section we consider the optimal stopping problem (5.1) where the gain function G_1 takes the following form

$$G_1(x) = (K - e^x)^+, \qquad \text{for all } x \in \mathbb{R}, \tag{5.2}$$

where K is some positive real constant. Clearly, the gain function $G_1 \in D^{1,2}(\{\log(K)\}, \emptyset\})$. Let $\tilde{G}_1 : \mathbb{R} \to \mathbb{R}$ be such that

$$\tilde{G}_1(x) = K - e^x, \qquad \text{for all } x \in \mathbb{R}. \tag{5.3}$$

Then $\tilde{G}_1$ is infinitely differentiable everywhere, and $\lim_{x\to-\infty} G_1(x) = K$. It has been shown in the literature, see [78] for example, an averaging function $A_{\tilde{G}_1}$ with respect to $\tilde{G}_1$ and $\underline{X}_{e_q}$ exists and takes the following form,

$$A_{\tilde{G}_1}(x) = K - \frac{e^x}{\mathbb{E}(e^{\underline{X}_{e_q}})} = K - \frac{q-\psi(1)}{\Phi(q)-1}\frac{\Phi(q)}{q}e^x, \qquad \text{for all } x \in \mathbb{R}. \tag{5.4}$$

By checking the derivatives of the $A_{\tilde{G}_1}$, we see that $\tilde{G}_1 \in \mathbf{G}_{G_1}$.

For all $x \in \mathbb{R}$ and $a \in \mathbb{R}$, define

$$\begin{aligned} h_{\tilde{G}_1}(x,a) &= \mathbb{E}_x\left(e^{-q\tau_a^-}\tilde{G}_1(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^-<\infty\}}\right) + \frac{q}{\Phi(q)}A_{\tilde{G}_1}(a)W^q(x-a) \\ &= K\mathbb{E}_x\left(e^{-q\tau_a^-}\mathbb{1}_{\{\tau_a^-<\infty\}}\right) - \mathbb{E}_x\left(e^{-q\tau_a^-}e^{X_{\tau_a^-}}\mathbb{1}_{\{\tau_a^-<\infty\}}\right) \\ &\quad + \frac{q}{\Phi(q)}A_{\tilde{G}_1}(a)W^q(x-a). \end{aligned}$$

Using Esscher transform and Theorem 8.1 in [49], we can rewrite $h_{\tilde{G}_1}$ as,

$$\begin{aligned} h_{\tilde{G}_1}(x,a) &= K(Z^q(x-a) - \frac{q}{\Phi(q)}W^q(x-a)) \\ &\quad -e^x(Z_1^{q-\psi(1)}(x-a) - \frac{q-\psi(1)}{\Phi(q)-1}W_1^{q-\psi(1)}(x-a)) \\ &\quad +\frac{q}{\Phi(q)}(K - \frac{q-\psi(1)}{\Phi(q)-1}\frac{\Phi(q)}{q}e^a)W^q(x-a) \\ &= KZ^q(x-a) - e^x Z_1^{q-\psi(1)}(x-a) \end{aligned}.$$

for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$. It can be verifies easily that G_1 satisfies the conditions in Assumption 4.4 in Chapter 4 as well with the sets $I^1 = \{\log(K)\}$ and $I^2 = \emptyset$, and the constant a_L in Assumption 4.4 takes the following form,

$$a_L = \min\{\log(K), \log\left(\frac{qK}{q-\psi(1)}\right)\}.$$

Define $a^\star$ and $b^\star$ be such that

$$a^\star = \sup\{a < a_L : h_{\tilde{G}_1}(x,a) \geq G_1(x) \text{ for all } x \in \mathbb{R}\} \tag{5.5}$$

$$b^\star = \begin{cases} \sup \mathcal{N} & \text{if } A_{\tilde{G}_1}(a^\star) > 0 \text{ and } \mathcal{N} \neq \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{5.6}$$

where $\mathcal{N} = \{b > a^\star : h_{\tilde{G}_1}(b, a^\star_{\tilde{G}_1}) = G_1(b)\}$. Then we have the following Theorem.

Theorem 5.1. *Consider the optimal stopping problem* (5.1) *for the gain function* G_1 (5.2). *Then* $V(x) = KZ^q(x-a^\star) - e^x Z_1^{q-\psi(1)}(x-a^\star)$ *for all* $x \in \mathbb{R}$, *where* $a^\star = \log\left(K\frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}\right)$. *And the optimal stopping time is* $\tau^\star = \{t \geq 0 : X_t < a^\star\}$.

Proof. We only have to prove that $a^\star = \log\left(K\frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}\right)$ and $b^\star = \infty$. and the rest follows from Theorem 4.6. Let $a_0 = \log\left(K\frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}\right)$. By using the convexity of ψ, we have $a_0 \le a_L$. Also note that $A_{\tilde{G}_1}(a_0) = 0$, so the function $h_{\tilde{G}_1}$ can be reduced to

$$\begin{aligned} h_{\tilde{G}_1}(x, a_0) &= \mathbb{E}_x\left(e^{-q\tau_{a_0}^-}\tilde{G}_1(X_{\tau_{a_0}^-})\mathbb{1}_{\{\tau_a^- < \infty\}}\right) \\ &= \mathbb{E}\left(A_{\tilde{G}_1}(\underline{X}_{e_q} + x)\mathbb{1}_{\{\underline{X}_{e_q} + x < a_0\}}\right) \end{aligned}$$

for all $x \in \mathbb{R}$, where the second equality is due to Theorem 3.1 in [78]. As $A_{\tilde{G}_1}(a) < 0$ for all $a > a_0$, we have

$$h_{\tilde{G}_1}(x, a_0) \ \ge \ \mathbb{E}\left(A_{\tilde{G}_1}(\underline{X}_{e_q} + x)\right) = \tilde{G}_1(x), \qquad \text{for all } x \in \mathbb{R},$$

Thus, $h_{\tilde{G}_1}(x, a_0) \ge \tilde{G}_1(x) = G_1(x)$ for all $x \le \log(K)$. On the other hand, as $A_{\tilde{G}_1}(a) > 0$ for all $a < a_0$, we have for all $x > a_0$

$$h_{\tilde{G}_1}(x, a_0) = \mathbb{E}\left(A_{\tilde{G}_1}(\underline{X}_{e_q} + x)\mathbb{1}_{\{\underline{X}_{e_q} + x < a_0\}}\right) > 0 = G_1(x),$$

which allows us to conclude that $h_{\tilde{G}_1}(x, a_0) \ge G_1(x)$ for all $x \in \mathbb{R}$. So we must have $a^\star \ge a_0$. Also for all $a > a_0$, $A_{\tilde{G}_1}(a) < 0$. So by Remark 3.23, $h_{\tilde{G}_1}(x, a) < G_1(x)$ for all x large enough and $a > a_0$. Therefore, $a^\star \le a_0$. So by definition of $a^\star$, we have $a^\star = a_0$. Finally, $b^\star = 0$ can be seen from definition of $b^\star$. □

5.2 Perpetual American call option

In this section we consider the optimal stopping problem (5.1) for the following gain function

$$G_2(x) = (e^x - K)^+, \qquad \text{for all } x \in \mathbb{R}, \tag{5.7}$$

where K is a strictly positive real number. Also in this section we assume that $q > \psi(1)$ where ψ is the Laplace exponent for the Lévy process X.

Clearly, the gain function $G_2 \in \widetilde{D}$ (see Definition 4.15 in Chapter 4). Define $S_2 : \mathbb{R} \to \mathbb{R}$ by setting

$$S_2(x) = e^{-\Phi(q)x}G_2(x) \qquad \text{for all } x \in \mathbb{R}. \tag{5.8}$$

From Lemma 4.18 in Chapter 4 that S_2 has at least one global maximum point, and

$$b^\star = \sup\{b \in \mathbb{R} : S_2(b) = \max\{S_2(x) : x \in \mathbb{R}\}\}, \tag{5.9}$$

is well defined. Following the standard calculation, the constant $b^\star$ (5.9) can be found for this specific gain function G_2 (5.7),

$$b^\star = \log\left(\frac{\Phi(q)}{\Phi(q)-1}K\right) > \log(K).$$

Then we have the following Theorem on the value function for perpetual American call options.

Theorem 5.2. *Consider the optimal stopping problem* (5.1) *for the gain function* G_2 (5.7). *Then* $(V_0, \tau^+_{b^\star})$ *is a solution, where*

$$\begin{aligned} V_0(x) &= \begin{cases} G_2(b^\star)e^{-\Phi(q)(b^\star - x)} & \text{for all } x \leq b^\star \\ G_2(x) & \text{otherwise,} \end{cases} \\ \tau^+_{b^\star} &= \{t \geq 0 : X_t > b^\star\}, \end{aligned}$$

and $b^\star = \log(\frac{\Phi(q)}{\Phi(q)-1}K)$.

Proof for Theorem 5.2. By Theorem 4.16, we only need to prove that the stochastic process $\{e^{-qt}V_0(X_t), t \geq 0\}$ is a supermartingale, then the proof is done. First, we show that $\mathbb{L}_X V_0(x) - qV_0(x) \leq 0$ for all $x \in \mathbb{R} \setminus \{b^\star\}$.

As $V_0(x) = G_2(b^\star)e^{-\Phi(q)(b^\star - x)}$ for all $x \leq b^\star$, so

$$\mathbb{L}_X V_0(x) - qV_0(x) = 0 \qquad \text{for all } x < b^\star.$$

Let $\tilde{G}_2(x) = e^x - K$ for all $x \in \mathbb{R}$, then $\tilde{G}_2(x) = V_0(x)$ for all $x > b^\star$. So for all $x > b^\star$

$$\begin{aligned} \mathbb{L}_X V_0(x) - qV_0(x) &= \mathbb{L}_X \tilde{G}_2(x) - q\tilde{G}_2(x) + \int_{-\infty}^0 (V_0(x+y) - \tilde{G}_2(x+y))\Pi(dy) \\ &= (\psi(1) - q)e^x + qK + \int_{-\infty}^0 (V_0(x+y) - \tilde{G}_2(x+y))\Pi(dy). \end{aligned}$$

Also by checking $\tilde{G}_2'(x)$ and $V_0'(x)$, we see that $V_0(x) - \tilde{G}_2(x)$ is decreasing in x. From Lemma 2.5, it follows that $\int_{-\infty}^0 (V_0(x+y) - \tilde{G}_2(x+y))\Pi(dy)$ is decreasing in x for all $x > b^\star$. Thanks to Lemma 4.19, we obtain that $\mathbb{L}_X V_0(x) - qV_0(x) \leq 0$ for all $x > b^\star$.

As $V_0 \in C^1(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{b^*\})$, we can apply Itô's formula and derive

$$e^{-qt}V_0(X_t) = V_0(X_0) + \int_0^t e^{-qs}(\mathbb{L}_X V_0(X_s) - qV_0(X_s))\mathbb{1}_{\{X_s \neq b^*\}}ds + M_t^{V_0}$$

for all $t \geq 0$ $\mathbb{P}$-a.s., where M^{V_0} is the local martingale term. As $q > \psi(1)$, it follows from Theorem 3.4 in Chapter 3 that M^{V_0} is a true martingale. Thus, for all $t \geq 0$ and $x \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{E}_x\left(e^{-qt}V_0(X_t)\right) \\
&= V_0(x) + \mathbb{E}_x\left(\int_0^t e^{-qs}(\mathbb{L}_X V_0(X_s) - qV_0(X_s))\mathbb{1}_{\{X_s \neq b^*\}}ds\right) \\
&\leq V_0(x).
\end{aligned}$$

Finally, by stationary and independent increments of Lévy processes, the stochastic process $\{e^{-qt}V_0(X_t), t \geq 0\}$ is a supermartingale. This completes the proof.

□

5.3 Perpetually American strangles

In this section we consider the optimal stopping problem (5.1) for the following gain function

$$G_3(x) = (K_1 - e^x)^+ + (e^x - K_2)^+ \tag{5.10}$$

where K_1 and K_2 are two strictly positive real numbers such that $K_1 < K_2$. We assume in this section that $q > \psi(1)$ where ψ is the Laplace exponent for the spectrally negative Lévy process X.

Clearly $G_3 \in D^{1,2}(\{\log(K_1), \log(K_2)\}, \emptyset)$. Let $\tilde{G}_3 : \mathbb{R} \to \mathbb{R}$ be such that

$$\tilde{G}_3(x) = K_1 - e^x \qquad \text{for all } x \in \mathbb{R}. \tag{5.11}$$

And the averaging function $A_{\tilde{G}_3}$ with respect to $\tilde{G}_3$ and $\underline{X}_{e_q}$ is well defined, and takes the following form,

$$A_{\tilde{G}_3}(x) = K_1 - \frac{\Phi(q)}{q}\frac{q - \psi(1)}{\Phi(q) - 1}e^x \qquad \text{for all } x \in \mathbb{R}. \tag{5.12}$$

By checking the derivatives of $A_{\tilde{G}_3}$, we see that $\tilde{G}_3 \in \mathbf{G}_{G_3}$. For all $x \in \mathbb{R}$ and $a \in \mathbb{R}$, define $h_{\tilde{G}_3}(x,a)$ to be

$$h_{\tilde{G}_3}(x,a) = \mathbb{E}_x\left(e^{-q\tau_a^-}\tilde{G}_3(X_{\tau_a^-})\mathbb{1}_{\{\tau_a^-<\infty\}}\right) + \frac{q}{\Phi(q)}A_{\tilde{G}_3}(a)W^q(x-a).$$

By applying the same calculation as in the perpetual American put option example, we obtain that

$$h_{\tilde{G}_3}(x,a) = K_1 Z^q(x-a) - e^x Z_1^{q-\psi(1)}(x-a) \qquad \text{for all } x \in \mathbb{R} \text{ and } a \in \mathbb{R}.$$

By using $h_{\tilde{G}_3}$, it can be verified easily that G_3 satisfies Assumption 4.4 with the sets $I^1 = \{\log(K_1), \log(K_2)\}$ and $I^2 = \emptyset$, and the constant a_L in Assumption 4.4 takes the following form,

$$a_L = \min\{\log(K_1), \log\left(\frac{qK_1}{q-\psi(1)}\right)\}.$$

Define $a^\star$ and $b^\star$ be such that

$$a^\star = \sup\{a < a_L : h_{\tilde{G}_3}(x,a) \geq G_3(x) \text{ for all } x \in \mathbb{R}\} \tag{5.13}$$

$$b^\star = \begin{cases} \sup\mathcal{N} & \text{if } A_{\tilde{G}_3}(a^\star) > 0 \text{ and } \mathcal{N} \neq \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{5.14}$$

where $\mathcal{N} = \{b > a^\star : h_{\tilde{G}_3}(b, a^\star) = G_3(b)\}$. Then we have the following Theorem.

Theorem 5.3. *Consider the optimal stopping problem* (5.1) *for the gain function* G_3 (5.10). *Then* $a^\star < a_p < \log(K_1)$ *where* $a_p = \log\left(K_1 \frac{q}{\Phi(q)}\frac{\Phi(q)-1}{q-\psi(1)}\right)$. *and* $b^\star \in (\log(K_2), \infty)$. *Furthermore,* $(V_1, \tau_{a^\star,b^\star})$ *is a solution, where*

$$V_1(x) = \begin{cases} h_{\tilde{G}_3}(x,a^\star) & \textit{for all } x \leq b^\star \\ G_3(x) & \textit{otherwise,} \end{cases}$$

$$\tau_{a^\star,b^\star} = \{t \geq 0 : X_t \notin [a^\star, b^\star]\},$$

where $a^\star$ *and* $b^\star$ *are as defined in* (5.13) *and* (5.14).

Proof for Theorem 5.3. First we prove that $a^\star < a_p$ and $b^\star > \log(K_2)$.

As $A_{\tilde{G}_3}(a) < 0$ for all $a > a_p$, so by Remark 3.23, $h_{\tilde{G}_3}(x,a) < G_3(x)$ for all x large enough. So, by definition, $a^\star \leq a_p$. If $a^\star = a_p$, then $h_{\tilde{G}_3}(\cdot, a^\star)$ converges to 0 as x goes to ∞, so we have again $h_{\tilde{G}_3}(x,a) < G_3(x)$ for all x large enough. Therefore, $a^\star < a_p$.

Next we prove that $b^\star > \log(K_2)$. Note that, from Theorem 5.1, it follows that

$$h_{\tilde{G}_3}(x, a_p) \geq (K_1 - e^x)^+ = G_3(x) \qquad \text{for all } x \in (-\infty, \log(K_2)]. \tag{5.15}$$

So, thanks to Lemma 3.19, we have for all $x \in (a^\star, \log(K_2)]$

$$\begin{aligned} G_3(x) &\leq h_{\tilde{G}_3}(x, a_p) \\ &= h_{\tilde{G}_3}(x, a^\star) + \int_{a^\star}^{a_p} W^q(x-a)(\mathbb{L}_X \tilde{G}_3(a) - q\tilde{G}_3(a))da \\ &< h_{\tilde{G}_3}(x, a^\star), \end{aligned}$$

where the second inequality is due to

$$\mathbb{L}_X \tilde{G}_3(a) - q\tilde{G}_3(a) = (q - \psi(1))e^a - qK < 0 \qquad \text{for all } a < a_p.$$

Therefore, we can conclude that $b^\star > \log(K_2)$.

Next we show that V_1 is the global solution. By Proposition 2.7 in Chapter 2, $\{e^{-qt}V_1(X_t), t \geq 0\}$ is a supermartingale. Then by guess and verification Lemma, $V_1(x) = V(x)$ for all $x \in \mathbb{R}$. And the optimal stopping time is $\tau_{a^\star, b^\star} = \{t \geq 0 : X_t \notin [a^\star, b^\star]\}$. □

Remark 5.4. *From Theorem 4.7, we have*

$$\begin{aligned} h_{\tilde{G}_3}(b^\star, a^\star) &= G_3(b^\star). \\ \frac{\partial h_{\tilde{G}_3}}{\partial x}(b^\star, a^\star) &= G_3'(b^\star). \end{aligned}$$

By Theorem 5.3, $a^\star < \log(K_1)$ *and* $b^\star > \log(K_2)$*, thus,*

$$\begin{aligned} h_{\tilde{G}_3}(b^\star, a^\star) &= e^{b^\star} - K_2, \\ \frac{\partial h_{\tilde{G}_3}}{\partial x}(b^\star, a^\star) &= e^{b^\star}. \end{aligned}$$

Therefore, the result obtained by applying the approach in Chapter 4 shows on contradiction with the result from Chapter 2.

5.4 Novikov-Shiryaev problem

In this section we consider the optimal stopping problem (5.1) for the following gain function,

$$G_4(x) = (x^+)^n \tag{5.16}$$

for all $x \in \mathbb{R}$, where n is some natural number.

Clearly, the gain function $G_4 \in \widetilde{D}$ (see definition 4.15 in Chapter 4), so the function

$$S_4(x) = e^{-\Phi(q)x} G_4(x), \qquad \text{for all } x \in \mathbb{R}$$

has at least one global maximum point. By the standard calculation, we derive that $S_4(x)$ is maximized at $x = b_0^\star$, where $b_0^\star = n/\Phi(q)$. Then we have the following Theorem.

Theorem 5.5. *Consider the optimal stopping problem* (5.1) *for the gain function* G_4 (5.16). *Define* $V_0 : \mathbb{R} \to \mathbb{R}$ *by setting*

$$V_0(x) = \begin{cases} G_4(x) e^{-\Phi(q)(b_0^\star - x)} & x \le b_0^\star \\ G_4(x) & x > b_0^\star, \end{cases} \tag{5.17}$$

where $b_0^\star = n/\Phi(q)$. *Then* $V(x) = V_0(x)$ *for all* $x \in \mathbb{R}$. *Furthermore, the optimal stopping time* $\tau^\star$ *is*

$$\tau^\star = \inf\{t \ge 0 : X_t > b_0^\star\}.$$

By Theorem 4.6, $(V_0, \tau^\star)$ is a closed left semi-solution up to the point $b_0^\star$. The supermartingale property of $\{e^{-qt} V_0(X_t), t \ge 0\}$ can be seen from Theorem 4.2 in [78]. Hence, the proof is omitted.

We remark here that the Novikov-Shiryaev problem has been studied by many authors. The problem is studied by Novikov and Shiryaev [61] for random walks, and conjectured that the results can be generalized to Lévy processes. Later on, Kyprianou and Surya [47] verified this conjecture by using fluctuation theory of Lévy processes. In both of the aforementioned articles, it is observed that the Appell polynomials Q_n, generated by the random variable $\overline{X}_{e_q}$, have been the key in finding the solution, as the optimal stopping boundary are found as the largest root of the following equation,

$$Q_n(x) = 0. \tag{5.18}$$

Our result shows no contradiction with the existing work. Indeed, $b_0^\star = n/\Phi(q)$ is the largest root of equation (5.18).

5.5 Linear combination of two perpetual American call options

In this section we consider the optimal stopping problem (5.1) for the following gain function,

$$G_5(x) = c(e^x - K_1)^+ + (1-c)(e^x - K_2)^+ \tag{5.19}$$

for all $x \in \mathbb{R}$, where K_1 and K_2 are two strictly positive real numbers such that $K_1 < K_2$, and $c \in (0,1)$. Furthermore, we suppose that $q > \psi(1)$.

Clearly, the gain function $G_5 \in \widetilde{D}$. Define

$$S_5(x) = G_5(x)e^{-\Phi(q)x}.$$

Let $b_0^\star$ be such that

$$b_0^\star = \sup\{b \in \mathbb{R} : S_5(b) = \max\{S_5(x) : x \in \mathbb{R}\}\}. \tag{5.20}$$

After some standard calculation, it can be shown that we have either one of the two cases below,

1. $b_0^\star = \log\left(\frac{\Phi(q)K_1}{\Phi(q)-1}\right)$ and $b_0^\star < \log(K_2)$,

2. $b_0^\star = \log\left(\frac{\Phi(q)(cK_1+(1-c)K_2)}{\Phi(q)-1}\right)$ and $b_0^\star > \log(K_2)$.

Then we have the following theorem.

Theorem 5.6. *Consider the optimal stopping problem* (5.1) *for the gain function* G_5 (5.19). *Then,*

(i) if $b_0^\star = \log\left(\frac{\Phi(q)(cK_1+(1-c)K_2)}{\Phi(q)-1}\right)$, *then the pair* $(V_0, \tau_0^\star)$ *is a solution to the opti-*

mal stopping problem (5.1), *where*

$$
\begin{aligned}
V_0(x) &= \begin{cases} G_5(x)e^{-\Phi(q)(b_0^\star - x)} & x \leq b_0^\star \\ G_5(x) & x > b_0^\star. \end{cases} \\
\tau_0^\star &= \inf\{t \geq 0 : X_t > b_0^\star\}.
\end{aligned}
\tag{5.21}
$$

where $b_0^\star$ *is as defined in equation* (5.20).

(ii) if $b_0^\star = \log\left(\frac{\Phi(q)K_1}{\Phi(q)-1}\right)$, *then the pair* $(V_0, \tau_0^\star)$ *is a closed left semi-solution up to the point* $b_0^\star$.

The following Lemma is need for the proof of Theorem 5.6.

Lemma 5.7.

(i) If $b_0^\star = \log\left(\frac{\Phi(q)K_1}{\Phi(q)-1}\right)$, *then* $\mathbb{L}_X V_0(x) - qV_0(x)$ *is decreasing in* $(b_0^\star, \log(K_2))$.

(ii) If $b_0^\star = \log\left(\frac{\Phi(q)(cK_1+(1-c)K_2)}{\Phi(q)-1}\right)$, *then* $\mathbb{L}_X V_0(x) - qV_0(x)$ *is decreasing in* $(b_0^\star, \infty)$.

Proof. Suppose that $b_0^\star = \log\left(\frac{\Phi(q)(cK_1+(1-c)K_2)}{\Phi(q)-1}\right)$, then $b_0^\star > \log(K_2)$. So for all $x > b_0^\star$,

$$
\mathbb{L}_X V_0(x) - qV_0(x) = \mathbb{L}_X G_0(x) - qG_0(x) + \int_{-\infty}^{0} (V_0(x+y) - G_0(x+y))\Pi(dy),
\tag{5.22}
$$

where $G_0(x) = e^x - (cK_1 + (1-c)K_2)$. By checking the first derivative of $V_0(x) - G_0(x)$, we see that $V_0(x) - G_0(x)$ is positive and decreasing for all $x \in \mathbb{R}$. Therefore, it follows from Lemma 2.5 that $\int_{-\infty}^{0}(V_0(x+y) - G_0(x+y))\Pi(dy)$ is decreasing in x. Also by checking $\mathbb{L}_X G_0(x) - qG_0(x)$, we can conclude that $\mathbb{L}_X V_0(x) - qV_0(x)$ is decreasing in x on $(b_0^\star, \infty)$.

For the other case, a similar argument as above can be made to get the result as required. □

Proof for Theorem 5.6. *(i).*

Using Lemma 5.7, by a similar argument as in the American call case, we can derive that $\mathbb{L}_X V_0(x) - qV_0(x) \leq 0$ for all $x \in \mathbb{R} \setminus \{b_0^\star\}$. Then, by using Itô's formula and Theorem 3.4, we have

$$
\mathbb{E}_x\left(e^{-qt}V_0(X_t)\right) \leq V_0(x)
$$

for all $x \in \mathbb{R}$. Then by the stationary and independent increments of Lévy processes, $\{e^{-qt}V_0(X_t), t \geq 0\}$ is a supermartingale. Then, the rest follows from Theorem 4.16.

(ii)

The second part of theorem is a direct application of the Theorem 4.16 in Chapter 4.

□

Now we consider the case where $b_0^\star = \log\left(\frac{\Phi(q)K_1}{\Phi(q)-1}\right)$ and $b_0^\star < K_2$. From Lemma 5.7 and Remark 4.5, it follows that V_0 (5.21) satisfies the Assumption 4.4 withe the sets $I_{V_0}^1 = \{\log(K_2)\}$ and $I_{V_0}^2 = \{b_0^\star\}$. Let us denote by a_{L,V_0} the constant a_L in Assumption 4.4 for V_0 (5.21). And let $\tilde{G}_5$ be a choice in $\mathbf{G}_{V_0}$, and $A_{\tilde{G}_5}$ be the unique type (L) averaging function w.r.t. $\tilde{G}_5$ and $\underline{X}_{e_q}$, and define the function $h_{\tilde{G}_5}$ by setting

$$h_{\tilde{G}_5}(x,a) \quad = \quad \mathbb{E}_x\left(A_{\tilde{G}_5}(\underline{X}_{e_q})\mathbb{1}_{\{\underline{X}_{e_q}<a\}}\right) + \frac{q}{\Phi(q)}A_{\tilde{G}_5}(a)W^q(x-a).$$

for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$. And let $a_1^\star$ and $b_1^\star$ be such that

$$a_1^\star \quad = \quad \sup\{a < a_{L,V_0} : h_{\tilde{G}_5}(x,a) \geq V_0(x) \text{ for all } x \in \mathbb{R}\}, \tag{5.23}$$

$$b_1^\star \quad = \quad \begin{cases} \sup \mathcal{N} & \text{if } A_{\tilde{G}_5}(a_1^\star) > 0 \text{ and } \mathcal{N} \neq \emptyset \\ \infty & \text{otherwise,} \end{cases} \tag{5.24}$$

where $\mathcal{N} = \{b > a_1^\star : h_{\tilde{G}_5}(b, a_1^\star) = V_0(b)\}$. Then we have the following Theorem for the case when $b_0^\star = \frac{\Phi(q)K_1}{\Phi(q)-1}$.

Theorem 5.8. *Consider the optimal stopping problem* (5.1) *for the gain function* G_5 (5.19), *and suppose that* $b_0^\star = \frac{\Phi(q)K_1}{\Phi(q)-1}$. *Then* $b_1^\star \in (\log(K_2), \infty)$, *and* $(V_1(x), \tau_1^\star)$ *is a solution pair, where*

$$V_1(x) \quad = \quad \begin{cases} h_{\tilde{G}_5}(x, a_1^\star) & x \leq b_1^\star \\ G_5(x) & x > b_1^\star. \end{cases} \tag{5.25}$$

$$\tau_1^\star \quad = \quad \inf\{t \geq 0 : X_t \notin (-\infty, b_0^\star] \cup [a_1^\star, b_1^\star]\}. \tag{5.26}$$

The following proposition is needed in proving Theorem 5.8.

Proposition 5.9. *Consider the optimal stopping problem* (5.1) *for a general gain function* g *such that* $g \circ \log$ *is well defined for all* $x > 0$ *and convex. Then the value function composed with* $\log$, *i.e.* $V \circ \log$ *is convex as well.*

Proof for Proposition 5.9. Define $Y_t = e^{X_t}$ for all $t \geq 0$, and $y = e^x$. Then we can rewrite V (5.1) as follows,

$$\bar{V}(y) := V(\log(y)) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_y \left(e^{-q\tau}\bar{g}(Y_\tau)\right) \tag{5.27}$$

for all $y = e^x$ and $x \in \mathbb{R}$, where $\bar{g}(x) = g \circ \log(x)$.

As the gain function $\bar{g}$ is convex in $(0, \infty)$, we have for all $0 < y_1 \leq y_2 < \infty$ and $c \in [0, 1]$,

$$\bar{g}(cy_1 + (1-c)y_2) \leq c\bar{g}(y_1) + (1-c)\bar{g}(y_2). \tag{5.28}$$

Thus,

$$\begin{aligned}
& c\bar{V}(y_1) + (1-c)\bar{V}(y_2) \\
&= c \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_{y_1}(e^{-q\tau}\bar{g}(Y_\tau)) + (1-c) \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_{y_2}(e^{-q\tau}\bar{g}(Y_\tau)) \\
&= \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_1(ce^{-q\tau}\bar{g}(y_1Y_\tau)) + \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_1((1-c)e^{-q\tau}\bar{g}(y_2Y_\tau)) \\
&\geq \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \left(\mathbb{E}_1(ce^{-q\tau}\bar{g}(y_1Y_\tau)) + \mathbb{E}_1((1-c)e^{-q\tau}\bar{g}(y_2Y_\tau))\right) \\
&= \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \left(\mathbb{E}_1(e^{-q\tau}\left(c\bar{g}(y_1Y_\tau) + (1-c)\bar{g}(y_2Y_\tau)\right)\right),
\end{aligned}$$

by equation (5.28), we obtain

$$\begin{aligned}
c\bar{V}(y_1) + (1-c)\bar{V}(y_2) &\geq \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \left(\mathbb{E}_1(e^{-q\tau}\left(\bar{g}(cy_1Y_\tau + (1-c)y_2Y_\tau)\right)\right) \\
&\geq \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \left(\mathbb{E}_{cy_1+(1-c)y_2}(e^{-q\tau}\left(\bar{g}(Y_\tau)\right)\right) \\
&= \bar{V}(cy_1 + (1-c)y_2),
\end{aligned} \tag{5.29}$$

as required. □

Proof for Theorem 5.8. As a result of Theorem 4.20 and Lemma 4.23, both $a_1^\star$ and $b_1^\star$ are well defined with $a_1^\star \in (b_0^\star, \log(K_2))$, and $V_1 = V$ on the set $(-\infty, b_1^\star]$. Next, we showe that $b_1^\star \in (\log(K_2), \infty)$.

If $b_1^\star = \infty$, then by part (vii) Lemma 4.23, $h_{\tilde{G}_5}(\cdot, a_1^\star) \to 0$ as $x \to \infty$. Therefore, $h_{\tilde{G}_5}(x, a_1^\star) < G_5(x)$ for all x large enough, which contradicts part (iv) in Lemma 4.23. Thus, we must have $b_1^\star < \infty$.

Now consider the case where $b_1^\star \leq \log(K_2)$. As $G_5 \circ \log$ is linear on the interval $[e^{a_1^\star}, e^{b_1^\star}]$, so it is the largest function on $[e^{a_1^\star}, e^{b_1^\star}]$ among all the convex functions $f : [e^{a_1^\star}, e^{b_1^\star}] \to \mathbb{R}$ such that $f = G_5 \circ \log$ on the set $\{e^{a_1^\star}, e^{b_1^\star}\}$. Also by

Proposition 5.9, $V_1 \circ \log$ is convex on the interval $[e^{a_1^\star}, e^{b_1^\star}]$ with $V_1 \circ \log = G_5 \circ \log$ on the set $\{e^{a_1^\star}, e^{b_1^\star}\}$. Thus, $V_1 \leq G_5$ on the set $(a_1^\star . b_1^\star)$. This clearly contradicts part (iv) in Lemma 4.23, which allows us to conclude that $b_1^\star > \log(K_2)$.

Finally, By applying the same calculation as in the American call example, we see that $\mathbb{L}_X V_1(x) - qV_1(x) \leq 0$ for all $x > b_1^\star$. Therefore, by Theorem 4.20, $V_1(x) = V(x)$ for all $x \in \mathbb{R}$. This completes the proof for the theorem.

□

5.6 An example where $a_i^\star$ and $b_i^\star$ diverge

In this section we consider the optimal stopping problem (5.1) where the underlying spectrally negative Lévy process X is pure Gaussian with coefficient $\sigma = 1$, and the gain function takes the following form,

$$G_6(x) = \begin{cases} \sin(x) & x \geq -\pi/2 \\ -1 & x < -\pi/2. \end{cases} \tag{5.30}$$

Clearly, $G_6 \in \widetilde{D}$ (see Definition 4.15 in Chapter 4). Let $S_6 : \mathbb{R} \to \mathbb{R}$ be such that

$$S_6(x) = e^{-\Phi(q)x} G_6(x).$$

Then by standard calculation, we see that the unique maximum point happens at $x = b_0^\star$, where $b_0^\star = \arctan\left(\frac{1}{\Phi(q)}\right) \in (0, \pi/2)$. So we have the following Theorem for the closed left semi-solution of the optimal stopping problem.

Theorem 5.10. *Consider the optimal stopping problem* (5.1) *for the gain function* G_6 (5.30) *where the underlying process* X *is the standard Brownian motion. Then the pair* $(V_0(x), \tau_0^\star)$ *is a closed left semi-solution up to the point* $b_0^\star$*, where*

$$V_0(x) = \begin{cases} \sin(b_0^\star) e^{-\Phi(q)(b_0^\star - x)} & x \leq b_0^\star \\ \sin(x) & x > b_0^\star. \end{cases} \tag{5.31}$$

$$\tau_0^\star = \{t \geq 0 : X_t > b_0^\star\}, \tag{5.32}$$

and $b_0^\star = \arctan\left(\frac{1}{\Phi(q)}\right)$.

Theorem 5.10 can be obtained as a direct application of Theorem 4.16 for the gain function G_6 .

Next we calculate the infinitesimal generator for the function V_0 for $x > b_0^\star$.

$$\mathbb{L}_X V_0(x) - qV_0(x) = -\frac{1}{2}\sin(x) - q\sin(x). \tag{5.33}$$

Thus, $\mathbb{L}_X V_0(x) - qV_0(x) > 0$ for all $x \in ((2n-1)\pi, 2n\pi)$, $n \in \mathbb{N}$. Therefore, from Remark 4.5, it follows that V_0 satisfies Assumption 4.4 with the sets $I^1_{V_0} = \emptyset$ and $I^2_{V_0} = \{b_0^\star\}$). Let $A_1 : \mathbb{R} \to \mathbb{R}$ be such that

$$A_1(x) \quad = \quad \frac{\Phi(q)}{q} e^{\Phi(q)x} \int_x^\infty e^{-\Phi(q)y} (qV_0(y) - \mathbb{L}_X V_0(y)) \mathbb{1}_{\{y \neq b_0^\star\}} dy,$$

for all $x \in \mathbb{R}$. Then A_1 is the unique type (L) averaging function w.r.t. V_0 and $\underline{X}_{e_q}$. And denote by a_{L,V_0} the constant a_L as specified in Assumption 4.4. Then, $a_{L,V_0} = \pi$. Define $h_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by setting

$$h_1(x,a) \quad = \quad \mathbb{E}_x \left(A_1(\underline{X}_{e_q}) \mathbb{1}_{\{\underline{X}_{e_q} < a\}} \right) + \frac{q}{\Phi(q)} A_1(a) W^q(x-a),$$

for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$. And let $a_1^\star$ and $b_1^\star$ be such that

$$a_1^\star \quad = \quad \sup\{a < a_{L,V_0} : h_1(x,a) \geq V_0(x) \text{ for all } x \in \mathbb{R}\} \tag{5.34}$$

$$b_1^\star \quad = \quad \begin{cases} \sup \mathcal{N}_1 & \text{if } A_1(a_1^\star) > 0 \text{ and } \mathcal{N}_1 \neq \emptyset \\ \infty & \text{otherwise.} \end{cases} \tag{5.35}$$

where $\mathcal{N}_1 = \{b > a_1^\star : h_1(b, a_1^\star) = V_0(b)\}$. Then we have the following Theorem.

Theorem 5.11. *Consider the optimal stopping problem* (5.1) *for the gain function* (5.30) *where the underlying process X is the standard Brownian motion. Then $b_1^\star < \infty$, $b_1^\star - a_1^\star > \pi$, and the pair $(V_1(x), \tau_1^\star)$ is a closed left semi-solution up to the point $b_1^\star$, where*

$$V_1(x) \quad = \quad \begin{cases} h_1(x, a_1^\star) & x \leq b_1^\star \\ g(x) & x > b_1^\star. \end{cases}$$

$$\tau_1^\star \quad = \quad \inf\{t \geq 0 : X_t \notin (-\infty, b_0^\star] \cup [a_1^\star, b_1^\star]\}.$$

Proof of Theorem 5.11. If $b_1^\star = \infty$, then it follows from part (vii) in Lemma 4.23 that $h_1(\cdot, a_1^\star) \to 0$ as $x \to \infty$, which clearly contradicts part (iv) in Lemma 4.23. Therefore, $b_1^\star < \infty$.

Next, we prove $b_1^\star - a_1^\star > \pi$. This is done by showing that $a_1^\star < \pi$ and $b_1^\star > 2\pi$. Indeed, it follows from part (iii) in Lemma 4.23 that $a_1^\star < a_{L,V_0} = \pi$. The fact that $b^\star > 2\pi$ can be seen from equation (5.33) and Corollary 4.25. The rest can be obtained directly from Theorem 4.20. This completes the proof. $\square$

By checking $\mathbb{L}_X V_1(x) - qV_1(x)$ for all $x > b_1^\star$, we see that V_1 satisfies Assumption 4.4 as well. We can apply the above construction again to V_1, and find a new closed left semi value function V_2 up to some point $b_2^\star \in (b_1^\star, \infty)$. In fact, because of the periodic property of the gain function G_6, the closed left semi value function V_i satisfies Assumption 4.4 for all $i \geq 1$. Furthermore, by using the same argument as in Theorem 5.11, we can show that $b_i^\star - a_i^\star > \pi$ holds true for all $i \in \mathbb{N}$. Thus, by repeating this construction above, we obtain two sequences $\{a_i^\star, i \in \mathbb{N}\}$ and $\{b_i^\star, i \in \mathbb{N}\}$ that are diverging to ∞.

5.7 An example where $a_i^\star$ and $b_i^\star$ converge

Throughout this section we assume that the spectrally negative Lévy process X is a standard Brownian motion $\{B_t, t \geq 0\}$ with Gaussian coefficient $\sigma = 1$. We first construct a twice continuously differentiable function $\hat{V} : \mathbb{R} \to \mathbb{R}$ such that $\{e^{-qt}\hat{V}(B_t), t \geq 0\}$ is a $\mathbb{P}_x$ supermartingale, and then find a function $g : \mathbb{R} \to \mathbb{R}$ such that $\hat{V}$ is the value function for the optimal stopping problem (5.1) for this gain function g. Finally, for this gain function g, we apply the approach as suggested in Chapter 3, and show that two strictly increasing countable sequences of $a_i^\star$ and $b_i^\star$ can be obtained, and they are converging.

First let us construct the function $\hat{V}$. Let $\{a_i \in \mathbb{R}, i \geq 1\}$ and $\{b_i \in \mathbb{R}, i \geq 1\}$ be two strictly increasing sequences such that $a_i < b_i < a_{i+1} < b_{i+1}$ for all $i = 1, 2, 3, ...$, and $\{a_i, i \geq 1\}$ and $\{b_i, i \geq 1\}$ are both converging to b. Let $l : \mathbb{R} \to \mathbb{R}$ be such that $l \in C^2(\mathbb{R})$, $l(x) = 0$ on the set on the set $\bigcup_{i=1}^{\infty}(a_i, b_i)$, and $l(x) < 0$ on $\left(\bigcup_{i=0}^{\infty}(a_i, b_i)\right)^c$, and the limits of $l(x)$ exist and are strictly negative at both ∞ and $-\infty$, and all $|l|$, $|l'|$ and $|l''|$ are bounded on $\mathbb{R}$. Define $\hat{V}$ to be the solution of the following Feynman Kac problem

$$\frac{1}{2}f''(x) - qf(x) = l(x), \qquad \text{for all } x \in \mathbb{R}. \tag{5.36}$$

where q is the strictly positive constant from the optimal stopping problem (5.1). It is well known in the literature that the Feynman Kac problem has the solution of the following form, see Kac [43], Kac [44] and page 271 in [45],

$$\hat{V}(x) = -\mathbb{E}_x\left(\int_0^\infty e^{-qt} l(B_t)dt\right), \qquad \text{for all } x \in \mathbb{R}.$$

Note that, as $|l'|$ and $|l''|$ are bounded on $\mathbb{R}$, then it follows from the bounded convergence theorem, that $\hat{V} \in C^2(\mathbb{R})$. Furthermore, as l is bounded in $\mathbb{R}$, we see that V is bounded on $\mathbb{R}$ as well, and limits of $\hat{V}$ at $-\infty$ and ∞ exist and are strictly positive. By Itô's formula we derive that

$$\begin{aligned} e^{-qt}\hat{V}(B_t) &= \hat{V}(B_0) + \int_0^t e^{-qs}\left(\mathbb{L}_X\hat{V}(B_s) - q\hat{V}(B_s)\right)ds + M_t^{\hat{V}} \\ &= \hat{V}(B_0) + \int_0^t e^{-qs}l(B_s)ds + M_t^{\hat{V}}, \end{aligned}$$

for all $t \geq 0$ $\mathbb{P}$-a.s., and $M^{\hat{V}}$ is a $\mathbb{P}_x$ local martingale. Moreover, as $\hat{V}$ is bounded on $\mathbb{R}$, $M^{\hat{V}}$ is a $\mathbb{P}_x$ martingale. Together with $\mathbb{L}_X\hat{V}(x) - q\hat{V}(x) = l(x) \leq 0$ for all $x \in \mathbb{R}$, we have for all $x \in \mathbb{R}$

$$\mathbb{E}_x\left(e^{-qt}\hat{V}(B_t)\right) \leq \hat{V}(x).$$

Finally, by stationary and independent increment of Brownian motion, we see that $\{e^{-qt}\hat{V}(B_t), t \geq 0\}$ is a supermartingale as required.

Next we construct the function $g : \mathbb{R} \to \mathbb{R}$. Define $l_1 : \mathbb{R} \to \mathbb{R}$ by

$$l_1(x) = \begin{cases} 0 & x < a_1 \\ -(x-a_1)^3 & x \in [a_1, a_1+\epsilon_1] \\ \bar{l}_1(x) & x \in (a_1+\epsilon_1, b_1) \\ 0 & x \geq b_1 \end{cases},$$

where $0 < \epsilon_1 < \frac{\left((a_1+\sqrt{3/q})\wedge b_1\right)-a_1}{2}$, and $\bar{l}_1 : \mathbb{R} \to \mathbb{R}$ is a function such that $l_1 \in C^2(\mathbb{R})$, and all $|l_1|$, $|l_1'|$ and $|l_1''|$ are bounded on $\mathbb{R}$ by some strictly positive constant D_1, and $l_1 < 0$ on the set $\mathbb{R}$. Similar, we can define $l_i : \mathbb{R} \to \mathbb{R}$ for all $i \geq 2$, $i \in \mathbb{N}$, as

$$l_i(x) = \begin{cases} 0 & x < a_i \\ -(x-a_i)^3 & x \in [a_i, a_i+\epsilon_i] \\ \bar{l}_i(x) & x \in (a_i+\epsilon_i, b_i) \\ 0 & x \geq b_i \end{cases},$$

where $0 < \epsilon_i < \max\{\frac{\left((a_i+\sqrt{3/q})\wedge b_i\right)-a_i}{2}, \epsilon_{i-1}\}$ for all $i \geq 2$. and $\bar{l}_i : \mathbb{R} \to \mathbb{R}$ is a function such that $l_i \in C^2(\mathbb{R})$, and all $|l_i|$, $|l_i'|$ and $|l_i''|$ are bounded on $\mathbb{R}$ by D_1, and

$l_i < 0$ on the set $\mathbb{R}$. Then, we define $g : \mathbb{R} \to \mathbb{R}$ by setting

$$g(x) = \hat{V}(x) + \sum_{i=1}^{\infty} l_i(x) \mathbb{1}_{\{x \in (a_i, b_i)\}}, \qquad x \in \mathbb{R}. \tag{5.37}$$

So $g \in C^2(\mathbb{R})$, $g(x) \le \hat{V}(x)$ on $\mathbb{R}$, $g(x) < \hat{V}(x)$ on the set $\bigcup_{i=1}^{\infty}(a_i, b_i)$, and all $|g|$, $|g'|$ and $|g''|$ are bounded on $\mathbb{R}$. Note that, by the construction of ϵ_1, we obtain that

$$\mathbb{L}_X g(x) - qg(x) < 0 \qquad \text{for all } x \in (a_i, a_i + \epsilon_i) \text{ and } i \in \mathbb{N}.$$

By the guess and verification lemma, we derive that for all $x \in \mathbb{R}$

$$\hat{V}(x) = V(x) = \sup_{\tau \in \mathcal{T}_{[0,\infty]}} \mathbb{E}_x(e^{-q\tau} g(B_\tau)),$$

and the optimal stopping time is $\tau^\star = \inf\{t \ge 0 : B_t \notin \bigcup_{i=1}^{\infty}[a_i, b_i]\}$.

Next, we study the optimal stopping problem (5.1) for the gain function g (5.37) using the approach suggested in Chapter 3. Clearly $g \in D^2(\emptyset)$ (see Definition 3.3 in Chapter 3). So we can define $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by setting

$$h(x,a) = \mathbb{E}_x\left(e^{-q\tau_a^-} g(B_{\tau_a^-}) \mathbb{1}_{\{\tau_a^- < \infty\}}\right) + \frac{q}{\Phi(q)} A_g(a) W^q(x-a)$$

for all $x \in \mathbb{R}$ and $a \in \mathbb{R}$, where

$$A_g(x) = \frac{\Phi(q)}{q} e^{\Phi(q)x} \int_x^{\infty} e^{-\Phi(q)y}(qg(y) - \mathbb{L}_X g(y))dy \tag{5.38}$$

for all $x \in \mathbb{R}$. Let $a_{L,1}$ be such that

$$a_{L,1} = \sup\{a \in \mathbb{R} : \mathbb{L}_X g(x) - qg(x) \le 0 \text{ for all } x \le a\}.$$

Then we have the following Lemma.

Lemma 5.12.

(i) $a_{L,1} \in (a_1 + \epsilon_1, b_1)$.

(ii) $h(x, a_1) = \hat{V}(x)$ *for all* $x \le b_1$.

(iii) $h(b_1, a) < g(b_1) = \hat{V}(b_1)$ *for all* $a \in (a_1, a_{L,1}]$, *and* $h(x, a_1) > \hat{V}(x) \ge g(x)$ *for all* $x > b_1$.

As a result of Lemma 5.12, g satisfies Assumption 3.24. So we can apply the approach in Chapter 3 and obtain the following result.

Theorem 5.13. *Consider the optimal stopping problem* (5.1) *for the gain function* g (5.37). *Define* $a_1^\star$ *and* $b_1^\star$ *to be*

$$a_1^\star = \sup\{a < a_{L,1} : h(x,a) \geq g(x) \text{ for all } x \in \mathbb{R}\} \quad (5.39)$$

$$b_1^\star = \begin{cases} \sup \mathcal{N}_1 & \text{if } A_g(a_1^\star) > 0 \text{ and } \mathcal{N}_1 \neq \emptyset \\ \infty & \text{otherwise.} \end{cases} \quad (5.40)$$

where $\mathcal{N}_1 = \{b > a_1^\star : h(b, a_1^\star) = g(b)\}$. *Then* $a_1^\star = a_1$ *and* $b_1^\star = b_1$. *and the pair* $(V_1, \tau_1^\star)$ *is a closed left semi-solution up to the point* b_1. *where*

$$V_1(x) = \begin{cases} h(x, a_1) & \text{if } x \in (-\infty, b_1] \\ g(x) & \text{otherwise,} \end{cases} \quad (5.41)$$

$$\tau_1^\star = \inf\{t \geq 0 : X_t \notin [a_1, b_1]\}. \quad (5.42)$$

Theorem 5.13 can be obtained as a result of Lemma 5.12 and Theorem 3.26.

Because of the structure of g, by using the same argument, we can verify that $V_1 \in D^2(I_{V_1})$ and satisfies Assumption 3.24 for some finite set I_{V_1}. Hence, we can repeat this construction again, and obtain a new pair $(a_2^\star, b_2^\star)$ for the stopping boundaries such that $a_2^\star = a_2$ and $b_2^\star = b_2$. Actually, thanks to the structure of g, we can repeat this approach for a countable number of times, and show that $a_i^\star = a_i$ and $b_i^\star = b_i$ for all $i \in \mathbb{N}$, and $\hat{V}(x) = V_i(x)$ for all $x \leq b_i$. Thus, we obtain two sequences $a_i^\star$ and $b_i^\star$ which are converging to b as required.

Proof for Lemma 5.12. *(i)*

By the construction of g, $\mathbb{L}_X g(x) - qg(x) < 0$ for all $x \leq a_1 + \epsilon_1$. So $a_{L,1} > a_1 + \epsilon_1$. The proof for $a_{L,1} < b_1$ is done by contradiction. Suppose that $a_{L,1} \geq b_1$, that is, $\mathbb{L}_X g(x) - qg(x) \leq 0$ for all $x \in (a_1, b_1)$. So, by Itô's formula, we derive

$$e^{-qt} g(B_t) = g(B_0) + \int_0^t e^{-qs} \left(\mathbb{L}_X g(B_s) - qg(B_s) \right) ds + M_t^g,$$

for all $t \geq 0$ $\mathbb{P}$-a.s., where M^g is the local martingale term. Let τ_n be the localization sequence for it. Then ,

$$\mathbb{E}_x \left(e^{-qt \wedge \tau_{a_1,b_1} \wedge \tau_n} g(B_{t \wedge \tau_{a_1,b_1} \wedge \tau_n}) \right) \leq g(x),$$

for all $x \in (a_1, b_1)$ and $t \geq 0$. As g is bounded on $[a_1, b_1]$, by letting t and n go to

∞ and applying the bounded convergence theorem, we get,

$$\mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}}g(B_{\tau_{a_1,b_1}})\right) \leq g(x) < \hat{V}(x), \tag{5.43}$$

for all $x \in (a_1, b_1)$, where the last inequality is due to the construction of g. Same calculation as above can be applied to $\hat{V}$ and obtain

$$\mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}}\hat{V}(B_{\tau_{a_1,b_1}})\right) = \hat{V}(x). \tag{5.44}$$

By combining equations (5.43) and (5.44) we derive for all $x \in (a_1, b_1)$

$$\mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}}\hat{V}(B_{\tau_{a_1,b_1}})\right) > \mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}}g(B_{\tau_{a_1,b_1}})\right). \tag{5.45}$$

On the other hand, as $g(x) = \hat{V}(x)$ for all $x \in \{a_1, b_1\}$ and $B_{\tau_{a_1,b_1}} \in \{a_1, b_1\}$ $\mathbb{P}$-a.s., we obtain

$$\mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}}\hat{V}(B_{\tau_{a_1,b_1}})\right) = \mathbb{E}_x\left(e^{-q\tau_{a_1,b_1}}g(B_{\tau_{a_1,b_1}})\right), \tag{5.46}$$

which clearly contradicts equation (5.45). Therefore, $a_{L,1} < b_1$ as required.

(ii)

The proof for (ii) can be done by using a similar argument as in Proposition 3.22 in Chapter 3.

(iii)

From part (ii), we know that $h(b_1, a_1) = \hat{V}(b_1) = g(b_1)$. Then, thanks to Lemma 3.19 and the fundamental theorem of calculus, we derive that for all $a \in (a_1, a_{L,1}]$

$$\begin{aligned} h(b_1, a) &= h(b_1, a_1) + \int_{a_1}^{a} W^q(b_1 - \tilde{a})\left(\mathbb{L}_X g(\tilde{a}) - qg(\tilde{a})\right) d\tilde{a} \\ &< h(b_1, a_1) \\ &= g(b_1), \end{aligned}$$

where the inequality is due to $a < a_{L,1}$, and $\mathbb{L}_X g(x) - qg(x) < 0$ for all $x \in (a_1, a_1 + \epsilon_1)$ (see the construction of l_1 for ϵ_1).

Next we show that $\hat{V}(x) < h(x, a_1)$ for all $x > b_1$. The proof is done by contradiction. Suppose that there exists $x_0 > b_1$ such that $\hat{V}(x_0) \geq h(x_0, a_1)$. Then by Proposition 3.18, we obtain that

$$\mathbb{E}_{b_1}\left(e^{-qt\wedge\tau_{a_1,x_0}}h(B_{t\wedge\tau_{a_1,x_0}}, a_1)\right) = h(b_1, a_1).$$

Then, by letting t go to ∞ and applying the bounded convergence theorem, we get

$$\mathbb{E}_{b_1}\left(e^{-q\tau_{a_1,x_0}}h(B_{\tau_{a_1,x_0}},a_1)\right) = h(b_1,a_1) = \hat{V}(b_1). \tag{5.47}$$

Also we can apply Itô's formula on $e^{-qt}\hat{V}(X_t)$ and derive for all $t \geq 0$

$$\begin{aligned}
&\mathbb{E}_{b_1}\left(e^{-qt\wedge\tau_{a_1,x_0}\wedge\tau_n}\hat{V}(B_{t\wedge\tau_{a_1,x_0}\wedge\tau_n})\right) \\
&= \hat{V}(b_1) + \mathbb{E}_{b_1}\left(\int_0^{t\wedge\tau_{a_1,x_0}\wedge\tau_n} e^{-qs}(\mathbb{L}_X\hat{V}(B_s) - q\hat{V}(B_s))ds\right) + \mathbb{E}_{b_1}\left(M^h_{t\wedge\tau_{a_1,x_0}\wedge\tau_n}\right) \\
&= \hat{V}(b_1) + \mathbb{E}_{b_1}\left(\int_0^{t\wedge\tau_{a_1,x_0}\wedge\tau_n} e^{-qs}(\mathbb{L}_X\hat{V}(B_s) - q\hat{V}(B_s))ds\right),
\end{aligned}$$

where M^h is the local martingale term, and τ_n is the localization sequence for it, and the second equality is due to optional sampling theorem. As $\hat{V}(x)$ and $\mathbb{L}_X\hat{V}(x) - q\hat{V}(x) = l(x)$ are both bounded for all $x \in \mathbb{R}$, so by letting t and n go to ∞ and applying the bounded convergence theorem, we derive

$$\begin{aligned}
\mathbb{E}_{b_1}\left(e^{-q\tau_{a_1,x_0}}\hat{V}(B_{\tau_{a_1,x_0}})\right) &= \hat{V}(b_1) + \mathbb{E}_{b_1}\left(\int_0^{\tau_{a_1,x_0}} e^{-qs}(\mathbb{L}_X\hat{V}(B_s) - q\hat{V}(B_s))ds\right) \\
&< \hat{V}(b_1),
\end{aligned} \tag{5.48}$$

where the second inequality is due to that $\mathbb{L}_X\hat{V}(x) - q\hat{V}(x) = l(x)$ for all $x \in \mathbb{R}$, and $l(x) \leq 0$ for all $x \in \mathbb{R}$ and $l(x) < 0$ on the set $(b_1, a_2 \wedge x_0)$. By combining equations (5.47) and (5.48) we obtain

$$\mathbb{E}_{b_1}\left(e^{-q\tau_{a_1,x_0}}\hat{V}(B_{\tau_{a_1,x_0}})\right) < \mathbb{E}_{b_1}\left(e^{-q\tau_{a_1,x_0}}h(B_{\tau_{a_1,x_0}},a_1)\right). \tag{5.49}$$

On the other hand, as $\hat{V}(x_0) \geq h(x_0,a_1)$, and $\hat{V}(a_1) = g(a_1) = h(a_1,a_1)$, we derive that

$$\mathbb{E}_{b_1}\left(e^{-q\tau_{a_1,x_0}}\hat{V}(B_{\tau_{a_1,x_0}})\right) \geq \mathbb{E}_{b_1}\left(e^{-q\tau_{a_1,x_0}}h(B_{\tau_{a_1,x_0}},a_1)\right),$$

which clearly contradicts equation (5.49). So there does not exists $x > b_1$ such that $\hat{V}(x_0) \geq h(x_0,a_1)$. Thus, $\hat{V}(x) < h(x,a_1)$ for all $x > b_1$ as required. □

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