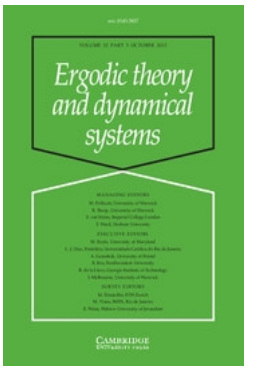

Periodic attractors of perturbed one-dimensional maps

O. KOZLOVSKI

Ergodic Theory and Dynamical Systems / Volume 33 / Issue 05 / October 2013, pp 1519 - 1541
DOI: 10.1017/etds.2013.28, Published online: 05 June 2013

Ergod. Th. & Dynam. Sys. (2013), **33**, 1519–1541
doi:10.1017/etds.2013.28

Periodic attractors of perturbed one-dimensional maps

O. KOZLOVSKI

Mathematics Institute, University of Warwick, UK
(e-mail: O.Kozlovski@warwick.ac.uk)

(*Received* 21 *October* 2012 *and accepted in revised form* 22 *March* 2013)

Abstract. In this paper we investigate how many periodic attractors maps in a small neighbourhood of a given map can have. For this purpose we develop new tools which help to make uniform cross-ratio distortion estimates in a neighbourhood of a map with degenerate critical points.

1. *Introduction*

Let $\mathcal{N}$ denote an interval or a circle and let $f : \mathcal{N} \to \mathcal{N}$ be a C^∞ map. In this paper we use the standard notions of a periodic point of f: its period, its immediate basin of attraction, etc. One can find all relevant definitions in [**dMvS93**].

The map f can have infinitely many periodic attractors. However, this is a non-generic situation: if all critical points of f are non-flat, the periods of the periodic attracting orbits are bounded from above, therefore if f has infinitely many periodic attracting orbits, they should accumulate on neutral periodic orbits and the periods of these neutral orbits are also bounded; see [**MdMvS92**] or [**dMvS93**, Theorem B, p. 268]. If f has a flat critical point, the periods of periodic attractors are not necessarily bounded; an example of such a map is given in [**KK11**]. As usual, we call a critical point c *non-flat* if in a neighbourhood of c the function $f(x)$ can be written as $\pm(\phi(x))^d$, where ϕ is a diffeomorphism and $d \in \mathbb{N}$, $d \geq 2$. For C^∞ maps it is equivalent to $D^d f(c) \neq 0$ for some $d \geq 2$.

In this paper we will study whether a small perturbation of f can have infinitely many periodic attractors and related questions. The simple answer to this problem is 'yes': one can construct an example of a C^∞ map f with a quadratic critical point which has a finite number of periodic attractors such that in any C^∞ neighbourhood of f there are maps which have infinitely many periodic attracting points and the periods of these points can be arbitrarily large; see [**Koz12**]. The source of these attractors is a parabolic fixed point, and our first theorem shows that if f does not have neutral periodic orbits and all critical points of f are quadratic, then this phenomenon of having an unbounded number of attractors for maps in an arbitrarily small neighbourhood of f is not possible.

THEOREM A. *Let $f : \mathcal{N} \to \mathcal{N}$ be a C^3 map with quadratic critical points. Suppose that f does not have neutral periodic orbits. Then there exist a neighbourhood $\mathcal{F} \subset C^3(\mathcal{N})$ of f and a natural number n_0 such that if $g \in \mathcal{F}$ and $\mathcal{O}$ is an attracting periodic orbit of g, then either the period of the orbit $\mathcal{O}$ is less than n_0 or there exists a critical point c of g whose iterates converge to $\mathcal{O}$ under iterations of the map g.*

In particular, all maps in $\mathcal{F}$ have finitely many periodic attractors and the number of these attractors is bounded by the number of attractors of f plus the number of critical points of f.

In [**dMvS93**, Theorem B′, p. 268], a stronger statement is given: though the conclusion in the statement is similar to Theorem A, it is not required that f has no neutral periodic points and it is not required that all critical points of f are quadratic. As the example above shows, this statement is not correct and the authors of [**dMvS93**] issued an erratum shortly after the book was published.

So, we see that there are situations when small perturbations of f can create an unbounded number of periodic attractors. If f has quadratic critical points, it is possible to prove that this is not typical. More precisely, the following is proven in [**Koz12**]: let $\mathcal{S}$ be a space of C^d, $d \geq 3$, maps of $\mathcal{N}$ with all critical points quadratic, and exclude diffeomorphisms of the circle from $\mathcal{S}$; then for a generic smooth family f_λ of maps in $\mathcal{S}$ there exists $M > 0$ such that the number of periodic attracting orbits of any map in this family f_λ is bounded by M. Interestingly enough, for a generic non-trivial smooth family of circle diffeomorphisms such a bound does not exist, that is, there are maps in a generic family with an arbitrarily large number of periodic attracting orbits.

The situation gets significantly more complicated if we allow degenerate (but non-flat) critical points. By a degenerate non-flat critical point we mean a point c of f such that $Df^k(c) = 0$ for $k = 1, 2, \ldots, m-1$ and $Df^m(c) \neq 0$, where $m \geq 3$.

Let us construct an example showing that Theorem A does not hold if we allow degenerate critical points of the map f. Let $f \in C^\omega$ be a map of a circle topologically equivalent to the doubling map $x \mapsto 2x \pmod 1$. Moreover, suppose that f has one critical point c of cubic type and the orbit of c is dense. Then there are maps arbitrarily close to f in C^ω topology such that they still have a cubic critical point and its iterates are attracted to a periodic attracting orbit of high period. We can perturb these maps so that the maps obtained do not have critical points at all, but still have periodic attractors. Thus, for any n_0 we can find a map g arbitrarily close to f which has a periodic attracting orbit of period larger than n_0 and no critical points. This obviously contradicts the first part of the conclusion of Theorem A.

One might think that if a map has critical points of even degree, then examples like the above are impossible because critical points of even degree cannot be destroyed by a small perturbation. Let us sketch an example showing that this is not the case. Let $f \in C^\omega$ be a unimodal map with a critical point c of degree four such that $a = f^{n_0}(c)$ is a repelling periodic point for some n_0 (so the map f is Misiurewicz). There exist an interval J_0 containing the critical point c, a sequence of intervals J_k, $k = 1, 2, \ldots$, such that $J_k \to a$ as $k \to \infty$, and a sequence n_k such that $f^{n_k}(J_k) = J_0$ and $f^{n_k}|_{J_k}$ is a diffeomorphism for all k. Moreover, under small perturbations of f the repelling periodic

point a and the intervals J_k persist, that is, if g is close enough to f, there exist a repelling periodic point a_g of g close to a and of the same period, and intervals $J_{g,k}$ such that $\lim_{k\to\infty} J_{g,k} = a_g$, $f^{n_k}(J_{g,k}) = J_0$ and $f^{n_k}|_{J_{g,k}}$ is a diffeomorphism. Using these intervals J_k, we can construct a sequence $g_{1,k}$ of perturbations of f in such a way that every map $g_{1,k}$ has two critical points $c^2_{1,k}$ and $c^3_{1,k}$ of degrees two and three such that the quadratic critical point $c^2_{1,k}$ is still mapped to $a_{g_{1,k}}$ by $g^{n_0}_{1,k}$ and the cubic critical point becomes a superattractor so that $g^{n_0}_{1,k}(c^3_{1,k}) \in J_{g_{1,k},k}$ and $g^{n_0+n_k}_{1,k}(c^3_{1,k}) = c^3_{1,k}$. In the same way we can perturb each of $g_{1,k}$ and obtain maps $g_{2,k}$ which still have two critical points of degree two and three, their cubic critical points are still superattractors of period $n_0 + n_k$ and the quadratic critical points become superattractors as well. Finally, we can break cubic critical points of maps $g_{2,k}$ and obtain a sequence of maps $g_{3,k}$ which satisfies the following properties: $\lim_{k\to\infty} g_{3,k} = f$, $g_{3,k}$ are unimodal maps with quadratic critical points, every map $g_{3,k}$ has two periodic attractors and the periods of these attractors tend to infinity as $k \to \infty$. Again, this contradicts the conclusion of Theorem A.

These examples show that a degenerate critical point of f can disappear under a perturbation or lose its degree, but the perturbed map g can have a periodic attractor related to this disappeared critical point. We conjecture that the second part of Theorem A holds in this case.

CONJECTURE. *Let $f : \mathcal{N} \to \mathcal{N}$ be a C^3 map with non-flat critical points. Suppose that f does not have neutral periodic orbits. Then there exists a neighbourhood $\mathcal{F} \subset C^3(\mathcal{N})$ of f such that for any $g \in \mathcal{F}$ the number of periodic attractors of g is bounded by the number of attractors of f plus the number of critical points of f counted with their multiplicities.*

By definition the *multiplicity* of a critical point c is $m - 1$, where m is such that $Df^k(c) = 0$, for $k = 1, 2, \ldots, m - 1$, and $Df^m(c) \neq 0$.

We have already mentioned that this conjecture does not hold if we allow neutral periodic orbits for the map f and there might be no upper bound on the number of attractors for maps close to f. The next theorem shows that nevertheless we can group these attractors in such a way that periodic attracting orbits in the same group are related to each other in a very simple way and there is a uniform bound on the number of these groups. To state this result we need a few definitions first.

If p is a periodic point of f and n is its period, then we will call the number $2n$ the *orientation preserving period* of p if $Df^n(p) < 0$, and if $Df^n(p) \geq 0$ then the *orientation preserving period* of p is just n.

We will call a closed interval $I \subset \mathcal{N}$ *periodic* if there is $n \in \mathbb{N}$ such that $f^n(I) = I$ and $f^n : I \to I$ is a bijection. Any periodic interval I of period n contains one or more periodic points of period n and if $f^n|_I$ is orientation reversing, it can contain periodic points of period $2n$. If n is even, I can contain a periodic point of period $n/2$ in its boundary. The interval I cannot contain periodic points of any other periods except n, $2n$, $n/2$.

A *pack of periodic points* is a collection of periodic points such that they all belong to some closed periodic interval (perhaps degenerate) and there is no larger periodic interval which contains more periodic points. A pack can consist of just one periodic point. All periodic points in a pack either have the same period, or there is one periodic point of period n which is orientation reversing and all other periodic points in the pack have period $2n$.

In other words, the orientation preserving period of all periodic points in a pack is the same. To every pack of periodic points one can associate a *pack of periodic orbits* in an obvious way.

We now state the main result of the paper.

THEOREM B. *Let $f : \mathcal{N} \to \mathcal{N}$ be a C^∞ map with non-flat critical points. There exist a neighbourhood $\mathcal{F} \subset C^\infty$ of f and $M > 0$, $\rho > 0$ such that for any $g \in \mathcal{F}$ there exist at most M exceptional packs of periodic orbits such that if p is a periodic point of g which is not a member of any of these exceptional packs, then*

$$|Dg^n(p)| > 1 + \rho,$$

where n is a period of p.

In other words, in the neighbourhood of f maps can possibly have many periodic attractors, but the set of the periods of these attractors has a uniformly bounded cardinality.

The theorem is stated for C^∞ maps. The only place where this is used is in the proof of Proposition 3.4 where a result of [**Ser76**] is used. One can state this theorem for C^k maps; however, in this case extra conditions should be put on the multiplicities of the critical points of the map f.

2. *Idea of the proofs and discussion*

Let us discuss the main problems that arise when we want to carry over some properties of a map f to the maps in a small neighbourhood of f. We will mainly keep in mind the following three results closely related to Theorems A and B: the Singer theorem about periodic attractors of maps with negative Schwarzian derivative [**Sin78**], a theorem about the Schwarzian derivative of the first entry map to a small neighbourhood of a critical value [**Koz00, VV04**], and [**dMvS93**, Theorem B, p. 268] which we have already mentioned several times. Let us remind the reader that the Schwarzian derivative of a function f is defined as $Sf(x) = D^3 f(x)/Df(x) - (3/2)(D^2 f(x)/Df(x))^2$. We will review some of the properties of the Schwarzian derivative in §3.

The maps we consider in this paper do not have wandering intervals and one of the consequences of this fact is the 'contraction principle': for every $\epsilon > 0$ there exists $\delta > 0$ such that if J is an interval with $|J| < \delta$ and not intersecting the immediate basin of a periodic attractor, then for any $n > 0$ each component of $f^{-n}(J)$ has length less than ϵ. Of course, this statement holds for maps in a neighbourhood of the map f, but then δ can depend on the choice of the map, and, in general, one cannot have a *uniform* version of this statement. On the other hand, this is an important lemma in the proof of Theorem B of [**dMvS93**] (see Lemma 10.3.2, p. 323) and in the proof of the fact that the first return map to a small interval around a critical value has negative Schwarzian derivative.

If one examines the proof of the contraction principle, it will be apparent that the only obstruction to the proof of its uniform version is the existence of parabolic points of f.

LEMMA 2.1. (Uniform contraction principle) *Let f be a $C^1(\mathcal{N})$ map and assume that f does not have wandering intervals and neutral periodic points. Then for any $\epsilon > 0$ there exist a neighbourhood $\mathcal{F} \subset C^1(\mathcal{N})$ of f and $\delta > 0$ such that if $g \in \mathcal{F}$ and J is an interval*

with $|J| < \delta$ and not intersecting the immediate basin of a periodic attractor of the map g, then for any $n > 0$ each component of $g^{-n}(J)$ has length less than ϵ.

The proof of this lemma is not hard and is given in Appendix A.

Using the uniform contraction principle, one can show that the first return map of g to a small interval around a critical value has negative Schwarzian derivative and the size of this interval is uniformly bounded by the following theorem.

THEOREM 2.2. *Let f be a $C^3(\mathcal{N})$ map of an interval or circle with quadratic critical points. Suppose that f does not have neutral periodic orbits. Let c be a critical point of f whose iterates do not converge to a periodic attractor. Then there exist a neighbourhood $\mathcal{F} \subset C^3(\mathcal{N})$ of f and a neighbourhood J of c such that if $g \in \mathcal{F}$ and $g^n(x) \in J$ for some $x \in \mathcal{N}$ and $n \geq 0$, then $Sg^{n+1}(x) < 0$.*

The proof of this theorem follows the same lines as the proof of its single map version; see [**Koz00, VV04**]. One should notice that if all critical points of f are quadratic, one can choose a neighbourhood of the critical points so that the Schwarzian derivative of a perturbed map g will be negative with a uniform estimate on it (see Appendix A). In particular, this implies that the cross-ratio distortion estimates similar to [**dMvS89**, Theorem 1.2] hold uniformly. We will see that this is not the case if f has degenerate critical points.

Now the proof of Theorem A is a straightforward consequence of the Singer and Mañé theorems.

Proof of Theorem A. Take a neighbourhood U of all critical points of f whose iterates do not converge to periodic attractors of f and so small that Theorem 2.2 holds; that is, if J is a connected component of this neighbourhood, $g \in \mathcal{F}$, and $g^n(x) \in J$, then $Sg^{n+1}(x) < 0$. We can also assume that boundary points of each connected component of U are some preimages of repelling periodic points of f. Decreasing $\mathcal{F}$ if necessary, we can assume that these periodic repellers persist for maps in $\mathcal{F}$ and, thus, the set U_g can be defined so that the boundary points of U_g are preimages of some repellers of g and continuously depend on g, $U_f = U$, and $Sg^{n+1}(x) < 0$ if $g^n(x) \in U_g$. Let $W \subset U$ be a smaller neighbourhood of critical points of f, and again by decreasing $\mathcal{F}$ we can assume that $W \subset U_g$ for all $g \in \mathcal{F}$.

Let $\mathcal{O}$ be an attracting periodic orbit of g of period n which intersects W. Let $p \in W \cap \mathcal{O}$, J be a connected component of U_g containing p, and $R : X \to J$ be the first entry map of g to J. The immediate basin of attraction B of $g(p)$ cannot contain preimages of repelling periodic points, therefore it is entirely contained in a connected component of X. This implies that $Sg^n(x) < 0$ for all $x \in B$ and Singer's argument shows that there is an iterate of a critical point of g in B.

The Mañé theorem [**Man85**] states that the set of points whose iterates under the map f never enter the domain W consists of a hyperbolic set, and attraction basins of non-degenerate periodic attracting orbits (because f does not have neutral periodic points). Thus, for small perturbations of f the number of periodic attractors whose orbits do not intersect W does not change. □

The statement about the negative Schwarzian derivative of the first return map for maps in the neighbourhood of f holds only if all critical points of f are quadratic.

Indeed, consider the function $\phi(x) = x^3$. This function has negative Schwarzian derivative everywhere and, moreover, the Schwarzian derivative of ϕ tends to minus infinity when x goes to zero.

Now consider small perturbations of ϕ of the form $\phi_\lambda(x) = x^3 + \lambda x$, where $|\lambda| \ll 1$. The Schwarzian derivative of ϕ_λ is

$$S\phi_\lambda(x) = 6\frac{\lambda - 6x^2}{(\lambda + 3x^2)^2}.$$

We see that for small values of λ at zero the Schwarzian derivative is $6/\lambda$, thus it is positive and very large and Theorem 2.2 cannot possibly hold if we drop the condition on the critical points being quadratic.

In fact, the cross-ratio distortion estimates we have mentioned above also do not hold uniformly if we allow degenerate critical points. To deal with this problem we will introduce a notion of critical intervals in §3. These critical intervals will capture some properties of the critical points when they cease to exist under a perturbation of the map. In particular, we will show that the attracting periodic points of sufficiently high period must have either a critical point or a definite part of a critical interval in their basin of attraction. This will be the main step in proving Theorem B.

Another application of the critical intervals is given in §4 where we prove a uniform version of the pullback estimates widely used in the literature. These estimates are also an important part in the proof of Theorem B. Since they might be independently useful and important in their own right we state them here. See §4 for more details.

THEOREM. (Theorem 4.5) *Let f be a $C^\infty(\mathcal{N})$ map with all critical points non-flat. There exist a neighbourhood $\mathcal{F}$ of f in $C^\infty(\mathcal{N})$ and a function $\rho(\epsilon, N)$ such that the following holds. Let g be in $\mathcal{F}$, $J \subset T$ be intervals such that $g^m|_T$ is a diffeomorphism and the intersection multiplicity of the intervals $g^k(T)$, $k = 0, \ldots, m-1$, is bounded by N. Then*

$$D(T, J) < \rho(D(g^m(T), g^m(J)), N),$$

where $D(T, J) = |T||J|/|L||R|$ denotes the cross-ratio.

Moreover, $\rho(\epsilon, N)$ tends to zero when ϵ goes to zero and N is fixed.

THEOREM. (Theorem 4.6) *Let f be a $C^\infty(\mathcal{N})$ map with all critical points non-flat. There exist a neighbourhood $\mathcal{F}$ of f in $C^\infty(\mathcal{N})$ and a function $\rho(\epsilon, N)$ such that the following holds. Let g be in $\mathcal{F}$, $\{J_k\}_{k=0}^m$ and $\{T_k\}_{k=0}^m$ be chains such that $J_k \subset T_k$ for all $0 \le k \le m$. Assume that the intersection multiplicity of $\{T_k\}_{k=0}^s$ is at most N and that T_m contains an ϵ-scaled neighbourhood of J_m. Then T_0 contains $\rho(\epsilon, N)$-scaled neighbourhood of J_0.*

Moreover, $\rho(\epsilon, N)$ tends to infinity when ϵ goes to infinity and N is fixed.

3. *Cross-ratio estimates in the presence of large positive Schwarzian*

There are many well-known estimates for the cross-ratio distortion of a map; however, these estimates often involve constants which implicitly depend on the map. In this section

we will give a few explicit estimates for the cross-ratio distortion. First, we start with the standard definitions of the cross-ratio and state a few of its well-known properties.

Let $J \subset T$ be two intervals and L and R are connected components of $T \setminus J$. The cross-ratio of these intervals is defined as

$$D(T, J) = \frac{|T||J|}{|L||R|}.$$

If $f : T \to \mathbb{R}$ is monotone on T, we define the cross-ratio distortion of f by

$$B(f, T, J) = \frac{D(f(T), f(J))}{D(T, J)}.$$

Let f be a real differentiable function and $\{T_j\}_{j=0}^m$ be a collection of intervals. The *intersection multiplicity* of $\{T_j\}_{j=0}^m$ is the maximal number of intervals with a non-empty intersection. The *order* of $\{T_j\}_{j=0}^m$ is the number of intervals containing a critical point of f. This sequence of intervals $\{T_j\}_{j=0}^m$ is called a *chain* if T_j is a connected component of $f^{-1}(T_{j+1})$.

If I is a real interval of the form $(a - b, a + b)$ and $\lambda > 0$ then we define $\lambda I = (a - \lambda b, a + \lambda b)$. By definition $(1 + 2\delta)I$ is called the δ-scaled neighbourhood of I. We say that I is δ-well-inside J if $J \supset (1 + 2\delta)I$.

Let f be a C^3 mapping. The Schwarzian derivative of f is defined as

$$Sf(x) = \frac{D^3 f(x)}{Df(x)} - \frac{3}{2}\left(\frac{D^2 f(x)}{Df(x)}\right)^2.$$

It is easy to check that the Schwarzian derivative of a composition of two maps has the form

$$S(f \circ g)(x) = Sf(g(x))Dg(x)^2 + Sg(x).$$

This formula implies that if a map has negative Schwarzian derivative then all its iterates also have negative Schwarzian derivatives.

It is also well known that maps having negative Schwarzian derivative increase cross-ratios. The next lemma gives an estimate of the cross-ratio distortion in terms of the map's Schwarzian derivative.

LEMMA 3.1. *Let $f : T \to f(T)$ be a C^3 diffeomorphism and suppose that $Sf(x) < C$ for all x in T and some constant $C > 0$. Moreover, suppose that $C|T|^2 < \pi^2/2$. Then, for any $J \subset T$,*

$$B(f, T, J) > \cos^2(\sqrt{C/2}|T|).$$

Remark. One does need a bound on the size of the interval (as in the lemma) in order to have a non-trivial estimate on the cross-ratio distortion from below. More precisely, for any $\epsilon > 0$ there exist a C^3 diffeomorphism $f : [0, 1] \to [0, 1]$ and an interval $J \in [0, 1]$ such that $Sf(x) < \pi^2$ and $B(f, [0, 1], J) < \epsilon$.

Proof. First, using rescaling, we can assume that $T = [0, 1]$. Let $J = [a, b]$. The Schwarzian derivative of a Möbius transformation is zero, therefore post-composing the map with a Möbius transformation does not change the cross-ratio distortion $B(f, T, J)$ and the map's Schwarzian derivative. By post-composing the map f with an appropriate

Möbius transformation we can assume that $f(0)=0$, $f(a)=a$ and $f(1)=1$. Since f is monotone, we can now assume that $Df(x)>0$.

The interval $[0, a]$ is mapped onto itself by f, therefore there exists a point $u_1 \in [0, a]$ such that $Df(u_1)=1$. If $f(b) \geq b$, then $B(f, T, J) \geq 1$ and we are done. Otherwise, there are points $v_1 \in [a, b]$, $v_2 \in [b, 1]$ such that

$$Df(v_1) = \frac{f(b)-a}{b-a} < 1 \quad \text{and} \quad Df(v_2) = \frac{1-f(b)}{1-b} > 1.$$

Hence, there exists a point $u_2 \in [v_1, v_2]$ such that $Df(u_2)=1$. Notice that $B(g, T, J) = Df(v_1)/Df(v_2)$ and in order to estimate the cross-ratio distortion from below we should estimate $Df(v_1)$ from below and $Df(v_2)$ from above.

By a direct computation one can check that another form for the Schwarzian derivative of f is

$$Sf(x) = -2\sqrt{Df(x)}D^2\frac{1}{\sqrt{Df(x)}}.$$

This implies that if we denote $1/\sqrt{Df(x)}$ by $\phi(x)$, then the function ϕ satisfies the linear second-order differential equation

$$\phi''(x) = -\tfrac{1}{2}Sf(x)\phi(x). \tag{1}$$

Moreover, we know that

$$\phi(u_1) = \phi(u_2) = 1. \tag{2}$$

Let us compare the solutions of this equation with the solutions of the equation

$$\psi''(x) = -\tfrac{1}{2}C\psi(x) \tag{3}$$

with the same boundary conditions

$$\psi(u_1) = \psi(u_2) = 1. \tag{4}$$

Claim. Suppose that $\phi : [0, 1] \to \mathbb{R}$ satisfies equations (1), (2) and $\phi(x) > 0$ for all $x \in [0, 1]$. Suppose that $\psi : [0, 1] \to \mathbb{R}$ satisfies (3), (4). Then for all $x \in [u_1, u_2]$ one has $\phi(x) \leq \psi(x)$ and for all $x \in [u_2, 1]$ one has $\phi(x) \geq \psi(x)$.

To prove this claim let us first notice that the inequality $C < \pi^2/2$ implies that $\psi(x) \geq 0$ for all $x \in [0, 1]$.

Next, one can easily check that ϕ and ψ satisfy the Picone identity

$$D\left(\phi(x)\psi(x)\, D\left(\frac{\phi(x)}{\psi(x)}\right)\right) = \frac{1}{2}(C - Sf(x))\phi(x)^2 + \left(D\phi(x) - \frac{\phi(x)}{\psi(x)}D\psi(x)\right)^2.$$

Notice that the right-hand side in the Picone identity is always positive.

Set $x_0 = \inf\{x \in [u_1, 1] : \phi(x) > \psi(x)\}$. By continuity we get $\phi(x_0) = \psi(x_0)$ and $D(\phi(x)/\psi(x))|_{x=x_0} \geq 0$. The Picone identity implies that, for all $x > x_0$,

$$\phi(x)\psi(x)D\left(\frac{\phi(x)}{\psi(x)}\right) > \phi(x_0)\psi(x_0)D\left(\frac{\phi(x)}{\psi(x)}\right)\bigg|_{x=x_0}.$$

In particular, we get $D(\phi(x)/\psi(x)) > 0$, and, therefore, $\phi(x)/\psi(x) > \phi(x_0)/\psi(x_0) = 1$ for all $x \geq x_0$.

If $x_0 < u_2$, we would have $1 = \phi(u_2) > \psi(u_2) = 1$ which is not possible, so $x_0 \geq u_2$ and we have proved the first part of the claim.

To prove the second part of the claim we should notice that since $\phi(u_2) = \psi(u_2) = 1$ and $\phi(x) \leq \psi(x)$ for $x \in [u_1, u_2]$ we get $D(\phi(x)/\psi(x))|_{x=u_2} \geq 0$. Using the Picone identity once more and arguing as before, we conclude that $\phi(x) \geq \psi(x)$ for all $x \in [u_2, 1]$ and the proof of the claim is finished.

Using this claim, we can estimate the cross-ratio distortion in terms of the function ψ:

$$B(f, T, J) = \frac{Df(v_1)}{Df(v_2)} = \left(\frac{\phi(v_2)}{\phi(v_1)}\right)^2 > \left(\frac{\psi(v_2)}{\psi(v_1)}\right)^2.$$

The solution of equations (3) and (4) is

$$\psi(x) = \frac{\cos(\sqrt{C/2}(x - (u_1 + u_2)/2))}{\cos(\sqrt{C/2}((u_1 - u_2)/2))}.$$

On the interval [0, 1] the function ψ reaches its maximum at the point $(u_1 + u_2)/2$ and its minimum at one of the boundary points 0 or 1. Hence,

$$\begin{aligned}\frac{\psi(v_2)}{\psi(v_1)} &\geq \min\left\{\cos\left(\sqrt{C/2}\left(\frac{u_1 + u_2}{2}\right)\right), \cos\left(\sqrt{C/2}\left(1 - \frac{u_1 + u_2}{2}\right)\right)\right\} \\ &\geq \cos(\sqrt{C/2})\end{aligned}$$

since $(u_1 + u_2)/2 \in [0, 1]$ and $\sqrt{C/2} < \pi/2$. □

If the Schwarzian derivative is strictly negative, the cross-ratio distortion is always greater than one. If it is negative and bounded away from zero by some constant, in general, one cannot improve this estimate on the cross-ratio distortion: the interval J can be small and close to one of the end points of the interval T. However, if J is situated exactly in the centre of T and not very small, we can get a definite increase of the cross-ratio as given below.

LEMMA 3.2. *Let $f : T \to f(T)$ be a C^3 diffeomorphism and suppose that $Sf(x) < -C$ for all x in T and some constant $C > 0$. Then, for any interval $J \subset T$ such that T is equal to the δ-scaled neighbourhood of J,*

$$B(f, T, J) > \frac{1 + 2\delta}{\sqrt{C/2}|T|} \sinh\left(\frac{\sqrt{C/2}|T|}{1 + 2\delta}\right) \geq 1 + \frac{1}{12}\frac{C|T|^2}{(1 + 2\delta)^2}.$$

Proof. Start by rescaling T to [0, 1]. Then $J = [a, 1 - a]$, where $a = \delta/(1 + 2\delta)$. By post-composing f with a Möbius transformation we can assume that $f(0) = 0$, $f(1) = 1$ and $f(a) + f(1 - a) = 1$. Since the Schwarzian derivative is negative on T we already know that $f(a) \leq a$.

Let us denote the ratio $f(a)/a$ by r. Notice that $(1 - f(1 - a))/a$ is equal to r as well and that $r \leq 1$. By the mean value theorem there exist points $u_1 \in [0, a]$ and $u_2 \in [1 - a, 1]$ such that

$$Df(u_1) = Df(u_2) = r.$$

As in the proof of the previous lemma let us denote $1/\sqrt{Df(x)}$ by $\phi(x)$. The function ϕ satisfies equation (1) with boundary conditions

$$\phi(u_1) = \phi(u_2) = \frac{1}{\sqrt{r}}. \tag{5}$$

We will compare the solution of this equation with the function ψ which satisfies

$$\psi'' = \tfrac{1}{2}C\psi \tag{6}$$

and the boundary conditions similar to (5). This equation is easy to solve and the solution is

$$\psi(x) = \frac{\cosh(\sqrt{C/2}(x - (u_1 + u_2)/2))}{\sqrt{r}\cosh(\sqrt{C/2}((u_1 - u_2)/2))}.$$

As in the proof of the previous lemma the following is true: $\phi(x) \le \psi(x)$ for all $x \in [u_1, u_2]$. Now, let us estimate the cross-ratio distortion

$$\begin{aligned} B(f, T, J) &= \frac{f(1-a) - f(a)}{(1-2a)r^2} \\ &= \frac{1}{(1-2a)r^2}\int_a^{1-a} Df(t)\,dt \\ &\ge \frac{\cosh(\sqrt{C/2}((u_1-u_2)/2))^2}{(1-2a)r}\int_a^{1-a}\frac{1}{\cosh(\sqrt{C/2}(t-(u_1+u_2)/2))^2}\,dt \\ &= \frac{\cosh(\sqrt{C/2}((u_1-u_2)/2))^2}{\sqrt{C/2}(1-2a)r}\tanh\left(\sqrt{C/2}\left(t - \frac{u_1+u_2}{2}\right)\right)\Bigg|_{t=a}^{1-a}. \end{aligned}$$

By an elementary consideration one can show that under the restrictions $u_1 \in [0, a]$ and $u_2 \in [1-a, 1]$ the last expression achieves its minimum when $u_1 = a$ and $u_2 = 1-a$. Thus,

$$\begin{aligned} B(f, T, J) &\ge 2\frac{\cosh(\sqrt{C/2}(\frac{1}{2} - a))^2}{\sqrt{C/2}(1-2a)}\tanh\left(\sqrt{C/2}\left(\frac{1}{2} - a\right)\right) \\ &= \frac{\sinh(\sqrt{C/2}(1-2a))}{\sqrt{C/2}(1-2a)}. \end{aligned}$$

□

In order to understand the cross-ratio distortion for maps in a neighbourhood of a map which has degenerate critical point we first study it in the case of the polynomials.

PROPOSITION 3.3. *For any polynomial f of degree d there exist at most $(d-1)/2$ intervals E_j (which we will call* critical intervals*), $j = 1, \ldots, d_E$ such that the following results hold.*

- *Let $\kappa \in (0, 1/4\sqrt{d_E})$ be a number, $T_1, \ldots, T_m$ be intervals and their intersection multiplicity be bounded by N. Moreover, suppose that $f|_{T_i}$ is a diffeomorphism and that*

$$|T_i \cap E_j| < \kappa|E_j|$$

for all $i = 1, \ldots, m$, $j = 1, \ldots, d_E$. Then

$$\prod_{i=1}^{m} B(f, T_i, J_i) > \exp(-16\kappa N d_E^2),$$

where $J_i \subset T_i$ are any intervals.

- *Let $\lambda > 1$, $\kappa \in (0, 1/13\sqrt{d_E})$ be numbers, $J \subset T$ be intervals such that the interval T is equal to the δ-scaled neighbourhood of J and $f|_T$ is a diffeomorphism. Moreover, assume that*

$$|T \cap E_j| < \kappa |E_j|/\lambda$$

for all $j = 1, \ldots, d_E$ and either there exists a critical point c of f contained in the interval λT or there exists $j_0 \in [1, d_E]$ such that

$$T \not\subset 2E_{j_0} \quad \text{and} \quad \lambda T \cap E_{j_0} \neq \emptyset.$$

Then

$$B(f, T, J) > 1 + \frac{1}{12}\left(\frac{16}{17(1+\lambda)^2} - 32\frac{\kappa^2 d_E}{\lambda^2}\right)\frac{1}{(1+2\delta)^2}.$$

Notice that there is no dynamics involved in this proposition.

Proof. The derivative of f is also a polynomial; let x_k, $k = 1, \ldots, d-1$, be its roots. Then the Schwarzian derivative of f can be written as

$$\begin{aligned} Sf(x) &= 2\sum_{1\le k<l\le d-1} \frac{1}{(x-x_k)(x-x_l)} - \frac{3}{2}\left(\sum_{k=1}^{d-1}\frac{1}{x-x_k}\right)^2 \\ &= -\sum_{k=1}^{d-1}\frac{1}{(x-x_k)^2} - \frac{1}{2}\left(\sum_{k=1}^{d-1}\frac{1}{x-x_k}\right)^2. \end{aligned}$$

Let $a_j \pm ib_j$, $j = 1, \ldots, d_E$, be all non-real roots of Df among $x_1, \ldots, x_{d-1}$. Then the formula for the Schwarzian derivative above implies

$$\begin{aligned} Sf(x) &\le -\sum_{j=1}^{d_E}\left(\frac{1}{(x-a_j-ib_j)^2} + \frac{1}{(x-a_j+ib_j)^2}\right) \\ &= -2\sum_{j=1}^{d_E}\frac{(x-a_j)^2 - b_j^2}{((x-a_j)^2+b_j^2)^2}. \end{aligned}$$

Define the critical intervals as $E_j = [a_j - 2b_j, a_j + 2b_j]$. It is easy to see that if x is a point which is not contained in any of the intervals E_j, then $Sf(x) < 0$. Otherwise, let E_j be a critical interval of minimal length containing the point x. The above estimate for the Schwarzian derivative implies that

$$Sf(x) < \frac{2d_E}{b_j^2}.$$

If an interval T_k is not contained in any of the critical intervals (but can have non-empty intersection with them), then $T_k \cap [a_j - b_j, a_j + b_j] = \emptyset$ for all $j = 1, \ldots, d_E$ because $|T_k \cap E_j| < \kappa|E_j| < |E_j|/4$, and therefore $Sf|_{T_k} < 0$ and $B(f, T_k, J_k) > 1$.

Fix a critical interval E_j. Let $T_{k_1}, \ldots, T_{k_{m'}}$ be all intervals which are contained in E_j but are not contained in a critical interval of length smaller than $|E_j|$. We have already argued that $Sf|_{T_{k_i}} < 2d_E/b_j^2$. By the choice of the number κ we know that

$$\begin{aligned} \sqrt{\frac{1}{2}\max_{x\in T_{k_i}} Sf(x)}\,|T_{k_i}| &< \frac{\sqrt{d_E}}{b_j}\kappa|E_j| \\ &< \frac{\pi}{2}. \end{aligned} \tag{7}$$

So, we can apply Lemma 3.1 and get

$$\log B(f, T_{k_i}, J_{k_i}) > \log\left(\cos^2\left(\frac{\sqrt{d_E}}{b_j}|T_{k_i}|\right)\right)$$
$$> -\frac{d_E}{b_j^2}|T_{k_i}|^2.$$

Here we have used the fact that $\cos(x) \geq 1 - x^2$ for all $x \in \mathbb{R}$.

Now we can estimate the contribution of the cross-ratio distortions on all the intervals T_{k_i}:

$$\sum_{i=1}^{m'} \log B(f, T_{k_i}, J_{k_i}) > -\frac{d_E}{b_j^2}\sum_{i=1}^{m'}|T_{k_i}|^2$$
$$> -\frac{d_E}{b_j^2}\kappa|E_j|\sum_{i=1}^{m'}|T_{k_i}|$$
$$> -\frac{d_E}{b_j^2}\kappa N|E_j|^2$$
$$= -16\kappa N d_E.$$

Thus, we get

$$\sum_{j=1}^{m} \log B(f, T_k, J_k) > -16\kappa N d_E^2$$

and the first part of the proposition is proved.

We now prove the second part. First, suppose that we are in the first case, that is, there exists a critical point c such that $c \in \lambda T$. Set $I = [c, T]$. Since $c \in \lambda T$ we get $(1+\lambda)/2\,|T| \geq |I|$.

If T is not contained in any critical interval, then arguing as before we get that $Sf(x) < 0$ for all $x \in T$ and

$$\min_{x\in T}(-Sf(x))\,|T|^2 \geq \frac{|T|^2}{|I|^2}$$
$$\geq \frac{4}{(1+\lambda)^2}.$$

If T is contained in some critical intervals, let E_j be such an interval of minimal length. Using estimate (7) and estimating the contribution to the Schwarzian derivative of critical intervals which contain T, we get

$$\min_{x\in T}(-Sf(x))\,|T|^2 \geq \frac{4}{(1+\lambda)^2} - 32\frac{\kappa^2 d_E}{\lambda^2}.$$

Notice that since $\kappa^2 < 1/13^2 d_E$ the right-hand side of the inequality above is positive.

Now consider the remaining case and assume that $\lambda T \cap E_{j_0} \neq \emptyset$ and $T \not\subset 2E_{j_0}$. Denote by A the interval $[a_{j_0}, T]$. Since $T \not\subset 2E_{j_0} = [a_{j_0} - 4b_{j_0}, a_{j_0} + 4b_{j_0}]$ we get

$$|A| > 4b_{j_0}.$$

On the other hand, the condition $\lambda T \cap E_{j_0} \neq \emptyset$ implies that

$$|A| - (1+\lambda)/2\,|T| < 2b_{j_0}.$$

These two inequalities combined give an estimate on the length of the interval $|T|$:

$$(1+\lambda)|T| > 4b_{j_0}.$$

Another inequality we will be using which is easy to check is

$$-2\frac{(x-a)^2-b^2}{((x-a)^2+b^2)^2} \leq -\frac{1}{(x-a)^2+b^2}$$

if $|x-a| \geq 2b$.

Using these inequalities leads to

$$\min_{x\in T}(-Sf(x))\,|T|^2 \geq \frac{|T|^2}{|A|^2+b_{j_0}^2} - 32\frac{\kappa^2 d_E}{\lambda^2}.$$

Let us estimate the term containing A and T:

$$\begin{aligned}\frac{|T|^2}{|A|^2+b_{j_0}^2} &\geq \frac{|T|^2}{(2b_{j_0}+(1+\lambda)/2|T|)^2+b_{j_0}^2}\\ &= \frac{1}{(2b_{j_0}/|T|+(1+\lambda)/2)^2+(b_{j_0}/|T|)^2}\\ &\geq \frac{1}{((1+\lambda)/2+(1+\lambda)/2)^2+((1+\lambda)/4)^2}\\ &= \frac{16}{17(1+\lambda)^2}.\end{aligned}$$

Applying Lemma 3.2 to the inequalities obtained concludes the proof. □

PROPOSITION 3.4. *Let f be a $C^\infty(\mathcal{N})$ map with all critical points non-flat. There exist a neighbourhood $\mathcal{F}$ of f in $C^\infty(\mathcal{N})$ and $d_f \geq 0$ such that for any $\epsilon > 0$, $N > 0$, $\delta > 0$, $\lambda > 1$ there exist $\kappa > 0$ and $\tau > 0$ with the following properties. For any $g \in \mathcal{F}$ there exist at most d_f critical intervals E_j, $j = 1, \ldots, d_g$ such that:*

- *if $J \subset T$ are intervals, $g^m|_T$ is a diffeomorphism, the intersection multiplicity of $\{g^k(T)\}_{k=0}^{m-1}$ is bounded by N, $|g^k(T)| < \kappa$, and $|g^k(T) \cap E_j| < \kappa|E_j|$ for all $k = 0, \ldots, m-1$ and $j = 1, \ldots, d_g$, then*

$$B(g^m, T, J) > 1-\epsilon;$$

- *if J is an interval, $T = (1+2\delta)J$, $g|_T$ is a diffeomorphism, $|T| < \kappa$,*

$$|T \cap E_j| < \kappa|E_j|/\lambda$$

for all $j = 1, \ldots, d_g$ and either there exists a critical point c of g contained in the interval λT or there exists $j_0 \in [1, d_g]$ such that

$$T \not\subset 2E_{j_0} \quad \text{and} \quad \lambda T \cap E_{j_0} \neq \emptyset,$$

then

$$B(g, T, J) > 1+\tau.$$

Proof. Fix a small neighbourhood U of the critical set of f. Take a connected component U_0 of U. Decreasing U_0 if necessary, we can assume that U_0 contains only one critical point of f of order d. In the domain U_0 the function f can be written as $f|_{U_0} = (\phi_0)^d$, where ϕ_0 is a diffeomorphism. Take $\mathcal{F}$ small enough so that the function $g \in \mathcal{F}$ can be decomposed as $g|_{U_0} = \psi \circ P \circ \phi$, where P is a polynomial of degree at most d, and ψ and ϕ are diffeomorphisms such that ψ is C^∞ close to the identity map and ϕ is C^∞ close to ϕ_0; see [**Ser76**]. So, the Schwarzian derivatives of ψ and ϕ are uniformly bounded. Now we can apply Lemma 3.1 to the functions ϕ, ψ and Proposition 3.3 to the polynomial P.

Take another neighbourhood W of the critical set of f so that W is compactly contained in U. Decrease $\mathcal{F}$ if necessary so that the Schwarzian derivative of maps in $\mathcal{F}$ is uniformly bounded from above outside W. Then Lemma 3.1 implies that there are constants C and κ such that, for all $g \in \mathcal{F}$,

$$B(g, g^k(T), g^k(J)) > 1 - C|g^k(T)|^2$$

if the interval $g^k(T)$ is disjoint from W and $|g^k(T)| < \kappa$.

Decrease κ so that if an interval of length κ has a non-empty intersection with W, then this interval is contained in U.

Now we can estimate the cross-ratio distortion:

$$\begin{aligned}
\log(B(g^m, T, J)) &= \sum_{k=1}^{m-1} \log(B(g, g^k(T), g^k(J))) \\
&= \Bigg(\sum_{g^k(T)\cap W=\emptyset} + \sum_{g^k(T)\cap W\neq\emptyset} \Bigg) \log(B(g, g^k(T), g^k(J))) \\
&> -C \sum_{k=1}^{m-1} |g^k(T)|^2 - 16\kappa N d_f^2 \\
&> -C\kappa N|\mathcal{N}| - 16\kappa N d_f^2.
\end{aligned}$$

The last expression can be made arbitrarily close to zero by decreasing κ. □

4. *Uniform pullback estimates*

We also want to know a bound from below on the cross-ratio distortion when there are no bounds on the length of the intervals $g^k(T)$. Such a bound exists, though it is not as good as in the proposition above. To prove this bound we need a few statements.

LEMMA 4.1. *There exists a function $\rho(\epsilon, d) > 0$ such that if f is a polynomial of degree less than or equal to d, $\hat{J} \subset \hat{T}$ are intervals, $\hat{T}$ contains the ϵ-scaled neighbourhood of $\hat{J}$, T is a connected component of $f^{-1}(\hat{T})$, and J is a connected component of $f^{-1}(\hat{J})$ which is contained in T, then the interval T contains the $\rho(\epsilon, d)$-scaled neighbourhood of J.*

Moreover, $\rho(\epsilon, d)$ tends to infinity when ϵ goes to infinity with fixed d.

LEMMA 4.2. *There exists a function $\rho(\epsilon, d) > 0$ such that if f is a polynomial of degree less than or equal to d, $J \subset T$ are intervals, and $f|_T$ is a diffeomorphism, then*

$$D(T, J) < \rho(D(f(T), f(J)), d).$$

Moreover, $\rho(\epsilon, d)$ tends to zero when ϵ goes to zero with fixed d.

The second lemma is a straightforward consequence of the first, and we will prove here only the first lemma.

Proof of Lemma 4.1. First, we can assume that $\hat{T}$ is equal to the ϵ-scaled neighbourhood of $\hat{J}$. Next, we can rescale the polynomial f and assume that $T = \hat{T} = [0, 1]$. Thus, $f([0, 1]) \subset [0, 1]$.

Let A_d be a set of polynomials of degree less than or equal to d such that for any $g \in A_d$ one has $g(x) \in [0, 1]$ for any $x \in [0, 1]$ and $g(y) \in \{0, 1\}$ for $y \in \{0, 1\}$. The set A_d is compact. Indeed, any polynomial d of degree less than or equal to d is uniquely determined by its values at the points $x_k = k/d$, where $k = 0, \ldots, d$. So,

$$g(x) = \sum_{k=0}^{d} g(x_k) N_k(x),$$

where N_k is a Newton polynomial

$$N_k(x) = \frac{\prod_{i \neq k}(x - x_i)}{\prod_{i \neq k}(x_k - x_i)}.$$

Since $g(x_k) \in [0, 1]$ for $g \in A_d$ and all $k = 0, \ldots, d$, we see that the set A_d is compact. Therefore, the maximum of the derivatives of polynomials in A_d is bounded:

$$\inf_{g \in A_d} \max_{x \in [0,1]} |Dg(x)| < K.$$

This implies that both components $T \backslash J$ are greater than $\epsilon/K(1 + 2\epsilon)$ and the function ρ exists. Using the compactness argument once again, it is easy to show that $\rho(\epsilon, d) \to \infty$ when $\epsilon \to \infty$. □

PROPOSITION 4.3. *Let f be a $C^\infty(\mathcal{N})$ map with all critical points non-flat. There exist a neighbourhood $\mathcal{F}$ of f in C^∞ and a function $\rho(\epsilon)$ such that the following holds.*

Let g be in $\mathcal{F}$, $\hat{J} \subset \hat{T}$ are intervals, $\hat{T}$ contains the ϵ-scaled neighbourhood of $\hat{J}$, T is a connected component of $g^{-1}(\hat{T})$, and J is a connected component of $g^{-1}(\hat{J})$ which is contained in T, then the interval T contains the $\rho(\epsilon)$-scaled neighbourhood of J.

Moreover, $\rho(\epsilon)$ tends to infinity when ϵ goes to infinity.

Proof. The proof of this proposition is similar to the proof of Proposition 3.4.

Fix two neighbourhoods $U \subset U'$ of the critical set, and let each connected component of U contain just one critical point of f and U' compactly contain U. Take $\mathcal{F}$ so small that the distortion of the derivative of maps $g \in \mathcal{F}$ on the complement to U is bounded and that inside every connected component of U' a map $g \in \mathcal{F}$ can be decomposed as $\psi \circ P \circ \phi$, where P is a polynomial of uniformly bounded degree and ψ, ϕ are diffeomorphisms; see the proof of Proposition 3.4. From the lemma above it follows that if $T \subset U'$ then the function ρ exists. Since the derivative distortion on the complement of U is uniformly bounded, the function ρ exists also when T belongs to the complement of U. In the remaining case the interval T must contain a component of $U' \backslash U$ and cannot be small. The set $\mathcal{F}$ is precompact in the C^1 topology and, using a compactness argument again, we get the function ρ in the remaining case too. □

Similarly, we obtain the following proposition.

PROPOSITION 4.4. *Let f be a $C^\infty(\mathcal{N})$ map with all critical points non-flat. There exist a neighbourhood $\mathcal{F}$ of f in C^∞ and a function $\rho(\epsilon)$ such that the following holds.*

Let g be in $\mathcal{F}$, $J \subset T$ are intervals, and $g|_T$ is a diffeomorphism, then

$$D(T, J) < \rho(D(g(T), g(J))).$$

Moreover, $\rho(\epsilon)$ tends to zero when ϵ goes to zero.

THEOREM 4.5. *Let f be a $C^\infty(\mathcal{N})$ map with all critical points non-flat. There exist a neighbourhood $\mathcal{F}$ of f in $C^\infty(\mathcal{N})$ and a function $\rho(\epsilon, N)$ such that the following holds.*

Let g be in $\mathcal{F}$, $J \subset T$ be intervals such that $g^m|_T$ is a diffeomorphism and the intersection multiplicity of the intervals $g^k(T)$, $k = 0, \ldots, m-1$, is bounded by N. Then

$$D(T, J) < \rho(D(g^m(T), g^m(J)), N).$$

Moreover, $\rho(\epsilon, N)$ tends to zero when ϵ goes to zero and N is fixed.

Proof. Let $\mathcal{F}$ be so small that Proposition 3.4 holds with $\epsilon = \frac{1}{2}$, and Proposition 4.4 holds as well. Let κ be the constant given by the first proposition and ρ is a function given by the second one. Fix $g \in \mathcal{F}$ and let $E_1, \ldots, E_d$ be the corresponding critical intervals.

Let $k_1 < \cdots < k_{m'}$ be all indexes such that for every k_i either $|g^{k_i}(T)| \geq \kappa$ or there is j such that $|g^{k_i}(T) \cap E_j| \geq \kappa |E_j|$. If $k \neq k_i$, then $|g^k(T)| < \kappa$ and $|g^k(T) \cap E_j| < \kappa |E_j|$ for all $j = 1, \ldots, d$, and the first part of Proposition 3.4 can be applied to such intervals. Clearly, the number m' of these indexes is bounded above by some constant which depends only on κ, the number of critical intervals d and the intersection multiplicity N (and is independent of the choice of g).

Due to Proposition 3.4,

$$D(g^{k_{m'}+1}(T), g^{k_{m'}+1}(J)) < 2D(g^m(T), g^m(J)).$$

Now we can apply Proposition 4.4 to the map $g : g^{k_{m'}}(T) \to g^{k_{m'}+1}(T)$ and get

$$D(g^{k_{m'}}(T), g^{k_{m'}}(J)) < \rho(2D(g^m(T), g^m(J))).$$

Denote $\psi(D) = \rho(2D)$. Then

$$D(T, J) < 2\psi^{m'}(D(g^m(T), g^m(J)))$$

and, since m' is uniformly bounded, the theorem is proved. □

THEOREM 4.6. *Let f be a $C^\infty(\mathcal{N})$ map with all critical points non-flat. There exist a neighbourhood $\mathcal{F}$ of f in $C^\infty(\mathcal{N})$ and a function $\rho(\epsilon, N)$ such that the following holds.*

Let g be in $\mathcal{F}$, and $\{J_k\}_{k=0}^m$ and $\{T_k\}_{k=0}^m$ be chains such that $J_k \subset T_k$ for all $0 \leq k \leq m$. Assume that the intersection multiplicity of $\{T_k\}_{k=0}^s$ is at most N and that T_m contains an ϵ-scaled neighbourhood of J_m. Then T_0 contains the $\rho(\epsilon, N)$-scaled neighbourhood of J_0.

Moreover, $\rho(\epsilon, N)$ tends to infinity when ϵ goes to infinity and N is fixed.

Proof. This time let $k_1 < \cdots < k_{m'}$ be all indexes such that the interval T_{k_i} contains at least one critical point. If $\mathcal{F}$ is small, the number of critical points of a map $g \in \mathcal{F}$ is uniformly bounded, and since the intersection multiplicity of the intervals $\{T_k\}_{k=0}^m$ is bounded by N, the number m' is uniformly bounded as well.

Now we can apply the previous theorem to maps $g^{k_{i+1}-k_i+1} : g(T_{k_i}) \to T_{k_{i+1}}$ and Proposition 4.3 to maps $g : T_{k_i} \to T_{k_{i+1}}$ and finish the proof. □

5. *Proof of Theorem B*

The proof of this theorem uses the same ideas as in [**MdMvS92**] or [**dMvS93**], but we will need to tweak that proof quite a bit. We will also follow the notation in the book [**dMvS93**] where possible.

We start the proof by making a few trivial observations. If f is a diffeomorphism of a circle, then the neighbourhood $\mathcal{F}$ of f can be taken so it consists only of diffeomorphisms. In this case the theorem trivially holds as all periodic orbits of a circle diffeomorphism form one pack.

If $\mathcal{N}$ is an interval, we can enlarge it and set $\tilde{\mathcal{N}} = 3\mathcal{N}$. We can also extend the map f smoothly to a map $\tilde{\mathcal{N}} \to \tilde{\mathcal{N}}$ so that no extra critical points are created. If $\mathcal{N}$ is a circle, we set $\tilde{\mathcal{N}} = \mathcal{N}$.

Take $\mathcal{F}$ such that Propositions 3.4, 4.3 and 4.4 and Theorems 4.5 and 4.6 hold. Fix some small $\kappa > 0$.

If $\mathcal{F}$ is small enough, the number of critical points of maps in $\mathcal{F}$ is uniformly bounded. Hence, there can be only a uniformly bounded number of periodic orbits of $g \in \mathcal{F}$ which contain critical points in their basins of attraction.

Fix a map $g : \tilde{\mathcal{N}} \to \tilde{\mathcal{N}}$ which is C^∞ close to f and let E_j, $j = 1, \ldots, d_g$, be the critical intervals of g given by Proposition 3.4. Let $\mathcal{O} \subset \mathcal{N}$ be a periodic orbit of g. Denote the orientation preserving period of $\mathcal{O}$ by n.

Let $p \in \mathcal{N}$ be a point of $\mathcal{O}$ and define $T_p \subset \tilde{\mathcal{N}}$ be a maximal interval containing p such that each component of $T_p \backslash p$ contains at most one point of $\mathcal{O}$. Thus, the closure of T_p contains five points of $\mathcal{O}$ if p is not one of the four points closest to the boundary of $\mathcal{N}$.

Now fix $p \in \mathcal{O}$ such that the corresponding interval T_p has minimal length. Set $U_n = 3T_p$. Obviously, the interval U_n is a subset of $\tilde{\mathcal{N}}$ and the closure of U_n can contain at most 13 points of the orbit $\mathcal{O}$ while U_n itself contains at most 11 points of $\mathcal{O}$ in its interior. Let $\{\hat{U}_k\}_{k=0}^n$ be a chain such that $g^k(p) \in \hat{U}_k$ for all $k = 0, \ldots, n$ and $\hat{U}_n = U_n$.

LEMMA 5.1. *The intersection multiplicity of the chain* $\{\hat{U}_k\}_{k=0}^n$ *is bounded by* 44.

This is almost the same as [**dMvS93**, Lemma 10.3(i), p. 323], where it is formulated for diffeomorphic pullbacks instead of chains. The proof is the same, though.

Proof. The interval $\hat{U}_n$ contains at most 11 points of the orbit $\mathcal{O}$, hence $\hat{U}_k$ can contain at most 11 points of $\mathcal{O}$ as well. Thus if an interval $\hat{U}_i$ contains a point x, there exist at most ten points of $\mathcal{O}$ between $g^i(p)$ and x.

Suppose that $x \in \hat{U}_{k_1} \cap \cdots \cap \hat{U}_{k_m}$ with $0 \le k_1 < \cdots < k_m \le n$. Arguing as in the previous paragraph, we see that $g^{k_i}(p)$ can be one of 22 points of $\mathcal{O}$ around x. □

We will denote by U_n^l and U_n^r the left and right components of $U_n \backslash p$ and by $\{\hat{U}_k^l\}_{k=0}^n$ and $\{\hat{U}_k^r\}_{k=0}^n$ the corresponding chains. Notice that the point $g^k(p)$ is always a boundary point of the intervals $\hat{U}_k^l$ and $\hat{U}_k^r$.

Let us inductively define intervals U_k^r, $k = 0, \ldots, n$, by the following rule. U_k^r is the maximal interval containing $g^k(p)$ as its boundary point and satisfying the following conditions:

- $g(U_k^r) \subset U_{k+1}^r$;
- $g|_{U_k^r}$ is a diffeomorphism;
- $|U_k^r| \leq \kappa/2$;
- if $g^k(p) \in 2E_j$ for some j, then $|U_k^r| \leq \kappa|E_j|/2$;
- if $g^k(p) \notin 2E_j$, then U_k^r is disjoint from E_j.

Notice that $|U_k^r \cap E_j| \leq \kappa|E_j|/2$ for all k and j.

We will call k a *cutting time* if $g(U_k^r) \neq U_{k+1}^r$. The cutting times can be of one of the following types:

- a *critical cutting time* if U_k^r contains a critical point of g in its boundary;
- an *internal cutting time* if $|U_k^r| = \kappa/2$ or there exists a critical interval E_j such that $g^k(p) \in 2E_j$ and $|U_k^r| = \kappa|E_j|/2$;
- a *boundary cutting time* if there exists a critical interval such that $g^k(p) \notin 2E_j$ and U_k^r contains a boundary point of E_j in its boundary.

Since the number of critical points and critical intervals of maps in $\mathcal{F}$ is uniformly bounded and the intersection multiplicity of $\{U_k^r\}_{k=1}^n$ is universally bounded, the number of critical and boundary cutting times is uniformly bounded.

The intervals U_k^l are defined in the same way. We also set $U_k = U_k^l \cup U_k^r$.

By the definition of the intervals U_k it follows that $g^n|_{U_0}$ is a diffeomorphism. A simple argument shows that $U_0 \subset T_p$; see [**dMvS93**, Lemma 10.2, p. 322].

Now consider two cases. First, suppose that $g^n(U_0^r)$ is strictly contained in U_0^r. In this case all periodic points in U_0^r belong to the same pack, and if a point in this interval is not periodic, then it is in the attraction basin of one of the attracting points of the pack. Let k_1 be the minimal cutting time in $\{U_k^r\}_{k=1}^n$. If k_1 is a critical time, then one of the iterates of a critical point is in U_0^r, and therefore this critical point is in the attraction basin of some periodic point in the pack. Since the number of critical points of maps in $\mathcal{F}$ is uniformly bounded and the same critical point cannot be in the attraction basins of two different orbits, the number of such packs is uniformly bounded. Similarly, if k_1 is a boundary cutting time, then a boundary point of one of the critical intervals is in the attraction basin of a point from the pack and the number of critical intervals is also uniformly bounded.

Now consider the case where k_1 is an internal cutting time and suppose that it corresponds to the critical interval E_j, that is, $g^{k_1}(p) \in 2E_j$ and $|U_{k_1}^r| = \kappa|E_j|/2$. Let p' be another periodic point and suppose that if we perform a similar construction for p', we get the first cutting time k_1' internal and $g^{n'}(U_0^{r'}) \subset U_0^{r'}$ where $U_k^{r'}$ are the corresponding intervals. Since every point in U_k^r as well as in $U_{k_1'}^{r'}$ is either periodic or its iterates are attracted to a periodic orbit, it follows that if the closures of these two intervals have non-empty intersection, then the points p and p' belong to the same pack of periodic points. The interval $U_{k_1}^r$ has length $\kappa|E_j|/2$; there are at most $2/\kappa + 1$ disjoint intervals

like this. If we take into account all critical intervals, then we see that there can exist at most $(1+d_f)(1+2/\kappa)$ packs of periodic orbits in this case.

Let us summarize. All periodic points p such that $g^n(U_0^r) \subset U_0^r$ belong to a finite number of packs of periodic orbits. The number of these packs is bounded by some constant which depends on κ, d_f and the number of critical points of maps in $\mathcal{F}$ and does not depend on the choice of $g \in \mathcal{F}$.

From now on we will assume that $U_0^r \subset g^n(U_0^r)$. Let U_{-k}^r denote the diffeomorphic pullback of U_0^r along the orbit of p, $g^k(U_{-k}^r) = U_0^r$.

LEMMA 5.2. *If the interval U_{-n}^r contains another periodic point p' with order preserving period $n' \le n$, then the periodic points p and p' belong to the same pack of periodic orbits.*

Proof. We know that the interval U_0^r is a subset of T_p, so U_0^r contains at most one point of $\mathcal{O}$ in its interior. Let q be this point if it exists, otherwise let $q = p$. If $q = p'$, we are done, so assume that $q \neq p'$.

Since $p' \in U_{-n}^r$ we get $g^n(p') \in U_0^r$, and therefore there are no periodic points from $\mathcal{O}$ in the interval $(q, g^n(p'))$. Let $q' \in \mathcal{O}$ be another periodic point from the orbit $\mathcal{O}$ such that $p' \in (q, q')$ and the open interval (q, q') does not contain any points of $\mathcal{O}$.

If $g^{n'}(q) = q$ or $g^{n'}(q) = q'$, then the interval $[q, p']$ or $[p', q']$ is periodic and the points q and p' belong to the same pack of periodic orbits. Otherwise, the interval $(g^{n'}(q), g^{n'}(p')) = (g^{n'}(q), p')$ contains a point from the orbit $\mathcal{O}$, and therefore, the interval $(q, g^n(p'))$ will contain a point from $\mathcal{O}$ as $g^n : (q, p') \to (q, g^n(p'))$ is a diffeomorphism. This is a contradiction. □

PROPOSITION 5.3. *There exist constants $\rho > 0$ and $\kappa_0 > 0$ such that for any $\kappa \in (0, \kappa_0)$ there exists $M \in \mathbb{N}$ such that the following holds.*

For every $g \in \mathcal{F}$ there are at most M exceptional packs of periodic orbits of g such that if $\mathcal{O}$ is not in one of the exceptional packs, then there is a point $\theta^r \in U_0^r$ such that

$$Dg^n(\theta^r) > 1 + 2\rho.$$

Proof. We can assume that $Dg^n(x) < 2$ for all $x \in U_0^r$, otherwise we have nothing to prove. Since $U_n = 3T_p$ and $U_0 \subset T_p$ the closure of the interval $g^n(U_0^r)$ is contained in the interior of U_n. In particular, this implies that there exists at least one cutting time for $\{U_k^r\}_{k=1}^n$. Let m be the minimal cutting time, that is, there is no cutting time m' with $m' < m$. Another property of the minimal cutting time is that $g^m(U_0^r) = U_m^r$.

Consider several cases now. First, suppose that m is critical or boundary cutting time. Let $M' = \frac{1}{3}U_m^r$, and $M \subset U_0^r$ is a preimage of M' under g^m. Due to the second part of Proposition 3.4 we know that $B(g, U_m^r, M') > 1 + \tau$ where $\tau > 0$ is a constant independent of the choice of $g \in \mathcal{F}$. Moreover, τ does not change if we decrease κ. We know that $|g^k(U_0^r) \cap E_j| \le \kappa |E_j|/2$ for all $k = 0, \dots, n-1$ and j, so we can apply the first part of Proposition 3.4 to maps $g^m : U_0^r \to U_m^r$ and $g^{n-m-1} : g(U_m^r) \to g^n(U_0^r)$. Decreasing κ if necessary, we can get $B(g^n, U_0^r, M) > 1 + \tau/2$.

Let L and R be connected components of $U_0^r \backslash M$ and let L contain the point p in its boundary. According to Theorem 4.6, the interval $g^n(R)$ cannot be very small compared to the interval $g^n(U_0^r)$. Indeed, if $g^n(R)$ is small, then it has a huge space inside U_n^r, that is, $Cg^n(R) \subset U_n^r$ for some large constant C. If we apply Theorem 4.6 to the

map $g^{n-m} : g^m(R) \subset \hat{U}_m^r \to g^n(R) \subset U_n^r$, we can see that the interval $g^m(R)$ would have a big space in U_m^r. However, one of the components of $U_m^r \backslash g^m(R)$ has length $2|g^m(R)|$, so the space is bounded. A similar argument holds for the interval $g^n(L)$; in this case we should consider $U_n^l \cup g^n(U_0^r)$ as a neighbourhood of $g^n(L)$.

Thus, there exists a constant $\beta > 0$ independent of the choice of $g \in \mathcal{F}$ such that

$$|g^n(L)| > \beta |g^n(U_0^r)| \quad \text{and} \quad |g^n(R)| > \beta |g^n(U_0^r)|.$$

Since $U_0^r \subset g^n(U_0^r)$ and $Dg|_{U_0^r} < 2$ it follows that

$$|L| > \tfrac{1}{2}\beta |U_0^r| \quad \text{and} \quad |R| > \tfrac{1}{2}\beta |U_0^r|.$$

Now we can apply the 'first expansion principle'; see [**dMvS93**, Theorem 1.3, p. 280] to the map $g^n : U_0^r \to g^n(U_0^r)$ and get a point $\theta^r \in U_0^r$ with $Dg^n(\theta^r) > 1 + 2\rho$, where ρ does not depend on $g \in \mathcal{F}$.

The remaining case we have to consider is when m is the internal cutting time. By definition we know that in this case either $|U_m^r| = \kappa/2$ or there exists a critical interval E_j such that $g^m(p) \in 2E_j$ and $|U_m^r| = \kappa |E_j|/2$. We will consider only the second case; the first can be dealt with in the exactly same way.

Consider an interval $U_{-6n}^r \subset U_0^r$. The derivative of g^n on U_0^r is bounded by 2, therefore $|U_{-6n}^r| > 2^{-7}|g^n(U_0^r)|$ and the interval $U_0^r \backslash U_{-6n}^r$ has a definite space inside the interval U_n^r. Applying Theorem 4.6 to the map $g^{n-m} : \hat{U}_m^r \to U_n^r$, we get a constant $\gamma > 0$ such that $|U_{-7n+m}^r| > \gamma |U_m^r| = \gamma\kappa |E_j|/2$. This constant is independent of $g \in \mathcal{F}$ and κ.

Define an interval $W \subset U_m^r$ so that $g^m(p)$ is the boundary point of W and $|W| = \frac{1}{8}\gamma\kappa |E_j|$. Clearly, $g^{6n}|_W$ is a diffeomorphism.

Let $\mathcal{O}'$ be another periodic orbit of g of period n'. Suppose that if for $\mathcal{O}'$ we repeat the construction we did for $\mathcal{O}$, then the corresponding first cutting time m' is also of internal type with the same interval E_j. Let W' be defined as W but for the orbit $\mathcal{O}'$.

Claim. If the intervals W and W' have a non-empty intersection, then the orbits $\mathcal{O}$ and $\mathcal{O}'$ belong to the same pack.

Without loss of generality we can assume that $n' \geq n$. Also observe that the intervals W and W' have the same length.

Let us consider several cases of how the intervals W and W' can intersect. First, let us suppose that $g^m(p) \in W'$. Then $g^{m+n'-m}(p) \in U_{-6n'}^{r'}$ and, due to Lemma 5.2, the points p and p' are in the same pack.

Another case is $g^m(p) \notin W'$ and $g^{m'}(p') \notin W$. In this case it is easy to see that since $|U_{-7n'+m'}^{r'}| > 4|W'|$, the interval $U_{-7n'+m'}^{r'}$ contains the point $g^m(p)$ and the same argument as above can be applied.

The last case is $g^m(p) \notin W'$ and $g^{m'}(p') \in W$. If $g^{m'+n}(p') = g^{m'}(p')$, then the interval $(g^m(p), g^{m'}(p'))$ is periodic and the points p and p' are in the same pack. If $g^{m'+n}(p')$ is in the interval $(g^m(p), g^{m'}(p'))$, then $(g^m(p), g^{m'}(p'))$ is mapped into itself and iterates of the point $g^{m'}(p')$ are attracted to some periodic attractor, which is impossible because $g^{m'}(p)$ is a periodic point.

So, $g^{m'+n}(p') \notin (g^m(p), g^{m'}(p'))$. This implies that $g^{m'+in}(p') \in U_m^r \backslash (g^m(p), g^{m'}(p'))$ for $i = 1, \ldots, 6$. The intervals U_m^r and $U_{m'}^{r'}$ have the same length $\kappa |E_j|/2$, therefore,

in this case

$$U_m^r \backslash (g^m(p), g^{m'}(p')) \subset U_{m'}^{r'}$$

and the interval $g^{n'-m'}(U_{m'}^{r'})$ contains six points from the orbit $\mathcal{O}'$ in its interior. This contradicts the fact that $U_n^r \supset g^{n'-m'}(U_{m'}^{r'})$ contains at most five points from $\mathcal{O}'$. The claim is proved.

Now we can finish the proof of the proposition. It follows from the claim that there are at most $8/(\gamma\kappa) + 1$ packs of periodic orbits such that a periodic point from such a pack can have the minimal cutting time of boundary type associated with the critical interval E_j. Since the number of the critical intervals is uniformly bounded, the lemma follows. □

The theorem easily follows from this proposition. Take $\kappa \in (0, \kappa_0)$ so small that Proposition 3.4 holds with

$$\epsilon = 1 - \left(\frac{1+\rho}{1+2\rho}\right)^{1/3}$$

and $N = 44$. For this choice of κ let $\theta^r \in U_0^r$ and $\theta^l \in U_0^l$ be given by the proposition, so

$$Dg^n(\theta^{r,l}) > 1 + 2\rho.$$

Set $T = (\theta^l, \theta^r)$. Since $T \subset U_0$, $|g^k(U_0)| \le \kappa$, and $|g^k(U_0) \cap E_i| \le \kappa |E_j|$ for all $k = 0, \ldots, n-1$ and j and the map $g^n : T \to g^n(T)$ is a diffeomorphism, Proposition 3.4 can be applied to all intervals $J^* \subset T^* \subset T$. We get

$$B(g^n, T^*, J^*)^3 > \frac{1+\rho}{1+2\rho}.$$

Now the 'minimum principle' (see [**dMvS93**, Theorem 1.1, p. 275]) can be applied and

$$Dg^n(x) > 1 + \rho$$

for all $x \in T$. In particular, $Dg^n(p) > 1 + \rho$.

A. *Appendix. Proof of the uniform contraction principle*

Following the suggestion of a referee, we outline here proofs of Lemma 2.1 and uniform bounds on the Schwarzian derivative used in the proof of Theorem 2.2.

Proof of Lemma 2.1. Suppose that the conclusion of the lemma is false. This means that there exist

- a map $f \in C^1(\mathcal{N})$,
- a constant $\epsilon > 0$,
- a sequence of maps $g_k \in C^1(\mathcal{N})$, $k = 1, 2, \ldots,$
- a sequence of intervals $I_k \subset \mathcal{N}$,
- and a sequence of positive integers n_k,

such that the following properties are satisfied:

(1) f does not have wandering intervals;
(2) f does not have neutral periodic points;
(3) $|I_k| > \epsilon$ for all k;
(4) $g_k \to f$ in C^1 norm as $k \to \infty$;

(5) $n_k \to +\infty$;
(6) $|g_k^{n_k}(I_k)| \to 0$;
(7) $g_k^{n_k}(I_k)$ does not intersect an immediate attraction basis of a periodic attractor of g_k.

By considering a subsequence we can assume that the intervals I_k converge to an interval I_0. This interval I_0 cannot be degenerate as its length is bounded by ϵ from below. Notice that at this point we cannot claim that $\liminf_{n\to+\infty} |f^n(I_0)| = 0$.

Claim 1. These are no periodic points of f in $\mathrm{int}(f^n(I_0))$, $n = 0, 1, \ldots$, where int denotes the interior of a set.

Indeed, if $a \in \mathrm{int}(f^{n_0}(I_0))$, for some $n_0 \geq 0$, is a periodic point, then a cannot be a neutral point of f because of property (2). Hence, under a small perturbation of f the point a persists and there exists k_0 such that $g_k^{n_0}(I_k)$ contains a periodic point of g_k for all $k \geq k_0$. If a is an attracting periodic point, then we get a contradiction with property (7). If a is repelling, then there exists $\epsilon_a > 0$ such that $|f^n(I_0)| > \epsilon_a$ for all n. This also holds for small perturbations of f, and it contradicts property (6).

Similarly, one can prove the following claim.

Claim 2. Intervals $f^n(I_0)$ cannot have a non-empty intersection with immediate basins of attraction of attracting periodic points of f.

Let $W = \bigcup_{n=0}^{\infty} \mathrm{int}(f^n(I_0))$. The set W is not necessarily forward invariant, but its closure is. Take a connected component U of W. If, for some $m > 0$, $f^m(U) \cap U \neq \emptyset$, then $f^m(U) \subset \bar{U}$, where $\bar{U}$ is the closure of U. Consider several cases.

(1) If U contains a periodic point of f, then one of the $f^n(I_0)$ contains a periodic point of f in its interior. This contradicts Claim 1.
(2) If U is an interval and there are no periodic points of f in U, then $f^m|_U$ is monotone and one of the boundary points a of U is an attracting periodic point of f. Moreover, the immediate basin of attraction of a contains U, and therefore some $f^n(I_0)$ has a non-empty intersection with it, which is impossible according to Claim 2.
(3) Let U be a circle. In this case $\mathcal{N} = W = U$ and the map f does not have periodic points. By compactness there are finitely many $0 \leq n_1 < \cdots < n_r$ such that $\mathcal{N} = \bigcup_{i=1}^{r} \mathrm{int}(f^{n_r}(I_0))$. It is easy to see that there exist $z \in \mathcal{N}$ and $l > 0$ such that the points z and $f^l(z)$ are in $\mathrm{int}(f^{n_0}(I_0))$ for some n_0. Let $\epsilon_0 = \min_{x\in\mathcal{N}} |f^l(x) - x|$. Obviously $\epsilon_0 > 0$ as f has no periodic points. Then $|g^l(x) - x| > \epsilon_0/2$ for all x if g is sufficiently close to f. So, for large k one has that $\{z, g_k^l(z)\} \subset g_k^{n_0}(I_k)$ and therefore $|g_k^n(I_k)| > \epsilon_0/2$ for all $n > n_0$. This contradicts property (6).

Finally, if the orbit of U is disjoint, then either U is a wandering interval of f or it is attracted to a periodic attractor. Both cases are impossible because of property (1) and Claim 2. □

Let c be a quadratic critical point of $f \in C^3(\mathcal{N})$ and let $B = Df^2(c)$. Fix a neighbourhood $\mathcal{F} \in C^3(\mathcal{N})$ of f and some interval T of c so f does not have other critical points in T. We can assume that all maps in $\mathcal{F}$ have one quadratic critical point in T. If $\mathcal{F}$ and T are small enough, we get $D^3g(x)Dg(x) - \frac{3}{2}(D^2g(x))^2 < -B^2$ for all $g \in \mathcal{F}$ and $x \in T$.

Let $c_g \in T$ denote the critical point of $g \in \mathcal{F}$. Due to the mean value theorem we get $Dg(x) = Dg(c_g) + D^2g(z)(x - c_g)$ for some $z \in [c_g, x]$. Therefore, $|Dg(x)| < A|x - c_g|$ for some $A > 0$ for all $g \in \mathcal{F}$ and $x \in T$. Combining these inequalities, we get

$$Sg(x) = \frac{D^3g(x)Dg(x) - \frac{3}{2}(D^2g(x))^2}{(Dg(x))^2} < -\frac{B^2}{A^2|x - c_g|^2}.$$

This is the required estimate. The rest of the proof of Theorem 2.2 literally follows the proof in [**Koz00**] or [**VV04**].

References

[dMvS89] W. de Melo and S. van Strien. A structure theorem in one dimensional dynamics. *Ann. of Math.* **129**(3) (1989), 519–546.

[dMvS93] W. de Melo and S. van Strien. *One-Dimensional Dynamics*. Springer, Berlin, 1993.

[KK11] V. Kaloshin and O. S. Kozlovski. A C^r unimodal map with an arbitrary fast growth of the number of periodic points. *Ergod. Th. & Dynam. Sys.* **32**(01) (2011), 159–165.

[Koz00] O. S. Kozlovski. Getting rid of the negative Schwarzian derivative condition. *Ann. of Math. (2)* **152**(3) (2000), 743–762.

[Koz12] O. S. Kozlovski. Hilbert–Arnold problem for one dimensional maps, 2012, in preparation.

[Man85] R. Mañé. Hyperbolicity, sinks and measure in one-dimensional dynamics. *Commun. Math. Phys.* **100** (1985), 495–524.

[MdMvS92] M. Martens, W. de Melo and S. van Strien. Julia–Fatou–Sullivan theory for real one-dimensional dynamics. *Acta Math.* **168** (1992), 273–318.

[Ser76] F. Sergeraert. Une extension d'un théoreme de fonctions implicites de Hamilton. *Mémoires de la SMF* **46** (1976), 163–171.

[Sin78] D. Singer. Stable orbits and bifurcation of maps of the interval. *SIAM J. Appl. Math.* **35**(2) (1978), 260–267.

[VV04] S. van Strien and E. Vargas. Real bounds, ergodicity and negative Schwarzian for multimodal maps. *J. Amer. Math. Soc.* **17**(4) (2004), 749–782.