CHARACTERS OF THE GENERIC HECKE ALGEBRA
OF A SYSTEM OF BN-PAIRS

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For Joy
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ABSTRACT.

This thesis is concerned with the construction of the characters of the generic Hecke algebra of a system of BN-pairs of type \((W,R)\). The approach used exploits the connection between these characters and those of the Coxeter group \(W\).

Part I of the thesis gives definitions, results and conjectures relevant to calculating the characters of any generic Hecke algebra and Part II applies these results to the problem of calculating the characters of certain generic Hecke algebras.

In more detail: Chapter 1 gives basic definitions and describes the connection between the characters of the generic Hecke algebra and those of the Coxeter group \(W\).

Chapter 2 divides the problem of calculating the characters into two parts and gives conjectures and results concerning one of these 'parts'. The results given solve this part of the problem (in a non-explicit manner) for some but not all generic Hecke algebras.

The last three chapters are all concerned with solving the other 'part' of the problem by an inductive method which uses the connection described in chapter 1.

Chapter 3 describes this method (the 'circle product'). Chapters 4 and 5 apply this method to a particular 'family' of generic Hecke algebras, culminating in an explicit formula (given in theorem 5.3.9) for certain character values.

The appendices contain tables of character values of some generic Hecke algebras.
STANDARD NOTATION.

\[ Z \] the ring of rational integers
\[ Q \] the field of rationals
\[ C \] the complex field
\[ Z_+ = \{ z \in Z \mid z > 0 \} \]
\[ Z_s = \{1, 2, \ldots, s\} \text{ for each } s \in Z_+. \text{ ( } Z_0 = 0 \text{ see below) } \]
\[ k \] a subfield of \( C \)
\[ u \] an indeterminate over \( C \)
\[ K_0 = Q(u) \text{ the field of rational functions in } u \text{ over } Q \]
\[ K \] a finite field extension of \( K_0 \) (see also below definition 1.4.3)
\[ I_0 = Q[u] \text{ the ring of polynomials in } u \text{ over } Q \]
\[ I \] the integral closure of \( I_0 \) in \( K \)
\[ \emptyset \] the empty set

\[ A \setminus B = \{ a \in A \mid a \notin B \} \] (A and B sets)

\[ \langle \ldots \rangle \] the group generated by a given set of elements

\[ (\alpha \cdot \beta)(d) = \alpha(\beta(d)) \text{ where } \alpha \text{ and } \beta \text{ are maps. } d \text{ is in the } \]
\[ \text{domain of } \beta \text{ and } \beta(d) \text{ is in the domain of } \alpha \]

\[ \langle \rho_1, \rho_2, \ldots, \rho_t \rangle \text{ a partition of } \sum_{i=1}^{t} \rho_i \text{ (} \rho_i \in Z_+ \cup \{0\} \text{ for each } i \in Z_t \rangle \]

\[ \langle \sigma_1, \sigma_2, \ldots, \sigma_n \rangle \text{ a partition of } n. \text{ (} n \in Z_+, \sigma_i \in Z_+ \text{ for each } i \in Z_n \rangle \text{ (see the beginning of } \S 4.1) \]

(iii)
PART I

CHAPTER I

THE GENERIC HECKE ALGEBRA AND ITS CHARACTERS.

Two equivalent definitions of the generic Hecke algebra are given in this chapter (see definition 1.1.8 and below lemma 1.3.4) and the significance of its characters is explained.

§1.1 COXETER GROUPS AND THE GENERIC HECKE ALGEBRA.

DEFINITION 1.1.1.

Given a group $W$ and a subset $R$ of involutions of $W$, the pair $(W,R)$ is a Coxeter system if the following condition holds:

For each $r,s$ in $R$ let $n_{rs}$ be the order of $rs$. Let $T$ be the set of pairs $(r,s)$ such that $n_{rs}$ is finite. $R$ is a set of generators for $W$ with defining relations

$$(rs)^{n_{rs}} = 1_W$$

for all pairs $(r,s)$ in $T$.

In which case $W$ is a Coxeter group with distinguished generators $R$.

NOTE: The relations $(rs)^{n_{rs}} = 1_W$ can be rewritten as follows:

$$(1.1.2) \quad r^2 = 1_W \quad \text{for all } r \text{ in } R.$$  
$$(1.1.3) \quad (rs \ldots)^{n_{rs}} = (sr \ldots)^{n_{rs}}$$

for all pairs $(r,s)$ in $T$ with $r \neq s$, where $(ab \ldots)^n$ means the product of the first $n$ terms in the alternating sequence $a,b,a,b,\ldots$.

Since every element $w \neq 1_W$ of $W$ is a product of elements
of $R$, we can make the following definition.

**DEFINITION 1.1.4.**

The length of the element $w$ of $W (w \neq 1_W)$, denoted by $l(w)$, is the least number of terms possible in an expression for $w$ as a product of elements of $R$. Any such expression with this number of terms is said to be reduced. Conventionally $l(1_W) = 0$.

**LEMMA 1.1.5.**

Let $w \in W$ and $r \in R$. $l(wr) = l(w) + 1$.

Proof:

By equations (1.1.2) and (1.1.3) it is clear that $l(wr) \neq l(w)$. If $l(wr) > l(w)$ then clearly $l(wr) = l(w) + 1$. $l(wr . r) = l(w)$ thus if $l(wr) < l(w)$ we must have $l(wr) = l(w) - 1$.

**LEMMA 1.1.6.**

Given two reduced expressions for $w$ in $W (w \neq 1_W)$, one can be transformed into the other using only the relations (1.1.3).

Proof: [2, chapter IV, exercises §1.13b].

**COROLLARY 1.1.7.**

Given a set of positive integers $\{c_r | r \in R\}$ such that if $r, s$ in $R$ are conjugate in $W$ then $c_r = c_s$, it follows that there exists a function $c: W \rightarrow \mathbb{Z}_+$ given by

$$c(w) = c(r_1) + c(r_2) + \ldots + c(r_t)$$

where $r_1 r_2 \ldots r_t$ is any reduced expression for $w \in W$.

Proof:

If $n_{rs}$ is odd then $(rs\ldots s)_{n_{rs}-1} r(sr\ldots r)_{n_{rs}-1} = s$
showing that \( r \) and \( s \) are conjugate in \( W \). Thus the result now follows from lemma 1.1.6.

**DEFINITION 1.1.8.**

The set \( \{ c_r \mid r \in R \} \) described in corollary 1.1.7 is an indexing system for \((W,R)\).

For the rest of this thesis we will consider only finite Coxeter groups.

**DEFINITION 1.1.9.**

The associative \( K \)-algebra with identity element \( h_1 \) generated by \( \{ h_r \mid r \in R \} \) with defining relations

\[
(1.1.10) \quad h_r^2 = u c_r h_1 + (u c_r - 1) h_r \quad \text{for all } r \in R.
\]

\[
(1.1.11) \quad (h_r h_s \ldots \ldots )_{n_{rs}} = (h_s h_r \ldots \ldots )_{n_{rs}} \quad \text{for all } r,s \in R \text{ with } r \neq s,
\]

is the generic Hecke algebra of type \((W,R)\) with indexing system \( \{ c_r \mid r \in R \} \) and will be denoted by \( H(W,R,c,K,u) \) abbreviated to \( H(K,u) \).

**THEOREM 1.1.12.**

\( H(K,u) \) has \( K \)-basis \( \{ h_w \mid w \in W \} \) where

\[
(1.1.13) \quad h_w = h_{r_1} h_{r_2} \ldots h_{r_t} \quad \text{for any reduced expression } r_1 r_2 \ldots r_t \text{ of } w.
\]

This theorem can be readily proved using a specialisation \( f \) of \( K \) with \( f(u) = 1 \) (see §1.4), but as the given \( K \)-basis arises naturally in the alternative definition of \( H(K,u) \) given in §1.3 we omit the proof.

Note that lemma 1.1.6 shows that \( h_w \) is well defined.
§1.2 SYSTEMS OF BN-PAIRS.

A BN-pair is defined in [4, 8.2]. The definition of a system of BN-pairs is due to Curtis, Iwahori and Kilmoyer [5].

DEFINITION 1.2.1.

Let \((W, R)\) be a Coxeter system with indexing system \(\{c_r \mid r \in R\}\) (see definitions 1.1.1 and 1.1.8). A system of BN-pairs of type \((W, R)\) is a set \(S\) of finite groups, indexed by an infinite set \(P\) of prime powers such that:

(i) Each \(G(q)\) in \(S (q \in P)\) has a BN-pair, say, \((B(q), N(q))\).

(ii) For each \(q\) in \(P\) there is a map \(n_q : W \rightarrow N(q)\) such that \(w \rightarrow n_q(w)(B(q) \cap N(q))\) defines an isomorphism from \(W\) onto the Coxeter group \(W(q) = N(q)/(B(q) \cap N(q))\) which maps \(R\) onto the set of distinguished generators of \(W(q)\).

(iii) For each \(q\) in \(P\) and \(r\) in \(R\),

\[|B(q):B(q) \cap (n_q(r))^{-1}B(q)n_q(r)| = q^{c_r}\]

Matsumoto [8, theorem 3] has shown that it is sufficient to specify that \(W\) is a finite group generated by a set \(R\) of involutions, since \(W\) is then necessarily a Coxeter group with distinguished generators \(R\).

Each of the families of finite Chevalley groups of a fixed type form a system of BN-pairs. In these cases \(P\) is the set of all prime powers and the integers \(c_r\) all have value 1. [6, §1]

Each of the families of 'twisted Chevalley groups' form a system of BN-pairs. In these cases the integers \(c_r\) are not in general all equal and in some of these cases
§1.3 HECKE ALGEBRAS.

Let \( G(q) \) be an element of the system of BN-pairs \( S \).
Define in the group algebra \( kG(q) \) the idempotent
\[
e = \frac{1}{|B(q)|} \sum_{g \in B(q)} g
\]

DEFINITION 1.3.1.
\( e kG(q) e \) is a Hecke algebra and will be denoted by \( E_k(q) \).

LEMMA 1.3.2.

The Hecke algebra \( E_k(q) = e kG(q) e \) is isomorphic as a \( k \)-algebra to the endomorphism algebra \( \text{End}_{kG(q)}(V(q)) \), where \( V(q) \) is the left ideal \( kG(q)e \) regarded as a left \( kG(q) \)-module.

Proof:
The map \( \alpha : E_k(q) \rightarrow \text{End}_{kG(q)}(V(q)) \) given by
\( \alpha(t) : v \mapsto vt \) for all \( t \) in \( E_k(q) \) and \( v \) in \( V(q) \), is an isomorphism.

\[\blacksquare\]

THEOREM 1.3.3 ([8, theorem 4])

(i) \( E_k(q) \) has \( k \)-basis \( \{d_w \mid w \in W\} \) where \( d_w = q^{c(w)} e_n q(w) e \) (see definition 1.2.1(ii) and corollary 1.1.7).

(ii) Let \( w \in W \) then \( d_w = d_{r_1} d_{r_2} \cdots d_{r_t} \) for any reduced expression \( r_1 r_2 \cdots r_t \) of \( w \).

(iii) \( E_k(q) \) is generated as a \( k \)-algebra with identity \( d_1 \) by \( \{d_r \mid r \in R\} \) with defining relations
\[
d_r^2 = q^{c_r} + (q^{c_r} - 1)d_r \quad \text{for all} \quad r \in R.
\]
for all \( r, s \in R \) with \( r \neq s \).

Tits (see \([2, \text{ page 55}]\)) is responsible for the idea of the generic Hecke algebra (see definition 1.1.9) which is seen to 'specialise' (see §1.4) to \( E_k(q) \) on putting \( u = q \) (\( k \) a suitable field of characteristic zero). This connection between \( H(K, u) \) and \( E_k(q) \) is described in detail below.

**Lemma 1.3.4.**

For all \( x, y, z \) in \( W \) there exist polynomials \( \sigma_{xyz}(u) \) in \( \mathbb{Z}[u] \) such that

(i) For any \( q \) in \( P \), \( d_x d_y = \Sigma_{z \in W} \sigma_{xyz}(q) d_z \)

(ii) \( \Sigma_{t \in W} \sigma_{xyt}(u) \sigma_{tzv}(u) = \Sigma_{s \in W} \sigma_{xsv}(u) \sigma_{yzs}(u) \) for all \( x, y, z, v \) in \( W \).

Proof:

(i) This follows immediately from theorem 1.3.3.

(ii) Let \( q \in P \). If \( u \) is replaced by \( q \) in the given equation it becomes equivalent to the associativity of \( E_k(q) \). Since \( P \) is an infinite set the result follows.

Lemma 1.3.4 enables us to define \( H(K, u) \) to be the \( K \)-algebra with \( K \)-basis \( \{ h_w \mid w \in W \} \) and multiplication given by

\[
    h_x h_y = \Sigma_{z \in W} \sigma_{xyz}(u) h_z
\]

for all \( x, y \) in \( W \).

Lemma 1.3.4(ii) shows that this algebra is associative. Theorem 1.3.3 clearly shows that this definition of \( H(K, u) \) is equivalent to definition 1.1.9 and that theorem 1.1.12 is correct.
§1.4 CHARACTERS OF THE GENERIC HECKE ALGEBRA.

\( K_0 \) and \( K \) are the fields of fractions of \( I_0 \) and \( I \) respectively. Given a prime ideal \( D \) of \( I \), let \( K_D = \{ \frac{\delta}{\alpha} \mid \delta \in I, \alpha \in I \setminus D \} \); this is a subring of \( K \) containing \( I \). Let \( H(K_D, u) \) be the subring \( \{ \sum_{w \in W} \lambda_w h_w \mid \lambda_w \in K_D \} \) of \( H(K, u) \).

**DEFINITION 1.4.1.**

A specialisation \( f \) of \( K \) with nucleus \( D \) is a ring homomorphism \( f : K_D \to C \) with \( f(1) = 1 \) and \( \ker(f) = DK_D \).

Note that \( D \) is determined by \( f \) since \( D = I \cap \ker(f) \).

The range \( k = f(K_D) \) of \( f \) is a subfield of \( C \) (see [6, §4]).

**LEMMA 1.4.2.**

(i) For each \( q \) in \( P \) there exists a specialisation \( f_q \) of \( K \) with nucleus \( D(q) \) such that \( f_q(u) = q \).

\( f_q \) can be extended to a ring epimorphism

\[ f_q : H(K_D(q), u) \to h_k(q)(q) \]

where \( k(q) \) is the range of \( f_q \) by setting

\[ f_q(\sum_{w \in W} \lambda_w h_w) = \sum_{w \in W} f_q(\lambda_w) d_w \]

for all \( \lambda_w \) in \( K_D(q) \).

(ii) There is a specialisation \( f_1 \) of \( K \) with nucleus \( D(1) \) and range \( k(1) \) such that \( f_1(u) = 1 \).

\( f_1 \) can be extended to a ring epimorphism

\[ f_1 : H(K_D(1), u) \to k(1)W \]

by setting

\[ f_1(\sum_{w \in W} \lambda_w h_w) = \sum_{w \in W} f_1(\lambda_w) w \]

for all \( \lambda_w \) in \( K_D(1) \).

Proof:

Follows immediately from [6, theorem 4.1 and lemma 4.2].

**DEFINITION 1.4.3.**

Given a specialisation \( f \) of \( K \) with nucleus \( D \)

(i) for \( \alpha \in K \) we say that \( 'f(\alpha) \) is defined'
if and only if $\alpha \in K_D$.

(ii) for $\delta \in H(K,u)$ we say that 'f(\delta) is defined' if and only if $\delta \in H(K_D,u)$.

Note that (ii) can be equivalently expressed by saying that for $\delta = \sum_{w \in W} \alpha_w w$, f(\delta) is defined if and only if f(\alpha_w) is defined for all w in W.

Tits has shown that $H(K,u)$ is semi-simple and hence separable because the characteristic of K is zero. (see [6, theorem 6.2]). Thus we can find a finite extension K of $K_0$ such that K is a splitting field for $H(K,u) \cong K \otimes_{K_0} H(K,u)$. We will from now on assume that K has this property.

A representation of $H(K,u)$ over K is a K-algebra homomorphism $\xi : H(K,u) \rightarrow \{ \tau \mid \tau \text{ is a K-linear transformation from } V \text{ to } V \}$ for some K-space V. The character of $\xi$ is the K-linear map $\eta : H(K,u) \rightarrow K$ given by $\eta(\delta) = \text{trace}(\xi(\delta))$ for all $\delta \in H(K,u)$. $\eta$ is an irreducible character if $\xi$ is an irreducible representation. (see [3, chapters I and II]).

The set of all functions $\lambda : H(K,u) \rightarrow K$ is an additive group with respect to the composition:

$$(\lambda_1 + \lambda_2)(\delta) = \lambda_1(\delta) + \lambda_2(\delta)$$

for all $\delta \in H(K,u)$. The subgroup generated by the characters of $H(K,u)$ is called the character group of $H(K,u)$ and is denoted by $X(H(K,u))$. The set of irreducible characters of $H(K,u)$ is a free Z-basis for $X(H(K,u))$.

**Lemma 1.4.4.**

If $\eta \in X(H(K,u))$ and $w \in W$ then

1. $\eta(h_w) \in I$
(ii) for any specialisation $f$ of $K$, $f(\eta(h_w))$ is defined.

Proof:

(i) follows from [6, lemma 7.2].

(ii) follows immediately from (i).

\begin{theorem}[Tits]
Let $X(E_C(q))$ be the character group of $E_C(q)$ over $C$.
Let $X(CW)$ be the character group of $CW$ over $C$.

(i) For each $q$ in $P$ there exists a bijection from $X(H(K,u))$ to $X(E_C(q))$, which maps the irreducible characters of $H(K,u)$ onto the set of irreducible characters of $E_C(q)$.
If $\eta \in X(H(K,u))$ maps to $\mu \in X(E_C(q))$ then

$$\mu(f_q(\delta)) = f_q(\eta(\delta))$$
for all $\delta \in H(K,u)$, in particular

$$\mu(d_w) = f_q(\eta(h_w))$$
for all $w \in W$. (see lemma 1.4.2).

(ii) There exists a bijection from $X(H(K,u))$ to $X(CW)$, which maps the set of irreducible characters of $H(K,u)$ to the set of irreducible characters of $CW$. If $\eta \in X(H(K,u))$ maps to $\chi \in X(CW)$ then $X(f_\chi(\delta) = f_\chi(\eta(\delta))$ for all $\delta \in H(K,u)$, in particular $X(w) = f_\chi(\eta(h_w))$ for all $w \in W$.

Proof:

This theorem follows from the proof of theorem 7.4 in [6].

J.A. Green has pointed out that the character group $X(H(K,u))$ is characterised in the following way:

\begin{theorem}

$$X(H(K,u)) = \{ \eta: H(K,u) \rightarrow K | \eta \text{ is } K-\text{linear,}$$

$$\eta(h_wh_v) = \eta(h_vh_w) \text{ for all } w, v \in W \}.$$

Proof:

Let $F$ be a splitting field for the semi-simple $F$-algebra
A. Using Wedderburn's theorem and reducing to the case of a total matrix algebra one can readily show that the character group $X(A) = \{ \sigma: A \to F \mid \sigma \text{ is } F\text{-linear, } \sigma(xy) = \sigma(yx) \text{ for all } x, y \in A \}$. Since $H(K,u)$ is semi-simple the theorem follows.

Note that for every pair $\alpha, \beta$ of linear transformations of some $K$-space that $\trace(\alpha \cdot \beta) = \trace(\beta \cdot \alpha)$. Hence it is clear that for any $\eta$ in $X(H(K,u))$ and $w, v$ in $W$ that $\eta(hwv) = \eta(hvw)$.

The material in §1.2, §1.3 and §1.4 is nearly all in [6]. In that paper the generic Hecke algebra is denoted by $A_K(u)$ and the Hecke algebra by $H_k(q)$.

§1.5 THE ALGEBRA $H_0$ AND ITS CHARACTERS.

The following chapters are independent of the material in this section.

By [6, theorem 4.1 and lemma 4.2] there exists a specialisation $f_0$ of $K$ with nucleus $D_0$ and range $k_0$ such that $f_0(u) = 0$.

Let $H_0$ be the $k_0$-algebra with $k_0$-basis $\{b_w \mid w \in W\}$ and multiplication given by

$$b_x b_y = \sum_{z \in W} \sigma_{xyz}(0)b_z$$

for all $x, y \in W$.

(see lemma 1.3.4).

It is clear from theorem 1.3.3 that $H_0$ is generated as a $k_0$-algebra with identity $b_1$ by $\{b_r \mid r \in R\}$ and that the following are defining relations for this set of generators:

$$(1.5.1) \quad b_r^2 = -b_r$$

for all $r \in R$. 

\[(1.5.2) \quad (b_r b_s \ldots )_{rs} = (b_s b_r \ldots )_{rs} \quad \text{for all} \quad r, s \ \text{in} \ R \ \text{with} \ r \neq s.\]

It is also clear that \(b_w = b_{r_1} b_{r_2} \ldots b_{r_t} \) where \(r_1 r_2 \ldots r_t\) is any reduced expression for \(w \in W\).

\(f_o\) can be extended to a ring epimorphism from \(H(K_{D_0}, u)\) to \(H_o\) by setting
\[f_o(\sum \lambda_w h_w) = \sum f_o(\lambda_w)b_w \quad \text{for all} \quad \lambda_w \in K.\]
(Compare lemma 1.4.2).

Unlike \(H(K, u)\), \(E_k(q)\) \((q \in P)\) and \(CW\) the algebra \(H_o\) is not semi-simple and there is hence no analogue of theorem 1.4.5 for this algebra. The irreducible characters of \(H_o\) are described in theorem 1.5.4.

**Lemma 1.5.3.**

Let \(N_o\) be the nilpotent radical of \(H_o\). \(H_o/N_o\) is commutative.

**Proof:**

We show that \(b_r b_s - b_s b_r \in N_o\) for all \(r, s \in R\) with \(r \neq s\). It is well known that it is sufficient to show that \(b_r b_s - b_s b_r\) is properly nilpotent, i.e. given any elements \(\lambda_w \in k_o\) \((w \in W)\) that
\[a = \sum_{w \in W} \lambda_w b_w (b_r b_s - b_s b_r) \quad \text{is nilpotent.}\]

Fix \(w \in W\) and \(r, s \in R\).

If \(l(wr) < l(w)\) and \(l(ws) < l(w)\) or \(l(wr) < l(w), l(ws) > l(w)\) and \(l(wsr) < l(ws)\) or \(l(ws) < l(w), l(wr) > l(w)\) and \(l(wrs) < l(wr)\) then using equation (1.5.1) one readily finds that
Thus $b_w(b_r b_s - b_s b_r) \neq 0$ implies that $b_w(b_r b_s - b_s b_r) = b_{w_1} + b_{w_2}$ for some $w_1, w_2 \in W$ with $l(w_1), l(w_2) > l(w)$. So it is clear that there exists $t \in \mathbb{Z}_+$ with $a^t = 0$.

**Theorem 1.5.4.**

Let $R = \{ r_1, r_2, \ldots, r_m \}$. The set 
\[ \{ \delta_{i_1 i_2 \ldots i_m} \mid i_j \in \{0,1\}, j \in \mathbb{Z}_m \} \]

of $k_0$-homomorphisms from $H_0$ to $k_0$ where 
\[ \delta_{i_1 i_2 \ldots i_m}(b_r) = -i_j \quad \text{for all } j \in \mathbb{Z}_m \]

and 
\[ \delta_{i_1 i_2 \ldots i_m}(b_1) = 1 \]

is the set of all irreducible characters of $H_0$ over $k_0$.

**Proof:**

Let $\delta$ be an irreducible character of $H_0$. By lemma 1.5.3 $\delta(b_1) = 1$. Thus $\delta$ is an irreducible character if and only if 
\[ \delta(b_r) = 1 \]
\[ (\delta(b_r))^2 = -\delta(b_r) \quad \text{for all } r \in R \]

and 
\[ (\delta(b_r)\delta(b_s)\ldots)_{n_{rs}} = (\delta(b_s)\delta(b_r)\ldots)_{n_{rs}} \]

for all $r, s \in R$ with $r \neq s$. The result is now clear.

**Corollary 1.5.5.**

\[ \dim_{k_0}(N_0) = |W| - 2|R| . \]

**Proof:**

\[ \dim_{k_0}(H_0) - \dim_{k_0}(N_0) = |\{ \delta \mid \delta \in X(H_0), \delta \text{ is irreducible} \}| . \]

Given $\eta \in X(H(K,u))$ it is clear that the $k_0$-linear map $f_0(\eta):H_0 \to k_0$ defined by 
\[ f_0(\eta):b_w \to f_0(\eta(h_w)) \]

is a character of $H_0$.
However, theorems 1.4.5(ii) and 1.5.4 show that even if $\eta$ is irreducible, $f_\circ(\eta)$ will not in general be irreducible. We thus make the following definition.

**DEFINITION 1.5.6.**

Let $|R| = m$. Let $\{\eta_j | j \in Z_s\}$ be the set of all irreducible characters of $H(K,u)$. (By theorem 1.4.5(ii) $s$ is the number of conjugacy class in $W$).

The **decomposition matrix** of $H_0$, denoted by $D_{H_0}$, is an $(s \times 2^m)$-matrix with columns indexed by the set $\Theta = \{(i_1, i_2, \ldots, i_m) | i_j \in \{0, 1\}, j \in Z_m\}$. The $(j, (i_1, \ldots, i_m))$th entry $d_{j, (i_1, \ldots, i_m)}$ of $D_{H_0}$ is defined by the equations

$$f_\circ(\eta_j) = \sum_{(i_1, \ldots, i_m) \in \Theta} d_{j, (i_1, \ldots, i_m)} \delta_{i_1, i_2, \ldots, i_m}$$

for all $j \in Z_s$.

Appendix 4 gives the decomposition matrix for some particular examples of $H_0$. The symmetry exhibited in these examples can be explained using the involutory semi-linear automorphism of $H(K,u)$ described in [6, §8].
CHAPTER 2

LINEAR DEPENDENCE OF THE CHARACTER VALUES.

The problem of evaluating the characters of the generic Hecke algebra is separated into two parts (see below corollary 2.1.5). Some results and conjectures concerning one of these parts are then given (see §2.2).

§2.1 THE CHARACTER TABLE.

DEFINITION 2.1.1.
Let $W$ have $m_w$ conjugacy classes. The character table, $T(H(K,u))$, of $H(K,u)$ is an $(m_w \times |W|)$-array of elements of $K$. Rows are indexed by the irreducible characters of $H(K,u)$ and the columns are indexed by the elements of $W$. Let $\eta$ be an irreducible character of $H(K,u)$ and $w$ an element of $W$. The $(\eta, w)^{th}$ entry of $T(H(K,u))$ is $\eta(h_w)$.

(Appendix 5 gives some examples of character tables).

REMARK 1. By lemma 1.4.4(i) all the entries of $T(H(K,u))$ are in $I$.

REMARK 2. Since any character of $H(K,u)$ is a $K$-linear function and $\{h_w \mid w \in W\}$ is a $K$-basis of $H(K,u)$, we see that the character table $T(H(K,u))$ completely determines the values of the irreducible characters of $H(K,u)$.

THEOREM 2.1.2.
If $B$ is a set of conjugacy class representatives for $W$ then the columns of $T(H(K,u))$ indexed by $B$ span the column space of $T(H(K,u))$.

Proof:
Let the column of $T(H(K,u))$ indexed by $w \in W$ be $\eta(h_w)$. 

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We can consider it to be an element of the $K$-space $K^m_w$.

Since the row rank of a matrix is equal to its column rank the theorem will follow if we can show that \( \{ \eta(h_b) | b \in B \} \) is a linearly independent set over $K$.

$I_0$ is a principal ideal domain and hence is a Dedekind domain. $K$ is a finite field extension of $K_0$, thus by [1, chapter VII, 2.5, propn 5, corollary 3] $I$ is a Dedekind domain.

Let $f$ be a specialisation of $K$ with the prime ideal $D$ of $I$ as its nucleus (see \( \text{§1.4} \)). [1, chapter VII, \( \text{§2.2} \), theorem 1(g) and chapter II, \( \text{§3.1} \), propn 2] show that $K_D$ is a principal ideal domain with $DK_D$ a prime ideal (since it is a maximal ideal).

Let $\Sigma_{b \in B} \sigma_b \eta(h_b) = 0$ where $\sigma_b \in K$ for all $b \in B$.

Since $K$ is the field of fractions of $I$ we can assume without loss of generality that each $\sigma_b \in I \subset K$. Further since we have shown that $K_D$ is a principal ideal domain, it is a unique factorization domain and we can assume by cancelling out any common divisor of the elements $\sigma_b \ (b \in B)$ that either $\sigma_b = 0$ for all $b \in B$, or that there exists at least one $b \in B$ with $\sigma_b \notin DK_D$. $\text{Ker}(f) = DK_D$, so in this latter case

$$\Sigma_{b \in B} f(\sigma_b)f_1(\eta(h_b))$$

is a non-trivial linear combination of $\{ f(\eta(h_b)) | b \in B \} \subset C^m_w$. ($f$ is applied component-wise to the vectors $\eta(h_b)$). In particular

$$0 = \Sigma_{b \in B} f'_1(\sigma_b)f'_1(\eta(h_b))$$

is a non-trivial linear combination over $C$ of the columns of the character table of $CW$ (see theorem 1.4.5(ii)), giving a contradiction since these columns are well known to be linearly independent over $C$. 

\[15\]
Immediately we have

**COROLLARY 2.1.3**

Let $B$ be a set of conjugacy class representatives for $W$. For each $w$ in $W$ there exist unique elements $\sigma(w,b)$ of $K$ ($b \in B$) such that

$$\eta(h_w) = \sum_{b \in B} \sigma(w,b)n(h_b) \quad \text{for all } \eta \in X(H(K,u)).$$

The coefficients $\sigma(w,b)$ are determined by the relations

$$\eta(h_w h_v) = \eta(h_v h_w) \quad \text{for all } \eta \in X(H(K,u))$$

and $w,v \in W$ (see theorem 1.4.6 and below), in the following way.

**LEMMA 2.1.5.**

Let $H_0(K,u)$ be the $K$-space spanned by $\{h_w h_v - h_v h_w \mid w,v \in W\}$.

(i) Let $\xi \in H(K,u)$. $\eta(\xi) = 0$ for all $\eta \in X(H(K,u))$ if and only if $\xi \in H_0(K,u)$.

(ii) The elements $\sigma(w,b)$ of $K$ are uniquely determined by the equations

$$h_w = \sum_{b \in B} \sigma(w,b)h_b \mod H_0(K,u) \quad \text{for all } w \in W.$$

Proof:

(i) This follows readily from the proof of theorem 1.4.6.

(ii) By theorem 1.4.6 and (i) we see that

$$X(H(K,u)) \cong H(K,u)/H_0(K,u)$$

as $K$-spaces. So by theorem 2.1.2 $H(K,u)/H_0(K,u)$ has $K$-basis $\{h_b + H_0(K,u) \mid b \in B\}$.

\[\Box\]
Corollary 2.1.3 shows that we can divide the problem of evaluating the characters of $H(K,u)$ into two parts in the following way:

**PART 1.** Determine the elements $\sigma(w,b)$ of $K$ defined in corollary 2.1.3 ($w \in W$, $b \in B$) for some suitable set $B$ of conjugacy class representatives of $W$.

**PART 2.** Calculate the character values $\eta(h_b)$ for all irreducible characters $\eta$ of $H(K,u)$ and all $b$ in $B$, where $B$ is the same set of class representatives as in 'Part 1'. (Note that the set of all irreducible characters of $H(K,u)$ is a free $Z$-basis for $X(H(K,u))$.)

$$\{2.2 \ \sigma(w,b)\}$$

**NOTATION**

$Y = \{w \in W \mid w$ is of minimal length in its conjugacy class $\}$

$Y^* = \{w \in W \mid l(w) \leq l(rwr)$ for all $r \in R\}$

$B$ a set of conjugacy class representatives of $W$ with $B \subseteq Y$.

$\{\sigma(w,b) \mid w \in W, b \in B\}$ the set of elements of $K$ defined by corollary 2.1.3. (Appendix 7 gives some examples of these).

Note that clearly $Y \subseteq Y^*$, but in general $Y \neq Y^*$.

For example in the Coxeter group $S_5$ (see Appendix 2) the 4-cycle $(1452) \in Y^* \setminus Y$.

**CONJECTURE 2.2.1.**

$\sigma(w,b) \in Z[u]$ for all $w \in W$ and $b \in B$.

**CONJECTURE 2.2.2.**

For each $w$ in $W$ and $y$ in $Y$ there exists $\alpha(w,y)$ in $Z[u]$. 
such that

\[
(2.2.3) \quad \eta(h_w) = \sum_{y \in Y} \alpha(w, y) \eta(h_y) \quad \text{for all } \eta \in X(H(K, u)).
\]

Note that equations (2.2.3) do not in general determine the coefficients \( \alpha(w, y) \) uniquely.

**Lemma 2.2.4.**

For each \( w \) in \( W \) and \( y \) in \( Y^* \) there is an element \( \delta(w, y) \) in \( \mathbb{Z}[u] \) such that

\[
\eta(h_w) = \sum_{y \in Y} \delta(w, y) \eta(h_y) \quad \text{for all } \eta \in X(H(K, u)).
\]

**Proof:**

We use induction on \( l(w) \). If \( w \in Y^* \) the result is trivial. Let \( w \in W \setminus Y^* \). There exists \( r \in R \) with \( l(rwr) < l(w) \), in fact by lemma 1.1.5, \( l(rwr) = l(w) - 2 \). Thus there exists \( v \in W \) such that \( l(v) = l(w) - 2 \) and \( w = rvr \). By (1.1.10), (1.1.13) and (2.1.4)

\[
\eta(h_w) = \eta(h_r h_v h_r)
= \eta(h_v h_r h_r)
= u^{c_r} \eta(h_r) + (u^{c_r} - 1) \eta(h_{rv})
\]

Since \( l(v), l(vr) < l(w) \) the result follows by induction.

**Conjecture 2.2.5.**

If \( y \) and \( z \) in \( Y \) are conjugate in \( W \) then

\[
\eta(h_y) = \eta(h_z) \quad \text{for all } \eta \in X(H(K, u)).
\]

Note that in general the condition: '\( y, z \in Y^* \), \( y \) conjugate to \( z \) in \( W' \) does not imply that \( \eta(h_y) = \eta(h_z) \) for all \( \eta \) in \( X(H(K, u)) \). For example in the Coxeter group \( S_5 \) (see Appendix 2) we have (1452) and (1234) in \( Y^* \). These two elements are clearly conjugate but \( \eta(h_{1452}) = u^5 \) and \( \eta(h_{1234}) = u^3 \) where \( \eta \) is the unit character of \( H(K, u) \). (see definition 4.1.3).
CONJECTURE 2.2.6.

Let \( w \) be an element of \( W \). The following two statements are equivalent.

(i) \( w \in Y \)

(ii) If \( w = tv \) \((t,v \in W)\) with \( l(t) + l(v) = l(w) \) then \( l(vt) = l(w) \).

(Clearly (i) implies (ii) as \( vt = t^{-1}(tv)t \).)

These four conjectures and lemma 2.2.4 are related in the following ways.

LEMMA 2.2.7.

(i) Conjecture 2.2.6 implies conjecture 2.2.2.

(ii) Conjecture 2.2.2 together with conjecture 2.2.5 implies conjecture 2.2.1.

(iii) Conjecture 2.2.1 implies conjecture 2.2.2 which implies lemma 2.2.4.

Proof:

(ii) and (iii) are immediate.

(i) We use induction on \( l(w) \). If \( w \in Y \) result is trivial. Let \( w \in W \setminus Y \). There exists a reduced expression \( r_1 \cdots r_t \) for \( w \) \((t = l(w))\) and \( i \in \mathbb{Z}_{t-1} \) such that

\[
r_ir_{i+1} \cdots r_tr_{i-1}\]

is reduced and \( r_{i+1} \cdots r_tr_{i-1} \) is not reduced. By (1.1.13) and (2.1.4) we can without loss of generality take \( i = 1 \). Thus \( l(rwr) = l(w) - 2 \) where \( r = r_1 \). The proof can now be completed exactly as the proof of lemma 2.2.4 was.

The rest of this section is concerned with proving some special cases of the conjectures above.

As noted in §1.2 each of the families of finite Chevalley groups of a fixed type form a system of BN-pairs.
Carter's book [4] describes the Coxeter groups for these systems (Note that these Coxeter groups are Weyl groups and are referred to as such in [4]). In particular it shows that

(i) The Coxeter group of type $A_{n-1}$ ($n > 1$) is isomorphic to the symmetric group $S_n$ ([4, page 124]) and an isomorphism can be found which maps the distinguished generators of $W$ onto \{$u_i = (i \ i+1) | \ i \in \mathbb{Z}_{n-1}$\} (see theorem A2.1). We will identify the Coxeter group of type $A_{n-1}$ with $S_n$ and hence $R$ with \{$u_i | \ i \in \mathbb{Z}_{n-1}$\}.

(ii) The Coxeter group of type $B_2$ is isomorphic to the group $\langle a, g | a^2 = g^2 = (ag)^4 = 1 \rangle$ which has order 8.

(iii) The Coxeter group of type $B_3$ is isomorphic to the group $\langle a, g, d | a^2 = g^2 = d^2 = (ag)^3 = (gd)^4 = (ad)^2 = 1 \rangle$ which has order 48.

(iv) The Coxeter group of type $G_2$ is isomorphic to the group $\langle a, g | a^2 = g^2 = (ag)^6 = 1 \rangle$ which has order 12.

**THEOREM 2.2.8.**

Conjectures 2.2.1, 2.2.2, 2.2.5 and 2.2.6 are true for $W$ of types $A_t$ ($1 \leq t$), $B_2$, $B_3$ and $G_2$.

Proof:

Lemma 2.2.7 shows that it is sufficient to prove that conjectures 2.2.5 and 2.2.6 hold. These can be checked for $W$ of types $B_2$, $B_3$ and $G_2$ by listing all the elements of $W$ in terms of reduced expressions and using the relations (2.1.4).

Let $W$ be of type $A_{n-1}$ ($n > 1$). $W = S_n$ by part (i) of the discussion above this theorem. Thus conjecture
2.2.6 follows immediately from theorem A2.7(i) and since for \( x \) an indeterminate over \( C \) and \( z \) in \( Z_+ \) the field \( Q(x) \) is a finite field extension of \( Q(x^z) \) it is clear that conjecture 2.2.5 follows from corollaries A2.6 and 4.2.10(ii).

REMARK. For a group \( W \) for which conjectures 2.2.5 and 2.2.6 hold the proof of lemma 2.2.7(i) gives a method of calculating the coefficients \( \sigma(w,b) \) \( (w \in W, \ b \in B \subset Y) \). This is illustrated in Appendix 1 for the case of \( W \) of type \( B_3 \).

**LEMMA 2.2.9.**

Let \( r, s \in R \) and \( \eta \in X(H(K,u)) \).

(i) If \( n_{rs} \) is odd then \( r \) and \( s \) are conjugate in \( W \).

(ii) If \( n_{rs} \) is odd then \( \eta(h_r) = \eta(h_s) \).

Note that (ii) is a special case of conjecture 2.2.5.

Proof:

(i) Equations (1.1.2) and (1.1.3) show that

\[ (rs\ldots s)_{n_{rs}-1} r(sr\ldots r)_{n_{rs}-1} = s \]

(ii) Since \( n_{rs} \) is odd we have

\[ \eta(h_{rs\ldots r}) = \eta(h_{sr\ldots s})_{n_{rs}} \]

By equations (1.1.13) and (2.1.4)

\[ \eta(h_{sr\ldots r})_{n_{rs}-1} h_r = \eta(h_s h_{sr\ldots r})_{n_{rs}-1} \]

By equations (1.1.10) and (1.1.13)

\[ u^c_r \eta(h_{sr\ldots s})_{n_{rs}-2} + (u^c_r-1) \eta(h_{sr\ldots r})_{n_{rs}-1} \]

\[ = u^c_s \eta(h_{rs\ldots r})_{n_{rs}-2} + (u^c_s-1) \eta(h_{sr\ldots r})_{n_{rs}-1} \]

By (i) \( r \) and \( s \) are conjugate in \( W \) hence \( c_r = c_s \) and

\[ \eta(h_{rs\ldots r})_{n_{rs}-2} = \eta(h_{sr\ldots s})_{n_{rs}-2} \]
The result now clearly follows by 'decreasing induction'.

**Lemma 2.2.10.**

Conjecture 2.2.6 is true for \( w \in \{ w_{\sigma} \in W \mid \sigma \in \Psi \} \), where \( W \) is a Weyl group with root system \( \Psi \) (see [3, Chapter 2]).

Proof:

Since it is true that in conjecture 2.2.6 (i) implies (ii) for any Coxeter group \( W \) we need only prove that (ii) implies (i) in this case.

Let \( \Pi \) be a fundamental system in \( \Psi \). \( \{ w_{\rho} \mid \rho \in \Pi \} \) is a set of distinguished generators of \( W \) (considered as a Coxeter group).

Fix \( w_{\sigma} (\sigma \in \Psi) \). Since \( w_{\sigma} = w_{-\sigma} \) we can assume that \( \sigma \in \Psi^+ \) the set of positive roots. Thus

\[
\sigma = \sum_{\rho \in \Pi} \lambda_{\rho} \rho \quad \text{where} \quad \lambda_{\rho} \in \mathbb{Z}_+ \cup \{0\} \text{ for all } \rho \in \Pi.
\]

There exists \( \tau \in \Pi \) such that the inner product \( (\tau, \sigma) > 0 \), otherwise \( (\sigma, \sigma) = \sum_{\rho \in \Pi} \lambda_{\rho} (\rho, \sigma) < 0 \) a contradiction.

If \( w_{\sigma} \) is not of minimal length in its class then clearly \( \sigma \notin \Pi \), thus at least two elements of \( \{ \lambda_{\rho} \mid \rho \in \Pi \} \) are non-zero. Hence

\[
w_{\sigma}(\tau) = -2(\sigma, \tau)_{\sigma} \in \Psi^- = \Psi \setminus \Psi^+\]

and

\[
(w_{\sigma}w_{\tau})^{-1} = w_{\tau}(w_{\sigma}(\tau))
\]

\[
= -2(\sigma, \tau)_{\sigma} + \left\{ \frac{4(\sigma, \tau)^2}{(\sigma, \sigma)(\tau, \tau)} - 1 \right\} \in \Psi^-.
\]

By [4, lemma 2.2.1 and theorem 2.2.2]

\[
l(w_{\tau}w_{\sigma}w_{\tau}) = l(w_{\sigma}) - 2
\]

Thus there exists \( x \in W \) with \( l(x) = l(w_{\sigma}) - 2 \) and \( w_{\sigma} = w_{\tau}w_{\tau}xw_{\tau} \).

\[
l((xw_{\tau})w_{\tau}) < l(w_{\sigma}) \quad \text{so the proof is complete.}
\]

We now give an explicit formula for the coefficients...
DEFINITION 2.2.11.
Let $e, f \in \mathbb{Z}$.

$$
(c_{e,f}) = \begin{cases} 
0 & \text{if } e < 0 \text{ or } f < 0 \\
0 & \text{if } 0 \leq e \text{ and } e < f \\
1 & \text{if } 0 \leq e \text{ and } f = 0 \\
e!/(f!(e-f)!) & \text{if } 0 < f < e
\end{cases}
$$

LEMMA 2.2.12.
Let $W$ be of type $A_{n-1}$ $(n > 1)$. By part (i) of the discussion above theorem 2.2.8, $W = S_n$. Corollary A2.6 shows that for any indexing system $\{c_r \mid r \in R\}$ (see definition 1.1.8) that $c_r = c_s$ for all $r, s \in R$. Let $c_r = c$ for $r \in R$. Let $\eta \in X(H,K,u)$ and $a, s, t \in \mathbb{Z}_{n-1}$ with $a > s > t$.

Then using cycle notation for elements of $S_n$ we have

(i) \(\eta(h(a s)) = \sum_{j=0}^{a-1-s} (a-1-s)^j u^j c_j (u^{c-1})^{a-1-s-j} \eta(h(a a-1 a-2 \ldots s+j))\)

(ii) \(\eta(h(a s s-1 s-2 \ldots t)) = \sum_{j=0}^{a-1-s} (a-1-s)^j u^j c_j (u^{c-1})^{a-1-s-j} \eta(h(a a-1 a-2 \ldots t+j))\)

(iii) \(\eta(h(a t t+1 t+2 \ldots s)) = \sum_{j=0}^{a-1-t} (a-1-t)^j u^j c_j (u^{c-1})^{a-1-t-j} \eta(h(s+j s+j+1 \ldots a))\)

Note that there exists a set of class representatives $B \subseteq Y$ for $S_n$ such that $(a a-1 \ldots s+j), (a a-1 \ldots t+j) \in B$ for all $j \in \mathbb{Z}_{a-1-r} \cup \{0\}$ ($r = s$ or $t$ as appropriate). Further, by theorem 2.2.8 (conjecture 2.2.5), $\eta(h(s+j s+j+1 \ldots a)) = \eta(h(a a-1 \ldots s+j))$. 

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Proof:

Let $u_i = (i \ i+1)$ for $i \in \mathbb{Z}_{n-1}$. Note that

$$(a \ s) = u_s u_{s+1} \ldots u_{a-1} u_a \ldots u_s$$

$$(a \ s \ s-1 \ldots t) = u_s u_{s+1} \ldots u_{a-1} u_a \ldots u_t$$

$$(a \ t \ t+1 \ldots s) = u_t u_{t+1} \ldots u_{a-1} u_a \ldots u_s$$

Let $e, f \in \mathbb{Z}_+$ with $a > e \geq f$. Using equations (A2.2), (A2.3), (A2.4), (1.1.13) and (2.1.4) we see that

$$\eta(h_{e} u_{e+1} \ldots u_{a-1} u_{a-2} \ldots u_{f}) = \eta(h_{e+1} u_{a-1} u_{a-2} \ldots u_{f} h_{e})$$

$$= \eta(h_{e+1} u_{a-1} u_{a-2} \ldots u_{e} u_{e-1} u_{e-2} \ldots u_{f})$$

$$= \eta(h_{e+1} u_{a-1} u_{a-2} \ldots u_{e} u_{e-1} u_{e-2} \ldots u_{f})$$

$$= \eta(h_{e-1} u_{e+1} u_{a-1} u_{a-2} \ldots u_{f})$$

$$= \eta(h_{e+1} u_{a-1} u_{a-2} \ldots u_{f} h_{e})$$

$$= \eta(h_{e+1} u_{a-1} u_{a-2} \ldots u_{f} h_{e})$$

$$= u_{e} \eta(h_{e+1} u_{a-1} u_{a-2} \ldots u_{f+1})$$

$$+ (u_{e-1}) \eta(h_{e+1} u_{a-1} u_{a-2} \ldots u_{f+1})$$

Using induction on $(a-e)$ we prove that

$$(2.2.13) \quad \eta(h_{e} u_{a-1} u_{a-2} \ldots u_{f}) = 
\sum_{j=0}^{a-1-e} (a-1-e) u_{j} c_{j} (u_{e-1})^{a-1-e-j} \eta(h_{a-1} u_{a-2} \ldots u_{f+j})$$

If $(a-e) = 1$ equation (2.2.13) is clearly valid.

Using the inductive hypothesis and the equation derived above, we have

$$\eta(h_{e} u_{a-1} u_{a-2} \ldots u_{f}) =$$

$$u_{c} \sum_{i=0}^{a-2-e} (a-2-e) u_{c} c_{i} (u_{c-1})^{a-2-e-i} \eta(h_{a-1} u_{a-2} \ldots u_{f+1+i})$$

$$+ (u_{c-1}) \sum_{i=0}^{a-2-e} (a-2-e) u_{c} c_{i} (u_{c-1})^{a-2-e-i} \eta(h_{a-1} u_{a-2} \ldots u_{f+1+i})$$

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Since for $x, y \in \mathbb{Z}_+$, \( \binom{x}{y} + \binom{x}{y+1} = \binom{x+1}{y+1} \) this shows equation (2.2.13) to be valid.

(i) follows immediately from equation (2.2.13) by putting $e = f = s$.

(ii) follows immediately from equation (2.2.13) by putting $e = s$ and $f = t$.

(iii) can clearly be proved in a way analogous to that used to prove (ii).
CHAPTER 3

THE CIRCLE PRODUCT.

In this chapter the basis of an inductive method for calculating the characters of $H(K,u)$ is described (see definition 3.2.5 and corollary 3.2.10) and its connection with an analogous method for calculating the characters of $W$ is displayed in §3.3.

§3.1 GROTHENDIECK GROUPS.

No proofs are given in this section.

Let $B$ be a ring and $M$ a category of finitely generated $B$-modules with $B$-homomorphisms as the morphisms. For any finitely generated $B$-module $V$ let $[V] = \{U \in M| U \cong V\}$.

DEFINITION 3.1.1.

The Grothendieck group $\chi(M)$ of the category $M$ is $F/S$ where $F$ is the free $\mathbb{Z}$-module on $\{[V]\}_{V \in M}$ and $S$ is the additive subgroup of $F$ generated by $\{[U] - [V] + [Y] | 0 \rightarrow U \rightarrow V \rightarrow Y \rightarrow 0$ is a short exact sequence in $M\}$. Denote the element $[V] + S$ by $\langle V \rangle$.

We abbreviate 'short exact sequence' to s.e.s.

LEMMA 3.1.2.

(i) Every element of $\chi(M)$ has the form $\sum_{V \in M} z_V V$ where each $z_V \in \mathbb{Z}$ and only a finite number of them are non-zero.

(ii) $\langle V \rangle U(+)Y(\text{ whenever } 0 \rightarrow U \rightarrow V \rightarrow Y \rightarrow 0$ is a s.e.s. in $M$. In particular $\langle V \rangle U(+)U(\text{.})$.
(iii) \( 0( = 0^X(M) \)

(iv) \( U \cong V \) implies that \( )U( = )V( . \)

**LEMMA 3.1.3.**

Given a map \( \psi \) from \( M \) to an abelian group \( L \) such that \( \psi(V) = \psi(U) + \psi(Y) \) whenever \( 0 \to U \to V \to Y \to 0 \) is a s.e.s in \( M \) then there exists a unique additive map

\[
g : \chi(M) \to L
\]

\[
g : )V( \mapsto \psi(V) \text{ for all } V \in M.
\]

**NOTATION**

\( BN \) is the category with the set of all finitely generated left \( B \)-modules as objects and for \( U, V \in BN \),

\[
\text{Mor}(U, V) = \{ \theta : U \to V | \theta \text{ is a } B\text{-homomorphism} \}.
\]

**LEMMA 3.1.4.** (compare [9, page 130]).

If \( B \) is an algebra over some field and \( U_1, U_2, \ldots, U_t \) is a complete set (up to isomorphism) of irreducible left \( B \)-modules, then \( \chi(BN) \) is freely generated as an abelian group by \( \{ )U_i( | i \in Z_t \} \).

**COROLLARY 3.1.5.**

If \( B \) is a semi-simple \( F \)-algebra (\( F \) a field) with character group \( X \) then the map \( \alpha : (BN) \to X \) given by \( \alpha : )V( \mapsto \text{character of } V \) for all \( V \in BN \) is an isomorphism of additive groups.

**§3.2 THE CIRCLE PRODUCT.**

**NOTATION.**

Let \( J \subset R \). By lemma A3.2 \( (W_J, J) \) is a Coxeter system
where $W_J$ is the parabolic subgroup of $W$ corresponding to $J$ (see definition A3.1).

Clearly we can regard the generic Hecke algebra $H(W_J,J,c|W,J,K,u)$ which we abbreviate to $H_J$ as a subalgebra of $H(W,R,c,K,u) = H_R$. We abbreviate $H_J(N)$ to $J(N)$ and $\mathcal{X}(H_J(N))$ to $\mathcal{X}(H_J)$. (see 3.1).

**DEFINITION 3.2.1.**

The statement $\{J_i | i \in S\}$ where $S$ is an indexing set and each $J_i$ is a subset of $R$ means that for $i \neq j$, $J_i \cap J_j = \emptyset$ and for all $r \in J_i$ and $t \in J_j$, $tr = rt$.

If $\{J_i | i \in S\}$ we say that the sets $J_i$ are mutually perpendicular.

**LEMMA 3.2.2.**

Let $J$ and $T$ be subsets of $R$.

(i) $\{J,T\}$ implies that $W_J W_T = W_{J \cup T}$ and for $w \in W_J$ and $v \in W_T$, $l(wv) = l(w) + l(v)$.

(ii) $\{J,T\}$ implies that $H_J \otimes H_T \cong H_{J \cup T}$ as $K$-algebras.

Proof:

(i) Trivial.

(ii) By part (i) $H_{J \cup T} = H_J H_T$. Define a map $g : H_J \times H_T \to H_{J \cup T}$ by $g : (\delta_J, \delta_T) \mapsto \delta_J \delta_T$ for all $\delta_J \in H_J$ and $\delta_T \in H_T$. $g$ is clearly a balanced map hence there exists a unique $K$-algebra homomorphism $g' : H_J \otimes_K H_T \to H_{J \cup T}$ given by $g' : \delta_J \otimes \delta_T \mapsto \delta_J \delta_T$.

$g'$ is clearly an epimorphism and since $\dim_K(H_J \otimes_K H_T) = \dim_K(H_{J \cup T}) = |W_J||W_T|$, $g'$ is an isomorphism.

**COROLLARY 3.2.3.**

Let $\{J,T\}$, $V_J \in J(N)$ and $V_T \in T(N)$. $V_J \otimes_K V_T$ can be made into an $H_{J \cup T}$-module by defining the following
action

\((\delta_j \chi_T)(v_j \cdot v_T) = \delta_j v_j \delta_T v_T\) for all \(\chi \in H_Y\) and \(v \in V_Y\)

\((Y = J, T)\).

Proof:

It is well known that \(V_J \otimes_K V_T\) is an \(H_J \otimes_K H_T\)-module with respect to the action \((\delta_j \otimes \delta_T)(v_j \otimes v_T) = \delta_j v_j \otimes \delta_T v_T\). Thus the result now follows immediately from the proof of lemma 3.2.2(ii).

\[\square\]

**Lemma 3.2.4.**

Let \(J \subseteq R\). \(H_R\) is a right \(H_J\)-module with respect to right multiplication by elements of \(H_J\).

Let \(X\) be a transversal for \(W_J\) in \(W\) then

\[H_R = \sum_{x \in X} x H_J\]

and each \(h x H_J\) is a right \(H_J\)-module.

Proof:

By equation (1.1.13) it is clear that \(H_R = \sum_{d \in D^R_J} d H_J\)

where \(D^R_J\) is the set of special coset representatives for \(W_J\) in \(W\) (see definition A3.4). Thus it is sufficient to show that if \(x W_J = d W_J\) then \(h x H_J = d H_J\).

If \(x W_J = d W_J\) then \(x = dw\) for some \(w \in W_J\) and \(h_x = h_d h_w\).

\(h_w H_J = H_J\) since \(h_1 \in h_w H_J\) by \(|\cdot|\), lemma 5.1. So \(h x H_J = h_d H_J\).

\[\square\]

Corollary 3.2.3 allows us to make the following definition

**Definition 3.2.5.**

Let \(T \subseteq R\) and \(J_i \subseteq T\) for \(i \in Z_s\) \((s > 1)\). Let \(\downarrow (J_i | i \in Z_s)\)

Given a module \(V_i \in J_i \mathcal{N}\) for each \(i\) in \(Z_s\).

\((i) \ V_1 \cdot_T V_2 \cdot_T \ldots \cdot_T V_s\) is the left \(H_R\)-module
$H_R \otimes_{H_{J_1} \cup_{J_2} \cup \ldots \cup J_s} (\ldots ((V_1 \otimes_{K_2} V_2) \otimes_{K_3} \ldots \otimes_{K_s} V_s))$.

($H_R$ is regarded as a right $H_{J_1} \cup_{J_2} \cup \ldots \cup J_s$-module. See lemma 3.2.4).

(i) If $\eta_i$ is the character of $V_i$ then

$\eta_1 \circ \eta_2 \circ \ldots \circ \eta_s$ is the character of $V_1 \circ T_2 \circ \ldots \circ T_s$.

In cases where no confusion will arise we will abbreviate $V_1 \circ \ldots \circ V_s$ to $V_1 \circ T_2 \circ \ldots \circ T_s$ and $\eta_1 \circ \ldots \circ \eta_s$ to $\eta_1 \circ \ldots \circ \eta_s$ when $T = R$.

We have immediately

**LEMMA 3.2.6.**

In the notation of definition 3.2.5

(i) $\eta_1 \circ \eta_2 \circ \ldots \circ \eta_s$ is the induced character

$(\eta_1 \cdot \eta_2 \cdot \ldots \cdot \eta_s)_{H_R}^{H_{J_1} \cup_{J_2} \cup \ldots \cup J_s}$ where $\eta_1 \cdot \eta_2 \cdot \ldots \cdot \eta_s$ is the character of $H_{J_1} \cup_{J_2} \cup \ldots \cup J_s$ given by

$$(\eta_1 \cdot \ldots \cdot \eta_s)(\delta_1 \cdot \delta_2 \cdot \ldots \cdot \delta_s) = \eta_1(\delta_1) \cdot \ldots \cdot \eta_s(\delta_s) \text{ for all } \delta_i \in H_{J_i} \text{ (i } \in \mathbb{Z};).$$

(ii) For any permutation $\sigma \in S_s$

$\eta(1) \circ \eta(2) \circ \ldots \circ \eta(s) = \eta_1 \circ \eta_2 \circ \ldots \circ \eta_s$ and hence $V(1) \circ \ldots \circ V(s) \cong V_1 \circ \ldots \circ V_s$.

**LEMMA 3.2.7.**

If

$$\begin{array}{c}
\begin{array}{c}
R \\
L \\
A \quad B \quad C
\end{array} & \begin{array}{c}
J
\end{array}
\end{array}$$

is a subset lattice diagram for $R$ with $\perp(A,B,C)$, $\perp(L,C)$, $\perp(A,J)$ and if $V_Y \in \mathbb{Y}^N$ ($Y = A,B,C$) then

(i) $(V_A \circ L_B \circ V_C) \circ R_Y \cong V_A \circ V_B \circ V_C \cong V_A \circ R(\eta_B \circ \eta_C)$ as left $H_R$-modules.

(ii) $(\eta_A \circ L_B \circ \eta_C) = \eta_A \circ \eta_B \circ \eta_C = \eta_A \circ R(\eta_B \circ \eta_C)$
where \( \eta_Y \) is the character afforded by \( V_Y \) \((Y = A, B, C)\).

Proof:

(ii) follows immediately from (i).

(i) We have \( W = D_{AUBUC}^R W_{AUBUC} = D_{LUC}^R D_{AUB}^L W_{AUBUC} \)
(see definition A3.4). Using the length function one finds that \( D_{AUBUC}^R = D_{LUC}^R D_{AUB}^L \).

By lemma 3.2.4
\[
V_A \circ V_B \circ V_C = \sum_{d \in D_{AUBUC}^R} h_d \otimes ((V_A \otimes V_B) \otimes V_C)
= \sum_{s \in D_{LUC}^R} \sum_{g \in D_{AUB}^L} h_s h_g \otimes ((V_A \otimes V_B) \otimes V_C)
\]
and
\[
(V_A \circ L_B \circ V_C) \circ V_C = \sum_{s \in D_{LUC}^R} \sum_{g \in D_{AUB}^L} h_s \otimes ((h_g \otimes (V_A \otimes V_B)) \otimes V_C)
= \sum_{s} \sum_{g} h_s \otimes ((h_g \otimes (V_A \otimes V_B)) \otimes V_C)
\]

Clearly the map \( \alpha : (V_A \circ L_B \circ V_C) \rightarrow V_A \circ V_B \circ V_C \)
given by \( \alpha : h_s \otimes ((h_g \otimes (v_A \otimes v_B)) \otimes v_C) \mapsto h_s h_g \otimes ((v_A \otimes v_B) \otimes v_C) \)
for all \( v_Y \in V_Y \) \((Y = A, B, C)\) is a \( K \)-isomorphism and also an \( H_R \)-map. Hence it is an \( H_R \)-isomorphism.

It is well known that the map \( \sigma : (V_A \otimes V_B) \otimes V_C \rightarrow V_A \otimes (V_B \otimes V_C) \) given by \( \sigma : (v_A \otimes v_B) \otimes v_C \mapsto v_A \otimes (v_B \otimes v_C) \) is an \( H_R \)-isomorphism hence the proof is complete.

\[ \Box \]

**THEOREM 3.2.8.**

If \( \lfloor J_i \rfloor \in \mathbb{Z}_t \) \((t > 1)\) where for each \( i \) in \( \mathbb{Z}_t \), \( J_i \subseteq R \) then there exists a multi-\( \mathbb{Z} \)-linear additive group monomorphism
\[
\theta^R_{J_1 J_2 \ldots J_t} : \chi(H_{J_1}) \otimes \mathbb{Z} \cdot \otimes \mathbb{Z} \chi(H_{J_t}) \rightarrow \chi(H_R)
\]
given by \( \theta^R_{J_1 \ldots J_t} : V_i (\otimes \ldots \otimes) V_t (\mapsto V_i \circ \ldots \circ V_t ( \)
for all \( V_i \in J_1 \cdot N \) \((i \in \mathbb{Z}_t)\).
Proof:

STEP 1: Let $J, T \subseteq R$ with $\mathbb{1}(J, T)$. It is sufficient to show that there exists a biadditive group monomorphism

$$\theta^R_{JT}: \kappa(H_J) \otimes_Z \kappa(H_T) \longrightarrow \kappa(H_R)$$
given by $\theta^R_{JT}(V_J \otimes V_T) \longmapsto V_J \circ V_T$, for all $V_J \in J \mathcal{N}$ and $V_T \in T \mathcal{N}$.

Proof:

Immediate from lemma 3.2.7(i) and lemma 3.1.2(iv).

STEP 2: For each $V_J$ in $J \mathcal{N}$ there exists an additive group homomorphism $b_{V_J}: \kappa(H_T) \longrightarrow \kappa(H_R)$ given by

$$b_{V_J}(V_T) \longmapsto V_J \circ V_T$$
for all $V_T \in T \mathcal{N}$.

Proof:

Let $V_J \in J \mathcal{N}$. Define a map $\alpha: T \mathcal{N} \longrightarrow \kappa(H_R)$ by $\alpha(V_T) = V_J \circ V_T$, for all $V_T \in T \mathcal{N}$. By lemma 3.1.3 it is sufficient to show that if $0 \rightarrow U \rightarrow V \rightarrow Y \rightarrow 0$ is a s.e.s. in $T \mathcal{N}$ then $\alpha(U) + \alpha(Y) = \alpha(V)$.

We show that there exists a s.e.s.

$$(3.2.9) \quad 0 \rightarrow V_J \circ U \rightarrow V_J \circ V \rightarrow V_J \circ Y \rightarrow 0$$

Using lemma 3.2.4 it can readily be shown that if $\delta: U \rightarrow V$ and $\delta: V \rightarrow Y$ are the maps in the s.e.s. $0 \rightarrow U \rightarrow V \rightarrow Y \rightarrow 0$ then the maps

$$1_{H_R} \otimes (1_{V_J} \otimes \delta) : V_J \circ U \longrightarrow V_J \circ V$$

make $(3.2.9)$ a s.e.s. in $R \mathcal{N}$. Thus $\alpha(U) + \alpha(Y) = \alpha(V)$ as required.

STEP 3: There exists a biadditive group homomorphism

$$\theta^R_{JT}: \kappa(H_J) \otimes_Z \kappa(H_T) \longrightarrow \kappa(H_R)$$
given by $\theta^R_{JT}(V_J \otimes V_T) \longmapsto V_J \circ V_T$, for all $V_J \in J \mathcal{N}$ and $V_T \in T \mathcal{N}$.

Proof:

By analogy with step 2, for each $V_T \in T \mathcal{N}$.
there exists an additive homomorphism $g_{V_T^j}: \chi(H_j) \rightarrow \chi(H_R)$ given by $g_{V_T^j}(V_J) \rightarrow V_J \cdot V_T^j$ for all $V_J \in jN$. Thus there exists a balanced map $b.g: \chi(H_J) \otimes \chi(H_T) \rightarrow \chi(H_R)$ given by $b.g: (V_J \otimes V_T^j) \rightarrow V_J \cdot V_T^j$ for all $V_J \in jN$ and $V_T \in T^jN$.

STEP 4: $\Theta_{HT}^R$ is a monomorphism.

Proof:

Let $\{U_s | s \in S\}$ be a full set of irreducible $H_J$-modules.

Let $\{Y_t | t \in I\}$ be a full set of irreducible $H_T$-modules.

By corollary 3.2.3 it is clear that $\{U_s \otimes Y_t | (s,t) \in S \times I\}$ is a full set of irreducible $H_{JT}$-modules.

Lemma 3.1.4 shows that $\chi(H_J) \otimes \chi(H_T)$ has $Z$-basis $\{U_s \otimes Y_t | (s,t) \in S \times I\}$. Suppose that

$$\Theta_{HT}^R((\sum_{(s,t) \in S \times I} z_{st} U_s \otimes Y_t) (s,t)) = 0$$

for some $z_{st} \in Z$ then

$$\sum_{(s,t) \in S \times I} z_{st} U_s \cdot V_t^j = 0$$

Let $U_s$ afford the character $\beta_s$, and let $Y_t$ afford the character $\lambda_t$. Then by lemma 3.2.6(i)

$$\sum_{(s,t) \in S \times I} z_{st} \beta_s \lambda_t^R = 0$$

and so

$$\sum_{(s,t) \in S \times I} z_{st} \beta_s \lambda_t = 0.$$ 

Since $\{U_s \otimes Y_t | (s,t) \in S \times I\}$ is a full set of irreducible $H_{JT}$-modules the characters $\beta_s \lambda_t$ ($s \in S, t \in I$) are linearly independent over $K$. Thus $z_{st} = 0$ for all $(s,t) \in S \times I$, and $\Theta_{HT}^R$ is a monomorphism.

COROLLARY 3.2.10.

If $\frac{1}{(J_i | i \in Z_t)} (t > 1)$ where for each $i$ in $Z_t$ $J_i \subset R$ then there exists a multi-$Z$-linear additive group
monomorphism

\[ \delta^R_{J_1, J_2, \ldots, J_t} : x(H_{J_1}) \otimes x(H_{J_2}) \otimes \cdots \otimes x(H_{J_t}) \rightarrow x(H_R) \]
given by

\[ \delta^R_{J_1, J_2, \ldots, J_t} : \eta_1 \otimes \cdots \otimes \eta_t \rightarrow \eta_1 \cdot \cdots \cdot \eta_t \quad \text{for} \quad \eta_i \in x(H_{J_i}) \quad (i \in Z_t). \]

Proof:

Let \( \rho_i : x(H_{J_i}) \rightarrow x(H_{J_i}) \quad (i \in Z_t) \)
and \( \rho : x(H_R) \rightarrow x(H_R) \)
be the isomorphisms described by corollary 3.1.5.

\[ \delta^R_{J_1, J_2, \ldots, J_t} = \rho \cdot \delta^R_{J_1, J_2, \ldots, J_t} \cdot (\rho_1 \otimes \cdots \cdot \rho_t)^{-1} \]

§3.3 CIRCLE PRODUCT AND THE COXETER GROUP.

NOTATION.

Let \( J \subset R \). Abbreviate \( \chi(CW_J N) \) to \( \chi(W_J) \) and \( x(CW_J) \) to \( x(W_J) \).

Let \( \rho_J : x(H_J) \rightarrow x(H_J) \)
and \( \lambda_J : x(W_J) \rightarrow x(W_J) \)
be the isomorphisms described in corollary 3.1.5.

Lemma 3.3.1.

(i) The map \( \delta_J : x(H_J) \rightarrow x(W_J) \) defined by

\[ \delta_J(\eta)(f_J(\delta)) = f_J(\eta(\delta)) \quad \text{for all} \quad \eta \in x(H_J) \quad \text{and} \quad \delta \in H(K,u) \]
is an additive group isomorphism.

(ii) The map \( \sigma_J : \chi(H_J) \rightarrow \chi(W_J) \) where

\[ \sigma_J = \lambda_J^{-1} \cdot \delta_J \cdot \rho_J \]
is an additive group isomorphism.

Proof:

(i) Immediate from theorem 1.4.5(ii)

(ii) Immediate from (i).
DEFINITION 3.3.2.
Let \( J_{i}^{(J_i | i \in Z_{t})} \ (t > 1) \) where for each \( i \) in \( Z_{t} \), \( J_{i} \subset R \).

(i) \( \psi_{J_{1}J_{2} \ldots J_{t}}^{R} : \bigotimes_{Z} X(W_{J_{1}}) \otimes \cdots \otimes \bigotimes_{Z} X(W_{J_{t}}) \rightarrow X(W_{R}) \)
is defined by
\[
\psi_{J_{1} \ldots J_{t}}^{R} = \sigma_{R} \cdot \psi_{J_{1} \ldots J_{t}}^{R} \cdot (\sigma_{J_{1}} \cdots \sigma_{J_{t}})^{-1}
\]
Let \( \psi_{J_{1} \ldots J_{t}}^{R} (U_{1} \otimes \cdots \otimes U_{t}(i) = U_{1} \circ \cdots \circ U_{t}(i) \text{ for all } U_{i} \in CX_{W_{J_{i}}}(i \in Z_{t}) \).

(ii) The map
\( \psi_{J_{1}J_{2} \ldots J_{t}}^{R} : X(W_{J_{1}}) \otimes \cdots \otimes \bigotimes_{Z} X(W_{J_{t}}) \rightarrow X(W_{R}) \)
is defined by
\[
\psi_{J_{1} \ldots J_{t}}^{R} = \delta_{R} \cdot \psi_{J_{1} \ldots J_{t}}^{R} \cdot (\delta_{J_{1}} \cdots \delta_{J_{t}})^{-1}
\]
Let \( \psi_{J_{1} \ldots J_{t}}^{R} (X_{1} \otimes \cdots \otimes X_{t}) = X_{1} \circ \cdots \circ X_{t} \text{ for all } X_{i} \in X(W_{J_{i}}) (i \in Z_{t}) \).

From theorem 3.2.8 we have immediately

LEMMA 3.3.3.

(i) \( \psi_{J_{1} \ldots J_{t}}^{R} \) and \( \psi_{J_{1} \ldots J_{t}}^{R} \) are multi-Z-linear additive group monomorphisms.

(ii) Let \( \{ \eta_{i}^{j} | i \in Z_{m_{W}} \} \) be the full set of irreducible characters of \( H_{R} \), so that \( \{ X^{i} | i \in Z_{m_{W}} \} \) is the set of all irreducible characters of \( CW \), where \( X^{i} = \delta_{R}(\eta_{i}^{j}) \) for all \( i \) in \( Z_{m_{W}} \). \( (m_{W} \) is the number of conjugacy classes in \( W) \).

If \( \eta_{j} \in X(H_{j}) \), \( X_{j} = \delta_{J_{j}}(\eta_{j}) \) for all \( i \in Z_{t} \)
and \( \eta_{1} \circ \cdots \circ \eta_{t} = \sum_{z_{i} \in Z_{m_{W}}} z_{i}\eta_{i}^{n} \) (\( z_{i} \in Z \) then
THEOREM 3.3.4.

Let \( \downarrow (J_i \mid i \in Z_t) \) \( (t > 1) \) where for each \( i \) in \( Z_t \), \( J_i \subseteq R \).

For each \( i \in Z_t \) let \( U_i \in CW_{J_i} \) (and \( x_i \in X(W_{J_i}) \)). Then

(i) \( U_1 \cdots U_t = (\cdots (U_1 \otimes U_2) \otimes U_t) \)

where \( CW \) is regarded as a right \( CW_{J_1} \cdots CW_{J_t} \)-module, the action being given by right multiplication.

(ii) \( x_1 \cdots x_t = (x_1 \cdot x_2 \cdots x_t)^W \) where \( x_1 \cdot x_2 \cdots x_t \) is the character of \( W_{J_1} \cdots W_{J_t} \) given by

\[
(x_1 \cdot x_2 \cdots x_t)(w_1 w_2 \cdots w_t) = x_1(w_1) \cdot x_2(w_2) \cdots x_t(w_t)
\]

for all \( w_i \in W_{J_i} \) \( (i \in Z_t) \)

(see definition 3.3.2 and lemma 3.2.2).

Proof:

(i) By lemma 3.1.2(iv) and the proof of corollary 3.2.10 this follows from (ii).

(ii) Lemma 3.2.7 shows that it is sufficient to prove the following two 'statements':

STATEMENT 1. Let \( J, T \subseteq R \) with \( \downarrow (J, T) \) and let \( x_J \in X(W_J) \), \( x_T \in X(W_T) \) then

\[
(x_J \cdot x_T)^W = (x_J \cdot x_T)^W
\]

STATEMENT 2. Let \( J_i \subseteq R \) and \( x_{J_i} \in X(W_{J_i}) \) for \( i = 1, 2 \) and 3. Let \( \downarrow (J_1, J_2, J_3) \) then

\[
(x_{J_1} \cdot x_{J_2} \cdot x_{J_3})^W = (x_{J_1} \cdot x_{J_2} \cdot x_{J_3})^W
\]

This latter statement can readily be proved using the well known formula for induced group characters.
Proof of statement 1:

Let \( \Delta^{-1}_J(X_J) = \eta_J \) and \( \Delta^{-1}_T(X_T) = \eta_T \), so

\[
X_J \cdot X_T = \Delta^R_{JT}(X_J \otimes X_T)
\]

\[
= (\Delta^R_J \cdot \Delta^R_T (\Delta^R_J \otimes \Delta^R_T)^{-1})(X_J \otimes X_T)
\]

\[
= \Delta^R_T (\eta_J \otimes \eta_T)
\]

\[
= \Delta^R_T (\eta_J \cdot \eta_T)
\]

\[
= \Delta^R_T ((\eta_J \cdot \eta_T)^H_R)
\]

Clearly \( \Delta^{-1}_J(\eta_J \cdot \eta_T) = X_J \cdot X_T \), thus it is sufficient to prove that if \( S \subset R \) and \( \beta \in X(H_S) \) then

\[
(3.3.5) \quad \Delta^H_R(\beta_R) = (\Delta_S(\beta))^W_R
\]

Let \( h_w h_d = h_{d'(w)} \gamma_w, d \) where \( w \in W, d,d'(w) \in D^R_S \) (see definition A3.4) and \( \gamma_w, d \in H_S \).

Clearly \( f_1(\gamma_w, d) \) is defined and belongs to \( CW_S \).

So

\[
f_1(h_w) f_1(h_d) = f_1(h_{d'(w)}) f_1(\gamma_w, d)
\]

which can be rewritten as

\[
w d = d'(w) f_1(\gamma_w, d).
\]

Thus by lemmas 3.2.4 and 3.3.1

\[
(\Delta^H_R(\beta_R))(w) = f_1(\beta_R(h_w))
\]

\[
= f_1\left( \sum_{d \in D^R_S} \psi_{d,d'(w)} \beta(\gamma_w, d) \right)
\]

where \( \psi_{d,d'(w)} = \begin{cases} 1 & \text{if } d=d'(w) \\ 0 & \text{if } d \neq d'(w) \end{cases} \)

\[
= \sum_d \psi_{d,d'(w)} f_1(\beta(\gamma_w, d))
\]

\[
= \sum_d \psi_{d,d'(w)} f_1(\Delta_S(\beta)) (f_1(\gamma_w, d))
\]

\[
= (\Delta_S(\beta))^W_R(w)
\]

which proves equation (3.3.5).

\( \checkmark \)

Clearly analogues of the results (3.3.1), (3.3.3) and (3.3.4) can be found with \( CW \) replaced by \( E_G(q) \) for each \( q \in P \) (see definition 1.2.1), but as these would be incidental to our study of the characters of \( H_R \) we omit them.
PART II

INTRODUCTION.

For the remaining chapters of this thesis we restrict our attention to a special case, namely that where $W$ is a Weyl group of type $A_{n-1}$ ($n > 1$) and $c : W \to \mathbb{Z}^+$ (see corollary 1.1.7) coincides with the length function i.e. $c_r = 1$ for all $r \in R$. Denote the generic Hecke algebra by $G_n$ in this case.

By part (i) of the discussion above theorem 2.2.8, we can identify $W$ with the symmetric group $S_n$ on $n$ symbols and $R$ with $\{\mu = (i, i+1) \mid i \in \mathbb{Z}_{n-1}\}$. (A2.2), (A2.3) and (A2.4) are a set of defining relations for this set $R$ of generators.

We take $G_n$ to be the $K$-algebra with $K$-basis $\{g_w \mid w \in S_n\}$. Thus denoting $g_{i_{S_n}}$ by $g_0$ and $g_{u_{i_{S_n}}}$ by $g_i$ we have that $g_w = g_{i_1}g_{i_2}\cdots g_{i_t}$ where $\mu_{i_1}\mu_{i_2}\cdots \mu_{i_t}$ is any reduced expression for $w$ ($w \in S_n$) and that $G_n$ is generated as $K$-algebra with identity $g_0$ by $\{g_i \mid i \in \mathbb{Z}_{n-1}\}$ with defining relations

$$g_i^2 = u g_0 + (u - 1) g_i$$

for all $i \in \mathbb{Z}_{n-1}$

$$g_i g_j = g_j g_i$$

for all $i, j \in \mathbb{Z}_{n-1}$ with $i+1 < j$

$$g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1}$$

for all $i \in \mathbb{Z}_{n-2}$

Since the conjugacy classes of $S_n$ can be indexed by the set of all partitions of $n$ (see appendix 2 and above lemma 4.1.1) we can use the notation $\{X^\alpha \mid \alpha \rightarrow n\}$ for the set of all irreducible characters of $S_n$ over $C$.

Further by theorem 1.4.5(ii) we can denote the set of all irreducible characters of $G_n$ over $K$ by $\{\eta^\alpha \mid \alpha \rightarrow n\}$ and stipulate that for any $w \in S_n$, that $X^\alpha(w) = f_1(\eta^\alpha(g_w))$.  

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CHAPTER 4

INDUCTION FORMULAE

In this chapter 'part 2' (see below corollary 2.1.5) of the problem of evaluating the characters of the generic Hecke algebra $G_n$ is solved (see theorems 4.2.8 and 4.3.8 and the discussion above definition 4.2.1) at least in a theoretical sense. (Chapter 5 gives a more practical method for calculating character values.)

§4.1 IRREDUCIBLE CHARACTERS.

NOTATION.

Let $n \in \mathbb{Z}_+$. A partition $\delta$ of $n$ (denoted $\delta \vdash n$) is a finite sequence $< \delta_1, \delta_2, \ldots, \delta_t >$ of non-negative integers such that

$$\sum_{i=0}^{t} \delta_i = n \text{ and } 0 \leq \delta_t < \delta_{t-1} < \ldots < \delta_1$$

We also use the alternative notation $\delta = < a_1 a_2 \ldots a_n >$ where $a_j = |\{ i | i \in \mathbb{Z}_t, \delta_i = j \}|$. $\delta_1, \ldots, \delta_t$ are the parts of $\delta$.

We can assume $t = n$ if we wish since some of the $\delta_i$ can be equal to zero.

With each partition $\lambda$ of $n$ we associate a parabolic subgroup (see definition A3.1) $S_{n}^{\lambda}$ of $S_n$ as follows:

Let $\lambda = < \lambda_1, \lambda_2, \ldots, \lambda_r >$ with $0 < \lambda_r < \ldots < \lambda_1$.

Put $\lambda^t = \sum_{i=1}^{t} \lambda_i$ for $t \in \mathbb{Z}_r$ and $J_{\lambda^t} = \{ \mu_j = (j, j+1) | j \in \mathbb{Z}_n-1, j \neq \lambda^1, \lambda^2, \ldots, \lambda^{r-1} \} \subset R$

Then $S_{n}^{\lambda} = (S_n)_{J_{\lambda_{\lambda^t}}}$ (see definition A3.1).

Also we denote the subalgebra $(G_n)_{J_{\lambda_{\lambda^t}}}$ of $G_n$ by $G_{n}^{\lambda}$.
LEMMA 4.1.1.

With \( \lambda \) and notation as above; Let \( J_t = \{ \mu_j | \gamma_t^{-1} < j < \lambda_t \} \) (\( t \in \mathbb{Z}_r \)) where conventionally \( \gamma_0 = 0 \). Let \( T_t \) be the parabolic subgroup \((S_n)_{J_t}\) and \( L_t \) be the subalgebra \((G_n)_{J_t}\) of \( G_n \). Then for \( t \) in \( \mathbb{Z}_r \)

(i) \( T_t \) is the subgroup of \( S_n \) consisting of all those elements fixing all the symbols other than \{ \( \lambda_t^{-1} + 1, \lambda_t^{-1} + 2, \ldots, \lambda_t \) \}.

(ii) \( T_t \cong S_t \)

(iii) \( L_t \cong G_t \) and the Coxeter group of \( L_t \) is \( T_t \)

(iv) \( J = J_1 \cup \ldots \cup J_r \) and \( \underline{\cup}(J_1, \ldots, J_r) \)

(v) \( S_n = T_1 T_2 \cdots T_r \)

(vi) \( G_n = L_1 L_2 \cdots L_r \)

Proof:

Clearly the image of the monomorphism \( \pi : T_t \rightarrow S_n \)
defined by \( \bar{\pi} : \mu_j \mapsto \mu_j \gamma_t^{-1} \) for all \( j \in J_t \) is isomorphic to \( S_{\lambda_t} \). Thus \( \bar{\pi} \) induces an isomorphism \( \bar{\pi} : T_t \rightarrow S_{\lambda_t} \).

The K-map \( \pi'' : L_t \rightarrow G_t \) given by \( \pi''(g_w) = g_{\pi''(w)} \) for all \( w \in T_t \) is clearly a K-isomorphism. This proves (ii) and (iii). Parts (i),(iv),(v) and (vi) are readily seen to be true.

\( \Box \)

LEMMA 4.1.2.

(i) The map K-linear map \( j_n : G_n \rightarrow K \) defined by \( j_n(g_w) = ul(w) \) for all \( w \) in \( S_n \) is a character of \( G_n \).

(ii) Under the bijection from \( X(G_n) \) to \( X(G_n) \) described in theorem 1.4.5, \( j_n \) maps to \( 1_n \) the unit character of \( S_n \).
(iii) Let $\lambda \vdash n$ be the partition in lemma 4.1.1. Denote the restriction of $\mathbb{J}_n$ to $\mathcal{S}_n^{\lambda}$ by $\mathbb{J}_n^{\lambda}$ and the restriction of $1_n$ to $\mathcal{S}_n^{\lambda}$ by $1_n^{\lambda}$. Identify the character group $\chi(L_t)$ with $\chi(G_t^{\lambda})$ and the character group $\chi(T_t)$ with $\chi(S_t^{\lambda})$ for all $t$ in $\mathbb{Z}_n$ (see lemma 4.1.1). Then

$$
(\mathbb{J}_n^{\lambda})^G = \mathbb{J}_1^{\lambda_1} \mathbb{J}_2^{\lambda_2} \cdots \mathbb{J}_r^{\lambda_r}
$$

and

$$
(1_n^{\lambda})^S = 1_1^{\lambda_1} 1_2^{\lambda_2} \cdots 1_r^{\lambda_r}.
$$

Proof:

(i) That $\mathbb{J}_n$ is a representation of $G_n$ is clear from the defining relations given for $G_n$ in the introduction to part II. Since $\mathbb{J}_n(g_0) = 1$, $\mathbb{J}_n$ is also a character.

(ii) Immediate.

(iii) Immediate from lemma 3.2.6(ii) and theorem 3.3.4(ii).

DEFINITION 4.1.3.

$\mathbb{J}_n$ defined in the above lemma is the **unit character** of $G_n$.

NOTATION.

Let $\delta = <\delta_1, \delta_2, \ldots, \delta_n> \vdash n$ and $w \in S_n$. Denote by $\delta_w$ the sequence whose terms are the elements of the following set arranged in decreasing order:

$$
\{ \delta_1 + w(1) - 1, \delta_2 + w(2) - 2, \ldots, \delta_n + w(n) - n \}.
$$

Note; $\delta_w$ may or may not be a partition.

From [7, chapter 5] one can readily derive

THEOREM 4.1.4. (Frobenius).

$$
\{ X^\delta = \sum_{w \in S_n} \text{sign}(w) (1_n^w)^S \mid \delta \vdash n \} \text{ where } 1_n^w = 0 \text{ if } n
$$
\( \gamma_w \) is not a partition, is the set of all irreducible characters of \( S_n \) over \( C \).

( \text{sign}(w) \) is +1 if \( w \) is an even permutation and is -1 if \( w \) is an odd permutation.

**Corollary 4.1.5.**

\[
\{ \eta \delta = \sum \text{sign}(w) \left( \bigwedge_{w \in S_n}^w G_n \right) | \delta \vdash n \} \quad \text{where } \bigwedge_{n}^{w} = 0
\]

if \( \delta_w \) is not a partition, is the set of all irreducible characters of \( G_n \) over \( K \).

**Proof:**

Immediate from theorem 4.1.4, lemma 4.1.2(iii) and lemma 3.3.3(ii).

**§ 4.2 \( \alpha \cdot \eta \)**

Fix \( n \) in \( Z_+ \) and \( t \) in \( Z_{n-1} \). Assume that \( t \geq n - t \)
and let \( \lambda = < t, n-t > \).

Put \( J_1 = \{ \mu_1, \mu_2, \ldots, \mu_{n+1} \} \) and \( J_2 = \{ \mu_{n+1}, \mu_{n+2}, \ldots, \mu_{n-1} \} \).

Set \( T_1 = (S_n)J_1 \) and \( L_1 = (G_n)J_1 \) for \( i = 1, 2 \).

Lemma 4.1.1 shows that \( S_n = T_1 T_2, G_n = L_1 L_2, T_1 \cong S_t, T_2 \cong S_{n-t}, L_1 \cong G_t \) and \( L_2 \cong G_{n-t} \).

We identify \( X(L_1) \) with \( X(G_t) \) and \( X(L_2) \) with \( X(G_{n-t}) \),
thus if \( \alpha \in X(G_t) \) and \( \eta \in X(G_{n-t}) \) we can form the character \( \alpha \cdot \eta \) of \( G_n \). (See definition 3.2.5(ii)).

By lemmas 4.1.2(iii), 3.2.7(ii) and corollary 4.1.5 it is clear that in theory at least, we can evaluate all of the irreducible characters of \( G_n \) once we can evaluate all products of the form \( \alpha \cdot \eta \). Accordingly the evaluation of \( (\alpha \cdot \eta)(g_w) \) for certain \( w \) in \( S_n \) is the subject of the rest of this chapter (see theorems 4.2.8, 4.3.8 and corollary...
4.2.10. The results obtained together with the remark below theorem 2.2.8 enable \((\alpha \cdot \eta)(g_w)\) to be evaluated for all \(w\) in \(S_n\), \(\alpha\) in \(X(G_t)\) and \(\eta\) in \(X(G_{n-t})\).

**DEFINITION 4.2.1.**

Let \(m \in \mathbb{Z}_+\).

(i) Given \(\delta \leftarrow m\), the **subclass** \((\delta)\) of \(S_m\) is the subset of elements \(w\) of \((\delta)\) (the conjugacy class of \(S_m\) associated with \(\delta\)) such that if \(w = c_1c_2 \ldots c_r\) is the decomposition of \(w\) into disjoint cycles then each \(c_i\) \((i \in \mathbb{Z}_r)\) is of the form \((j j-1 j-2 \ldots)\) for some \(j \in \mathbb{Z}_m\).

(ii) A function \(f: G_m \rightarrow K\) satisfying \(f(g_w) = f(g_v)\) whenever \(w\) and \(v\) are in the same subclass of \(S_m\) is called a subclass function on \(G_m\).

**NOTATION.**

Let \(m \in \mathbb{Z}_+\). Denote the subclass \((\frac{1^m}{1^m})\) of \(S_m\) by \(\text{I}^0\). Conventionally let \(\emptyset = (\frac{1^m}{1^m})\).

Given a subclass function \(f: G_m \rightarrow K\) and \(\delta \leftarrow m\), for any \(w \in (\delta)\) denote \(f(g_w)\) by \(f_\delta\).

Denote \(f(\frac{1^m}{1^m})\) by \(f_0\). \((f_0 = f(g_0))\).

**LEMMA 4.2.2.**

Let \(m \in \mathbb{Z}_+\), \(\delta \leftarrow m\) and \(w \in (\delta) \subseteq S_m\). Let \(w\) be of minimal length in its conjugacy class and \(f: G_m \rightarrow K\) be a subclass function for which

\[ f(g_xg_y) = f(g_yg_x) \quad \text{for all } x, y \in S_m \]

then \(f(g_w) = f_\delta\). \((\text{Note that } \{w \in (\delta)\} \text{ w is of minimal length} \} \text{ contains (\delta) but is not in general equal to it})\)

**Proof:** Let \(z \in S_m\). If \(z = xy\) \((x, y \in S_m)\) with \(1(x) + 1(y) = \)
l(yx) = l(y) + l(x) then we call yx a rotamer of z. Since yx = x^{-1}yx, all rotamers of z are conjugates of z. g_z = g_x g_y and g_yx = g_y g_x thus if z' is any rotamer of z then f(g_z) = f(g_z').

Let w = \mu_{i_1} \mu_{i_2} \cdots \mu_{i_r} (r = l(w)) be a reduced expression for w. Theorem A2.7(i) shows that all the elements of \{i_1, i_2, \ldots, i_r\} are distinct. Let \{j_1, \ldots, j_r\} = \{i_1, \ldots, i_r\} with j_1 > j_2 > \ldots > j_r and let

w' = \mu_{j_1} \mu_{j_2} \cdots \mu_{j_r}. Since w' \in \langle \beta \rangle it is sufficient to prove that w' is a rotamer of w.

We use induction on l(w). By relation (A2.3) it is clear that w has a reduced expression in terms of

\{\mu_j \mid j \in \{i_1, i_2, \ldots, i_r\}\} either beginning or ending with \mu_{j_1}. Thus there exists \sigma \in S_r with \sigma(1) = 1 such that

\mu_{j_1} \mu_j \sigma(2) \cdots \mu_j \sigma(r) = \mu_{j_1} \nu \text{ say}

is a rotamer of w. By the inductive hypothesis \mu_{j_2} \mu_{j_3} \cdots \mu_{j_r} is a rotamer of \nu and as \mu_j commutes with every element of the set \{\mu_{j_3}, \mu_{j_4}, \ldots, \mu_{j_r}\} it is clear that \mu_{j_1} \cdots \mu_{j_r}

is a rotamer of w.

We see from theorem A3.8 that the elements of the special transversal \mathcal{D}_{J_1 \cup J_2}^R of S_n^\Lambda in S_n can be indexed by the set of subsets of Z_n of order t:

\mathcal{D}_{J_1 \cup J_2}^R = \{ w_A \mid A \subset Z_n, |A| = t \} where if

A = \{a_1, a_2, \ldots, a_t\} with a_1 \leq a_2 \leq \ldots \leq a_t and

Z_n \setminus A = \{b_1, \ldots, b_{n-t}\} with b_1 \leq \ldots \leq b_{n-t} then

w_A = (1 \ 2 \ \ldots \ t \ t+1 \ \ldots \ n-t)

Abbreviate \mathcal{E}_{w_A} as \mathcal{E}_A.
Using lemmas A2.2, A3.5 and corollary A3.6 we can readily prove

**LEMMA 4.2.3.**

Given \( j \in \mathbb{Z}_{n-1} \) and \( A \subset \mathbb{Z}_n \) with \( |A| = t \)

(i) if \( j, j+1 \in A \) then

\[
\varepsilon_j \varepsilon_A = \varepsilon_A \varepsilon^{-1}_{w_A}(j) \quad \text{and} \quad w_A^{-1}(j) \in T_1
\]

(see above definition 4.2.1)

(ii) if \( j, j+1 \in \mathbb{Z}_n \setminus A \) then

\[
\varepsilon_j \varepsilon_A = \varepsilon_A \varepsilon^{-1}_{w_A}(j) \quad \text{and} \quad w_A^{-1}(j) \in T_2
\]

(iii) if \( j \in A \) and \( j+1 \in \mathbb{Z}_n \setminus A \) then

\[
\varepsilon_j \varepsilon_A = \varepsilon_{\mu_j w_A} \quad \text{and} \quad \mu_j w_A \in D_{j+1}^R \cup \mu_2
\]

(iv) if \( j \in \mathbb{Z}_n \setminus A \) and \( j+1 \in A \) then

\[
\varepsilon_j \varepsilon_A = u \varepsilon_{\mu_j w_A} + (u-1) \varepsilon_A \quad \text{and} \quad \mu_j w_A \in D_{j+1}^R \cup \mu_2
\]

**COROLLARY 4.2.4.**

Let \( i \in \mathbb{Z}_n \setminus \{1\} \) and \( s \in \mathbb{Z}_{n-1} \cup \{0\} \). Put \( x = (i+s, i-1+s, \ldots, 1+s) = \mu_{i-1+s} \mu_{i-2+s} \cdots \mu_{1+s} \in i \subset S_n \) (see below definition 4.2.1). Let \( A \subset \mathbb{Z}_n \) with \( |A| = t \), then

\[
\varepsilon_x \varepsilon_A \in G_n \setminus \varepsilon_A G_n \quad \text{unless}
\]

either

(i) \( 1+s, 2+s, \ldots, i+s \in A \) when

\[
\varepsilon_x \varepsilon_A = \varepsilon_A \varepsilon_v \quad \text{where} \quad v \in i \subset T_1
\]

or

(ii) \( 1+s, 2+s, \ldots, i+s \in \mathbb{Z}_n \setminus A \) when

\[
\varepsilon_x \varepsilon_A = \varepsilon_A \varepsilon_w \quad \text{where} \quad w \in i \subset T_2
\]

or

(iii) There exists \( r \in \mathbb{Z}_{i-1} \) with

\( 1+s, 2+s, \ldots, r+s \in \mathbb{Z}_n \setminus A \) and \( r+1+s, r+2+s, \ldots, i+s \in A \) when

\[
\varepsilon_x \varepsilon_A = (u-1) \varepsilon_A \varepsilon_v \varepsilon_w + q \quad \text{where} \quad v \in i-r \subset T_1, \quad w \in \mathbb{Z}_r \subset T_2 \quad \text{and} \quad q \in G_n \varepsilon_A G_n
\]
Proof:

Put \( x = x_i \). We use induction on \( i \). The case \( i = 2 \) follows immediately from lemma 4.2.2. Note that if \( p, p+1 \) \( \in A \) then \( w^{-1}_A(p+1) = w^{-1}_A(p) + 1 \) and \( w^{-1}_A(p+1) \in Z_t \). Similarly if \( z, z+1 \in Z_n \setminus A \) then \( w^{-1}_A(z+1) = w^{-1}_A(z) + 1 \) and \( w^{-1}_A(z), w^{-1}_A(z+1) \in Z_n \setminus Z_t \). These facts together with the equation \( e_{x_i} = e_{u_i-1+s} e_{x_{i-1}} \) (\( i > 2 \)) and lemma 4.2.3 are clearly sufficient to complete the proof.

\[ \times \]

**THEOREM 4.2.5.**

Let \( i \in Z_n \setminus \{1\} \). If \( \alpha \in X(G_t) \) and \( \eta \in X(G_{n-t}) \) are subclass functions (see definition 4.2.1) and if \( x, y \in i \subset S_n \) then

(i) \( (\alpha \circ \eta)(g_x) = (\alpha \circ \eta)(g_y) \)

(ii) \( (\alpha \circ \eta)(g_x) = (n-i) \alpha_{i-1} \eta_i + (u-1) \sum_{j=1}^{i-1} (n-i) \alpha_{j-1} \eta_{i-j} \)

\( + (n-i) \alpha_1 \eta_i \) (see definition 2.2.11 and the notation below definition 4.2.1)

Note that \( \alpha_1 = \alpha(g_o) \) and \( \eta_1 = \eta(g_o) \).

Proof:

Let \( A, B \subset Z_n \) with \( |A| = |B| = t \). By lemma 3.2.4 there exist unique elements \( \sigma_{x,B,A} \) in \( G_n \) such that

\[ e_x e_A = \sum_{B \subset Z_n, |B|=t} e_{B} \sigma_{x,A,B} \]

(4.2.7)

By lemmas 3.2.4 and 3.2.6(i)

\( (\alpha \circ \eta)(g_x) = \sum_{A \subset Z_n, |A|=t} (\alpha \circ \eta)(\sigma_{x,A,A}) \)

(4.2.6)

Since \( x \in i \subset S_n \), there exists \( s \in Z_{n-1} \cup \{0\} \) with \( x = (i+s \ i-1+s \ldots \ 1+s) \). Using corollary 4.2.4 to evaluate...
the algebra elements $\sigma_{x,A}^\perp$ and counting the number of sets $A$ satisfying the various conditions it is clear that formula (4.2.6) is correct. Thus (ii) is proved.

(i) follows from (ii) since the right hand side of (4.2.6) is independent of $x$.

There is a sense in which we can 'multiply' together expressions of the type on the right hand side of equation (4.2.6) (see theorem 4.2.8(ii) and the example below theorem 4.2.12) to obtain a formula for $(\alpha \cdot \eta)_\delta$ where $\delta \vdash n$. (see below definition 4.2.1). In order to be able to describe this 'multiplication' we introduce the following algebras and maps:

$\mathcal{B}_{n,t}$ is the associative commutative free $\mathbb{Q}(u)$-algebra with $\mathbb{Q}(u)$-basis $\{x(e,f) \mid e,f \in \mathbb{Z}\}$ multiplication given by

$x(e,f), x(s,r) = x(e+s-n, f+r-t)$ for all $e, f, s, r \in \mathbb{Z}$.

Let $T = \{ \delta \mid \delta \vdash m, m \in \mathbb{Z}_+ \} \cup \{ <0 > \}$. Define a map from $T$ to $\mathbb{Z}_+$ by $\delta \mapsto n_\delta$ where for $\delta = < 1^{a_1} 2^{a_2} \ldots m^{a_m}>$

$n_\delta = \sum_{i=2}^{m} ia_i \cdot (n < 1 > = n < 0 > = 0 ).$

Define a map from $\{f \mid f : \Gamma \rightarrow K$ is a subclass function $\}$ to $\{r \mid r : T \rightarrow K\}$ by $f \mapsto f^T$ where for $\delta = < 1^{a_1} \ldots m^{a_m}>$

$f^T(\delta) = \begin{cases} 0 & \text{if } n_\delta > n \\ f_\delta, & \text{if } n_\delta < n, \text{ where } \delta^\prime = < n-n_\delta a_2 a_3 \ldots m^{a_m}> \\ \end{cases}$

and $f^T(< 0 >) = f_0 = f(g_0)$.

$\mathcal{M}_{n,t}$ is the associative commutative free $\mathcal{B}_{n,t}$-algebra with $\mathcal{B}_{n,t}$-basis $T \times T$ and multiplication given by

$(\delta_1, \delta_2)(\delta_3, \delta_4) = (\delta_1 \delta_3, \delta_2 \delta_4)$ for all $\delta_1, \delta_2, \delta_3, \delta_4 \in T$

where for $\delta = < 1^{a_1} 2^{a_2} >$ and $\zeta = < 1^{b_1} 2^{b_2} >$ in $T$

$\delta \zeta = < 1^{a_1+b_1} 2^{a_2+b_2} >$ and $\delta < 0 > = < 0 > \delta = \delta$. 

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\( \mathcal{J} : B_{n,t} \rightarrow Q(u) \) is the \( Q(u) \)-linear map defined by
\[ \mathcal{J} : x(e,f) \mapsto (e \uparrow f) \] for all \( e,f \in Z \) (see definition 2.2.11). Given subclass functions \( \alpha \in X(G_t) \) and \( \eta \in X(G_{n-t}) \) we define the map \( \mathcal{J}_{\alpha,\eta} : M_{n,t} \rightarrow K \) by

\[ \mathcal{J}_{\alpha,\eta} : \sum_{(\delta,\tau)} y_{\delta,\tau} (\delta,\tau) \mapsto \sum_{(\delta,\tau)} J(y_{\delta,\tau}) \alpha^T(\delta) \eta^T(\tau) \]

where \( (\delta,\tau) \) runs over any finite subset of \( T \times T \) and each \( y_{\delta,\tau} \in B_{n,t} \).

Using the above definitions and notation we are able to state and prove

**THEOREM 4.2.8.**

For each \( i \in \mathbb{Z}_n \setminus \{1\} \) define the element \( m(n,t,i) \) of \( M_{n,t} \) by

\[ m(n,t,i) = x(n-i,t-i)(< i >, < 0 >) \]

\[ + (u-1) \sum_{j=1}^{i-1} x(n-i,t-j)(< j >, < i-j >) \]

\[ + x(n-i,t)(< 0 >, < i >) \]

Let \( \alpha \in X(G_t) \) and \( \eta \in X(G_{n-t}) \) be subclass functions.

(i) If \( y \in B_n \) then \( (\alpha \circ \eta)(g_y) = \mathcal{J}_{\alpha,\eta}(m(n,t,i)) \)

(ii) Let \( \delta = < \delta_1, \delta_2, \ldots, \delta_s > \vdash n \) with \( \delta \neq < 1^n > \).

Let \( r \) be the greatest integer such that \( \delta_r > 1 \). Then if \( w \in (\delta) \)

\[ (\alpha \circ \eta)(g_w) = \mathcal{J}_{\alpha,\eta}(m(n,t,\delta_1) \cdot m(n,t,\delta_2) \cdots m(n,t,\delta_r)) \]

(Note that the right hand side of this equation is independent of the choice of \( w \) in \( (\delta) \).)

(iii) \( (\alpha \circ \eta)(g_0) = (\frac{n}{t} \alpha \cdot \sum_{< 1^n >, < n-t >} \eta \cdot < 1^n >) \)

Proof:

(i) This is a restatement of theorem 4.2.5.

(ii) Immediate from definition 3.2.5(ii).

(iii) Let \( z \in S_n \) be of minimal length in its class.
If \( p_1 p_2 \ldots p_s \) is a reduced expression for \( z \) then

\[ g_z = g_{i_1} g_{i_2} \ldots g_{i_s} \]

and by theorem A2.7(i)

\[ |\{i_1, i_2, \ldots, i_s\}| = s. \]

Let \( g_v g_A = \sum_{B \in Z_n} q_{g_B, 0} g_{v, B, A} \) for all \( A \subset Z_n \) with \( |A| = t \)

and \( v \in S_n \), where \( q_{g_v, B, A} \in \mathcal{G}_n \) (of equation (4.2.7)).

By lemma 4.2.3 it is clear that

\[ (4.2.9) \quad \sigma_{Z, A, A} = \sigma_{u_{i_1}, A}^{A} \sigma_{u_{i_2}, A}^{A} \cdots \sigma_{u_{i_s}, A}^{A} \]

for all \( A \subset Z_n \) with \( |A| = t \).

Let \( w = c_1 c_2 \ldots c_r \) be the decomposition of \( w \) into a product of disjoint cycles. We can assume that \( c_j \in \mathcal{G}_j \) for each \( j \) in \( Z_r \).

By equation (4.2.9)

\[ \sigma_{w, A, A} = \sigma_{c_1, A}^{A} \sigma_{c_2, A}^{A} \cdots \sigma_{c_r, A}^{A} \quad \text{for all} \quad A \subset Z_n \quad \text{with} \quad |A| = t. \]

Fix \( A \). By corollary 4.2.4 for each \( j \in Z_r \)

\[ \sigma_{c_j, A}^{A} = p_j(u) g_{v_j} g_{x_j} \]

where either \( p_j(u) = 0 \) or if \( p_j(u) \neq 0 \) then there exists \( i_j \in Z_{\mathcal{G}_j} \cup \{0\} \) with \( v_j \in \mathcal{G}_j \subset T_1 \) and \( x_j \in \mathcal{G}_{j-1} \subset T_2 \), in which case \( p_j(u) = \begin{cases} 1 & \text{if } i_j = 0 \text{ or } \mathcal{G}_j \\ (u-1) & \text{otherwise.} \end{cases} \)

\[ g_{v_1} g_{x_1} \ldots g_{v_r} g_{x_r} = g_{v_1} g_{v_2} \ldots g_{v_r} g_{x_1} \ldots g_{x_r} \]

and by the proof of corollary 4.2.4 it is clear that

\[ l(v_1 v_2 \ldots v_r) = l(v_1) + l(v_2) + \ldots + l(v_r) \]

and

\[ l(x_1 \ldots x_r) = l(x_1) + \ldots + l(x_r). \]

Thus

\[ \sigma_{w, A, A} = p_1(u) \ldots p_r(u) g_{v_1} \ldots g_{v_r} g_{x_1} \ldots g_{x_r} \]

and \( v_1 \ldots v_r \in T_1, x_1 \ldots x_r \in T_2 \).

If \( \sum_{j=1}^r i_j < t \) let \( \mathcal{C} \) be the partition of \( t \) who's parts are \( i_1, i_2, \ldots, i_r \) with as many '1's added as necessary.
Similarly if \( r \sum (\delta_j-i_j) \leq n-t \) let \( \mathcal{S} \) be the partition of \( n-t \) whose parts are \((\delta_1-i_1), \ldots, (\delta_r-i_r)\) with as many '1's added as is necessary.

We have that \(|\{A \mid \sigma_{\mathcal{S},A,A} = q(u)g_{v_j}g_{x_j}, q(u) \in Q(u), q(u) \neq 0, v_j \in 1, x_j \in \delta_j-i_j \subset T_2\}| = (t_{-i_j}) \)

and if \( S = \{ A \mid \sigma_{\mathcal{S},A,A} = q'(u)g_{v_j}g_{x_j}, q'(u) \in Q(u), q'(u) \neq 0, v \in (\mathcal{S}) \subset T_1, x \in (\mathcal{S}) \subset T_2\} \) then

\[ |S| = (t_{-i_1-i_2-1}) \]

Thus denoting \(|\{j \mid j \in Z, i_j \neq 0 \text{ or } \delta_j\}| \) by \( m(\tau) \),

\[ \Sigma(\alpha, \eta)(\sigma_{\mathcal{S},A,A}) = (t_{-i_1-1}) \ldots (t_{-i_r}) \]

\[ = \mathcal{M}(x(n-\delta_1, t-1) \ldots x(n-\delta_2, t-2) \ldots) \]

\[ \ldots x(n-\delta_r, t-i_r) (u-1)^m(\tau) \alpha^T(\tau)n^T(\mathcal{S}) \]

(see the notation below definition 4.2.1 and theorem 4.2.5).

\[ (\alpha, \eta)(g_\mathcal{S}) = \Sigma(\alpha, \eta)(\sigma_{\mathcal{S},A,A}) \] thus clearly

from the above formula

\[ (\alpha, \eta)(g_\mathcal{S}) = \mathcal{M}(x(n-t, \delta_1) \ldots x(n-t, \delta_r)) \]

COROLLARY 4.2.10.

(i) The characters of \( G_n \ (n > 1) \) are subclass functions.

(ii) Given \( \beta \in X(G_n) \), \( \delta \vdash n \) and \( w \) of minimal length in \( (\delta) \), then \( \beta(g_w) = \beta g_\mathcal{S} \).

(iii) Let \( \alpha \in X(G_t) \), \( \eta \in X(G_{n-t}) \) and let \( \delta = < \delta_1, \delta_2, \ldots, \delta_s > \vdash n \). Assume that \( \delta \neq < 1^n > \) and let \( r \) be the greatest integer such that \( \delta_r > 1 \). Then

\[ (\alpha, \eta)(\gamma) = \mathcal{M}(x(n-t, \delta_1) \ldots x(n-t, \delta_r)) \]

Proof:

(ii) follows from (i), lemma 4.2.2 and relations (2.1.4).

(iii) follows from (i) and theorem 4.2.8(ii).
The unit character \( \zeta_m \) of \( G_m \) (\( m \in \mathbb{Z}_+ \)) is a subclass function (see definition 4.1.3), thus the result follows from lemmas 32.6(ii) 3.2.7(ii), 4.1.2(iii), corollary 4.1.5 and theorem 4.2.8(ii). (Note the remark below theorem 4.2.8(ii)).

Let \( \lambda = <\lambda_1, \ldots, \lambda_s> \rightarrow n \) with \( 0 < \lambda_s < \cdots < \lambda_1 \). Let \( \lambda_j \in X(G_n) \) for each \( j \) in \( Z_s \). Lemma 4.1.1 shows that we can form the character \( ^1\lambda \cdot ^2\lambda \cdot \cdots \cdot ^s\lambda \) of \( G_n \). In a way analogous to that used to prove theorem 4.2.5 we can find a formula for \( ^1\lambda \cdot \cdots \cdot ^s\lambda \rightarrow i \) \( (i \in Z_n \setminus \{1\}) \). Since by corollary 3.2.7(ii) no 'new' information is gained we omit the proof and merely state the result.

THEOREM 4.2.12.

Using the above notation,
\[
\left( ^1\lambda \cdot \cdots \cdot ^s\lambda \right) \rightarrow i = \sum (u-1)^i \left\langle \delta_1, \delta_2, \ldots, \delta_s \right\rangle^{-1} \left( ^1\lambda \right) \left( ^2\lambda \right) \delta_2 \cdots \delta_s \frac{(n-i)!}{s! \left( \lambda_1 - \delta_1 \right)! \cdots \left( \lambda_s - \delta_s \right)!}
\]
where the summation is over all ordered \( s \)-tuples \( \left\{ \delta_1, \delta_2, \ldots, \delta_s \right\} \) such that \( 0 \leq \delta_j < \lambda_j \) for all \( j \in Z_s \) and \( \sum_{j=1}^{s} \delta_j = i \).

Clearly an analogue of formula (4.2.11) must exist for \( \left( ^1\lambda \cdot \cdots \cdot ^s\lambda \right) \rightarrow i \) \( (i \rightarrow n) \).

THEOREM 4.2.13.

\( \eta(w) \in \mathbb{Z}[u] \) for all \( \eta \in X(G_n) \) and \( w \in S_n \).

Proof:

By induction on \( n \), it follows from lemma 3.2.7(ii),
lemma 4.1.2(iii), corollary 4.1.5 and theorem 4.2.8 that 
\( \eta_0 \in Z[u] \) for all \( \eta \in X(G_n) \) and \( \delta \vdash n \). In the notation of §2.2 one readily finds from the remark below theorem 2.2.8 that \( \sigma(w,b) \in Z[u] \) for all \( w \in S_n \) and \( b \in B \). Thus the result follows from corollary 4.2.10.

EXAMPLE ILLUSTRATING THEOREM 4.2.8.

Let \( n = 5, l = 3, \delta = < 2.2.1> \), \( \alpha \in X(G_3) \) and \( \eta \in X(G_2) \). By theorems 4.2.5 and 4.2.8:

\[
(\alpha \cdot \eta)_2 = \left( \frac{5-2}{3-2} \right) \alpha_2 \eta_1 + (u-1) \left( \frac{5-2}{3-1} \right) \alpha_1 \eta_1 + \left( \frac{5-2}{3} \right) \alpha_1 \eta_2
\]

\[
= \mathcal{J}_{\alpha, \eta} (m(5,3,2))
\]

\[
(\alpha \cdot \eta)_0 = \mathcal{J}_{\alpha, \eta} \left( (m(5,3,2))^2 \right)
\]

\[
= \left( \frac{5-2-2}{3-2-2} \right) \alpha < 2^2 \eta_1 + (u-1) \left( \frac{5-2-2}{3-2-1} \right) \alpha_2 \eta_1 + \left( \frac{5-2-2}{3-2} \right) \alpha_2 \eta_2
\]

\[
+ (u-1) \left( \frac{5-2-2}{3-1-2} \right) \alpha_2 \eta_1 + (u-1) \left( \frac{5-2-2}{3-1-1} \right) \alpha_1 \eta_1 + (u-1) \left( \frac{5-2-2}{3-1} \right) \alpha_1 \eta_2
\]

\[
+ \left( \frac{5-2-2}{3} \right) \alpha_2 \eta_2 + (u-1) \left( \frac{5-2-2}{3} \right) \alpha_1 \eta_2 + \left( \frac{5-2-2}{3} \right) \alpha_1 \eta_1 < 2^2>
\]

\[
= (u-1) \alpha_2 \eta_1 + \alpha_2 \eta_2 + (u-1) \alpha_2 \eta_1 + (u-1) \alpha_1 \eta_1 + \alpha_2 \eta_2
\]

\[
= 2(u-1) \alpha_2 \eta_1 + 2 \alpha_2 \eta_2 + (u-1) \alpha_1 \eta_1
\]

§4.3 \( (\sigma, \delta, \mathcal{L}) \)-RECTANGLES.

Let \( n \in Z_+ \) and \( t \in Z_{n-1} \). By equation (4.2.11) and lemma 3.2.6(ii) we see that given \( \delta \vdash n \) there exist elements \( e(\delta, \delta, \mathcal{L}) \) of \( Q(u) \) (\( \delta \vdash t, \mathcal{L} \vdash n-t \)) such that

\[
(4.3.1) \quad (\alpha \cdot \eta)_\delta = \sum_{\delta \vdash t} e(\delta, \delta, \mathcal{L}) \alpha_\delta \eta_\mathcal{L} \quad \text{for all } \delta \vdash t, \mathcal{L} \vdash n-t \\alpha \in X(G_t) \text{ and } \eta \in X(G_{n-t}).
\]

Corollary 4.3.9 gives a formula for \( e(\delta, \delta, \mathcal{L}) \).
LEMMA 4.3.2.

The coefficients \( e(\delta, \delta', \tau) \) are uniquely determined by equation (4.3.1).

Proof:

Let \( \{ \alpha^a | a \leftarrow t \} \) and \( \{ \eta^b | b \leftarrow n-t \} \) be the sets of all irreducible \( G_t \) and \( G_{n-t} \) characters respectively. Fix \( \delta \leftarrow n \) and define the following three matrices:

\[
M_1 = \{(\alpha^a)_{\delta} \}_{a, \delta \leftarrow t}
\]

\[
M_2 = \{(\eta^b)_{\tau} \}_{\tau, b \leftarrow n-t}
\]

\[
M_3 = \{(\alpha^a \eta^b)_{\delta} \}_{a, b \leftarrow t, n-t}
\]

By theorem 2.1.2 \( M_1 \) and \( M_2 \) are invertible. Equation (4.3.1) shows that \( e(\delta, \delta', \tau) \) is the \((\delta, \tau)^{th}\) entry of \( M_1^{-1}M_2M_3^{-1} \).

DEFINITION 4.3.3.

Let \( \delta = < 1, \delta_2, \ldots, \delta_n \leftarrow n \)

\( \delta = < 1, \delta_2, \ldots, \delta_t \leftarrow t \)

\( \tau = < 1, \tau_2, \ldots, (n-t) \leftarrow n-t \)

(i) A \((\delta, \delta', \tau)\)-rectangle is a \((t+1) \times (n-t+1)\) array of non-negative integers, say \( r_{ij} \) in the \((i, j)^{th}\) position \((i \in \{0, 1, \ldots, t\}, j \in \{0, 1, \ldots, n-t\})\) such that

\[
(a) \quad \sum_{j=0}^{n-t} r_{ij} = \delta_i \quad \text{for} \quad i \in \{1, 2, \ldots, t\}
\]

\[
(b) \quad \sum_{i=0}^{t} r_{ij} = \tau_j \quad \text{for} \quad j \in \{1, 2, \ldots, (n-t)\}
\]

\[
(c) \quad \sum_{a=0}^{b} \frac{r_{ab} b-a}{b} = \delta_b \quad \text{for} \quad b \in \{1, 2, \ldots, n\}
\]

\[
(d) \quad r_{00} = 0
\]

(ii) The \((\delta, \delta', \tau)\)-rectangle \( \mathcal{R} \) with entries \( r_{ij} \)

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has sum

\[ s(\mathbf{1}_R) = \sum_{i=1}^{t} \sum_{j=1}^{n-t} r_{ij} \]

and for each \( a \in \mathbb{Z}^n \) it has \( a \)-value

\[ \mathbf{1}_a = \frac{\sigma_a}{r_{oa}! r_1 a_1! \cdots r_{ao}!} \]

REMARK.

Given \( \sigma \vdash n, \delta \vdash t \) and \( \iota \vdash n-t \) there may exist none, one or more \((\sigma, \delta, \iota)\)-rectangles.

DEFINITION 4.3.4.

Let \( m \in \mathbb{Z}^+ \) and \( d \in \mathbb{Z}_{m-1} \). A \((d+1) \times (m-d+1)\) array of elements of \( \mathbb{Z} \) is called an \((m,d)\)-rectangle if it is a \((\beta, \mathcal{E}, \mathcal{F})\)-rectangle for some \( \beta \vdash m, \mathcal{E} \vdash d \) and \( \mathcal{F} \vdash m-d \).

DEFINITION 4.3.5.

Let \( m \in \mathbb{Z}^+ \), \( d \in \mathbb{Z}_{m-1} \) and \( \beta = < \beta_1, \ldots, \beta_r > \) with \( 0 < \beta_r < \cdots < \beta_1 \).

A \( d \)-cutting \( \mathcal{J} \) of \( \beta \) (denoted by \( \mathcal{J}(d)\beta \)) is an ordered pair of ordered \( r \)-tuples \( (\{z_1, \ldots, z_r\}, \{y_1, \ldots, y_r\}) \) of non-negative integers such that

\[ \sum_{i=1}^{r} z_i = d \quad \text{and} \quad z_i + y_i = \beta_i \quad \text{for all} \quad i \in \mathbb{Z}_r. \]

LEMMA 4.3.6.

Given \( m \in \mathbb{Z}^+ \) and \( d \in \mathbb{Z}_{m-1} \) there exists a surjection \( \mathfrak{V}_m: \{ \kappa | \kappa(d) \mu, \mu \vdash m \} \rightarrow \{ \mathbf{1}_R | \mathbf{1}_R \text{ is an } (m,d)\text{-rectangle} \} \) given by:

If \( \mathcal{J} \) is as given in definition 4.3.5, then \( \mathfrak{V}_m(\mathcal{J}) \) is the
(β,z,y)-rectangle with entries \( q_{ij} = |\{ x \in \mathbb{Z}_r \mid z_x = i, y_x = j \}| \) for all \( (i,j) \in \{0,1,..,d\} \times \{0,1,..,m-d\} \setminus \{(0,0)\} \), \( q_{00} = 0 \). 

\( z \) is the partition of \( d \) whose parts are \( z_1,z_2,..,z_r \) and
\( y \) is the partition of \( m-d \) whose parts are \( y_1,y_2,..,y_r \).

Proof:

That \( \{ q_{ij} \mid (i,j) \in \{0,..,d\} \times \{0,..,m-d\} \} \) is an \((m,d)\)-rectangle is readily proved by checking that conditions (i)(a),(b),(c) and (d) of definition 4.3.3 hold.

Let \( \lambda = < \lambda_1,..,\lambda_f > \leftarrow \mathbb{Z} \), \( \xi \leftarrow \mathbb{Z} \), \( \mathcal{f} \leftarrow \mathbb{Z} \), \( \mathcal{d} \leftarrow d \), \( \mathcal{m} \leftarrow m-d \) and let \( \mathcal{R} \) be a \((\lambda,\xi,\mathcal{f})\)-rectangle with entries \( r_{ij} \). For \( v \in \mathbb{Z} \) let \( x_v \) be the number of \( i \in \mathbb{Z} \) with \( \lambda_i = v \) (so that \( \lambda = < 1^{x_1}..m^{x_m} > \)). By definition 4.3.3(i)(c)

\[ \sum_{a=0}^{v} r_{a,v-a} = x_v \text{ for each } v \in \mathbb{Z}. \]

thus it is clear that there exists a \( d \)-cutting \( \kappa = (\{d_1,..,d_f\} \times \{b_1,..,b_f\}) \) of \( \lambda \) such that for each \( v \in \mathbb{Z} \) and \( a \in \mathbb{Z} \cup \{0\} \) there are precisely \( r_{a,v-a} \) values of \( i \in \mathbb{Z} \) with \( d_i = v \) and \( d_i + b_i = v \). Clearly \( \mathcal{Y}_m(\kappa) = \mathcal{R} \), showing that \( \mathcal{Y}_m \) is a surjection.

**Lemma 4.3.7.**

Let \( \mathcal{R} \) be an \((m,d)\)-rectangle \((d \in \mathbb{Z}_{m-1})\) then

\[ |\bigcup_{\mathcal{J}(d) \beta} \mathcal{Y}_m(\mathcal{J})| = |\mathcal{R}| = \sum_{a=1}^{m} |\mathcal{R}_a| \]

(see definition 4.3.3(ii)).

Proof:

Immediate for the part of the proof of lemma 4.3.6 which shows that \( \mathcal{Y}_m \) is surjective.

**Theorem 4.3.8.**

Let \( \alpha \in \mathcal{X}(G_t) \), \( \eta \in \mathcal{X}(G_{n-t}) \) and \( \delta \leftarrow n \) then
\((\alpha \cdot n) \cdot \eta = \sum_{\delta} \left( \sum_{\mathcal{R}} (u-1)^{s(\mathcal{R})} \prod_{a=1}^{n} \mathcal{R}_a \right) \alpha \cdot \delta \cdot \eta \cdot \tau \)

(see definition 4.3.3(ii)).

Proof:

If \(\delta = \langle 1^n \rangle\) the result is readily checked.

Let \(\delta = \langle \delta_1, \delta_2, \ldots, \delta_s \rangle \neq \langle 1^n \rangle \) let \(n\) and \(r\) be the greatest integer such that \(\delta_r > 1\).

For each \(b \in Z_r\) choose a term, say \(V_b\), of \(m(n, t, \delta_b)\) (see theorem 4.2.8). So

\[ V_b = p_b x(n-\delta_b, t-d_b) (\langle z_b \rangle, \langle \delta_b-z_b \rangle) \]

for some \(z_b \in Z_{\delta_b} \cup \{0\}\) and \(p_b \in Q(u)\).

Let \(\sum_{b=1}^{r} z_b = d\). We note that \(\sum_{b=1}^{r} \delta_b-z_b = n-d-(s-r)\).

Let \(z\) and \(y\) be the partitions of \(d\) and \(n-d-(s-r)\) with parts \(z_1, z_2, \ldots, z_r\) and \(\delta_1-z_1, \ldots, \delta_r-z_r\) respectively. Thus

\[ V_1V_2 \cdots V_r = p_1p_2 \cdots p_r x(n-\delta_1-\ldots-\delta_r, t-z_1-\ldots-z_r) (\langle z \rangle, \langle y \rangle) \]

Define

\[ S_{V_1} \cdots V_r = \{ f = (\{a_1, a_2, \ldots, a_s\}, \{b_1, \ldots, b_s\}) | f(t) \delta, a_i=z_i \text{ for all } i \in Z_r \} \]

Clearly \(|S_{V_1} \cdots V_r| = (s-r)\) and if for each \(b \in Z_r\) \(U_b\) is a term of \(m(n, t, \delta_b)\) then \(S_{V_1} \cdots V_r \cap S_{U_1} \cdots U_r \neq \emptyset\) if and only if \(V_b = U_b\) for all \(b \in Z_r\).

Let \(\chi = (\{d_1, \ldots, d_s\}, \{i_1, \ldots, i_s\})\) \(t) \delta\). For each \(b \in Z_r\) put \(Y_b = q_b x(n-\delta_b, t-d_b) (\langle d_b \rangle, \langle i_b \rangle)\) where

\[ q_b = \begin{cases} 1 & \text{if } d_b = 0 \text{ or if } i_b = 0 \\ (u-1) & \text{otherwise} \end{cases} \]

\(Y_b\) is a term of \(m(n, t, \delta_b)\) and clearly \(\chi \in S_{Y_1} \cdots Y_r\).

Thus \(\{ f | f(t) \delta \} \) is equal to the disjoint union:

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possible choices of $V_b$ for each $b \in Z_r$.

Let $\mathcal{J} = \{a_1, \ldots, a_s, b_1, \ldots, b_s\} \in S_{V_1 \cdots V_r}$ and let $a$ and $b$ be the partitions of $t$ and $n-t$ with parts $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ respectively. By lemma 4.3.6 it is clear that

$$J_{\alpha, \eta}(V_1 \cdots V_r) = (u-1)^s(\sum_{\mathcal{J}} (s-t-d) \alpha^T(a) \eta^T(b))$$

Hence

$$J_{\alpha, \eta}(m(n, t, \delta_1) \cdots m(n, t, \delta_r)) = \left( \sum_{\mathcal{J}} (u-1)^s(\sum_{\mathcal{J}} (s-t-d) \alpha^T(a) \eta^T(b)) \right)$$

where $a$ and $b$ are the partitions with parts $a_1, \ldots, a_s$ and $b_1, \ldots, b_s$ respectively.

Thus by lemmas 4.3.6 and 4.3.7, we see that

$$J_{\alpha, \eta}(m(n, t, \delta_1) \cdots m(n, t, \delta_r)) = \sum_{\mathcal{J}} \left( \sum_{\mathcal{J}} (u-1)^s(\prod_{a=1}^{n} \prod_{R_a}) \alpha \eta \right)$$

where the second sum is over all $(\delta, \delta, \tau)$-rectangles.

Equation (4.2.11) now shows that the proof is complete.

**COROLLARY 4.3.9.**

Let $e(\delta, \delta, \tau)$ be defined by equation (4.3.1) (see lemma 4.3.2) then

$$e(\delta, \delta, \tau) = \sum_{\mathcal{J}} (u-1)^s(\prod_{R_a}) \alpha \eta$$

where $R$ is $(\delta, \delta, \tau)$-rectangle.

**Proof:**

Immediate from theorem 4.3.8.
CHAPTER 5

CHARACTERISTICS.

In this chapter a formula is derived for the values of the irreducible characters of $G_n$ in terms of the values of the irreducible characters of $S_n$. (see theorem 5.3.9).

§5.1 THE ALGEBRA $F$.

DEFINITION 5.1.1.

Let $m \in \mathbb{Z}^+$. The function $\rho \mapsto |\rho|$ from $S_m$ to $\mathbb{Z}^+$ is defined as follows:

If $\rho = \langle 1^{\rho_1}2^{\rho_2}\ldots m^{\rho_m} \rangle$ then $|\rho| = \sum_{i=1}^{n} \rho_i$

DEFINITION 5.1.2.

$F$ is the graded $K$-algebra $F_1 \oplus F_2 \oplus \ldots = \sum_{m \geq 1} F_m$

where $F_m$ is the $K$-space the $K$-basis $\{ x_\delta | \delta \vdash m \}$ and multiplication is given by:

If $\delta \vdash m$ and $\tau \vdash d$ ($m,d \in \mathbb{Z}^+$) then

(5.1.3) $x_\delta x_\tau = \sum_{\sigma \vdash m+d} e(\sigma, \delta, \tau) x_\sigma$

(recall that $e(\sigma, \delta, \tau)$ is defined by equation (4.3.1). See lemma 4.3.2).

NOTATION. We will abbreviate $x_{< r >}$ to $x_r$ for all $r \in \mathbb{Z}^+$.

DEFINITION 5.1.4.

Let $m \in \mathbb{Z}^+$ and $\eta \in X(G_m)$. The following element of $F_m$

$\phi_\eta = \sum_{\sigma \vdash m} \eta_\sigma x_\sigma$

is the $F$-characteristic of $\eta$.

(recall that $\eta_\sigma = \eta(g_w)$ for any $w \in (\mathfrak{g})$. See definition 5.3.9).
4.2.1 and below).

THEOREM 5.1.5.

(i) \( F \) is commutative.

(ii) \( F \) is associative.

(iii) If \( \alpha \in X(G_m) \) and \( \eta \in X(G_d) \) \((m,d \in \mathbb{Z}^+)\) then

\[ \phi_{\alpha \circ \eta} = \phi_{\alpha} \phi_{\eta} \]

Proof:

(i) Using the notation of definition 5.1.2: if \( R \) is a \((\sigma, \delta, \tau)\)-rectangle then its transpose \( R^t \) is clearly a \((\delta, \tau, \sigma)\)-rectangle. Further \( s(R) = s(R^t) \) and for all \( a \in \mathbb{Z}_{m+d} \) \((R)_a = (R^t)_a \) (see definition 4.3.3). Thus corollary 4.3.9 shows that \( F \) is commutative.

(iii) \[ \phi_{\alpha \circ \eta} = \sum_{\delta - m + d} \sum_{\sigma - m} \sum_{\tau - d} \phi_{(\sigma \circ \eta) \sigma \tau} \]

\[ = \sum_{\delta - m} \sum_{\sigma - m} \phi_{(\sigma \circ \eta) \sigma \tau} \]

\[ = \phi_{\alpha} \phi_{\eta} \]

By equation (4.3.1)

\[ = \phi_{\alpha} \phi_{\eta} \]

(ii) By (iii) and lemma 3.2.7(ii), if \( \xi \in X(G_r) \)

\[(r \in \mathbb{Z}^+) \text{ then} \]

\[ (\phi_{\alpha} \phi_{\eta}) \phi_{\xi} = \phi_{\alpha \circ \eta} \phi_{\xi} = \phi_{(\alpha \circ \eta) \circ \xi} = \phi_{\alpha \circ (\eta \circ \xi)} = \phi_{\alpha} \phi_{\eta} \phi_{\xi} \]

\[ = \phi_{\alpha} \phi_{\eta} \phi_{\xi} \]

From which equation and theorem 2.1.2 the associativite of \( F \) can be readily proved.

\[ \times \]

LEMMA 5.1.6.

Let \( \sigma = < 1 \delta_1 \ldots \delta_m > \rightarrow m \) \((m \in \mathbb{Z}^+)\) and let \( Z(\sigma) = \{ j \in \mathbb{Z}^+_+ | \delta_j \neq 0 \} \). Then for each \( d \in \mathbb{Z}^+ \)

\[ x_{\delta \circ \xi} = (\delta_{d+1}) \phi_{j(d)} + (u-1) \sum_{j \in Z(\sigma) \delta_{d+j} + 1} \phi_{j(d)} \]

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where
\[ \rho_0(d) = \langle 1, 1, 2, \ldots, d \rangle^{m+d} \]
and
\[ \rho_j(d) = \langle 1, 1, \ldots, j-1, (d+j), \ldots \rangle^{m+d} \]
for all \( j \in \mathbb{Z}(\sigma) \).
Note that \( |\rho_0(d)| = |\sigma| + 1 \) and that \( |\rho_j(d)| = |\sigma| \) for all \( j \in \mathbb{Z}(\sigma) \).

Proof:

We use corollary 4.3.9. If \( \sigma \mapsto m+d \) and a \((\sigma, \sigma, \langle \sigma \rangle)\)-rectangle exists then \( \sigma = \rho_0(d) \) or \( \rho_j(d) \) for some \( j \in \mathbb{Z}(\sigma) \).

For each of these values for \( \sigma \) there exists a unique rectangle:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & \ldots & 0 & 1 \\
\sigma_1 & 0 & \ldots & \ldots & \ldots & \ldots \\
\sigma_2 & 0 & \ldots & \ldots & \ldots & \ldots \\
& & & \ddots & \ddots & \ddots \\
\sigma_m & 0 & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

is the \((\rho_0(d), \sigma, \langle \sigma \rangle)\)-rectangle.

\[
\begin{array}{ccccccc}
0 & 0 & 0 & \ldots & 0 & 0 \\
\sigma_1 & 0 & \ldots & \ldots & \ldots & \ldots \\
& & & \ddots & \ddots & \ddots \\
\sigma_{j-1} & 0 & \ldots & 0 & 1 & 0 \\
\sigma_j & 0 & \ldots & 0 & \ldots & \ldots \\
\sigma_m & 0 & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

is the \((\rho_j(d), \sigma, \langle \sigma \rangle)\)-rectangle.

(\( j \in \mathbb{Z}(\sigma) \)).

\[
\text{LEMMA 5.1.8.}
\]

Let \( \beta = \langle 1, \ldots, m \rangle \mapsto m \) (\( m \in \mathbb{Z}_+ \))

There exist unique elements \( \lambda_\rho \) in \( K (\rho \mapsto m) \)
such that

\[
\begin{align*}
(i) & \quad (x_1)^{\beta_1} (x_2)^{\beta_2} \cdots (x_m)^{\beta_m} = \sum_{\rho \mapsto m} \lambda_\rho x_\rho \\
(ii) & \quad \{ \rho \mid \lambda_\rho \neq 0, |\beta| < |\rho| \} = \{ \beta \}
\end{align*}
\]
Proof:

(i) Immediate as $F$ is a graded algebra.

(ii) We use induction on $|\beta|$. Let $r$ be the greatest integer such that $\beta_r \neq 0$. Let $\sigma = < 1^{\beta_1} 2^{\beta_2} \ldots r^{\beta_r} \ldots > \vdash m-r$.

By (i) there exist elements $\lambda_\pi$ in $K$ ($\pi \vdash m-r$) with

$$(x_1)^{\beta_1}(x_2)^{\beta_2} \ldots (x_r)^{\beta_r} = \sum_{\pi \vdash m-r} \lambda_\pi x_\pi$$

By the inductive hypothesis

$$\{ \pi \mid \lambda_\pi \neq 0, |\sigma| < |\pi| \} = \{ \sigma \}$$

so the result now follows from lemma 5.1.6, since in the notation of that lemma $\beta = \rho_0(r)$.

\[\square\]

NOTATION.

Let $m \in \mathbb{Z}^+$ and $\sigma = < 1^{\sigma_1} \ldots m^{\sigma_m} > \vdash m$, then we denote the following element of $F_m$

$$(x_1)^{\sigma_1} \ldots (x_m)^{\sigma_m}$$

by $x^\sigma$.

THEOREM 5.1.9.

(i) $\{ x_r \mid r \in \mathbb{Z}^+ \}$ is a set of free generators for $F$.

(ii) $\{ x^\delta \mid \delta \vdash m \}$ is a $K$-basis for $F_m$. ($m \in \mathbb{Z}^+$).

Proof:

(i) In the notation of lemma 5.1.8 and below

$$x_\beta = \frac{1}{\lambda_\beta} (x^\beta - \sum_{|\rho| < |\beta|} \rho x_\rho)$$

Thus induction on $|\beta|$ shows that the given set generate $F$.

Suppose that $\sum \delta_\pi x_\pi = 0$ where $\pi$ runs over a finite non-empty set $T$ of partitions. Let $\rho \in \{ \sigma \mid \sigma \in T, |\pi| < |\sigma| \}$ for all $\pi \in T$, then it is clear by lemma 5.1.8 that $\delta_\rho = 0$.

Hence $\delta_\pi = 0$ for all $\pi \in T$.

(ii) Immediate from (i) and its proof.

\[\square\]
NOTATION. Fix $n \in \mathbb{Z}_+$. Let $T_n = \{ \sigma_j \mid j \in J \}$ be the set of all partitions of $n$ ($J$ some indexing set) indexed in such a way that $i < j$ implies that $|\sigma_i| \geq |\sigma_j|$.

By theorem 5.1.9(ii) there exist elements $v_{ij}$ of $K$ $(i, j \in J)$ such that
\[(5.2.1) \quad x^i = \sum_{j \in J} v_{ij} x^j \quad \text{for all } i \in J.\]

**DEFINITION 5.2.2.**

$V_n$ is the $(|J| \times |J|)$-matrix with $(i, j)^{th}$ entry $v_{ij}$.

(Appendix 8 gives $V_2, V_3$ and $V_4$ explicitly).

**DEFINITION 5.2.3.**

Let $\beta = <\beta_1, \ldots, \beta_t>$ with $0 < \beta_1 < \ldots < \beta_t$ and $\rho = <\rho_1, \ldots, \rho_s>$ with $0 < \rho_1 < \ldots < \rho_s$ be elements of $T_n$. Then

(i) $\equiv$ is the relation defined on $T_n$ as follows: $\beta \equiv \rho$ if and only if there exist subsets $I_j$ of $Z_t$ ($j \in Z_s$) satisfying
\[(a) \quad I_j \cap I_i = \emptyset \quad \text{for all } i, j \in Z_s \text{ with } i \neq j.\]
\[(b) \quad \bigcup_{j \in Z_s} I_j = Z_t\]
\[(c) \quad \rho_j = \sum_{r \in I_j} \beta_r \quad \text{for all } j \in Z_s.\]

(ii) $N(\beta, \rho)$ is the number of sets $\{I_j \mid j \in Z_s\}$ of subsets of $Z_t$ satisfying conditions (i)(a), (b) and (c).

We have immediately the following two lemmas

**LEMMA 5.2.4.**

Let $\beta, \rho \in T_n$. If $\beta \equiv \rho$ then $|\beta| \geq |\rho|$ with equality if and only if $\beta = \rho$. 

\[
\begin{aligned}
\end{aligned}

Remark. Lemma 5.2.4 shows that

\[ N(\sigma_i, \sigma_j) = 0 \text{ if } i > j. \]

**Lemma 5.2.5.**

Let \( \beta \) and \( \rho \) be as in definition 5.2.3. Fix \( r \in \mathbb{Z}_t \).
For each \( z \) in \( \mathbb{Z}_s \) with \( \rho_z \geq \beta_r \) define \( \pi_z \) to be the partition of \( n-\beta_r \) with the same parts as \( \rho \) except that \( \rho_z \) is replaced by \( \rho_z - \beta_r \). Let \( \beta' \) be the partition of \( n-\beta_r \) with the same parts as \( \beta \) except that \( \beta_r \) is omitted. Then

\[ N(\beta, \rho) = \sum_{z \in \mathbb{Z}_s} N(\beta', \pi_z) m_z \]
where \( m_z = |\{i \mid i \in \mathbb{Z}_s, \rho_i = \rho_z\}|. \)

**Theorem 5.2.6.**

(i) \( v_{ij} = N(\sigma_i, \sigma_j)(u-1)|\sigma_i|-|\sigma_j| \)

(ii) \( v_{ii} = \sigma_{i1}! \sigma_{i2}! \cdots \sigma_{in}! \) where \( \sigma_i = \langle 1^{\sigma_{i1}} \cdots n^{\sigma_{in}} \rangle \)

(iii) \( V_n \) is upper triangular, i.e. \( v_{ij} = 0 \) if \( i > j \).

(see the ordering of the partitions of \( n \) given at the beginning of this section.)

**Proof:**

(iii) Immediate from (i) and the remark below lemma 5.2.4.

(ii) Immediate from (i) and definition 5.2.3(ii).

(i) Let \( \beta \) and \( \rho \) be as in definition 5.2.3 then it is sufficient to show that \( \lambda_{\beta, \rho} = N(\beta, \rho)(u-1)|\beta|-|\rho| \) where

\[ x^\beta = \sum_{\sigma \in \mathbb{Z}_n} \lambda_{\beta, \sigma} x^\sigma \]

We use induction on \( |\beta| \). In the notation of lemma 5.2.5: By theorem 5.1.5(i) \( x^\beta = x^{\beta'} x^{\beta_r} \). So if

\[ x^{\beta'} = \sum_{\ell \in \mathbb{Z}_n-\beta_r} \lambda_{\beta', \ell} x^\ell \]
then \( \lambda_{\beta', \rho} \) is the coefficient
of $x$ in $\sum_{z \in \mathbb{Z}_+} \lambda^\beta x x^r$ which by equation (5.1.7) is the coefficient of $x$ in $\sum_{z \in \mathbb{Z}_+} \beta^r x x^r \rho_z \beta^r$.

Thus by the inductive hypothesis and equation (5.1.7)

$$\lambda \beta, \rho = \sum_{z \in \mathbb{Z}_+} N(\beta^r, \pi_z) (u-1)^{1-\rho_z} r_{rz}$$

where $\rho_z = \begin{cases} 0 & \text{if } \rho_z > \beta^r \\ 1 & \text{if } \rho_z = \beta^r \end{cases}$

by lemma 5.2.5.

\[ \text{by lemma 5.2.5.} \]

\[ (5.3) \quad \mathcal{V}_n, X(G_n) \text{ and } X(CS_n). \]

NOTATION.

Fix $n \in \mathbb{Z}_+$. Let \{ $\eta^\alpha$ | $\alpha \vdash n$ \} be the set of all irreducible characters of $G_n$ over $K$ and \{ $X^\alpha$ | $\alpha \vdash n$ \} be the set of all irreducible characters of $S_n$ over $C$. (see the introduction to Part II). By theorem 1.4.5(ii) we can assume that $X^{\alpha}(w) = f_1(n^{\alpha}(g_w))$ for all $w \in \mathcal{W}$, where $f_1$ is the specialisation of $K$ with $f_1(u) = 1$ which is shown to exist by lemma 1.4.2(ii).

Let $\tau_n = \{ \sigma_j = < 1, \sigma_j^1, 2, \sigma_j^2, \ldots, n, \sigma_j^n > | j \in J \}$ be the set of all partitions of $n$, indexed in such a way that $i < j$ implies that $|\sigma_i| \geq |\sigma_j|$ for all $i, j \in J$. (see definition 5.1.1).

Define two $(|J| \times |J|)$-matrices as follows:

$\bar{\Gamma}_n$ is the matrix with $(i, j)^{th}$ entry $\eta^{\sigma_i^1} \sigma_j^*$.

(see corollary 4.2.10(i) and the notation below definition 4.2.1).

$\Theta_n$ is the matrix with $(i, j)^{th}$ entry $\theta_{ij}$ defined by the equations.
\[ \emptyset \eta \sigma_i = \sum_{j \in J} \theta_{ij} x_j \sigma_j \] for all \( i \in J \). (see definition 5.1.4 and theorem 5.1.9(ii)). Appendix 8 gives \( \mathbb{H}_2, \mathbb{H}_3 \) and \( \mathbb{H}_4 \) explicitly.

**DEFINITION 5.3.1.**

Let \( \sigma = < 1, \sigma_1, \ldots, \sigma_n > \rightarrow n \). The polynomial of \( \sigma \) is

\[ p(\sigma) = (u+1)^\sigma_2 (u^2+u+1)^\sigma_3 \ldots (u^{n-1}+u^{n-2}+\ldots+1)^\sigma_n \]

**NOTATION.**

Let \( m, d \in \mathbb{Z}_+ \). If \( \lambda = < \lambda_1, \ldots, \lambda_n > \rightarrow m \) and \( \tau = < \tau_1, \ldots, \tau_n > \rightarrow d \) then we denote the partition \( < \lambda_1+\tau_1, \lambda_2+\tau_2, \ldots > \) of \( m+d \) by \( \lambda+\tau \).

We have immediately

**LEMMA 5.3.2.**

Let \( m, d \in \mathbb{Z}_+ \), \( \lambda \rightarrow m \) and \( \tau \rightarrow d \).

(i) \( p(\lambda+\tau) = p(\lambda)p(\tau) \)

(ii) \( |\lambda+\tau| = |\lambda| + |\tau| \)

**LEMMA 5.3.3.**

(i) \( \mathbb{H}_n \mathbb{V}_n = \emptyset \_n \)

(ii) \( x_i(\theta_{ij}) = (x^{\sigma_i})^{\sigma_j}/\sigma_j! \sigma_{j+1}! \ldots \sigma_{j+n}! \)

for all \( i, j \in J \).

**Proof:**

(i) \( \sum_{j \in J} (\eta \sigma_i) x_j \sigma_j = \emptyset \eta \sigma_i \) by definition 5.1.4

\[ = \sum_{r \in J} \theta_{ir} x_r \sigma_r \]

\[ = \sum_{j \in J} \sum_{r \in J} \theta_{ir} x_j \sigma_j \] by equation (5.2.1)

(ii) Let \( Y \) be the graded \( \mathbb{C} \)-algebra \( \mathbb{E}^{\mathbb{Y}}_m \) where \( Y_m \) is the
C-space with C-basis \{ y_\delta \mid \delta \vdash m \} and multiplication is given by:

For \delta \vdash m and \tau \vdash d \ (m,d \in \mathbb{Z}_+)

\[ y_\delta y_\tau = \sum_{\delta \vdash m+d} f_1(e(\sigma,\delta,\tau))y_\delta \]

(Note that \( e(\sigma,\delta,\tau) \) is defined by equation (4.3.1)).

Clearly \( f_1 \) can be extended to a ring homomorphism

\[ f_1 : F_{D_1} \rightarrow Y \] where \( F_{D_1} = \{ \sum \lambda_\delta x_\delta \mid \lambda_\delta \in K_{D_1}, \delta \) runs over any finite set of partitions} (see §1.4) by

\[ f_1 : \sum \lambda_\delta x_\delta \mapsto \sum f_1(\lambda_\delta)y_\delta \]

Let \( \sigma = < 1^{\sigma_1} \ldots m^{\sigma_m} \vdash m \) \ (m \in \mathbb{Z}_+) \). By equation (5.1.7)

\[ y_\delta y_\tau < t > = (\delta_{t+1})y_\sigma < t > \] for all \( t \in \mathbb{Z}_+ \). Thus if we denote

\[(y_\delta < 1 >) \ldots (y_\delta < m >) = y_\delta \delta_1 \ldots \delta_m y_\delta \]

(5.3.4)

Since \( \xi \sigma_i \eta = \sum_{j \in J} \theta_{ij} x^{\sigma_j} = \sum_{j \in J} \xi (n_{ij}) \sigma_j \sigma_j y_\delta \)

we have that

\[ \sum_{j \in J} f_1(\theta_{ij})y^{\sigma_j} = \sum_{j \in J} (x^{\sigma_i}) \sigma_j y^{\sigma_j} \]

So by equation (5.3.4)

\[ f_1(\theta_{ij}) = (x^{\sigma_i})^{\sigma_j} / \sigma_j ! \ldots \sigma_j ! \]

To prove the main result in this section - theorem 5.3.9 - we need the following identity in the terms \( N(\beta,\rho) \)

**LEMMA 5.3.5.**

Let \( \rho = < \rho_1 \ldots \rho_s > \) and \( \sigma = < 1^{\sigma_1} \ldots n^{\sigma_n} > \) be elements of \( T_n \). Denote the partition \( < \rho_2, \ldots, \rho_s > \) of \( n-\rho_1 \) by \( \rho' \), then

\[ N(\sigma,\rho) = \sum_{\tau \vdash \rho_1} \frac{\sigma_1 ! \ldots \sigma_n !}{\tau_1 ! \ldots \tau_n ! \lambda_1! \ldots \lambda_{n-\rho_1}!} N(\tau, < \rho_1 >) N(\lambda, \rho') \]

\( \lambda \vdash n-\rho_1 \)

\( \tau+\lambda = \sigma \)

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where \( \tau = < 1 \tau_1 \ldots \tau_p > \) and \( \lambda = < 1 \lambda_1 \ldots (n-\rho_1) \ldots n-\rho_1 > \)

Proof:

Let \( a = < 1 a_1 \ldots a_m > \) and \( b = < b_r, b_2, \ldots, b_t > \) be partitions of \( m \) (\( m \in \mathbb{Z}_+ \)). Define

\[ S = \{ \{ \delta_1, \delta_2, \ldots, \delta_t \} \mid \delta_j \leftarrow b_j \text{ for all } j \in \mathbb{Z}_t, \delta_1 + \delta_2 + \ldots + \delta_t = a \} \]

and put \( M(a, b) = |S| \).

Clearly \( N(a, b) = M(a, b) \).

where \( \delta_r = < 1 r_1 \ldots r_2 > \) for all \( r \in \mathbb{Z}_t \) and if \( b_r = < b_r \ldots b_r > \) then

\[ M(a, b) = \sum_{\delta} M(\delta, < b_1 >)M(\delta, b') \]

\[ \delta \leftarrow b_1 \]

\[ \delta \leftarrow m-b_1 \]

\[ \delta + \delta = a \]

Equation (5.3.6) can now readily be derived.

**Lemma 5.3.7.**

\[ \Theta = \frac{(x^{n<\lambda>})_{\sigma_j} \cdot \rho(\sigma_j)}{\delta_{j_1} ! \ldots \delta_{j_n} !} \cdot \frac{1}{\sigma_{j_1} ! \ldots \sigma_{j_n} !} \]

\[ \Theta \mid J \mid J = \frac{\rho(\sigma_j)}{\delta_{j_1} ! \ldots \delta_{j_n} !} \cdot \sigma_{j_1} ! \ldots \sigma_{j_n} ! \cdot \sigma_{j_1} ! \ldots \sigma_{j_n} ! \]

Proof:

With the given ordering of the partitions we have \( \sigma \mid J \mid = < n > \) and by corollary 4.1.5 \( \eta \mid J \mid = j_n \) the unit character of \( G_n \) (see definition 4.1.3).

Thus by lemma 5.3.3(ii) and the fact that \( V_n \) is non-singular it is sufficient to show that

\[ \sum_{j \in J} (x^{n<\lambda>})_{\sigma_j} \rho(\sigma_j) \cdot \sigma_{j_1} ! \ldots \sigma_{j_n} ! \cdot \sigma_{j_1} ! \ldots \sigma_{j_n} ! \cdot v_{j_i} = (j_n)_{i} \sigma_i \]

for all \( i \in J \).

or equivalently (by theorem A2.7(ii) and lemma 4.1.2(ii) ) that

\[ \sum_{\sigma = < 1 \sigma_1 \ldots n \sigma_n >} \frac{\rho(\sigma) N(\sigma, \rho) \cdot (u-1) \mid \sigma \mid \mid \rho \mid}{\sigma_1 ! \ldots \sigma_n ! \cdot \sigma_1 ! \ldots \sigma_n !} = u^{n} \mid \rho \mid \]

for all \( \rho \in T_n \)

where \( \sigma = < 1 \sigma_1 \ldots n \sigma_n > \)
We use induction on \( n \):

Fix \( \rho = < \rho_1, \rho_2, \ldots, \rho_n > \) and let \( \rho' = < \rho_2, \ldots, \rho_n > \)

By equation (5.3.61) and Lemma 5.3.2

\[
\sum_{\sigma} p(\sigma) N(\sigma, \rho) (u-1) |\sigma| - |\rho| = \sum_{\lambda} p(\lambda) N(\lambda, \rho') (u-1) |\lambda| - |\rho'|
\]

By the inductive hypothesis

\[
\sum_{\tau} p(\tau) N(\tau, <\rho_1>) (u-1) |\tau| - |\rho_1| = \rho_1 - |\rho_1|
\]

and

\[
\sum_{\lambda} p(\lambda) N(\lambda, \rho') (u-1) |\lambda| - |\rho'| = u - \rho_1 - |\rho'|
\]

Since \( u - |\rho_1| = u - |\rho| \) the proof is complete.

**Lemma 5.3.8.**

\[
\theta_{ij} = \frac{(x_{ij})_{\sigma_j} p(\sigma_j)}{\sigma_{j1} \cdots \sigma_{jn} 1^{\sigma_{j1}} \cdots n^{\sigma_{jn}}}
\]

**Proof:**

By lemma 5.3.3(ii) it is sufficient to show that \( \theta_{ij} = q_{ij} p(\sigma_j) \) for some \( q_{ij} \in \mathbb{Q} \).

Lemmas 4.1.2(iii), 3.2.7(ii) and corollary 4.1.5 show that \( \varnothing \sigma_{ij} \) is a linear combination over \( \mathbb{Z} \) of characteristics of the type \( \varnothing_{\alpha \cdot \eta} \) where \( \alpha \in X(G_t) \) and \( \eta \in X(G_{n-t}) \) for some \( t \in \mathbb{Z}_{n-1} \) and of \( \varnothing \sigma_{ij} \) (note that \( \sigma_{ij} = < n > \)). Thus by lemma 5.3.7 it is sufficient to...
prove that if \( \phi_{\alpha \cdot \eta} = \sum_{\sigma \in \mathcal{S}_n} \theta_{\alpha \cdot \eta} \sigma \chi^{\sigma} \) then \( \theta_{\alpha \cdot \eta} \sigma = q_{\alpha \cdot \eta} \phi_{\sigma}(\sigma) \) for some \( q_{\alpha \cdot \eta} \sigma \in \mathbb{Q} \).

We use induction on \( n \):

By the inductive hypothesis

\[
\phi_{\alpha} = \sum_{t \in \mathcal{T}} q_{\alpha \cdot t} \phi_t(\sigma) \chi^{\sigma} \quad \text{and} \quad \phi_{\eta} = \sum_{t \in \mathcal{T} - \eta} q_{\eta \cdot t} \phi_t(\delta) \chi^{\delta}
\]

for some \( q_{\alpha \cdot t}, q_{\eta \cdot t} \in \mathbb{Q} \).

By theorem 5.1.5(iii) \( \phi_{\alpha \cdot \eta} = \phi_{\alpha} \phi_{\eta} \), so by lemma 5.3.2(i)

\[
\phi_{\alpha \cdot \eta} = \sum_{t \in \mathcal{T}} q_{\alpha \cdot t} q_{\eta \cdot t} \phi_t(\tau + \delta) \chi^{\tau + \delta}
\]

Thus \( \theta_{\alpha \cdot \eta} \sigma = \sum_{t \in \mathcal{T}} q_{\alpha \cdot t} q_{\eta \cdot t} \phi_t(\sigma) \).

We are now able to state and prove a theorem which gives a complete solution to 'Part 2' (see below lemma 2.1.5) of the problem of evaluating the irreducible characters of \( G_n \) over \( K \) in terms of the values of the irreducible characters of \( S_n \) over \( \mathbb{C} \).

**THEOREM 5.3.9.**

\[
(\eta^i)_{\sigma_j} = \sum_{t \in \mathcal{T}} (X^i)_{\sigma_t} \chi^{p(\sigma_t)} \frac{N(\sigma_t, \sigma_j)}{\sigma_t! \cdot \sigma_{t_1}! \cdot \ldots \cdot \sigma_{t_{n-1}}! \cdot \sigma_1 \cdot \sigma_{t_2} \cdot \ldots \cdot \sigma_{t_n} \cdot 1} \sigma_{t_1}^t \sigma_{t_2} \cdot \ldots \cdot \sigma_{t_n}^t \cdot 1}
\]

for all \( i, j \in \mathcal{J} \).

**Proof:**

Immediate from lemmas 5.3.3(i), 5.3.7 and theorem 5.2.6(i).

The notation used to state theorem 5.3.9 can be found...
in theorem 4.1.4, corollary 4.1.5 and definitions 5.3.1 (and above), 5.2.3(ii) and 5.1.1.

REMARK.

Theorem 5.3.9 and the remark below theorem 2.2.8 enable one to calculate the value of $\eta(\omega)$ for any irreducible character $\eta$ of $G_n$ and any element $w$ of $S_n$, provided that the values of the irreducible characters of $S_n$ over $C$ are known.

Appendix 6 gives the values of $\eta^{(6)}$ for all irreducible characters $\eta$ of $G_n$ and all partitions $\sigma$ of $n$ for $n \in \mathbb{Z}_6$.

Appendix 5 gives the values of $\eta(\omega)$ for all irreducible characters $\eta$ of $G_n$ and all elements $w$ of $S_n$ for $n \in \mathbb{Z}_4$.

Appendix 8 gives (implicitly) the values of $N(\beta, \rho)$ for all partitions $\beta, \rho$ of 2, 3 and 4. (see theorem 5.2.6(i)).
REFERENCES.


4 CARTER, R.: Simple groups of Lie type. Wiley (1972)


AN EXAMPLE OF THE LINEAR DEPENDENCE OF CHARACTER VALUES.
(See the remark below theorem 2.2.8).

Let \( W \) be of type \( B_3 \) (see [4, chapter 3]). Thus

\[
W = \langle a, g, d | a^2 = g^2 = d^2 = (ag)^3 = (gd)^4 = (ad)^2 = 1_W \rangle
\]

We will identify \( W \) with this group. Let \( c(a) = c(g) = c(d) = 1 \) where \( c \) is the function from \( W \) to \( \mathbb{Z}_+ \) described in Corollary 1.1.6.

The corresponding generic Hecke algebra \( H(K, u) \) is thus generated as a \( K \)-algebra with identity \( h_1 \) by
\[
\{ h_a, h_g, h_d \}
\]
with the following defining relations;

\[
\begin{align*}
(A.1.1) & \quad h_x^2 = uh_1 + (u-1)h_x \quad \text{for all } x \in \{a, g, d\} \\
(A.1.2) & \quad h_a h_g h_a = h_g h_a h_g \\
(A.1.3) & \quad h_g d h_g d = h_d g h_d h_g \\
(A.1.4) & \quad h_a h_d = h_d h_a
\end{align*}
\]

Also for any \( w \in W \) we have

\[
\begin{align*}
(A.1.5) & \quad h_w = h_{r_1} h_{r_2} \ldots h_{r_f} \quad \text{where } r_1 r_2 \ldots r_f \text{ is any reduced expression for } w. (r_1, \ldots, r_f \in \{a, g, d\}).
\end{align*}
\]

Let \( \eta \) belong to \( \mathbb{X}(H(K, u)) \) then

\[
\begin{align*}
\eta(h_{agdgag}) &= \eta(h_{agdaga}) \quad \text{by (A.1.2)} \\
&= \eta(h_a h_g d g h_a) \quad \text{by (A.1.5)} \\
&= \eta(h_{gdag} h_a^2) \quad \text{by equations (2.1.4)} \\
&= u\eta(h_{gdag}) + (u-1)\eta(h_{gdaga}) \quad \text{by (A.1.1)}
\end{align*}
\]

In the same manner we find that

\[
\begin{align*}
\eta(h_{gdag}) &= u\eta(h_{da}) + (-1)\eta(h_{dag}) \\
\eta(h_{gdaga}) &= u\eta(h_{dag}) + (u-1)\eta(h_{dgag})
\end{align*}
\]

and

\[
\eta(h_{dgag}) = u\eta(h_{dg}) + (u-1)\eta(h_{dag})
\]

Thus
\[ \eta(h_{agdag}) = u^2 \eta(h_{da}) + (u-1)u \eta(h_{dg}) + (u-1)(u^2+1) \eta(h_{dag}) \]

We note (without proof) that a set of class representatives of \( W \) consisting of elements of minimal length in their conjugacy classes can be chosen such that \( \{da, dg, dag\} \) is a subset of it.
THE SYMMETRIC GROUP

The symmetric group $S_n$ on $n$ symbols is the set of all permutations of the elements of $\mathbb{Z}_n$ with composition given by

$$ (\beta \sigma)(i) = \beta(\sigma(i)) \quad \text{for all } \beta, \sigma \in S_n \text{ and } i \in \mathbb{Z}_n. $$

It is well known that each conjugacy class of $S_n$ is the set of all elements of $S_n$ of a fixed cycle-type. Denote the class of elements of cycle-type $\alpha_1 \alpha_2 \ldots \alpha_n$ by $(\alpha)$ where $\alpha$ is the partition $<1^{\alpha_1} 2^{\alpha_2} \ldots n^{\alpha_n}>$ of $n$ (see lemma 4.1.1).

**THEOREM A2.1.**

$S_n$ is a Coxeter group. $\{p_i = (i \ i+1) \mid i \in \mathbb{Z}_{n-1}\}$ is a set of distinguished generators and the following are defining relations for this set of generators.

(A2.2) $p_i^2 = 1_{S_n}$ for all $i \in \mathbb{Z}_{n-1}$

(A2.3) $p_i p_j = p_j p_i$ for all $i, j \in \mathbb{Z}_{n-1}$ with $i+2 < j$.

(A2.4) $p_i p_{i+1} p_i = p_{i+1} p_i p_{i+1}$ for all $i \in \mathbb{Z}_{n-2}$.

Note that (A2.3) and (A2.4) can be expressed in the more usual forms; $(p_i p_j)^2 = 1_{S_n}$ and $(p_i p_{i+1})^3 = 1_{S_n}$.

**Proof:**

It is well known that $S_n$ is generated by the set $\{p_i \mid i \in \mathbb{Z}_{n-1}\}$. Thus it is sufficient to show that the group $U_n = \langle a_1, a_2, \ldots, a_{n-1} \mid a_i^2 = 1, (a_i a_j)^2 = 1 \text{ for all } i, j \in \mathbb{Z}_{n-1}, (a_i a_{i+1})^3 = 1 \text{ for all } i \in \mathbb{Z}_{n-2} \rangle$ has order less than or equal to $n!$. The set $\{a_i a_{i+1} \ldots a_{n-1} U_{n-1} \} \cup \{U_{n-1}\}$ is closed under left multiplication by elements of $U_n$, thus the result can be readily proved using induction.
LEMMA A2.5.

All the distinguished generators of $S_n$ are conjugate in $S_n$.

COROLLARY A2.6.

If $\{c_{i_j} | i \in \mathbb{Z}_{n-1}\}$ is an indexing system for $S_n$ (see definition 1.1.8) then $c_{i_j} = c_{i_j}$ for all $i,j \in \mathbb{Z}_{n-1}$.

Our main result in this appendix is

THEOREM A2.7.

(i) An element $\rho$ in $S_n$ is of minimal length in its conjugacy class if and only if in any reduced expression for $\rho$ each $\mu_j (j \in \mathbb{Z}_{n-1})$ appears at most once.

(ii) The minimal length of an element of the conjugacy class $(\alpha)$ is $n - |\alpha|$. (see definition 5.1.1).

Proof:

Let $\rho \in (\alpha)$. Let $\rho$ have a reduced expression in terms of the elements of $\{\mu_j | j \in J \subset \mathbb{Z}_{n-1}\}$. Define an equivalence relation $T$ on $J$ by

$i\equiv j$ if and only if either $\{i,i+1,\ldots,j\} \subset J, (i < j)$ or $\{i,i-1,\ldots,j\} \subset J, (i \geq j)$.

and denote the equivalence classes with respect to this relation by $J_1,J_2,\ldots,J_t$.

Since for $i \in J_r$ and $j \in J_s$, $\mu_i \mu_j = \mu_j \mu_i$ provided that $r \neq s$, we can write the reduced expression of $\rho$ in the form

$\delta_1 \delta_2 \ldots \delta_t$ where $\delta_r$ is a product of elements of $\{\mu_i | i \in J_r\} (r \in \mathbb{Z}_t)$.

Clearly $\delta_r$ moves at most $|J_r|+1$ symbols (i.e. has
at least $n-|J_r|-1$ fixed symbols). Thus if $\rho$ moves $\lambda$ points

$$\sum_{r=1}^{t} (|J_r|+1) = \sum_{r=1}^{t} |J_r| + t = |J| + t$$

Clearly $t$ is less than or equal to the number of non-$1$-cycles of $\rho$ which equals $|\alpha| - \alpha_1$, where $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$. Also $|J| \leq 1(\rho)$. So

$$\lambda \leq 1(\rho) + |\alpha| - \alpha_1$$

But $\lambda = n - \alpha_1$, thus $n - |\alpha| \leq 1(\rho)$ with equality only if $|J| = 1(\rho)$ i.e. each $u_j$ appears at most once in the reduced expression for $\rho$.

(ii) and the 'only if' part of (i) now follow since $\delta$ in $(\alpha)$ given by

$$\delta = (1)(2) \ldots (\alpha_1)(\alpha_1+1 \alpha_1+2) \ldots (\alpha_1+2\alpha_2-1 \alpha_1+2\alpha_2) \ldots$$

has length $\alpha_2+2\alpha_3+\ldots+(n-1)\alpha_n = n - |\alpha|$. The proof is completed by showing that the product

$$\prod_{i \in J_r} u_i$$

taken in any order is a $(|J_r|+1)$-cycle.

Clearly it is sufficient to prove

$$(A2.8) \prod_{j=1}^{n-1} u_{\sigma(j)}$$

is an $n$-cycle for any $\sigma$ in $S_{n-1}$.

We proceed by induction on $n$.

Let $\rho_0 = \prod_{j=1}^{n-2} \rho_{\sigma(j)}$, $\rho_1 = \prod_{j=1}^{n-2} \rho_{\sigma(j)}$ and $\rho_2 = \prod_{j=1}^{n-2} \rho_{\sigma(j)}$ where $\sigma(j) < \sigma(n-1)$ and $\sigma(j) > \sigma(n-1)$.

Clearly $\rho_0 = \rho_1 \rho_2$ and by the inductive hypothesis $\rho_1$ is a $\sigma(n-1)$-cycle containing the symbol $\sigma(n-1)$ and $\rho_2$ is an $(n-\sigma(n-1))$-cycle containing the symbol $\sigma(n-1)+1$.

Since $\rho = \rho_0 \rho_{\sigma(n-1)}$, $\rho$ is an $n$-cycle and $(A2.8)$ is proved.

The length function on $S_n$ can be evaluated using the following result

**Lemma A2.9.**

Let $\rho = (1 \ldots 2 \ldots n) \in S_n$. 

\[7\]
Put $m_i = |\{ j \mid j < i, o_j > o_i \}|$ for each $i \in Z_n$. Then

$$ l(\rho) = \sum_{i=1}^{n} m_i $$

Proof:

$S_n$ is isomorphic to the Weyl group of type $A_{n-1}$ (see [4, page 124]). Using [4, lemma 2.2.1 and theorem 2.2.2] one can show that

$$ l(\rho P_i) = \begin{cases} l(\rho) + 1 & \text{if } \rho_i < \rho_{i+1} \\ l(\rho) - 1 & \text{if } \rho_i > \rho_{i+1} \end{cases} $$

The lemma now follows by induction on $l(\rho)$. •

\[77\]
APPENDIX 3

SPECIAL TRANSVERSALS IN COXETER GROUPS.

Let \((W, R)\) be a Coxeter system (see definition 1.1.1).

DEFINITION A3.1.

Let \(J \subseteq R\). The subgroup \(W_J = \langle r \mid r \in J \rangle\) is the parabolic subgroup of \(W\) associated with \(J\).

LEMMA A3.2.

\((W_J, J)\) is a Coxeter system for each \(J \subseteq R\).

Proof: \([2, \text{chapter 4, \S 1, theorem 2(1)}]\).

THEOREM A3.3.

Let \(J \subseteq R\). In each left coset of \(W_J\) in \(W\) there exists a unique element of minimal length. Further if \(d\) is such an element then

\[ l(dw) = l(d) + l(w) \quad \text{for all } w \in W_J. \]

Proof:

It is sufficient to prove that \(l(dw) = l(d) + l(w)\) for all \(w \in W_J\), since this clearly implies that \(d\) is the unique element of minimal length in \(dW_J\).

We use induction on \(l(w)\). Let \(w \in W_J\), \(r \in R\) and assume that \(l(dw) = l(d) + l(w)\) and \(l(wr) = l(w) + 1\). Suppose that \(l(dw) \neq l(d) + l(w)\). By lemma 1.1.4,
\[ l(dw) = l(dw) - 1 \] and by the 'Exchange condition' (see \([, \text{chapter 4, \S 1.5}]\))

either \(dw = dw'\) where \(w' \in W_J\) and \(l(w') = l(w) - 1\), which gives a contradiction since \(l(wr) = l(w) + 1\).

or \(dw = d'w\) where \(l(d') = l(d) - 1\). In which case
\( dW_j = d'W_j \) and we again get a contradiction since \( d \) is of minimal length in \( dW_j \).

Thus \( l(dwr) = l(d) + l(wr) \) and the theorem follows by induction.

**DEFINITION A3.4.**

The special transversal of \( W_j \) in \( W \) is the set of left coset representatives each of which is of minimal length in its coset (see theorem A3.3). This transversal will be denoted by \( D^R_j \).

**LEMMA A3.5.**

Given \( J \subset R \), \( d \in D^R_j \) and \( r \in R \) then

- either \( rd \in dW_j \)
- or \( rd \in D^R_j \)

**Proof:**

Assume that \( rd \not\in D^R_j \). There exists \( w \in W_j \) with \( l(rdw) < l(rd) + l(w) \). In fact \( l(rdw) = l(dw) - 1 \) and by the 'Exchange condition'

- either \( rdw = dw' \) where \( w' \in W_j \), so that \( rd \in dW_j \).
- or \( rdw = d'w \) where \( l(d') = l(d) - 1 \). In which case \( rd = d' \) and if \( y \in W_j \) then

\[ l(rdy) \geq l(dy) - 1 = l(d) - 1 + l(y) = l(d') + l(y) \geq l(d'y) \]

So \( l(rdy) = l(rd) + l(y) \) showing that \( rd \in D^R_j \), a contradiction.

**COROLLARY A3.6.**

Let \( J \subset R \), \( d \in D^R_j \) and \( r \in R \). If \( l(rd) < l(d) \) then \( rd \in D^R_j \).

**Proof:**

Immediate from lemma A3.5 and definition A3.4.
**Lemma A3.7.**

Fix $r \in R$ and $J \subset R$. Let $C(r)$ be the conjugacy class of $r$ in $W$. Let

$$S_1 = \{ d \in D_J^R \mid rd \in dW_J \}$$

$$S_2 = \{ d \in D_J^R \mid rd \in D_J^R, l(rd) < l(d) \}$$

$$S_3 = \{ d \in D_J^R \mid rd \in D_J^R, l(rd) > l(d) \}$$

Then

$$|S_1| = \frac{|C(r) \cap W_J|}{|C(r)|}$$

$$|S_2| = \frac{1}{2} \frac{|C(r) \setminus W_J|}{|C(r)|} = |S_3|$$

**Proof:**

By lemma A3.5 $|S_1| + |S_2| + |S_3| = |W|/|W_J|$. Since $S_1$ clearly has the given order it is thus sufficient to show that $|S_2| = |S_3|$. Let $d \in S_2 \cup S_3$. Since $r(rd) = d \in D_J^R$ we have that $rd \in S_2 \cup S_3$. Thus $r(S_2 \cup S_3) = S_2 \cup S_3$. By lemma 1.1.4, for any $w \in W$, $l(rw) = l(w) \pm 1$ and so

$$|S_2| = |S_3|.$$

We now give an explicit description of $D_J^R$ for a particular Coxeter group $W$.

**Theorem A3.8.**

Let $W = S_n$ ($n \in \mathbb{Z}^+ \cup \{0\}$) and $R = \{ u_1 = (i \ i+1) \mid i \in \mathbb{Z}_{n-1} \}$ (see theorem A2.1). Given $\lambda = < \lambda_1, \lambda_2, \ldots, \lambda_r >$ with $0 < \lambda_r < \lambda_{r-1} < \ldots < \lambda_1$, let $J_\lambda$ be the subset of $R$ described above lemma 4.1.1 then

$$D_J^R = \{ (a_{11}a_{12} \ldots a_{1\lambda_1}a_{21} \ldots a_{2\lambda_2} \ldots a_{r1}a_{r\lambda_r}) \in S_n \mid a_{i1} < a_{i2} < \ldots < a_{ii} \text{ for all } i \in \mathbb{Z}_r \}$$
Proof:

The given set of elements of $S_n$ has order $|S_n|/|S_n^\lambda|$ (see lemma 4.1.1(ii) and (v)) and since for $\rho, \sigma \in S_n$, $\rho S_n = \sigma S_n$ is equivalent to $\rho^{-1}\sigma \in S_n$, it is clear that the given set is a transversal for $S_n^\lambda$ in $S_n$. That it is the special transversal is clear from lemma A2.9.
THE DECOMPOSITION MATRIX OF $H_0$.

The decomposition matrix (see definition 1.5.6) of $H_0$ (see 1.5) is given below in the cases where $W$ is $S_1, S_2, S_3, S_4, S_5$ and $S_6$. (see theorem A2.1). We denote the algebra $H_0$ by $\Gamma_1, \ldots, \Gamma_6$ in these cases respectively.

The notation for the irreducible characters of $G_1$ (see the introduction to Part II) $(n \in Z_6)$ is that described by corollary 4.1.5.

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The character tables (see definition 2.1.1) of the generic Hecke algebras $G_1$, $G_2$, $G_3$ and $G_4$ (see introduction to Part II) are given below. (That of $G_4$ is transposed). The notation used for the irreducible characters is that described in corollary 4.1.5 and cycle notation is used for elements of the Weyl groups.

The character table of the generic Hecke algebra whose Weyl group is of type $B_2$ (see part (ii) of the discussion above theorem 2.2.8) is also given. The algebra is denoted by $H_{B_2}$ in this case.
\[
\begin{array}{c|cccccccc}
H_{B_2} & 1 & a & g a g & g & a g a & a g & g a & a g a \\
\hline
\eta^1 & 1 & u & u^3 & u & u^3 & u^2 & u^2 & u^4 \\
\eta^2 & 1 & u & u & -1 & -u^2 & -u & -u & u^2 \\
\eta^3 & 2 & u-1 & u(u-1) & u-1 & u(u-1) & 0 & 0 & -2u^2 \\
\eta^4 & 1 & -1 & -u^2 & u & u & -u & -u & u^2 \\
\eta^5 & 1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c}
G_{x_1} & 1 \\
\hline
\eta^{(3)} & 1 \\
\end{array}
\]

\[
\begin{array}{c|c}
G_{x_2} & 1 & (12) \\
\hline
\eta^{(2)} & 1 & u \\
\eta^{(2)} & 1 & -1 \\
\end{array}
\]

\[
\begin{array}{c|cccccccc}
G_{x_3} & 1 & (12) & (23) & (13) & (123) & (132) \\
\hline
\eta^{(3)} & 1 & u & u & u^3 & u^2 & u^2 \\
\eta^{(4)} & 2 & u-1 & u-1 & 0 & -u & -u \\
\eta^{(5)} & 1 & -1 & -1 & -1 & 1 & 1 \\
\end{array}
\]
<table>
<thead>
<tr>
<th>$G_4$</th>
<th>$\eta^{(4)}$</th>
<th>$\eta^{(13)}$</th>
<th>$\eta^{(2^*)}$</th>
<th>$\eta^{(2^*, 2)}$</th>
<th>$\eta^{(1^*)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>(12)</td>
<td>$u$</td>
<td>$2u-1$</td>
<td>$u-1$</td>
<td>$u-2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>(23)</td>
<td>$u$</td>
<td>$2u-1$</td>
<td>$u-1$</td>
<td>$u-2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>(34)</td>
<td>$u$</td>
<td>$2u-1$</td>
<td>$u-1$</td>
<td>$u-2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>(13)</td>
<td>$u^3$</td>
<td>$u^3$</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>(24)</td>
<td>$u^3$</td>
<td>$u^3$</td>
<td>0</td>
<td>$-1$</td>
<td>$-1$</td>
</tr>
<tr>
<td>(14)</td>
<td>$u^5$</td>
<td>$u^4$</td>
<td>$-u^2(u-1)$</td>
<td>$-u$</td>
<td>$-1$</td>
</tr>
<tr>
<td>(123)</td>
<td>$u^2$</td>
<td>$u(u-1)$</td>
<td>$-u$</td>
<td>$-u+1$</td>
<td>1</td>
</tr>
<tr>
<td>(132)</td>
<td>$u^2$</td>
<td>$u(u-1)$</td>
<td>$-u$</td>
<td>$-u+1$</td>
<td>1</td>
</tr>
<tr>
<td>(243)</td>
<td>$u^2$</td>
<td>$u(u-1)$</td>
<td>$-u$</td>
<td>$-u+1$</td>
<td>1</td>
</tr>
<tr>
<td>(234)</td>
<td>$u^2$</td>
<td>$u(u-1)$</td>
<td>$-u$</td>
<td>$-u+1$</td>
<td>1</td>
</tr>
<tr>
<td>(143)</td>
<td>$u^4$</td>
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<td>$-u^2$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
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<td>$u^4$</td>
<td>0</td>
<td>$-u^2$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
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<td>0</td>
<td>$-u^2$</td>
<td>0</td>
<td>1</td>
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<td>$u^4$</td>
<td>0</td>
<td>$-u^2$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
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<td>$-u^2$</td>
<td>0</td>
<td>$u$</td>
<td>$-1$</td>
</tr>
<tr>
<td>(1234)</td>
<td>$u^3$</td>
<td>$-u^2$</td>
<td>0</td>
<td>$u$</td>
<td>$-1$</td>
</tr>
<tr>
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<td>0</td>
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</tr>
<tr>
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<td>$-u^2$</td>
<td>0</td>
<td>$u$</td>
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</tr>
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<td>(1324)</td>
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<td>$-u^3$</td>
<td>$-u^2(u-1)$</td>
<td>$u^2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>(1423)</td>
<td>$u^5$</td>
<td>$-u^3$</td>
<td>$-u^2(u-1)$</td>
<td>$u^2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>(1234)</td>
<td>$u^2$</td>
<td>$u(u-2)$</td>
<td>$u^2+1$</td>
<td>$-2u+1$</td>
<td>1</td>
</tr>
<tr>
<td>(1324)</td>
<td>$u^4$</td>
<td>$-u^2$</td>
<td>$u^3+u$</td>
<td>$-u^2$</td>
<td>1</td>
</tr>
<tr>
<td>(1423)</td>
<td>$u^6$</td>
<td>$-u^4$</td>
<td>$2u^3$</td>
<td>$-u^2$</td>
<td>1</td>
</tr>
</tbody>
</table>
APPENDIX 6

CHARACTER TABLES (Part 2).

Let \( \eta \) be an irreducible character of \( G_n \) (see the introduction to Part II) and let \( \sigma \vdash \eta \). Then since \( \eta \) is a subclass function (see corollary 4.2.10(i) and definition 4.2.1(ii)) we can define

\[ \eta_{\sigma} = \eta(g_v) \text{ for any } v \text{ in the subclass } (\sigma) \]

(see definition 4.2.1(i)). Further by corollary 4.2.10(ii)

\[ \eta_{\sigma} = \eta(g_v) \text{ for any } v \text{ of minimal length in the conjugacy class } (\sigma). \]

The values of \( \eta_{\sigma} \) for all irreducible characters of \( G_n \) and all partitions \( \sigma \) of \( n \) are given below for \( n \in \mathbb{Z}_6 \).

The values are displayed in tables, one for each \( n \), as follows:
The rows are indexed by their irreducible characters of \( G_n \) (in the notation of corollary 4.1.5) and the columns are indexed with the partitions of \( n \). The \( (\eta, \sigma)^{th} \) entry is an ordered tuple say \( a_r, a_{r-1}, \ldots, a_0 \) of non-negative integers with the following significance:

\[ \eta_{\sigma} = \delta_r a_r u^r + \delta_{r-1} a_{r-1} u^{r-1} + \ldots + \delta_1 a_1 u + \delta_0 a_0 \]

where \( \delta_i \) is the sign ( + or - ) at the head of the column containing \( a_i \) (\( i \in \mathbb{Z}_r \)).

With this interpretation the columns of the tables are columns of the character table of \( G_n \) (see definition 2.1.1), in fact they are the columns corresponding to a set of class representatives of \( S_n \) each of which is of minimal length in its class. Theorem 2.1.2 shows that these columns span the column space of the character table.

That the entries of the given tables are all integers.
follows from theorem 4.2.13.

There is an obvious symmetry in each of the given tables; namely that in each column the first entry is the reverse of the last entry and the second entry is the reverse of the penultimate entry etc. This symmetry can readily be explained using the involutory semi-linear automorphism of the generic Hecke algebra described in [6, §8].
\[ \begin{array}{c|c|c}
G_1 & \langle 1 \rangle & \langle 2 \rangle \\
\eta & + & + \\
\end{array} \]

\[ \begin{array}{c|c|c|c}
G_2 & \langle 1^2 \rangle & \langle 1 \rangle & \langle 2 \rangle \\
\eta & + & + & - \\
\eta & 1 & 1 & 0 \\
\eta & 1 & 0 & 1 \\
\end{array} \]

\[ \begin{array}{c|c|c|c|c}
G_3 & \langle 1^3 \rangle & \langle 1^2 \rangle & \langle 1 \rangle & \langle 2 \rangle \\
\eta & + & + & - & + & - & + \\
\eta & 1 & 1 & 0 & 1 & 0 & 0 \\
\eta & 2 & 1 & 1 & 0 & 1 & 0 \\
\eta & 1 & 0 & 1 & 0 & 0 & 1 \\
\end{array} \]

\[ \begin{array}{c|c|c|c|c|c|c|c}
G_4 & \langle 1^4 \rangle & \langle 1^2 \rangle & \langle 1^2 \rangle & \langle 1 \rangle & \langle 2 \rangle & \langle 2 \rangle \\
\eta & + & + & - & + & - & + & - \\
\eta & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\eta & 3 & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 0 \\
\eta & 2 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
\eta & 3 & 1 & 2 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 1 \\
\eta & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
\end{array} \]
<table>
<thead>
<tr>
<th>$C_5$</th>
<th>$\langle 1^5 \rangle$</th>
<th>$\langle 1^2 2 \rangle$</th>
<th>$\langle 1^2 3 \rangle$</th>
<th>$\langle 14 \rangle$</th>
<th>$\langle 12^2 \rangle$</th>
<th>$\langle 23 \rangle$</th>
<th>$\langle 5 \rangle$</th>
</tr>
</thead>
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<tr>
<td>$\eta_{(5)}$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\eta_{(14)}$</td>
<td>4</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$\eta_{(23)}$</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\eta_{(2^2 3)}$</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\eta_{(123)}$</td>
<td>5</td>
<td>2</td>
<td>3</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\eta_{(1^2 2^2)}$</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\eta_{(1^3 2)}$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>G&lt;sub&gt;76&lt;/sub&gt;</td>
<td>〈1&lt;sup&gt;6&lt;/sup&gt;〉</td>
<td>〈1&lt;sup&gt;4&lt;/sup&gt;2〉</td>
<td>〈1&lt;sup&gt;3&lt;/sup&gt;3〉</td>
<td>〈1&lt;sup&gt;2&lt;/sup&gt;4〉</td>
<td>〈1&lt;sup&gt;2&lt;/sup&gt;2&lt;sup&gt;2&lt;/sup&gt;〉</td>
<td>〈1&lt;sup&gt;2&lt;/sup&gt;3〉</td>
<td>〈15〉</td>
</tr>
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<td>-----------------</td>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>〈2&lt;sup&gt;5&lt;/sup&gt;〉</td>
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<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>〈2&lt;sup&gt;4&lt;/sup&gt;〉</td>
<td>5</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>〈2&lt;sup&gt;3&lt;/sup&gt;〉</td>
<td>9</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>〈2&lt;sup&gt;2&lt;/sup&gt;〉</td>
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<td>6</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>〈2&lt;sup&gt;1&lt;/sup&gt;〉</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>0</td>
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</tr>
<tr>
<td>〈2&lt;sup&gt;0&lt;/sup&gt;〉</td>
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<td>6</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>〈1&lt;sup&gt;9&lt;/sup&gt;〉</td>
<td>1</td>
<td>4</td>
<td>6</td>
<td>3</td>
<td>3</td>
<td>0</td>
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</tr>
</tbody>
</table>
APPENDIX 7.

LINEAR DEPENDENCE IN THE CHARACTER TABLE.

Let $B$ be a set of conjugacy class representative in $W$. Corollary 2.1.3 shows how the linear dependence of the columns of the character table of the generic Hecke algebra (see definition 2.1.1) can be described in terms of the elements $\sigma(w,b)$ of $K$. The values of $\sigma(w,b)$ are given below for the cases where $W$ is $S_1, S_2, S_3$ and $S_4$ (see theorem A2.1) thus the generic Hecke algebra is $G_1, \ldots, G_4$ respectively (see the introduction to Part II). In each case $B$ is chosen to be a set of class representative such that each element of it is of minimal length in its class. By corollary 4.2.10(ii) the values of $\sigma(w,b)$ obtained are independent of the particular choice of $B$.

The values are given in the form of tables, one for each $n$ ($n \in \mathbb{Z}_4$). The rows are indexed by the elements of $W$ and the columns are indexed by the elements of $B$. The $(w,b)\text{th}$ entry is $\sigma(w,b)$. (All omitted entries are zero).
| \( G_1 \) | 1 |  \\
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

| \( G_2 \) | 1 | (12) |  \\
|---|---|---|
| 1 | 1 |  \\
| (12) | 1 |  \\

| \( G_3 \) | 1 | (12) | (123) |  \\
|---|---|---|---|
| 1 | 1 |  \\
| (12) | 1 |  \\
| (23) | 1 |  \\
| (123) | 1 |  \\
| (132) | 1 |  \\
| (13) | \( u \) | \( u - 1 \) |  \\

95
<table>
<thead>
<tr>
<th>$G_4$</th>
<th>1</th>
<th>(12)</th>
<th>(123)</th>
<th>(1234)</th>
<th>(12X34)</th>
</tr>
</thead>
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<tr>
<td>1</td>
<td>1</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(12)</td>
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<tr>
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<td></td>
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<td></td>
</tr>
<tr>
<td>(34)</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(13)</td>
<td>$u$</td>
<td>$u-1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(24)</td>
<td>$u$</td>
<td>$u-1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(14)</td>
<td>$u^2$</td>
<td>$2u(u-1)$</td>
<td>$(u-1)^2$</td>
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</tr>
<tr>
<td>(123)</td>
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</tr>
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<td>(234)</td>
<td>1</td>
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<td></td>
</tr>
<tr>
<td>(143)</td>
<td>$u$</td>
<td>$u-1$</td>
<td></td>
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<td></td>
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<tr>
<td>(134)</td>
<td>$u$</td>
<td>$u-1$</td>
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<tr>
<td>(142)</td>
<td>$u$</td>
<td>$u-1$</td>
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<td></td>
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</tr>
<tr>
<td>(124)</td>
<td>$u$</td>
<td>$u-1$</td>
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<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1324)</td>
<td>$u(u-1)$</td>
<td>$u^2-u+1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(1423)</td>
<td>$u(u-1)$</td>
<td>$u^2-u+1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(12X34)</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(13X24)</td>
<td>$u-1$</td>
<td>$u$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(14X23)</td>
<td>$u(u-1)^2$</td>
<td>$(u-1)(u^2+1)$</td>
<td>$u^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX 8.

$V_n$ and $\otimes_n$

$V_n$ is defined by 5.2.2. $\otimes_n$ is defined at the beginning of 5.3. They are both matrices over $\mathbb{Q}[u]$ (see theorem 5.2.6 and lemma 5.3.8) and have the property that the $(i,j)$th entry of their product $\otimes_n V_n$ is $(\eta_i \sigma_j)$ (see lemma 5.3.3). $\eta_i$ is an irreducible character of $G_n$ (see the introduction to Part II) and $\sigma_j$ is a partition of $n$ (see corollary 4.2.10(i) and below definition 4.2.1). In fact for each $n \in \mathbb{Z}_6$ the matrix $\mathbb{I}_n = \{(\eta_i \sigma_j)_{ij}\}$ is the 'table' for $G_n$ given in appendix 6 after the correct rearrangement of rows and of columns (and interpreting the tables of appendix 6 as described there).

$V_2, V_3, V_4$ and $\otimes_2, \otimes_3, \otimes_4$ are given below and in each cases the order in which the partitions of $n$ have been used to index the rows and columns is stated.
\[ \mathbf{v}_2 = \begin{pmatrix} 2 & u-1 \\ 0 & 1 \end{pmatrix} \quad \mathbf{\Theta}_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} (u+1) \\ \frac{1}{2} & -\frac{1}{2} (u+1) \end{pmatrix} \]

Order of partitions: \( \langle 1^2 \rangle, \langle 2 \rangle. \)

\[ \mathbf{v}_3 = \begin{pmatrix} 6 & 3(u-1) & (u-1)^2 \\ 0 & 1 & (u-1) \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{\Theta}_3 = \begin{pmatrix} \frac{1}{6} & \frac{1}{2} (u+1) & \frac{1}{3} (u^2 + u+1) \\ \frac{1}{3} & 0 & -\frac{1}{3} (u^2 + u+1) \\ \frac{1}{6} & -\frac{1}{2} (u+1) & \frac{1}{3} (u^2 + u+1) \end{pmatrix} \]

Order of partitions: \( \langle 1^3 \rangle, \langle 12 \rangle, \langle 3 \rangle. \)

\[ \mathbf{v}_4 = \begin{pmatrix} 24 & 12(u-1) & b(u-1)^2 & 4(u-1)^2 & (u-1)^3 \\ 0 & 2 & 2(u-1) & 2(u-1) & (u-1)^2 \\ 0 & 0 & 2 & 0 & (u-1) \\ 0 & 0 & 0 & 1 & (u-1) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ \mathbf{\Theta}_4 = \begin{pmatrix} \frac{1}{24} & \frac{1}{4} (u+1) & \frac{1}{8} (u^2 + 2u+1) & \frac{1}{4} (u^3 + u+1) & \frac{1}{8} (u^3 + u^2 + u+1) \\ \frac{1}{8} & -\frac{1}{4} (u+1) & -\frac{1}{8} (u^2 + 2u+1) & 0 & -\frac{1}{4} (u^3 + u^2 + u+1) \\ \frac{1}{12} & 0 & \frac{1}{4} (u^2 + 2u+1) & -\frac{1}{3} (u^2 + u+1) & 0 \\ \frac{1}{8} & \frac{1}{4} (u+1) & -\frac{1}{8} (u^2 + 2u+1) & 0 & \frac{1}{4} (u^3 + u^2 + u+1) \\ \frac{1}{24} & -\frac{1}{4} (u+1) & \frac{1}{8} (u^2 + 2u+1) & \frac{1}{4} (u^3 + u^2 + u+1) & -\frac{1}{4} (u^3 + u^2 + u+1) \end{pmatrix} \]

Order of partitions:

\[ \langle 1^4 \rangle, \langle 1^2 2 \rangle, \langle 2^2 \rangle, \langle 1 3 \rangle, \langle 4 \rangle. \]