Hereditary properties of permutations are strongly testable

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Abstract

We show that for every hereditary permutation property $\mathcal{P}$ and every $\varepsilon_0 > 0$, there exists an integer M such that if a permutation π is ε_0-far from $\mathcal{P}$ in the Kendall's tau distance, then a random subpermutation of π of order M has the property $\mathcal{P}$ with probability at most ε_0. This settles an open problem whether hereditary permutation properties are strongly testable, i.e., testable with respect to the Kendall's tau distance, which is considered to be the edit distance for permutations. Our method also yields a proof of a conjecture of Hoppen, Kohayakawa, Moreira and Sampaio on the relation of the rectangular distance and the Kendall's tau distance of a permutation from a hereditary property.

1 Introduction

Property testing is a topic with growing importance with many connections to various areas of mathematics (e.g., see [14,15,42] for relation to graph limits) and computer science. A property tester is an algorithm that decides whether a large input object has the considered property by querying only a small sample of it. Since the tester is presented with a part of the input structure, it is necessary to allow an error based on the robustness of the tested property of the input. Following [29,31,34,43,45,46], we say that a property $\mathcal{P}$ of a class of structures (e.g., functions, graphs) is testable if for every ε, there exists a randomized algorithm $\mathcal{A}$ such that the number of queries made by $\mathcal{A}$ is bounded by a function of ε independent of the input and such that if the input has the property $\mathcal{P}$, then $\mathcal{A}$ accepts with probability at least $1-\varepsilon$, and if the input is ε-far from $\mathcal{P}$, then $\mathcal{A}$ rejects with probability at least $1-\varepsilon$. The exact notion depends on the studied class $\mathcal{C}$ of combinatorial structures, the considered properties $\mathcal{P}$ and the chosen metric on $\mathcal{C}$. There are also some variants of this notion. For example, one can allow only a one-sided error, i.e., $\mathcal{A}$ is required to accept whenever the input has the property $\mathcal{P}$, or the size of the sample may also depend (in

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a sublinear way) on the input size (for example as in testing monotonicity of functions [1, 24, 27, 30]).

A well-investigated area of property testing is testing properties of dense graphs, i.e., those with quadratically many edges. One of the most significant results in this area is that of Alon and Shapira [6] asserting that every hereditary graph property, i.e., a property preserved by taking induced subgraphs, is testable, which extends several earlier results [7, 31, 44]; a characterization of testable graph properties can be found in [3]. A logic perspective of graph property testing was addressed in [2, 26] and the connection to graph limits was explored in [42].

Testing properties of other objects have also been intensively studied. For example, results on testing properties of strings can be found in [4, 40], results related to constraint satisfaction problems in [5], and to more algebraically oriented properties in [9,11,12,46–48]. Besides the dense case, a property testing in sparse structures has also attracted substantial attention. The bounded degree graph case was introduced in [33]. Unlike in the dense case, not all hereditary properties are testable [13] though many properties can be tested [10, 19, 22, 32], also see surveys [21, 29]. In this work, we study testing properties of permutations in a property testing model analogous to the dense graph setting, and we fill a gap related to testing hereditary properties with respect to the counterpart of the edit distance.

To state our results, we need to introduce some terminology. A *permutation of order* N is a bijective mapping from $[N]$ to $[N]$ where $[N]$ denotes the set $\{1, \ldots, N\}$. The order of a permutation π is denoted by $|\pi|$. If π is a permutation of order N and $X \subseteq [N]$, then the *subpermutation* of π *induced* by X, denoted by $\pi \restriction X$, is the permutation π' of order $|X|$ such that $\pi'(i) < \pi'(j)$ if and only if $\pi(x_i) < \pi(x_j)$ for all $i, j \in [|X|]$ where $X = \{x_1, \ldots, x_{|X|}\}$ and $x_1 < \cdots < x_{|X|}$. We note that subpermutations are often referred to as *patterns*, in particular, in extremal combinatorics. However, we follow the terminology from previous papers related to testing permutation properties and to permutation limits, which also makes the terminology closer to the case of graphs.

A *permutation property* $\mathcal{P}$ is a set of permutations. If $\pi \in \mathcal{P}$, we say that a permutation π has the property $\mathcal{P}$. Since we are interested only in permutation properties, we often refer to permutation properties just as properties. A property $\mathcal{P}$ is *hereditary* if it is closed under taking subpermutations, i.e., if $\pi \in \mathcal{P}$, then any subpermutation of π is in $\mathcal{P}$. An example of a hereditary property is the set of all permutations not containing a fixed permutation as a subpermutation.

There are several notions of distances between permutations, see [23]. The rectangular distance and the Kendall's tau distance will be of most interest to us. Let π and σ be two permutations of the same order N. The *rectangular distance* of π and σ, which is denoted by $\mathrm{dist}_\square(\pi, \sigma)$, is defined as

$$\max_{S,T} \frac{\left| |\pi(S) \cap T| - |\sigma(S) \cap T| \right|}{N}$$

where the maximum is taken over all subintervals S and T of $[N]$. The *Kendall's* *tau distance* $\mathrm{dist}_K(\pi, \sigma)$ is defined as

$$\frac{|\{(i,j) \mid \pi(i) < \pi(j), \sigma(i) > \sigma(j), i,j \in [N]\}|}{\binom{N}{2}} .$$

Alternatively, the Kendall's tau distance of two permutations is the minimum number of swaps of consecutive elements transforming π to σ normalized by $\binom{N}{2}$.

It can be shown that if two permutations are close in the Kendall's tau distance, then they are close in the rectangular distance. The converse is not true: the rectangular distance of two random permutation is concentrated around 0 but their Kendall's tau distance is concentrated around 1/2. So, testing permutation properties with respect to the Kendall's tau distance is more difficult than with respect to the rectangular distance (at least in the sense that every tester designed for testing with respect to the Kendall's tau distance also works for testing with respect to the rectangular distance but not vice versa in general). The Kendall's tau distance is considered to correspond to the edit distance of graphs which appears in the hereditary graph property testing, while the rectangular distance is considered to correspond to the cut norm appearing in the theory of graphs limits, see [41]. The latter is demonstrated in the notion of regularity decompositions of permutations developed by Cooper [17, 18] and permutation limits introduced by Hoppen et al. [35,36] (also see [17,39] for relation to quasirandom permutations).

If $\mathcal{P}$ is a property, the rectangular distance of a permutation π from $\mathcal{P}$ is

$$\mathrm{dist}_\square(\pi, \mathcal{P}) = \min_{\sigma \in \mathcal{P}, |\pi| = |\sigma|} \mathrm{dist}_\square(\pi, \sigma)$$

and the Kendall's tau distance is

$$\mathrm{dist}_K(\pi, \mathcal{P}) = \min_{\sigma \in \mathcal{P}, |\pi| = |\sigma|} \mathrm{dist}_K(\pi, \sigma) .$$

In case that $\mathcal{P}$ contains no permutation of order $|\pi|$, we define the distance of π and $\mathcal{P}$ to be ∞. One of our results asserts that if a permutation is close to a *hereditary* property $\mathcal{P}$ in the rectangular distance, it is also close in the Kendall's tau distance.

One of the results in [25] can be reformulated as a tester for monotonicity of a permutation with $O(\log n/\varepsilon)$ queries. The distance considered in [25] is the minimum number of insertions and deletions to transform one permutation to another, which is incomporable with Kendall's tau distance. It is shown in [25] that a logarithmic number of queries is needed. In particular, the hereditary properties of permutations cannot be tested with constant sample size with respect to the distance considered in [25].

Testing hereditary permutation properties in full generality was addressed by Hoppen, Kohayakawa, Moreira and Sampaio [37]. They considered testing properties *through subpermutations* where the tester is presented with a random subpermutation of the input permutation (the size of the subpermutation depends on the tested property and the required error). In particular, if an input permutation π has order N, then a random subset $X \subseteq [N]$ is chosen and the tester is presented with $\pi \restriction X$. The main result of [37] is the following.

Theorem 1. *Let $\mathcal{P}$ be a hereditary property. For any real $\varepsilon > 0$, there exists M such that every permutation π of order at least M with $\mathrm{dist}_{\square}(\pi, \mathcal{P}) > \varepsilon$ satisfies that a random subpermutation of π of order M has the property $\mathcal{P}$ with probability at most ε.*

Theorem 1 implies that hereditary properties are testable through subpermutations with respect to the rectangular distance with one-sided error: the tester accepts if the random subpermutation has the property $\mathcal{P}$ and thus the tester always accepts permutations having the property $\mathcal{P}$.

Kohayakawa [38] asked whether hereditary properties of permutations are also testable through subpermutations with respect to the Kendall's tau distance, which he refers to as *strong testability*. We resolve this problem in the positive way. We prove an analogue of Theorem 1 with the rectangular distance replaced with the Kendall's tau distance (Theorem 6). Hence, we establish that hereditary properties are testable through subpermutations with respect to the Kendall's tau distance with one-sided error. This result was also announced in [8] but the authors informed us that their argument is incorrect. Since the Kendall's tau distance is the counterpart of the edit distance for graphs, our result was proposed in [37] as a possible permutation analogue of the result of Alon and Shapira [6]. It is also worth noting that our arguments are not based on regularity decompositions or on the analysis of limit structures.

Hoppen et al. [37] observed that the strong testability of hereditary properties through subpermutations would be implied by the following statement.

Conjecture 1. *Let $\mathcal{P}$ be a hereditary property. For every positive real ε_0, there exists δ_0 such that any permutation π satisfying $\mathrm{dist}_{\square}(\pi, \mathcal{P}) < \delta_0$ also satisfies $\mathrm{dist}_K(\pi, \mathcal{P}) < \varepsilon_0$.*

The conjecture is an analogue of the known relation between the rectangular distance and the edit distance to hereditary graph properties from [42]. Our method actually gives the proof of this conjecture which we state as Theorem 7. However, we included the proof of Theorem 6 instead of just stating that it can be derived from Theorem 7 for completeness.

1.1 Overview of our results

Here, we briefly summarize our contribution.

- We establish that hereditary permutation properties are strongly testable (Theorem 6), i.e., testable with respect to the Kendall's tau distance.
- We prove Conjecture 1 asserting that every permutation close to a hereditary property in the rectangular distance is also close in the Kendall's tau distance.

- Our methods are purely combinatorial and they do not rely on the limit approach or the regularity decompositions.

To demonstrate the importance of the last aspect of our proof, let us mention a recent breakthrough on the graph removal lemma by Fox [28] and its induced variant by Conlon and Fox [16]. In the case of the non-induced graph removal lemma, the direct combinatorial proof of Fox yields a much better dependence on the parameters than the standard proof using Szemerédi Regularity Lemma. In particular, the number of parts in Szemerédi Regularity Lemma for graphs (as well as Cooper's regularity lemma for permutations from [17]) is a tower function with height $O(\varepsilon^{-5})$. However, Fox's proof yields a bound for the removal lemma where the height is reduced to $O(\log \varepsilon^{-1})$. In the induced case, the proof of Conlon and Fox yields a bound where the height of the tower function is polynomial in ε^{-1}, improving on the previous wowzer type bound following from the strong regularity lemma of Alon et al. [3]. However, it is not clear to what extent these improvements can directly translate to permutations.

In our setting, consider $\mathcal{P}$ to be the set of permutations avoiding a fixed permutation π as a subpermutation, which is analogous to avoiding a fixed graph as a subgraph in the graph removal lemma. It can be shown that the depth d of a k-branching for $\mathcal{P}$ (as used in the proof of Lemma 4) is at most $(k|\pi|)^k$ and the weight w of its root is at most $(k|\pi|)^d$. Plugging these bounds into our arguments, we obtain (assuming π is fixed) that the size M_0 of the sample from Theorem 6 is only double exponential in $\varepsilon_0^{-O(1)}$ where ε_0 is the error parameter. Having seen this slow dependence on ε_0, it is natural to ask whether our approach can also be applied in the case of graphs but this seems not to be the case (at least in a straightforward way).

1.2 Overview of the proof

The tester for a property $\mathcal{P}$ is presented with a random subpermutation (sample) of the input permutation π (the sample size depends on $\mathcal{P}$ and the required precision ε) and it accepts iff the sample has the property $\mathcal{P}$. So, the proof lies in showing that if the input permutation π is ε-far from $\mathcal{P}$, then a random sample of sufficiently large size is in $\mathcal{P}$ with probability at most ε.

To prove the statement above, we view the input permutation as a bijection from $[N]$ to $[N]$ and divide the domain into K almost equal parts $X_1, \dots, X_K$ and the range into k almost equal parts $Y_1, \dots, Y_k$. In this way, we form $K \times k$ boxes and we think of the box corresponding to X_i and Y_j as "dense" if the number of $x \in X_i$ with $\pi(x) \in Y_j$ exceeds $\varepsilon'|X_i|$. The particular choice of k, K and ε' depends on $\mathcal{P}$ and ε; the choice will satisfy that for every i there will be at least one j such that the box corresponding to X_i and Y_j is dense.

The proof uses a notion of M-sequences, which are sequences of non-empty subsets of $[M]$. In the proof, we use k-sequences and K-sequences which decompose rows and columns of the $K \times k$ box grid, respectively. We define notions of being approximate and witnessing for pairs of such sequences. Let us describe these two notions informally. A K-sequence $B_1, \dots, B_\ell$ is approximate with respect to a k-sequence $A_1, \ldots, A_\ell$ if for every $i \in B_m$, the decomposed permutation is dense in the i-th column only in rows with indices in A_m. This notion is used to restrict the boxes where the decomposed permutation has non-trivial density. A K-sequence $B_1, \ldots, B_\ell$ is witnessing with respect to a k-sequence $A_1, \ldots, A_\ell$ if for every $i \in B_m$ and $j \in A_m$, the box in the i-th column and in the j-th row contains enough points of the permutation to deduce that a random subpermutation does not belong to $\mathcal{P}$ with probability at least $1 - \varepsilon$.

The proof strategy is the following. During the proof, we keep a K-sequence approximate with respect to a k-sequence. The proof starts with the K-sequence $[K]$ and the k-sequence $[k]$, i.e., the input permutation π can be dense in any of the boxes. At each step of the proof, one of the following three cases happens. If the K-sequence is witnessing, then we conclude that the tester rejects with probability at least $1 - \varepsilon$. If every large permutation approximated by the K-sequence is in $\mathcal{P}$, then the distance of π from $\mathcal{P}$ is at most ε. If neither of the two cases applies, we construct a new k-sequence, which we call a reduction of the original k-sequence, and a new K-sequence such that the new K-sequence is approximate with respect to the new k-sequence. Since we argue that there is no infinite sequence of k-sequences each being a reduction of the previous one, the proof eventually stops with one of the first two outcomes.

A natural question is whether the regularity lemma for permutations from [18] can lead to significant simplifications of the proof. The lemma guarantees the existence of decomposition to boxes such that almost every dense box is almost uniformly dense. The uniform density of boxes would lead to removing the middle part of the condition in the definition of witnessing K-sequences (as this would follow from the uniform density of dense boxes). This would simplify some parts of the proof but the main challenge lying in relating the Kendall's tau distance to the decomposition would remain.

2 Notation

We present now notation used in the paper. Let us start with a formal definition of k-sequences. If k is an integer, then a *k-sequence* A for an integer k is a sequence $A_1, \ldots, A_\ell$ of non-empty subsets of $[k]$. We refer to ℓ as the *length* of A and we write $|A|$ for the length of A. The *basic* k-sequence is the k-sequence of length one comprised of the set $[k]$. A k-sequence A is *simple* if each A_i has size one. Finally, a k-sequence A is *monotone* if every pair $x \in A_i$ and $x' \in A_{i'}$ with $1 \le i < i' \le |A|$ satisfies that $x < x'$.

Before we proceed further, we have to introduce some auxiliary notation. If A is a k-sequence, then we write $|A|_i$ for the sum $|A_1| + \cdots + |A_i|$. For completeness, we define $|A|_0 = 0$. If a and b are integers, then $a \bmod b$ is equal to the integer $x \in [b]$ with the same remainder as a after division by b.

Fix a k-sequence A. Let $A_i = \{x^i_1, \ldots, x^i_{|A_i|}\}$ where $x^i_1 < \cdots < x^i_{|A_i|}$. For an integer m, we define a function $g^{A,m} : [m \cdot |A|_{|A|}] \to [k]$ as

$$g^{A,m}(j) := x^i_{(j - m \cdot |A|_{i-1})) \bmod |A_i|}$$

where i is the largest integer such such that $m \cdot |A|_{i-1} < j$. For example, if $A = \{1,2,3\},\{1,4\},\{3\}$, then

$$g^{A,4}(1),\ldots,g^{A,4}(24) =$$

$$1,2,3,1,2,3,1,2,3,1,2,3,1,4,1,4,1,4,1,4,3,3,3,3\,.$$

Note that the sequence $g^{A,m}(1)g^{A,m}(2)\ldots g^{A,m}(m \cdot |A|_{|A|})$ has $|A|$ blocks such that the i-th block consists of m parts each containing the elements of A_i in the increasing order.

A permutation π is an *m-expansion* of a k-sequence A if the following holds:

- the order of π is $m \cdot |A|_{|A|}$, and
- if $g^{A,m}(j) < g^{A,m}(j')$ for $j,j' \in \{1,\ldots,m|A|_{|A|}\}$, then $\pi(j) < \pi(j')$.

For example, one of 3-expansions of the 2-sequence $\{1,2\},\{1\}$. is the permutation $4,8,2,7,3,9,1,6,5$. In other words, if a permutation π is an m-expansion of A, then the range of π can be viewed as partitioned into k parts such the following holds: the permutation π consists of $|A|$ groups (in the example, these are $4,8,2,7,3,9$ and $1,6,5$) where the i-th group has m blocks of length $|A_i|$ each and the values of π in each block belong to the parts of the range of π with indices in A_i in the increasing order. The number of m-expansions of a k-sequence A is equal to

$$\prod_{j=1}^{k} (m \cdot |\{i \text{ such that } i \in [|A|] \text{ and } j \in A_i\}|)!\,.$$

Let $\mathcal{P}$ be a hereditary property. A k-sequence A is *$\mathcal{P}$-good* if there exists an m-expansion of A in $\mathcal{P}$ for every integer m. Otherwise, the k-sequence A is *$\mathcal{P}$-bad*. So, if A is $\mathcal{P}$-bad, there exists an integer m such that no m-expansion of A is in $\mathcal{P}$. The smallest such integer m is called the *$\mathcal{P}$-order* of A and it is denoted by $\langle A\rangle_{\mathcal{P}}$; if $\mathcal{P}$ is clear from the context, we just write $\langle A\rangle$. Observe that if A is $\mathcal{P}$-bad, then no m-expansion of A is in $\mathcal{P}$ for every $m \ge \langle A\rangle$ (here, we use that $\mathcal{P}$ is hereditary).

If A is a $\mathcal{P}$-bad k-sequence, then any k-sequence A' obtained from A by replacing one element, say A_i, by a sequence of at least one and at most $|A_i|\langle A\rangle$ proper subsets of A_i is called a *$\mathcal{P}$-reduction* of A. For example, if the 3-sequence $A = \{1\},\{2,3\},\{1,3\}$ is $\mathcal{P}$-bad and its $\mathcal{P}$-order is two, then one of its $\mathcal{P}$-reductions is $\{1\},\{2\},\{2\},\{3\},\{1,3\}$.

The *k-branching* of a hereditary property $\mathcal{P}$ is a rooted tree $\mathcal{T}$ such that

- each node u of $\mathcal{T}$ is associated with a k-sequence A^u,
- the root of $\mathcal{T}$ is associated with the basic k-sequence,
- if the k-sequence A^u of a node u is $\mathcal{P}$-good or simple, then u is a leaf, and

- if the k-sequence A^u of a node u is $\mathcal{P}$-bad and it is not simple, then the number of children of u is equal to the number of $\mathcal{P}$-reductions of A and the children of u are associated with the $\mathcal{P}$-reductions.

Note that the k-branching, i.e., the tree and the association of its nodes with k-sequences, is uniquely determined by the property $\mathcal{P}$ and the integer k.

Let us argue that the k-branching of every hereditary property $\mathcal{P}$ is *finite*. We define the *score* of a k-sequence A to be the sequence $m_1, \ldots, m_k$ where m_i is the number of A_i's of cardinality $k+1-i$. Observe that the score of a $\mathcal{P}$-reduction of a $\mathcal{P}$-bad k-sequence A is always lexicographically smaller than that of A. Since the lexicographic ordering on the scores is a well-ordering, the k-branching is finite for every hereditary property $\mathcal{P}$.

Let $\mathcal{T}$ be the k-branching of a hereditary property $\mathcal{P}$. We now assign to every node u of the k-branching of $\mathcal{P}$ an integer weight w_u. The weight of a leaf node u is one if A^u is $\mathcal{P}$-good. Otherwise, the weight of a leaf node u is $k\langle A^u\rangle$. If u is an internal node, then w_u is equal to $\langle A^u\rangle km$ where m is the maximum weight of a child of u. In particular, the weight of u is at least the weight of any of its children.

3 Decompositions

In this section, we introduce a grid-like way of decomposing permutations which we use in our proof. The domain of a permutation will be split into K equal size parts and the range into k such parts with $k \leq K$.

We start with some auxiliary notation. Recall that $[a]$ denotes all integers from 1 to a. We extend this notation by writing $[a]_{i/b}$ for the set of all integers $k \in [a]$ such that $i-1 < k/\lfloor a/b\rfloor \leq i$, i.e., $[a]_{i/b}$ is the i-th part after dividing $[a]$ into b equal-size parts (with $b+1$-st part containing the remaining elements). For example, $[25]_{2/6} = \{5,6,7,8\}$. Observe that $|[a]_{1/b}| = \cdots = |[a]_{b/b}| = \lfloor a/b\rfloor$ and $|[a]_{b+1/b}| \leq b-1$.

Fix now a permutation π of order N and integers $K \in [N]$, $i \in [K]$, $k \in [K]$ and $j \in [k]$. We define $R_{i,j}(\pi)$ as

$$R_{i,j}(\pi) := \{x \in [N]_{i/K} \text{ such that } \pi(x) \in [N]_{j/k}\}$$

and we set

$$\rho_{i,j}(\pi) := \frac{|R_{i,j}(\pi)|}{\lfloor N/K\rfloor} \; .$$

Vaguely speaking, $\rho_{i,j}(\pi) \in [0,1]$ is the density of π in the part of the $K \times k$ grid at the coordinates (i,j). The values of K and k will always be clear from the context.

To get used to the definition of the sets $R_{i,j}$ and the quantities $\rho_{i,j}$, we now prove a simple auxiliary lemma.

Lemma 2. *Let k and K be positive integers and let $\varepsilon' \leq 1/(k+1)$ be a positive real. For every permutation π of order at least $k(k+1)K$ and every $x \in [K]$, there exists $y \in [k]$ such that $\rho_{x,y}(\pi) \geq \varepsilon'$.*

Proof. Observe that

$$|R_{x,1}(\pi)| + \cdots + |R_{x,k}(\pi)| \geq \lfloor |\pi|/K \rfloor - k$$
$$\geq \left(1 - \frac{1}{k+1}\right) \lfloor |\pi|/K \rfloor \ .$$

Since $\varepsilon' \leq 1/(k+1)$, there must exist y such that $\rho_{x,y}(\pi) \geq \varepsilon'$ by the pigeonhole principle. □

Fix a permutation π, integers k, K and M such that $1 \leq k \leq K \leq |\pi|$, and a real $0 \leq \varepsilon' < 1$. If A is a k-sequence, then we say that a K-sequence B is (A, M, ε')-*approximate* for π if the following holds:

- the length of B is $|A|$,
- B is monotone,
- $|B|_{|B|} = \sum_{i=1}^{|B|} |B_i| \geq K - M$, and
- for every $i \in [|A|]$, if $x \in B_i$ and $y \in [k] \setminus A_i$, then $\rho_{x,y}(\pi) < \varepsilon'$.

In other words, an (A, M, ε')-approximate K-sequence B decomposes the whole index set $[K]$ except for at most M indices into $|A|$ parts such that the indices contained in the parts determined by B are in the increasing order and for $x \in B_i$, the only dense sets $R_{x,y}(\pi)$ are those with $y \in A_i$.

Suppose that a k-sequence A is $\mathcal{P}$-bad for a hereditary property $\mathcal{P}$. We say that a K-sequence B is (A, ε')-*witnessing* for π if the following holds:

- the length of B is $|A|$,
- there exist integers $1 \leq x_1 < \ldots < x_{|A|_{|A|} \cdot \langle A \rangle} \leq K$ such that $x_j \in B_i$ if $|A|_{i-1}\langle A \rangle < j \leq |A|_i \langle A \rangle$, and
- $\rho_{x_j, g^{A,\langle A \rangle}(j)}(\pi) \geq \varepsilon'$ for every $j \in [|A|_{|A|} \cdot \langle A \rangle]$ (the definition of the function g can be found in Section 2).

In other words, a K-sequence B which decomposes the index set $[K]$ is (A, ε')-witnessing, if it is possible to find indices such that there are $|A_i|\langle A \rangle$ indices x_j in each B_i and all the sets $R_{x_j, g^{A,\langle A \rangle}(j)}(\pi)$ are dense. The motivation for this definition is the following: if B is (A, ε')-*witnessing*, then each set $R_{x_j, g^{A,\langle A \rangle}(j)}(\pi)$ has at least $\varepsilon' \lfloor |\pi|/K \rfloor$ elements and consequently at least $(\varepsilon' \lfloor |\pi|/K \rfloor)^{|A|\langle A \rangle}$ subsets of $[|\pi|]$ induce subpermutations that are $\langle A \rangle$-expansions of A. This will allow us to deduce that a random subpermutation of sufficiently large order does not have the property $\mathcal{P}$ with high probability.

We now state a lemma saying that if a K-sequence B is approximate but not witnessing with respect to a k-sequence A for a permutation π, then there exists a reduction A' of A and a K-sequence B' such that B' is approximate with respect to A'.

Lemma 3. *Let $\mathcal{P}$ be a hereditary property, let k, K, m and M be positive integers and let $\varepsilon' \leq 1/(k+1)$ be a positive real. Suppose that a $\mathcal{P}$-bad k-sequence A and a monotone K-sequence B with $|A| = |B|$. If the K-sequence B is (A, M, ε')-approximate for a permutation π, $|\pi| \geq k(k+1)K$, B is not (A, ε')-witnessing for π and $|B_i| \geq mk\langle A\rangle$ for every $i \in [|B|]$, then there exist a $\mathcal{P}$-reduction A' of A and a monotone K-sequence B' such that*

- *the lengths of A' and B' are the same,*
- *B' is $(A', M + mk\langle A\rangle, \varepsilon')$-approximate for π, and*
- *$|B'_i| \geq m$ for every $i \in [|B'|]$.*

Proof. If B is not (A, ε')-witnessing for π, then there exists an index $j \in [|B|]$ such that there is no $|A_j|\langle A\rangle$-tuple $x_1 < \cdots < x_{|A_j|\langle A\rangle}$ in B_j satisfying $\rho_{x_i,y_i}(\pi) \geq \varepsilon'$ where $y_i = g^{A,\langle A\rangle}(|A|_{j-1}\langle A\rangle + i)$. Fix such an index j for the rest of the proof.

If $|A_j| = 1$, then an $\langle A\rangle$-tuple with the properties given in the previous paragraph is formed by any $\langle A\rangle$ elements of B_j by Lemma 2. So, we assume that $|A_j| \geq 2$ in the rest of the proof. Define x_1 to be the smallest index in B_j such that $\rho_{x_1,y_1)}(\pi) \geq \varepsilon'$. Suppose that we have defined the indices $x_1, \ldots, x_i$ and define x_{i+1} to be the smallest index in B_j that is larger than x_i such that $\rho_{x_{i+1},y_{i+1}}(\pi) \geq \varepsilon'$. If no such index exists, we stop constructing the sequence. Let ℓ be the number of the indices defined. By the choice of j, $\ell < |A_j|\langle A\rangle$. For completeness, set $x_0 = 0$ and $x_{\ell+1} = K + 1$.

Define C_i, $i \in [\ell + 1]$, to be the set of the elements of B_j strictly between x_{i-1} and x_i. If the subset C_i has size less than m, remove it from the sequence and let $C'_1, \ldots, C'_{\ell'}$ be the resulting sequence. Observe that

$$\begin{array}{rcl} |B_j| - \sum_{i=1}^{\ell'} |C'_i| & \leq & \ell + (\ell+1)(m-1) \\ & \leq & (\ell+1)m - 1 \\ & \leq & m|A_j|\langle A\rangle - 1 \\ & \leq & mk\langle A\rangle - 1 \end{array} \qquad (1)$$

since the sets $C'_1, \ldots, C'_{\ell'}$ contain all the elements of B_j except for the elements $x_1, \ldots, x_\ell$ and the elements contained in the sets $C_1, \ldots, C_{\ell+1}$ with cardinalities at most $m - 1$. In particular, we can infer from $|B_j| \geq mk\langle A\rangle$ that $\ell' \geq 1$.

Next, define C''_i, $i \in [\ell']$, to be the set of $y \in [k]$ such that there exists $x \in C'_i$ with $\rho_{x,y}(\pi) \geq \varepsilon'$. Lemma 2 implies that the sets $C''_1, \ldots, C''_{\ell'}$ are non-empty. We infer from the way we have chosen the indices $x_1, \ldots, x_\ell$ that each set C''_i is a proper subset of A_j. Finally, define the k-sequence A' to be the K-sequence A with A_j replaced with $C''_1, \ldots, C''_{\ell'}$ and the K-sequence B' to be the K-sequence B with B_j replaced with $C'_1, \ldots, C'_{\ell'}$. By the definition of $C''_1, \ldots, C''_{\ell'}$ and by (1), the K-sequence B' is $(A', M + mk\langle A\rangle, \varepsilon')$-approximate for π. By the choice of $C'_1, \ldots, C'_{\ell'}$, we have that $|B'_i| \geq m$ for every $i \in [|B'|]$. Finally, since $\ell' \leq \ell \leq |A_j|\langle A\rangle$ and every C'_i, $i \in [\ell']$, is a proper subset of B_j, A' is $\mathcal{P}$-reduction of A. ☐

We finish this section with the following lemma on approximating the structure of a sufficiently large permutation π with respect to a hereditary property.

Lemma 4. *Suppose $\mathcal{P}$ is a hereditary property. For all integers k and reals ε and ε', $0 < \varepsilon \leq 1$ and $0 < \varepsilon' \leq 1/(k+1)$, there exists an integer K such that for every permutation π of order at least $k(k+1)K$, there exist a k-sequence A and a K-sequence B with the same lengths such that*

- *A is $\mathcal{P}$-bad and B is (A, ε')-witnessing for π, or*
- *A is $\mathcal{P}$-good and B is $(A, \lfloor \varepsilon K \rfloor, \varepsilon')$-approximate for π.*

Proof. Let $\mathcal{T}$ be the k-branching with respect to $\mathcal{P}$. Let d be the depth of $\mathcal{T}$, i.e., the maximum number of vertices on a path from the root to a leaf, and let w_0 be the weight of the root of $\mathcal{T}$. We show that $K := \lceil dw_0/\varepsilon \rceil$ has the properties claimed in the statement of the lemma.

Let π be a permutation of order at least $k(k+1)K$. Based on π, we define a path from the root to one of the nodes in $\mathcal{T}$ in a recursive way. In addition to choosing the nodes u^i on the path, we also define monotone K-sequences B^i such that B^i is $(A^{u^i}, i \cdot w_0, \varepsilon')$-approximate for π and $|B^i_j| \geq w_{u^i}$ for every $j \in [|B^i|]$.

Let u_0 be the root of $\mathcal{T}$ and set B^0 to be the basic K-sequence. Clearly, B^0 is $(A^{u^0}, 0, \varepsilon')$-approximate for π. Suppose that the node u^i on the path has already been chosen and we now want to choose the next node. If u^i is a leaf node, we stop. If u^i is not a leaf node, then the k-sequence A^{u^i} must be $\mathcal{P}$-bad. If B^i is (A^{u^i}, ε')-witnessing for π, we also stop. Otherwise, Lemma 3 applied with m equal to the maximum weight of a child of u^i (note that $|B^i_j| \geq mk\langle A^{u^i}\rangle$ for every $j \in [|B^i|]$) implies that there exist a $\mathcal{P}$-reduction A' of A^{u^i} and a K-sequence B^{i+1} such that B^{i+1} is $(A', i \cdot w_0 + mk\langle A^{u^i}\rangle, \varepsilon')$-approximate for π and $|B^{i+1}_j| \geq m$ for every $j \in [|B^{i+1}|]$. Choose u^{i+1} to be the child of u^i such that $A^{u^{i+1}} = A'$. Since $mk\langle A^{u^i}\rangle \leq w_0$, we obtain that B^{i+1} is $(A^{u^{i+1}}, (i+1)w_0, \varepsilon')$-approximate for π.

Let ℓ be the length of the constructed path. We claim that the k-sequence A^{u^ℓ} and the K-sequence B^ℓ have the properties described in the statement of the lemma.

If u^ℓ is not a leaf node, then A^{u^ℓ} is $\mathcal{P}$-bad and B^ℓ is $(A^{u^\ell}, \varepsilon')$-witnessing for π (since we have stopped at u^ℓ). If u^ℓ is a leaf node and A^{u^ℓ} is $\mathcal{P}$-bad, then B^ℓ is $(A^{u^\ell}, \varepsilon')$-witnessing for π by Lemma 3 applied for $m = 1$ (A^{u^ℓ} cannot have a $\mathcal{P}$-reduction because it is simple). Finally, if u^ℓ is a leaf node and A^{u^ℓ} is $\mathcal{P}$-good, B^ℓ is $(A^{u^\ell}, \lfloor \varepsilon K \rfloor, \varepsilon')$-approximate for π since $dw_0 \leq \lfloor \varepsilon K \rfloor$. ☐

4 Testing

In this section, we establish our main result. The next lemma, which says that every permutation that is far from a hereditary property $\mathcal{P}$ in the Kendall's tau distance has a witnessing K-sequence for a suitable choice of k and K, is the core of our proof.

Lemma 5. *Let $\mathcal{P}$ be a hereditary property of permutations. For every real $\varepsilon_0 > 0$, there exist integers k, K and M, and a real $\varepsilon' > 0$ such if π is a permutation of order at least M with $\mathrm{dist}_K(\pi, \mathcal{P}) \geq \varepsilon_0$, then there exist a $\mathcal{P}$-bad k-sequence A and a K-sequence B with the same length such that B is (A, ε')-witnessing for π.*

Proof. Without loss of generality, we can assume that $\varepsilon_0 < 1$. Set $k = \lceil 10/\varepsilon_0 \rceil$, $\varepsilon = \varepsilon_0/10$ and $\varepsilon' = \varepsilon_0/(10k + 10) \leq 1/(k + 1)$. Let K be the integer from the statement of Lemma 4 applied for $\mathcal{P}$, k, ε and ε'. Using this value, set

$$M = \max\left\{ k(k+1)K,\ \left\lceil \frac{10k}{\varepsilon_0} \right\rceil,\ \left\lceil \frac{10K}{\varepsilon_0} \right\rceil \right\}.$$

We show that this choice of k, K, M and ε' satisfies the assertion of the lemma.

Let π be a permutation of order $N \geq M$. Apply Lemma 4 to π. Let A be the k-sequence and B the K-sequence as in the statement of the lemma. Either A is $\mathcal{P}$-bad and B is (A, ε')-witnessing for π, which is the conclusion of the lemma, or A is $\mathcal{P}$-good and B is $(A, \varepsilon K, \varepsilon')$-approximate for π. Hence, we assume the latter and deduce that $\mathrm{dist}_K(\pi, \mathcal{P}) < \varepsilon_0$.

To reach our goal, we define two auxiliary functions $f_B : [N] \to [|B|]$ and $f_A : [N] \to [k]$. Informally speaking, when searching for a permutation in $\mathcal{P}$ close to π, we consider an m-expansion of A for a very large integer m and we show that one of its subpermutations is close to π. As explained after the definition of an m-expansion, every m-expansion can be viewed as consisting of $|A| = |B|$ blocks where the i-th block has $m \cdot |A_i|$ elements. In the subpermutation we construct, we choose the element corresponding to $x \in [N]$ in the $f_B(x)$-th block of an m-expansion of A and the value of $g^{A,m}$ for this element will be the $f_A(x)$-th smallest element of $A_{f_B(x)}$.

Let us now proceed in a formal way. First, we define the function f_B. Let $x \in [N]$ and let i be the integer such that $x \in [N]_{i/K}$. Let j be the largest integer such that i is smaller than all the elements of B_j; if no such set exists, let $j = |B| + 1$. Set $f_B(x) := \max\{1, j - 1\}$. Clearly, f_B is non-decreasing and if $i \in B_j$, then $f_B(x) = j$ for every $x \in [N]_{i/K}$. We now proceed with defining the function f_A. If $i \in B_{f_B(x)}$, $\pi(x) \in [N]_{i'/k}$ such that $i' \in [k]$ and $\rho_{i,i'}(\pi) \geq \varepsilon'$, set $f_A(x) = i''$ where i'' is the number of elements of $A_{f_B(x)}$ smaller or equal to i'. Otherwise, set $f_A(x) = 1$.

Since A is $\mathcal{P}$-good, there exists an N-expansion σ of A that is in $\mathcal{P}$. Set

$$z_x := |A|_{f_B(x)-1} N + x|A_{f_B(x)}| + f_A(x) \quad \text{for } x \in [N].$$

Observe that $1 \leq z_1 < \cdots < z_N \leq N \cdot |A|_{|A|}$. In the rest of the proof, we establish that the subpermutation π' of σ induced by $\{z_1, \ldots, z_N\}$ satisfies $\mathrm{dist}_K(\pi, \pi') \leq \varepsilon_0$. Since $\mathcal{P}$ is hereditary and $\sigma \in \mathcal{P}$, this implies $\mathrm{dist}_K(\pi, \mathcal{P}) \leq \varepsilon_0$.

We now define five types of pairs (x, x'), $1 \leq x < x' \leq N$. Suppose that $x \in [N]_{i/K}$, $\pi(x) \in [N]_{j/k}$, $x' \in [N]_{i'/K}$ and $\pi(x') \in [N]_{j'/k}$.

- The pair (x, x') is of *Type I* if $i = K + 1$ or $i' = K + 1$.
- The pair (x, x') is of *Type II* if $j = k + 1$ or $j' = k + 1$.
- The pair (x, x') is of *Type III* if it is not of Type I and $i \notin B_{f_B(x)}$ or $i' \notin B_{f_B(x')}$.
- The pair (x, x') is of *Type IV* if it is neither of Type I nor of Type II, and $\rho_{i,j} < \varepsilon'$ or $\rho_{i',j'} < \varepsilon'$.
- The pair (x, x') is of *Type V* if it is not of Type II and $j = j'$.

We now estimate the number of pairs (x, x'), $1 \leq x < x' \leq N$, of each of the five types. The number of pairs of Type I is at most $K(N-1) \leq \varepsilon_0 N(N-1)/10$ since $|[N]_{K+1/K} \cap [N]| \leq K$. Similarly, the number of pairs of Type II is at most $k(N-1) \leq \varepsilon_0 N(N-1)/10$ since $|[N]_{k+1/k} \cap [N]| \leq k$. The number of pairs of Type III is at most $\varepsilon N(N-1) = \varepsilon_0 N(N-1)/10$ since $K - (|B_1| + \cdots + |B_{|B|}|) \leq \varepsilon K$.

For $i \in [K]$ and $j \in [k]$ with $\rho_{i,j}(\pi) < \varepsilon'$, the number of the choices of $x \in [N]_{i/K}$ with $\pi(x) \in [N]_{j/k}$ is at most $\varepsilon' N/K$. Hence, the number of x with this property for some i and j is at most $\varepsilon' kN < \varepsilon_0 N/10$. Consequently, the number of pairs of Type IV is strictly less than $\varepsilon_0 N(N-1)/10$. Finally, for x with $\pi(x) \in [N]_{j/k}$, the number of choices of $x' \neq x$ with $\pi(x') \in [N]_{j/k}$ is at most $N/k - 1$. Hence, the number of pairs of Type V is strictly less than $N(N/k - 1) \leq N(N-1)/k \leq \varepsilon_0 N(N-1)/10$.

We conclude that the number of pairs (x, x'), $1 \leq x < x' \leq N$, that are of at least of one of Types I–V is at most $\varepsilon_0 N(N-1)/2$.

We claim that if the pair (x, x'), $1 \leq x < x' \leq N$, is not of any of the Types I–V, then $\pi(x) < \pi(x')$ if and only if $\pi'(x) < \pi'(x')$. Let i, i', j and j' be chosen as in the previous paragraph. Suppose $\pi(x) < \pi(x')$. If (x, x') is not of any of the Types I–V, then it holds that $i \in B_{f_B(x)}$, $i' \in B_{f_B(x')}$, $j \neq j'$, $\rho_{i,j}(\pi) \geq \varepsilon'$ and $\rho_{i',j'}(\pi) \geq \varepsilon'$. This implies that the $f_A(x)$-th smallest element of $A_{f_B(x)}$ is smaller than the $f_A(x')$-th smallest element of $A_{f_B(x')}$. Consequently, $\pi'(x) < \pi'(x')$ by the choice of z_x and $z_{x'}$. Analogously, one can show that if $\pi(x) > \pi(x')$, then $\pi'(x) > \pi'(x')$.

Since the number of pairs (x, x'), $1 \leq x < x' \leq N$, of at least one of the five types is at most $\varepsilon_0 N(N-1)/2$, we get that $\mathrm{dist}_K(\pi, \pi') < \varepsilon_0$ as desired. $\square$

We are now ready to prove our main theorem. Note that Theorem 6 implies that hereditary properties of permutations are strongly testable through subpermutations: for $\varepsilon > 0$, the tester take a random subpermutation of order M_0 from the statement of Theorem 6 and it accepts if the random subpermutation has the tested property.

Theorem 6. *Let $\mathcal{P}$ be a hereditary property. For every positive real ε_0, there exists M_0 such that if π is a permutation of order at least M_0 with $\mathrm{dist}_K(\pi, \mathcal{P}) \geq \varepsilon_0$, then a random subpermutation π of order M_0 has the property $\mathcal{P}$ with probability at most ε_0.*

Proof. Without loss of generality, we assume that $\varepsilon_0 < 1$. Apply Lemma 5 to $\mathcal{P}$ and ε_0. Let k, K and M be the integers and let ε' be the real as in the statement of the lemma. Note that we can also assume that $\varepsilon' < 1$. Set M_0 as

$$M_0 = \max\left\{M, K(K+1), \frac{\log \frac{kK}{\varepsilon_0}}{\log \frac{K+1}{K+1-\varepsilon'}}\right\} .$$

Let π be a permutation of order $N \geq M_0$. Note that the probability that a random M_0-element subset X of $[N]$ contains no element of a set $R_{i,j}(\pi)$ with $\rho_{i,j}(\pi) \geq \varepsilon'$ is at most

$$\left(1 - \frac{|R_{i,j}(\pi)|}{N}\right)^{M_0} = \left(1 - \rho_{i,j}(\pi)\left\lfloor\frac{N}{K}\right\rfloor\frac{1}{N}\right)^{M_0}$$

$$\leq \left(1 - \frac{\varepsilon'}{K+1}\right)^{M_0} \leq \frac{\varepsilon_0}{kK} .$$

By the union bound, the probability that there exists $i \in [K]$ and $j \in [k]$ with $\rho_{i,j}(\pi) \geq \varepsilon'$ such that X contains no element from the set $R_{i,j}(\pi)$ is at most ε_0. This implies that with probability at least $1 - \varepsilon_0$ a random M_0-element subset X of $[M_0]$ contains at least one element from each set $R_{i,j}(\pi)$ with $\rho_{i,j}(\pi) \geq \varepsilon'$.

By Lemma 5, if $\mathrm{dist}_K(\pi, \mathcal{P}) \geq \varepsilon_0$, there exists a k-sequence A and a K-sequence B such that A is $\mathcal{P}$-bad and B is (A, ε')-witnessing for π. Since a random M_0-element subset of $[N]$ contains an element from each $R_{i,j}(\pi)$ with $\rho_{i,j}(\pi) \geq \varepsilon'$ with probability at least $1 - \varepsilon_0$, a random M_0-element subpermutation of π contains an $\langle A\rangle$-expansion of A as a subpermutation with probability at least $1 - \varepsilon_0$. Consequently, a random M_0-element subpermutation of π is not in $\mathcal{P}$ with probability at least $1 - \varepsilon_0$. ☐

We are also in a position to prove that, for hereditary properties $\mathcal{P}$, the function $\mathrm{dist}_K(\pi, \mathcal{P})$ is continuous with respect to the metric given by $\mathrm{dist}_\square$ in the sense considered in [37].

Theorem 7. *Let $\mathcal{P}$ be a hereditary property. For every $\varepsilon_0 > 0$, there exists $\delta_0 > 0$ such that any permutation π satisfying $\mathrm{dist}_\square(\pi, \mathcal{P}) < \delta_0$ also satisfies $\mathrm{dist}_K(\pi, \mathcal{P}) < \varepsilon_0$.*

Proof. Apply Lemma 5 to $\mathcal{P}$ and ε_0. Let k, K and M be the integers and let ε' be the real as in the statement of the lemma. Set M_0 to be the maximum of M and $K+1$, and set δ_0 to be the minimum of $1/M_0$ and $\frac{\varepsilon'}{4K}$.

Suppose that there exists a permutation $\sigma \in \mathcal{P}$ with $|\pi| = |\sigma|$ and $\mathrm{dist}_\square(\pi, \sigma) < \delta_0$. If the order of π is smaller than M_0, then π and σ must be the same which yields $\mathrm{dist}_\square(\pi, \mathcal{P}) = \mathrm{dist}_K(\pi, \mathcal{P}) = 0$. So, we can assume that the order of π is at least M_0.

Assume to contrary that $\mathrm{dist}_K(\pi, \mathcal{P}) \geq \varepsilon_0$. By Lemma 5, there exists a $\mathcal{P}$-bad k-sequence A and a K-sequence B such that B is (A, ε')-witnessing for π. By the choice of δ_0, B is $(A, \varepsilon'/2)$-witnessing for σ (recall that the order of π

is at least $K+1$). This yields that $R_{x_j,g^{A,\langle A\rangle}(j)}(\sigma) \neq \emptyset$ for every $j \in [|A|_\ell \cdot \langle A\rangle]$ where x_j are chosen as in the definition of $(A, \varepsilon'/2)$-witnessing. In particular, σ contains a subpermutation not in $\mathcal{P}$ (choose one element from each of the sets $R_{x_j,g^{A,\langle A\rangle}(j)}$ and consider the subpermutation induced by the chosen elements) which is impossible since $\mathcal{P}$ is hereditary. □

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