Safety Criteria for Aperiodic Dynamical Systems

Zahir Bishnani

Submitted for the degree of
PhD in Interdisciplinary Mathematics

Mathematics Institute,
University of Warwick

July 1997

Abstract

The use of dynamical system models is commonplace in many areas of science and engineering. One is often interested in whether the attracting solutions in these models are robust to perturbations of the equations of motion. This question is extremely important in situations where it is undesirable to have a large response to perturbations for reasons of safety. An especially interesting case occurs when the perturbations are aperiodic and their exact form is unknown. Unfortunately, there is a lack of theory in the literature that deals with this situation. It would be extremely useful to have a practical technique that provides an upper bound on the size of the response for an arbitrary perturbation of given size. Estimates of this form would allow the simple determination of safety criteria that guarantee the response falls within some pre-specified safety limits. An excellent area of application for this technique would be engineering systems. Here one is frequently faced with the problem of obtaining safety criteria for systems that in operational use are subject to unknown, aperiodic perturbations.

In this thesis I show that such safety criteria are easy to obtain by using the concept of *persistence of hyperbolicity*. This persistence result is well known in the theory of dynamical systems. The formulation I give is functional analytic in nature and this has the advantage that it is easy to generalise and is especially suited to the problem of unknown, aperiodic perturbations. The proof I give of the persistence theorem provides a technique for obtaining the safety estimates we want and the main part of this thesis is an investigation into how this can be practically done.

The usefulness of the technique is illustrated through two example systems, both of which are forced oscillators. Firstly, I consider the case where the unforced oscillator has an asymptotically stable equilibrium. A good application of this is the problem of ship stability. The model is called the *escape equation* and has been argued to capture the relevant dynamics of a ship at sea. The problem is to find practical criteria that guarantee the ship does not capsize or go through large motions when there are external influences like wind and waves. I show how to provide good criteria which ensure a safe response when the external forcing is an arbitrary, bounded function of time. I also consider in some detail the *phased-locked loop*. This is a periodically forced oscillator which has an attracting periodic solution that is synchronised (or phase-locked) with the external forcing. It is interesting to consider the effect of small aperiodic variations in the external forcing. For hyperbolic solutions I show that the phase-locking persists and I give a method by which one can find an upperbound on the maximum size of the response.

Contents

Acknowledgements

First and foremost I would like to thank my supervisor Robert MacKay. He introduced me to this fascinating area of research and has constantly been a source of enthusiasm, advice and brilliant ideas. He always made time for me and made sure my work progressed in the right directions. I am privileged to have had such an excellent supervisor.

I was fortunate enough to have spent a year at the Laboratoire de Topologie, Université de Bourgogne and this was an extremely enjoyable experience. I would like to thank the staff there for making me feel so welcome and the two guys I shared an office with, Daniel Panazzollo and Gregory Mendausse, for being great company throughout the year.

I would also like to thank the Department of Applied Mathematics and Theoretical Physics, University of Cambridge for hosting me during my final two years of research. While there, I benefitted enormously from discussions with my colleagues in the Nonlinear Centre. Jacques-Alexandre Sepulchre introduced me to some crucial mathematical techniques and made useful comments concerning the presentation of my thesis. Swan Kim was always enthusiastic about my work and gave me excellent advice on how to be a more effective researcher. Mark Johnston and Sinisa Slijepčević were always around to bounce ideas off and frequently returned them with added value. Allan McRobie from the Engineering Department also made useful comments about my work.

There are a few other people I would like to thank for helpful discussions. I benefitted from numerous conversations with Mark Muldoon whilst at Warwick. Also, Jack Hale and Steve Montgomery-Smith were extremely enthusiastic when I met them during the "Finite to Infinite Dimensional Dynamical Systems" programme at the Newton Institute.

My primary source of funding has been a studentship from EPSRC. Financial assistance has also been provided by awards from Trinity College and the ERASMUS scheme.

I am extremely grateful for the support of all my friends who have kept in contact with me during my research and provided me with an active life outside of mathematics. Without this I would surely have gone crazy. I especially appreciate the efforts of the following people: Andrew Finch, Jason Annadani, Javier Fuente, Catherine Gasson, Tom Ash, Katherine Pierce, Shoaib Alam, Alistair Caunt, Fiz, Sylvia, Jim, Rudeboy Randall, the 'Rider, and last but not least MC Zebra.

Finally, I am eternally grateful to my parents for being so supportive and encouraging and who have made it all worthwhile by being so proud of my academic achievements.

Declaration

All the work described in this thesis is believed to be original, except where explicit reference is made to other sources. This thesis is the result of my own work and includes nothing which is the outcome of work doen in collaboration. No part of this thesis has been, or is being, submitted for any degree other than that of Doctor of Philosophy at the University of Warwick.

Chapter 1

Introduction

1.1 Motivation

One of the most common and successful ways of modelling interesting phenomena in science is to use *dynamical systems*. There is a great history to this area of mathematics much of which originates from the use of *differential equations* in physics. Differential equations arise quite naturally, for example, as a result of applying Newton's laws of motion to physical bodies. For a long time there was an emphasis on searching for exact solutions to differential equations and although this produced a useful theory for certain classes of linear equations, it proved to be an unsuccessful approach for most nonlinear equations. Starting with Poincaré's discovery in the late nineteenth century of non-integrability in celestial mechanical models, it became clear that for typical nonlinear systems, including those with only a few degrees of freedom, exact solutions could not be found. Thus the emphasis shifted to the question of qualitative behaviour in dynamical systems. Poincaré himself developed much of this qualitative theory applying many more geometric ideas than previously used. This approach was continued by Birkhoff, Liapunov and many others in the first half of this century and from these important early works sprang an exciting branch of mathematics. In the Sixties many of the geometrical ideas of Poincaré and Birkhoff were formalised and developed further by, amongst others, Smale, Bowen and Arnold. I refer to this qualitative theory of nonlinear systems as traditional dynamical systems theory. The books by Guckenheimer & Holmes [14], Katok [20] and Wiggins [57], for example, are good introductory works. Since then nonlinear dynamics has exploded into a very significant area of research not only in mathematics but also in many other areas of science and engineering.

Reasons for this include the massive increase in the use of computers and numerical simulation techniques in the applications and also the popularisation of 'chaos theory' in recent years.

The usual starting point is a dynamical system coming from a *mapping* or from an *ordinary differential equation* (ODE). I concentrate on ODE's in this thesis as they are of greater interest in the applications. It is much more common to start with a continuous-time system when modelling real world phenomena. However, on some occasions a discrete-time model is actually more natural. For example, in many ecological situations, especially where breeding is seasonal, one need only consider the population of a given species once per generation. More examples can be found in economics, where decisions or transactions are often made at discrete times like weekly or quarterly. Mappings are also important because they can appear naturally in the analysis or the numerical simulation of continuous-time systems. Most of the work in this thesis has a straightforward extension to mappings and I give a brief sketch of how this works.

There is also great interest in the use of partial differential equations (PDE's) which are systems with more than one independent variable. However, the theory for PDE's is somewhat different in nature and does not usually come under the umbrella of dynamical systems theory. I concentrate on systems where time is the only independent variable although, in principle, some of the results obtained here could be generalised to PDE's. Also of interest in many applications are models with some stochastic element in the equations of motion. Again this theory is different in nature and I do not deal with it in this thesis.

The main issue I address is the *robustness* of dynamical behaviour in *aperiodic*, nonlinear ODE's. The principle motivation for treating this problem is the need for rigourous and effective safety criteria in physical systems where robustness to unknown perturbations is required. This is a largely undeveloped area of dynamical systems theory and there are two main reasons for this.

Firstly, the traditional theory of dynamical systems usually starts with autonomous systems, that is, systems with no explicit time-dependence in the equations of motion. Much of the theory that follows from this starting point can be extended to non-autonomous systems but there are some important differences especially when the system is aperiodic.

Secondly, even where there is a theory applicable to aperiodic systems it is usually of a qualitative nature and not suitable for obtaining quantitative information. For example, structural stability can be used to deduce the existence of topologi-

cally conjugate solutions in 'close' dynamical systems, but the standard proofs do not naturally give estimates of the degree of closeness required.

The approach I take addresses both of these problems. I have presented an extension of some the well developed theory of autonomous systems to the non-autonomous case, taking care to make as few assumptions as possible about the time-dependence. The main purpose in doing this is to enable one to provide useful quantitative information relevant to the problems of robustness and stability.

An example of the type of application I have in mind is the problem of ship stability which I treat in some detail in chapter 5. The usual dynamical model is a simple forced oscillator system known as the *escape equation* or *capsize* equation, The external forcing comes from wind and waves. Typically this external forcing is assumed to be periodic. In [51, 52, 28, 29] for example, Thompson and colleagues give some very interesting results regarding the dynamical behaviour of solutions under increasing amplitudes of sinusoidal excitation. It is fully recognised in these works that steady-state analysis and linear approximation techniques are insufficient in highly non-linear situations and thus cannot account for the real danger to stability facing a ship at sea. The conclusion is that one needs to account for dynamical behaviour under perturbation and protect against *transient capsize.* Thompson and colleagues have proposed new safety measures such as an *index of capsizability.* The methods used are geometrical or topological in nature and rely heavily on the fact that for periodic systems one can reduce to a Poincare mapping and then perform a good phase-space analysis based on numerical simulation. Doing this, one finds a variety of phenomena including saddle-nodes, period doubling cascades, indeterminate jumps, fractal basin boundaries and homoclinic tangles. The qualitative behaviour of such low-dimensional systems is of course of considerable mathematical interest.[i] However, the issue that is of fundamental importance in this situation is the practical determination of criteria which guarantee that the ship does not capsize or undergo large motions. When there is no external excitation, there is an asymptotically stable equilibrium where the ship is vertical. When there is forcing, we would like to know the form of the response and more importantly, an upper bound on it's size. In reality of course, the external forces affecting a ship at sea are aperiodic and largely unknown in form. It is not clear at this stage whether the bifurcation scenario when the forcing is 'nearly' periodic resembles closely the periodic case as detailed in [51, 52]. Johnson's paper [17], for example, suggests that it

[i] Indeed, many of the more exotic types of dynamics have been originally investigated in forced oscillator systems such as the van de Pol oscillator. See for example, [14].

does not although there is very little known in general about aperiodic bifurcations. More importantly for present purposes, there seems to be no theory so far in the literature which provides good upper bounds for the size of the response to a given size of forcing without overly restricting the type of forcing considered.

The ship stability problem is one which has had considerable attention in recent years. It is however, only one of the many dynamical systems used in engineering applications where robustness is an important issue. For example, McRobie [28] gives many more examples from marine technology. The area of electrical and electronic engineering is also rich with applications of ODE's. Two interesting ones in this area are the *swing equation* in power generation and *phase-locked loops* in analogue to digital conversion [21]. Of course, in addition to engineering, there are numerous other fields in which one finds questions of robustness and stability in relation to dynamical system models. For example, many biological systems are modelled by ODE's, and one is often concerned whether the behaviour of the models is robust to external perturbations. Murray [31] is a good introduction to this area.

In analysing the periodically forced, escape equation for ship stability there is the advantage that one is essentially dealing with a two dimensional system. Thus geometrical and topological methods are reasonably well suited. However, it is not clear how easily the techniques generalise to higher dimensional systems.

The general problem of robustness I am interested in can be stated as follows. Consider the first-order non-autonomous system

$$\dot{x} = f(x,t)$$

where $x \in \mathbb{R}^n$. Note that one can always write an nth-order system in this way so it is quite a general form. Suppose that $\dot{x} = f_0(x,t)$ is the unperturbed system and has an attracting solution $x_0(t)$, for example, an equilibrium or periodic orbit. If f is thought of as a perturbation of f_0, the problem is to determine the form of the response of the system and in particular whether it satisfies some pre-defined safety conditions, for example belonging to some pre-specified region of phase-space. When x_0 satisfies a condition called *uniform hyperbolicity*, it will be shown that, for $f - f_0$ small enough, there is a uniformly close attracting solution $x_f(t)$ of the perturbed system. One also has this persistence property for unstable uniformly hyperbolic solutions and in fact for uniformly hyperbolic invariant sets.

If we call $(f - f_0)$ the perturbation and $(x_f - x_0)$ the response then a useful safety estimate to look for is an upperbound on the size of the response, to an arbitrary perturbation of given size.

It is a standard approach to the problem of robustness to assume that the perturbation is periodic. For example, for external driving it is often assumed that excitation is sinusoidal. Doing this makes it relatively simple to determine the response to various amplitudes and frequencies of driving force. One obtains a frequency-response curve this way and this is an extremely important measure which is commonly found in the engineering literature.

There are a number of reasons for assuming periodicity.

- It is sometimes a realistic assumption that the system will be exposed to periodic or roughly periodic external perturbation.
- It is a simplifying procedure and the analysis can become more tractable. There is already a great deal of literature on the subject of periodic systems and the methods are well proven.
- It is relatively easy to do numerical simulations.
- In linear systems the optimal, 'resonant' driving forces are periodic.

However there are also a number of reasons why I believe this approach can be unsatisfactory.

- Safety criteria based on periodic perturbation are often used in cases when one really does expect perturbations to be aperiodic. These criteria cannot then be rigourous.
- In this thesis I present an analysis which is reasonably easy to perform and which does not require periodicity. This shows that assuming periodicity is not the only tractable approach.
- In numerical simulations of periodic systems, one often discovers a wealth of interesting dynamical behaviour. Whether this behaviour manifests itself in the real systems being modelled is a fairly open question. The analysis in this thesis could be used to answer this question in the case of non-bifurcation behaviour which is an important first step.
- For typical nonlinear systems, the optimal, resonant driving forces are *not* periodic.

So for these reasons I claim that the only way of obtaining rigourous safety estimates in a wide variety of cases is to treat aperiodic systems. The results I

present here are one way of doing this and follow naturally from a dynamical systems perspective although I present a different formulation to much of the traditional dynamical systems literature.

Another safety estimate of obvious importance is a lower bound for the size of the basin of attraction for asymptotically stable solutions. Here the motivation is safety with respect to perturbations in initial condition. This is often done numerically, for example [44], but I would like to highlight some situations for which the standard methods are not satisfactory and for which the analytic approach I develop could prove more fruitful.

- When there is high dimensional or infinite dimensional phase space, numerics could be computationally expensive and thus prohibitive. I develop a technique which works in arbitrarily large spaces and which can give estimates that are uniform in system size.

- High dimensional parameter space. Again, computationally this is problematic so analytic estimates could be extremely useful.

- Aperiodic systems often require computation over long time scales. My approach can often give estimates which are uniform in time.

- When there are 'unknown' perturbations one cannot numerically investigate the basins of attraction of the response. With the methodology I present here there is a natural way of estimating the basin size of perturbed solutions even though they are essentially unknown.

1.2 Different approaches to the problem of robustness in aperiodic systems

Here I summarise some approaches that can been be found in the literature which in some way are relevant to the problem of robustness of aperiodic systems.

1.2.1 Dynamical systems

It is typical in the dynamical systems community to treat only autonomous systems and moreover look only for qualitative features of the dynamics. Much of the theory is expected to have extensions to non-autonomous systems but there is certainly a lack of theory in this direction.

The concept which is of fundamental importance to almost every dynamical systems approach to the robustness problem is that of uniform hyperbolicity. Robustness to perturbations is a natural consequence of uniformly hyperbolic solutions or uniformly hyperbolic invariant sets. It is also usually the case that a lack of uniform hyperbolicity implies some form of non-robustness so the concepts are to some extent equivalent. In the next chapter I treat this theory in some detail.

It might be expected that from the ideas contained in the standard proofs of robustness and stability in dynamical systems theory there are ways of obtaining quantitative information like estimates of response size. However this particular issue is very rarely addressed.

Time-dependent structural stability

One of the first results from the dynamics community concerning robustness in aperiodic systems is the concept of time-dependent structural stability.

An autonomous dynamical system is said to be structurally stable if every 'nearby' system has a qualitatively similar phase portrait. Greater details of this concept are given in the following chapter. The concept of time-dependent structural stability is a generalisation that allows non-autonomous perturbations.

A diffeomorphism f is called *time-dependent structurally stable* if there is some neighbourhood U of f in the space of diffeomorphisms such that for any $n \in \mathbb{Z}$, f^n is topologically conjugate to the composition of an arbitrary sequence of diffeomorphisms $f_0 \circ f_1 \circ \cdots \circ f_n$ each of which is picked from U. Since time-dependent perturbations of f can be thought of as a sequence such as this, one could derive useful safety criteria by estimating the size of the allowable neighbourhood, although to my knowledge this has not been done.

This concept has been introduced by Franks in [12] where he shows that uniformly hyperbolic systems are time-dependent structurally stable.

Shadowing

Another result which is relevant to aperiodic perturbations of autonomous systems is the *shadowing lemma* of Bowen and Anosov. More details of the discrete-time version can be found in Lanford [22], but here is the basic idea.

Roughly speaking, given a reference system, a *pseudo-orbit* is an orbit obtained by allowing any 'small enough' time-dependent perturbation.[ii]

[ii] Alternatively one could define a pseudo-orbit to be an orbit which solves the reference ODE except at an arbitrary sequence of times where it is allowed to have 'small' jumps.

The *shadowing* property is said to hold if, for every small enough pseudo-orbit there is a uniformly close true orbit. This true orbit is called a *shadowing orbit.* The *shadowing lemma* says that any uniformly hyperbolic invariant set has the shadowing property.

One can use the shadowing property to show robustness of a solution to aperiodic perturbations in the following way. From the sketch definition above, the unperturbed solution is a pseudo-orbit of the perturbed system. Thus if the shadowing property holds, there is a true orbit of the perturbed system uniformly-close. Thus the unperturbed solution can be seen to be robust. What is required for this idea to be of use in the problem of safety criteria is a clean proof which gives good estimates. I believe this can be obtained most satisfactorily using the framework I present in this thesis although I attack the problem of robustness more directly. For an idea of the use of shadowing estimates see Sauer & Yorke [43], or Coomes, Kocak & Palmer [33]. These papers treat the problem of determining global error estimates for numerical integration of ODE's by using a functional analytic characterisation of hyperbolicity.

Skew-Product Systems

An idea in fairly common use is to consider a non-autonomous system as being the product of a forced system and a forcing system with the property that the forcing dynamics are independent of the forced dynamics.

Definition 1.1 *A* skew-product system *is a dynamical system of the form*

$$\dot{x} = f_1(x, y) \tag{1.1}$$
$$\dot{y} = f_2(y) \tag{1.2}$$

(1.1) is the forced dynamics and (1.2) is the forcing dynamics.

The most simple way to treat the differential equation $\dot{x} = f(x, t)$ as a skew-product system is to consider the modified system

$$\dot{x} = f(x, y) \tag{1.3}$$
$$\dot{y} = 1 \tag{1.4}$$

This gives us an autonomous skew-product flow but unless the time-dependence is periodic or has some other restrictive structure we have a non-compact phase-space for the forcing dynamics and the approach gains us very little.

An extension of this approach is to consider the following modified system.

$$\dot{x} = F(x, y) \tag{1.5}$$

$$\dot{y} = g(y) \tag{1.6}$$

Here the time dependence has been treated as the 'output' of some forcing dynamical system on a compact phase-space Y. This is useful if the forcing dynamics are known to come from a low-dimensional autonomous system but this is not often the case for general perturbations of physical systems. For example, the unknown perturbations affecting a building or a ship could only realistically be assumed to come from a high-dimensional or stochastic dynamical system.

However, given this assumption, much can be said about such systems including interesting things about robustness and stability of solutions. Again, the important condition for robustness is hyperbolicity. For some theory in this vein see Stark [47]. I do not consider this situation since I am interested in allowing as general a time-dependence in the system as possible.

A second and perhaps more useful way of using the skew-product approach is to first consider the linearisation of $\dot{x} = f(x, t)$ about some known solution x_0. This gives us a linear non-autonomous equation

$$\dot{\xi} = Df_{x_0(t),t}\xi \tag{1.7}$$

For reasons I will discuss in detail in the following chapter, study of this equation can tell us most of the essential things about robustness and stability of x_0.

We can treat (1.7) as a skew-product system by considering the matrix-valued function $Df_{x_0(t),t}$ as the forcing dynamic and ξ as the forced dynamic. This is essentially the approach taken by Sacker & Sell in a series of papers [38, 39, 40, 41, 42].

The approach is most effective when the asymptotic behaviour of $Df_{x_0(t),t}$ is known since in that case one only needs to consider the 'limit points' of $Df_{x_0(t),t}$. When these are not known the situation is more complicated.

I treat this theory in some detail in the following chapter since the characterisation of robustness obtained this way is related to the one I have favoured in this thesis. However, another weakness of the skew-product approach is that the topological methods used do not readily give quantitative results. By placing it in a Banach space context one might be able to remedy this. I prefer however to completely avoid the formalism of skew-product systems and treat the problem of aperiodic ODE's more directly.

1.2.2 Control theory / Systems theory

Another approach to the problem of deriving safety criteria can be obtained from a control theoretic viewpoint. The idea is to treat the perturbation or external forcing as a control variable. Then the *objective* is to induce the largest response for a given size of forcing. Thus the relevant optimal control problem is to search for a *worst case scenario*. Once the optimal perturbation is found, it is then easy to obtain the maximum response numerically.

For additive forcing it is well known that, for a given maximum amplitude, the control strategy which induces the largest C^k-norm response is of *bang-bang* type. That is, the control always takes the maximum amplitude but switches direction from time to time. Typically this has a non-trivial switching locus so apart from a few exceptional cases it cannot be found explicitly.

If we are interested in (preventing) responses which have large 'energy gains', then some answers are given in the work of Hubler, (see for example [5, 56]), who has formulated *the principle of the dynamical key*. This work seeks to bound the energy gain of a system in terms of the L_2-norm of the external forcing function. One can formulate this as a variational problem and this way find the optimal control strategy. The surprising result is that the optimal forcing is just the time-reflected dynamics of the unperturbed system. Since this can be readily determined numerically, one can easily find an upper-bound for the energy gain of the response. This result appears to rely on the self-adjoint nature of the problem when defined on L_2 functions and may not generalise easily to more typical cases.

1.2.3 The approach I take

In this thesis, I develop a functional analytic approach to aperiodic systems and suggest that it is the most useful and general one to take. Although it is not a new idea to characterise dynamical systems using functional analysis it is certainly less common than the traditional geometric viewpoint and in consequence is rarely used in applications.

The advantages of my approach will become obvious as the theory is presented. In particular, the methods are used to obtain rigourous safety criteria for a variety of nonlinear systems in which there are unknown perturbations and this is something that I have not found in the existing literature.

1.3 Outline of thesis

As I have mentioned, it is a well known result in traditional dynamical systems theory that a solution which is 'hyperbolic' is robust to perturbations. It is also true to say that robustness, in the sense of unique nearby continuation, is only present when there is uniform hyperbolicity. This idea extends to uniformly hyperbolic invariant sets, an example of which is the Smale horseshoe. Although the theory of uniformly hyperbolic sets is one of the cornerstones of dynamical systems theory, a formulation of the theory which naturally incorporates aperiodic systems is not easily found in the literature. In chapter 2, I present an extension of the standard theory to aperiodic systems. In particular, I present a characterisation based on linear operators acting on Banach spaces and argue that it is a unified approach which generalises easily and is especially suited to the problem of obtaining safety criteria when there are arbitrary, bounded perturbations. There are two very important results in this chapter. Firstly, I show the equivalence between hyperbolicity of a solution and invertibility of a certain linear operator $\mathcal{L}$. Secondly, I give a theorem which states that hyperbolic solutions persist under small enough perturbations of the vector field. This by proved by using the implicit function theorem and the invertibility characterisation of hyperbolicity. All of the results in chapter 2 can be deduced or found in the current literature and full references are given. However, a suitable, unified exposition of the theory is not available in the literature.

In chapter 3, I tackle the problem of obtaining good safety criteria. The method I use is based on the persistence of hyperbolicity. The crucial element of this method is estimating the response to perturbations by finding an upperbound for the norm of an associated linear operator. This can be practically achieved in a variety of circumstances and I discuss how one should do this.

An important extension of the method is given in the section on adapted estimates. It is clear that some perturbations will be more harmful than others in terms of the response they induce. Thus the region of 'safe' perturbations is very far from being shaped like a ball in the space of perturbations. By adapting the norm one uses in this space one can take account of this effect to some extent. This gives us a better estimate of the shape of the region of safe perturbations and thus more effective safety criteria.

Another useful safety criteria I investigate is a lower bound on the size of the basin of attraction for asymptotically stable solutions. Although there are some obvious numerical methods one could use to do this, I show how they can be obtained

analytically using the persistence of hyperbolicity property. This is especially useful for high dimensional systems since the results are uniform in system size. In particular the method can be used to find basin of attraction estimates for solutions of perturbed systems even when the precise form of the perturbation is unknown.

In chapter 4, I consider generalisations of the theory. Firstly, there is the important area of discrete-time systems. Although less common in applications, they are still a very important part of dynamical systems theory and I show that the results for continuous-time systems immediately generalise to the discrete-time case.

The second important generalisation is to systems with discontinuous perturbations. For example, external forcing which is bounded in L_p-norm or even impulses. Although in previous chapters I have dealt with spaces of functions and vector-fields that are continuous with respect to time, one can extend the theory to more general function spaces. To do this I present the relevant ideas from functional analysis, in particular, theory concerning the L_p spaces, the Sobolev spaces and spaces of distributions. Then a generalisation of persistence of hyperbolicity theorem is given and used to obtain safety criteria.

In chapter 5, I present some applications of the theory. Firstly I consider robustness and stability in a simple aperiodically forced oscillator system. This has particular relevance to the problem of ship stability and more generally escape phenomena. I show in detail how one can obtain safety criteria and I compare these to results from numerical simulations.

A second application is the robustness of 'phase locking' in an aperiodically forced oscillator. I show that the concept of phase-locking makes sense for aperiodic systems and derive in detail some lower bounds on the size of perturbation required to break the phase-locking effect. An interesting technique I develop in this chapter is the use of a re-parametrisation of time. This is neccessary in order to incorporate *frequency modulation* into the general framework.

In chapter 6, I summarise the findings of the thesis, evaluate the usefulness of the approach and discuss some of its limitations. I also give some directions for further research.

Chapter 2

Hyperbolicity

A good introduction to the theory of uniformly hyperbolic invariant sets is given by Lanford [22]. Like most of the approaches to this subject he treats only autonomous, discrete-time dynamical systems. However, it is one of the few expositions which does not rely solely on a geometric approach. More details, including the theory for continuous-time systems, can be found in the books by Robinson [36] and Shub [46].

A good formulation of the theory which naturally incorporates aperiodic systems is not available in the literature so I present here in detail the key definitions and theorems. I first give non-autonomous versions of the standard definitions and properties of hyperbolicity. Then I consider a characterisation of hyperbolicity based on operators acting on Banach spaces and show it is a unified approach to the theory and allows for clean proofs and greater generality.

Some early examples of a related approach can be found in the theory of exponential dichotomy for linear time-varying ODE's. This was developed by Massera & Schaffer [26], Coppel [7], Daleckii & Krein [9] and Sacker & Sell [39, 40, 41, 42], in the 60's and 70's although it stems from the ideas of Perron and Bohl in the early part of the century. Later in this chapter I present the relevant aspects of this theory.

Mather [27], has also given a functional analytic characterisation of hyperbolicity for autonomous dynamical systems. Another example can be found in Aubry, MacKay & Baesens [2], MacKay [24] and Sepulchre & MacKay [45], where they have successfully used essentially the same characterisation of hyperbolicity to investigate the dynamics of *networks* of coupled units. Networks are discrete-space dynamical systems with high or infinite dimensional phase space so are naturally suited to a functional analytic approach.

2.0.1 Notation

Let X be an arbitrary Banach space, with the norm $|\ |$. For an interval $\mathcal{I} \subset \mathbb{R}$ and $k \in \mathbb{N}$, let $\mathcal{C}^k(\mathcal{I}, X)$ denote the Banach space of k-times continuously differentiable functions $x : \mathcal{I} \to X$ which are bounded and have k bounded derivatives. This is given the standard $\mathcal{C}^k$ norm

$$||x||_{\mathcal{C}^k} = \max\{\ ||x||_\infty,\ ||\dot{x}||_\infty, \ldots,\ ||x^{(k)}||_\infty\ \}$$

where $||x||_\infty = \sup_{t\in\mathcal{I}} |x(t)|$.

For convenience, I abbreviate $\mathcal{C}^k(\mathbb{R}, \mathbb{R}^n)$ to $\mathcal{C}^k$ as it will be the primary space of interest. Also, I abbreviate $\mathcal{C}^k(\mathbb{R}^+, \mathbb{R}^n)$ to $\mathcal{C}^k_+$ and $\mathcal{C}^k(\mathbb{R}^-, \mathbb{R}^n)$ to $\mathcal{C}^k_-$.

In applications one is often interested in obtaining uniform bounds for the norm of the state variable and its derivatives. This is why the $\mathcal{C}^k$ norm is the most natural choice. In principle one could use a different norm on $\mathcal{C}^k$ as long as it remains complete. For example, one could use the weighted norm

$$||x||_{\mathcal{C}^k} = \max\{\ w_0\,||x||_\infty,\ w_1\,||\dot{x}||_\infty, \ldots,\ w_k\,||x^{(k)}||_\infty\ \}$$

with an arbitrary choice of weights, $w_i \in \mathbb{R}^+ \setminus \{0\}$. In particular, if one is more interested in $||x||$ than the norm of its derivatives then a good choice would be to make $w_i/w_0 << 1$ for $i \neq 0$.

Denote by BL^n, the space of bounded linear maps $A : \mathbb{R}^n \to \mathbb{R}^n$. This is just the space of $n \times n$ matrices. With the linear operator norm

$$|A| = ||A||_{\mathbb{R}^n \to \mathbb{R}^n} = \sup_{|v|=1} |Av|$$

this is a Banach space.

For an element x_0 in the Banach space X and $\mu \in \mathbb{R}$, let $B_X(\mu, x_0)$ be the size μ open-ball in X centred on x_0. Formally

$$B_X(\mu, x_0) = \{\ x \in X \mid ||x - x_0|| < \mu\ \}$$

Let the symbol D denote $\frac{\partial}{\partial x}$ and the symbol $\dot{}$ denote $\frac{d}{dt}$.

2.0.2 Solutions to non-autonomous systems

Consider the non-autonomous dynamical system

$$\dot{x} = f(x, t) \tag{2.1}$$

where x lies in the Banach space $\mathbb{R}^n$ and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a bounded vector field which satisfies the following conditions.

- f is C^1 with respect to x for fixed t, and the Jacobian satisfies $|Df_{x,t}| < M$ with M independent of t
- f is C^0 with respect to t for fixed x.

The theory generalises to dynamical systems defined on differentiable manifolds but for notational and presentational convenience I just consider systems defined on $\mathbb{R}^n$.

Definition 2.1 *A (bounded)* solution *of (2.1), is a function $x \in C^1$, such that $x(t)$ satisfies (2.1) for all $t \in \mathbb{R}$.*

Note that I do not include an initial condition but instead consider any solution bounded for all time. One could treat initial value problems by considering solutions on the positive half-line only. I treat this case to some extent in a later chapter where estimates of basins of attraction are obtained, although, it could also be useful to extend all the hyperbolicity theory in this chapter to initial value problems and solutions on half-lines. I give some ideas of how this can be done later.

2.1 Standard definitions of uniform hyperbolicity

Definition 2.2 *The* linearisation, *or* variational equation *of $\dot{x} = f(x,t)$ about a bounded solution x_0, is defined by*

$$\dot{\xi} = Df_{x_0(t),t}\, \xi \tag{2.2}$$

Clearly, $Df_{x_0(t),t}$ is continuous and uniformly bounded. Let $X_0(s)$ denote the principal matrix solution of (2.2). That is, the matrix solution for which $X_0(0) = I$. Define the evolution operators by $X_t(s) = X_0(s)X_0^{-1}(t)$. These are matrix solutions of (2.2) which satisfy $X_t(t) = I$. They are also known as *fundamental solutions.*

2.1.1 Equilibria and periodic orbits

The most simple solution one can consider is an equilibrium. The following are standard definitions.

Definition 2.3 *An equilibrium solution x_0, of an autonomous system $\dot{x} = f(x)$ is* hyperbolic *if the Jacobian Df_{x_0}, has no purely imaginary eigenvalues. It is* linearly attracting *if each eigenvalue has strictly negative real part.*

The next type of solution I consider is a periodic orbit.

Floquet theory

Suppose $x_0(t)$ is a periodic solution (of period kT) for the T-periodic system $\dot{x} = f(x,t)$. Clearly, the variational equation (2.2), will be kT-periodic. Without loss of generality assume $k = 1$.

Theorem 2.1 (Floquet) *Every fundamental solution $X(t)$, of the variational equation is of the form*

$$X(t) = P(t)\exp(tB)$$

where $P(t)$ is a T-periodic matrix and B is a constant matrix.

Definition 2.4 *The* monodromy map *(or monodromy matrix) is defined by $M = \exp(TB)$.*

Since the monodromy map satisfies $X(T) = MX(0)$, the qualitative behaviour of solutions can be deduced from the monodromy map.

Remark 2.1

Definition 2.5 *The eigenvalues $\{\rho_i\}$, of M, are called* Floquet multipliers *or* characteristic multipliers. *If $\rho_i = \exp(\lambda_i T)$ then λ_i is called a* characteristic exponent.

Remark 2.2 *M depends upon the choice of fundamental solution. The Floquet multipliers are unique since different monodromy matrices are similar and thus have the same eigenvalues. The characteristic exponents have unique real part but one can add $2\pi in/T$ for any $n \in \mathbb{Z}$ to yield another characteristic exponent.*

For periodic orbits of time-periodic systems, the standard definition of hyperbolicity is

Definition 2.6 *For a time-periodic system, $\dot{x} = f(x,t)$, of period T, we say a solution, $x_0(t)$, of period kT, $k \in \mathbb{N}$, is* hyperbolic *if none of its Floquet multipliers lie on the unit circle. It is* linearly attracting *if all the Floquet multipliers have modulus strictly less than one.*

The key property of hyperbolic equilibria and periodic orbits is that the tangent space along the solution admits a continuous splitting into a forward-time contracting subspace and a backward-time contracting subspace. The existence of a splitting motivates the definition of hyperbolicity for arbitrary bounded solutions.

2.1.2 Arbitrary bounded solutions

Definition 2.7 *We say that a bounded solution x_0, is* strongly uniformly hyperbolic *if there exist constants $K, \alpha > 0$ and for each $t \in \mathbb{R}$ a splitting, $\mathbb{R}^n = E_t^+ \oplus E_t^-$,*

$$\xi^+ \in E_t^+ \implies |X_t(s)\xi^+| \leq Ke^{-\alpha(s-t)} |\xi^+| \qquad s \geq t \tag{2.3}$$

$$\xi^- \in E_t^- \implies |X_t(s)\xi^-| \leq Ke^{-\alpha(t-s)} |\xi^-| \qquad s \leq t \tag{2.4}$$

Note that $E_t^\pm$ are invariant with respect to the linearised dynamics in the sense that $\xi \in E_t^\pm \implies X_t(s)\xi \in E_s^\pm$.

Usually one includes in the definition of uniform hyperbolicity, a condition which prevents the 'angle' between E_t^+ and E_t^- from approaching 0, for example, requiring the projection onto E_t^+ to be uniformly bounded or requiring, for some $J \in \mathbb{R}$, that

$$|\xi^+ + \xi^-| \geq J > 0 \tag{2.5}$$

for arbitrary unit vectors $\xi^+ \in E^+, \xi^- \in E^-$.

Lemma 2.1 *The angle condition (2.5) is automatically satisfied if (2.3) and (2.4) are satisfied.*

Proof Since $Df_{x_0(t),t}$ is continuous and uniformly bounded, say $||Df_{x_0(t),t}||_\infty \leq M_0$, we know that

$$||X_t(t+T)|| \leq e^{\left(\int_t^{t+T} ||Df_{x_0(t),t}||_\infty ds\right)} \leq e^{TM_0} \tag{2.6}$$

From (2.3) and (2.4) we see that for unit vectors $\xi^+ \in E^+$ and $\xi^- \in E^-$,

$$\begin{aligned} |X_t(t+T)\xi^+| &\leq Ke^{-\alpha T} \\ |X_t(t+T)\xi^-| &\geq K^{-1}e^{\alpha T} \end{aligned}$$

Using (2.6) and the triangle inequality we deduce that

$$\begin{aligned} |\xi^- + \xi^+| &\geq e^{-TM_0} |X_t(t+T)\xi^- + X_t(t+T)\xi^+| \\ &\geq e^{-TM_0} \left(|X_t(t+T)\xi^-| - |X_t(t+T)\xi^+|\right) \\ &\geq e^{-TM_0} \left(K^{-1}e^{\alpha T} - Ke^{-\alpha T}\right) \\ &\geq J > 0 \qquad \text{for } T \text{ large enough} \end{aligned}$$

as required ■

It is described as *uniform* because K and α can be chosen independently of t. There is an interesting non-uniform generalisation of hyperbolicity but as I do not consider it in this thesis I will drop the word uniform.

Note that this definition encompasses the previous definitions.

Proposition 2.1 *An equilibrium or periodic orbit x_0 is hyperbolic according to definition 2.7 above if and only if the Jacobian Df_{x_0} has eigenvalues off the imaginary axis (equilibrium) or characteristic exponents off the imaginary axis (periodic orbit).*

Proof For equilibria, E_t^+ and E_t^- are the eigenspaces corresponding to the eigenvalues in the left and right half-planes respectively. For periodic orbits, one can make the periodic change of variables $\psi = P(t)\xi$ to change the periodic variational equation into the constant coefficient equation

$$\dot{\psi} = B\psi \tag{2.7}$$

The variational equation (2.2) has a hyperbolic splitting if and only if (2.7) has. But, the characteristic exponents are the eigenvalues of B which are clearly off the imaginary axis if and only if there is a hyperbolic splitting for (2.7). ■

2.1.3 Autonomous systems

For autonomous systems the only *strongly* hyperbolic solutions are equilibria.

To see this notice that if x_0 is a non-equilibrium solution of $\dot{x} = f(x)$ then, by the chain rule, we have

$$\frac{d\dot{x}_0}{dt} = Df_{x_0}\dot{x}_0$$

Thus the span of $\dot{x}_0$ gives non-trivial bounded solutions of the variational equation. Clearly $\dot{x}_0$ is neither in E_t^+ nor E_t^- since it does not contract exponentially in either forward or backward time.

This degeneracy results from the continuous time-translation invariance of autonomous systems. For periodic orbits of autonomous systems, notice that there is always a simple Floquet multiplier of 1.

To cope with autonomous systems one usually extends definition 2.7 to allow a splitting $\mathbb{R}^n = E_t^+ \oplus E_t^- \oplus \mathbb{R}$ and some adjustments need to be made to the hyperbolicity theory. For example, a periodic orbit $x_0(t)$, is called hyperbolic if it corresponds to a hyperbolic fixed point of a Poincaré map. To treat this case in the style of this thesis one could allow a reparametrisation of time, or treat the orbit as a *normally hyperbolic* manifold. Because of the differences in theory I make a distinction between the *strong* hyperbolicity defined above and the weaker autonomous version. Since I only treat the strongly hyperbolic case I will take hyperbolic to mean strongly hyperbolic.

2.1.4 Forward and backward contracting subspaces

E_t^+ consists of tangent vectors which contract exponentially in forward time. E_t^- consists of tangent vectors which contract exponentially in backward time. More correctly they could be written $E^{\pm}_{x_0(t),t}$ since they are in the tangent space at $x_0(t)$ but I use the simplified notation when there is no ambiguity. They are commonly called the stable and unstable subspaces but stable and unstable are somewhat misleading descriptions since the unstable subspace is exponentially attracting while the stable subspace is exponentially repelling. I will refer to them as the forward and backward contracting subspaces.

In contrast to the equilibrium case, if the linearisation is time-dependent there is not always a simple correspondence between the (time-dependent) eigenvalues of the Jacobian and the hyperbolic splitting. The following example demonstrates this. Consider the linear ODE, $\dot{\xi} = U(t)^{-1}AU(t)\ \xi$, where

$$U(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \quad , \quad A = \begin{pmatrix} -1 & -5 \\ 0 & -1 \end{pmatrix}$$

This has eigenvalues $\{-1,-1\}$ for each $t \in \mathbb{R}$ which might suggest that 0 was asymptotically stable. However, a fundamental solution is given by

$$X(t) = \begin{pmatrix} e^t(\cos t + 1/2 \sin t) & e^{-3t}(\cos t - 1/2 \sin t) \\ e^t(\sin t - 1/2 \cos t) & e^{-3t}(\sin t + 1/2 \cos t) \end{pmatrix}$$

so 0 is unstable with $\dim E^- = 1$. By a simple change of variables we can turn this into an example of a non-hyperbolic system with the time-dependent eigenvalues bounded away from the imaginary axis.

However, see Coppel [7] for some ways of relating the eigenvalues to hyperbolicity.

Remark 2.3 *The dimensions of E_t^+ and E_t^- are constant along hyperbolic solutions.*

The theory of hyperbolicity gives us the existence, for the nonlinear system, of forward and backward contracting manifolds $W^+_{x_0(t),t}$ and $W^-_{x_0(t),t}$ tangent to E_t^+ and E_t^- at $x_0(t)$. One can also deduce that the local dynamics are governed (up to topological conjugacy) by the linearisation. Thus near to $x_0(t)$ there is exponential contraction in forward and backward time along $W^+_{x_0(t),t}$ and $W^-_{x_0(t),t}$ respectively.

2.1.5 Stability type

Definition 2.8 *The* stability type *of a hyperbolic solution is given by the pair* $(\dim E_t^+, \dim E_t^-)$, *for any* $t \in \mathbb{R}$. *If* $\dim E_t^- = 0$ *then we say that* x_0 *is* linearly attracting.

For example, the stability type of a hyperbolic equilibrium is (l, m) where l is the number of eigenvalues of Df_{x_0} with negative real part and m is the number with positive real part. For a hyperbolic periodic orbit, l is the number of Floquet multipliers inside the unit circle and m is the number outside the unit circle.

2.2 Exponential dichotomy

This splitting property for hyperbolic solutions was noticed a long time ago and first appeared implicitly in the works of Bohl and Perron. It was termed *exponential dichotomy* by Massera & Schaffer [26] who developed the formal theory for linear differential equations along with Daleckii & Krein [9], Coppel [7] and others in the 50's and 60's. The study of exponential dichotomy in linear differential equations has continued to be an active area of research since then.

I present here some of the basic theory of exponential dichotomy in relation to the linear time-varying differential equation (2.2). For a more detailed treatment, see Coppel [7], Daleckii & Krein [9] or Massera & Schaffer [26].

Definition 2.9 *(2.2) is said to have an* exponential dichotomy *on the interval* $\mathcal{I} \subset \mathbb{R}$ *if there exist a projection*[i] $\mathcal{P}_0$ *and constants* $K, \alpha > 0$ *such that*

$$||X_0(t)\ \mathcal{P}_0\ X_0^{-1}(s)|| \leq Ke^{-\alpha(t-s)} \qquad t \geq s \tag{2.8}$$

$$||X_0(t)\ (I - \mathcal{P}_0)\ X_0^{-1}(s)|| \leq Ke^{-\alpha(s-t)} \qquad t \leq s \tag{2.9}$$

The interval can be finite or infinite, for example, $[0, 1]$, $\mathbb{R}^+$, $\mathbb{R}^-$ or $\mathbb{R}$. Unless otherwise specified I will assume $\mathcal{I} = \mathbb{R}$.

Theorem 2.2 *A bounded solution* x_0, *is strongly uniformly hyperbolic if and only if the variational equation (2.2), has an exponential dichotomy on* $\mathbb{R}$.

Proof Suppose the variational equation has an exponential dichotomy on $\mathbb{R}$. Define the continuous projection operator $\mathcal{P}_t = X_0(t)\mathcal{P}_0 X_0^{-1}(t)$. Then the forward and backward contracting subspaces are given by

$$\begin{aligned} E_t^+ &= \text{Range}\,\mathcal{P}_t \\ E_t^- &= \text{Ker}\,\mathcal{P}_t = \text{Range}(\mathcal{I} - \mathcal{P}_t) \end{aligned}$$

[i] A projection is a linear operator $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}^n$ such that $\mathcal{P}^2 = \mathcal{P}$.

$\mathbb{R}^n = E_t^+ \oplus E_t^-$ for each $t \in \mathbb{R}$ and clearly (2.3) and (2.4) are satisfied for the same K and α.

Conversely, given a splitting $\mathbb{R}^n = E_t^+ \oplus E_t^-$ satisfying (2.3) and (2.4), taking $\mathcal{P}_0$ to be the projection with range E_0^+ and kernel E_0^- clearly gives an exponential dichotomy. ■

2.2.1 Exponential dichotomies on half lines

For an exponential dichotomy on $\mathbb{R}^+$ with projection $\mathcal{P}_0$, the forward contracting subspace is given by $E_t^+ = \text{Range}\,\mathcal{P}_t$. This contains all solutions bounded on $\mathbb{R}^+$. Ker $\mathcal{P}_t$ can be any complementary subspace. Similarly, for an exponential dichotomy on $\mathbb{R}^-$, with projection $\mathcal{Q}_0$, the backward contracting subspace is given by $E_t^- =$ Ker $\mathcal{Q}_t$ which contains all solutions bounded on $\mathbb{R}^-$. Range $\mathcal{Q}_t$ is any complementary subspace. So for an exponential dichotomy on $\mathbb{R}$ the projection is uniquely defined. This is summed up in the following theorem

Theorem 2.3 *(2.2) has an exponential dichotomy on $\mathbb{R}$ if and only if it has an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$ such that $\mathbb{R}^n = E_t^+ \oplus E_t^-$.*

As an example of a system for which there are exponential dichotomies on $\mathbb{R}^+$ and $\mathbb{R}^-$ but not on $\mathbb{R}$ consider the planar autonomous differential equation, $\dot{x} = f(x)$ sketched in figure 2.1.

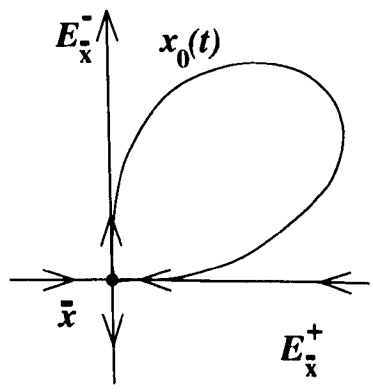


Figure 2.1: $x_0(t)$ is a solution which is homoclinic to the hyperbolic saddle $\bar{x}$.

$x_0(t)$ is a bounded solution which is homoclinic to a hyperbolic saddle point $\bar{x}$. Since $x_0(t) \to \bar{x}$ as $t \to \pm\infty$, the variational equation $\dot{\xi} = Df_{x_0}\xi$ converges to $\dot{\xi} = Df_{\bar{x}}\xi$ in both forward and backward time so must have an exponential

dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$.[ii] However, since the (one dimensional) forward and backward contracting manifolds of $\bar{x}$ coincide along x_0, their tangent spaces, $E^+_{x_0(t),t}$ and $E^-_{x_0(t),t}$ must also coincide. But these are the forward and backward contracting subspaces for the linearisation and so cannot have direct sum $\mathbb{R}^2$. Thus there is no exponential dichotomy on $\mathbb{R}$.

If however we perturb to get $\dot{x} = f(x)+\varepsilon g(t)$ then we can look for solutions $x_\varepsilon(t)$ homoclinic to the nearly-fixed point $\bar{x}_\varepsilon$. If $x_\varepsilon(t)$ exists, then generically, it will have an exponential dichotomy and thus correspond to a *transversal intersection* of the stable and unstable manifolds of $\bar{x}_\varepsilon$. It is then known as a *transversal homoclinic solution.* The existence of transversal homoclinic solutions allows one to deduce many interesting results. For example, if there are many such intersections then the nearby dynamics can be extremely complex. For a treatment of the case where g is periodic the reader is referred to Palmer [32], who uses the theory of exponential dichotomy to derive a Melnikov type function for the existence of x_ε and also to deduce the existence of random dynamics near to x_ε. This can also be done for aperiodic forcing. Gruendler [13] has some answers in this direction and in principle one could use the theory developed in this thesis to provide an elegant quantitative approach to this problem.

2.3 A useful characterisation of hyperbolicity

The study of hyperbolicity for solutions of (2.1) or equivalently exponential dichotomy for (2.2) can be reduced to the study of a linear differential operator on suitable function spaces. This is fundamentally the approach I take in order to treat aperiodic systems in an appropriate manner.

It will prove advantageous to think of $\dot{x} = f(x,t)$ as a differential equation on $\mathcal{C}^1$ functions $x : \mathbb{R} \to \mathbb{R}^n$ rather than on vectors $x \in \mathbb{R}^n$. By definition, for $x \in \mathcal{C}^1$ we have $\dot{x} \in \mathcal{C}^0$. It follows from the hypotheses on the vector field $f(x,t)$ that for $x \in \mathcal{C}^1$, the right hand side $f(x(t),t)$, is a bounded $\mathcal{C}^0$ function. Thus the vector field can be regarded as the operator $f : \mathcal{C}^1 \to \mathcal{C}^0$, $f(x)(t) \mapsto f(x(t),t)$. Moreover, since $f(x,t)$ is uniformly $\mathcal{C}^1$ with respect to $x \in \mathbb{R}^n$, it follows that the operator f is $\mathcal{C}^1$ with respect to functions $x \in \mathcal{C}^1$ as well.

Define $\tilde{\mathcal{C}}^1$, the space of $\mathcal{C}^1$*-vector fields*, by $\tilde{\mathcal{C}}^1 = \mathcal{C}^1(\mathcal{C}^1, \mathcal{C}^0)$. This clearly contains the vector fields we are interested in but in fact it is much larger and contains

[ii] This actually follows from an argument given in the 'splitting index' theory detailed later in the chapter.

operators which are not really vector fields at all. For our purposes however, this is not a problem. In fact, it allows us to show robustness to a much more general class of perturbations, for example, where the time dependence is more complicated such as differential-delay equations.

The norm I will use on $\tilde{\mathcal{C}}^1$ is

$$||f||_{\tilde{\mathcal{C}}^1} = \sup_{x \in \mathcal{C}^1} \max \{ ||f(x(\cdot),\cdot)||_{\mathcal{C}^0} \ , \ ||Df_x||_{\mathcal{C}^1 \to \mathcal{C}^0} \}$$

This is a generalisation of the norm on $\mathcal{C}^1$ vector fields used in standard dynamical systems theory.[iii] Note that $Df_x(t) = Df_{x(t),t}$.

Consider the nonlinear operator, $G : \mathcal{C}^1 \to \mathcal{C}^0$, defined by

$$G(x)(t) = \dot{x}(t) - f(x(t), t)$$

In this context we have the following reformulation of definition 2.1.

Definition 2.10 *$x \in \mathcal{C}^1$ is a* bounded solution *of (2.1) if $G(x) = 0$.*

The *linearisation* of G about a solution $x_0 \in \mathcal{C}^1$, is $\mathcal{L} = DG_{x_0} : \mathcal{C}^1 \to \mathcal{C}^0$, given by

$$\mathcal{L} = \frac{d}{dt} - Df_{x_0}$$

Note that all the $\mathcal{C}^1$ solutions of the variational equation (2.2) are obtained by solving $\mathcal{L}\xi = 0$.

2.3.1 Green's functions and the inhomogeneous variational equation

It is well known that the exponential dichotomy of the variational equation is closely related to the solvability of the inhomogeneous equation

$$\dot{\xi} = Df_{x_0}\xi + \varphi \tag{2.10}$$

where $\varphi \in \mathcal{C}^0$. All the $\mathcal{C}^1$ solutions of (2.10) are given by solving $\mathcal{L}\xi = \varphi$. To do this we look for a two-point function of time associated with $\mathcal{L}$ called a *Green's function*. As we are interested in obtaining bounded solutions rather than particular initial value problems or boundary value problems I only consider the relevant Green's function.

[iii] See for example Robinson [36].

Definition 2.11 *For the first order linear differential operator $\mathcal{L} : \mathcal{C}^1 \to \mathcal{C}^0$ defined above, a* Green's function *an essentially bounded matrix-valued operator, $W : \mathbb{R} \times \mathbb{R} \to BL(\mathbb{R}^n, \mathbb{R}^n)$, such that for each $s \in \mathbb{R}$, W satisfies*

1. $\mathcal{L}W(t,s) = 0$ *for* $t \neq s$
2. $\lim_{t\to s^+} W(t,s) - \lim_{t\to s^-} W(t,s) = I$

There is also a more direct way of characterising W. Let δ_t be the distribution defined by $\delta_t[\varphi] = \varphi(t)$, for suitable $\mathbb{R}^n$-valued test functions, φ. For some details of the theory of distributions[iv] see section 4.2. δ_t is the *Dirac delta function* or *unit impulse* (at time t).

Proposition 2.2 *W is a Green's function for $\mathcal{L}$, if and only if, for each s, $W(\cdot, s)$ is an essentially bounded solution of the distributional equation*

$$\mathcal{L}W(\cdot, s) \;=\; \delta_s$$

For this reason the Green's function is also known as the *impulse response*.

Proof To see that this is equivalent to Definition 2.11 notice that property 1 is trivially satisfied and property 2 follows from

$$\begin{aligned} I \;=\; \delta_s[I] \;&=\; \mathcal{L}W(t,s)[I] \\ &=\; \lim_{h\to 0} \int_{s-h}^{s+h} \frac{d}{dt}W(t,s) - Df_{x_0}(t)W(t,s)\; dt \\ &=\; \lim_{t\to s^+} W(t,s) - \lim_{t\to s^+} W(t,s) \end{aligned}$$

■

When there is a Green's function there is an explicit expression for solutions of (2.10). This is given by

$$(\mathcal{L}^{-1}\varphi)(t) \;=\; \int_{\mathbb{R}} W(t,s)\varphi(s)ds \tag{2.11}$$

To see (2.11) gives a solution of (2.10) we can differentiate to give

$$\begin{aligned} \frac{d}{dt}(\mathcal{L}^{-1}\varphi)(t) \;&=\; \int_{\mathbb{R}} \frac{d}{dt}W(t,s)\varphi(s)ds \\ &=\; Df_{x_0}(t) \int_{\mathbb{R}} W(t,s)\varphi(s)ds \;+\; \varphi(t) \\ &=\; Df_{x_0}(t)\; (\mathcal{L}^{-1}\varphi)(t) \;+\; \varphi(t) \end{aligned} \tag{2.12}$$

[iv] For now the details of the theory of distributions are unimportant. Basically, distributions are functionals defined by their action on a suitable space of 'test' functions (with compact support). The action of a distribution g, on these test functions is denoted by $g[\varphi]$. They are a generalisation of the usual concept of function. Standard functions can also be treated as distributions by the rule $g[\varphi] = \int_{\mathbb{R}} g(u)\varphi(u)du$. The classic example of a distribution which is not a function in the usual sense is the δ distribution.

where $X_s(t) = X_0(t)X_0^{-1}(s)$ is the fundamental solution of the homogeneous variational equation satisfying $X_s(s) = I$.

For $t > s$, $W(t,s) \in E_t^+$ and for $t < s$, $W(t,s) \in E_t^-$.

To see that this satisfies definition 2.11 notice that it is bounded and

$$\lim_{t\to s^+} W(t,s) - \lim_{t\to s^-} W(t,s) \;=\; X_0(t)\mathcal{P}_0X_0^{-1}(t) + X_0(t)(I-\mathcal{P}_0)X_0^{-1}(t) = I$$

By construction it satisfies $|W(t,s)| \leq Ke^{\alpha|t-s|}$ and thus $W(t,\cdot) \in L_1$ with the uniform bound $||W(t,\cdot)||_1 \leq 2K/\alpha$. ■

Some more details of this method of solving (2.10) are given in the next chapter where it is a crucial part in providing persistence estimates for hyperbolic solutions.

Perron was the first to notice that questions about the homogeneous equation, $\dot{\xi} = A(t)\xi$, were closely related to the solvability of the inhomogeneous equation, $\dot{\xi} = A(t)\xi + \psi$. Massera & Schaffer [26] first developed this line of attack formally. They called the pair of function spaces $(\mathcal{A},\mathcal{B})$, *admissible* for $\mathcal{L}$ if $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ was surjective. One of the basic theorems they proved is that $\mathcal{L} : \mathcal{C}^1(\mathcal{I}) \to \mathcal{C}^0(\mathcal{I})$ is surjective if and only if the variational equation (2.2) has an exponential dichotomy on $\mathcal{I}$.

For half-lines the theorem is stated as

Theorem 2.4 *(2.2) has an exponential dichotomy on $\mathbb{R}^\pm$ if and only if $\mathcal{L} : \mathcal{C}^1_\pm \to \mathcal{C}^0_\pm$ is surjective.*

Proof

[ED $\Longrightarrow$ Surj]

Suppose there is an exponential dichotomy on $\mathbb{R}^+$ with projection $\mathcal{P}_s$. Then we can define a Green's function on $\mathbb{R}^+$ as follows.

$$W(t,s) \;=\; \begin{cases} X_s(t)\mathcal{P}_s & t > s > 0 \\ -X_s(t)(I-\mathcal{P}_s) & 0 < t < s \end{cases} \tag{2.15}$$

A $\mathcal{C}^1_+$ solution of $\mathcal{L}\xi = \varphi$ for any $\varphi \in \mathcal{C}^0_+$ is then given by

$$(\mathcal{L}^{-1}\varphi)(t) \;=\; \int_{\mathbb{R}^+} W(t,s)\varphi(s)ds$$

[Surj $\Longrightarrow$ ED]

Let V_1 be the subspace of $\mathbb{R}^n$ which contains all the initial conditions which generate bounded solutions (on $\mathbb{R}^+$) of the variational equation. That is,

$$V_1 \;=\; \{\, \xi \in \mathbb{R}^n \mid X_0\xi \in \mathcal{C}^0_+ \,\}$$

If for each $t \in \mathbb{R}$, we have[v] $W(t,\cdot) \in L_1$, and for some M *independent* of t,

$$||W(t,\cdot)||_1 = \int_{\mathbb{R}} |W(t,s)|\, ds \leq M < \infty$$

then (2.11) gives a $\mathcal{C}^1$ solution of (2.10).

To see this, observe that uniform boundedness of $\mathcal{L}^{-1}\varphi$ follows from Hölders inequality[vi] which gives

$$||\mathcal{L}^{-1}\varphi||_\infty \leq ||W(t,\cdot)||_1 \; ||\varphi||_\infty$$

From (2.12) it follows that $\frac{d}{dt}(\mathcal{L}^{-1}\varphi)$ is bounded if $\mathcal{L}^{-1}\varphi$ is. Clearly,

$$\left|\left|\frac{d}{dt}(\mathcal{L}^{-1}\varphi)\right|\right|_\infty \leq (||Df_{x_0}||\, ||W(t,\cdot)||_1 + 1)\, ||\varphi||_\infty$$

So $\mathcal{L}^{-1}$ is an integral operator with Green's function kernel. If W is the unique, L_1 Green's function then $\mathcal{L}^{-1}$ gives the unique $\mathcal{C}^1$ solution of (2.10) and satisfies

$$||\mathcal{L}^{-1}||_{C^1} \leq \max\{||W(t,\cdot)||_1 \;,\; ||Df_{x_0}||\, ||W(t,\cdot)||_1 + 1\}$$

The Green's function for a hyperbolic solution

Proposition 2.3 *When x_0 is hyperbolic, there is a unique, L_1 Green's function for $\mathcal{L}$. Moreover it satisfies the estimate*

$$|W(t,s)| \leq Ke^{\alpha|t-s|}$$

where K, α are the hyperbolicity constants.

Proof For each s, $W(\cdot,s)$ should be a bounded matrix solution of

$$\left(\frac{d}{dt} - Df_{x_0}\right) W(\cdot,s) = \delta_s \tag{2.13}$$

By hyperbolicity, for any column ξ, of $W(\cdot,s)$ we have

$$|\mathcal{P}_t\xi(t)| \geq Ke^{\alpha(t-s)}\, |\xi(s^+)|$$
$$|(I-\mathcal{P}_t)\xi(t)| \geq Ke^{\alpha(s-t)}\, |\xi(s^-)|$$

Thus for bounded solutions of (2.13) we require $\mathcal{P}_t\xi(t) = 0$ for $t < s$ and $(I-\mathcal{P}_t)\xi(t) = 0$ for $t > s$. This is enough to see that W_t should be

$$W(t,s) = \begin{cases} X_s(t)\mathcal{P}_s & t > s \\ -X_s(t)(I-\mathcal{P}_s) & t < s \end{cases} \tag{2.14}$$

[v] considered as a the matrix-valued function $W(t,\cdot) : \mathbb{R} \to BL(\mathbb{R}^n, \mathbb{R}^n)$

[vi] The relevant formulation is: For $f \in L_1$ and $g \in \mathcal{C}^0$ it follows that $f \cdot g \in L_1$ and $||f \cdot g||_1 \leq ||f||_1\, ||g||_\infty$. See section 4.2 for details.

Let V_2 be any complementary subspace and let $\mathcal{P}_0$ be the projection which satisfies $\text{Range}\,\mathcal{P}_0 = V_1$ and $\ker \mathcal{P}_0 = V_2$. Then we can define a Green's function $W(t,s)$ by (2.15) above. This is clearly bounded and satisfies definition 2.11

The proof is completed by showing that when $\mathcal{L}$ is surjective, $|W(t,s)| \leq Ke^{-\alpha|t-s|}$, from which it follows that $\mathcal{P}_0$ is the projection on to the forward contracting subspace. The details, which are messy, can be found in [9] or [7]. ■

This result is also true for $\mathbb{R}$ but we can say a little more.

Theorem 2.5 *The following are equivalent*

- x_0 *is hyperbolic.* *[Hyp]*
- $\mathcal{L} : \mathcal{C}^1 \to \mathcal{C}^0$ *is surjective.* *[Surj]*
- $\mathcal{L} : \mathcal{C}^1 \to \mathcal{C}^0$ *is invertible.* *[Inj + Surj]*

Proof

[Hyp $\Longrightarrow$ Surj]

If there is an exponential dichotomy on $\mathbb{R}$ then by proposition 2.3 there is a unique, L_1 Green's function given by (2.14) and thus a unique, $\mathcal{C}^1$ solution of $\mathcal{L}\xi = \varphi$ for any $\varphi \in \mathcal{C}^0$.

[Hyp $\Longrightarrow$ Inj]

If (2.2) has an exponential dichotomy on $\mathbb{R}$ (with projection $\mathcal{P}_0$) then it has no non-trivial bounded solutions since the evolution of any initial condition with non-zero component in $\mathcal{P}_0$ (resp. $I - \mathcal{P}_0$) is unbounded in backward (resp. forward) time. Thus $\mathcal{L}$ is injective.

[Surj $\Longrightarrow$ Hyp]

Clearly, if $\mathcal{L}$ is surjective on $\mathcal{C}^0$ then it is also surjective on $\mathcal{C}^0_+$ and $\mathcal{C}^0_-$. By theorem 2.4, the variational equation (2.2) must have an exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$ and clearly the projections are the same. By theorem 2.3 we deduce that (2.2) has an exponential dichotomy on $\mathbb{R}$. ■

2.3.2 The reformulated definition of hyperbolicity

Because of theorem 2.5, we can take instead the invertibility of $\mathcal{L}$ as a definition of hyperbolicity.

Definition 2.12 *A bounded solution* x_0 *of (2.1) is* (strongly) uniformly hyperbolic *if* $\mathcal{L} = DG_{x_0} : \mathcal{C}^1 \to \mathcal{C}^0$ *is invertible.*

This definition encompasses the previous definitions of hyperbolic solution, 2.3, 2.6 and 2.7, and is especially suited to aperiodic systems. For most purposes, it is advantageous to use this definition. Exploiting the invertibility of $\mathcal{L}$ allows for clear and elegant proofs of most of the powerful properties of hyperbolic systems. I prove persistence and structural stability this way and one can also prove shadowing like this, for example, Lanford [22], Meyer & Hall [30], Sauer & Yorke [43] and Coomes, Kocak & Palmer [33].

Remark 2.4 *It is a basic result of linear analysis, for example [11], that when the inverse of a bounded linear operator exists, it is also a bounded linear operator.*

This means that for hyperbolic solutions $||\mathcal{L}^{-1}|| < \infty$. In fact, $||\mathcal{L}^{-1}||$ will be used as the principle measure of the hyperbolicity of x_0.

For the linearisation about an equilibrium of an autonomous system or a periodic solution we have the following simplification of theorem 2.12.

Theorem 2.6 *When Df_{x_0} is constant or periodic then the following are equivalent.*

- *x_0 is hyperbolic.* *[Hyp]*
- *$\mathcal{L}$ is surjective.* *[Surj]*
- *$\mathcal{L}$ is injective.* *[Inj]*

Proof

[Inj $\Longrightarrow$ Hyp]

We have seen from Floquet's theorem that $X(t) = P(t)e^{tB}$, so any characteristic exponent $e^{i\theta}$, on the imaginary axis generates a bounded solution $p(t)e^{i\theta}$ of the variational equation with $p(t)$ having the same period as Df_{x_0}. Thus if $\mathcal{L}$ is injective, x_0 must have all characteristic exponents off the imaginary axis. Thus x_0 is hyperbolic.

The rest of the proof follows from theorem 2.5. ■

2.3.3 Splitting index

Although the absence of non-trivial bounded solutions of the homogeneous equation implies hyperbolicity when Df_{x_0} is periodic, it may not be enough when Df_{x_0} is aperiodic. A simple example which has injective linearisation but no exponential dichotomy is given by the linear scalar equation

$$\dot{\xi} = \tan^{-1}(t)\xi \tag{2.16}$$

Notice that solutions expand in both forward and backward time so there is no forward or backward contracting subspace and thus x_0 is not hyperbolic.

The question of whether the absence of non-trivial bounded solutions of a certain homogeneous equation is equivalent to hyperbolicity can be answered to some extent using the idea of the *splitting index*. The theory was developed in the general context of skew-product systems by Sacker & Sell [38, 39, 40, 41, 42], where full details can be found.

The splitting index theory is helpful in understanding better the functional analytic characterisation of hyperbolicity although it is not of immediate use in providing quantitative information. For this purpose, working solely with the invertibility criterion is sufficient and in many ways preferable to working in the skew-product framework. However for completeness I present here the relevant ideas of this theory.

Definition 2.13 *Let $\varphi : X \times Y \times \mathbb{R} \to X$ and $\sigma : Y \times \mathbb{R} \to Y$ be autonomous flows. Then $\pi : X \times Y \times \mathbb{R} \to X \times Y$ defined by*

$$\pi(x, y, t) = (\varphi(x, y, t), \sigma(y, t))$$

is a skew-product flow.

φ is called the forced dynamics and σ is called the forcing dynamics.

In relation to the (non-autonomous) linearisation $\mathcal{L}\xi = \dot{\xi} - Df_{x_0}\xi = 0$, we should think of Df_{x_0} as belonging to a space of bounded matrix-valued functions, A suitable space is $\mathcal{A} = C^0(\mathbb{R}, BL(\mathbb{R}^n, \mathbb{R}^n))$.[vii] This is a Banach space if we use sup-norm.

The way to treat the linearisation as a linear skew-product flow is to consider $Df_{x_0} \in \mathcal{A}$ as a forcing variable and ξ as the forced variable. This gives us the skew-product flow

$$\begin{aligned} \pi : X \times \mathcal{A} \times \mathbb{R} &\longrightarrow X \times \mathcal{A} \\ (\ \xi\ ,\ A\ ,\ s\ ) &\longmapsto (\ X_0(s)\xi\ ,\ \sigma(A, s)\ ) \end{aligned}$$

where X_0 is the principal solution operator of $\dot{\xi} = A(t)\xi$ and $\sigma(A, s)(t) = A_s(t) = A(t + s)$ is the translation operator in $\mathcal{A}$.

In order to give a good characterisation of hyperbolicity using this formalism we need some definitions.

Definition 2.14 *The* hull *of $A \in \mathcal{A}$ is the closure in $\mathcal{A}$ of all its translates. That is*

$$H(A) = \text{cl}\ \{\sigma(A, s) \mid s \in \mathbb{R}\}$$

[vii] $BL(\mathbb{R}^n, \mathbb{R}^n)$ is the space of $n \times n$ matrices.

The topology on $\mathcal{A}$ in which this closure is taken is very important. The topology inherited from sup-norm is that of uniform convergence on the whole of $\mathbb{R}$. This is not always appropriate. An alternative is to use the topology of uniform convergence on *compact subsets*. This can be obtained by using the metric

$$d(A_{ij}, B_{ij}) = \sum_{k=1}^{\infty} 2^{-k} \sup_{t\in[-k,k]} |A_{ij}(t) - B_{ij}(t)|$$

on each matrix element of $A, B \in \mathcal{A}$.

Proposition 2.4 *(i) In $\mathcal{A}$ with the topology of uniform convergence on compact subsets, $H(Df_{x_0})$ is compact if and only if Df_{x_0} is bounded and uniformly continuous with respect to t.*

(ii) In $\mathcal{A}$ with sup-norm, $H(Df_{x_0})$ is compact if and only if Df_{x_0} is almost-periodic.

For C^1 solutions x_0 we expect Df_{x_0} to be continuous and bounded so uniform continuity is a little stronger than we would want. By working in spaces of uniformly continuous functions however, we could guarantee Df_{x_0} is uniformly continuous. For the remainder of this section on the splitting index I assume that the appropriate topology is chosen so that $H(Df_{x_0})$ is compact. This is a crucial assumption for the theory that follows.

Definition 2.15 *$A^+ \in \mathcal{A}$ is an* ω-limit point *of A if there exists a sequence of times $t_k \to \infty$ with $\sigma(A, t_k) \to A^+$.*

$A^- \in \mathcal{A}$ is an α-limit point of A if there exists a sequence of times $t_k \to -\infty$ with $\sigma(A, t_k) \to A^-$.

The ω-limit set *of $A \in \mathcal{A}$ is the set of ω-limit points of A and is denoted by $\omega(A)$.*

The α-limit set *of $A \in \mathcal{A}$ is the set of α-limit points of A and is denoted by $\alpha(A)$.*

Clearly $\alpha(A)$ and $\omega(A)$ are subsets of $H(A)$.

Definition 2.16 *For $A \in \mathcal{A}$ let $E_0^+(A)$ (resp. $E_0^-(A)$) be the forward (resp. backward) contracting subspace (at time $t = 0$) of the equation $\dot{\xi} = A(t)\xi$. Define the* bounded set *by*

$$\mathcal{B}(A) = \{ \xi \in C^0 \mid \dot{\xi} = A(t)\xi \}$$

Using this notation the forward and backward contracting subspaces for the variational equation are $E_0^{\pm} = E_0^{\pm}(Df_{x_0})$. Notice also that $\mathcal{B}(Df_{x_0}) = \{0\}$ is another way of saying $\mathcal{L}$ is injective.

Condition (*): For each $A^+ \in \omega(Df_{x_0})$ and $A^- \in \alpha(Df_{x_0})$, suppose that

$$\mathcal{B}(A^+) = \{0\} \qquad \text{and} \qquad \mathcal{B}(A^-) = \{0\}$$

This condition asks for the absence of non-trivial bounded solutions to the homogeneous equations corresponding to each element of the α and ω-limit sets of Df_{x_0}.

Definition 2.17 *If Condition (*) holds, then for any $A^+ \in \omega(Df_{x_0})$ and $A^- \in \alpha(Df_{x_0})$ define*

$$\begin{aligned} d_\omega &= \dim E_0^+(A^+) \\ d_\alpha &= \dim E_0^+(A^-) \end{aligned}$$

d_ω and d_α are well defined, that is independent of the choices of A^+ and A^-, since $\omega(Df_{x_0}), \alpha(Df_{x_0})$ are compact and connected and $\dim E_0^+(\cdot)$ is continuous.

Remark 2.5 *If Condition (*) holds then there are exponential dichotomies on both half-lines $\mathbb{R}^+$ and $\mathbb{R}^-$ and*

$$\begin{aligned} \dim E_0^+(Df_{x_0}) &= d_\omega \\ \dim E_0^-(Df_{x_0}) &= n - d_\alpha \end{aligned}$$

where n is the dimension of the phase space for the system.

Definition 2.18 *The* splitting index *of $\mathcal{L}$ is defined as*

$$\kappa = d_\omega - d_\alpha$$

Remark 2.6 *Clearly,*

$$-n \leq \kappa \leq n$$

Theorem 2.7 *When Condition (*) holds, or equivalently, $\mathcal{L}\xi = 0$ has exponential dichotomy on both half-lines, $\mathcal{L}$ is Fredholm and its Fredholm index is κ.*

Recall that a linear operator T on Banach spaces is said to be Fredholm if range T is closed and both dim ker T and dim coker T are finite. The Fredholm index is given by the difference in these dimensions.

Note that Condition (*) and $\kappa = 0$ can hold without there being an exponential dichotomy on $\mathbb{R}$. We have already come across a simple example of this. An orbit $x_0(t)$, homoclinic to a hyperbolic saddle point $\bar{x}$, in an autonomous system. In this case, $\alpha(Df_{x_0}) = \omega(Df_{x_0}) = Df_{\bar{x}}$ and clearly there is an exponential dichotomy on both half-lines. However, although $\kappa(Df_{x_0}) = 0$, we have non-trivial bounded solutions of the linearisation and there fails to be a splitting over the whole of $\mathbb{R}$. Thus knowledge of just the asymptotic behaviour of the system is not enough in general to determine hyperbolicity.

Condition (**): For each $A \in H(Df_{x_0})$, suppose that $\mathcal{B}(A) = \{0\}$.

Note that this is stronger than condition (*).

Remark 2.7 *If Condition (**) holds then*

$$d_\omega \leq d_\alpha$$

and it follows that

$$-n \leq \kappa \leq 0$$

The main characterisations of hyperbolicity using this approach are given by the following two theorems.

Theorem 2.8 *x_0 is hyperbolic if and only if Condition (**) holds and $\kappa = 0$.*

Theorem 2.9 *x_0 is hyperbolic if and only if Condition (**) holds and for every $A \in H(Df_{x_0})$ we have $\dim E_0^+(A)$ independent of A.*

To determine hyperbolicity using these two theorems we must at least show that there are no non-trivial bounded solutions of the homogeneous equations associated with elements of the compact hull $H(Df_{x_0})$. In general this is stronger than injectivity of $\mathcal{L}$. However this can be seen as a generalisation of theorem 2.6 to aperiodic systems since it relates the behaviour of solutions to homogeneous equations to the solvability of inhomogeneous equations.

When Df_{x_0} is periodic, theorem 2.6 follows quickly using these arguments since in this case the ω-limit set, the α-limit set and the hull $H(Df_{x_0})$ in $\mathcal{A}$ with sup-norm is just the set of translates of Df_{x_0}.[viii] The hull is compact by proposition 2.4 above. If $\mathcal{L}$ is injective then only the zero solution is bounded for all translates of the homogeneous equation so Condition (**) clearly holds. Also it is clear that the dimensions of the forward and backward contracting subspaces must remain the same. Thus $\kappa(Df_{x_0}) = 0$ and we deduce that x_0 is hyperbolic.

Our example of an injective linearisation without exponential dichotomy, equation (2.16), clearly has $\kappa(Df_{x_0}) = -1$. To see this notice that $\omega(\tan^{-1}(\cdot)) = 1$ and $\alpha(\tan^{-1}(\cdot)) = -1$. Then using the fact that $E_0^+(1) = 0$ and $E_0^+(-1) = \mathbb{R}$ we have $d_\omega - d_\alpha = -1$. So although in this case $\mathcal{L}$ is injective, it is not invertible.

2.3.4 Spectral properties of $\mathcal{L}$

For completeness I detail some properties of the spectrum of $\mathcal{L}$. Again, although this is useful in understanding better the invertible operator characterisation of

[viii] The set of translates of Df_{x_0} is closed since sup-norm is translationally invariant. To be more precise one can consider $\mathcal{A}$ quotiented by the equivalence relation identifying translates. The set of translates of Df_{x_0} map to a single point under the quotient map and since it is continuous (using the quotient topology), the pre-image of this point must be closed.

hyperbolicity but not of direct applicability to the problem of quantitative safety estimates.

The spectrum of $\mathcal{L}$ is closely related to what is known as the Mather spectrum. More details about this approach can be found in Mather [27] where autonomous systems, both discrete and continuous time, are considered. Proofs of all the theorems below relating to spec $\mathcal{L}$ can be found in Van Minh [53, 54] where there are also generalisations of the results for finite dimensional phase spaces to Banach spaces.

For a linear operator on a Banach space $T : B \to B$, let the spectrum be denoted by spec T. The spectrum is also well defined when the domain of T is only dense in B. It follows that $\mathcal{L}$ has a well defined spectrum since $\mathcal{C}^1$ is dense in $\mathcal{C}^0$. Let $\mathcal{L}_\lambda = \frac{d}{dt} - (Df_{x_0} + \lambda I)$. Let $\rho(\mathcal{L})$ be the resolvent set of $\mathcal{L}$. Notice that $\lambda \in \rho(\mathcal{L})$ if and only if $\mathcal{L}_\lambda$ is invertible or equivalently $\mathcal{L}_\lambda \xi = 0$ has an exponential dichotomy.

The following theorem tells us that spec $\mathcal{L}$ is translationally invariant with respect to the imaginary axis.

Theorem 2.10 $\lambda \in \operatorname{spec} \mathcal{L} \implies \lambda + i\omega \in \operatorname{spec} \mathcal{L} \quad \forall \omega \in \mathbb{R}$.

More specifically, it can be shown that the spec $\mathcal{L}$ consists of a finite number of bands $\{[a_j, b_j] + i\omega,\ \omega \in \mathbb{R}\}$. For equilibria and periodic solutions the bands are degenerate in that they are actually just lines. The real values are given by the eigenvalues of the Jacobian and the Floquet exponents respectively. However, this is typically not the case for aperiodic systems where examples can be constructed which have bands of non-zero width.

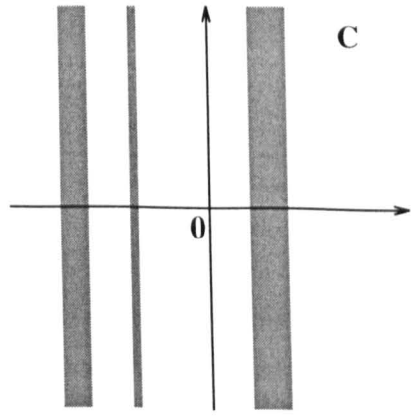


Figure 2.2: Typical form of spec $\mathcal{L}$ for aperiodic systems.

Note that spec $-Df_{x_0}$ considered as an operator $-Df_{x_0} : \mathcal{C}^0 \to \mathcal{C}^0$ also has this same translation invariance. For example, if x_0 is an equilibrium, then $-\mu + i\omega \in$ spec Df_{x_0}, for each eigenvalue μ, of the Jacobian matrix Df_{x_0} and any $\omega \in \mathbb{R}$. It

follows that $-\mu + i\omega \in \operatorname{spec} \mathcal{L}$.

Theorem 2.11 *$\mathcal{L}$ is the infinitesimal generator of the (strongly continuous) group of evolution operators $\{\mathcal{T}^h : \mathcal{C}^0 \to \mathcal{C}^0 \ ; \ h \in \mathbb{R}\}$ with $\mathcal{T}^h$ defined by*

$$(\mathcal{T}^h \xi)(t) \ = \ X_{t-h}(t) \ \xi(t-h)$$

To see associativity notice that

$$\begin{aligned}(\mathcal{T}^h \circ \mathcal{T}^g \ \xi)(t) &= X_{t-h}(t) \ X_{t-h-g}(t-h) \ \xi(t-h-g) \\ &= X_{t-h-g}(t) \ \xi(t-h-g) \\ &= (\mathcal{T}^{h+g} \xi)(t)\end{aligned}$$

The following is a spectral mapping theorem relating $\mathcal{T}^h$ and $\mathcal{L}$.

Theorem 2.12 $\operatorname{spec} \mathcal{T}^h \ = \ \exp(h \operatorname{spec} \mathcal{L})$.

It follows that $\operatorname{spec} \mathcal{T}^h$ is rotationally invariant and consists of finitely many annuli centred at the origin.

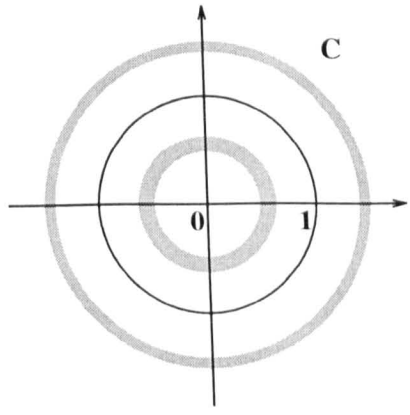


Figure 2.3: Typical form of $\operatorname{spec} \mathcal{T}^1$ for aperiodic systems.

$\operatorname{spec} \mathcal{T}^1$ is essentially the same as the Mather spectrum [27]. He first proved it was rotationally invariant and used it as a characterisation of hyperbolicity for both discrete time and continuous time autonomous systems.

Recall that an operator is said to be hyperbolic if its spectrum is disjoint from the unit circle. It follows from the previous theorem that

Corollary 2.1 *x_0 is hyperbolic if and only if $\mathcal{T}^h$ is hyperbolic for $h \neq 0$.*

Remark 2.8 *For a non-equilibrium solution of a continuous-time autonomous system, the spectrum of the linearised evolution operator* $\operatorname{spec} \mathcal{T}^h$*, always contains the unit circle. Thus it cannot be (strongly) hyperbolic.*

2.4 Persistence of hyperbolic solutions

The key property of hyperbolic solutions that I will exploit is their robustness to perturbations of the dynamical system. The robustness of the exponential dichotomy property of linear time-varying ODE's is well known and theorems relating to this can be found in [7, 9]. From this property one can easily deduce the robustness of hyperbolic solutions of nonlinear ODE's. Below I prove this in a Banach space setting using the implicit function theorem. This approach has three main advantages.

- Firstly the analysis is clean and relies on a well known theory.
- Secondly, it generalises very easily since many problems of robustness can be treated as root-finding of operators on Banach spaces.
- Thirdly, the proof gives a natural way to estimate the max size of the response when the perturbations are unknown and to compute the response numerically when the perturbation is known.

Given a reference system $\dot{x} = f_0(x,t)$, with solution $x_0 \in \mathcal{C}^0$, consider the perturbed system

$$\dot{x} = f(x,t) \tag{2.17}$$

where f is a $\tilde{\mathcal{C}}^1$-perturbation of f_0.

In the function space setting we consider the operator $G : \tilde{\mathcal{C}}^1 \times \mathcal{C}^1 \to \mathcal{C}^0$, defined by

$$G_f(x)(t) = \dot{x}(t) - f(x(t),t)$$

Roots of G_f are bounded solutions of (2.17). Writing G_0 in place of G_{f_0} we see that $G_0(x_0) = 0$ since x_0 is the solution of the unperturbed system.

Theorem 2.13 (Persistence of hyperbolic solutions) *Let x_0 be a hyperbolic solution of $\dot{x} = f_0(x,t)$. Then for $\tilde{\mathcal{C}}^1$-close f, there is a locally unique continuation, $x_f \in \mathcal{C}^1$, solving $\dot{x} = f(x,t)$. Moreover, x_f is also hyperbolic with the same stability type as x_0 and varies smoothly with f.*

Proof To prove theorem 2.13 we apply the implicit function theorem (IFT) to G. Details of the IFT can be found in appendix A. The conditions for the IFT are:

1. G is $\mathcal{C}^1$ with respect to x in a neighbourhood of (f_0, x_0).

2. G is $\mathcal{C}^1$ with respect to f in a neighbourhood of (f_0, x_0). In fact continuous suffices for the existence of a unique continuation x_f, but for the stronger conclusion that x_f is hyperbolic and smooth in f we require G to be smooth in f.

3. DG_{0,x_0} is invertible.

These are satisfied since:

1. G is $\mathcal{C}^1$ in x since f is $\mathcal{C}^1$ in x.

2. G is $\mathcal{C}^1$ in f since for $\delta f \in \tilde{\mathcal{C}}^1$,

$$\frac{\partial G_f}{\partial f}|_{f,x_f}(\delta f) = \delta f(x_f(\cdot), \cdot) \tag{2.18}$$

3. The linearisation about x_0 is

$$\mathcal{L}_0 = DG_{0,x_0} = \left(\frac{d}{dt} - Df_{0,x_0}\right) \tag{2.19}$$

The hypothesis that x_0 is hyperbolic guarantees that $\mathcal{L}_0$ is invertible with bounded inverse.

So for any f in some $\tilde{\mathcal{C}}^1$ neighbourhood of f_0, the IFT guarantees the existence of a locally unique continuation $x_f \in \mathcal{C}^1$ satisfying $G_f(x_f) = 0$ and $x_{f_0} = x_0$.

Since invertibility of operators on Banach spaces is an open condition,[ix] $\mathcal{L}_f$ defined by

$$\mathcal{L}_f = DG_{f,x_f} = \left(\frac{d}{dt} - Df_{x_f}\right)$$

is also invertible for close enough f. To see this, notice that $||\mathcal{L}_f - \mathcal{L}_0||$ can be made as small as we like since, by the $\mathcal{C}^1$ assumption on G, $\mathcal{L}_f$ is continuous with respect to f at f_0. Thus x_f is also hyperbolic.

From the construction of the splitting for $\mathcal{L}_f$ invertible we see that the splitting varies continuously. Thus the dimensions are conserved and x_f retains stability type. ■

Corollary 2.2 *While $\mathcal{L}_f$ remains invertible, x_f is at least $\mathcal{C}^1$ in f and*

$$\begin{aligned} \frac{dx_f}{df} : \tilde{\mathcal{C}}^1 &\longrightarrow \mathcal{C}^1 \\ \delta f &\longmapsto \mathcal{L}_f^{-1}\, \delta f(x_f(\cdot), \cdot) \end{aligned} \tag{2.20}$$

[ix] See lemma 3.1 for details.

Proof Recall that DG is given by (2.19) and that $\frac{\partial G}{\partial f}$ is given by (2.18). It follows from the implicit function theorem and the chain rule for differentiation[x] that $\frac{dx_f}{df} = -DG^{-1} \circ \frac{\partial G}{\partial f}$. ■

If G_f is only continuous in f then the IFT can still be applied to guarantee a unique continuation, x_f. However, the differential equation above is no longer valid and some extra work is required in order to provide safety estimates. Some more comments on this can be found in section 3.5.

Using uniform norms in $\tilde{\mathcal{C}}^1$ and $\mathcal{C}^1$ is necessary in order to guarantee a uniformly close response. A situation which does *not* fit into this framework is additive forcing with small frequency modulation, for example, $\dot{x} = F(x) + \sin(\omega_\varepsilon(t))$, with $\omega_\varepsilon(t) \approx \omega_0 t$. In this case, $\sin(\omega_\varepsilon(t))$ is not continuous (in $\mathcal{C}^0$ with sup-norm) with respect to ε so we do not expect a locally unique continuation for small ε. However, by allowing a reparametrisation of time we *can* make the perturbed system uniformly close and then apply the theorem above. For details of this see section 5.2.

2.5 More properties of hyperbolicity

Now I detail some more of the well known properties of hyperbolicity in the context of non-autonomous systems. It is instructive to show how they are easily and elegantly obtained using the invertible linear operator characterisation of hyperbolicity. It will become clear that estimates can be obtained not just for robustness to perturbation but also for asymptotic stability of attractors, structural stability of hyperbolic invariant sets and shadowing of pseudo-orbits. These are some of the most powerful notions in dynamical systems theory and constructive proofs that apply to aperiodic systems should prove useful in applications.

2.5.1 Stability

Theorem 2.14 *If x_0 is linearly attracting then it is uniformly asymptotically stable.*

This is proved in detail in section 3.6. The proof is used to obtain estimates of the size of the basin of attraction of a given linearly attracting solution x_0. The approach can be extended to provide basin of attraction estimates for the unique response to $\tilde{\mathcal{C}}^1$-small perturbations. This is very straightforward with the invertible operator characterisation of hyperbolicity but is not easily achieved using other techniques since the perturbed system is essentially unknown.

[x] See appendix A for details.

2.5.2 Hyperbolic sets

In this section I show that the definition of a hyperbolic solution can be generalised to sets of solutions. Hyperbolic sets provide a good basis for the analysis of more complicated dynamical behaviour. For example, the hyperbolic set found in Smale's horseshoe map is an ubiquitous source of chaotic dynamics as it is present whenever there are transverse homoclinic intersections.

I should point out first that most of the chaotic attractors found in applications are not uniformly hyperbolic, for example, the Henon attractor. However, the theory of uniform hyperbolicity is an important first step towards understanding the more complicated typical situations. It is only recently that non-uniform generalisations of hyperbolicity have been treated in any depth. See Katok & Hasselblatt's book [20] for an example of this.

As I previously mentioned, for continuous time autonomous systems, the only (strongly) hyperbolic objects are equilibria. There is a modification which allows one to work with more interesting hyperbolic sets for autonomous systems but I do not include that here.

First I give the definition of a hyperbolic set which is best suited to aperiodic systems. Then I state the standard definition and show that they are equivalent. As usual, by hyperbolic I mean strongly uniformly hyperbolic.

A definition of hyperbolic set

Usually one identifies a *solution* of (2.1) with its *orbit* in the extended phase space, However, I will need to draw a distinction between them. Recall that $x_0 \in \mathcal{C}^1$ is a *bounded solution* of (2.1) if $G(x_0) = 0$.

Definition 2.19 *A compact set of bounded solutions* $\Omega = \{x_j\} \subset \mathcal{C}^1$ *is called a* hyperbolic set *if each solution* $x_j \in \Omega$ *is hyperbolic and* $\left\|\mathcal{L}_{x_j}^{-1}\right\| < M$ *for some* $M \in \mathbb{R}$ *independent of* x_j

Standard definition

Definition 2.20 *The* orbit *of a bounded solution,* x_0*, is the set*

$$\{(x_0(t), t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^n \times \mathbb{R}$$

Definition 2.21 $\Lambda \subset \mathbb{R}^n \times \mathbb{R}$ *is a* bounded invariant set *if for each* $(x^0, t^0) \in \Lambda$ *there is a solution* $x_0 \in \mathcal{C}^1$ *satisfying* $x_0(t^0) = x^0$ *whose orbit* $\{(x_0(t), t) \mid t \in \mathbb{R}\}$ *is contained in* Λ.

Thus Λ is just a set of orbits of bounded solutions.

Definition 2.22 *A bounded invariant set, Λ, is said to be a* hyperbolic invariant set *if there are constants, $K, \alpha, J > 0$, and for each $(x^0, t^0) \in \Lambda$ a continuous splitting $\mathbb{R}^n = E^+_{(x^0,t^0)} \oplus E^-_{(x^0,t^0)}$ such that*

$$\begin{aligned} \xi^+ \in E^+_{(x^0,t^0)} &\Longrightarrow \left|X_{t^0}(t)\xi^+\right| \leq Ke^{-\alpha(t-t^0)} \left|\xi^+\right| \qquad t \geq s \\ \xi^- \in E^-_{(x^0,t^0)} &\Longrightarrow \left|X_{t^0}(t)\xi^-\right| \leq Ke^{-\alpha(t-t^0)} \left|\xi^-\right| \qquad t \leq s \\ \xi^+ \in E^+_{(x^0,t^0)},\ \xi^- \in E^-_{(x^0,t^0)} &\Longrightarrow \left|\xi^+ + \xi^-\right| \leq J \end{aligned}$$

where $X_{t^0}(t)$ is the evolution operator of the linearisation about the solution passing through (x^0, t^0).

Equivalence of the definitions

Theorem 2.15 *$\Omega = \{x_j\}$ is a hyperbolic set if and only if the bounded invariant set $\Lambda = \{(x_j(t), t) \mid t \in \mathbb{R}\}$ is hyperbolic.*

Proof This follows from theorem 2.2 and the uniformity of M, K, α ■

2.5.3 Structural stability

The notion of structural stability is extremely important when dealing with systems subject to unknown, aperiodic perturbations. Roughly speaking, a system is structurally stable if all 'nearby' systems display 'similar' behaviour. By 'nearby' it is meant a system which is close in some norm or topology in a space of vector fields. By 'similar' it is typically meant homeomorphic (topological) conjugacy or diffeomorphic (smooth) conjugacy. An important conclusion to draw is that if one can find a structurally stable model of a given phenomenon then any conclusions made will be robust to small inaccuracies and small perturbation in the actual model used.

Asking for structural stability of the whole dynamical system is often too strong a restriction since one is usually only interested in a subset of the dynamics, for example the attracting invariant sets. We say that an invariant set is structurally stable if it has has a topologically conjugate invariant set nearby for all close enough systems. When the set is just the orbit of a single hyperbolic solution then theorem 2.13 already tells us it is structurally stable. Below I deal with the case of general hyperbolic sets.

Even more generally, one could ask for structural stability of the *chain recurrent set*. It is beyond the scope of this thesis to deal with this notion but Robinson [36], for example, contains an introduction to the topic.

A word of warning is in order; it is natural to expect that since the structurally stable systems form an open set, the non-structurally stable systems are 'rare' and thus ignorable. This is not the case however. Structural stability is not usually a generic property of dynamical systems and in some sense it is rare itself. Thus restriction to models which are robust in this sense is not always a good thing to do a priori. Finding better general notions of robustness is still an open problem however.

An interesting first step in the question of structural stability of aperiodic systems was the concept of *time-dependent stability* of diffeomorphisms. The reader is referred to Franks [12] for details but roughly speaking a C^k diffeomorphism f, on a compact manifold is said to be time-dependent stable if there is a neighbourhood $U \supset f$, in the space of C^k diffeomorphisms such that for any n, f^n is topologically conjugate to any sequence of diffeomorphisms $f_1 \circ \cdots \circ f_n$ with each f_i picked from U. It turns out that for C^1 diffeomorphisms the notions of time-dependent stability, structural stability, hyperbolicity and shadowability are all in some sense equivalent. The situation for aperiodic continuous-time systems is analogous.

Definition 2.23 *A hyperbolic set, $\Omega_0 \subset C^1$, for $\dot{x} = f_0(x,t)$, is said to be $\tilde{C}^1$-*structurally stable *if, for each $\tilde{C}^1$-close system $\dot{x} = f(x,t)$, there is a C^1-close set of solutions, Ω_f, given by a homeomorphism $h_f : \Omega_0 \to \Omega_f$.*

Theorem 2.16 *A bounded set of solutions, Ω, is $\tilde{C}^1$-structurally stable if and only if it is hyperbolic.*

Proof [$\Longrightarrow$] The proof that $\tilde{C}^1$-structurally stable invariant sets are hyperbolic is extremely difficult and has only recently been given in the context of diffeomorphisms. I do not include it here but the reader is referred to Mane [25] or Robinson [36] for more details. It would however, be useful to have a formulation of this proof which used the invertible operator characterisation of hyperbolicity. Note that it is believed not to be true in $\tilde{C}^2$.

[$\Longleftarrow$] I prove now that hyperbolic sets are $\tilde{C}^1$-structurally stable.

The following lemma tells us that the solutions in hyperbolic sets are locally unique.

Lemma 2.2 (Expansivity) *If $\Omega = \{x^j\}$ is a hyperbolic set then there is a $\delta > 0$ such that for each $x^1, x^2 \in \Omega$, $||x^1 - x^2||_{C^1} < \delta \implies x^1 = x^2$.*

Proof **(Lemma)** This can be proved using the exponential dichotomy property of the linearisation but it is more direct to use the fact that hyperbolic solutions

are locally unique. This follows from the proof of the implicit function theorem.[xi] The lemma follows by noticing that the size of the neighbourhood of uniqueness for solutions in the hyperbolic set can be bounded below by some δ depending only on $||\mathcal{L}^{-1}||$ and $||\mathcal{L}||$. ■**(Lemma)**

This is called *expansivity* since it follows that any two distinct solutions $x^1, x^2 \in \Omega$ cannot have $|x^1(t) - x^2(t)| < \delta$ for all time. Expansivity actually holds in a small enough neighbourhood of Ω although this requires a bit more work in proving. See Lanford [22] for details.

By persistence of hyperbolic solutions (theorem 2.13), we have for each $x^j \in \Omega_0$ a locally unique, C^1-close, x_f^j, which solves $G_f(x_f^j) = 0$. Setting $\Omega_f = \{x_f^j\}$, I show that the mapping $\Omega_0 \to \Omega_f$, defined by $x^j \mapsto x_f^j$, is a homeomorphism.

It is surjective by construction.

To show it is injective, consider $x^1, x^2 \in \Omega_0$. Choose $\varepsilon = ||f - f_0||$ small enough so that, for each $x^j \in \Omega_0$, $||x_f^j - x^j|| < \delta/2$, with δ given by the expansivity lemma 2.2. Now if $x_f^1 = x_f^2$ then it follows from the triangle inequality that $||x^1 - x^2|| < \delta$ which, by lemma 2.2, means that $x^1 = x^2$ as required.

To complete the proof I show continuity. Now choose ε small enough so that, for each $x^j \in \Omega_0$, $||x_f^j - x^j|| < \delta$, with δ smaller than the size of the neighbourhood of uniqueness of x_f.

Now consider any sequence of the form $x^j \to x^*$, as $j \to \infty$ where $x^j \in \Omega_0$ and suppose $y = \lim_{j\to\infty} x_f^j$. Note that $G_f(y) = 0$.

Since $||x_f^j - x^j|| < \delta$ for each j it follows that $||y - x^*|| < \delta$ as well. But this means $y = x_f^*$, since x_f^* is the locally unique continuation of x^*. This proves that the bijection $\Omega_0 \mapsto \Omega_f$ is continuous. Since h_f is a bijection from a compact space into a Hausdorff space it is a homeomorphism and Ω_0 and Ω_f are topologically conjugate. ■

Corollary 2.3 *It follows from smoothness of x_f, with respect to f, that $h_f : \Omega_0 \to \Omega_f$ can be made as close to the identity as required by choosing ε small enough.*

Remark 2.9 *Ω_f is also a hyperbolic set since for ε small enough we can find a uniform bound* $\left|\left|\mathcal{L}^{-1}_{f,x_f^j}\right|\right| < M(\varepsilon)$

Theorem 2.17 *For small enough ε every set of bounded solutions close enough to Ω_0 must be a hyperbolic set.*

[xi] See appendix A for the details of the IFT and section 3.5 for how to estimate the size of the neighbourhood of uniqueness.

Proof The basic idea is that hyperbolicity (as characterised by invertibility of $\mathcal{L}_0$) is a stable property. Invertibility is an open condition in the space of bounded linear operators[xii] and it is clear that C^1-close solutions have $||\mathcal{L}_{f,y} - \mathcal{L}_{0,y_0}||$ as small as required. If ε is chosen small enough then for each solution in the nearby set $\{y^i\}$, a uniform bound can be chosen for $||\mathcal{L}_f - \mathcal{L}_0||$. In this way $\left|\left|\mathcal{L}^{-1}_{f,y^i}\right|\right|$ can be uniformly bounded which proves hyperbolicity of $\{y^i\}$. ■

The definition of structural stability I have presented here is equivalent to the standard definition which deals with hyperbolic invariant sets satisfying definition 2.22.

Definition 2.24 *A bounded invariant set, Λ_0, for the system $\dot{x} = f_0(x,t)$, is said to be $\tilde{C}^1$-*structurally stable *if, for each $\tilde{C}^1$-close system $\dot{x} = f(x,t)$, there is a homeomorphic, bounded, invariant set, Λ_f with topologically conjugate dynamics. That is, there is a homeomorphism $g_f : \Lambda_0 \to \Lambda_f$ such that on Λ_f, the following relation holds*

$$\varphi_f \circ g_f = g_f \circ \varphi_0$$

where φ_f is the (non-autonomous) flow associated with $\dot{x} = f(x,t)$.

An invariant set is hyperbolic if and only if it is $\tilde{C}^1$-structurally stable. The proof that hyperbolicity implies structural stability given for theorem 2.16 can be adapted to show this. The topological conjugacy is given by $\Lambda_f = g_f(\Lambda_0)$ with g_f defined by

$$g_f(x^0, t^0) = (x_f(t^0), t^0)$$

where x_f is the unique continuation of the (hyperbolic) solution passing through (x^0, t^0). The proof that g_f is a homeomorphism is essentially the same as in theorem 2.16.

[xii] See lemma 3.1

Chapter 3

Safety Criteria

The main objective of this chapter is to show how the persistence theorem 2.13 can give useful rigourous safety criteria for very general systems of ODE's.

3.1 A useful definition of safety criterion

It is helpful at this stage to have a specific idea of what is meant by a safety criterion. The system we are interested in is

$$\dot{x} = f(x,t) \tag{3.1}$$

where f is a $\tilde{C}^1$-small perturbation of f_0.

Recall that $B_A(\mu, a_0)$ is the open μ-ball in A centred on $a_0 \in A$.

Definition 3.1 *Let x_0 be a hyperbolic solution for $\dot{x} = f_0(x,t)$.*

For any given $\bar{\eta} \in \mathbb{R}$, let the set of safe responses *be the ball $B_{C^1}(\bar{\eta}, x_0)$,*

f is said to be a safe perturbation *of f_0 if x_0 has unique hyperbolic continuation $x_f \in B_{C^1}(\bar{\eta})$, solving (3.1).*

The region of safe perturbations is then $\mathcal{F}_{\bar{\eta}} \subset \tilde{C}^1$, such that

$$\mathcal{F}_{\bar{\eta}} = \left\{ f \in \tilde{C}^1 \mid \exists!\ \textit{hyp. contn. } x_f \in B_{C^1}(\bar{\eta}, x_0) \right\}$$

Given a safety margin $\bar{\eta} \in \mathbb{R}$, any condition on f guaranteeing $f \in \mathcal{F}_{\bar{\eta}}$ is called a vigorous safety criterion.

Note that $\mathcal{F} = \mathcal{F}_\infty$ is the set of perturbations which have a unique bounded hyperbolic continuation.

An additional concern in safety issues is stability. However, while there is a unique hyperbolic continuation, stability type is conserved so no extra conditions need to be imposed in the definitions above.

From theorem 2.13 we can deduce the following.

Corollary 3.1 *Let x_0 be a hyperbolic solution of $\dot{x} = f_0(x,t)$. Then for each $\bar{\eta} > 0$, there exists an $\varepsilon_{\bar{\eta}} > 0$ such that $B_{\tilde{C}^1}(\varepsilon_{\bar{\eta}}, f_0) \subset \mathcal{F}_{\bar{\eta}}$. That is,*

$$f \in B_{\tilde{C}^1}(\varepsilon_{\bar{\eta}}, f_0) \implies \exists!\ x_f \in B_{C^1}(\bar{\eta}, x_0)$$

Proof This follows from continuity of x_f with respect to f. ■

Thus the persistence theorem for hyperbolic solutions guarantees the existence of a non-empty ball of safe perturbations centred on f_0 as we would expect.

3.1.1 Interpretation

One is usually interested in robustness of *attracting* solutions. The following argument makes it clear why definition 3.1 is useful in this case. For simplicity I consider only additive forcing but the same reasoning holds in the general case.

Consider the dynamical system defined by

$$\dot{x} = f_0(x,t) + u(t) \tag{3.2}$$

with x_0 a linearly attracting solution for $\dot{x} = f_0(x,t)$.

Proposition 3.1 *Suppose that $f_0 + u \in \mathcal{F}$ and that $u(t) = 0$ for $t \leq t_0$. Then the unique linearly attracting continuation, $x_u \in C^1$, satisfies $x_u(t) = x_0(t)$ for $t \leq t_0$.*

Proof This follows from the fact that x_0 is attracting for the unperturbed system. Suppose $x_u(t) \neq x_0(t)$ for some $t \leq t_0$ then locally, $|x_u(t) - x_0(t)|$ must grow exponentially in backwards time. But this is a contradiction as x_u is supposed to be a uniformly close continuation of x_0. ■

Later in the chapter we will see that for linearly attracting solutions, $\frac{dx_u}{du}$ is causal. This follows from the existence of a causal Green's function for the linearisation. This is another way of seeing the above proposition is true.

The proposition can be interpreted as follows. Suppose that for $t < t_0$ the system is following the hyperbolic solution $x_0(t)$. Now suppose that, from time t_0, some continuous (with respect to time) forcing u is applied. As long as $f_0 + u \in \mathcal{F}$, it follows that the 'response' of the system will be $x_u \in B$, even though there may be other unsafe attracting solutions for the system.

This is a non-autonomous extension of the idea that while continuable, under *quasi-static* variation of the vector-field the solution will simply move along non-singular portions of the bifurcation diagram. It is typically assumed that there is a continuous relaxation process occurring as the vector field is varied.

Since time is explicitly present in the theory presented here, we have in effect a vigorous verification of this 'folklore' principle. Moreover, one could provide reasonable estimates of how quickly this relaxation process took place. For example when going from $\dot{x} = f_1(x)$ to $\dot{x} = f_2(x)$ we just construct a sensible time dependent equation $\dot{x} = f(x,t)$ such that $f \to f_1$ $(resp. f_2)$ as $t \to -\infty$ (resp. $+\infty$).

Note that if we perturb discontinuously, for example, an impulse or a step function at time t_0, then theorem 2.13 is not strictly applicable. I extend the approach to allow for these cases in section 4.2.

When x_0 is hyperbolic but with nonempty backward contracting subspace then naturally this idea breaks down. We cannot think of $\mathcal{F}$ as being a safety region in the usual sense. Although there is still a unique bounded continuation x_u, solving (3.2), for each u small enough, it is given by a non-causal operator so the above proposition does not hold. In fact, perturbations, however small, lead to responses which do *not* track x_u, unless they happen to lie in it's forward contracting manifold. Thus x_u is effectively unobtainable for physical systems. Nonetheless I will still refer to x_u, when it exists, as the response.

Although in this case x_u will not be tracked by the dynamics or by any numerical simulation of the dynamics one could still perform the continuation if u is known and x_u is of interest. The best procedure would be to use Newton's method in the space $\mathcal{C}^1$ although one could also use the differential equation (2.20).

3.2 Persistence estimates

Now I present the basic strategy for providing estimates of $||x_f - x_0||$ and $\varepsilon_{\bar{\eta}}$. The idea will be to use the differential equation (2.20) to establish an upper bound for the size of the response of the system for a given size of perturbation.

Recall from (2.20) that

$$\frac{dx_f}{df}\,\delta f \;=\; \mathcal{L}_f^{-1}\,\delta f(x_f(\cdot),\cdot)$$

Let $\eta = ||x - x_0||_{\mathcal{C}^1}$ and $\varepsilon = ||f - f_0||_{\tilde{\mathcal{C}}^1}$. Then we have the differential inequality

$$\frac{d\eta}{d\varepsilon} \;\leq\; ||\mathcal{L}_f^{-1}|| \tag{3.3}$$

3.2.1 Inverting $\mathcal{L}$

Having characterised hyperbolicity of solutions in terms of the invertibility of the linearisation $\mathcal{L}$ it is important to find out as much as we can about the form

Thus we can now obtain (3.5) and (3.6) at any given $t \in \mathbb{R}$ by computing the single fundamental matrix solution $X_s(t)$, and inverting to get $X_s(t)$.

In fact $X_t(s)^{-1} = X_t^*(s)$, where X_t^* is the fundamental matrix solution of the adjoint equation

$$\dot{X}_t^* = -X_t^* Df_{x_0} \quad ; \quad X_t^*(t) = I$$

The Green's function (3.4) can now be written

$$W(t,s) = \begin{cases} X_t^*(s)\mathcal{P}_s & t > s \\ -X_t^*(s)(I - \mathcal{P}_s) & t < s \end{cases} \tag{3.7}$$

and (3.5) can be written

$$(\mathcal{L}^{-1}\varphi)(t) = \int_t^\infty X_t^*(s)\mathcal{P}_s\varphi(s)ds - \int_{-\infty}^t X_t^*(s)(I - \mathcal{P}_s)\varphi(s)ds \tag{3.8}$$

Bounds for $||\mathcal{L}^{-1}||$

To find a bound for $||\mathcal{L}^{-1}||$ we use Hölders inequality.[i] Since $W(t,\cdot) \in L_1$ and $\varphi \in L_\infty$, we deduce that

$$\begin{aligned} |(\mathcal{L}^{-1}\varphi)(t)| &\le ||W(t,\cdot)\varphi||_1 \le ||W(t,\cdot)||_1 \; ||\varphi||_\infty \\ \text{and} \quad \left|\frac{d}{dt}(\mathcal{L}^{-1}\varphi)(t)\right| &\le ||Df_{x_0}|| \; ||W(t,\cdot)\varphi||_1 + ||\varphi||_\infty \\ &\le (||Df_{x_0}|| \, ||W(t,\cdot)||_1 + 1) \; ||\varphi||_\infty \end{aligned}$$

In fact we can replace by equalities and get the stronger result

Proposition 3.2

$$||\mathcal{L}^{-1}|| = \max\left\{ \sup_t ||W(t,\cdot)||_1 \; , \; ||Df_{x_0}|| \sup_t ||W(t,\cdot)||_1 + 1 \right\}$$

Proof For any $t \in \mathbb{R}$, we can always choose a $\varphi_1 \in L_\infty$, with $||\varphi_1||_\infty = 1$, such that

$$|W(t,s)\varphi_1(s)| = |W(t,s)| \qquad \forall s \in \mathbb{R}, \; s \neq t$$

where $|W(t,s)|$ is the matrix norm for $W(t,s)$ with respect to $|\;|_{\mathbb{R}^n}$. In this case

$$\begin{aligned} |(\mathcal{L}^{-1}\varphi_1)(t)| &= \int_{\mathbb{R}} |W(t,s)|\, ds \\ &= ||W(t,\cdot)||_1 \end{aligned}$$

[i] See section 4.2 for details

of $\mathcal{L}^{-1}$. In particular, to obtain persistence estimates, we should investigate how to bound $||\mathcal{L}^{-1}||$. Note that this section applies to both the unperturbed and perturbed linearisations $\mathcal{L}_0$ and $\mathcal{L}_f$.

Suppose that x_0 is hyperbolic, with projection $\mathcal{P}_s$ on to the forward contracting subspace E_s^+. Recall that $\mathcal{L}^{-1}$ is the integral operator, with kernel given by the Green's function

$$W(t,s) = \begin{cases} X_s(t)\mathcal{P}_s & t > s \\ -X_s(t)(I-\mathcal{P}_s) & t < s \end{cases} \tag{3.4}$$

$\mathcal{L}^{-1}$ thus takes the form

$$\begin{aligned}(\mathcal{L}^{-1}\varphi)(t) &= \int_{\mathbb{R}} W(t,s)\varphi(s)ds \\ &= \int_t^{\infty} X_s(t)\mathcal{P}_s\varphi(s)ds - \int_{-\infty}^t X_s(t)(I-\mathcal{P}_s)\varphi(s)ds \end{aligned} \tag{3.5}$$

By differentiating we see that

$$\begin{aligned}\frac{d}{dt}(\mathcal{L}^{-1}\varphi)(t) &= (Df_{x_0}\mathcal{L}^{-1}\varphi)(t) + \varphi(t) \\ &= Df_{x_0}(t)\left(\int_t^{\infty} X_s(t)\mathcal{P}_s\varphi(s)ds - \int_{-\infty}^t X_s(t)(I-\mathcal{P}_s)\varphi(s)ds\right) + \varphi(t)\end{aligned} \tag{3.6}$$

so it gives a C^1 solution as required.

Computing $\mathcal{L}^{-1}$

Computing (3.5) and (3.6) seems somewhat troublesome. Recall that X_s is the fundamental matrix solution of (2.2) satisfying

$$\dot{X}_s = Df_{x_0}X_s \quad ; \quad X_s(s) = I$$

So computing the solution (3.5) and (3.6) for a *single* value of t involves finding a different fundamental matrix solution, X_s, for *each* $s \in \mathbb{R}$ and evaluating it at time t. Of course, since we have exponential localisation around $s = t$ we only need to consider s 'close' to t, where close could mean a few characteristic time lengths away. This is still very inefficient.

However, there is a way round this problem. Recall that $X_s(t) = X_t(s)^{-1}$ where X_t is the fundamental matrix solution of (2.2) which satisfies

$$\frac{d}{ds}X_t(s) = Df_{x_0}X_t(s) \quad ; \quad X_t(t) = I$$

and we have $||\mathcal{L}^{-1}\varphi_1||_\infty = \sup_t ||W(t,\cdot)||_1$.

Similarly, for any $t \in \mathbb{R}$, we can choose a $\varphi_2 \in L_\infty$, with $||\varphi_2||_\infty = 1$, such that

$$\begin{aligned}\left|\frac{d}{dt}(\mathcal{L}^{-1}\varphi_2)(t)\right| &= ||Df_{x_0}|| \int_{\mathbb{R}} |W(t,s)|\, ds + 1 \\ &= ||W(t,\cdot)||_1 + 1\end{aligned}$$

giving $\left|\left|\frac{d}{dt}(\mathcal{L}^{-1}\varphi_2)\right|\right|_\infty = \sup_t \left|\left|\dot{W}(t,\cdot)\right|\right|_1 + 1$.

Recall that, by definition,

$$||\mathcal{L}^{-1}|| = \sup_{\varphi\in\mathcal{C}^0,\ ||\varphi||=1} ||(\mathcal{L}^{-1}\varphi)||_{\mathcal{C}^1} = \sup_{\varphi\in\mathcal{C}^0,\ ||\varphi||=1} \max\left\{||\mathcal{L}^{-1}\varphi||_\infty, \left|\left|\frac{d}{dt}(\mathcal{L}^{-1}\varphi)\right|\right|_\infty\right\}$$

Since $\varphi \in \mathcal{C}^0$ can be chosen as close as we like to $\varphi_1, \varphi_2 \in L_\infty$, the proposition follows. ■

This proposition shows that we do not lose anything in this estimate if we can determine the Green's function accurately.

If we cannot find the Green's function accurately but we can find estimates for the hyperbolicity constants K, α, then we can use

$$\sup_t ||W(t,\cdot)||_1 \leq 2K/\alpha$$

although this is a fairly crude estimate.

Equilibria

Inverting $\mathcal{L}$ when x_0 is an equilibrium is easy because $\mathcal{L}$ has constant coefficients and is thus translationally invariant. The principal matrix solution is $X_0(t) = \exp(tDf_{x_0})$ which can be written down easily in terms of the eigenvalues and eigenvectors of Df_{x_0}. The evolution operator is given by $X_s(t) = X_0(t-s) = \exp((t-s)Df_{x_0})$. It is also easy to find E^+ and E^-. They are just the eigenspaces corresponding to eigenvalues in the left half-plane and right half-plane respectively. It follows that $\mathcal{P}_0$ is the projection with range E^+ and null space E^-.

The Green's function is $W(t,s) = W_0(t-s)$ with

$$W_0(t) = \begin{cases} \exp(tDf_{x_0}(t))\mathcal{P}_0 & t > 0 \\ -\exp(tDf_{x_0}(t))(I-\mathcal{P}_0) & t < 0 \end{cases}$$

Then $\mathcal{L}^{-1}\varphi$ is the convolution of W_0 and φ

$$\begin{aligned}(\mathcal{L}^{-1}\varphi)(t) &= (W_0 * \varphi)(t) \\ &= \int_{\mathbb{R}} W_0(t-s)\varphi(s)ds \\ &= \int_{\mathbb{R}} W_0(s)\varphi(t-s)ds \\ &= \int_{-\infty}^{0} -\exp(sDf_{x_0}(s))(I-\mathcal{P}_0)\varphi(t-s)ds \\ &\qquad + \int_0^{\infty} \exp(sDf_{x_0}(s))\mathcal{P}_0\varphi(t-s)ds \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\frac{d}{dt}(\mathcal{L}^{-1}\varphi)(t) &= Df_{x_0}(t)(W_0 * \varphi)(t) + \varphi(t) \\ &= Df_{x_0}(t)\left(\int_{-\infty}^{0} \exp(sDf_{x_0}(s))(I-\mathcal{P}_0)\varphi(t-s)ds \right. \\ &\qquad \left. + \int_0^{\infty} \exp(sDf_{x_0}(s))\mathcal{P}_0\varphi(t-s)ds\right) + \varphi(t)\end{aligned}$$

Since $\varphi \in \mathcal{C}^0$ and $W_0 \in L_1$, we deduce that $W_0 * \varphi \in \mathcal{C}^0$ and by Young's inequality, $||(W_0 * \varphi)||_\infty \leq ||W_0||_1 \, ||\varphi||_\infty$. From previous comments there is always a choice of φ such that we have the following equality

$$||\mathcal{L}^{-1}|| = \sup_{||\varphi||=1} ||W_0 * \varphi||_{\mathcal{C}^1} = \max\{\, ||W_0||_1 \,,\, ||Df_{x_0}||\,||W_0||_1 + 1 \,\}$$

Periodic solutions

Assume now that x_0 is a hyperbolic periodic solution of least period T.

From the basic Floquet theorem, 2.1, we see that there is a decomposition of the fundamental matrix solution

$$X_0(t) = P(t)\exp(tB)$$

where $P(t)$ has period T and B is constant. By finding the Monodromy operator, $\mathcal{M} = \exp(TB)$, one can obtain qualitative information about the solutions of the variational equation. Specifically, if the characteristic multipliers are off the unit circle then the variational equation has an exponential dichotomy. $\mathcal{M}$ generally cannot be obtained in closed form but is easy to calculate numerically.

Knowing $\mathcal{M}$ is not enough however, to find estimate the bound for the unique $\mathcal{C}^1$ solution of the inhomogeneous equation $\mathcal{L}\xi = \varphi$, we must know $P(t)$ as well.

Since the non-uniqueness of the characteristic exponents hinders us, the Floquet decomposition proves to be unhelpful in estimating $||\mathcal{L}^{-1}||$.

To get $\mathcal{L}^{-1}$ one should instead look for the Green's function defined by (2.14). This involves finding the fundamental solution matrix of the adjoint equation X_t^*. The Green's function $W(t,s)$ is periodic in t since Df_{x_0} is periodic. Proposition 3.2 then gives the bound for $||\mathcal{L}^{-1}||$ we are after. Due to the periodicity of $W(t,s)$ however, we only need to consider one period. To be specific, if the period is T then

$$||\mathcal{L}^{-1}|| = \max \left\{ \sup_{t\in[0,T]} ||W(t,\cdot)||_1 \ , \ ||Df_{x_0}|| \sup_{t\in[0,T]} ||W(t,\cdot)||_1 + 1 \right\}$$

Aperiodic solutions

For aperiodic solutions we again need to find for each t the fundamental matrix solution X_t^* for the adjoint equation. Equivalently, we can consider impulses at each time t and evaluate the impulse response. The Green's function is then obtained and a bound for $||\mathcal{L}^{-1}||$ can be found.

The problem here is that for aperiodic solutions with no simplifying structure one cannot practically perform this operation for all times $t \in \mathbb{R}$. If there are some additional features of the solution then there is some hope. For example, if the asymptotic behaviour is known, say x is eventually periodic or decays for large positive and negative times then one can effectively restrict to a finite number of computations.

As mentioned above, $||\mathcal{L}^{-1}||$ can also be estimated by bounding the hyperbolicity constants. Generally, this will be as difficult as finding the Green's function but in some cases may prove easier.

Another option would be to just consider a finite time and ignore the far distant future and past. This is not technically allowed in the theory I have presented so far but in principle one could formulate a definition of hyperbolicity for finite time within the same framework. An additional complication introduced is that we lose uniqueness of continuation unless we specify initial and final conditions on the correct contracting manifolds. This could be a very useful generalisation since in applications one is often interested only in finite time safety estimates.

3.2.2 Inverting $\mathcal{L}_f$

In the case that x_0 is an equilibrium or a periodic solution of the unperturbed system we can find a good upper bound for $||\mathcal{L}_0^{-1}||$. When x_0 is aperiodic things are

trickier as we have seen but one may still be able to achieve this. Say $||\mathcal{L}_0^{-1}|| \leq K_0$. Unless we have some restrictions or symmetries on the perturbed system we expect x_f to be aperiodic. If we are continuing x_0 numerically for a given perturbation of the system then we can find $||\mathcal{L}_f^{-1}||$ by the methods above. If however the perturbation is unknown we probably cannot use these methods. In certain cases we may be able to find the minimum exponential decay rate for the Green's function (and thus the hyperbolicity constants) in terms of $||x_f - x_0||$. See section 5.1 for a tractable case. However, the following lemma gives an a priori upper bound on $||\mathcal{L}_f^{-1}||$.

Lemma 3.1 *If $\mathcal{L}_0 : X \to Y$ is a bounded linear operator with bounded inverse and $\mathcal{L}_1 : X \to Y$ is a bounded linear operator such that $||\mathcal{L}_1 - \mathcal{L}_0|| < ||\mathcal{L}_0^{-1}||^{-1}$, then $\mathcal{L}_1$ is also invertible with*

$$||\mathcal{L}_1^{-1}|| \leq \left[||\mathcal{L}_0^{-1}||^{-1} - ||\mathcal{L}_1 - \mathcal{L}_0||\right]^{-1}$$

The proof can be found in many standard texts in linear analysis, for example [11].

So we look for $\beta(\varepsilon, \eta)$ such that

$$||\mathcal{L}_f - \mathcal{L}_0|| = ||Df_{x_f} - Df_{0,x_0}|| \leq \beta(\varepsilon, \eta)$$

and deduce that

$$||\mathcal{L}_f^{-1}|| \leq [K_0^{-1} - \beta(\varepsilon, \eta)]^{-1}$$

This is valid while $\beta(\varepsilon, \eta) < K_0^{-1}$.

3.2.3 Upper bounds for $||x_f - x_0||$ and estimates of safe perturbations

lemma 3.1 combined with (3.3) gives

$$\frac{d\eta}{d\varepsilon} \leq [K_0^{-1} - \beta(\varepsilon, \eta)]^{-1} \tag{3.10}$$

We can integrate this equation starting from $(\varepsilon, \eta) = (0, 0)$ until we hit the line $\beta(\varepsilon, \eta) = K_0^{-1}$, say at ε_0. Then $\varepsilon < \varepsilon_0$ guarantees a unique hyperbolic continuation and moreover $||x_f - x_0|| \leq \eta(\varepsilon)$.

To find a safe region of perturbation for some safety requirement $\bar{\eta}$, we integrate until $\eta(\varepsilon) = \min\{\ \bar{\eta}\ ,\ \eta(\varepsilon_0)\ \}$ say at $\varepsilon_{\bar{\eta}}$. Then clearly

$$\varepsilon < \varepsilon_{\bar{\eta}} \Rightarrow f \in \mathcal{F}_{\bar{\eta}}$$

Formally, we can use a dummy variable to split (3.10) into the two differential equations

$$\frac{d\eta}{ds} = 1 \quad ; \quad \frac{d\varepsilon}{ds} = K_0^{-1} - \beta(\varepsilon, \eta)$$

and integrate until $\frac{d\varepsilon}{ds}$ becomes 0. Figure 3.1 represents the integration schematically.

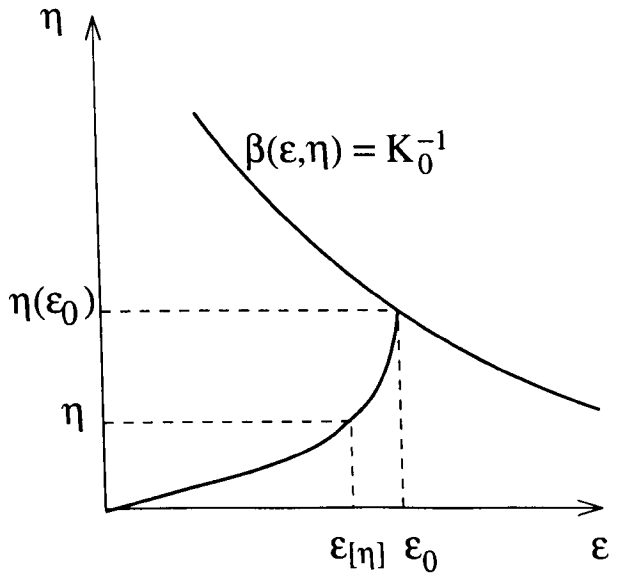


Figure 3.1: Continuation estimates (schematic)

Remark 3.1 *The slope at* $(0,0)$ *is* $||\mathcal{L}_0^{-1}||$ *This is accurate to first order for a 'worst-case' perturbation.*

For example, if only the size of the perturbation is known, we should assume a worst-case scenario in order to get rigourous safety criteria. If more is known about the perturbation then the results of section 3.4 could be used to adapt the estimates.

Remark 3.2 *The slope at* $(\varepsilon_0, \eta(\varepsilon_0))$ *diverges representing possible non-invertibility and consequently loss of continuation.*

Note that even if our estimates for $||\mathcal{L}_f^{-1}||$ are optimal we do not expect the safety estimates $\varepsilon(\eta)$ to be optimal. The reason for this can be seen in figures 3.2 and 3.3. Suppose that we can evaluate $||\mathcal{L}_f^{-1}||$ accurately in terms of η and ε. Suppose also that we find the smallest $(\varepsilon_0, \eta(\varepsilon_0))$ for which $\mathcal{L}_f$ could lose invertibility. Then although there must be some perturbation of size ε_0 which has a response of

size $\eta(\varepsilon_0)$, this response will probably not be the function which makes $\mathcal{L}_f$ non-invertible. Moreover the function which does this may not even be a solution of the system for any $\tilde{\mathcal{C}}^1$ perturbation. Thus the estimates must stop at ε_0 even though continuation is still available in principle. This results from the fact that all we are considering are the sizes of $f - f_0$ and $x - x_0$ instead of the functions themselves in the (infinite dimensional) spaces $\mathcal{C}^1$ and $\tilde{\mathcal{C}}^1$. This is inevitable if all we assume about the perturbations is that they are bounded.

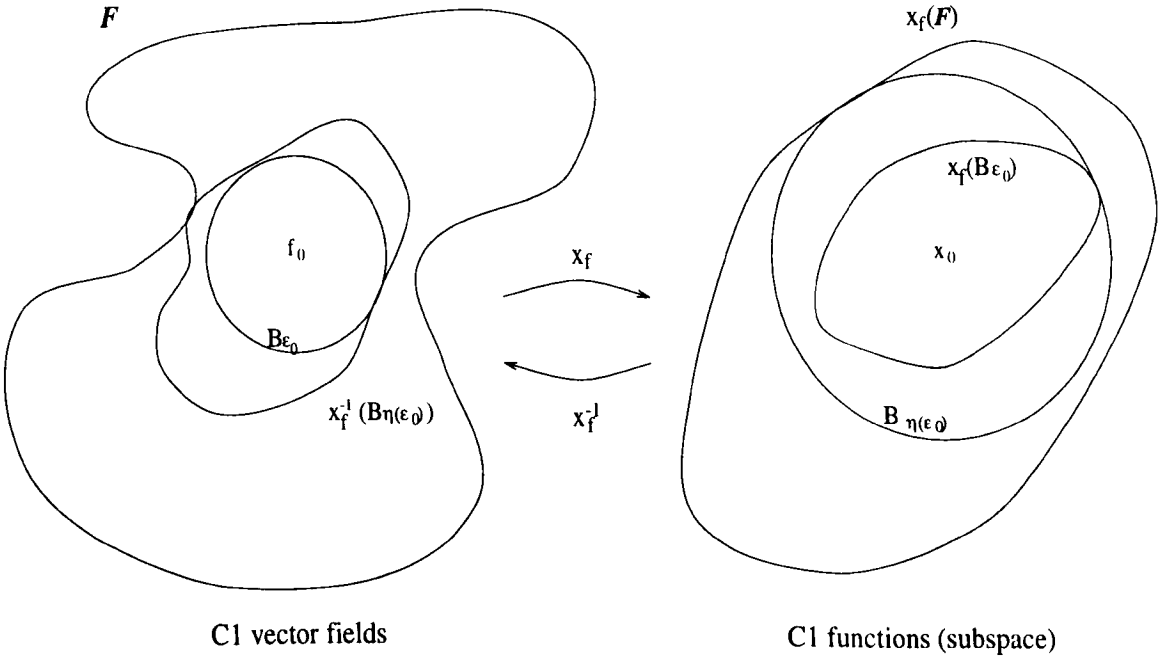


Figure 3.2: Determining an ε_0 such that $|\varepsilon| < \varepsilon_0$ guarantees unique $\mathcal{C}^1$ continuation. $F \subset \tilde{\mathcal{C}}^1$ is the true safe region of vector fields, that is, those for which x_0 has a unique continuation. This maps to $x_f(F) \subset \mathcal{C}^1$. Let $\eta(\varepsilon_0)$ be size of the largest ball (centred on x_0) contained in $x_f(F)$. We can only hope to continue in η until the boundary of this ball is reached. Let $x_f^{-1}(B_{\eta(\varepsilon_0)})$ be the subset of vector fields which have continuation in this ball. Continuation in ε will stop at $\varepsilon = \varepsilon_0$ giving B_{ε_0} as our estimate of the safe region. Typically, the smallest perturbation f for which $|x_f - x_0| = \eta(\varepsilon_0)$ will not be the one which reaches the boundary of F.

The problem is made worse by the fact that our estimates for $||\mathcal{L}_f^{-1}||$ are usually only general in nature so are only accurate for small perturbations. Thus it may well be the case that no perturbation of size ε_0 has response of size $\eta(\varepsilon_0)$ and that $||\mathcal{L}_f^{-1}||$ diverges only for $\eta >> \eta(\varepsilon_0)$ and $\varepsilon >> \varepsilon_0$.

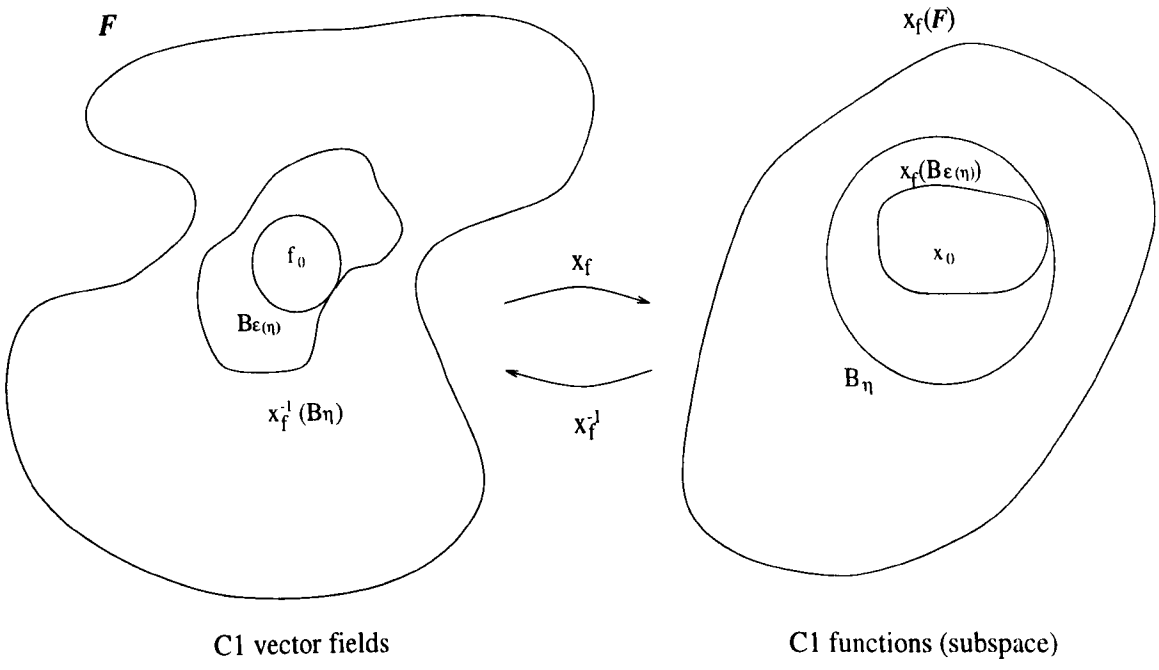


Figure 3.3: Determining an $\varepsilon(\eta)$ such that $|\varepsilon| < \varepsilon(\eta)$ guarantees unique C^1 continuation satisfying $||x_f - x_0|| < \eta$. As with the previous figure, there is an inevitable loss in treating unknown perturbations since we only measure using sup-norm. Clearly, typical perturbations of size $\varepsilon(\eta)$ do not have a response of size η. It is only the worst case which does.

3.3 Example: Additive forcing of equilibria

To illustrate some of the ideas developed so far I show how to obtain the estimates when x_0 is an attracting hyperbolic equilibrium and the perturbation is additive forcing. The operator we are concerned with is

$$G_u(x) = \dot{x} - f_0(x) - u(t)$$

The linearisation about x_0 is $\mathcal{L}_0\xi = \dot{\xi} - Df_{0,x_0}\xi$ which has constant coefficients. Since Df_{0,x_0} has all eigenvalues in the left half-plane, the projection onto E^+ is trivial, $\mathcal{P}_0 = I$. Following section 3.2.1, we see that the Green's function is given by

$$W_0(t) = \begin{cases} \exp(tDf_{0,x_0}) & t > 0 \\ 0 & t < 0 \end{cases}$$

The solution of $\mathcal{L}_0\xi = \varphi$ for each $\varphi \in C^0$ is given by the convolution

$$\begin{aligned} (\mathcal{L}_0^{-1}\varphi)(t) &= (W_0 * \varphi)(t) \\ &= \int_{\mathbb{R}^+} \exp(sDf_{0,x_0})\varphi(t-s)ds \end{aligned}$$

Remark 3.3 *$\mathcal{L}_0^{-1}$ is causal.*

Remark 3.4 *When x_0 is not linearly attracting but has non-trivial backward time contracting subspace we use the general form (3.9) but in this case we get a non-causal impulse 'response'.*

Let $\varepsilon = ||u||$ and let $K_0 = \left|\left|\mathcal{L}_0^{-1}\right|\right| = \max\left\{||W||_1, 1 + \left|\left|\dot{W}\right|\right|_1\right\}$. We can find $\beta(\eta)$ such that $||Df_{0,x} - Df_{0,x_0}|| \leq \beta(\eta)$. Assume, for example, that L is a Lipschitz constant for f_0 so we can take $\beta(\eta) = L\eta$. Using lemma 3.1 and equation (3.10) we get the inequality

$$\frac{d\eta}{d\varepsilon} \leq \frac{1}{K_0^{-1} - L\eta} \tag{3.11}$$

There is a unique continuation, x_u, while $L\eta < K_0^{-1}$ and integrating (3.11) we find $||x_u - x_0|| \leq \eta(\varepsilon)$ where $\eta(\varepsilon)$ is the smallest solution of

$$\varepsilon = K_0^{-1}\eta - L\eta^2/2$$

which is valid as long as $\varepsilon \leq \varepsilon_0 = K_0^{-2}/2L$. Figure 3.4 is a a schematic representation of the situation. Again the slope at $(0,0)$ is K_0.

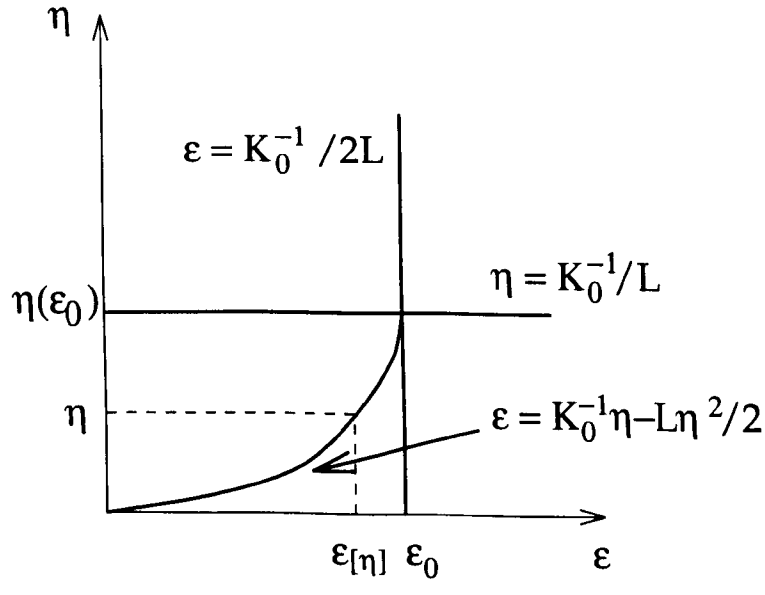


Figure 3.4: Continuation estimates for additive forcing (schematic)

3.4 Adapted safety measures

The strategy so far has given us an estimate of safe region in the space of vector fields, $\{f \mid |\varepsilon| \leq \varepsilon_{\bar{\eta}}\} \in \mathcal{F}_{\bar{\eta}}$. This is of course a subset of the true safe region because

we have used sub-optimal estimates. In estimating η we have used the differential equation

$$\frac{d\eta}{d\varepsilon} = ||\mathcal{L}_f^{-1} \circ I_{\tilde{C}^1,C^0}||_{\tilde{C}^1 \to C^1} \leq ||\mathcal{L}_f^{-1}||_{C^0 \to C^1}$$

But $\frac{d\eta}{d\varepsilon}$ may be much smaller than this especially if the perturbation does not induce a big response in the system relative to $\varepsilon = ||f - f_0||_{\tilde{C}^1}$. An example of this is additive forcing of a damped forced oscillator.[ii] Here we find the biggest response comes from forcing close to the 'natural' frequency of the oscillator. It is well known that away from this frequencies even forcing which is big in sup-norm may not do much to the system. This is not catered for in the present formulation.

The result we obtain is a $\tilde{C}^1$-ball of safe vector fields when in fact the true safe region is probably very differently shaped. This can be seen from figures 3.2 and 3.3.

3.4.1 Adapted norm

One can take this into account to some extent by using an adapted norm on perturbations.

To do this we first adapt the norm on C^0. Suppose that $\mathcal{L}_0$ is the linearisation about a hyperbolic solution x_0.

Definition 3.2 *For $u \in C^0$,* $||u||_* = ||\mathcal{L}_0^{-1}u||_{C^1} / ||\mathcal{L}_0^{-1}||_{C^0 \to C^1}$

Instead of using C^0-norm, we measure C^0 functions by the (normalised) response induced in the linearisation at x_0. For example, when x_0 is a linearly attracting equilibrium we have the explicit inverse $\mathcal{L}_0^{-1}u = W_0 * u$ so

$$||u||_* = ||W_0 * u||_{C^1} / ||W_0||_1$$

Theorem 3.1 *For $\mathcal{L}_0 : C^1 \to C^0$ invertible, $||\cdot||_*$ is equivalent to $||\cdot||_\infty$. It follows that C^0_*, the space C^0 with norm $||\cdot||_*$, is complete.*

Proof Since $\mathcal{L}_0$ and $\mathcal{L}_0^{-1}$ are bounded linear operators with

$$||\mathcal{L}_0||_{C^1 \to C^0} = \left|\left|\frac{d}{dt} - Df_{0,x_0}\right|\right| \leq 1 + ||Df_{0,x_0}|| = k_0 \quad \text{and} \quad ||\mathcal{L}_0^{-1}||_{C^0 \to C^1} \leq K_0$$

it follows that $(K_0 k_0)^{-1} ||u||_\infty \leq ||u||_* \leq ||u||_\infty$. ∎

Remark 3.5 $||u||_* \leq ||u||_\infty$

Remark 3.6 $||\mathcal{L}_0^{-1}||_{C^0_* \to C^1} = ||\mathcal{L}_0^{-1}||_{C^0 \to C^1}$. *In fact, modulo a $||\mathcal{L}_0^{-1}||_{C^0 \to C^1}$ scaling factor, $\mathcal{L}_0^{-1} : C^0_* \to C^1$ is an isometry.*

[ii] For more detail see section 5.1.

3.4.2 New estimates

Let $\tilde{\mathcal{C}}^1_*$ be the space $\tilde{\mathcal{C}}^1$ with the norm inherited from $||\ ||_*$. That is,

$$||f||_{\tilde{\mathcal{C}}^1_*} = \sup_{x\in\mathcal{C}^1} \max\left\{||f(x(\cdot),\cdot)||_* \ , \ ||Df_x||_{\mathcal{C}^1\to\mathcal{C}^0_*}\right\}$$

$\tilde{\mathcal{C}}^1_*$ and $\tilde{\mathcal{C}}^1$ have equivalent norms so $\tilde{\mathcal{C}}^1_*$ is also a Banach space.

As before let $\varepsilon = ||f - f_0||_{\tilde{\mathcal{C}}^1}$. Define ε_* by

$$\varepsilon_* = ||f - f_0||_{\tilde{\mathcal{C}}^1_*}$$

The advantage of using $\tilde{\mathcal{C}}^1_*$ is that perturbations to the vector field are measured in terms of the response they induce in the linearisation at x_0.

Remark 3.7 *$\varepsilon_* < \varepsilon$ since $||\ ||_* < ||\ ||_\infty$.*

If we consider now

$$\frac{dx_f}{df} : \tilde{\mathcal{C}}^1_* \longrightarrow \mathcal{C}^1 \tag{3.12}$$

$$\delta f \longmapsto \mathcal{L}_f^{-1}\ \delta f(x_f(\cdot),\cdot) \tag{3.13}$$

the estimates we want can be obtained from

$$\frac{d\eta}{d\varepsilon_*} \leq ||\mathcal{L}_f^{-1}||_{\mathcal{C}^0_*\to\mathcal{C}^1}\ ||I||_{\tilde{\mathcal{C}}^1_*\to\mathcal{C}^0_*} = ||\mathcal{L}_f^{-1}||_{\mathcal{C}^0_*\to\mathcal{C}^1}$$

which is near-optimal for small perturbations.

To obtain an estimate of $||\mathcal{L}_f^{-1}||_{\mathcal{C}^0_*\to\mathcal{C}^1}$ we look for a $\beta_*(\varepsilon_*,\eta)$ satisfying

$$||\mathcal{L}_f - \mathcal{L}_0||_{\mathcal{C}^1\to\mathcal{C}^0_*} = ||Df_{x_f} - Df_{0,x_0}||_{\mathcal{C}^1\to\mathcal{C}^0_*} \leq \beta_*(\varepsilon_*,\eta)$$

The differential inequality used to estimate the size of response is

$$\frac{d\eta}{d\varepsilon_*} \leq ||\mathcal{L}_f^{-1}||_{\mathcal{C}^0_*\to\mathcal{C}^1} \leq \left[K_0^{-1} - \beta_*(\varepsilon_*,\eta)\right]^{-1} \tag{3.14}$$

This can be integrated to provide new estimates $\hat{\eta}_*(\varepsilon_*)$.

Say $\varepsilon = \tau(\varepsilon_*)$. Also let $\beta(\varepsilon,\eta) \leq ||Df_{x_f} - Df_{0,x_0}||_{\mathcal{C}^1\to\mathcal{C}^0}$ as used previously. Then the estimate

$$\begin{aligned} ||Df_{x_f} - Df_{0,x_0}||_{\mathcal{C}^1\to\mathcal{C}^0_*} &\leq ||\mathcal{L}_0^{-1}\circ[Df_{x_f} - Df_{0,x_0}]||_{\mathcal{C}^1\to\mathcal{C}^1} \,/\, ||\mathcal{L}_0^{-1}||_{\mathcal{C}^0\to\mathcal{C}^1} \\ &\leq ||Df_{x_f} - Df_{0,x_0}||_{\mathcal{C}^1\to\mathcal{C}^0} \\ &\leq \beta(\varepsilon,\eta) \end{aligned} \tag{3.15}$$

suggests we could choose $\beta_*(\varepsilon_*,\eta) = \beta(\tau(\varepsilon_*),\eta)$. However, this may not be a good estimate to use since it doesn't make use of the adapted norm.

Proposition 3.3 *The adapted safety estimates $\hat{\eta}_*$, are at least as good as $\hat{\eta}$, the estimates obtained from integrating (3.10).*

Proof We can always use estimate (3.15) and if we do I show that

$$\hat{\eta}_*(\tau(\varepsilon_*)) \leq \hat{\eta}(\tau(\varepsilon_*)) = \hat{\eta}(\varepsilon)$$

Consider the perturbed vector field to be parametrised by a straight-line path in $\tilde{C}^1$ starting at f_0. Clearly $\tau(\varepsilon_*)$ is a straight line. Also since $\varepsilon_* < \tau(\varepsilon_*)$ the slope satisfies $\tau'(\varepsilon_*) = \tau' > 1$. Thus taking $\beta_*(\varepsilon_*, \eta) = \beta(\tau(\varepsilon_*), \eta)$ from (3.15) gives

$$\begin{aligned} \frac{d\hat{\eta}_*}{d\varepsilon} &= \frac{d\hat{\eta}_*}{d\varepsilon_*} \circ \frac{d\eta_*}{d\varepsilon} \\ &\leq \left[K_0^{-1} - \beta(\varepsilon, \eta)\right]^{-1} / \tau' = \frac{d\hat{\eta}}{d\varepsilon} \end{aligned}$$

Thus $\hat{\eta}_*(\tau(\varepsilon_*)) \leq \hat{\eta}(\varepsilon)$ as required. [iii] ■

If $\beta(\varepsilon, \eta)$ is independent of ε then estimate (3.15) can be used to good effect. This suggests we use $\beta_*(\eta) = \beta(\eta)$ which gives $\hat{\eta}_*(\varepsilon_*) = \hat{\eta}(\varepsilon_*)$. Since $\varepsilon_* \leq \varepsilon$ this is an obvious improvement. This case arises for example when the perturbation is an external forcing. In a later chapter I treat the externally forced damped oscillator and the adapted estimates are put to good effect.

So one should expect the adapted estimates to give a much better idea of the 'bad' forcing functions and thus allow for much larger safety regions. Figure 3.5 gives a schematic idea of how our estimates will typically look after adapting the norm.

Example: additive forcing of a hyperbolic equilibrium

Let x_0 be a hyperbolic equilibrium of $\dot{x} = f(x)$. The linearisation at x_0 has constant coefficients so can always be inverted explicitly. Thus $|| \; ||_*$ can be calculated easily.

Now consider the perturbed system

$$\dot{x} = f_0(x) + g(t) = f(x, t)$$

where $g \in C^0$. Clearly, $\varepsilon = ||f - f_0||_{\tilde{C}^1} = ||g||$ and using the adapted norm $\varepsilon_* = ||f - f_0||_{\tilde{C}^1_*} = ||g||_*$. Since we are in the case where $\left|\left|Df_{x_f} - Df_{0,x_0}\right|\right|_{C^1 \to C^0} \leq \beta(\eta)$ is independent of ε, we can use the estimate $\left|\left|Df_{x_f} - Df_{0,x_0}\right|\right|_{C^1 \to C^0_*} \leq \beta(\eta)$. This gives us a the equation

$$\frac{d\eta}{d\varepsilon_*} \leq \left[K_0^{-1} - \beta(\eta)\right]^{-1}$$

which is easy to integrate and provides improved safety estimates.

[iii] I take the straight-line path since the proof is easier but since $\tau(\varepsilon_*) > \varepsilon_*$. the result holds for any path.

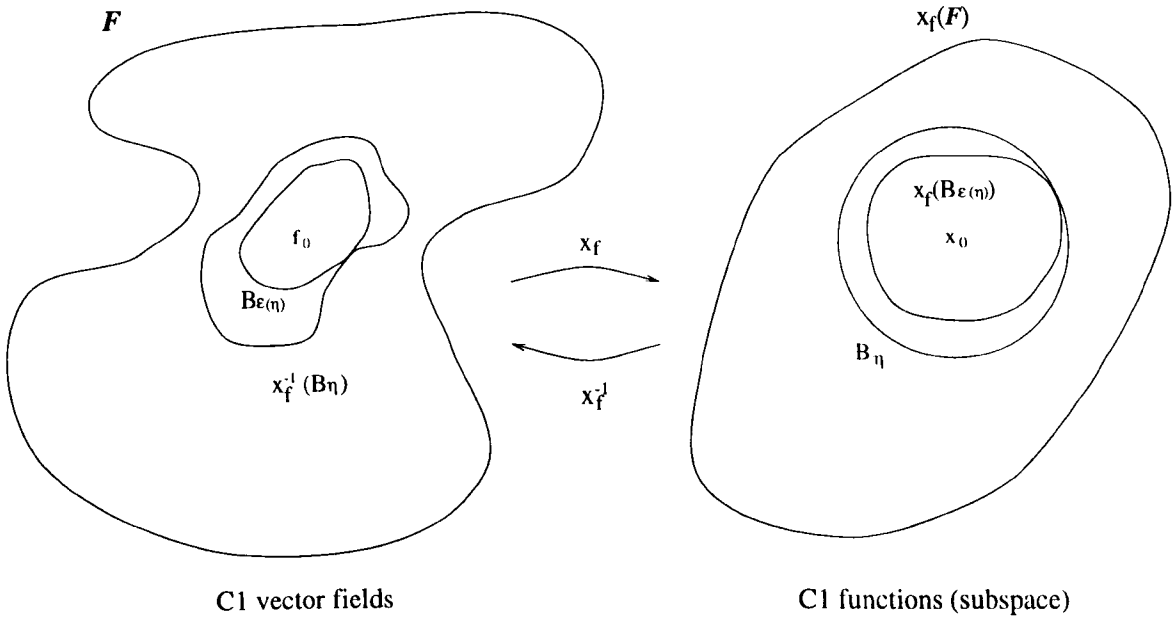


Figure 3.5: By using a different norm on $\tilde{\mathcal{C}}^1$ we can adapt the shape of $\tilde{\mathcal{C}}^1$ balls to take into account the response induced. This will usually give a better idea of the safe region.

3.4.3 Further refinements

In principle we can take this idea one step further. Rather than use $\mathcal{L}_0^{-1}$ to obtain a $\mathcal{C}^0$-norm which is optimal at $f = f_0$ we could use a norm which changes with (x, f). Ideally we would use

$$||u||_{*,f} = ||\mathcal{L}_f^{-1}u||_{\mathcal{C}^1} / ||\mathcal{L}_f^{-1}||$$

This is a family of norms each equivalent to $||\cdot||_\infty$. They provide the optimal estimate at each x_f. Clearly, for each f such that the continuation x_f exists, $\mathcal{L}_f^{-1}$ is an isometry between $\mathcal{C}^0_{*,f}$ and $\mathcal{C}^1$ and $||\mathcal{L}_f^{-1}||_{\mathcal{C}^0_{*,f}\to\mathcal{C}^1} = 1$.

What we do now is to consider the subset $\mathcal{F} \subset \tilde{\mathcal{C}}^1$ of vector fields with unique continuation. For each $f \in \mathcal{F}$ we use a different norm on tangent vectors $\delta f \in \tilde{\mathcal{C}}^1$.

$$||\delta f||_{T_f\tilde{\mathcal{C}}^1} = \sup_{x\in\mathcal{C}^1} \max\left\{||f(x(\cdot),\cdot)||_{*,f}\ ,\ ||Df_x||_{\mathcal{C}^1\to\mathcal{C}^0_{*,f}}\right\}$$

Consider a path γ in $\tilde{\mathcal{C}}^1$ such that $\gamma_0 = f_0$ and $\gamma_1 = f$. Then integrating (3.12) gives us

$$x_f - x_0 = \int_0^1 \mathcal{L}_f^{-1}\frac{d\gamma_s}{ds}(x_{\gamma_s}(\cdot),\cdot)ds$$

Thus we have the estimate

$$\eta \le \int_0^1 \left|\left|\frac{d\gamma_s}{ds}(x_{\gamma_s}(\cdot),\cdot)\right|\right|_{\mathcal{C}^0_{*,f}} ds$$

If one is interested in safety with respect to some set of paths A in $\tilde{\mathcal{C}}^1$ then one could use

$$\eta \leq \sup_{\gamma \in A} \int_0^1 \left|\left| \frac{d\gamma_s}{ds}(x_{\gamma_s}(\cdot), \cdot) \right|\right|_{\mathcal{C}^0_{*,f}} ds$$

One could go even further and make $\mathcal{F} \in \tilde{\mathcal{C}}^1$ a (Finsler) manifold by defining the metric

$$d_*(f,g) = \inf_{\gamma: \substack{\gamma_0 = f \\ \gamma_1 = g}} \int_0^1 \left|\left| \frac{d\gamma_s}{ds}(x_{\gamma_s}(\cdot), \cdot) \right|\right|_{\mathcal{C}^0_{*,f}} ds$$

Note that these ideal methods are somewhat unfeasible in practice as we do not expect to be able to explicitly invert $\mathcal{L}_f$ when x_f is essentially unknown. However, a good idea would be to use an approximation to $\mathcal{L}_f^{-1}$ which changes with f and x say, $\mathcal{M}_f \approx \mathcal{L}_f^{-1}$. Then one could use

$$||u||_{*,f} = ||\mathcal{M}_f u||_{C^1} / ||\mathcal{M}_f||$$

I believe this would be an interesting idea to investigate for some specific systems where more information about the perturbed solution is available.

3.5 Neighbourhood of uniqueness

theorem 2.13 guarantees a unique continuation in some $\mathcal{C}^1$ neighbourhood of x_0 for ε small enough. It could be useful to have a measure of how large this neighbourhood is. This can easily be done within the current framework and in this section I present the strategy for the general case. In later chapters some more detail is added and some numerics are presented for the hyperbolic equilibrium and limit cycle cases.

This section relies heavily on the proof of the IFT which is detailed in Appendix A. The idea behind the proof of theorem 2.13 is to use the IFT on the operator G. Applying the IFT to G shows that for hyperbolic solutions x_0, there is a unique continuation $x_f \in \mathcal{C}^1$. This follows from the fact that the operator N_f is a contraction and by the contraction mapping theorem, has a unique fixed point x_f. Here I derive estimates of the size of the contraction neighbourhood.

We are interested in the operator $N_f(x) : \mathcal{C}^1 \to \mathcal{C}^1$ defined by

$$N_f(x) = x - \mathcal{L}_0^{-1} \circ G_f(x)$$

Consider the derivative

$$\begin{aligned} DN_{f,x} &= I - \mathcal{L}_0^{-1} \circ \mathcal{L}_{f,x} \\ &= \mathcal{L}_0^{-1} \circ [\mathcal{L}_0 - \mathcal{L}_{f,x}] \end{aligned}$$

We can find neighbourhoods of $f = f_0$ and x_0 on which DN_ε is a strict contraction with a given contraction rate $\lambda < 1$. As before let $\eta = ||x - x_0||$ and $\varepsilon = ||f - f_0||$.

$$\begin{aligned} ||DN_{\varepsilon,x}|| &\leq K_0 \; ||Df_{0,x_0} - Df_x|| \\ &\leq K_0 \; \beta(\varepsilon, \eta) \\ &\leq \lambda < 1 \qquad\qquad \text{for } \beta(\varepsilon,\eta) \leq \lambda K_0^{-1} \end{aligned}$$

Let $\eta(\varepsilon, \lambda)$ be the first value of η such that $\beta(\varepsilon, \eta) = \lambda K_0^{-1}$ as in figure 3.6(a). Now we find a neighbourhood which is mapped into itself. This means finding, in

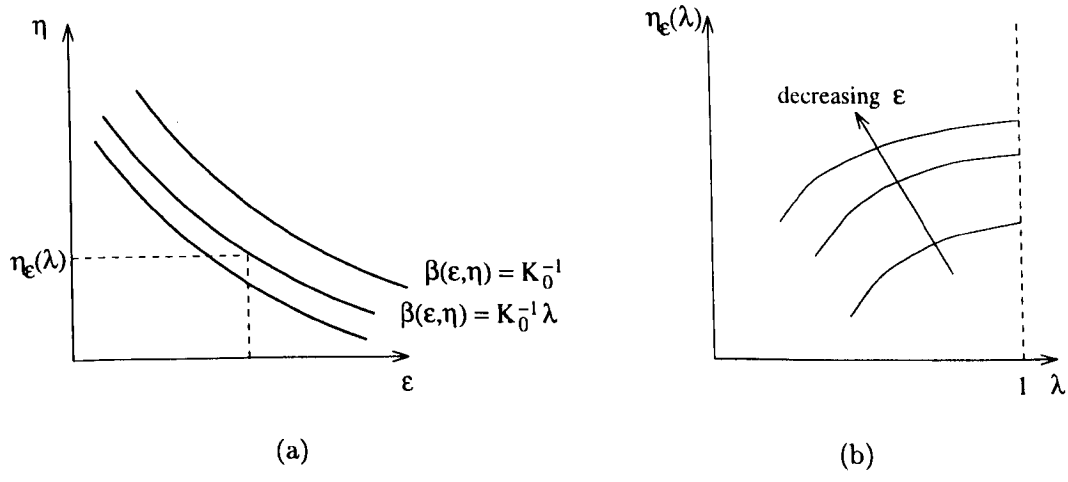


Figure 3.6: Determining $\eta(\varepsilon, \lambda)$, a neighbourhood of x_0 for which $||DN_{f,x}|| \leq \lambda$.

a neighbourhood of $\varepsilon = 0$, a neighbourhood of x_0 such that

$$|N_f(x) - x_0| \; < \; |x - x_0| \; = \; \eta$$

Figure 3.7 shows how we can calculate $|N_f(x) - x_0|$.
We can deduce that

$$\begin{aligned} |N_f(x) - x_0| &\leq |N_f(x) - N_f(x_0)| \; + \; |N_f(x_0) - x_0| \\ &\leq \lambda\eta + K_0 \; ||G_f(x_0)|| \qquad\qquad \text{for } \eta < \eta(\varepsilon, \lambda) \;\; \text{(MVT)} \\ &= \lambda\eta + K_0 \; ||f_0(x_0) - f(x_0)|| \\ &< \eta \end{aligned}$$

as long as

$$\frac{K_0 \; ||f_0(x_0) - f(x_0)||}{1 - \lambda} \; < \; \eta \; < \; \eta(\varepsilon, \lambda)$$

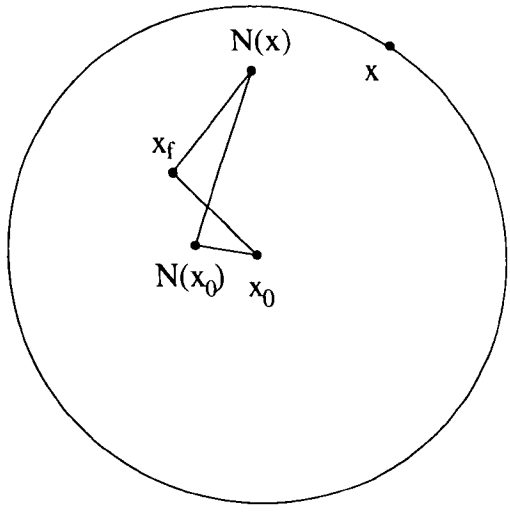


Figure 3.7: Neighbourhood of contraction.

For ε too great we see the inequality cannot hold. For ε small enough however we expect some interval $[\lambda_1, \lambda_2]$ such that these inequalities can be satisfied. We find this interval by solving

$$\eta(\varepsilon, \lambda) = \frac{K_0 \ ||f_0(x_0) - f(x_0)||}{1 - \lambda} \tag{3.16}$$

λ_2 is the largest solution of (3.16). λ_1 is the smallest. $\lambda < \lambda_1$ means the fixed point may not lie inside the ball $B_\eta(\varepsilon, \lambda)$. $\lambda > \lambda_2$ and the furthest x may not be being mapped closer to x_0.

$\eta(\varepsilon, \lambda_2)$ gives the largest contraction neighbourhood and thus a useful lower bound for the neighbourhood of uniqueness.

Notice that we can also deduce the upper bound $||x_f - x_0|| < \eta(\varepsilon, \lambda_1)$. In fact this is equivalent to the estimate (see figure (?))

$$\begin{aligned} ||x_f - x_0|| &\leq (N_f(x_0) - x_0)(1 + \lambda + \lambda^2 + \ldots) \\ &\leq \frac{K_0 \ ||f_0(x_0) - f(x_0)||}{1 - \lambda} \end{aligned}$$

I expect this to be only slightly worse than the estimates given in the previous section so could make a good first approximation.

3.6 Basins of attraction

In this section I develop a technique of determining basins of attraction for linearly attracting solutions.

3.6.1 Theory

Again we consider the ODE

$$\dot{x} = f(x,t) \tag{3.17}$$

where f is C^1 with respect to x and C^0 with respect to t.

As we will be interested in the forward time behaviour of solutions with a given initial condition, say at some arbitrary t_0, we can consider the space of functions bounded on the half-line $[t_0, \infty)$. Specifically, we let C^k_+ denote the Banach space of bounded, C^k functions $x : [t_0, \infty) \to \mathbb{R}^n$, with derivatives bounded and with the C^k-norm defined previously.

Recall that a solution $x_0 \in C^1$ is *linearly attracting* if the linearisation, $\dot{\xi} = Df_{x_0}\xi$, has exponential dichotomy on $\mathbb{R}$ with trivial backward contracting subspace. That is, $E^+_t = \mathbb{R}^n$ for every $t \in \mathbb{R}$. This extends to a definition for half-lines.

Definition 3.3 *We say $x_0 \in C^1_+$ is* linearly attracting on $[t_0, \infty)$ *if the linearisation, $\dot{\xi} = Df_{x_0}\xi$, has exponential dichotomy on $[t_0, \infty)$ with trivial backward contracting subspace. That is, $E^+_t = \mathbb{R}^n$ for all $t \in [t_0, \infty)$.*

Remark 3.8 *If $x_0 \in C^1$ is linearly attracting on $[t_0, \infty)$ then it is linearly attracting on every half-line $[t_1, \infty)$. However, the exponential dichotomy constants may depend on the choice of half-line. See for example [7].*

Proposition 3.4 *If $x_0 \in C^1$ is linearly attracting then for each $t_0 \in \mathbb{R}$ the restriction of x_0 to C^1_+, is linearly attracting on $[t_0, \infty)$.*

Note that the converse of this is not true in general since exponential dichotomy on every right half-line does not guarantee exponential dichotomy on the whole of $\mathbb{R}$. For example, consider the solution to a one-dimensional system which is forward asymptotic to a hyperbolic sink and backward asymptotic to a hyperbolic source. This is non-hyperbolic but has exponential dichotomy on every right half-line.

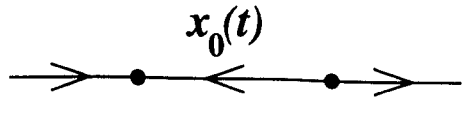


Figure 3.8: $x_0(t)$ is linearly attracting on every positive half-line but not on the whole of $\mathbb{R}$.

Characterisation of linearly attracting solutions

Consider the operator $H : \mathbb{R}^n \times \mathcal{C}^1_+ \longrightarrow \mathcal{C}^0_+ \times \mathbb{R}^n$

$$H_c(x) \;=\; (\; \dot{x} - f(x,t) \;,\; x(t_0) - c \;) \tag{3.18}$$

where initially, we use the norm $||(\psi, b)|| \;=\; ||\psi|| + |b|$ for $(\psi, b) \in \mathcal{C}^0_+ \times \mathbb{R}^n$.

$H_c(x) = 0$ gives solutions of (3.17).

Let $\hat{\mathcal{L}}_{c,x} = DH_c(x) : \mathcal{C}^1_+ \longrightarrow \mathcal{C}^0_+ \times \mathbb{R}^n$ be defined by

$$\hat{\mathcal{L}}_{c,x}\xi \;=\; \Big(\dot{\xi} - Df_x\xi \;,\; \xi(t_0) \Big)$$

For convenience I use the abbreviations $\hat{\mathcal{L}}_0 = \hat{\mathcal{L}}_{0,x_0}$ and $\hat{\mathcal{L}}_c = \hat{\mathcal{L}}_{c,x_c}$.

Theorem 3.2 *$x_0 \in \mathcal{C}^1_+$ is linearly attracting on $[t_0, \infty)$ if and only if $\hat{\mathcal{L}}_0$ is invertible.*

Proof [$\Longrightarrow$] The homogeneous problem

$$\hat{\mathcal{L}}_0\xi = 0 \iff \begin{cases} \dot{\xi} = Df_{x_0}\xi \\ \xi(t_0) = 0 \end{cases}$$

has the unique $\mathcal{C}^1_+$ solution $\xi(t) = 0$ so $\hat{\mathcal{L}}_0$ is injective.

To show surjectivity note that

$$\xi = \hat{\mathcal{L}}_0^{-1}(\psi, b) \iff \begin{cases} \dot{\xi} = Df_{x_0}\xi + \psi \\ \xi(t_0) = b \end{cases}$$

This is the inhomogeneous initial value problem and has general solution given by the variation of parameters formula, [15],

$$\begin{aligned} \hat{\mathcal{L}}_0^{-1}(\psi, b)(t) \;&=\; \xi_{\mathrm{H}}(t) \;+\; \xi_{\mathrm{P}}(t) \\ &= X_0(t, t_0)\; b \;+\; \int_{t_0}^{t} X_0(t,\tau)\; \psi(\tau)\; d\tau \end{aligned} \tag{3.19}$$

where $X_0(t,\tau)$ is the fundamental matrix solution for $\dot{\xi} = Df_{x_0}\xi$. Since x_0 is linearly attracting we know that

$$||X_0(t,\tau)|| \;\leq\; D\exp(-\alpha(t-\tau)) \qquad \text{for } t > \tau > t_0$$

Thus there is a constant K_0 such that $||\xi|| \;\leq\; K_0\,||(\psi, b)||$ giving $\hat{\mathcal{L}}_0^{-1} \leq K_0$ as required.

[$\Longleftarrow$] The converse follows easily from theorem 2.5. ■

Corollary 3.2 *$x_0 \in \mathcal{C}^1$ is linearly attracting if and only if for each $t_0 \in \mathbb{R}$, the operator $\hat{\mathcal{L}}_0$ is invertible and $\left\|\hat{\mathcal{L}}_0^{-1}\right\| \leq K_0$ with K_0 independent of t_0.*

Proof [$\Longrightarrow$] The invertibility of $\hat{\mathcal{L}}_0$ follows easily from Proposition 3.4. Since x_0 is uniformly hyperbolic the hyperbolicity constants D and α in the above proof and thus K_0 can be chosen independently of t_0.

[$\Longleftarrow$] The converse follows easily from theorem 2.5. ■

A important concept in the theory of ODE's is that of stability. There are many standard definitions of stability but here is the most useful for our purposes.

Definition 3.4 *We say a solution $x \in \mathcal{C}^1$ is* uniformly asymptotically stable *(on $\mathbb{R}$) if there exist constants $N, \bar{C}$ such that, for any $t_1 \in \mathbb{R}$,*

$$|y(t_1) - x(t_1)| < \bar{C} \;\Rightarrow\; \begin{cases} |y(t) - x(t)| < N\ |y(t_1) - x(t_1)| \quad \text{for all} \quad t > t_1 \\ y(t) \to x(t) \quad \text{as} \quad t \to +\infty \end{cases} \tag{3.20}$$

where $y(t)$ is the solution of (3.17) with initial condition $y(t_1)$.

Analogously, we say that a solution $x \in \mathcal{C}^1_+$ is uniformly asymptotically stable on $[t_0, \mathbb{R})$ *if there exist constants $N, \bar{C}$ such that (3.20) holds for each $t_1 \geq t_0$.*

It is called uniform because $N, \bar{C}$ can be chosen independently of t_1.

Definition 3.5 *For x_0 a uniformly asymptotically stable solution, we say $(c, t_0) \in \mathbb{R}^n \times \mathbb{R}$ is in the* basin of attraction *$\mathcal{BA}$, of x_0, if there is an $x_c \in \mathcal{C}^1_+$, solving (3.17), with $x_c(t_0) = c$, and such that $x_c(t) \to x_0(t)$ as $t \to +\infty$.*

For reasons I have already mentioned, it is often useful to find, analytically, lower bounds for the size of $\mathcal{BA}$. The main result I give is that linearly attracting solutions are uniformly asymptotically stable and thus have a basin of attraction. The proof will be used to provide estimates of the size of $\mathcal{BA}$.

Theorem 3.3 *Let $x_0 \in \mathcal{C}^1$ be a solution of (3.17). Then if x_0 is linearly attracting it is uniformly asymptotically stable. In particular, there is a $\bar{C}$ such that, for any $t \in \mathbb{R}$, if $|c - x_0(t)| \leq \bar{C}$ there exists a unique $\mathcal{C}^1_+$ continuation x_c, with $x_c(t) = c$, solving (3.17) and for which $x_c(t) \to x_0(t)$ as $t \to +\infty$.*

The theorem shows that we can find $B_{\bar{C}}(v)$, a $\bar{C}$-sized ball centred at $v \in \mathbb{R}^n$ such that

$$\{\ (\ B_{\bar{C}}(x_0(t))\ ,\ t\ )\ \mid t \in \mathbb{R}\ \} \subset \mathcal{BA}(x_0)$$

which guarantees that a cylinder around x_0, in the extended phase space, with uniform width $\bar{C}$, is attracted to x_0 in forward time.

Proof Assume without loss of generality that $x_0(t_0) = 0$. Then $H_0(x_0) = 0$. As with the persistence of hyperbolicity theorem I apply the implicit function theorem to H but this time I am continuing with respect to changes of initial condition.

Since, x_0 is linearly attracting, it follows that $\hat{\mathcal{L}}_0$ is invertible. So by the IFT, for some $\bar{C} > 0$, there is a unique continuation $x_c \in \mathcal{C}^1_+$ solving $H_c(x_c) = 0$ whenever $|c| < \bar{C}$.

Remark 3.9 *x_c is linearly attracting on $[t_0, \infty)$ and while $\hat{\mathcal{L}}_c$ remains invertible*

$$\frac{dx_c}{dc} = -\hat{\mathcal{L}}_c^{-1} \circ \frac{\partial H_c}{\partial c}(x_c) \tag{3.21}$$

We have shown bounded continuation but in order to prove theorem 3.3 we also need to show $x_c \to x_0$ as $t \to +\infty$. To this end notice that (3.21) gives us

$$\frac{dx_c}{dc} = -\hat{\mathcal{L}}_c^{-1} \circ [0, -I_{\mathbb{R}^n}] \tag{3.22}$$

$$\frac{dx_c}{dc}(\delta c)(t) = X_c(t, t_0)\delta c \tag{3.23}$$

where $X_c(t, \tau)$ is the evolution operator for $\dot{\xi} = Df(x_c)\xi$. Invertibility of $\hat{\mathcal{L}}_c$ guarantees that $X_c(t, t_0)$ is exponentially decaying and it follows that $|x_c(t) - x_0(t)| \to 0$ (exponentially) as $t \to +\infty$.

To complete the proof of uniform asymptotic stability we just choose an upper bound for $\left|\left|\hat{\mathcal{L}}_0^{-1}\right|\right|$ independent of t_0. This can be done by Corollary 3.2.

■

Note that for non-hyperbolic solutions which have linear attraction on right half-lines $[t_0, \mathbb{R})$, uniform bounds cannot be found for the whole of $\mathbb{R}$ so uniform asymptotic stability on the whole of $\mathbb{R}$ is not obtained. However there is an analogous theorem to 3.3.

Theorem 3.4 *For some $t_0 \in \mathbb{R}$, let $x_0 \in \mathcal{C}^1_+$ be a solution of (3.17). Then if x_0 is linearly attracting on $[t_0, \mathbb{R})$ it is uniformly asymptotically stable for $[t_0, \mathbb{R})$. In particular, there is a $\bar{C}$ such that, for any $t > t_0$, if $|c - x_0(t)| \leq \bar{C}$ there exists a unique $\mathcal{C}^1_+$ continuation x_c, with $x_c(t) = c$, solving (3.17) and for which $x_c(t) \to x_0(t)$ as $t \to +\infty$.*

Proof The proof above works here as well. ■

Thus we also expect a cylindrical regions lying in the basin of attraction for solutions which are linearly attracting on right half-lines. However, the width of the cylinder depends on the choice of t_0. If the solution is not linearly attracting on the whole of $\mathbb{R}$ then the width inevitably goes to zero as $t_0 \to -\infty$.

3.6.2 Estimates

One can use the proof of 3.3 directly to obtain estimates of $\bar{C}$. This could be a useful rigourous safety criterion. In fact one might have further restrictions to impose on the behaviour of the system, for example, given some 'safety time' T, one might want to determine the subset of initial conditions in $\mathcal{BA}$ which converges to within some tolerance of x_0, before time T has elapsed. This kind of safety criterion is simple to formulate within the current setup. Here I show how to find $\bar{C}$ in the general case and suggest two important adaptations which improve the estimates.

First we need to estimate $\left|\left|\hat{\mathcal{L}}_c^{-1}\right|\right|$. To do this we examine (3.19) once more. It tells us that

$$\begin{aligned} |\xi(t)| &\leq ||X_0(t,t_0)||\ |b| + \int_{t_0}^{t} ||X_0(t,\tau)||\, d\tau\ ||\psi|| \\ &= K_{\mathrm{H}}(t)\ |b| + K_{\mathrm{P}}(t)\ ||\psi|| \end{aligned}$$

It is useful at this stage to consider a weighted norm[iv] on $\mathcal{C}_+^0 \times \mathbb{R}^n$.

$$||(\psi, b)|| = ||\psi|| + \frac{|b|}{w_{\mathrm{H}}}$$

where $w_{\mathrm{H}} > 0$. This will be used to improve estimates. Using this norm it follows that

$$\begin{aligned} |\xi(t)| &\leq w_{\mathrm{H}} K_{\mathrm{H}}(t) \frac{|b|}{w_{\mathrm{H}}} + K_{\mathrm{P}}(t)\, ||\psi|| \\ &\leq \max\{w_{\mathrm{H}} K_{\mathrm{H}}(t), K_{\mathrm{P}}(t)\}\ ||(\psi, b)|| \\ &= K_0(t)\ ||(\psi, b)|| \end{aligned}$$

Letting $K_H = ||K_H(\cdot)||_\infty$, $K_P = ||K_P(\cdot)||_\infty$ and $K_0 = ||K_0(\cdot)||_\infty$ we get

$$\begin{aligned} \left|\left|\hat{\mathcal{L}}_0^{-1}\right|\right| &\leq K_0 \\ &= \max\{w_{\mathrm{H}} K_{\mathrm{H}}, K_{\mathrm{P}}\} \end{aligned}$$

Let $\eta = ||x - x_0||$ and assume we can find $\beta(\eta) \geq ||Df(x_c) - Df(x_0)||$. Then

$$\begin{aligned} \left|\left|\hat{\mathcal{L}}_c - \hat{\mathcal{L}}_0\right|\right| &= ||(Df(x_c) - Df(x_0), 0)|| \\ &\leq \beta(\eta) \end{aligned}$$

lemma 3.1 gives an upper bound for $\left|\left|\hat{\mathcal{L}}_c^{-1}\right|\right|$ valid while $\beta(\eta) < K_0^{-1}$.

$$\left|\left|\hat{\mathcal{L}}_c^{-1}\right|\right| \leq \left[K_0^{-1} - \beta(\eta)\right]^{-1}$$

[iv] It is only the *relative* weight of $|b|$ compared to $||\psi||$ that matters.

From (3.22) we have the estimate

$$\begin{aligned}\frac{dx_c}{dc} &= \left|\left|\hat{\mathcal{L}}_c^{-1}\right|\right| \, |||[(0, I_{\mathbb{R}^n \to \mathbb{R}^n})]||| \\ &\leq \left[K_0^{-1} - \beta(\eta)\right]^{-1} \; w_{\mathrm{H}}^{-1}\end{aligned}$$

Let $\hat{\eta}$ be the first value of η such that $\beta(\eta) = K_0^{-1}$. Then we can estimate $\bar{C}$ by

$$\bar{C} = w_{\mathrm{H}} \int_0^{\hat{\eta}} \left[K_0^{-1} - \beta(\eta)\right] d\eta \tag{3.24}$$

Optimising over w_{H}

To make best use of the weighted norm one should choose w_{H} so as to obtain the smallest estimate of $\left|\left|\frac{dx_c}{dc}\right|\right|$ or equivalently the largest estimate of $\bar{C}$. From (3.24) one might think that choosing w_{H} as big as possible would yield the best estimates but this is not the case since it affects $K_0 = \max\{w_{\mathrm{H}}K_{\mathrm{H}}, K_{\mathrm{P}}\}$. Consider both possibilities for K_0.

- $K_{\mathrm{P}} \geq w_{\mathrm{H}}K_{\mathrm{H}}$: This gives $K_0 = K_{\mathrm{P}}$ and increasing w_{H} *does* improve $\bar{C}$. However, when w_{H} crosses $K_{\mathrm{P}}K_{\mathrm{H}}^{-1}$ we move into the next case.

- $K_{\mathrm{P}} \leq w_{\mathrm{H}}K_{\mathrm{H}}$: In this case $K_0 = w_{\mathrm{H}}K_{\mathrm{H}}$ and $\bar{C}$ is given by

$$\begin{aligned}\bar{C} &= w_{\mathrm{H}} \int_0^{\hat{\eta}} \left[w_{\mathrm{H}}^{-1}K_{\mathrm{H}}^{-1} - \beta(\eta)\right] d\eta \\ &= \int_0^{\hat{\eta}} \left[K_{\mathrm{H}}^{-1} - w_{\mathrm{H}}\beta(\eta)\right] d\eta\end{aligned}$$

  so *decreasing* w_{H} improves $\bar{C}$.

It follows that the 'optimal' choice is $w_{\mathrm{H}} = K_{\mathrm{P}}K_{\mathrm{H}}^{-1}$. This gives

$$\left|\left|\frac{dx_c}{dc}\right|\right| \leq K_{\mathrm{H}}\left[1 - K_{\mathrm{P}}\beta(\eta)\right]^{-1}$$

and

$$\bar{C} = K_{\mathrm{H}}^{-1} \int_0^{\hat{\eta}} \left[1 - K_{\mathrm{P}}\beta(\eta)\right] d\eta$$

One of the advantages of this scheme is that we can estimate the size of the basin of attraction for solutions of the perturbed system. Since the perturbation is only assumed to be bounded in some norm, standard numerical methods cannot be used to do this. A brief description of how to do this can be found in the next chapter.

Chapter 4

Generalisations

4.1 Discrete time systems

So far I have dealt exclusively with continuous-time dynamical systems, in particular those arising from differential equations. This is the most useful context for applications although it could also be useful to have a discrete time analogue. In many ways the theory for discrete time aperiodic systems is easier. The problem of infinitesimal phase-shift degeneracy in autonomous systems does not appear. Also more convenient is the fact that there is no differentiation involved so the linearisation is a mapping from a space to itself.

In this section I give a brief summary of how one can obtain an analogous theory of hyperbolicity and thereby provide safety estimates for perturbed systems.

Of interest are solutions of the non-autonomous system

$$x_{n+1} = f(x_n, n) \tag{4.1}$$

Define $l^\infty = X^{\mathbb{Z}}$ to be the space of uniformly bounded bi-infinite sequences with elements in the Banach space X. This is a Banach space with the norm

$$||x||_\infty = \sup_{i\in\mathbb{Z}} |x_i|$$

Definition 4.1 *A* bounded solution *of (4.1) is a fixed point of the operator $F : l^\infty \to l^\infty$ defined by*

$$(Fx)_{n+1} = f(x_n, n)$$

4.1.1 Hyperbolicity

The linearisation of F about x is given by

$$(DF_x\xi)_{n+1} = Df_{x_n,n}\xi_n$$

Definition 4.2 *A bounded solution is* uniformly hyperbolic *if $I - DF_x$ is invertible.*

There is a standard definition of hyperbolicity which asks, for each n, the existence of a continuous splitting $X = E_n^+ \oplus E_n^-$ satisfying

$$\begin{aligned} \xi^+ \in E_m^+ &\Longrightarrow \left|Df_{x_n,n} \circ \cdots \circ Df_{x_m,m}\xi^+\right| \leq K\lambda^{n-m} \left|\xi^+\right| \qquad n \geq m \\ \xi^- \in E_m^- &\Longrightarrow \left|Df^{-1}_{x_n,n} \circ \cdots \circ Df^{-1}_{x_m,m}\xi^-\right| \leq K\lambda^{m-n} \left|\xi^-\right| \qquad n \leq m \end{aligned}$$

with $K > 0$, $0 < \lambda < 1$ independent of n.

As before, they are equivalent definitions and to prove this one could adapt the proof for the continuous time case.

Note that spec$(I - DF_x)$ is rotationally invariant and thus hyperbolicity of x is equivalent to hyperbolicity of $I - DF_x$. For more details of this in the context of autonomous systems the reader is referred to Lanford, [22].

4.1.2 Persistence of hyperbolic solutions

Consider the perturbed system

$$x_{n+1} = f(x_n, n) \tag{4.2}$$

where f is a C^1 perturbation of f_0

We are thus interested in fixed points of the operator F_f defined on l^∞

$$(F_f x)_{n+1} = f(x_n, n)$$

Theorem 4.1 *Let $x(0)$ be a hyperbolic solution of (4.2) for $f = f_0$. Then for C^1-close f, there is a locally unique $x(f)$ solving (4.2). Moreover it is hyperbolic with the same stability type as $x(0)$ and is as smooth in f as F is.*

This is proved by the contraction mapping theorem on F_f or equivalently the implicit function theorem on $G_f = I - F_f$.

4.1.3 Hyperbolic sets

A (uniformly) hyperbolic set is a a set of hyperbolic solutions, $\Omega = \{x^i\}$, with $||(I - DF^{-1}_{x^i}|| < M$, for some M independent of x^i.

As with the continuous time case, most of the well known properties of hyperbolic sets can be proved easily with this definition. This gives a nice way of obtaining estimates for aperiodic systems.

4.1.4 Estimates

To find estimates, we essentially look for bounds on $||(I - DF_{f,x})^{-1}||$. There is a theory analogous to the Green's function methods we have used for the continuous time case. Again the implicit function theorem gives a differential equation for the continuation and this can be used to effectively estimate the size of the response and thus obtain safety criteria.

4.2 Discontinuous perturbation

Here I detail how the theory of chapters 2 and 3 can be applied to more general types of differential equation, in particular those which allow for discontinuous perturbations or driving forces.

When studying ODE's of the form $\dot{x} = f(x,t)$ it is natural to seek C^1 (or *classical*) solutions. This requires f to be at least continuous with respect to time and for uniqueness f should be at least Lipschitz with respect to x. However, while this is mathematically the most natural choice, for many physical systems we may not be able to guarantee continuity in the equations of motion. This is especially the case when there are unknown driving forces or perturbations. For example, bang-bang controls switch discontinuously at certain times. Also of interest are perturbations which are unbounded or large in sup-norm but only integrally bounded. One might also need to consider impulses (δ-functions) which are not even functions in the usual sense. In these cases we expect solutions to lose their differentiability and possibly their continuity. To treat them we require a generalisation of the notions of solution and derivative. Note that I only consider cases of discontinuity in the time variable and impose the usual Lipschitz condition with respect to x. This is enough to cover the situations identified above.

One such generalisation is known as the *Carathèodory conditions* for existence of solutions to the initial value problem.

Theorem 4.2 *Let $f(x,t)$ be measurable in t for each fixed $x \in \mathbb{R}^n$ and Lipschitz in x for each fixed t. Suppose also that for fixed x, f is integrally bounded on compact intervals of t. Then for any $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$ there exists a unique absolutely continuous function $x(t; x_0, t_0)$ passing through (x_0, t_0) and satisfying $\dot{x} = f(x,t)$ Lebesgue almost everywhere.*

See, for example, Hale's book, [15], for the standard proof of this although, in principle, the theory also fits into the framework I develop below.

The characterisation of hyperbolicity I have developed so far has been essentially concerned with zeroes of the operator $G : \tilde{\mathcal{C}}^1 \times \mathcal{C}^1 \to \mathcal{C}^0$. To extend the theory to discontinuous right hand sides one should just look for good function spaces in which to pose the root finding problem. In this way one could also look for greater smoothness by posing the problem in spaces containing smoother functions.

The functional analytic nature of the theory means that one can make use of the large body of literature concerned with differential operators on abstract function spaces. I will detail some of these ideas and show how they can be used to generalise the idea of hyperbolic solutions. Then I show that the persistence result is still valid and can be used to provide robustness and stability estimates in an analogous manner to previous chapters.

4.2.1 Some useful functional analysis

In order to allow for discontinuous perturbations it will be necessary to use some more ideas from functional analysis, in particular the theory of L_p spaces, functionals, weak derivatives, distributions and Sobolev spaces. Here I present the basic concepts and techniques. Although some of the theory is standard in the study of partial differential equations and Fourier transformation it is not presented in a form which is of direct use for ODE problems and cannot be found in the dynamical systems literature. For more detailed coverage the reader is referred to [1, 4, 8].

Unless otherwise specified, the function spaces under consideration have domain space $\mathbb{R}$ and target space $\mathbb{R}^n$. Thus $\mathcal{A}(\mathbb{R}, \mathbb{R}^n)$ for example will be shortened to $\mathcal{A}$. I will also write $|\ |$ to denote $||\ ||_X$. For $x \in \mathcal{A}$, I will write $\frac{dx}{dt}$ or $\dot{x}$ to denote the derivative and $x^{(k)}$ to denote the kth derivative.

For simplicity, I use the abbreviation $\int f$ to denote $\int_{\mathbb{R}} f(s)ds$. The measure on $\mathbb{R}$ is the Lebesgue measure and the integration is *Bochner integration*, an extension of Lebesgue integration to Banach-space-valued functions.

For a given function space $\mathcal{A}$ denote the subset of functions in $\mathcal{A}$ with compact support by $\mathcal{A}_o$. It is given the same norm as $\mathcal{A}$.

Another simplification I employ is to write $\mathcal{A} = \mathcal{B}$ instead of the usual $\mathcal{A} \cong \mathcal{B}$ to represent isometrically isomorphic spaces.

Definition 4.3 *For two Banach spaces X, Y, we say that there is an* embedding *of X into Y if X is isometrically isomorphic to a subspace X' of Y, and the trivial injection $I : X' \to Y$ is continuous. This is denoted by $X \sqsubset Y$.*

$\mathcal{C}^k$ Spaces

$\mathcal{C}^0$ is the space of continuous functions bounded on $\mathbb{R}$. With the norm $||f||_\infty = \sup_{t\in\mathbb{R}} |f(t)|$ it is a Banach space.

By $\mathcal{C}^k$ it is meant the space of bounded k-times continuously differentiable functions with k bounded derivatives. With the norm $||f||_{\mathcal{C}^k} = \max_{0\le j\le k} ||f^{(j)}||_\infty$ it is a Banach Space.

By $\mathcal{C}^\infty$ it is meant the space of continuous bounded functions which have continuous, bounded derivatives of all orders.

Clearly $\mathcal{C}^\infty \sqsubset \mathcal{C}^{k+1} \sqsubset \mathcal{C}^k$ for every k.

Denote by $\mathcal{D}$ the space $\mathcal{C}_o^\infty$ also called the *test functions*.

L_p Spaces

For $p \in [1,\infty)$, L_p is defined to be the space of equivalence classes of measurable functions f, for which $\int |f|^p < \infty$, with the equivalence relation identifying functions that are equal almost everywhere (a.e.). L_p is a Banach Space with the norm $||f||_p = \left(\int |f|^p\right)^{1/p}$. The space L_1 is called the *integrable* functions.

A measurable function f, is said to be *essentially bounded* if there is a K such that $|f(t)| < K$ a.e. The greatest lower bound over such K is the *essential supremum* of f and is written $||f||_\infty = \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(t)| = \inf\{K \mid |f(t)| < K \text{ a.e.}\}$. There is no ambiguity in the notation $||\ ||_\infty$, since for $\mathcal{C}^0$ functions the essential supremum is also the supremum.

Define L_∞ to be the space of equivalence classes of measurable functions which are essentially bounded. With the norm defined above, L_∞ is a Banach space.

Definition 4.4 *A measurable function f, is said to be* locally integrable *if $\int_J |f| < \infty$ for all compact intervals $J \subset \mathbb{R}$.*

Define L_1^{loc} to be the space of equivalence classes of locally integrable functions f, for which

$$||f||_{L_1^{\text{loc}}} = \sup_t \int_t^{t+1} |f| < \infty$$

With this norm, L_1^{loc} is a Banach space.

Definition 4.5 *Let μ be a measurable function. Then μ is* absolutely continuous *(with respect to Lebesgue measure) if there is a locally integrable f_μ such that*

$$\mu(K) = \int_K f_\mu \qquad \forall K \text{ compact}$$

I denote by $\mathcal{AC}$, the Banach space consisting of those absolutely continuous functions μ, which have f_μ (as defined above) in L_1^{loc}. $||\mu||_{\mathcal{AC}} = ||f_\mu||_{L_1^{\text{loc}}}$

Let p^* be the *conjugate exponent* of p, defined by

$$\begin{aligned} 1/p + 1/p^* &= 1 && \text{for } 1 \leq p < \infty \\ p^* &= \infty && \text{for } p = 1 \\ p^* &= 1 && \text{for } p = \infty \end{aligned}$$

Here are some useful theorems concerning the L_p spaces.

Theorem 4.3 (Hölder's Inequality) *For $f \in L_p, g \in L_{p^*}$ we have $f \cdot g \in L_1$ and*

$$||f \cdot g||_1 = \int |f \cdot g| \leq ||f||_p \; ||g||_{p^*}$$

Theorem 4.4 *$L_p \sqsubset L_1^{loc}$ for all p and $\mathcal{C}^k \sqsubset L_1^{loc}$ for all k. In fact $\mathcal{D}$ is dense in L_p and $\mathcal{C}^k$ for all $p \neq \infty$ and any k. On compact domains we also have the embeddings $\mathcal{C}^k \sqsubset L_p$ for any k, p and $L_p \sqsubset L_q$ for $p \geqq$.*

Theorem 4.5 *It can be deduced that, for $1 \leq p < \infty$, L_p is the completion of $\mathcal{C}^0$ with respect to the norm $|| \; ||_p$.*

This characterisation does not apply to L_∞ since $\mathcal{C}^0$ is already complete with $|| \; ||_\infty$

Dual Spaces

For X an arbitrary normed vector space we denote by X^* the space of bounded linear functionals on X. That is

$$X^* = \left\{ T \in L(X, \mathbb{R}) \mid ||T|| = \sup_{||x||=1} |T[x]| < \infty \right\}$$

Note that I reserve the notation $T[x]$ for the action of bounded linear functionals, so as to distinguish them from other functions and operators.

Remark 4.1 *X^* is always a Banach space, and if X, Y are Banach spaces such that $X \sqsubset Y$, then $Y^* \sqsubset X^*$.*

For all $g \in L_{p^*}$ we can define a linear functional, T_g on L_p by

$$T_g[f] = \int |f \cdot g| \qquad \forall f \in L_p \tag{4.3}$$

By Hölder's inequality, $|T_g[f]| = ||f \cdot g||_1 \leq ||f||_p \; ||g||_{p^*}$. This gives $||T_g|| \leq ||g||_{p^*}$ and by careful choice of $f \in L_p$ we can make this an equality. Thus we have

an isometric injection of L_{p^*} into L_p^*. The obvious question to ask is whether all the functionals in L_p^* can be written in the form (4.3). An answer to this is given by the following theorem.

Theorem 4.6 (Riesz Representation Theorem) *Let $1 \leq p < \infty$. Then for any $T \in L_p^*$ there is a $g \in L_{p^*}$ such that*

$$T[f] = \int |f \cdot g| \qquad \forall f \in L_p$$

Moreover, $||T|| = ||g||_{p^*}$. Thus $L_p^* = L_{p^*}$.

Remark 4.2 *Although $L_1^* = L_\infty$, the converse is not true. L_∞^* is bigger than L_1. Thus for $1 < p < \infty$, it follows that L_p is reflexive, that is, $L_p^{**} = L_p$. It also follows that only L_2 is self-adjoint.*

Distributions

Definition 4.6 *$\mathcal{D}^*$, the dual space of $\mathcal{D} = C_o^\infty$, is called the space of* distributions.

The most general notion of function we have defined so far has been the locally integrable functions. In fact, for each locally integrable f, we can define the functional $T_f \in \mathcal{D}^*$ by

$$T_f[\varphi] = \int |f \cdot \varphi| \qquad \forall \varphi \in \mathcal{D}$$

This is an injection $f \mapsto T_f \in \mathcal{D}^*$. Note that for $f \in L_1^{\text{loc}}$ we have $||T|| = ||f||$ so we can formally identify functions $f \in L_1^{\text{loc}}$ with the induced distribution T_f. In fact, two locally integrable functions define the same distribution if and only if they are equal almost everywhere. That is, for $f_1, f_2 \in L_1^{\text{loc}}$ we have,

$$T_{f_1}[\varphi] = T_{f_2}[\varphi] \qquad \forall \varphi \in \mathcal{D} \qquad \Longleftrightarrow \qquad f_1 = f_2 \quad \text{a.e.}$$

$\mathcal{D}^*$ is not a space of functions in the usual sense since $T \in \mathcal{D}^*$ is only defined by its action on the test functions $\mathcal{D}$. To show this is not a vacuous notion I give an example of a distribution not identifiable with any $f \in L_1^{\text{loc}}$.

Theorem 4.7 *For any $t \in \mathbb{R}$, there is no $f \in L_1^{\text{loc}}$ such that*

$$\int f \cdot \varphi = \varphi(t) \qquad \forall \varphi \in \mathcal{D}$$

However, the functional δ, defined by $\delta[\varphi] = \varphi(0)$, $\forall \varphi \in \mathcal{D}$ is a well defined distribution.

δ_t is the *delta distribution* and is well known to engineers and physicists as the *unit impulse* or the *Dirac delta.* It is traditional to refer to delta type distributions as functions and to write them as functions, $\delta(s)$ and $\delta(s-t)$, or to place them under integral signs, for example, $\int \delta(s-t)\cdot\varphi(s)dt = \varphi(t)$, but to avoid any confusion I will not use this notation.

Weak Derivatives

With the notion of distribution firmly grounded it is possible to introduce the concept of weak derivatives.

For $f \in \mathcal{C}^1$ one can use integration by parts to get the identity

$$\int \dot{f}\cdot\varphi \;=\; -\int f\cdot\dot{\varphi} \qquad \forall\varphi\in\mathcal{D}$$

Similarly, for $f \in \mathcal{C}^k, k \geq m$, integration by parts m times gives

$$\int f^{(m)}\cdot\varphi \;=\; (-1)^m\int f\cdot\varphi^{(m)} \qquad \forall\varphi\in\mathcal{D}$$

This motivates the following definition of *distributional* or *weak derivative.*

Definition 4.7 *For $f \in \mathcal{D}^*$ and any $m \in \mathbf{N}$ define the mth* distributional derivative *$f^{(m)}$, by*

$$f^{(m)}[\varphi] \;=\; (-1)^m f[\varphi^{(m)}] \qquad \forall\varphi\in\mathcal{D} \tag{4.4}$$

Remark 4.3 *Since $\varphi^{(m)} \in \mathcal{D}$, it follows that $f^{(m)} \in \mathcal{D}^*$. Thus distributions have derivatives of all orders.*

Example 1 $H_t \in L_\infty$ is the *Heaviside function* defined by

$$H_t(s) \;=\; \begin{cases} 1 & s \geq t \\ 0 & s < t \end{cases}$$

Then $\dot{H}_t[\varphi] \;=\; -H_t[\dot{\varphi}] \;=\; -\int H_t\cdot\dot{\varphi} \;=\; \varphi(t) \;=\; \delta_t[\varphi]$. Thus $\dot{H}_t = \delta_t$.

Example 2 $\delta_t^{(m)}[\varphi] \;=\; -\delta_t[\varphi^{(m)}] \;=\; (-1)^m\varphi^{(m)}(t)$

Remark 4.4 *For $\mathcal{C}^k$ functions, the distributional derivatives up to order k coincide with the standard derivatives.*

Theorem 4.8 *The operator $\frac{d}{dt} : \mathcal{D}^* \to \mathcal{D}^*$ is surjective and*

$$\mathrm{Ker}\,\frac{d}{dt} \;=\; \{g_0 \in \mathcal{D}^* \mid g_0 \textit{ is constant}\}$$

Definition 4.8 *For $f \in \mathcal{D}^*$, any distribution g, such that $\dot{g} = f$ is called a* primitive *of f.*

It follows from the previous theorem that to any primitive we can add a constant distribution and get another primitive.

The spaces $\mathcal{C}^{-k}$

One can generalise the $\mathcal{C}^k$ spaces in order to allow weak derivatives.

Definition 4.9 *A distribution (or function) x is said to be in $\mathcal{C}^{-k}$ if $x = y^{(k)}$ for some $y \in \mathcal{C}^0$.*

Notice that $x \in \mathcal{C}^{-k} \implies \dot{x} \in \mathcal{C}^{-(k+1)}$. The converse is also true on compact domains. That is, $y \in \mathcal{C}^{-(k+1)}$ has primitive which is locally $\mathcal{C}^{-k}$ although it may be unbounded when considering the whole of $\mathbb{R}$.

The Sobolev spaces $W^{m,p}$

For $\mathcal{C}^m$ functions I have used the norm $|| \; ||_{\mathcal{C}^m}$ with which $\mathcal{C}^m$ is complete. We can also define on $\mathcal{C}^m$ functions the norms

$$||u||_{m,p} = \left(\sum_{j \leq m} ||u^{(j)}||_p^p \right)^{1/p}$$

Note that $\mathcal{C}^m$ is not complete with respect to $|| \; ||_{m,p}$ for $p \neq \infty$

Definition 4.10 *Define the* Sobolev spaces $W^{m,p}$ *by*

$$W^{m,p} = \{u \in L_p \mid u^{(j)} \in L_p \;\; j \leq m\}$$

where $u^{(j)}$ is a weak derivative.

Theorem 4.9 *With the norm $|| \; ||_{m,p}$, $W^{m,p}$ is a Banach space. Moreover, for $p \neq \infty$, $W^{m,p}$ is the completion of $\mathcal{C}^m$ with respect to this norm.*

$W^{m,\infty}$ is not the completion of $\mathcal{C}^m$ with respect to $|| \; ||_{m,\infty} = || \; ||_{\mathcal{C}^m}$. As we have seen, $\mathcal{C}^m$ is complete with this norm and clearly $W^{m,\infty} \neq \mathcal{C}^m$

Of most interest to us will be $W^{1,p}$ since we are dealing with first order systems. In this case we have some helpful embeddings which follow from the celebrated Sobolev embedding theorems.

Theorem 4.10

$$W^{1,p} \sqsubset L_q \qquad \forall p \leq q \leq \infty$$
$$W^{1,p} \sqsubset \mathcal{C}^0 \qquad \forall p$$

The spaces $(W^{m,p})^*$ and W^{-m,p^*}

We can think of the Sobolev spaces $W^{m,p}$ as follows

$$W^{m,p} \subset L_p \times \cdots \times L_p = L_p^{m+1}$$

In particular there is an isometry $P : W^{m,p} \to W \subset L_p^{m+1}$ defined by $Pu = (u^{(m)}, ..., u)$. The norm on L_p^{m+1} is given by

$$||(g_m, \ldots, g_0)||_{L_p^{m+1}} = \begin{cases} \left(\sum_{j=0}^{m} ||g_j||_p^p\right)^{1/p} & p \neq \infty \\ \max_{0 \leq j \leq m} ||g_j||_\infty & p = \infty \end{cases}$$

Remark 4.5 *For $1 \leq p < \infty$ we have $(L_p^{m+1})^* = L_{p^*}^{m+1}$.*

Theorem 4.11 *For each element $f \in (W^{m,p})^*$ there is a $g = (g_m, \ldots, g_0) \in L_{p^*}^{m+1}$ such that for each $u \in W^{m,p}$*

$$f[u] = \int u^{(m)} g_m + \cdots + u g_0 \tag{4.5}$$

Moreover, $||f|| = \min ||g||_{L_{p^*}^{m+1}}$, where min *is taken over all g satisfying (4.5).*

Remark 4.6 *For $1 < p < \infty$ the g satisfying (4.5) is unique.*

Remark 4.7 *For $1 \leq p < \infty$, every $f \in (W^{m,p})^*$ is the unique norm-preserving extension to $W^{m,p}$ of a distribution $T_g \in \mathcal{D}^*$ obtained from some g satisfying (4.5).*

Proof Find g satisfying (4.5) then define $T_{g_j}, T_g \in \mathcal{D}^*$ by $T_{g_j}[\varphi] = \int g_j \varphi$ and

$$T_g[\varphi] = \sum (-1)^j T_{g_j}^{(j)}[\varphi] = \int \varphi^{(m)} g_m + \cdots + \varphi g_0 \tag{4.6}$$

which agrees with (4.5) for $\varphi \in \mathcal{D}$. Clearly f is an extension of T_g to $W^{m,p}$. That it is unique follows from the fact that $W^{m,p} = W_o^{m,p}$ for $p \neq \infty$ and that there is a unique extension of T_g to functions in $W_o^{m,p}$. The details are too technical to go into here but see Adams' book (??) for a way of proving this. ■

Definition 4.11 *Define the space W^{-m,p^*} as the space of distributions T having the form (4.6) for some $g \in L_{p^*}^{m+1}$. It is given the norm*

$$||T|| = \min ||g||$$

where min *is taken over all g for which (4.6) holds.*

The extension of T to $W^{m,p}$ is unique so we have effectively proved

Theorem 4.12 *For $1 \leq p < \infty$ we have $(W^{m,p})^* = W^{-m,p^*} \supset L_{p^*}^{m+1}$ and for $1 < p < \infty$ we have $(W^{m,p})^* = W^{-m,p^*} = L_{p^*}^{m+1}$.*

In particular, for $m = p = 1$ we have $(W^{1,1})^* = W^{-1,\infty} \supset L_\infty \times L_\infty$.

Example $\delta_t \in (W^{1,1})^*$ defined by $\delta_t[u] = u(t)$ is clearly in $W^{-1,\infty}$ since it is the unique norm-preserving extension to $W^{1,1}$ of the distribution

$$T_{(H_t,0)}[\varphi] = \int H_t\dot{\varphi} + \int 0\varphi = \varphi(t) = \delta_t[\varphi]$$

where $(H_t, 0) \in L_\infty \times L_\infty$. By construction it satisfies (4.5). Moreover, it is obvious that $||\delta_t|| = ||(H_t, 0)|| = 1$.

Remark 4.8 *$(W^{1,1})^* \neq L_\infty \times L_\infty$ since $f \in (W^{1,1})^*$ may correspond to many different $g \in L_\infty \times L_\infty$ although each such g must of course give the same distribution T.*

Remark 4.9 *$(W^{1,\infty})^* \neq W^{-1,1}$ since distributions given by (4.6) have possibly non-unique continuous extensions to $W^{1,\infty}$.*

4.2.2 Generalisation of hyperbolicity

We are now in a position to generalise hyperbolicity to solutions x_0, which are not neccessarily C^1. To do this we look for the invertibility of the operator $\mathcal{L}_0 = DG_{x_0}$ acting between a suitable pair of function spaces.

Admissible spaces

Definition 4.12 *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach function spaces of the type discussed above. Then $(\mathcal{A}, \mathcal{B})$ is said to be* admissible *if*

1. $$x \in \mathcal{A}_o \iff \dot{x} \in \mathcal{B}_o$$

   where $\dot{x}$ can be a weak derivative.

2. *If $\lim_{t\to\infty} x = \infty$ or $\lim_{t\to-\infty} x = \infty$ then $x \notin \mathcal{A}$.*

Property 1 basically asks for the derivative of $\mathcal{A}$ functions to lie in $\mathcal{B}$ and for the primitives of $\mathcal{B}$ functions to lie in $\mathcal{A}$. Checking only functions defined on compact intervals is sufficient since primitives of $\mathcal{B}$ functions may be locally in $\mathcal{A}$ but unbounded in the $\mathcal{A}$ norm when considered on the whole of $\mathbb{R}$. Property 1 guarantees that the operator $\mathcal{L}_0 : \mathcal{A} \to \mathcal{B}$ is well defined.

Property 2 restricts to non-divergent functions which is quite natural for most physical systems.

Obviously $(\mathcal{C}^k, \mathcal{C}^{k-1})$ is admissible and quite a natural choice to use. However, to allow for perturbations which are discontinuous we must look for suitable alternatives.

Examples of the kinds of right hand sides we would like to consider are functions with steps, integrally bounded functions, and impulses. These correspond to the pairs, $(W^{1,\infty}, L_\infty)$, $(W^{1,1}, L_1)$, and $(L_\infty, (W^{1,1})^*)$ respectively.

The following diagram gives some more examples of admissible pairs in descending order of generality.

$$
\begin{array}{ccc}
x \in \mathcal{A} & \longleftrightarrow & \dot{x} \in \mathcal{B} \\
\\
\mathcal{C}^k & \longleftrightarrow & \mathcal{C}^{k-1} \\
\sqcap & & \sqcap \\
\vdots & & \vdots \\
\sqcap & & \sqcap \\
\mathcal{C}^1 & \longleftrightarrow & \mathcal{C}^0 \\
\sqcap & & \sqcap \\
W^{1,\infty} & \longleftrightarrow & L_\infty \\
\sqcap & & \sqcap \\
\vdots & & \vdots \\
\sqcap & & \sqcap \\
W^{1,1} & \longleftrightarrow & L_1 \\
\sqcap & & \sqcap \\
\mathcal{AC} & \longleftrightarrow & L_1^{\mathrm{loc}} \\
\sqcap & & \sqcap \\
\mathcal{C}^0 & \longleftrightarrow & \mathcal{C}^{-1} \\
\sqcap & & \sqcap \\
L_\infty & \longleftrightarrow & (W^{1,1})^* = W^{-1,\infty} \sqsupset L_\infty \times L_\infty \\
\sqcap & & \sqcap \\
\vdots & & \vdots \\
\sqcap & & \sqcap \\
L_1 & \longleftrightarrow & (W_o^{1,\infty})^* \neq W^{-1,1} \\
\sqcap & & \sqcap \\
\mathcal{C}^{-1} & \longleftrightarrow & \mathcal{C}^{-2} \\
\sqcap & & \sqcap \\
\vdots & & \vdots \\
\sqcap & & \sqcap \\
\mathcal{D}^* & \longleftrightarrow & \mathcal{D}^*
\end{array}
$$

Note that the embeddings suggested in the diagram are actually only valid on compact intervals but are shown so as to represent the relative sizes of the spaces.

Definition of hyperbolicity

Let $(\mathcal{A}, \mathcal{B})$ be an admissible pair of spaces. Consider the operator

$$G : \mathcal{A} \to \mathcal{B} \qquad G(x)(t) = \dot{x}(t) - f(x(t), t)$$

where $f \in \tilde{\mathcal{C}}^1_{\mathcal{A},\mathcal{B}} = \mathcal{C}^1(\mathcal{A}, \mathcal{B})$.

If we are interested in some ODE $\dot{x} = f(x,t)$ defined on $\mathbb{R}^n$ then it follows that as an operator $f \in \tilde{\mathcal{C}}^1_{\mathcal{A},\mathcal{B}}$ if the following conditions hold.

- $x \in \mathcal{A} \implies f(x(\cdot), \cdot) \in \mathcal{B}$
- f is uniformly $\mathcal{C}^1$ with respect to $x \in \mathbb{R}^n$ at fixed t. That is, the Jacobian matrix is bounded

  $$|Df_{x,t}| \leq C_0$$

  with C_0 independent of x and t.

These are enough to show that

$$||f||_{\tilde{\mathcal{C}}^1_{\mathcal{A},\mathcal{B}}} = \sup_{x \in \mathcal{A}} \max \{ ||f(x(\cdot), \cdot)||_{\mathcal{B}} \ , \ ||Df_x||_{\mathcal{A} \to \mathcal{B}} \}$$

is bounded. It follows that G is $\mathcal{C}^1$ with respect to $x \in \mathcal{A}$. Note also that Df_x is a continuous uniformly bounded matrix-valued function.

$x_0 \in \mathcal{A}$ is said to be a *solution* of $\dot{x} = f(x,t)$ if $G(x_0) = 0$. The *linearisation* about x_0 is given by $\mathcal{L} = DG_{x_0} : \mathcal{A} \to \mathcal{B}$ defined by $\mathcal{L} = \frac{d}{dt} - Df_{x_0}$.

Analogously to definition 2.7 we have the following generalisation.

Definition 4.13 *A solution $x_0 \in \mathcal{A}$ is* hyperbolic *if there exist constants $K, \alpha > 0$ and for each $t \in \mathbb{R}$ a splitting, $\mathbb{R}^n = E_t^+ \oplus E_t^-$, such that*

$$\xi^+ \in E_t^+ \implies |X_t(s)\xi^+| \leq Ke^{-\alpha(s-t)} \ |\xi^+| \qquad s \geq t \tag{4.7}$$

$$\xi^- \in E_t^- \implies |X_t(s)\xi^-| \leq Ke^{-\alpha(t-s)} \ |\xi^-| \qquad s \leq t \tag{4.8}$$

Since Df_{x_0} is continuous and uniformly bounded the variational equation $\dot{\xi} = Df_{x_0}\xi$ has the same form as before. The fundamental matrix solutions $X_t(s)$, are well defined and again have exponentially bounded growth. Thus the above definition makes sense.

The following conjecture makes it clear why much of the theory of hyperbolicity generalises immediately to any admissible space.

Conjecture 4.1 *For $(\mathcal{A}, \mathcal{B})$ admissible it follows that a solution $x_0 \in \mathcal{A}$ is hyperbolic if and only if $\mathcal{L} : \mathcal{A} \to \mathcal{B}$ is invertible.*

See Massera & Schaffer [26] and Rabiger & Schnaubelt [34] for two slightly different approaches to exponential dichotomy of linear non-autonomous systems. The details contained there make it clear that the above conjecture is true modulo some technical restrictions on the class of Banach space allowed. In [26], for example, there is a restriction to locally integrable spaces although this is only really neccessary if one is looking for absolutely continuous solutions. In [34], there is a restriction to translation invariant spaces of measurable functions which again does not cover right hand sides which are distributions but I expect this not to be a problem.

So at least for the admissible pairs given above that are locally integrable the conjecture is proved. It is not clear to me at this stage how to give a full proof of the conjecture but I will show that it holds for the pairs which I have identified to be of interest.

The admissible pair $(W^{1,p}, L_p)$

Of most interest will be the pairs $(W^{1,1}, L_1)$ and $W^{1,\infty}, L_\infty)$ but the theory applies to any $1 \leq p \leq \infty$.

First I examine the conditions under which the ODE $\dot{x} = f(x,t)$ defined on $\mathbb{R}^n$ gives us a well defined vector field $f \in \tilde{\mathcal{C}}^1_{W^{1,p},L_p}$. The conditions above translate to:

- $x \in W^{1,p} \implies f(x(\cdot),\cdot) \in L_p$
- The Jacobian matrix is uniformly bounded. That is for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$ we have $|Df_{x,t} \leq C_0|$ with C_0 independent of x and t.

Since we have the embedding $W^{1,p} \sqsubset \mathcal{C}^0$, it follows that we will only be interested in solutions in some bounded region of the phase space $\Omega \subset \mathbb{R}^n$. This means we only need check the conditions above for $\Omega \subset \mathbb{R}^n$.

An example of this is the system

$$\dot{x} = g(x) + h(t)$$

where $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$, $g(0) = 0$ and $h \in L_p$. Clearly, the conditions above are satisfied. This arises if we were interested in robustness of the equilibrium solution $x_0 = 0$ of the autonomous system $\dot{x} = g(x)$ with respect to external L_p-forcing.

More generally, consider the unperturbed ODE

$$\dot{y} = g_0(y,t)$$

with solution $y_0(t)$. Assume as usual that $g_0 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$. By making the change of variables

$$x = y - y_0(t)$$

we get the equivalent system

$$\dot{x} = f_0(x,t) = g_0(x + y_0(t), t) - \dot{y}_0(t)$$

with equilibrium solution $x_0 = 0$. The conditions above are satisfied since by the $\mathcal{C}^1$ hypothesis on g_0 we have $x \in W^{1,p} \implies g_0(x(\cdot) - y_0(\cdot), \cdot) \in L_p$.

Now consider the perturbed system

$$\dot{y} = g(y,t)$$

where $g(y - y_0, \cdot) - g(y - y_0, \cdot) \in L_p$ for y in some neighbourhood of $y_0 \in W^{1,p}$.

It follows then that

$$\dot{x} = f(x,t) = g(x + y_0(t), t) - \dot{y}_0$$

satisfies the neccessary conditions to get $f \in \tilde{\mathcal{C}}^1_{W^{1,p},L_p}$.

Now assume that the neccessary conditions are satisfied. The following facts about the spectrum of $\mathcal{L}$ will be useful. Proofs can be found in [3] and (Schnaubelt ??).

Lemma 4.1 $\operatorname{spec} \mathcal{L} : W^{1,p} \to L_p = \operatorname{spec} \mathcal{L} : \mathcal{C}^1 \to \mathcal{C}^0$

As before, we have the group of translation operators $\mathcal{T}^h : L_p \to L_p$ defined by

$$(\mathcal{T}^h \xi)(t) = X_{t-h}(t)\xi(t-h)$$

Lemma 4.2 $\operatorname{spec} \mathcal{T}^h : L_p \to L_p = \operatorname{spec} \mathcal{T}^h : \mathcal{C}^0 \to \mathcal{C}^0$

The spectral mapping theorem then follows immediately.

Lemma 4.3 $\exp(h \operatorname{spec} \mathcal{L}) = \operatorname{spec} \mathcal{T}^h$

The main theorem can now be given.

Theorem 4.13 *The solution $x_0 \in W^{1,p}$ is hyperbolic if and only if $\mathcal{L} : W^{1,p} \to L_p$ is invertible.*

Proof The theorem follows immediately from the fact that $\operatorname{spec} \mathcal{L} : W^{1,p} \to L_p = \operatorname{spec} \mathcal{L} : \mathcal{C}^1 \to \mathcal{C}^0$ and that $\mathcal{L} : \mathcal{C}^1 \to \mathcal{C}^0$ is invertible if and only if $\dot{\xi} = \xi Df_{x_0}$ has exponential dichotomy. To obtain estimates however, it is important to bound $\mathcal{L}^{-1}$ and so I prove this part of the theorem in more detail.

[$\Rightarrow$] It follows from the existence of the hyperbolic splitting that any non-zero initial condition diverges exponentially in either forwards or backwards time. Thus the only L_p solution of $\mathcal{L}\xi = 0$ is $\xi = 0$ and so $\mathcal{L}$ is injective.

Let $\mathcal{P}_t$ be the projection onto the forward contracting subspace E_t^+. We define the Green's function as before by

$$W(t,s) = \begin{cases} X_s(t)\mathcal{P}_s & t > s \\ -X_s(t)(I - \mathcal{P}_s) & t < s \end{cases}$$

Recall that $|W(t,s)| \leq Ne^{\alpha|t-s|}$ so $W(t,\cdot) \in L_1(\mathbb{R}, L(\mathbb{R}^n, \mathbb{R}^n)$. For each $\varphi \in L_p$, one has a solution of $\dot{\xi} = Df_{x_0}\xi + \varphi$ given by the integral operator

$$(\mathcal{L}^{-1}\varphi)(t) = \int W(t,s)\varphi(s)ds$$

Lemma 4.4 *For integral operators of the form $Tf = \int K(t,s)f(s)ds$, if*

$$\sup_s ||K(t,\cdot)||_1 \leq K_0 < \infty$$

$$\sup_t ||K(\cdot,s)||_1 \leq K_0 < \infty$$

then $T : L_p \to L_p$. Moreover, it is continuous and we have the estimate

$$||T||_{L_p \to L_p} \leq K_0$$

See Dunford & Schwartz [11] for details.

Since $\sup_t ||W(t,\cdot)||_1 = \sup_s ||W(\cdot,s)||_1 < \infty$ it follows that $\mathcal{L}^{-1}\varphi \in L_p$ and that $||\mathcal{L}^{-1}\varphi||_p \leq \sup_t ||W(t,\cdot)||_1 \, ||\varphi||_p$. Thus

$$||\mathcal{L}^{-1}||_{L_p \to L_p} \leq \sup_t ||W(t,\cdot)||_1$$

If N and α are the hyperbolicity constants then we can use the fact that $|W(t,s)| \leq Ne^{\alpha|t-s|}$. This gives the simple estimate

$$||\mathcal{L}^{-1}||_{L_p \to L_p} \leq 2N/\alpha$$

Since $\frac{d}{dt}(\mathcal{L}^{-1}\varphi) = Df_{x_0}\mathcal{L}^{-1}\varphi + \varphi$ it follows that $\frac{d}{dt}(\mathcal{L}^{-1}\varphi) \in L_p$ as well and we get the bound

$$\left|\left|\frac{d}{dt}\mathcal{L}^{-1}\right|\right|_{L_p \to L_p} \leq \sup_t ||W(t,\cdot)||_1 \, ||Df_{x_0}|| + 1$$

So $\mathcal{L} : W^{1,p} \to L_p$ is invertible and

$$||\mathcal{L}^{-1}|| \leq \max\{\sup_t ||W(t,\cdot)||_1 \ , \ \sup_t ||W(t,\cdot)||_1 \, ||Df_{x_0}|| + 1\}$$

■

The admissible pair $(L_\infty, (W^{1,1})^*)$

Conjecture 4.2 *The solution $x_0 \in L_\infty$ is hyperbolic if and only if $\mathcal{L} : L_\infty \to (W^{1,1})^*$ is invertible.*

A proof of this is not available at present but the generalisation should be valid since the Green's function methods have a distributional analogue.

4.2.3 Persistence theorem

For hyperbolic solutions coming from $\mathcal{A}$ where $(\mathcal{A}, \mathcal{B})$ is admissible there is an analogous persistence theorem. The operator of interest is $G_f : \mathcal{A} \to \mathcal{B}$ defined by

$$G_f(x)(t) \;=\; \dot{x}(t) - f(x(t), t)$$

where f is a $\tilde{\mathcal{C}}^1_{\mathcal{A},\mathcal{B}}$ perturbation of f_0.

Theorem 4.14 *If x_0 is a hyperbolic solution of $\dot{x} - f_0(x,t)$ then for $\tilde{\mathcal{C}}^1_{\mathcal{A},\mathcal{B}}$-close f, there is a locally unique continuation $x_f \in \mathcal{A}$, solving $\dot{x} = f(x,t)$. Moreover, x_f is hyperbolic, of the same stability type as x_0 and is smooth in f.*

Proof As in theorem 2.13, this follows from the application of the IFT to G. ■

This can now be used as before to provide good safety estimates for a much more general class of perturbations. Typically, one should start with a simple known solution x_0, like an equilibrium or a limit cycle and then consider perturbations drawn from a choice of spaces. The choice of perturbation space will dictate which space we expect the response to be in and the relevant estimates can be easily obtained.

There are some interesting uses of this generalisation apart from this obvious one. One idea is to think of a perturbation in initial condition as being the response to an appropriate impulse at time t_0. Specifically, by adding the function $c\delta_{t_0}$ for some $c \in \mathbb{R}^n$, we are simply the solution by c at time t_0. Thus a lower bound on the size of impulse neccessary to break hyperbolicity would also give a lower bound on the size of the basin of attraction. It is not clear at this stage whether this method could improve the previous basin estimates.

4.3 Further generalisations

Now I give some more ideas for generalising the theory. These represent potentially very fruitful directions of research.

4.3.1 Finite time estimates

It is often the case that one is only interested in solutions in finite-time interval. In this case the theory I have presented might be seen as not very useful since it deals with solutions defined on the whole of $\mathbb{R}$. However, it is relatively easy to treat only finite-time intervals if neccessary.

One way of doing this is to restrict the function spaces to ones which contain only decaying functions. For example, one could use exponentially growing norms on $\mathcal{C}^k$ functions.

$$||x||_{\mathcal{C}^k,\alpha} = \max_{m\in[0,k]} \{ \ |e^{\alpha t}x^{(m)}(t)| \ \}$$

Or more simply, one could use the Banach spaces $\mathcal{C}^k_c$ which are the spaces of $\mathcal{C}^k$ functions which converge to 0 at $\pm\infty$.

However, this is is to some extent unneccessary since it follows from the theory I have presented that the response to exponentially decaying perturbations is also exponentially decaying. This follows from the fact that the Green's function and thus $\mathcal{L}_f^{-1}$ is exponentially decaying. Thus is one starts off with a solution only defined on a finite interval, one can guarantee an exponentially decaying response to finite time perturbations. This is strong enough for most purposes. Moreover, one could formalise this idea by looking for decay rates explicitly when this is useful.

4.3.2 Basins of attraction for perturbed solutions

We have seen that linearly attracting solutions are uniformly asymptotically stable and thus have basins of attraction. It also has been shown that linearly attracting solutions persist under small perturbations. It is fairly simple to combine the proofs of these theorems and obtain a method for estimating the basin of attraction for the perturbed solution.

To do this we consider the operator

$$\begin{array}{rccl} G: & (\tilde{\mathcal{C}}^1_+ \times \mathbb{R}^n) \times \mathcal{C}^1_+ & \longrightarrow & (\mathcal{C}^0_+ \times \mathbb{R}^n) \\ & (f, c, x) & \longmapsto & (\ \dot{x}(\cdot) - f(x(\cdot), \cdot)\ ,\ x(0) - c\ ) \end{array}$$

If $G(f_0, 0, x_0^+) = 0$ for some linearly attracting solution $x_0 \in \mathcal{C}^1$ then one can apply the IFT to G and deduce the existence of a unique continuation $x_{f,c}$ for small enough perturbations of f and c. Moreover, the differential equation

$$\frac{dx_{f,c}}{d(f,c)}(\delta f, \delta c) = \mathcal{L}^{-1}_{f,c}(\delta f(x_{f,c}(\cdot), \cdot), \delta c)$$

can be used to obtain estimates of the basin of attraction for the perturbed solution.

4.3.3 Autonomous systems and normally hyperbolic manifolds

For autonomous systems, we have seen that the only strongly hyperbolic solutions are equilibria. There is a standard definition in use of hyperbolicity for solutions of autonomous systems. One just looks for a splitting $\mathbb{R}^n = E_t^+ \oplus E_t^- \oplus \{\lambda \dot{x}\}$. which allows a neutral direction tangent to the orbit.

A situation which would be interesting to deal with is non-autonomous perturbations of an autonomous system which has an attracting periodic solution x_0. Due to the neutral direction, there is a phase-shift degeneracy and one does not expect the orbit to persist under generic perturbations. However, there is an invariant cylinder in the extended phase-space consisting of all the time translations of x_0 which will persist. This can be seen to be a *normally hyperbolic manifold.* This is an invariant manifold in which contraction and expansion transverse to the manifold is stronger than any contraction or expansion in the manifold.

There is a theory for these manifolds, for example Wiggins [57], which shows that they persist under perturbations to the vector field. What would be extremely useful is a formulation which allows for non-autonomous perturbations and which naturally gives good estimates of the response so one can easily derive safety criteria.

There are number of ways one might attempt to do this.

- Lyapunov-Schmidt reduction: $\mathcal{L}$ is not invertible for most interesting solutions on normally hyperbolic manifolds, for example, a periodic orbit of an autonomous system. The basic idea behind the Lyapunov-Schmidt reduction is to split the problem of inverting $\mathcal{L}$ into a part which is invertible and a part which deals only with the kernel of $\mathcal{L}$. This second part should be finite dimensional and is known as the reduced problem. See [6] for details. However, this method fails if the problem is posed in function spaces allowing for aperiodic functions, for example C^k. This is because, if $\mathcal{L}$ has a finite dimensional kernel, the co-kernel is generically infinite dimensional so $\mathcal{L}$ is not a Fredholm operator and one cannot reduce to a finite dimensional problem. This problem is intrinsic to aperiodic systems and seems not appear in periodic systems.

- Partial differential equation method: This method is perhaps the most closely related to the invertible operator characterisation of hyperbolicity and potentially the most useful for the purposes of obtaining safety estimates. For autonomous systems, the method has been formulated by Sacker [37]. The idea is to use the coordinates (x, θ) where θ are coordinates on the manifold

and x are coordinates transverse to the manifold. Then one can parametrise the invariant manifold by a function $x = f(\theta)$ and look for a continuation of $f(\theta)$ for small perturbations. However, a problem with this method is that the linearisation associated with the manifold is a hyperbolic partial differential operator and does not appear to be invertible. In [37], an elliptic regularisation term is added to get round this problem and although this method works for autonomous systems, it is not clear if it would work for those which are time-dependent.

- Contraction mapping method: This is an extension of the ideas of Bogoliubov and Mitropolskii and a proof is given in Hale [15]. The idea is again to parametrise the manifold by a function $x = f(\theta)$ but this time to only work in the phase space rather than a function space. Then one shows persistence of $f(\theta)$ by making it a fixed point solution of a certain contraction mapping. Although this is a direct method and successfully answers the problem, there are a number of inadequacies. Firstly, it is not very easy to generalise and secondly, the analysis works only in a small neighbourhood of the unperturbed solution and thus one cannot continue very far from this limit.

Chapter 5

Applications

Here I consider some specific classes of examples and show how they can be made amenable to the analysis of the previous chapters.

5.1 Forced damped oscillators

5.1.1 Equations of motion

I consider the second order damped oscillator with the following equation of motion.

$$\ddot{x} + \gamma\dot{x} + V'(x) = u(t) \tag{5.1}$$

$\gamma > 0$ is the coefficient of damping. For simplicity I take linear damping but non-linear damping could be taken without any problems. $V'(x)$ comes from a potential energy function $V(x)$ which I assume is $\mathcal{C}^2$. Duffing's equation, for example, comes from $V(x) = -x^2/2 + x^4/4$ which is an archetypal twin well oscillator. Bounded solutions of (5.1) are functions $x \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$.

This is a very common class of examples both in dynamical systems theory and in the applications. They can be made to display many very interesting dynamical behaviours, for example, phase-locking, quasi-periodicity, period doubling and chaos. Indeed forced oscillators have been used in the past as good systems in which to explore these behaviours. See for example Guckenheimer & Holmes [14].

The second order equation (5.1) can be written as a first order system.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\gamma\dot{x} - V'(x) + u(t) \end{aligned} \tag{5.2}$$

Now we look for solutions $\binom{x}{\dot{x}} \in C^1(\mathbb{R}, \mathbb{R}^2)$. Care should be taken to use the correct norm when doing this however. In $C^1(\mathbb{R}, \mathbb{R}^2)$ we have the norm

$$\left|\left|\begin{pmatrix} x \\ \dot{x} \end{pmatrix}\right|\right|_{C^1} = \max\left\{ \sup_t \left|\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}\right| , \sup_t \left|\begin{pmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{pmatrix}\right| \right\}$$

Since we are interested in $||x||_{C^2}$, we need to use $\left|\binom{x}{y}\right|_\infty = \max\{|x|, |y|\}$ instead of the usual Euclidean norm, $\left|\binom{x}{y}\right|_2$. This gives the isometric isomorphism we want between $C^2(\mathbb{R}, \mathbb{R})$ and a subspace of $C^1(\mathbb{R}, \mathbb{R}^2)$.

I will be particularly interested in the perturbation of a linearly attracting equilibrium. Without loss of generality I assume this is $x_0 = 0$. Taking $V'(0) = 0$ and $V''(0) > 0$ guarantees this. The characteristic polynomial then has the two roots

$$\lambda = -\gamma/2 \pm \sqrt{\gamma^2/4 - V''(0)}$$

When $\gamma^2 > 4V''(0)$ we have two real negative roots. This is known as 'overdamping'. A more interesting situation is encountered when $\gamma^2 < 4V''(0)$. Then we have a complex conjugate pair of roots with negative real part. In this case we expect oscillations about $x = 0$ which for small amplitudes have frequency approaching $\omega_0 = \sqrt{V''(0) - \gamma^2/4}$.

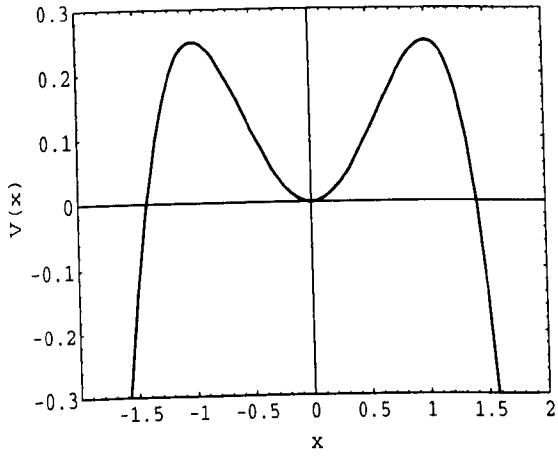
This model is frequently encountered in the applications and it is natural to ask how robust the equilibrium is when the system is subjected to external driving forces. One would also like to know how the basin of attraction changes. For periodic forcing there is an extremely well developed theory. Below I summarise some findings of this theory in the context of a simple ship capsize model.

5.1.2 Ship stability and the 'escape equation'

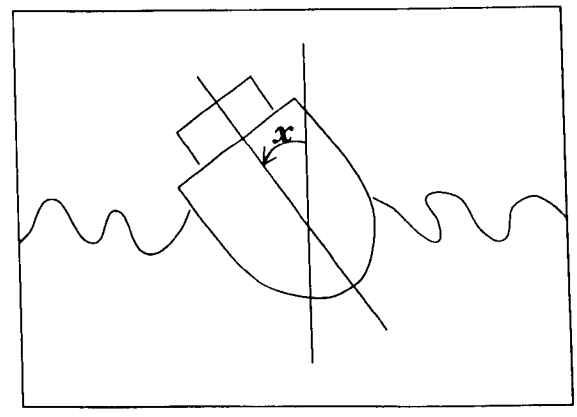
I am particularly interested in the study of what is termed the 'escape equation' or 'capsize equation'. It is given by a potential energy function of the form

$$V(x) = ax^2 - bx^4$$

with $a, b > 0$. Large or unbounded solutions represent escape. This model has frequently been used in the study of ship stability especially where one is interested in preventing capsize or large motions. x represents the angle of the ship to the vertical and the perturbations represent wind and waves in the lateral direction. Since x is an angle we could take $V(x)$ to be periodic, for example, $V(x) = a\cos(x) - b\cos(2x)$, but since I am only interested in the potential well containing $x = 0$, the quartic potential is adequate and allows for easier analysis.



(a) Potential function for the escape equation. $V(x) = ax^2 - bx^4$



(b)

More realistic models of ship stability have been developed but it has been argued that they are reasonably well approximated by a second order ODE of the form above. However, a modification which is often included is some parametric excitation to allow for vertical displacement of the centre of mass of the ship. This also fits in with the methodology of this thesis and I will give some estimates for this case as well.

Unforced dynamics

The phase portrait for the unforced dynamics is shown in figure 5.1. I consider only the case where the eigenvalues are complex conjugate since the over-damped case is not as physically realistic or interesting.

By linearising about the equilibrium, $x = 0$, we see that the frequency of small amplitude oscillations is approximately $\omega_0 = \sqrt{2a - \gamma^2/4}$. Due to the nonlinearity, the frequency decreases with increasing amplitude. This is typical of physical systems and is known as 'softening'.

Since it is only the ratio a/b that is important, (up to a rescaling), I will usually take $a = 1/2$, giving $\omega_0 = 1$. The nonlinearity can then be controlled by adjusting b.

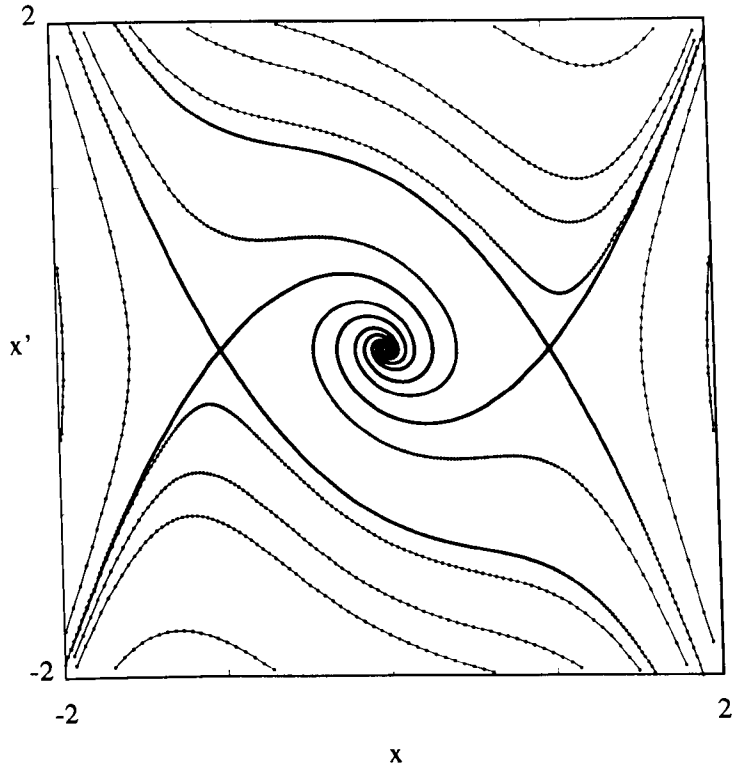


Figure 5.1: Phase portrait for the unforced escape equation.

Sinusoidal forcing

Now consider the system with sinusoidal forcing.

$$\ddot{x} + \gamma\dot{x} + 2ax + 4bx^3 = \varepsilon \sin(\omega t)$$

This has three dimensional phase space $(x, \dot{x}, t)$, due to the time dependence although since the forcing is periodic with period $T = 2\pi/\omega$, it is common to consider instead the two dimensional Poincare (or time T) map. However, since I am interested in the actual size of motions rather than just a qualitative description of the dynamics it is neccessary to consider the behaviour of the system at all times to avoid the loss of information involved in sampling periodically. Some typical orbits, projected onto the $(x, \dot{x})$-plane, are shown in figure 5.2.

For an idea of the response of the system to forcing with a range of frequencies and amplitudes see figure 5.3. I have used an initial condition close to the unforced equilibrium, and then measured the sup-norm of the response. For small amplitude forcing we see the frequency response peaks bending slightly to the left. This is what we expect in softening systems. As amplitude of forcing is increased we observe 'resonant hysteresis'. This is an important phenomenon when one is concerned

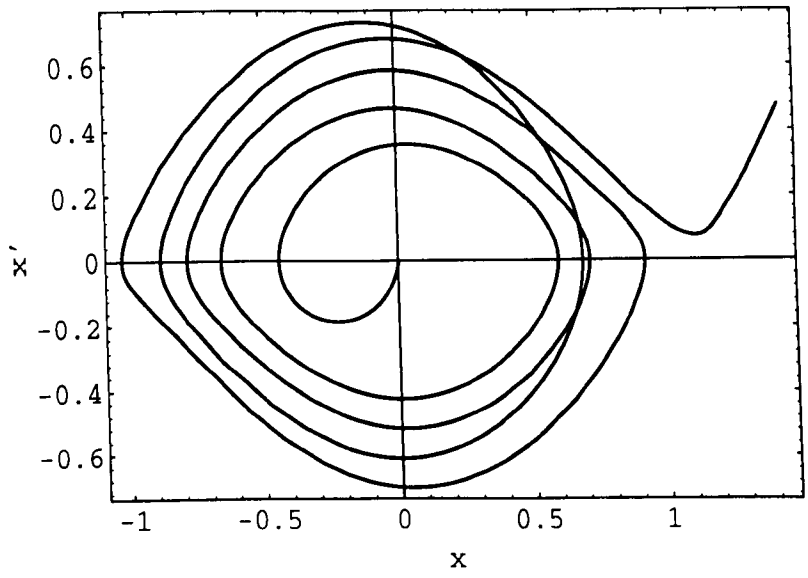


Figure 5.2: A typical escaping orbit of the sinusoidally forced escape equation. Note that this is a projection onto the $(x, \dot{x})$ plane. Forcing parameters are $\varepsilon = 0.3$, $\omega = 0.6$. Other parameters are $\gamma = 0.5$, $a = 0.5$, $b = 0.25$. Initial conditions are $(0, 0)$.

about robustness of solutions. When the excitation exceeds a certain amplitude, for certain excitation frequencies there can be two stable attracting solutions for the system. In this case increasing the frequency of the forcing leads to a saddle-node bifurcation and the system jumps to the upper limb. See figure 5.4 for a schematic frequency response diagram. These jumps are also evident in figure 5.3 where they appear as vertical slopes in the response at around $\omega = 0.6, \varepsilon = 0.3$. These jumps are not neccessarily escape phenomena since they can still have a bounded or safe response. When frequency is decreased again the system follows the upper limb and thus exhibits the bigger of the two responses. It is not unreasonable to expect a slow evolution of parameters like the amplitude and frequency of external forcing even if we expect them to be essentially periodic. Thus resonant hysteresis is a potentially dangerous phenomenon since escape could occur at frequencies for which the response is expected to be small. Figure 5.3 does not display the upper limb since it cannot be obtained from fixed frequency forcing with small initial conditions. This is another reason why it is important to consider aperiodic forcing when deriving safety criteria.

Many more details of this case can be found in [51, 52, 28, 29], where safety issues are discussed.

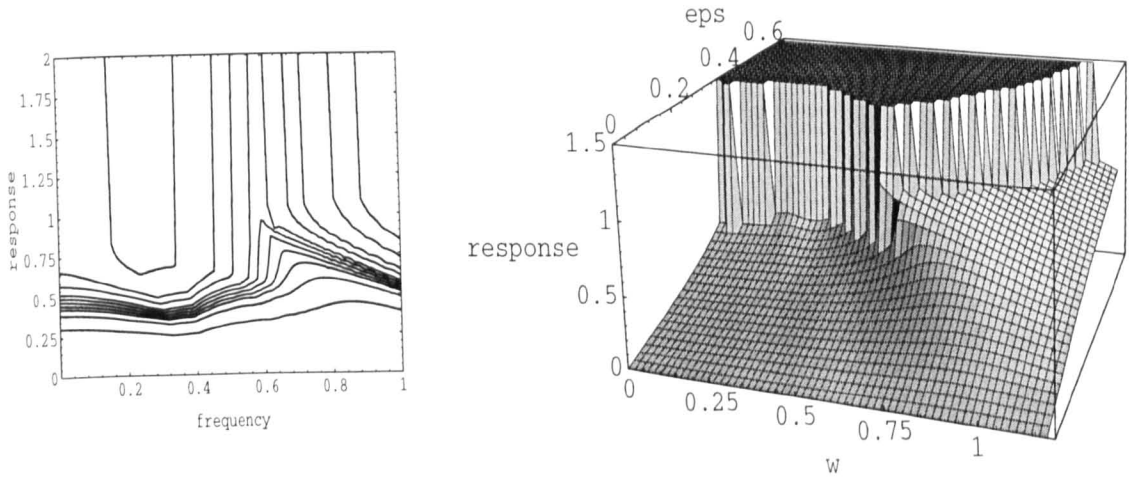


Figure 5.3: Frequency-response diagram obtained from sinusoidal forcing at a range of frequencies and amplitudes. Equation parameters are $\gamma = 0.5$, $a = 0.5$, $b = 0.5$. The forcing amplitudes in the first figure in order of magnitude are $\varepsilon = 0.2, 0.25, 0.27, 0.28, 0.29, 0.3, 0.31, 0.33, 0.35, 0.4$ in order of magnitude.

Aperiodic forcing

Unfortunately, the methodology, observations and most importantly the estimates relating to the periodically forced system do not apply in general to the aperiodic case and so a fresh approach is required.

From a mathematical perspective it is natural to ask what happens when the forcing is quasiperiodic, chaotic or stochastic. However, although there are some interesting results no-one has really addressed the issue of quantitative safety criteria nor do the techniques generalise easily to arbitrary forcing.

5.1.3 Persistence estimates

I now apply theorem 2.13 to the system in order to obtain estimates for arbitrary, bounded external forcing. The theorem applies immediately and although it is very general in nature with a little work some good results can be obtained.

Rather than write (5.1) as the system of two first order equations (5.2), it is convenient and illuminating to extend the theory to second order systems. Thus

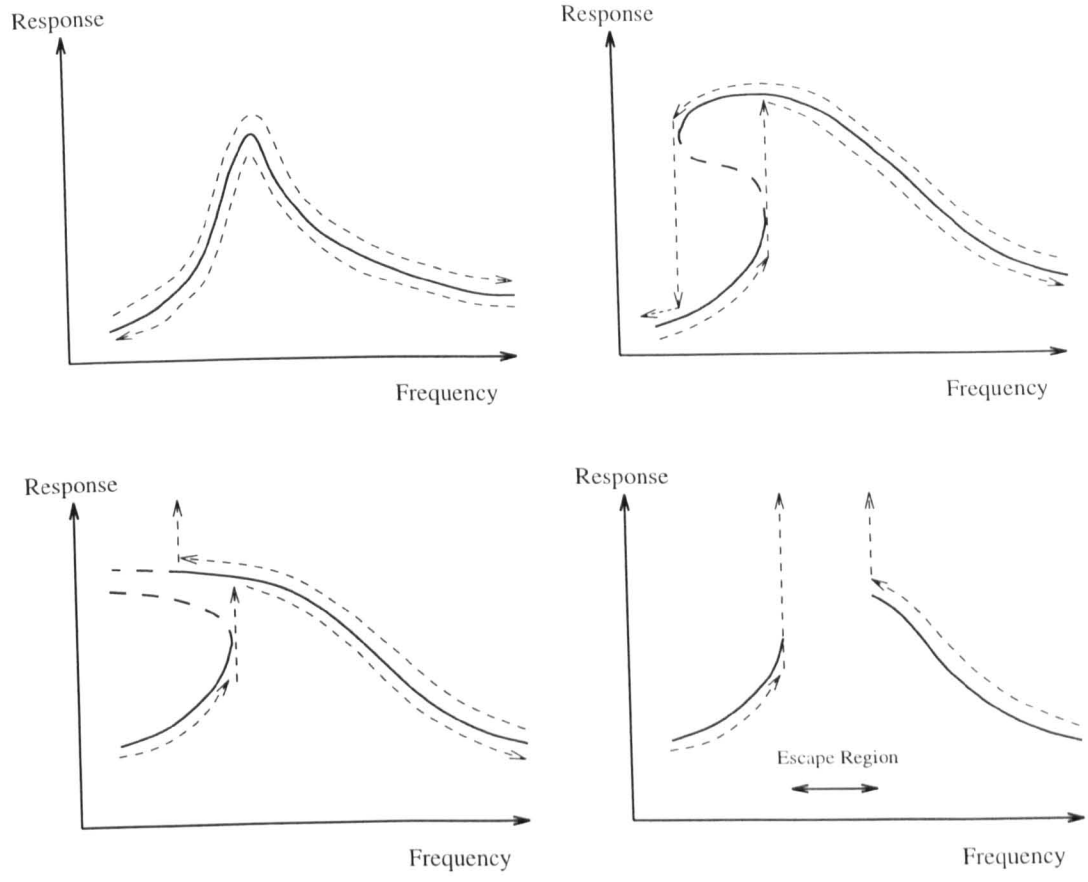


Figure 5.4: Schematic frequency-response diagrams for typical softening systems like the escape equation. Figures are in order of increasing forcing amplitude, showing linear-like response followed by resonant hysteresis and finally regions of escape (loss of safe solution).

we consider the operator

$$\begin{aligned} G_u : \ \mathcal{C}^2 &\longrightarrow \mathcal{C}^0 \\ x &\longmapsto \ddot{x} + \gamma\dot{x} + V'(x) - u \end{aligned} \tag{5.3}$$

For a given $u \in \mathcal{C}^0$, a function $x \in \mathcal{C}^2$ is a solution of (5.1) if $G_u(x) = 0$. It is hyperbolic if $DG_{u,x}$ is invertible and it is linearly attracting if the projection onto the stable subspace is the identity. The linearly attracting equilibrium $x_0 - 0$, satisfies $G_u(0) = 0$. To perform the continuation, consider the linearisation $\mathcal{L}_{u,x} = DG_{u,x} : \mathcal{C}^2 \to \mathcal{C}^0$

$$\mathcal{L}_{u,x}\xi \ = \ \ddot{\xi} + \gamma\dot{\xi} + V''(x)\xi \tag{5.4}$$

As usual I write $\mathcal{L}_0$ instead of DG_{0,x_0} and $\mathcal{L}_u$ instead of DG_{u,x_u}. As noted earlier, the linearisation at a fixed point has constant coefficients so can be solved explicitly

by the standard methods. When $\lambda_1 \neq \lambda_2$,[i] the impulse response or Green's function is given by $W(t,s) = W_0(t-s)$ where

$$W(t,0) \;=\; W_0(t) \;=\; \begin{cases} 0 & t < 0 \\ (\lambda_2 - \lambda_1)^{-1}\left(e^{\lambda_2 t} - e^{\lambda_1 t}\right) & t \geq 0 \end{cases}$$

Figure 5.5 shows a typical Green's function for the system when the roots of the characteristic polynomial are complex.

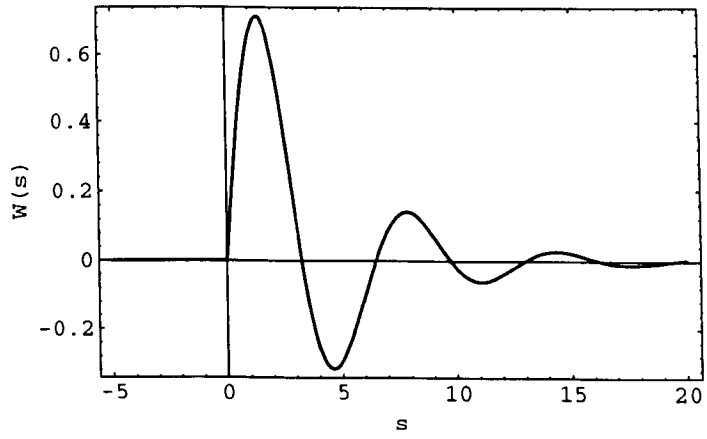


Figure 5.5: Green's function for a stable equilibrium with roots, $\lambda = a \pm bi$.

For each $\varphi \in \mathcal{C}^0$, the unique $\mathcal{C}^2$ solution of $\mathcal{L}_0\xi = \varphi$ is given by the convolution

$$(\mathcal{L}_0^{-1}\varphi)(t) \;=\; (W_0 * \varphi)(t) \;=\; (\lambda_2 - \lambda_1)^{-1}\int_0^\infty \left(e^{\lambda_2 s} - e^{\lambda_1 s}\right)\varphi(t-s)ds$$

$\mathcal{L}_0^{-1}\varphi$ is twice differentiable and it's derivatives are easily determined. They are given by

$$\begin{aligned} \frac{d}{dt}(\mathcal{L}_0^{-1}\varphi)(t) &= (\dot{W}_0 * \varphi)(t) \\ \frac{d^2}{dt^2}(\mathcal{L}_0^{-1}\varphi)(t) &= \varphi(t) - \gamma(\dot{W}_0 * \varphi)(t) - V''(0)(W_0 * \varphi)(t) \end{aligned}$$

Thus we have the estimate,

$$\begin{aligned} \left|\left|\mathcal{L}_0^{-1}\right|\right| &= \sup_{||\varphi||=1} ||W_0 * \varphi||_{C^2} \\ &\leq \max\left\{ ||W_0||_1 \,,\; \left|\left|\dot{W}_0\right|\right|_1 ,\; 1 + \gamma\left|\left|\dot{W}_0\right|\right|_1 + V''(0)\,||W_0||_1 \right\} \end{aligned}$$

Figure 5.6 shows how $\left|\left|\mathcal{L}_0^{-1}\right|\right|$ varies with damping γ, and nonlinearity $V''(0)$.

[i] If $\lambda_1 = \lambda_2$ then there is a slightly different (but unproblematic) formulation.

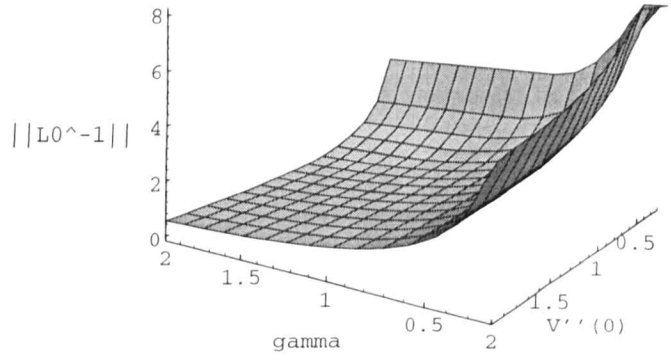


Figure 5.6: $||\mathcal{L}_0^{-1}||$ as a function of damping, γ, and nonlinearity, $V''(0)$.

Theorem 2.13 now guarantees a locally unique continuation $x_u \in \mathcal{C}^2$ for $||u||$ small enough. We now look for the neighbourhood of $u = 0 \in \mathcal{C}^0$ for which there is a unique bounded response lying in some predefined safety region. To do this we need to find a good estimate of $||\mathcal{L}_u^{-1}||$. This is a linear ordinary differential operator but it has unknown, bounded, time-varying coefficients and so cannot be easily inverted.

General bound for $||\mathcal{L}_u^{-1}||$

As in section 3.3, let $\varepsilon = ||u||$, $\eta = ||x_u||$ and $K_0 = ||\mathcal{L}_0^{-1}||$. From (5.4) it is easy to see that

$$\mathcal{L}_u - \mathcal{L}_0 \;=\; V''(x_u) - V''(0)$$

Given a bound $\beta(\eta) \geq ||V''(x_u) - V''(0)||_\infty$ we deduce from lemma 3.1

$$||\mathcal{L}_u^{-1}|| \;\leq\; [K_0^{-1} \;-\; \beta(\eta)]^{-1} \tag{5.5}$$

which is valid while $\beta(\eta) < K_0^{-1}$.

A more direct method of treating $\mathcal{L}_u$

As we have seen in section 3.2, $\mathcal{L}_u^{-1}$ is an integral operator whose kernel is the principal Green's function $W(t,s)$. Since x_u is stable this is given by

$$W(t,s) \;=\; \begin{cases} 0 & t < s \\ X_s(t) & t > s \end{cases}$$

where $X_s(t)$ is the fundamental solution of $\mathcal{L}_u\xi = 0$ satisfying $X_s(s) = I$. From (5.4) we see that

$$\mathcal{L}_u\xi = \ddot{\xi} + \gamma\dot{\xi} + (V''(0) + p(t))\xi \tag{5.6}$$

where $p(t) = V''(x_u) - V''(0)$.

If p is periodic then (5.6) is a standard damped Hill's equation. Typically however, p is an unknown *aperiodic* excitation although we can find a bound $||p|| \leq \Delta p$, where Δp depends on $\eta = ||x_u||$.

Some, details about Hill's equation can be found in [15]. An important property of the undamped Hill's equation is that arbitrarily small parametric excitation at certain frequencies can render the zero equilibrium unstable. The introduction of viscous damping creates a buffer zone so that for small enough excitation, $\Delta p < \varepsilon^*$ say, $x = 0$ becomes asymptotically stable. Figure 5.7 shows the zones of instability for both the undamped and damped cases. When the excitation is aperiodic we

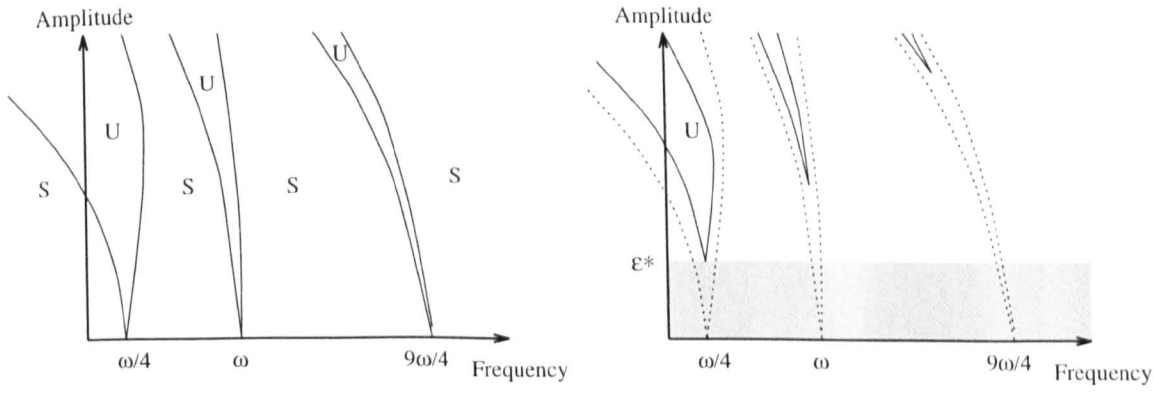


Figure 5.7: Schematic stability regions for a Hill's equation. First is the undamped case from which we can see that arbitrarily small excitation at certain frequencies cause instability. The second figure shows the damped case. The tongues get smaller and move off the axis and there is an ε^* such that excitation amplitudes below this value cannot cause instability.

expect the existence of a similar $\varepsilon^* \in (0, \bar{\varepsilon})$ such that $\Delta p < \varepsilon^*$ guarantees asymptotic stability of $x = 0$. This is of vital importance since while $x = 0$ is an asymptotically stable solution of $\mathcal{L}_u\xi = 0$, the Green's function will be exponentially decaying and thus integrally bounded. Clearly $||\mathcal{L}_u^{-1}||$ will be bounded while this is true. Rahn & Mote [35] looked at the problem of determining a minimum decay rate for solutions of $\mathcal{L}_u\xi = 0$ when p is an arbitrary bounded function. Their ideas can be extended

to provide a good estimate of $||\mathcal{L}_u^{-1}||$. The basic idea is determine the worst case excitation for a given Δp and thus deduce the minimum decay rate $b^* < 0$. Solutions will then satisfy $|\xi(t)| \leq Ne^{b^*(t-\tau)}|\xi(\tau)|$ for some $N > 0$. From this we can easily deduce that $||W(t,\cdot)||_1 \leq -N/b^*$ although this is a crude bound and we can do much better. As η increases we expect b^* to pass through 0 at which point we lose invertibility. For simplicity, I assume that $||\mathcal{L}_u^{-1}|| = \sup_t ||W(t,s)||_1$ which will be the case if $|| \; ||_{C^2}$ is weighted towards $|| \; ||_\infty$.

Calculating b^*

This section follows closely the paper of Rahn & Mote [35].

First we transform $\mathcal{L}_u\xi = 0$ into a canonical form. Using a change of variables and a rescaling of time

$$\begin{aligned} t &= 2s/\sqrt{V''(0) - \gamma^2} = \alpha s \\ y(s) &= e^{\gamma t/2}x(t) \\ q(s) &= 4p(t)/(4V''(0) - \gamma^2) \end{aligned}$$

we get

$$\ddot{y} + [1 + q(s)]y = 0$$

where additionally I assume that $\Delta p < V''(0) - \gamma^2/4$ so that $\Delta q = ||q|| < 1$.

Then we look for δ^*, N, depending on Δq, such that $|y(s)| \leq Ne^{\delta^*(s-s')}|y(s')|$. From this we see that

$$|\xi(t)| \leq Ne^{(\delta^*\sqrt{V''(0)-\gamma/4}-\gamma/2)(t-\tau)} |\xi(s)| = Ne^{b^*t} |\xi(\tau)| \tag{5.7}$$

$b^* < 0$ guarantees asymptotic stability of $\xi = 0$. This translates to the condition

$$\delta^* < 2\gamma/(4V''(0) - \gamma^2)$$

To find δ^* we look for the worst-case excitation $q^*(t)$. This is a standard problem in optimal control theory. From Pontyragin's maximum principle we learn that the worst case excitation is a 'bang-bang' control of the form $q^*(t) = \text{sign}(yz_2)\Delta q$ where $z = \binom{z_1}{z_2}$ is the co-state vector whose dynamics is given by the ODE adjoint to $\mathcal{L}_u\xi = 0$.

Remark 5.1 *The dynamics are scale invariant thus switching points lie on rays emanating from the origin. One switching ray is $y = 0$. If the $z_2 = 0$ switching ray is at angle θ^* in the right half plane then by symmetry it is at angle $\theta^* - \pi$ in the left half plane.*

Remark 5.2 *The optimal excitation is periodic with four switches per cycle. This follows from the fact that in between switches the response is that of a harmonic oscillator. The time between switches is thus amplitude independent and so a cycle consisting of four switches has a particular period S.*

Since the optimal excitation is periodic it follows from Floquet theory that the response is exponentially bounded but to find the maximum growth rate δ^* we should solve the equations of motion in terms of θ and find a growth rate $\delta(\theta)$. Then we maximise over θ to get δ^* and θ^*.

Clearly $0 \leq \theta \leq \pi/2$ since the response to $q(s) = -\Delta q$ excitation has a longer period than the response to $q(s) = +\Delta q$ excitation.

Note that we are looking for maximum growth *rate* rather than the maximum gain in phase space. By maximum phase space gain, it is meant the largest growth in amplitude per revolution. Clearly, the maximum phase space gain for the canonical

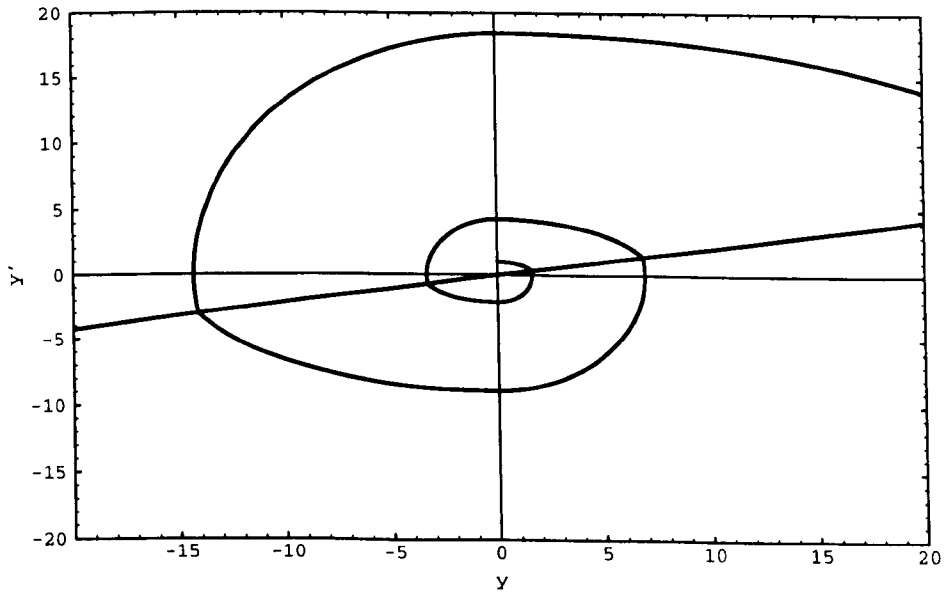


Figure 5.8: An optimally excited, canonical Hill's equation. Initial conditions are $y = 0, \dot{y} = 1$. t goes from 0 to 15. The relevant parameters are $\Delta q = 0.65$, $\theta^* = 0.210$. The growth rate in this case is 0.213.

equation is achieved by switching at $y = 0$ and $\dot{y} = 0$ and for the original equation the switches are at $\xi = 0$ and $\dot{\xi} = 0$. However, $||W(t, \cdot)||_1$ is not maximised by these excitations.

Let $\omega_+ = \sqrt{1 + \Delta q}$ and $\omega_- = \sqrt{1 - \Delta q}$. Solutions of $\ddot{y} + \omega_\pm^2 y = 0$ are given by

$$\begin{aligned} y(s) &= A \sin(\omega_\pm s - \varphi) \\ \dot{y}(s) &= A\omega_\pm \cos(\omega_\pm s - \varphi) \end{aligned}$$

The line in phase space at angle θ is given by $\dot{y} = my$ where $\tan\theta = m$. Note that between switches, energy remains constant. That is, $\omega_\pm y^2 + \dot{y}^2 = (A\omega_\pm)^2$.

Assume that the initial condition is $y = 0, \dot{y} = 1$. Then the solution until the first switch is

$$\begin{aligned} y(s) &= (1/\omega_-)\sin(\omega_- s) \\ \dot{y}(s) &= \cos(\omega_- s) \end{aligned}$$

The switching ray at angle θ is hit at time s_1 when $\tan(\omega_- s_1) = \omega_-/m$ giving

$$s_1 = (1/\omega_-)\tan^{-1}(\omega_-/m)$$

At the first switch we have

$$\begin{aligned} y(s_1) &= (1/\omega_-)\ \sin(\omega_- s_1) &= A_1 \sin(\omega_+ s_1 - \varphi_1) \\ \dot{y}(s_1) &= \qquad\qquad \cos(\omega_- s_1) &= A_1\omega_+ \cos(\omega_+ s_1 - \varphi_1) \end{aligned}$$

Solving for A_1 and φ_1 gives

$$\begin{aligned} (A_1\omega_+)^2 &= (\omega_+/\omega_-)^2 \sin^2(\omega_- s_1) + \cos^2(\omega_- s_1) \\ \tan(\omega_+ s_1 - \varphi_1) &= (\omega_+/\omega_-)\tan(\omega_- s_1) \end{aligned}$$

which simplifies to

$$\begin{aligned} (A_1\omega_+)^2 &= (m^2 + \omega_+^2)/(m^2 + \omega_-^2) \\ \varphi_1 &= -\tan^{-1}(\omega_+/m) + (\omega_+/\omega_-)\tan^{-1}(\omega_-/m) \end{aligned}$$

where $\tan^{-1}$ is chosen in the interval $[0, \pi/2]$.

The second switch occurs at time s_2 when $y(s_2) = A_1 \sin(\omega_+ s_2 - \varphi_1) = 0$. Clearly at this time $\dot{y}(s_2) = -A_1\omega_+$. Since $\omega_+ s_2 - \varphi_1 = \pi$ we have

$$s_2 = (\varphi_1 + \pi)/\omega_+$$

By symmetry and scale invariance this process is repeated in the left half-plane. A cycle is thus periodic with period

$$S = 2s_2 = 2(\varphi_1 + \pi)/\omega_+$$

The growth factor of solutions between $s = 0$ and $s = s_2$ is $A_1\omega_+$ so for a whole cycle solutions grow by a factor of $(A_1\omega_+)^2$. To find the growth rate we solve $|y(S)| = e^{\delta S}\,|y(0)|$, that is $(A_1\omega_+)^2 = e^{\delta}S$. This gives

$$\delta = (1/S)\log(\omega_+ A_1)^2 \tag{5.8}$$

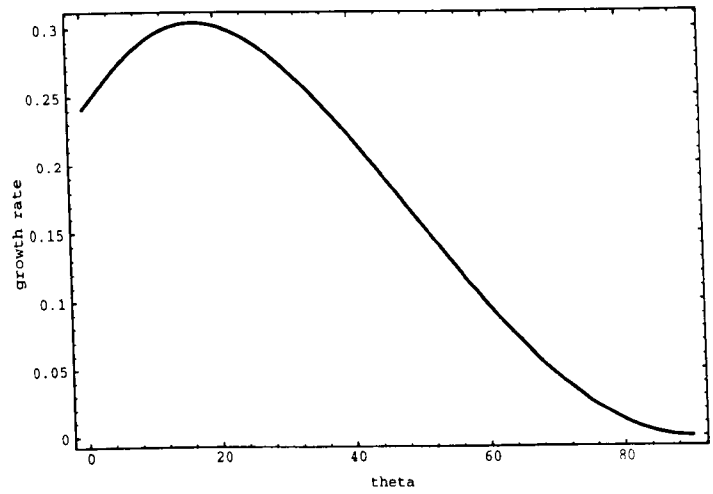
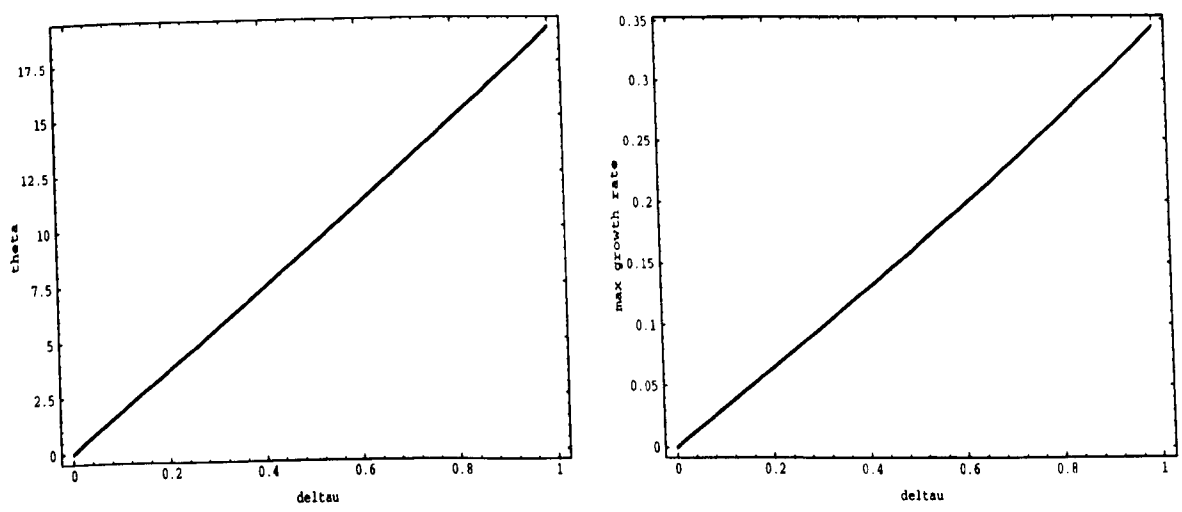


Figure 5.9: Growth rate δ, against switching angle θ (measured in degrees). $\Delta q = 0.9$.

Figure 5.9 shows how δ varies with θ.

To find δ^* we maximise (5.8) with respect to θ. Let θ^* be such that δ is maximal for this angle of switching ray. If we let $m^* = \tan\theta^*$ then it turns out that $\delta^* = m^*$.

Figure 5.10 shows how θ^* and δ^* vary with Δq.



(a) Optimal switching angle θ^*, versus Δq.

(b) Maximum growth rate δ^*, versus Δq.

Figure 5.10:

It follows from the linearity of the system that the fastest growing solution with initial conditions $y = 0, \dot{y} = 1$ provides an upperbound on the growth rate for any solutions of the canonical equation.

We can now see what the minimum decay rate is for the original variational equation. From (5.7) we can see it is given by

$$b^* = \delta^* \sqrt{V''(0) - \gamma^2/4} - \gamma/2$$

Clearly $|X_s(t)| \leq Ne^{b^*(s-t)}$ and since $\mathcal{L}_u^{-1}$ is the integral operator with this as kernel, $\mathcal{L}_u$ will be invertible while $b^* < 0$.

Crude estimate of $||\mathcal{L}_u^{-1}||$

Given δ^* we can find the smallest N such that

$$|y(s)| \leq Ne^{\delta^*(s-s')} |y(s')|$$

Note that we must use δ^* since anything smaller would not be a rigourous bound for all $s, s' > 0$ and anything bigger would give worse estimates.

To do this we find some time $\bar{s} \in [0, s_2]$ such that $y^*(s)$ is tangent to $Ne^{\delta^* s}$ at $\bar{s}$. Solving

$$\begin{aligned} Ne^{\delta^* \bar{s}} &= (1/\omega_-) \sin(\omega_- \bar{s}) \\ \delta^* Ne^{\delta^* \bar{s}} &= \cos(\omega_- \bar{s}) \end{aligned}$$

gives $\tan(\omega_- \bar{s}) = \omega_-/b^*$. Since $\delta^* = m^*$ it follows that the tangency occurs at the first switch and $\bar{s} = s_1$. Figure 5.11 shows this clearly.

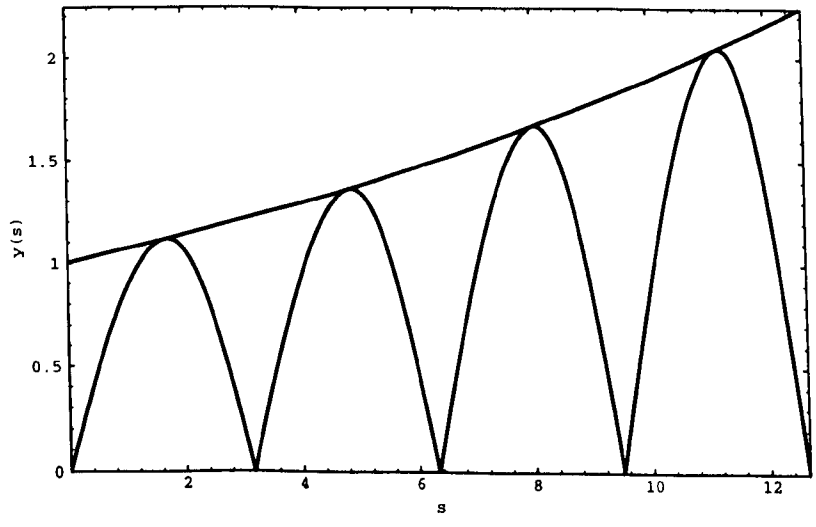


Figure 5.11: Plot of worst case solution y^*, and smallest bounding exponential, $Ne^{\delta^* s}$. Amplitude of excitation is $\Delta q = 0.2$ and the tangencies occur whenever y^* hits the switching locus at angle θ^*. The first of these is at time s_1=1.6766.

Thus we have

$$N^2 = e^{-2ms_1}/(m^2+\omega_-^2)$$

Given N a simple but effective estimate can be obtained.

$$|\xi(t)| \leq Ne^{b^*(t-\tau)}|\xi(\tau)|$$

so using the inequality

$$\begin{aligned}\int_{-\infty}^{\infty}|W(t,s)|\,ds &= \int_0^{\infty}|X_s(t)|\,ds \\ &\leq \int_0^{\infty} Ne^{b^*s}ds \end{aligned} \tag{5.9}$$

we see immediately that

$$||\mathcal{L}_u^{-1}|| \leq -N/b^* \tag{5.10}$$

Note that this is not accurate even when $\Delta p = 0$ because (5.9) is not a good estimate. Since the solution oscillates its integral is likely to be much lower than the integral of the exponential which bounds it.

Figure 5.12 shows how this estimate compares with the general estimate given by (5.5). It at least has the advantage that one can continue further. In fact one can continue until the true $\hat{\eta}$ where one is bound to lose invertibility if the excitation is of the right form.

Better estimates for $||\mathcal{L}_u^{-1}||$

Instead of finding a bounding exponential for solutions, $\xi(t)$, of the variational equation we could just integrate $|\xi^*|$ itself. Since it has the minimum decay rate of any solution it also has the biggest integral.

The worst case solution to the canonical equation is given by

$$y^*(s) = \begin{cases} (A_1\omega_+)^n(1/\omega_-)\sin(\omega_- s) & nS/2 < s < nS/2+s_1 \\ (A_1\omega_+)^n A_1\cos(\omega_+ s-\varphi_1) & nS/2+s_1 < s < (n+1)S/2 \end{cases}$$

Recall that we have used the transformation $y(s) = e^{\gamma t/2}\xi(t)$, $t = \alpha s$, where $\alpha = 2/\sqrt{4V''(0)-\gamma^2}$.

The estimate for $||\mathcal{L}_u^{-1}||$ we want is then

$$
\begin{aligned}
||\mathcal{L}_u^{-1}|| &\leq \int_0^\infty |\xi^*(t)|\ dt \\
&= \alpha \int_0^\infty \left|e^{-\alpha s\gamma/2} y^*(s)\right|\ ds \\
&= \alpha \int_0^{S/2} \left|e^{-\alpha s\gamma/2} y^*(s)\right| ds\ + \alpha A_1 \omega_+ \int_{S/2}^{S} \left|e^{-\alpha s\gamma/2} y^*(s - S/2)\right| ds\ + \cdots \\
&= \alpha \int_0^{S/2} \left|e^{-\alpha s\gamma/2} y^*(s)\right| ds\ + \alpha A_1 \omega_+ e^{-\alpha S\gamma/4} \int_0^{S/2} \left|e^{-\alpha s\gamma/2} y^*(s)\right| ds\ + \cdots \\
&= \alpha M \sum_0^\infty (A_1 \omega_+ e^{-\alpha S\gamma/4})^n \\
&= \alpha M/(1 - A_1 \omega_+ e^{-\alpha S\gamma/4}) \qquad (5.11)
\end{aligned}
$$

where $M = \int_0^{S/2} \left|e^{-\alpha s\gamma/2} y^*(s)\right|\ ds$.

Figure 5.12 compares the estimates of $||\mathcal{L}_u^{-1}||$ obtained from (5.5), (5.10) and (5.11). Note that (5.10) is not good at $\eta = 0$ or for small η since it does not take

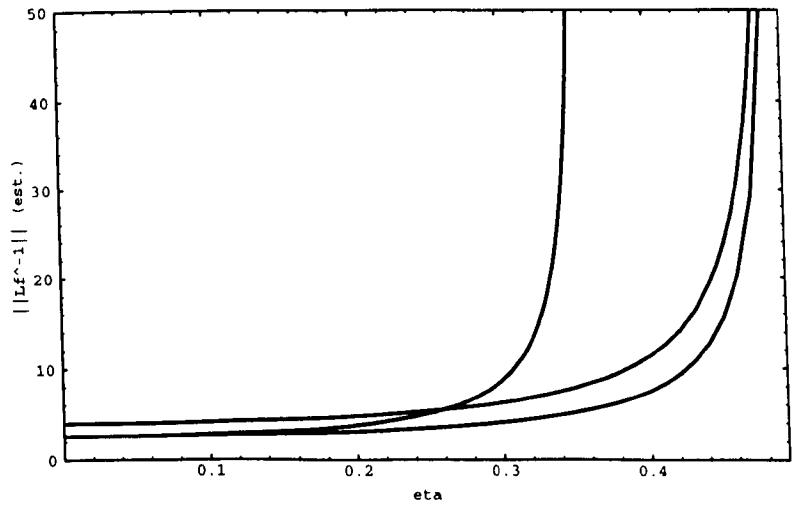


Figure 5.12: Upperbound for $||\mathcal{L}_u^{-1}||$ obtained from (5.5), (5.10) and (5.11), (going left to right).

into account the oscillatory nature of solutions. However, it eventually gives an improvement over the general estimate (5.5). Clearly (5.11) gives the best results although they are still suboptimal. This is because the worst case parametric excitation in the linearised equation does not result from any allowable external forcing of the nonlinear equation. Since we have the additional information that $p(t)$ in (5.6) must be $V''(x_u)$ for some x_u that satisfies the original ODE, in principle one could improve the estimates even further although this is tricky and time consuming.

Persistence estimates

So using one or all of the above methods we can find an upper bound $||\mathcal{L}_u^{-1}|| \leq \delta(\eta)$.

Since $\frac{\partial G_u}{\partial u} = -I_{C^0 \to C^0}$ we can see that $||\frac{\partial G_u}{\partial u}|| \leq \alpha(\varepsilon, \eta) = 1$.

This gives us the differential inequality

$$\frac{d\eta}{d\varepsilon} \leq \delta(\eta) \tag{5.12}$$

Following sections 3.2 and 3.3 we make this an equality and integrate to obtain the estimates $\eta(\varepsilon)$ and $\hat{\varepsilon}$. For any $\varepsilon < \hat{\varepsilon}$, calculating $\eta(\varepsilon)$ gives an upper bound on the size of the response. Equivalently, calculating $\varepsilon(\eta)$ gives an estimate of the largest forcing amplitude which has response below η.

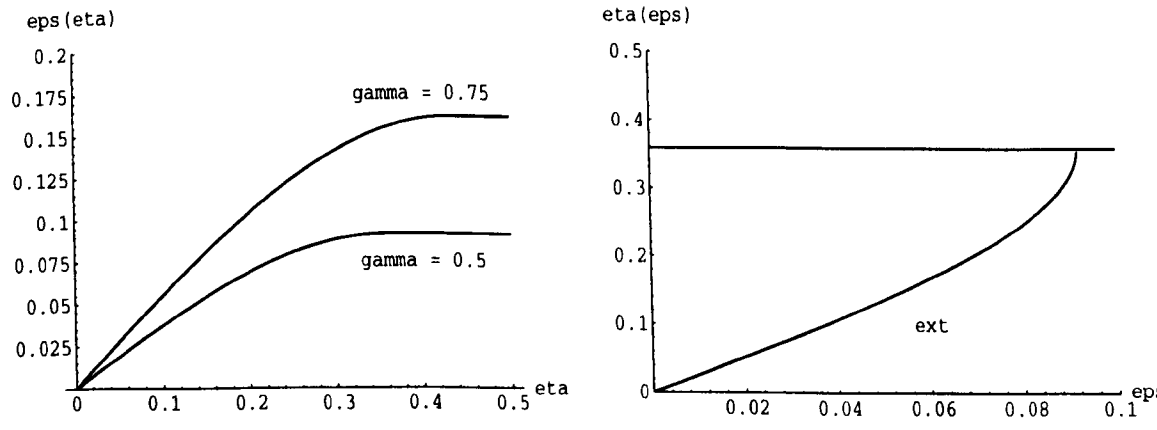
For the escape equation, if we use the basic estimate (5.5), obtaining estimates for the response is particularly simple. $V''(x) = 2a - 12bx^2$ so we can take $\beta(\eta) = 12b\eta^2$. This gives

$$\frac{d\eta}{d\varepsilon} \leq [K_0^{-1} - 12b\eta^2]^{-1}$$

which gives a unique continuation x_u, while $\eta^2 < \hat{\eta}^2 = K_0^{-1}/(12b)$. Integration gives $||x_u|| \leq \eta(\varepsilon)$ where $\eta(\varepsilon)$ is the smallest solution of

$$\varepsilon = K_0^{-1}\eta - 4b\eta^3$$

If we use estimate (5.11) we get a better estimate of $\eta(\varepsilon)$. Figure 5.13 shows $\eta(\varepsilon)$ and $\varepsilon(\eta)$ for this case.



(a) $\varepsilon(\eta)$ obtained by integrating (5.12). The two values of damping are $\gamma = 1/2$ and $\gamma = 3/4$. $a = 1/2$, $b = 1/4$.

(b) $\eta(\varepsilon)$ obtained by integrating (5.12). $\gamma = 1/2$, $a = 1/2$, $b = 1/4$.

Figure 5.13:

Remark 5.3 *Notice that $\eta(\varepsilon)$ goes vertical at $(\hat{\varepsilon}, \hat{\eta})$ as continuation is no longer guaranteed for $\varepsilon > \hat{\varepsilon}$. Equivalently, $\varepsilon(\eta)$ levels off at this point.*

Worst case forcing

For the forced damped oscillator the switching locus for the optimal bang-bang control is easily obtained. Consider the effect on the phase portrait of adding a constant forcing term. $u(t) = +\varepsilon$ shifts the equilibrium to the right and $u(t) = -\varepsilon$ shifts it to the left. Note that $\frac{dy}{dx} = \dot{y}/\dot{x}$ is just the slope of the trajectory. In the upper half-plane we have $\dot{x} = y > 0$ so increasing $\frac{dy}{dx}$ gives bigger solutions. Similarly, in the lower half-plane, decreasing $\frac{dy}{dx}$ gives bigger solutions. Since $\frac{dy}{dx} = -\gamma - V'(x)/y \pm \varepsilon/y$ we set

$$u^*(t) = \begin{cases} \varepsilon\,, & y > 0 \\ -\varepsilon\,, & y < 0 \end{cases}$$

The switching locus is the $y = 0$ axis. Figure 5.14 shows a solution with this forcing scheme. The amplitude is just large enough to induce escape.

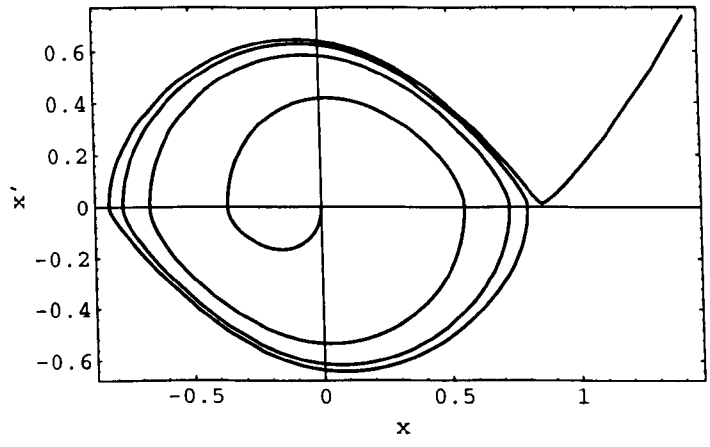


Figure 5.14: A solution which just escapes for system with optimal bang-bang control.

Comparisons of persistence estimates with numerical data

The natural question to ask about this procedure is how accurate are the estimates. By numerically integrating the system (5.1) with various types of forcing function, including the optimal bang-bang control, one can make a comparison which gives some answers to this question.

Figure 5.15 shows the response of the system for various forcing functions and the hyperbolicity estimates of the response given by integrating (5.12). The periodic forcing frequency for both the sinusoidal and 'square wave' forcing is chosen so as

to give the biggest response ($\omega \approx 0.6$) but it is clearly less efficient at producing large motions than the bang-bang control. We would expect this effect to increase with nonlinearity.

As already discussed, the hyperbolicity estimates are optimal for small forcing amplitudes. This is borne out in the figures where it is clear that the slopes of the hyperbolicity estimates and the numerics for the bang-bang control are equal at the origin. Figure 5.16 is a blow up of this region.

A key point to note here is that the hyperbolicity estimates are of the same order as the values obtained from numerical simulation while we have invertibility. This is a clear indication that the method is of some use.

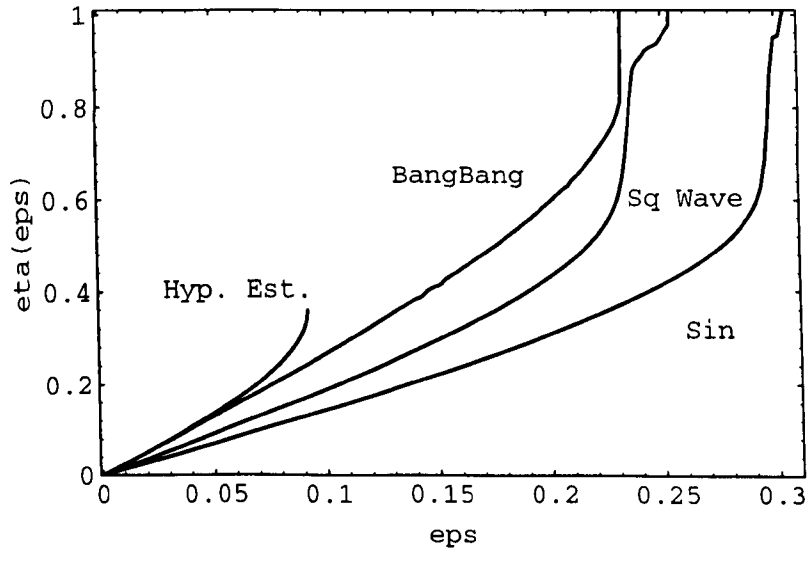


Figure 5.15: Comparison of safety estimates obtained from persistence theorem and from numerical simulation.

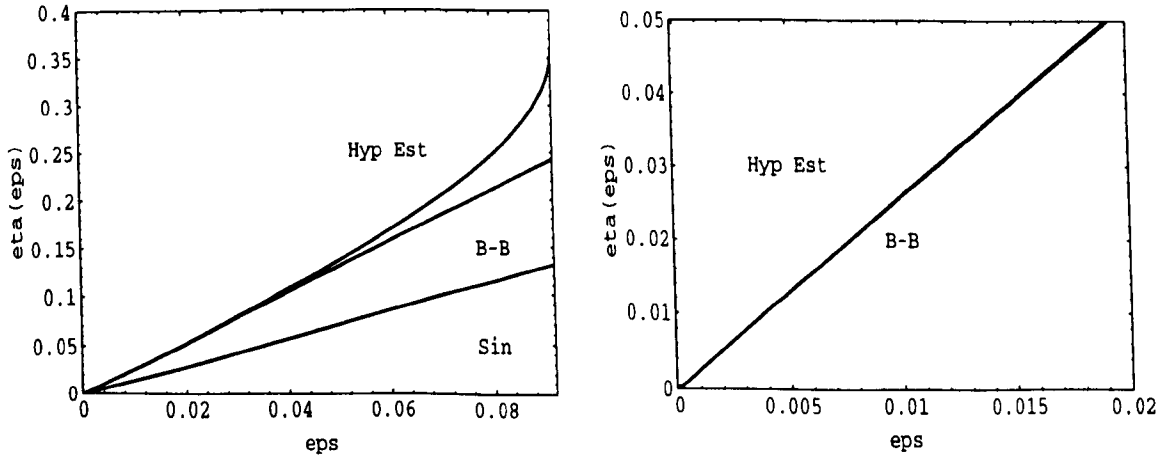
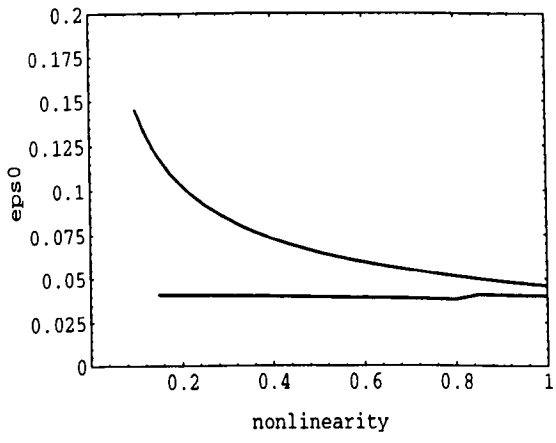


Figure 5.16: Blow up of figure 5.15.

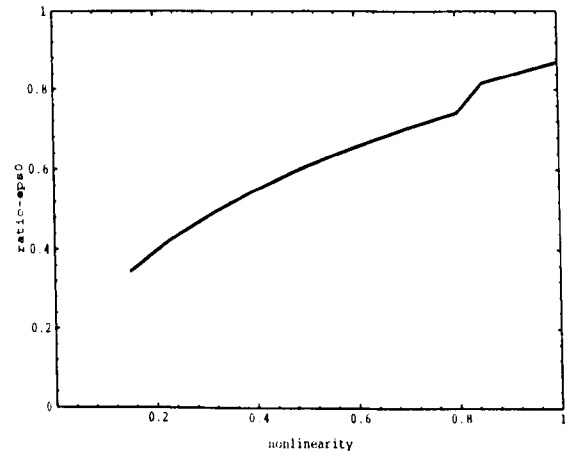
Effect of nonlinearity

By varying b we can investigate how the quality of the estimates are affected by the amount of nonlinearity in the system.

From figure 5.17(a) compares the persistence estimate of the size of the smallest escape inducing forcing to the actual value. We see that the estimates do not degrade with nonlinearity but in fact improve. Figure 5.17(b) gives the ratio which clearly gets closer to 1 as b increases.



(a) Smallest amplitude of forcing which induces escape, ε_0, for bang-bang control (top curve), and using persistence estimates (bottom curve), as a function of nonlinearity b.



(b) Ratio of the two values as a function of b.

Figure 5.17:

5.1.4 Adapted estimates

One can also provide adapted estimates based on the analysis of section 3.4.

For the unforced oscillator we have the Green's function W_0. The adapted norm we use on forcing functions $u \in \mathcal{C}^0$ is

$$||u||_* = ||W_0 * u||_{C^2} / ||W_0||_1$$

This norm makes $\mathcal{L}_0^{-1} : \mathcal{C}_*^0 \to \mathcal{C}^2$ an isometry.

The differential equation satisfied by the response x_u is

$$\frac{dx_u}{du} = \mathcal{L}_u^{-1} \circ \frac{\partial G_u}{\partial u} \quad : \mathcal{C}_*^0 \to \mathcal{C}^2$$

Let $\eta = ||x_u||_{C^2}$ and $\varepsilon = ||u||_\infty$ as usual. Define $\varepsilon_* = ||u||_*$. Then estimates for the size of the response can be obtained by solving

$$\frac{d\eta}{d\varepsilon_*} \leq ||\mathcal{L}_u^{-1}||_* \left|\left|\frac{\partial G_u}{\partial u}\right|\right|_*$$

Clearly,

$$||\mathcal{L}_u^{-1}||_* \leq ||\mathcal{L}_u^{-1}||_{C^0 \to C^1} \leq \delta(\eta)$$

Since,

$$\frac{\partial G_u}{\partial u} = -I_{C_*^0 \to C_*^0}$$

it follows that $||\frac{\partial G_u}{\partial u}||_* = 1$. Integrating

$$\frac{d\eta}{d\varepsilon_*} = \delta(\eta)$$

gives the estimates we require. Clearly $\eta(\varepsilon_*) = \eta_{\text{old}}(\varepsilon)$ where $\eta_{\text{old}}(\varepsilon)$ is the general estimate obtained previously.

$||u||_* < \varepsilon_*(\eta)$ guarantees that the response satisfies $||x_u|| < \eta$. Continuation can be performed until $\eta = \hat{\eta}$ say at $\varepsilon_* = \hat{\varepsilon}_*$. This is a useful safety criterion since it conveys the additional information about bad forcing functions in the form of a computable quantity. That is, given some u we can evaluate $||u||_*$ and decide, for any predefined safety region, if a unique 'safe' response is guaranteed.

As an example, consider periodic forcing, say $u = \varepsilon \sin(\omega t)$. Then

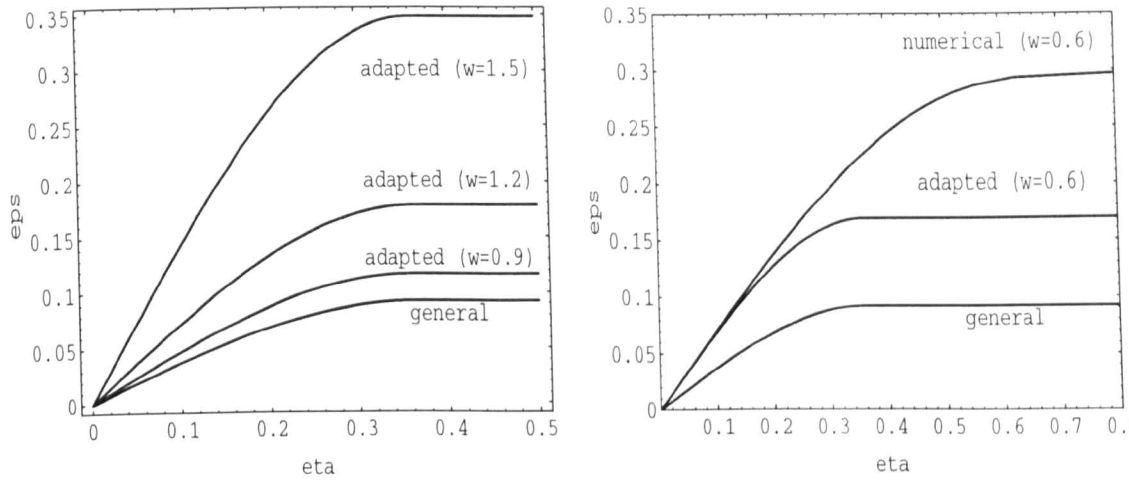
$$||u||_* = ||W_0 * u||_{C^2} / ||W_0||_1 = \left|\left|\varepsilon \int_0^\infty W_0(s) \sin(\omega(t-s))ds\right|\right|_{C^2} / ||W_0||_1$$

$W_0 * u$ is periodic with periodic $2\pi/\omega$ so this simplifies to

$$\begin{aligned} ||u||_* = \max \Bigg\{ & \sup_{t\in[0,2\pi/\omega)} \left|\varepsilon \int_0^\infty W_0(s) \sin(\omega(t-s))ds\right|, \\ & \sup_{t\in[0,2\pi/\omega)} \left|\varepsilon \int_0^\infty \dot{W}_0(s) \sin(\omega(t-s))ds\right|, \\ & \sup_{t\in[0,2\pi/\omega)} \left|\varepsilon \left(\sin(\omega t) + \int_0^\infty \ddot{W}_0(s) \sin(\omega(t-s))ds\right)\right| \Bigg\} \end{aligned}$$

which can be computed easily.

Figure 5.18 shows how $\eta_{\text{new}}(\varepsilon)$ compares to $\eta_{\text{old}}(\varepsilon)$. In 5.18(a) we see that the adapted estimates allow for much bigger forcing although the η at which invertibility is lost is not improved as we would expect.



(a) (b)

Figure 5.18: Adapted estimates of $\varepsilon(\eta)$ when $u(t) = \sin(\omega t)$

In 5.18(b) we see that the adapted estimates give a much better idea of the response and are clearly close to optimal for small amplitude forcing. Figure 5.19 gives $\varepsilon(\eta)$ for a range of frequencies. Notice the similarity to the inverse of a frequency response diagram.

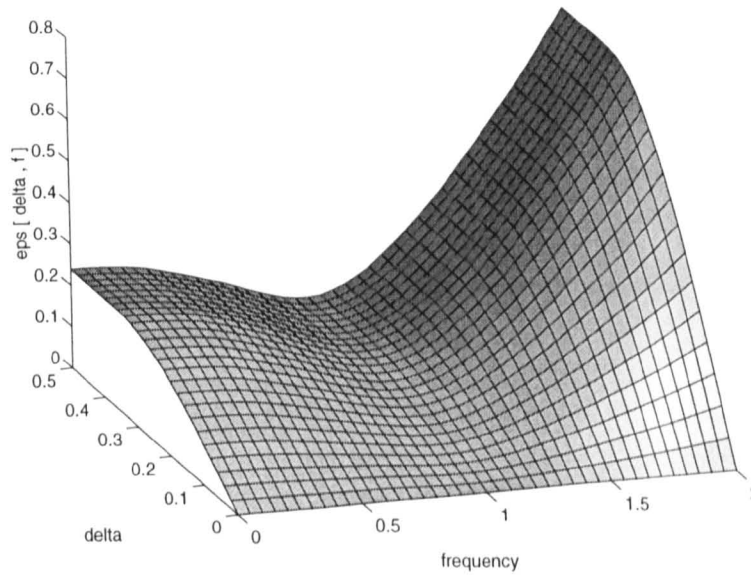


Figure 5.19: Adapted estimates of $\varepsilon(\eta)$ when $u(t) = \sin(\omega t)$

5.1.5 Parametrically forced oscillator

Now I consider the damped forced oscillator with some unknown parametric excitation in addition to some external forcing. The relevant ODE is now

$$G_{u,v} = \ddot{x} + \gamma\dot{x} + (1 + v(t))V'(x) - u = 0$$

where $u, v \in \mathcal{C}^0$. Again I will focus on the application of this equation to the ship capsize problem. The potential function is $V(x) = ax^2 - bx^4$ as before.

In [52] some detailed physical arguments are used to derive a ship capsize equation of this form. However, they use the assumption that the perturbations are sinusoidal waves which leads to the external and parametric excitation both being sinusoidal. This assumption can be dropped by using the analysis I have described. A more realistic assumption about waves and other perturbations is that they are bounded and continuous but otherwise arbitrary.

One can perform the continuation analysis in the usual way. The origin is a linearly attracting equilibrium for the unforced system. The linearisation of $G_{u,v}$ about a solution $x \in \mathcal{C}^2$ is given by

$$\mathcal{L}_{u,v,x}\xi = \ddot{\xi} + \gamma\dot{\xi} + (1 + v(t))V''(x)\xi$$

When $u = v = 0$, this is the same constant coefficient operator as previously considered. Thus we have $||\mathcal{L}_0|| = K_0$. It is easy to see that

$$||\mathcal{L}_0 - \mathcal{L}_{u,v}|| = ||V''(0) - (1 + v(t))V''(x)||_\infty \leq (1 - ||v||)12b\eta^2 + 2a\,||v||$$

We can continue while $||\mathcal{L}_0 - \mathcal{L}_{u,v}|| < K_0^{-1}$. This will be true while

$$\eta^2 < \eta_1(||v||)^2 = \frac{K_0^{-1} - 2a\,||v||}{(1 - ||v||)\ 12b}$$

We then have the estimate

$$||\mathcal{L}_{u,v}^{-1}|| \leq \beta(\eta, ||v||)^{-1} = [K_0^{-1} - ||\mathcal{L}_0 - \mathcal{L}_{u,v}||]^{-1}$$

and by the persistence theorem it follows that for u, v small enough there exists a unique continuation $x_{u,v} \in \mathcal{C}^2$ satisfying

$$\frac{dx_{u,v}}{d\binom{u}{v}} = -\mathcal{L}_{u,v}^{-1} \circ (V'(x_{u,v}), -I)$$

To provide estimates we use

$$\frac{d\eta}{d\binom{u}{v}} \leq \beta(\eta, ||v||)^{-1}\ (\alpha(\eta), 1)$$

where $\alpha(\eta) = 2a\eta + 12b\eta^2 \geq ||V'(x_{u,v})||$. Thus in a neighbourhood of $(u = 0, v = 0)$, the persistence theorem guarantees $\eta(u, v)$ is a smooth function. If we have a path $(u(s), v(s))$, parametrised by $s \in \mathbb{R}$, starting at $(0, 0)$ then we can integrate

$$\frac{d\eta}{ds} = \beta(\eta, ||v(s)||) \cdot (\alpha(\eta), 1) \cdot \begin{pmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{pmatrix}$$

to provide an upperbound for the response. More importantly, if we want an upperbound for the response, η, in terms of $(||u||, ||v||)$ then this is guaranteed to be a smooth surface, while uniquely continuable.

$\eta(||u||, ||v||)$ can be found using the following method. We choose rays going through the origin $(||u||, ||v||) = (0, 0)$, say at angle θ, and parametrise these by $s \in \mathbb{R}$. Thus we have $(||u||, ||v||) = (s\cos\theta, s\sin\theta)$. Then we compute $\eta^\theta(s)$ using

$$\frac{d\eta^\theta(s)}{ds} = \beta(\eta, s\sin\theta)\ (\alpha(\eta), 1) \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

This provides a mesh for the surface $\eta(||u||, ||v||)$.

In fact an easier method to obtain the surface in this case is to choose lines of fixed $||v||$ and integrate from $||u|| = 0$ since $\eta(0, v) = 0$. This follows from the fact that parametric forcing induces no response unless there is some external forcing as well. Figure 5.20 shows the surface for the capsize equation.

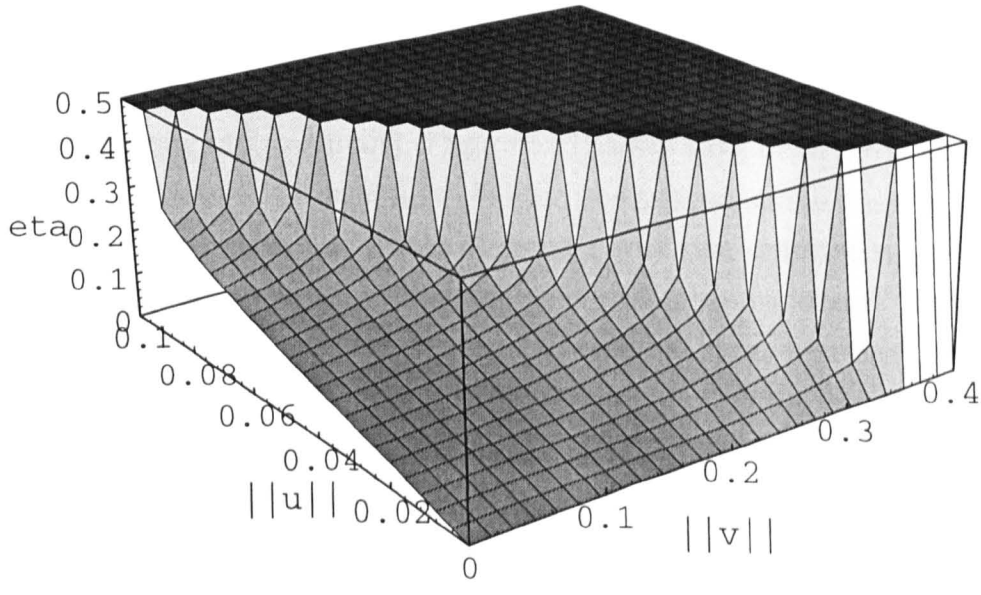


Figure 5.20: Estimates of the response for the parametrically driven system.

5.2 Phase-locked loops

Now I apply the theory to an attracting hyperbolic periodic orbit. Recall that a time periodic solution, $x_0(t)$, of period T, of the period mT system, $\dot{x} = f_0(x,t)$, is called hyperbolic if it's Floquet multipliers lie off the unit circle. It is linearly attracting if they are strictly contained in the unit circle. As noted earlier, for an autonomous system this renders a limit cycle weakly hyperbolic[ii] due to the Floquet multiplier at 1 corresponding to infinitesimal phase shift. However, there are numerous examples of strongly hyperbolic limit cycles and here I consider in detail the phase-locked loop.

The phase-locked loop is obtained when an oscillator which has natural frequency $\bar{\omega}$ is forced periodically at a frequency ω_0 close to $\bar{\omega}$. Under certain conditions, the oscillator 'locks' on to the forcing and the response is periodic with frequency ω_0. This is called (1-1) *frequency locking* or *entrainment* although it is equally well described as (1-1) *phase-locking* since the phase of the response tracks the phase of the forcing. In fact, when the forcing is close to rational multiples of the natural frequency of the oscillator, say $n\omega_0$ $m\bar{\omega}$, then one can obtain (n:m) phase locking displaying a periodic response with frequency $n\omega_0/m$.

An ideal system in which to study this effect is the Van-der-Pol oscillator an example of which is

$$\begin{aligned} \dot{x} &= y - x^3/3 + x \\ \dot{y} &= -x + \beta u(t) \end{aligned}$$

Consider the forcing $u(t) = \sin(\omega t)$. Figure 5.21 shows a region of $\beta - \omega$-space which supports $1-1$ phase-locked solutions. Within the tongue, small enough perturbations to frequency and/or amplitude of forcing will give qualitatively similar responses. It is interesting to ask whether there is an analogous form of phase-locking for aperiodic perturbations of the forcing function. In most practical applications of the theory one would not expect exactly periodic forcing but instead some small modulation in the frequency and amplitude. Using persistence of hyperbolicity I show here that small enough perturbations preserve the phase-locking effect and moreover that one can estimate the extent of the safe region of perturbations.

[ii] Note that one should be able to use the theory of normally hyperbolic manifolds to deal with this case.

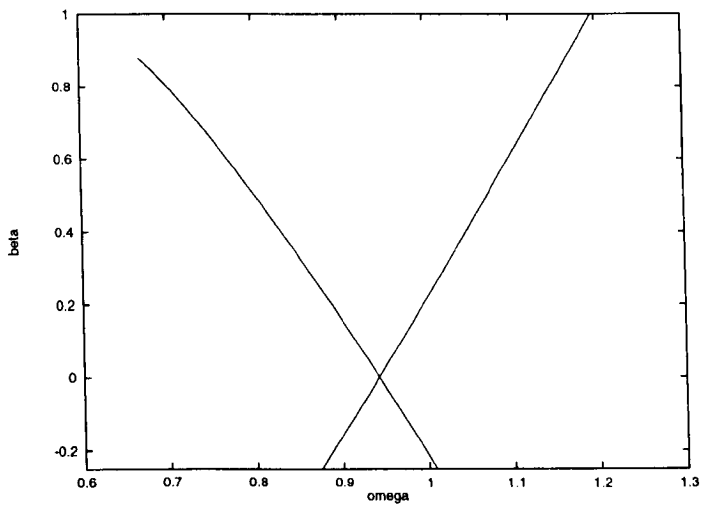


Figure 5.21: 1-1 phase-locking tongue

5.2.1 Robustness of phase-locked loops

As far as the analysis of previous chapters are concerned, only $\tilde{C}^1$-small perturbations are allowed for C^1-close responses. However, as I have already noted, while this covers amplitude modulation, it does not allow any frequency modulation. The reason for this is clear. Even smooth changes in frequency are discontinuous in $\tilde{C}^1$.

A way to deal with this is to allow a near-identity re-parametrisation of time. Formally, suppose we can find a C^2 function, $t_\varepsilon : \mathbb{R} \to \mathbb{R}$, with $t_0 = I$, such that $f_\varepsilon(x(\cdot), t_\varepsilon(\cdot))$ is uniformly close to $f_0(x(\cdot), t_\varepsilon(\cdot))$. That is,

$$\sup_{x \in C^1} \left\| \frac{\partial}{\partial \varepsilon} f_\varepsilon\left(x(\cdot), t_\varepsilon(\cdot)\right) \right\|_\infty < \infty \tag{5.13}$$

Theorem 5.1 *Suppose that the system $\dot{x} = f_0(x, s)$ has a hyperbolic periodic solution x_0. and that $f_\varepsilon(x, t_\varepsilon)$ is $\tilde{C}^1$-close to $f_0(x, s)$ in the sense of (5.13) above. Then x_0 has a locally unique continuation, $x_\varepsilon(t_\varepsilon) \in C^1$, solving $\dot{x} = f_\varepsilon(x, t_\varepsilon)$. It is hyperbolic with the same stability type as x_0 and varies smoothly (at least C^1) with ε.*

Proof Writing $'$ for $\frac{d}{ds}$ and $\dot{}$ for $\frac{d}{dt_\varepsilon}$ we get, using the chain rule, $x' = \dot{x}\, t'_\varepsilon$. So

$$G_\varepsilon(x) = \dot{x} - f_\varepsilon(x, t_\varepsilon) = 0 \iff H_\varepsilon(x) = x' - f_\varepsilon(x, t_\varepsilon) t'_\varepsilon = 0$$

Note that $H_0(x_0) = 0$. Also

$$DH_{\varepsilon,x} = \xi' - Df_\varepsilon(x, t_\varepsilon) t'_\varepsilon \xi$$

and

$$\frac{\partial}{\partial \varepsilon} H_\varepsilon(x(t_\varepsilon)) = \frac{\partial}{\partial \varepsilon} f_\varepsilon(x, t_\varepsilon) + f_\varepsilon(x, t_\varepsilon) \frac{\partial}{\partial \varepsilon} t'_\varepsilon$$

which (by the hypotheses on t_ε) are both bounded so $H_\varepsilon(x)$ is $\mathcal{C}^1$ at $(\varepsilon, x) = (0, x_0)$. $DH_{0,x_0} = DG_{0,x_0}$ is invertible since x_0 is hyperbolic so we can apply the IFT to $H_\varepsilon(x)$ to get a unique continuation, $x_\varepsilon \in \mathcal{C}^1$, solving $H_\varepsilon(x_\varepsilon) = 0$ and thus by construction solving $\dot{x} = f_\varepsilon(x, t_\varepsilon)$. ■

To provide estimates we use the differential equation

$$\frac{dx_\varepsilon}{d\varepsilon} = -DH^{-1}_{\varepsilon,x_\varepsilon} \circ \frac{\partial}{\partial\varepsilon} H_\varepsilon(x_\varepsilon) \tag{5.14}$$

5.2.2 Persistence estimates

Since $\mathcal{L}_0 = DH_{0,x_0}$ is the linearisation about a periodic solution, we can use the theory in section 3.2.1 to estimate $||DH^{-1}_{0,x_0}||$. Briefly speaking, we find the fundamental solution matrix X_t for each $t \in [0, T]$. Note that it is periodic in t. This is not usually available analytically but can easily be found numerically. This gives us the Green's function $W(t, s)$ defined by (2.14). This is also periodic in t. The bound we are looking for is then given by

$$||\mathcal{L}_0^{-1}|| = \max\left\{ \sup_{t\in[0,T]} ||W(t,\cdot)||_1 \ , \ 1 + \sup_{t\in[0,T]} \left|\left|\dot{W}(t,\cdot)\right|\right|_1 \right\}$$

Say we find a K_0 such that $||\mathcal{L}_0^{-1}|| < K_0$.

Suppose also that we find the following estimates.

$$\begin{aligned} ||DH_{\varepsilon,x} - DH_{0,x_0}|| &= ||Df_\varepsilon(x, t_\varepsilon)t'_\varepsilon - Df_0(x_0, s)|| \\ &\leq ||Df_\varepsilon(x, t_\varepsilon) - Df(x_0, s)|| + ||Df_\varepsilon(x, t_\varepsilon)|| \ ||t'_\varepsilon - 1|| \\ &\leq \beta_1(\varepsilon, \eta) + \beta_2(\varepsilon, \eta) \ \beta_3(\varepsilon) \\ &= \beta(\varepsilon, \eta) \end{aligned}$$

Using lemma 3.1 we can then deduce the bound

$$||DH^{-1}_{0,x_0}|| \leq \left[K_0^{-1} - \beta(\varepsilon, \eta)\right]^{-1}$$

which is valid while $\beta(\varepsilon, \eta) < K_0^{-1}$.

Suppose we also find $\alpha_1(\varepsilon, \eta), \alpha_2(\varepsilon, \eta), \alpha_3(\varepsilon)$ such that

$$\begin{aligned} \left|\left|\frac{\partial}{\partial\varepsilon} H_\varepsilon(x_\varepsilon)\right|\right| &\leq \left|\left|\frac{\partial}{\partial\varepsilon} f_\varepsilon(x, t_\varepsilon)\right|\right| + ||f_\varepsilon(x, t_\varepsilon)|| \ \left|\left|\frac{\partial}{\partial\varepsilon} t'_\varepsilon\right|\right| \\ &\leq \alpha_1(\varepsilon, \eta) + \alpha_2(\varepsilon, \eta) \ \alpha_3(\varepsilon) \\ &= \alpha(\varepsilon, \eta) \end{aligned}$$

Then defining $\eta = ||x_\varepsilon(t_\varepsilon) - x_0(s)||$ we have

$$\frac{d\eta}{d\varepsilon} \leq \frac{\alpha(\varepsilon,\eta)}{K_0^{-1} - \beta(\varepsilon,\eta)} \tag{5.15}$$

which can be integrated as before to obtain $\eta(\varepsilon), \hat{\varepsilon}$ and $\varepsilon(\delta)$.

Remark 5.4 $|\varepsilon| \leq \varepsilon(\delta)$ *guarantees* x_ε *remains within a* δ*-sized tubular neighbourhood of* x_0.

5.2.3 Example: External forcing with amplitude and frequency modulation

An interesting case which fits into this framework is an autonomous system with additive periodic forcing of frequency ω_0 which has a hyperbolic limit cycle. We then incorporate some aperiodic amplitude and frequency modulation. Specifically,

$$\dot{x} = f(x) + C_\varepsilon(t)\sin\left(\int_0^t \omega_\varepsilon(\tau)\, d\tau\right)$$

with $\omega_0(t) = \omega_0$ and $C_0(t) = C_0$. The re-parametrisation to use in this case is

$$\omega_0\, ds = \omega_\varepsilon(t)\, dt$$

so that $\sin\left(\int_0^{t_\varepsilon} \omega_\varepsilon(\tau)\, d\tau\right) = \sin(\omega_0 s)$. We find the bounds α and β such that

$$\begin{aligned}
||DH_{\varepsilon,x} - DH_{0,x_0}|| &\leq ||Df_x - Df_{x_0}|| + ||Df_x||\, \left|\left|1 - \frac{\omega_0}{\omega_\varepsilon(t_\varepsilon)}\right|\right| \\
&\leq \beta_1(\eta) + \beta_2(\eta)\, \beta_3(\varepsilon) \\
&= \beta(\varepsilon,\eta)
\end{aligned}$$

$$\begin{aligned}
\left|\left|\frac{\partial}{\partial\varepsilon} H_\varepsilon(x_\varepsilon)\right|\right| &\leq \left|\left|\frac{\partial}{\partial\varepsilon} C_\varepsilon(t_\varepsilon(s))\sin(\omega_0 s)\right|\right| \\
&\quad + ||f(x) + C_\varepsilon(t_\varepsilon(s))\sin(\omega_0 s)||\, \left|\left|\frac{\omega_0 \frac{\partial}{\partial\varepsilon}\omega_\varepsilon(t_\varepsilon(s))}{\omega_\varepsilon^2(t_\varepsilon(s))}\right|\right| \\
&\leq \alpha_1(\varepsilon) + \alpha_2(\varepsilon,\eta)\, \alpha_3(\varepsilon) = \alpha(\varepsilon,\eta)
\end{aligned}$$

and integrate (5.15) to obtain safety estimates.

For the periodically forced Van-der-Pol system with frequency modulation, the perturbed system is

$$\begin{aligned}
\dot{x} &= y - x^3/3 + x \\
\dot{y} &= -x + \beta\sin\left(\int_0^t \omega_\varepsilon(\tau)\, d\tau\right)
\end{aligned}$$

With bounds on the size of the frequency modulation we can easily find α and β above and integrate (5.15).

Chapter 6

Concluding Remarks

6.1 Summary

I have argued that for many interesting dynamical system, including a number of important engineering systems, it is essential to consider the effect of aperiodic perturbations. In particular, it is often neccessary to consider the effect of arbitrary perturbations drawn from a given space of functions and bounded in the appropriate norm. I have concentrated on $\mathcal{C}^1$ vector fields with uniform norms as this is most natural, although I have also presented a generalisation which allows a much wider class of perturbations

I have showed that uniformly hyperbolic solutions are robust to every small enough such perturbation by proving the existence of a unique nearby continuation. This is essentially a persistence of hyperbolicity type theorem set in the context of general non-autonomous systems. The reason for proving persistence was to demonstrate that it allowed one to derive a set of useful safety criteria in a unified manner. This is in fact the main thrust of my thesis and I have gone into great detail in showing how one can do this in practical way.

Evidence of the practicality of the technique is given by its use on the forced oscillator equations, in particular, the application to ship stability. In this case, one can get close to optimal estimates for small or medium levels of forcing. For larger values of forcing, bifurcations occur, so naturally, the approach breaks down.

An interesting generalisation which I developed was the use of adapted estimates. Instead of measuring the perturbations by their sizes in sup-norm we measure instead the response they induce in the linearisation of the unperturbed system. This is an important concept which improves the safety criteria obtained and could benefit from even more refinement.

6.2 Limitations of the approach

There are a number of reasons for which one could claim that the techniques I have developed might fail to be useful. Here I discuss whether these problems are serious and when they might apply.

Neccessity of hyperbolicity

The first and most obvious limitation of the approach is that the safety estimates are based on hyperbolicity of solutions and thus fail when non-hyperbolicity is encountered. It could be argued that many bifurcations are benign and pose no threat when one is concerned about safety or engineering integrity of a system.

However, although many common bifurcations like saddle-nodes and period doublings are not catastrophic in themselves, one often finds they are a sign of impending danger. For example, after a period doubling bifurcation occurs one typically finds a sequence of further period doublings leading to chaotic motion. From universality and renormalisation arguments these occur at a geometrically increasing rate as parameters are varied. The question of whether the transition from stable solutions to unstable motion is a loss of safety is of course a complicated one to which there are no obvious general answers.

Another argument in favour of using hyperbolicity estimates is that for aperiodic systems one has very little understanding of even the most typical bifurcation scenarios that might occur when non-hyperbolicity is encountered. It seems desirable to exclude such behaviour until more is known.

Another related limitation is that some interesting dynamical behaviour, including most known chaotic attractors, do not appear to be uniformly hyperbolic. However, one might be able to generalise the theory to non-uniform hyperbolicity. and obtain non-uniform or measure-theoretic safety estimates.

Quality of estimates

It could be argued that the estimates one can derive from the analysis I have presented are too conservative in that they take no account of the 'typical' perturbations that are encountered in real systems and only protect against rare worst possible case situations.

In response to this argument I would claim that for physical systems where some degree of safety is required, criteria protecting against worst case scenarios are always useful guidelines even if they not always adhered to. It should also

be possible to incorporate some more probabilistic ideas into the theory if one is interested in likelihoods of failure and if there is extra information about typical perturbations. The adapted estimates I have given is a first step in this direction as it gives some quantitative idea of the 'harmful' perturbations.

The need for accurate models

It could be argued that the dynamical systems viewpoint fails to give effective quantitative information about real dynamical systems precisely because the starting point is a model of the dynamics. Typically we do not expect to have an accurate model of the dynamics; only some 'good' fit to experimental data within a certain class of models. For example, the 2nd order ODE ship capsize model is only an approximation although there is some semi-rigourous justification involved.

One could deal with this problem in two ways. Firstly, one could model he uncertainty as a perturbation. Secondly, and perhaps more interestingly, one could attempt to use the theory I have presented using only experimental data. The essential requirement of the technique is the determination of the Green's function or impulse response. This can be obtained or 'learnt' experimentally for many systems by using well controlled 'impulses' and measuring the response of the system.

6.3 Directions for further research

There are a number of directions which I believe could prove fruitful if investigated.

- Some interesting and important applications which could easily be treated using the theory I have presented here are the *swing equation*, [21], and the *stable inverted pendulum*, [23]. These are both essentially forced damped oscillators with subtle differences to the ones I have looked at already. Again the main idea is to show that the interesting solutions are uniformly hyperbolic and are thus robust to small-enough, aperiodic perturbations.

- The problem of aperiodic perturbations to non-equilibrium solutions of autonomous systems could be treated if one was able to formulate a good non-autonomous proof of *persistence of normal hyperbolicity*. To obtain good safety estimates one should look for the correct invertible operator characterisation of normal hyperbolicity but this appears to be a fairly tricky, unsolved problem.

- As I have mentioned above, for real systems, one does not usually have a good model for the dynamics. There are a number of ways in which one might use the data coming from a real system in order to generate a model. In particular, to obtain safety criteria, one should learn the Green's function and I believe this could be done to good effect in a number of situations.

- In this thesis I have specifically looked at non-bifurcational behaviour. Bifurcation theory for aperiodic systems is a virgin area and I believe needs some serious investigation. The main problem seems to be that when the relevant linear operator $\mathcal{L}$ becomes non-invertible, indicating a loss of hyperbolicity, it does not typically stay Fredholm. For the basic bifurcations in periodic and autonomous systems, the Fredholm property holds and can thus one can always 'reduce' to a finite dimensional bifurcation problem. Some preliminary numerical investigations I have made suggest that there are some important differences between bifurcations in autonomous systems and those in aperiodic systems and I believe it would be fruitful to pursue this direction.

Appendix A

Implicit Function Theorem

Here I state and prove the implicit function theorem for Banach spaces. The theorem and its proof are crucial to much of the theory presented in this thesis.

Theorem A.1 *Let Σ, X, Y be Banach spaces. Let $G : \Sigma \times X \longmapsto Y$ be such that $G(\sigma_0, x_0) = 0$. Suppose also that G is C^1 in a neighbourhood of (σ_0, x_0) and that $D_xG(\sigma_0, x_0)$ has a bounded inverse. Then there exist open neighbourhoods U of σ_0 in Σ and V of x_0 in X and a locally unique C^1 function $x : U \longmapsto V$ such that $x_{\sigma_0} = x_0$ and $G(\sigma, x_\sigma) = 0$.*

Proof For convenience let D_xG be denoted by DG and $G(\sigma, \cdot)$ by $G_\sigma(\cdot)$. We are interested in finding solutions to $G_\sigma(x) = 0$ for σ close to σ_0.

Consider the mapping

$$\begin{array}{rccl} N_\sigma : & X & \longrightarrow & X \\ & x & \longmapsto & x - J \circ G_\sigma(x) \end{array}$$

where $J = DG_{\sigma_0}(x_0)^{-1}$. Since J is a bounded linear operator we can find $k \in \mathbb{R}$ such that $||J|| \leq k$.

Notice that $G_\sigma(\hat{x}) = 0 \iff N_\sigma(\hat{x}) = \hat{x}$.

We prove that for σ close to σ_0, N_σ is a contraction on a neighborhood of x_0 and so by the contraction mapping theorem has a unique fixed point. In proving this we can provide estimates for the sizes of these neighbourhoods. This will allow us to give bounds for how much the implicit solution can change and also an estimate of the neighbourhood in which it is the unique solution.

To use the contraction mapping theorem we need to show:

1. $N_\sigma : B(x_0, \eta_1) \hookleftarrow$ for σ in a neighbourhood of σ_0, and some $\eta_1 > 0$. Here $B(x, \eta)$ denotes the size η ball around x.

2. $|N_\sigma(x_1) - N_\sigma(x_2)| \leq \lambda |x_1 - x_2|$ for some $\lambda < 1$ and for all $x_1, x_2 \in B(x_0, \eta_1)$

Let $\eta = |x - x_0|$ and $\varepsilon = |\sigma - \sigma_0|$. Then consider

$$DN_\sigma(x) \;=\; I - J\, DG_\sigma(x) \;=\; J\,[DG_{\sigma_0}(x_0) - DG_\sigma(x)]$$

This gives

$$\begin{aligned} ||DN_\sigma(x)|| &\leq k\; ||DG_{\sigma_0}(x_0) - DG_\sigma(x)|| \\ &\leq \lambda < 1 \qquad \text{for } \eta < \bar{\eta}(\varepsilon, \lambda)\ ,\ \varepsilon < \bar{\varepsilon} \end{aligned}$$

We can then deduce that

$$\begin{aligned} |N_\sigma(x) - x_0| &\leq |N_\sigma(x) - N_\sigma(x_0)| + |N_\sigma(x_0) - x_0| \\ &\leq \lambda\eta + k|G_\sigma(x_0)| && \text{for } \eta < \bar{\eta}\ ,\ \varepsilon < \bar{\varepsilon} \ \text{ (by MVT)} \\ &< \eta_1 && \text{for } \eta < \eta_1\ ,\ \varepsilon < \varepsilon_1 \end{aligned}$$

by choosing $\eta_1 < \bar{\eta}$, $\varepsilon_1 < \bar{\varepsilon}$ and $\lambda < 1$ so that $|G_\sigma(x_0)| < k^{-1}(1 - \lambda)\eta_1$. This guarantees $N_\sigma : B(x_0, \eta_1) \hookleftarrow$ for $\sigma \in B(\sigma_0, \varepsilon_1)$ and so proves 1 above.

2 follows immediately by applying the mean value theorem.

So for $|\sigma - \sigma_0|$ small enough N_σ is a contraction on a neighbourhood of x_0 and by the contraction mapping theorem we can deduce the existence of a unique fixed point $x_\sigma \in B(x_0, \eta_1) \subset X$ such that $N_\sigma(x_\sigma) = x_\sigma$ and thus $G_\sigma(x_\sigma) = 0$.

By uniqueness $x_{\sigma_0} = x_0$. ■

Remark A.1 *It follows from the chain rule that*

$$\frac{dx_\sigma}{d\sigma} \;=\; -DG_\sigma(x_\sigma)^{-1} \circ \frac{\partial}{\partial\sigma} G_\sigma(x_\sigma)$$

Bibliography

[1] R. A. Adams. *Sobolev spaces.* Academic, 1975.

[2] S. Aubry, R. S. MacKay, and C. Baesens. Equivalence of uniform hyperbolicity for symplectic twist maps and phonon gap for Frenkel-Kontorova models. *Physica D*, 56:123–134, 1992.

[3] B. Aulbach, N. Van Minh, and P. Zabreiko. Groups of weighted translation operators in L_p and linear nonautonomous differential equations. *Journal of mathematical analysis and applications*, 193:622–631, 1995.

[4] E. J. Beltrami and M. R. Wohlers. *Distributions and the boundary values of analytic functions.* Academic, 1966.

[5] K. Chang and A. Hubler. Entrainment of chaotic systems with resonant driving forces. In M. Yamaguti, editor, *Towards the harnessing of chaos*, 1994.

[6] S.-N. Chow and J. K. Hale. *Methods of Bifurcation Theory.* Springer-Verlag, 1982.

[7] W. A. Coppel. *Dichotomies in stability theory*, volume 629 of *LN*. Springer-Verlag, 1978.

[8] R. Curtain and A. J. Pritchard. *Functional analysis in modern applied mathematics.* Academic Press, 1977.

[9] J. L. Daleckii and M. G. Krein. *Stability of solutions of differential equations in Banach space.* A. M. S. translations, 1974.

[10] M. Ding, E. Ott, and C. Grebogi. Crisis control: preventing chaos-induced capsizing of a ship. *Phys. Rev. E*, 50(5):4228–4230, 1994.

[11] N. Dunford and J. T. Schwartz. *Linear operators: general theory*, volume 1. Interscience, 1958.

[12] J. M. Franks. Time dependent stable diffeomorphisms. *Inventiones math*, 24:163–172, 1974.

[13] J. Gruendler. Homoclinic solutions for autonomous ordinary differential equations with nonautonomous perturbations. *J. Diff. Eq.*, 122:1–26, 1995.

[14] J. Guckenheimer and P. Holmes. *Nonlinear oscillations, dynamical systems and bifurcations of vector fields.* Springer-Verlag, 1983.

[15] J. K. Hale. *Ordinary differential equations.* Wiley, 1969.

[16] P. Hartman. *Ordinary differential equations.* John Wiley, 1964.

[17] R. A. Johnson. Random differential equations and applications. In *??*, pages 163–181. ??, 1995.

[18] M. Kan and H. Taguchi. Chaos and fractals in ship capsizing. In M. Yamaguti, editor, *Towards the harnessing of chaos*, pages 365–368, 1994.

[19] T. Kato. *Perturbation theory for linear operators.* Springer, 1966.

[20] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems.* CUP, 1995.

[21] J. Kim and J. Stringer. *Applied chaos.* John Wiley, 1992.

[22] O. E. Lanford. Introduction to hyperbolic sets. In G. Velo and A. S. Wightman, editors, *Regular and chaotic motions*, pages 73–102. Plenum, 1985.

[23] M. Levi. Stability of the inverted pendulum. *SIAM review*, 30(4):639–644, 1988.

[24] R. S. MacKay. Dynamics of networks: features which persist from the uncoupled limit. In S.J. van Strien and S. M. Verduyn Lunel, editors, *Stochastic and spatial structures of dynamical systems*, pages 81–104. North-Holland, 1996.

[25] Mane. A proof of the C^1 stability conjecture. *Pub. Math. de l'IHES*, 66:161–210, 1987.

[26] J. Massera and J. J. Schaffer. *Linear differential equations and function spaces.* Academic, 1966.

[27] J. N. Mather. Characterisation of Anosov diffeomorphisms. *Prok Kon Nederl Akad Wet Indag Math*, 30:479–483, 1968.

[28] F. A. McRobie. The engineering design of nonlinear systems. In M. Yamaguti, editor, *Towards the harnessing of chaos*, pages 187–203, 1994.

[29] F. A. McRobie and J. M. T. Thompson. Global integrity in engineering dynamics - methods and applications. In J. H. Kim, editor, *EPRI workshop on applications of chaos*, 199?

[30] K.R. Meyer and G.R. Hall. *Introduction to Hamiltonian dynamical systems and the N-body Problem.* Springer-Verlag, 1991.

[31] J. Murray. *Mathematical biology.* Springer, 1989.

[32] K. J. Palmer. Exponential dichotomy and transversal homoclinic points. *J. Diff. Eq.*, 55:225–256, 1984.

[33] K. J. Palmer. A shadowing theorem for ordinary differential equations. *ZAMP*, 46:85–106, 1995.

[34] F. Rabiger, A. Rhandi, and R. Schnaubelt. Perturbation and an abstract characterisation of evolution semigroups. *J. Math. Anal. Appl.*, 198:516–533, 1996.

[35] C. D. Rahn and C. D. Mote, Jr. On the stability of the damped Hill's equation with arbitrary, bounded parametric excitation. *Transactions of the ASME*, 60:366, 1993.

[36] C. Robinson. *Dynamical systems.* CRC Press, 1995.

[37] R. J. Sacker. A new approach to the perturbation theory of invariant surfaces. *Comm. Pure. Appl. Math.*, 18:717–732, 1965.

[38] R. J. Sacker. The splitting index for linear differential systems. *J. Diff. Eq.*, 33:368–405, 1979.

[39] R. J. Sacker and G. R. Sell. Existence of dichotomies and invariant splittings for linear differential equations I. *J. Diff. Eq.*, 15, 1974.

[40] R. J. Sacker and G. R. Sell. Existence of dichotomies and invariant splittings for linear differential equations II. *J. Diff. Eq.*, 22, 1976.

[41] R. J. Sacker and G. R. Sell. Existence of dichotomies and invariant splittings for linear differential equations III. *J. Diff. Eq.*, 22, 1976.

[42] R. J. Sacker and G. R. Sell. Existence of dichotomies and invariant splittings for linear differential equations IV. *J. Diff. Eq.*, 27, 1978.

[43] T. Sauer and J. A. Yorke. Rigorous verification of trajectories for computer simulation of dynamical systems. *Nonlinearity*, 4:961–979, 1991.

[44] W. Schielen. Stability numbers for nonlinear systems. In J. M. T. Thompson and S. R. Bishop, editors, *Nonlinearity and chaos in engineering dynamics*, pages 345–352. John Wiley, 1994.

[45] J. A. Sepulchre and R. S. MacKay. Localized oscillations in conservative or dissipative networks of weakly coupled autonomous oscillators. *Nonlinearity*, 10:679–713, 1997.

[46] M. Shub. *Global stability of dynamical systems.* Springer-Verlag, 1987.

[47] J. Stark. Invariant graphs for forced systems. to appear in Physica D, 1997.

[48] H. B. Stewart, J. M. T. Thompson, A. N. Lansbury, and Y. Ueda. Generic patterns of bifurcation governing escape fom potential wells. *Int. J. of Bifn. Chaos*, 1(1):265–267, 1991.

[49] W. Szemplinska and J. Rudowski. On analytic estimates of safe impulsive velocity in the driven escape oscillator. In J. M. T. Thompson and S. R. Bishop, editors, *Nonlinearity and chaos in engineering dynamics*, pages 353–363. John Wiley, 1994.

[50] Z. Szmydt. *Fourier transformation and linear differential equation.* D. Reidel, 1977.

[51] J. M. T. Thompson, R. C. T. Rainey, and M. S. Soliman. Ship stability criteria based on chaotic transients from incursive fractals. *Phil. Trans. R. Soc. Lond. A*, 332:149–167, 1990.

[52] J. M. T. Thompson, R. C. T. Rainey, and M. S. Soliman. Mechanics of ship capsize under direct and parametric wave excitation. *Phil. Trans. R. Soc. Lond. A*, 338:471–490, 1992.

[53] N. Van Minh. Semigroups and stability of nonautonomous differential equations in Banach spaces. *Transactions of the american mathematical society*, 345(1):223–241, 1994.

[54] N. Van Minh. Spectral theory for linear nonautonomous differential equations. *Journal of mathematical analysis and applications*, 187:339–351, 1994.

[55] L. Virgin. Approximate criterion for capsize based on deterministic dynamics. *Dynamics and stability of systems*, 4(1):55–70, 1989.

[56] C. Wargitsch and A. Hubler. Resonances of nonlinear oscillators. *Phys. Rev. E*, 51(2):1508–1519, 1995.

[57] S. Wiggins. *Normally hyperbolic invariant manifolds in dynamical systems.* Springer, 1995.