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CRITICAL SLOPE P-ADIC L-FUNCTIONS OF CM MODULAR FORMS

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ABSTRACT

For ordinary modular forms, there are two constructions of a p-adic L-function attached to the non-unit root of the Hecke polynomial, which are conjectured but not known to coincide. We prove this conjecture for modular forms of CM type, by calculating the the critical-slope L-function arising from Kato's Euler system and comparing this with results of Bellaïche on the critical-slope L-function defined using overconvergent modular symbols.

1. Setup

1.1. Introduction. Let f be a cuspidal new modular eigenform of weight ≥ 2, and p a prime not dividing the level of f. It has long been known that if α is any root of the Hecke polynomial of f at p such that $v_p(\alpha) < k-1$, then there is a p-adic L-function $L_{p,\alpha}(f)$ interpolating the critical L-values of f and its twists by Dirichlet characters of p-power conductor; see [12, 1, 16].

If f is *non-ordinary* (the Hecke eigenvalue of f at p has valuation > 0) then both roots of the Hecke polynomial satisfy this condition, but if f is ordinary, then there is one root with valuation $k-1$ ("critical slope"), to which the classical modular symbol constructions do not apply. Two approaches exist to rectify this injustice to the ordinary forms by constructing a critical-slope p-adic L-function. Firstly, there is an approach using p-adic modular symbols [15, 14, 2]. Secondly, there is an approach using Kato's Euler system [9] and Perrin-Riou's p-adic regulator map [13] (cf. [4, Remarque 9.4]). Although it is natural to conjecture that the objects arising from these two constructions coincide (cf. [14, Remark 9.7]), and the results of [10] are strong evidence for this conjecture, prior to the present work this was not known in a single example.

In this paper, we show that the two critical-slope L-functions coincide for modular forms of CM type. In this case, Bellaïche has shown [3] that the "modular symbol" critical-slope p-adic L-function is related to the Katz p-adic L-function for the corresponding imaginary quadratic field. We show here that

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the same relation holds for the Kato critical slope p-adic L-function, by comparing Kato's Euler system with another Euler system: that arising from elliptic units. Using the results of [18] and [5] relating elliptic units to Katz's L-function, we obtain a formula (Theorem 3.2) for the Kato L-function, which coincides with Bellaïche's formula for its modular symbol counterpart (up to a scalar factor corresponding to the choice of periods). This establishes the equality of the two critical-slope p-adic L-functions for ordinary eigenforms of CM type (Theorem 3.4).

1.2. NOTATION. Let K be a finite extension of either $\mathbb{Q}$ or $\mathbb{Q}_p$, where p is an odd prime. We write $K_\infty = K(\mu_{p^\infty})$, $\overline{K}$ for an algebraic closure of K and K^{ab} for the maximal abelian extension of K in $\overline{K}$. A p-adic representation of the absolute Galois group $\mathrm{Gal}(\overline{K}/K)$ is a finite-dimensional $\mathbb{Q}_p$-vector space with a continuous linear action of $\mathrm{Gal}(\overline{K}/K)$.

A Galois extension L of K will be called a p-adic Lie extension if $G = \mathrm{Gal}(L/K)$ is a compact p-adic Lie group of finite dimension. In this case, we denote by $\Lambda(G)$ its Iwasawa algebra; it is defined to be the completed group ring

$$\Lambda(G) = \varprojlim \mathbb{Z}_p[G/U],$$

where U runs over all open normal subgroups of G. We write $Q(G)$ for the total quotient ring of $\Lambda(G)$. If R is a p-adically complete $\mathbb{Z}_p$-algebra, we shall write $\Lambda_R(G)$ for $R \mathbin{\widehat{\otimes}} \Lambda(G)$, the Iwasawa algebra with coefficients in R.

If L is a complete discretely valued subfield of $\mathbb{C}_p$, we write $\mathcal{H}_L(G)$ for the algebra of L-valued distributions on G (the continuous dual of the space of locally L-analytic functions). This naturally contains $\Lambda_L(G)$ as a subalgebra. When G is the cyclotomic Galois group Γ (isomorphic to $\mathbb{Z}_p^\times$), and $i \in \mathbb{Z}$, we shall write ℓ_i for the element $\frac{\log(\gamma)}{\log \chi(\gamma)} - i$ of $\mathcal{H}_{\mathbb{Q}_p}(\Gamma)$ (where γ is any element of Γ of infinite order, and χ is the cyclotomic character).

Assume now that K is a number field, and let S be a finite set of places of K (which we shall always assume to contain the infinite places). Let K^S be the maximal extension of K which is unramified outside S, and let V be a p-adic representation of $\mathrm{Gal}(K^S/K)$. For an extension L of K contained in K^S, write $H^1_S(L,V)$ for the Galois cohomology group $H^1(\mathrm{Gal}(K^S/L),V)$. Let T be a $\mathrm{Gal}(\overline{K}/K)$-stable lattice in V. If $L \subset K^S$ is a p-adic Lie extension of K, define

$$H^1_{\mathrm{Iw},S}(L,T) = \varprojlim H^1_S(L_n,T),$$

where L_n is a sequence of finite Galois extensions of K such that $L = \bigcup_n L_n$ and the inverse limit is taken with respect to the corestriction maps. Note that $H^1_{\mathrm{Iw},S}(L,T)$ is equipped with a continuous action of $G = \mathrm{Gal}(L/K)$, which extends to an action of $\Lambda(G)$. We also define $H^1_{\mathrm{Iw},S}(L,V) = H^1_{\mathrm{Iw},S}(L,T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, which is independent of the choice of lattice T.

Similarly, let F be a finite extension of $\mathbb{Q}_p$, V a p-adic representation of $\mathrm{Gal}(\overline{F}/F)$ and T a $\mathrm{Gal}(\overline{F}/F)$-invariant lattice in V. For a p-adic Lie extension L of F such that $L = \bigcup L_n$ with L_n/F finite Galois, define

$$H^1_{\mathrm{Iw}}(L,T) = \varprojlim H^1(L_n,T) \qquad \text{and} \qquad H^1_{\mathrm{Iw}}(L,V) = H^1_{\mathrm{Iw}}(L,T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

For a finite extension K of $\mathbb{Q}$, denote by $\mathbb{A}_K$ the ring of adèles of K. If $\mathfrak{f}$ is an integral ideal of K, write $K(\mathfrak{f})$ for the ray class field modulo $\mathfrak{f}$. Let $K(\mathfrak{f}p^\infty) = \bigcup_n K(\mathfrak{f}p^n)$, and define the Galois group $G_{\mathfrak{f}p^\infty} = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)$.

1.3. Grössencharacters. Let K be an imaginary quadratic field. We fix an embedding $K \hookrightarrow \mathbb{C}$. An algebraic Grössencharacter of K of infinity-type (m,n) is a continuous homomorphism $\psi : K^\times\backslash\mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ whose restriction to $\mathbb{C}^\times$ is given by $z \mapsto z^m \bar{z}^n$.

Let θ be the Artin map $\widehat{K}^\times/K^\times \longrightarrow \mathrm{Gal}(K^{\mathrm{ab}}/K)$. We choose the normalizations such that

$$\theta(\varpi_{\mathfrak{q}}) = [\mathfrak{q}]^{-1} \bmod I_{\mathfrak{q}},$$

where $\varpi_{\mathfrak{q}}$ is a uniformizer at the prime $\mathfrak{q}$, $I_{\mathfrak{q}}$ is the inertia group and $[\mathfrak{q}]$ is the arithmetic Frobenius element at $\mathfrak{q}$. Then we have the following well-known result:

Theorem 1.1 (Weil, [17]): *Let ψ be an algebraic Grössencharacter of K, and let L be the finite extension of $\mathbb{Q}$ inside $\mathbb{C}$ generated by $\psi(\widehat{K}^\times)$. Then for any prime λ of L, there is a (clearly unique) continuous character*

$$\psi_\lambda : \mathrm{Gal}(\overline{K}/K) \longrightarrow L_\lambda^\times$$

with the property that

$$\psi_\lambda \circ \theta = \psi|_{\widehat{K}^\times}.$$

The character ψ_λ is unramified outside the primes dividing $\ell\mathfrak{f}$, where ℓ is the prime of $\mathbb{Q}$ below λ and $\mathfrak{f}$ is the conductor of ψ.

The choice of normalization for the Artin map implies that

$$\psi_\lambda([\mathfrak{a}]) = \psi(\mathfrak{a})^{-1}$$

for each $\mathfrak{a}$ coprime to $\ell\mathfrak{f}$. With these conventions, the Hodge–Tate weights[1] of ψ_λ are given as follows. Let λ be a prime of L, and μ a *split* prime of K, which lie above the same prime of $L \cap K$. Then the decomposition groups of μ and $\overline{\mu}$ in $\mathrm{Gal}(K^{\mathrm{ab}}/K)$ are each isomorphic to $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{ab}}/\mathbb{Q}_p)$, and the Hodge–Tate weight of ψ_λ is m at μ and n at $\overline{\mu}$.

2. Comparison of Euler systems

2.1. Elliptic units. As above, let K be an imaginary quadratic field, with a fixed choice of embedding $K \hookrightarrow \mathbb{C}$. We shall fix, for the remainder of this paper, an embedding $\overline{K} \hookrightarrow \mathbb{C}$ compatible with this choice. In particular, for each integral ideal $\mathfrak{f}$, we regard the ray class field $K(\mathfrak{f})$ as a subfield of $\mathbb{C}$, and we write $K(\mathfrak{f})^+$ for its real subfield[2].

Definition 2.1: *If L is a subfield of $\mathbb{C}$, a* CM-pair *of modulus $\mathfrak{f}$ over L is a pair (E, α) consisting of an elliptic curve E/L and a point $\alpha \in E(L)_{\mathrm{tors}}$, such that*

- *there is an isomorphism $\mathrm{End}_{KL}(E) \cong \mathcal{O}_K$, such that the resulting action of $\mathrm{End}_{KL}(E)$ on $\mathrm{coLie}(E/KL) \cong KL$ is the natural action of K;*
- *the annihilator of α in $\mathcal{O}_K$ is exactly $\mathfrak{f}$;*
- *there is an isomorphism $E(\mathbb{C}) \longrightarrow \mathbb{C}/\mathfrak{f}$ mapping α to 1.*

Note that we do not assume that $L \supseteq K$ here, hence the slightly convoluted statement of the first condition.

Theorem 2.2: *Let $\mathfrak{f}$ be such that $\mathcal{O}_K^\times \cap (1+\mathfrak{f}) = \{1\}$, $\bar{\mathfrak{f}} = \mathfrak{f}$, and the smallest integer in $\mathfrak{f}$ is ≥ 5. Then there exists a CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})^+$, and for any field L containing $K(\mathfrak{f})^+$, this CM-pair is the unique CM-pair of modulus $\mathfrak{f}$ over L up to unique isomorphism.*

Proof. Consider the canonical CM-pair $(\mathbb{C}/\mathfrak{f}, 1)$ over $\mathbb{C}$. This corresponds to a point $P_{\mathfrak{f}}$ on the modular curve $Y_1(N)(\mathbb{C})$, where N is the smallest integer in $\mathfrak{f}$.

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[1] We adopt the convention that the cyclotomic character has Hodge–Tate weight $+1$; this is, of course, the Galois character attached to the norm map $\mathbb{A}_K^\times \longrightarrow \mathbb{R}^\times$, which has infinity-type $(1, 1)$.

[2] We stress that $K(\mathfrak{f})$ is not a CM field in general, so the definition of $K(\mathfrak{f})^+$ depends on the choice of embedding, and in particular $K(\mathfrak{f})^+$ is not a totally real field.

Since $N \geq 5$ by assumption, the curve $Y_1(N)$ has a canonical model over $\mathbb{Q}$ such that $Y_1(N)(L)$ parametrises elliptic curves over L with a point of order N for each $L \subseteq \mathbb{C}$. Our claim is then precisely that $P_{\mathfrak{f}} \in Y_1(N)(K(\mathfrak{f})^+)$.

It is clear that $P_{\mathfrak{f}} \in Y_1(N)(\mathbb{R})$, since there is a canonical isomorphism from $\mathbb{C}/\mathfrak{f}$ to the elliptic curve $E_{\mathbb{R}} = \{y^2 = 4x^3 - g_2 x - g_3\}$ where g_2 and g_3 are the usual weight 4 and 6 Eisenstein series, given by $z \mapsto (\wp(z, \mathfrak{f}), \wp'(z, \mathfrak{f}))$. Since $\mathfrak{f} = \bar{\mathfrak{f}}$, the coefficients g_2 and g_3 are real, so $E_{\mathbb{R}}$ is indeed defined over $\mathbb{R}$; and as $\overline{\wp(z, \Lambda)} = \wp(\bar{z}, \bar{\Lambda})$, this uniformization maps $1 \in \mathbb{C}/\mathfrak{f}$ to a real point of $E_{\mathbb{R}}$. Hence $P_{\mathfrak{f}} \in Y_1(N)(\mathbb{R})$.

On the other hand, it is well known that there exists a CM-pair of modulus $\mathfrak{f}$ over $K(\mathfrak{f})$ (whether or not $\bar{\mathfrak{f}} = \mathfrak{f}$), so $P_{\mathfrak{f}} \in Y_1(N)(K(\mathfrak{f}))$. Hence $P_{\mathfrak{f}} \in Y_1(N)(K(\mathfrak{f})^+)$. ∎

Remark 2.3: It follows from this construction that the canonical CM pair (E, α) over $K(\mathfrak{f})^+$ becomes isomorphic over $\mathbb{R}$ to ($E_{\mathbb{R}}$, image of $1 \in \mathbb{C}$). So the complex conjugation automorphism of $E(\mathbb{C})$ arising from this $K(\mathfrak{f})^+$-model corresponds to the natural complex conjugation on $\mathbb{C}/\mathfrak{f}$.

We recall the theory of elliptic units, as described in [9, §15.5-6].

Theorem 2.4: *For each pair $(\mathfrak{f}, \mathfrak{a})$ of ideals of K such that $\mathcal{O}_K^\times \cap (1 + \mathfrak{f}) = \{1\}$ and $\mathfrak{a}$ is coprime to $6\mathfrak{f}$, there is a canonical element*

$$ {}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}} \in K(\mathfrak{f})^\times, $$

the elliptic unit *of modulus $\mathfrak{f}$ and twist $\mathfrak{a}$. If $\mathfrak{f}$ has at least two prime factors, ${}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}} \in \mathcal{O}_{K(\mathfrak{f})}^\times$; and for any two ideals $\mathfrak{a}, \mathfrak{b}$ coprime to $6\mathfrak{f}$, we have*

$$ (N(\mathfrak{b}) - [\mathfrak{b}]) \cdot {}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}} = (N(\mathfrak{a}) - [\mathfrak{a}]) \cdot {}_{\mathfrak{b}}\mathbf{e}_{\mathfrak{f}}, \tag{1} $$

where $[\mathfrak{a}] = \left(\frac{\mathfrak{a}}{K(\mathfrak{f})/K}\right) \in \mathrm{Gal}(K(\mathfrak{f})/K)$ is the arithmetic Frobenius element at $\mathfrak{a}$.

Vital for our purposes is the following complex conjugation symmetry of the elliptic units:

Proposition 2.5: *If $\mathfrak{f}$ satisfies the hypotheses of Theorem 2.2, then we have*

$$ \overline{{}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}}} = {}_{\bar{\mathfrak{a}}}\mathbf{e}_{\mathfrak{f}}. $$

Proof. This follows from the construction of the elliptic units. We have

$$ {}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}} = {}_{\mathfrak{a}}\theta_E(\alpha)^{-1} $$

where (E, α) is the canonical CM pair over $K(\mathfrak{f})$, and ${}_{\mathfrak{a}}\theta_E$ is the element of the function field of E constructed in [9, §15.4].

By Theorem 2.2, E admits a model over $K(\mathfrak{f})^+$, and it is clear that if ι is the nontrivial element of $\mathrm{Gal}(K(\mathfrak{f})/K(\mathfrak{f})^+)$ arising from complex conjugation, we have $\iota({}_{\mathfrak{a}}E) = {}_{\bar{\mathfrak{a}}}E$ and hence (by the uniqueness of ${}_{\mathfrak{a}}\theta_E$) we have $({}_{\mathfrak{a}}\theta_E)^\iota = {}_{\bar{\mathfrak{a}}}\theta_E$. Since $\alpha \in E(K(\mathfrak{f})^+)$, we deduce that

$$\overline{{}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}}} = ({}_{\mathfrak{a}}\theta_E)^\iota(\alpha)^{-1} = {}_{\bar{\mathfrak{a}}}\theta_E(\alpha)^{-1} = {}_{\bar{\mathfrak{a}}}\mathbf{e}_{\mathfrak{f}}$$

as required. ∎

Remark 2.6: Modulo differing choices of conventions, this is the formula labelled "Transport of Structure" in §2.5 of [7].

2.2. Elliptic units in Iwasawa cohomology. Let p be a rational prime which splits in K. For fixed $\mathfrak{f}$ (which we shall assume prime to p), the ideal $\mathfrak{g} = \mathfrak{f}p^n$ satisfies the condition $\mathcal{O}_K^\times \cap (1+\mathfrak{g}) = \{1\}$ for all $n \gg 0$, so if $(\mathfrak{a}, 6p\mathfrak{f}) = 1$ we may define the elements ${}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^n}$. These are *norm-compatible* (c.f. [9, §15.5]), and we may extend their definition to all $n \geq 0$ using the norm maps.

Remark 2.7: Since $\mathfrak{f}p^n$ has at least two prime factors for $n \geq 1$, we have ${}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^n} \in \mathcal{O}_{K(\mathfrak{f}p^n)}^\times$.

Let S be a set of places of K containing the infinite places and the primes above p. Then we have the Kummer maps

$$\kappa_L : \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathcal{O}_{L,S}^\times \xrightarrow{\cong} H^1_S(L, \mathbb{Z}_p(1)).$$

Since the sequence of elements ${}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^\infty} = ({}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^n})_{n\geq 0}$ is a norm-compatible sequence of units, their images under the Kummer maps are corestriction-compatible, so we obtain an element

$${}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^\infty} \in H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)) = \varprojlim_n H^1_S(K(\mathfrak{f}p^n), \mathbb{Z}_p(1)).$$

Theorem 2.8: *If $\mathfrak{f}$ is Galois-stable, then we have*

$$\iota_*\left({}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^\infty}\right) = {}_{\bar{\mathfrak{a}}}\mathbf{e}_{\mathfrak{f}p^\infty},$$

where ι_* is the involution of $H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1))$ induced by complex conjugation.

Proof. Immediate from Proposition 2.5, since $\mathfrak{f}p^n$ satisfies the conditions of Theorem 2.2 for all $n \gg 0$. ∎

Definition 2.9: *We also define the element*

$$\mathbf{e}_{\mathfrak{f}p^\infty} = (N(\mathfrak{a}) - [\mathfrak{a}])^{-1} \cdot {}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{f}p^\infty} \in Q(G_{\mathfrak{f}p^\infty}) \otimes_{\Lambda(G_{\mathfrak{f}p^\infty})} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)),$$

where $\Lambda(G_{\mathfrak{f}p^\infty})$ *is the Iwasawa algebra of* $G_{\mathfrak{f}p^\infty} = \mathrm{Gal}(K(\mathfrak{f}p^\infty)/K)$ *and* $Q(G_{\mathfrak{f}p^\infty})$ *its total ring of quotients.*

Remark 2.10: The element $\mathbf{e}_{\mathfrak{f}p^\infty}$ is independent of the choice of $\mathfrak{a}$, by equation (1).

Corollary 2.11: *We have* $\iota_*(\mathbf{e}_{\mathfrak{f}p^\infty}) = \mathbf{e}_{\mathfrak{f}p^\infty}$.

Proof. The automorphism ι_* of $H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1))$ is $\Lambda(G_{\mathfrak{f}p^\infty})$-semilinear, with the action of ι on $G_{\mathfrak{f}p^\infty}$ being given by conjugation in $\mathrm{Gal}(\overline{K}/\mathbb{Q})$; hence ι_* extends canonically to the tensor product with $Q(G_{\mathfrak{f}p^\infty})$; and since $\iota[\mathfrak{a}]\iota = [\bar{\mathfrak{a}}]$, this finishes the proof by Theorem 2.8 and Remark 2.10. ∎

Let W be any continuous representation of $G_{\mathfrak{f}p^\infty}$ on a one-dimensional vector space over some finite extension L of $\mathbb{Q}_p$. Then we have an isomorphism

$$H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} W \xrightarrow{\cong} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), W(1)). \tag{2}$$

Definition 2.12: *For an element* $w \in W$, *let* $\mathbf{e}_{\mathfrak{f}p^\infty}(w)$ *be the image of* $\mathbf{e}_{\mathfrak{f}p^\infty} \otimes w$ *under* (2), *which is an element of*

$$Q(G_{\mathfrak{f}p^\infty}) \otimes_{\Lambda(G_{\mathfrak{f}p^\infty})} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), W(1)).$$

Define

$$\mathbf{e}_\infty(w) \in Q(\Gamma) \otimes_{\Lambda(\Gamma)} H^1_{\mathrm{Iw},S}(K_\infty, W(1))$$

to be the image of $\mathbf{e}_{\mathfrak{f}p^\infty}(w)$ *under the corestriction map*

$$H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), W(1)) \longrightarrow H^1_{\mathrm{Iw},S}(K_\infty, W(1)).$$

Lemma 2.13: *If* W *has no fixed points under* $\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K_\infty)$, *then we have*

$$\mathbf{e}_\infty(w) \in H^1_{\mathrm{Iw},S}(K_\infty, W(1)).$$

Proof. Suppose $G_{\mathfrak{f}p^\infty}$ acts on W via the character $\tau : G_{\mathfrak{f}p^\infty} \longrightarrow L^\times$. Then we must show that the ideal in $\Lambda(\Gamma)$ generated by the elements

$$\{(N\mathfrak{a} - \tau([\mathfrak{a}])^{-1}[\mathfrak{a}]) : \mathfrak{a} \text{ is an integral ideal coprime to } 6\mathfrak{f}\}$$

contains a power of p. However, if this is not the case, it must consist of elements of $\Lambda(\Gamma)$ which all vanish at some character η of Γ. Then $\chi([\mathfrak{a}])\tau([\mathfrak{a}]) - \eta([\mathfrak{a}])$

vanishes for every $\mathfrak{a}$. By the Chebotarev density theorem, we must have $\tau = \chi^{-1}\eta$, which contradicts the assumption that τ does not factor through Γ. ∎

We write ιW for the representation of $G_{\mathfrak{f}p^\infty}$ that acts on $\{\iota w : w \in W\}$ via $g \cdot (\iota w) = \iota(\iota g \iota) \cdot w$.

Theorem 2.14: *If W has no fixed points under $\mathrm{Gal}(K(\mathfrak{f}p^\infty)/K_\infty)$, the element*

$$\mathbf{e}_\infty(w) \in H^1_{\mathrm{Iw},S}(K_\infty/K, W(1))$$

satisfies

$$\iota_*(\mathbf{e}_\infty(w)) = \mathbf{e}_\infty(\iota w)$$

where ι_* is induced from the maps

$$H^1_S(K(\mathfrak{f}p^n), W(1)) \longrightarrow H^1_S(K(\mathfrak{f}p^n), (\iota W)(1))$$

sending a cocycle τ to the cocycle $g \mapsto \iota\tau(\iota g \iota)$, for each $n \geq 0$.

We split the proof of the theorem into a number of steps.

Definition 2.15: *Let $\Lambda^\sharp(G_{\mathfrak{f}p^\infty})(1)$ denote $\Lambda(G_{\mathfrak{f}p^\infty})(1)$ endowed with the action of $\mathrm{Gal}(K^S/K)$ via the product of the cyclotomic character with the inverse of the canonical character $\mathrm{Gal}(K^S/K) \twoheadrightarrow G_{\mathfrak{f}p^\infty} \hookrightarrow \Lambda(G_{\mathfrak{f}p^\infty})^\times$, i.e. $g.\omega = \chi(g)\bar{g}^{-1}\omega$ for any $g \in \mathrm{Gal}(K^S/K)$ and $\omega \in \Lambda^\sharp(G)$. Here, $\bar{g}$ denotes the image of g in $G_{\mathfrak{f}p^\infty}$.*

Lemma 2.16: *We have a commutative diagram*

$$\begin{array}{ccc} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} W & \xrightarrow{\cong} & H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), W(1)) \\ \Big\downarrow{\scriptstyle \iota_* \otimes \iota} & & \Big\downarrow{\scriptstyle \iota_*} \\ H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \iota W & \xrightarrow{\cong} & H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), (\iota W)(1)) \end{array} \tag{3}$$

where the left-hand vertical map is the tensor product of the automorphism ι_* of $H^1_{\mathrm{Iw},S}(K_\infty, \mathbb{Z}_p(1))$ and the canonical map $\iota : W \longrightarrow \iota W$, and the right-hand vertical map is as defined in the statement of Theorem 2.14.

Proof. We will deduce this isomorphism by using an alternative definition of the Iwasawa cohomology which renders the horizontal maps in the diagram easier to handle. By Shapiro's lemma, we have a canonical isomorphism of $\Lambda(G_{\mathfrak{f}p^\infty})$-modules

$$H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), M(1)) \cong H^1_S(K, M \otimes_{\mathbb{Z}_p} \Lambda^\sharp\big(G_{\mathfrak{f}p^\infty})(1)\big)$$

for any $\mathrm{Gal}(K^S/K)$-module M which is finite-rank over $\mathbb{Z}_p$ or $\mathbb{Q}_p$.

Let τ be the character by which $G_{\mathfrak{f}p^\infty}$ acts on W, and define $\tau_* : \Lambda^\sharp(G) \to \Lambda^\sharp(G)$ to be the map induced by $g \to \tau(g)^{-1}g$. Then the natural twisting map

$$j : H^1_S\big(K, \Lambda^\sharp(G)(1)\big) \otimes W \xrightarrow{\cong} H^1_S\big(K, \Lambda^\sharp(G)(1) \otimes W\big),$$

is explicitly given as follows: if $c : \mathrm{Gal}(K^S/K) \to \Lambda^\sharp(G)(1)$ is a cocycle and $w \in W$, define

$$j(c \otimes w)(g) = \tau_*(c(g)) \otimes w.$$

We check that $j(c \otimes w)$ is a cocycle. Let $h, g \in \mathrm{Gal}(K^S/K)$. Then

$$\begin{aligned} j(c \otimes w)(gh) &= \tau_*(c(gh)) \otimes w \\ &= \tau_*(g.c(h)) \otimes w + \tau_* c(g) \otimes w \\ &= \chi(g)\tau_*(g^{-1}c(h)) \otimes w + \tau_* c(g) \otimes w \\ &= \chi(g)\tau(g)\ \ g^{-1}[\tau_*(c(h))] \otimes w + \tau_*(c(g)) \otimes w \\ &= g.[j(c \otimes w)(h)] + j(c \otimes w)(g) \end{aligned}$$

Rewrite the diagram (3) as

$$\begin{array}{ccc} H^1_S(K, \Lambda^\sharp(G)(1)) \otimes_{\mathbb{Z}_p} W & \xrightarrow{j_W} & H^1_S(K, \Lambda^\sharp(G)(1) \otimes W) \\ \downarrow{\scriptstyle \iota_* \otimes \iota} & & \downarrow{\scriptstyle \iota_*} \\ H^1_S(K, \Lambda^\sharp(G)(1)) \otimes_{\mathbb{Z}_p} \iota W & \xrightarrow{j_{\iota W}} & H^1_S(K, \Lambda^\sharp(G)(1) \otimes \iota W) \end{array} \tag{4}$$

It is then immediate from the description of j that the diagram commutes, which finishes the proof. ∎

Proof of Theorem 2.14. By Corollary 2.11 and Lemma 2.16, we have

$$\iota_*(\mathbf{e}_{\mathfrak{f}p^\infty}(w)) = \mathbf{e}_{\mathfrak{f}p^\infty}(\iota w).$$

The action of ι_* is clearly compatible with corestriction, so we have a commutative diagram

$$\begin{array}{ccc} H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), W(1)) & \longrightarrow & H^1_{\mathrm{Iw},S}(K_\infty, W(1)) \\ \downarrow \iota^* & & \downarrow \iota_* \\ H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), (\iota W)(1)) & \longrightarrow & H^1_{\mathrm{Iw},S}(K_\infty, \iota W(1)) \end{array}$$

which implies that $\iota_*(\mathbf{e}_\infty(w)) = \mathbf{e}_\infty(\iota w)$, completing the proof. ∎

Lemma 2.17: *Let V be any p-adic representation of* $\mathrm{Gal}(K^S/\mathbb{Q})$. *Then the restriction map induces an isomorphism*

$$H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, V) \longrightarrow H^1_{\mathrm{Iw},S}(K_\infty, V)^{\mathrm{Gal}(K_\infty/\mathbb{Q}_\infty)}.$$

Proof. The restriction map is induced from the restriction maps on finite level, which fit into the exact sequence

$$\begin{aligned} 0 \longrightarrow H^1\big(\mathrm{Gal}(K_n/\mathbb{Q}_n), V^{\mathrm{Gal}(K^S/K_n)}\big) &\longrightarrow H^1_S(\mathbb{Q}_n, V) \\ \longrightarrow H^1_S(K_n, V)^{\mathrm{Gal}(K_n/\mathbb{Q}_n)} &\longrightarrow H^2\big(\mathrm{Gal}(K_n/\mathbb{Q}_n), V^{\mathrm{Gal}(K^S/K_n)}\big). \end{aligned}$$

Since $\mathbb{Q}_p$ has characteristic 0, the higher cohomology groups of any $\mathbb{Q}_p$-linear representation of the cyclic group of order 2 are zero. This gives the claim at each finite level, and hence in the inverse limit. ∎

Let α be the unique nontrivial element of $\mathrm{Gal}(K_\infty/\mathbb{Q}_\infty)$.

Lemma 2.18: *We have $\alpha = \delta\iota$, where δ is the unique element of* $\mathrm{Gal}(K_\infty/K)$ *which acts on $\mathbb{Q}_\infty$ as complex conjugation. In particular, δ is of order* 2.

Corollary 2.19: *If α is the unique nontrivial element of* $\mathrm{Gal}(K_\infty/\mathbb{Q}_\infty)$, *then for any $w \in W$,*

$$\alpha_*\left(\mathbf{e}_\infty(w)\right) = \delta \cdot \mathbf{e}_\infty(\iota w).$$

Proof. As above, write $\alpha = \delta\iota$. By Lemma 2.17, we have $\iota^* \cdot e_\infty(w) = \mathbf{e}_\infty(\iota w)$. Hence $\alpha_*\left(\mathbf{e}_\infty(w)\right) = \delta \cdot \iota_*\left(\mathbf{e}_\infty(w)\right) = \delta \cdot \mathbf{e}_\infty(\iota w)$. ∎

2.3. The two-variable L-function of K.

We recall the construction (originally due to Yager [18]) of a two-variable p-adic L-function from the elliptic units.

Let $\mathfrak{p}$ be one of the two primes of K above p. We choose an embedding $\overline{K} \hookrightarrow \overline{\mathbb{Q}}_p$ inducing the $\mathfrak{p}$-adic valuation on K. Then for any finite extension L/K, and any $\operatorname{Gal}(\overline{K}/K)$-module M, we may define

$$Z^1_{\mathfrak{p}}(L, M) = \bigoplus_{\mathfrak{q}|\mathfrak{p}} H^1(L_{\mathfrak{q}}, M) = H^1(K_{\mathfrak{p}}, \operatorname{Ind}_L^K M).$$

which is a $\operatorname{Gal}(L/K)$-module. We also define

$$Z^1_{\mathrm{Iw},\mathfrak{p}}(K(\mathfrak{f}p^\infty), M) = \varprojlim_L Z^1_{\mathfrak{p}}(L, M)$$

where the limit is taken over finite extensions L/K contained in $K(\mathfrak{f}p^\infty)$.

We now recall the theory of two-variable Coleman series, as introduced, under certain additional hypotheses, by Yager [18], and generalized to the semi-local situation here by de Shalit [5, §II.4.6]. Let $\zeta = (\zeta_{p^n})_{n \ge 0}$ be a compatible system of p-power roots of unity in $\overline{K}$; and let $\widehat{F}_\infty$ be the completion of $K(\mathfrak{f}\bar{\mathfrak{p}}^\infty)$ with respect to the prime $\mathfrak{P}$ of $\overline{K}$ above $\mathfrak{p}$ induced by our choice of embedding $\overline{K} \hookrightarrow \overline{\mathbb{Q}}_p$, and $\widehat{\mathcal{O}}_\infty$ the ring of integers of $\widehat{F}_\infty$. (Thus $\widehat{\mathcal{O}}_\infty$ is a complete discrete valuation ring with maximal ideal generated by p, and its residue field is a finite extension of the unique $\mathbb{Z}_p$-extension of $\mathbb{F}_p$.)

Proposition 2.20: *There is a unique morphism of $\Lambda(G_{\mathfrak{f}p^\infty})$-modules*

$$\operatorname{Col}^\zeta : Z^1_{\mathrm{Iw},\mathfrak{p}}(K(\mathfrak{f}p^\infty), \mathbb{Z}_p(1)) \longrightarrow \Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$$

with the following property:

For each finite-order character η of $G_{\mathfrak{f}p^\infty}$ which is not unramified at $\mathfrak{p}$, we have

$$\operatorname{Col}^\zeta(u)(\eta) = \tau(\eta, \zeta)^{-1} \eta(\tilde{\varphi})^n \left(\sum_{\sigma \in G_{\mathfrak{f}p^m}} \eta(\sigma)^{-1} \log_{\mathfrak{P}}(u_m^\sigma) \right).$$

Here $\tilde{\varphi}$ is the unique lifting of the arithmetic Frobenius of $\operatorname{Gal}(K(\mathfrak{f}\bar{\mathfrak{p}}^\infty)/K)$ to $\operatorname{Gal}(K(\mathfrak{f}p^\infty)/K_\infty)$, m is any integer such that η factors through the quotient $G_{\mathfrak{f}p^m} = \operatorname{Gal}(K(\mathfrak{f}p^m)/K)$, $\log_{\mathfrak{P}}$ is the logarithm map

$$\mathcal{O}^\times_{K(\mathfrak{f}p^n),\mathfrak{P}} \longrightarrow K(\mathfrak{f}p^n)_{\mathfrak{P}},$$

and

$$\tau(\eta, \zeta) = \sum_{\sigma \in \operatorname{Gal}(K(\mathfrak{f}\bar{\mathfrak{p}}^\infty)(\mu_{p^n})/K(\mathfrak{f}\bar{\mathfrak{p}}^\infty))} \omega(\sigma)^{-1} \zeta_{p^n}^\sigma,$$

where n is the exact power of $\mathfrak{p}$ dividing the conductor of η.

Definition 2.21: *We let*

$$\mathbb{L}_{\mathfrak{f}p^\infty} = \mathrm{Col}^{\zeta}(\mathbf{e}_{\mathfrak{f}p^\infty}) \in \widehat{\mathcal{O}}_\infty \,\widehat{\otimes}_{\mathbb{Z}_p} Q(G_{\mathfrak{f}p^\infty}).$$

Proposition 2.22: *The element $\mathbb{L}_{\mathfrak{f}p^\infty}$ lies in $\Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$, and it coincides with the measure $\mu(\mathfrak{f}\bar{\mathfrak{p}}^\infty)$ in* [5, Theorem II.4.14].

Proof. We have $(N\mathfrak{a} - [\mathfrak{a}]) \cdot \mathbb{L}_{\mathfrak{f}p^\infty} \in \Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$ for all $\mathfrak{a}$. Since the ideal generated by $N\mathfrak{a} - [\mathfrak{a}]$ for all integral ideals $\mathfrak{a}$ coprime to $6\mathfrak{f}$ has height 2, this implies that $\mathbb{L}_{\mathfrak{f}p^\infty} \in \Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$ (cf. [5, §II.4.12]).

To show that the resulting measure coincides with de Shalit's $\mu(\mathfrak{f}\bar{\mathfrak{p}}^\infty)$, we compare the defining property of the map Col above with [5, Theorem II.5.2]. For a finite-order character η of $G_{\mathfrak{f}p^n}$, whose conductor $\mathfrak{g}$ is divisible by $\mathfrak{p}$ and satisfies $\mathcal{O}_K^\times \cap (1+\mathfrak{g}) = \{1\}$, de Shalit shows that

$$\eta(\mu(\mathfrak{f}\bar{\mathfrak{p}}^\infty)) = \frac{-1}{12g} G(\eta) \sum_{\mathfrak{c} \in \mathrm{Cl}(\mathfrak{g})} \eta^{-1}([\mathfrak{c}]) \log \phi_{\mathfrak{g}}(\mathfrak{c}),$$

where g is the smallest rational integer in $\mathfrak{g}$, $\phi_{\mathfrak{g}}(\mathfrak{c})$ is Robert's invariant and the quantity $G(\eta)$ coincides with what we have called $\tau(\eta,\zeta)^{-1}\eta(\tilde{\varphi})^n$. Since

$$(N(\mathfrak{a}) - [\mathfrak{a}])\phi_{\mathfrak{g}}(\mathfrak{c}) = [\mathfrak{c}] \cdot \left({}_{\mathfrak{a}}\mathbf{e}_{\mathfrak{g}}\right)^{-12g},$$

this shows that the two measures coincide at every finite-order character, and hence they are equal in $\Lambda_{\widehat{\mathcal{O}}_\infty}(G_{\mathfrak{f}p^\infty})$. ∎

Remark 2.23: If one identifies $G(\mathfrak{f}p^\infty)$ with the ray class group modulo $\mathfrak{f}p^\infty$ via the Artin map, normalized as in §1.3 above, then this measure coincides with the pullback of the Katz two-variable L-function of K (cf. [8, §4]) up to a difference of signs. This remark will be important in the proof of Theorem 3.4 below.

2.4. Kato's zeta element. Let $f = \sum a_n q^n$ be a modular form of CM type, corresponding to a Grössencharacter ψ of K with infinity-type $(1-k, 0)$ where k is the weight of f. It is clear that the coefficient field $F = \mathbb{Q}(a_n : n \ge 1)$ of f is contained in the finite extension L/K contained in $\mathbb{C}$ generated by $\psi(\widehat{K}^\times)$.

Following [9, §6.3], we write $S(f)$ and $V(f)$ for the subspaces of the de Rham and Betti cohomology of the Kuga–Sato variety attached to f. Note that both of these are F-vector spaces, and $S(f)$ is 1-dimensional over F while $V(f)$ is 2-dimensional. For a commutative ring A over F, define $S_A(f) = S(f) \otimes_F A$ and $V_A(f) = V(f) \otimes_F A$. If λ is a place of F above p, we may identify $V_{F_\lambda}(f)$

with the p-adic representation associated to f of Deligne [6] and $S_{F_\lambda}(f)$ may be identified with $\mathrm{Fil}^1\,\mathbb{D}_{\mathrm{cris}}(V_{F_\lambda}(f))$.

Definition 2.24: *Let χ be a Dirichlet character of conductor p^n. We define the maps $\theta^{\pm}_{\chi,f}$ by*

$$\begin{array}{rclcl} \theta^{\pm}_{\chi,f} & : & S(f)\otimes_{\mathbb{Q}}\mathbb{Q}(\mu_{p^n}) & \longrightarrow & V_{\mathbb{C}}(f)^{\pm} \\ & & x\otimes y & \longmapsto & \sum_{\sigma\in G_n}\chi(\sigma)\sigma(y)\,\mathrm{per}_f(x)^{\pm} \end{array}$$

where $G_n=\mathrm{Gal}(\mathbb{Q}(\mu_{p^n})/\mathbb{Q})$, $\mathrm{per}_f : S(f)\longrightarrow V_{\mathbb{C}}(f)$ is the period map as defined in [9, §6.3] *and $\gamma\mapsto\gamma^{\pm}$ is the projection from $V_{\mathbb{C}}(f)$ to its (1-dimensional) ±1-eigenspace for the complex conjugation.*

Theorem 2.25 ([9, Theorem 12.5(1)]): *We have a L_λ-linear map*

$$\begin{array}{rcl} V_{L_\lambda}(f) & \longrightarrow & H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, V_\lambda(f)) \\ \gamma & \longmapsto & \mathbf{z}^{\mathrm{Kato}}_{\gamma} \end{array}$$

which satisfies the following. Let χ be a Dirichlet character of conductor p^n, $\gamma\in V_L(f)$ and $1\le r\le k-1$, then

$$\theta^{\pm}_{\chi,f}\circ\exp^*\left(\mathbf{z}^{\mathrm{Kato}}_{\gamma}\otimes(\zeta_{p^n})^{\otimes(k-r)}\right)=(2\pi i)^{k-r-1}L_{\{p\}}(f^*,\chi,r)\cdot\gamma^{\pm}$$

where $\pm=(-1)^{k-r-1}\chi(-1)$.

Let $\mathfrak{f}$ be an ideal of $\mathcal{O}_K$ satisfying the conditions in Theorem 2.2 which is contained in the conductor of ψ. Let (E,α) be the canonical CM-pair over $K(\mathfrak{f})$. Following [9, §15.8], we define $V_L(\psi)=H^1(E(\mathbb{C}),\mathbb{Q})^{\otimes(k-1)}\otimes_K L$ and $S(\psi)=H^0(\mathrm{Gal}(K(\mathfrak{f})/K),\mathrm{coLie}(E)^{\otimes(k-1)}\otimes_K L)$, where the action of $\mathrm{Gal}(K(\mathfrak{f})/K)$ on the space $\mathrm{coLie}(E)^{\otimes(k-1)}\otimes_K L$ is as described in *op.cit.*. Both of these are 1-dimensional L-vector spaces. For any commutative ring A over L, we write $V_A(\psi)=V_L(\psi)\otimes_L A$ and $S_A(\psi)=S(\psi)\otimes_L A$. The Galois group $\mathrm{Gal}(\overline{K}/K)$ acts on $V_L(\psi)\otimes_L L_\lambda$ via ψ_λ, and there exists a period map

$$\mathrm{per}_\psi : S(\psi)\longrightarrow V_{\mathbb{C}}(\psi)$$

induced by passing to the $(k-1)$-st tensor power from the comparison isomorphism per_∞ described above.

We now recall Kato's results on the relation between this zeta element and the elliptic units.

LEMMA 2.26 ([9, Lemma 15.11]): *Fix a choice of isomorphism of L-vector spaces*

$$s : S(\psi) \xrightarrow{\sim} S_L(f).$$

(a) *There exists a unique isomorphism of representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *over* L_λ

$$\widetilde{V_{L_\lambda}(\psi)} \longrightarrow V_{L_\lambda}(f)$$

such that the isomorphism $S_{L_\lambda}(\psi) \longrightarrow S_{L_\lambda}(f)$ induced by the functoriality of $\mathbb{D}_{\mathrm{dR}}$ is compatible with s.

(b) *There exists a unique isomorphism of representations of* $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ *over* L

$$\widetilde{V_L(\psi)} \longrightarrow V_L(f)$$

for which the diagram

$$\begin{array}{ccc} S(\psi) & \xrightarrow{\mathrm{per}_\psi} & \widetilde{V_{\mathbb{C}}(\psi)} \\ \downarrow & & \downarrow \\ S_L(f) & \xrightarrow{\mathrm{per}_f} & V_{\mathbb{C}}(f) \end{array}$$

commutes.

Note that the isomorphism of part (b) implies an isomorphism $V_{L_\lambda}(\psi) \xrightarrow{\cong} V_{L_\lambda}(f)$ on extending scalars to L_λ, but one does not know that this coincides with the isomorphism of part (a), as remarked in [9, §15.11].

DEFINITION 2.27: *We write $\Phi_{\psi,f}$ for the canonical map*

$$H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), V_{L_\lambda}(\psi)) \longrightarrow H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, V_{L_\lambda}(f))$$

as defined in [9, (15.12.1)].

Concretely, this map can be defined as follows:

$$H^1_{\mathrm{Iw},S}(K(\mathfrak{f}p^\infty), V_{L_\lambda}(\psi)) \longrightarrow H^1_S(K, \Lambda^\sharp(\Gamma) \otimes V_{L_\lambda}(\psi)) \longrightarrow$$
$$H^1_S(\mathbb{Q}, \mathrm{Ind}_K^{\mathbb{Q}}\left(\Lambda^\sharp(\Gamma) \otimes V_{L_\lambda}(\psi)\right)) \xrightarrow{\cong} H^1_S(\mathbb{Q}, \Lambda^\sharp(\Gamma) \otimes V_{L_\lambda}(f)).$$

THEOREM 2.28: *Let $\gamma \in V_L(\psi)$ and write γ' for its image in $V_L(f)$ under the map given by Lemma 2.26(b). Then we have*

$$\Phi_{\psi,f}\left(\mathbf{e}_\infty(\gamma) \otimes (\zeta_{p^n})^{\otimes(-1)}\right) = \mathbf{z}^{\mathrm{Kato}}_{\gamma'}.$$

Proof. This is [9, (15.16.1)]; it is immediate from a comparison the interpolating properties of the two zeta elements, since an element of $H^1_{\mathrm{Iw}}(\mathbb{Q}_\infty/\mathbb{Q}, V_{L_\lambda}(f))$ is uniquely determined by its images under the dual exponential maps at each finite level in the tower $\mathbb{Q}_\infty/\mathbb{Q}$. ∎

PROPOSITION 2.29: *We have a commutative diagram*

$$\begin{array}{ccc} H^1_{\mathrm{Iw},S}(K_\infty, V_{L_\lambda}(\psi)) & \xrightarrow{\Phi_{\psi,f}} & H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, V_{L_\lambda}(f)) \\ \downarrow & \swarrow_{\cong} & \\ H^1_{\mathrm{Iw},S}(K_\infty, V_{L_\lambda}(\psi) \oplus \iota V_{L_\lambda}(\psi))^{\alpha=1} & & \end{array}$$

where the left-hand vertical map sends x to $x \oplus \delta \cdot \iota_*(x)$, and the diagonal isomorphism is given by restriction.

Proof. Clear. ∎

3. Critical-slope L-functions

Let f be a modular form of CM type, as above, and ψ the corresponding Grössencharacter. We choose a basis γ of $V_L(\psi)$, and let γ' be its image in $V_L(f)$ under the isomorphism of Lemma 2.26(b).

We fix an embedding $\overline{K} \hookrightarrow \overline{\mathbb{Q}}_p$ which induces the λ-adic valuation on L. This gives an embedding $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \hookrightarrow \mathrm{Gal}(\overline{K}/\mathbb{Q})$, whose image is contained in the subgroup $\mathrm{Gal}(\overline{K}/K)$. This gives a localization map

$$\mathrm{loc}_p : H^1_{\mathrm{Iw},S}(\mathbb{Q}_\infty, M) \longrightarrow H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, M)$$

for each $\mathrm{Gal}(K^S/\mathbb{Q})$-module M. Moreover, we have a map

$$\mathrm{loc}_{\mathfrak{p}} : H^1_{\mathrm{Iw},S}(K_\infty, M) \longrightarrow H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, M)$$

for each $\mathrm{Gal}(K^S/K)$-module M, and we clearly have $\mathrm{loc}_p = \mathrm{loc}_{\mathfrak{p}} \circ \mathrm{res}_{K/\mathbb{Q}}$.

Via the isomorphism of Lemma 2.26(a), the space $V_{L_\lambda}(f)$ is isomorphic as a representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ to $V_{L_\lambda}(\psi) \oplus \iota\,(V_{L_\lambda}(\psi))$. Note that ι does not normalize the image of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, so the two factors are non-isomorphic; indeed $V_{L_\lambda}(\psi)$ has Hodge–Tate weight $1-k$, while $\iota\,(V_{L_\lambda}(\psi))$ has Hodge–Tate weight 0. Hence we have

$$\mathrm{loc}_p(\mathbf{z}^{\mathrm{Kato}}_{\gamma'}) \in H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, V_{L_\lambda}(\psi)) \oplus H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, \iota(V_{L_\lambda}(\psi))).$$

Let us write pr_1 and pr_2 for the projections to the two direct summands above. By Corollary 2.28, the projection $\mathrm{pr}_1 \mathrm{loc}_p(\mathbf{z}^{\mathrm{Kato}}_{\gamma'})$ to $H^1_{\mathrm{Iw}}(\mathbb{Q}_{p,\infty}, V_{L_\lambda}(\psi))$ is

$$\mathrm{loc}_{\mathfrak{p}}\left(\mathbf{e}_\infty(\gamma) \otimes (\zeta_{p^n})^{\otimes(-1)}\right).$$

By Proposition 2.29, we see that the projection of $\mathrm{loc}_p(\mathbf{z}^{\mathrm{Kato}}_{\gamma'})$ to the other direct summand is

$$\delta \cdot \mathrm{loc}_{\mathfrak{p}}\left[\iota_*\left(\mathbf{e}_\infty(\gamma) \otimes (\zeta_{p^n})^{\otimes(-1)}\right)\right] = \left[\delta \cdot \mathrm{loc}_{\mathfrak{p}}\left(\iota_*(\mathbf{e}_\infty(\gamma))\right)\right] \otimes (\zeta_{p^n})^{\otimes(-1)}.$$

We have

$$\iota_*\left(\mathbf{e}_\infty(\gamma)\right) = \mathbf{e}_\infty(\iota\gamma),$$

so this simplifies to

$$\mathrm{pr}_2\left(\mathrm{loc}_p \mathbf{z}^{\mathrm{Kato}}_{\gamma'}\right) = \delta \cdot \left[\mathrm{loc}_{\mathfrak{p}}\left(\mathbf{e}_\infty(\iota\gamma)\right)\right] \otimes (\zeta_{p^n})^{\otimes(-1)}.$$

Definition 3.1: *Let $L^\gamma_{p,1} \in \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} \mathbb{D}_{\mathrm{cris}}(V_{L_\lambda}(\psi)(k-1))$ and $L^\gamma_{p,2} \in \Lambda(\Gamma) \otimes_{\mathbb{Z}_p} \mathbb{D}_{\mathrm{cris}}(\iota V_{L_\lambda}(\psi)(k-1))$ be the unique elements such that*

$$\mathcal{L}^\Gamma_{V_{L_\lambda}(f)(k-1)}\left(\mathbf{z}^{\mathrm{Kato}}_{\gamma'} \otimes (\zeta_{p^n})^{\otimes(k-1)}\right) = L^\gamma_{p,1} \oplus L^\gamma_{p,2}.$$

We shall see below that if $g = \bar{f}$ is the complex conjugate of f, then $L^\gamma_{p,1}$ will be the ordinary p-adic L-function of g, and $L^\gamma_{p,2}$ is the critical-slope p-adic L-function of g.

Theorem 3.2: *For every character η of Γ, we have*

$$L^\gamma_{p,1}(\eta) = \mathbb{L}_{\mathfrak{f}p^\infty}(\eta\left(\psi_\lambda \chi^{k-2}\right)^{-1}) \cdot t^{k-1}\gamma,$$

and

$$L^\gamma_{p,2}(\eta) = (\ell_0 \dots \ell_{k-2}\delta\mathbb{L}_{\mathfrak{f}p^\infty})\left(\eta\left(\psi^\iota_\lambda \chi^{k-2}\right)^{-1}\right) \cdot \iota\gamma.$$

Proof. For brevity, we shall write e_j for $(\zeta_{p^n})^{\otimes j}$, considered as a basis vector of $\mathbb{Q}_p(j)$.

It is easy to see that if ξ is a character of $G_{\mathfrak{f}p^\infty}$ of the form $\chi^j\tau$, where τ is unramified and $j \geq 0$, and V is any crystalline representation with non-negative Hodge-Tate weights, then for any $x \in H^1_{\mathrm{Iw}}(K(\mathfrak{f}p^\infty), V)$ and any choice of basis e_ξ of $\mathbb{Q}_p(\xi)$ we have

$$\mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{V(\xi)}(x \otimes e_\xi)(\eta) = (\ell_0 \dots \ell_{j-1})(\eta) \cdot \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_V(x)(\eta\xi^{-1}) \otimes t^{-j} e_\xi.$$

Note that if ξ takes values in the finite extension $L/\mathbb{Q}_p$, this is an equality of two elements of $L \otimes \widehat{F}_\infty \otimes \mathbb{D}_{\mathrm{cris}}(V(\xi))$: the element $t^{-j}e_\xi \in \mathbb{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} L(\xi)$

transforms via τ under $G_{\mathbb{Q}_p}$, and hence lies in $\widehat{F}_\infty \otimes \mathbb{D}_{\mathrm{cris}}(L(\xi))$, since the periods of unramified characters lie in $\widehat{F}_\infty \subseteq \mathbb{B}_{\mathrm{cris}}$.

We apply this result with $V = \mathbb{Q}_p$ (the trivial representation), $x = \mathbf{e}_{\mathfrak{f}p^\infty} \otimes e_{-1}$, and various values of ξ. Firstly, taking ξ to be the cyclotomic character, we have

$$\mathbb{L}_{\mathfrak{f}p^\infty} = \ell_0^{-1} \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{\mathbb{Q}_p(1)}(\mathbf{e}_{\mathfrak{f}p^\infty}),$$

and thus

$$\mathbb{L}_{\mathfrak{f}p^\infty}(\eta) = \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{\mathbb{Q}_p}(\mathbf{e}_{\mathfrak{f}p^\infty} \otimes e_{-1})(\chi^{-1}\eta) \otimes t^{-1} e_1. \tag{5}$$

On the other hand we have

$$\begin{aligned} L^\gamma_{p,1}(\eta) &= \mathcal{L}^{\Gamma}_{V_{L_\lambda}(\psi)(k-1)} \left(\mathrm{pr}_1(\mathbf{z}^{\mathrm{Kato}}_{\gamma'}) \otimes e_{k-1}\right)(\eta) \\ &= \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{V_{L_\lambda}(\psi)(k-1)} \left(\mathbf{e}_\infty(\gamma) \otimes e_{k-2}\right)(\eta) \end{aligned}$$

The group $G_{\mathbb{Q}_p}$ acts on $V_{L_\lambda}(\psi)(k-1)$ via the unramified character $\chi^{k-1}\psi_\lambda$, so this is

$$L^\gamma_{p,1}(\eta) = \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{\mathbb{Q}_p} \left(\mathbf{e}_\infty \otimes e_{-1}\right) \left((\chi^{k-1}\psi_\lambda)^{-1}\eta\right) \otimes (t^{k-1}\gamma) \otimes (t^{1-k} e_{k-1}).$$

Comparing this with (5), we deduce that

$$L^\gamma_{p,1}(\eta) = \mathbb{L}_{\mathfrak{f}p^\infty} \left((\chi^{k-2}\psi_\lambda)^{-1}\eta\right) \otimes (t^{k-1}\gamma) \otimes (t^{2-k} e_{k-2}).$$

If we identify $\mathbb{D}_{\mathrm{cris}}(\mathbb{Q}_p(k-2))$ with $\mathbb{Q}_p$ in the usual way, $t^{2-k} e_{k-2}$ is sent to 1. As remarked above, the element $t^{k-1}\gamma \in \mathbb{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V_{L_\lambda}(\psi)$ lies in $\widehat{F}_\infty \otimes_{\mathbb{Q}_p} \mathbb{D}_{\mathrm{cris}}(V_{L_\lambda}(\psi))$. So if ω is a K-basis of $S(\psi)$, then the image of ω under the crystalline comparison isomorphism is a basis of $\mathbb{D}_{\mathrm{cris}}(V_{L_\lambda}(\psi))$, and if we define $\Omega_p = (\gamma \otimes e_{1-k})/\omega$, this will lie in $\widehat{F}_\infty$ and our result becomes

$$L^\gamma_{p,1}(\eta) = \mathbb{L}_{\mathfrak{f}p^\infty} \left((\chi^{k-2}\psi_\lambda)^{-1}\eta\right) \cdot \Omega_p \omega.$$

We now turn to $L^\gamma_{p,2}$. We have

$$\begin{aligned} L^\gamma_{p,2}(\eta) &= \mathcal{L}^{\Gamma}_{\iota(V_{L_\lambda}(\psi))(k-1)} \left(\mathrm{pr}_2(\mathbf{z}^{\mathrm{Kato}}_{\gamma'}) \otimes e_{k-1}\right)(\eta) \\ &= \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{\iota(V_{L_\lambda}(\psi))(k-1)} \left((\delta \cdot \mathbf{e}_\infty(\iota\gamma)) \otimes e_{k-2}\right)(\eta) \\ &= (-1)^{k-2}\eta(\delta) \mathcal{L}^{G_{\mathfrak{f}p^\infty}}_{\iota(V_{L_\lambda}(\psi))(k-1)} \left(\mathbf{e}_\infty(\iota\gamma) \otimes e_{k-2}\right)(\eta). \end{aligned}$$

The group $G_{\mathbb{Q}_p}$ acts on $\iota(V_{L_\lambda}(\psi))$ by the character ψ_λ^ι, which is unramified; so this is

$$\begin{aligned} L_{p,2}^{\gamma}(\eta) &= (-1)^{k-2}\eta(\delta)(\ell_0\dots\ell_{k-2})(\eta)\cdot\mathcal{L}_{\mathbb{Q}_p}^{G_{\mathfrak{f}p^\infty}}(\mathbf{e}_\infty\otimes e_{-1})\left((\chi^{k-1}\psi_\lambda^\iota)^{-1}\eta\right) \\ &\qquad\qquad\otimes t^{1-k}e_{k-1}\otimes\iota\gamma. \\ &= (-1)^{k-2}\eta(\delta)(\ell_0\dots\ell_{k-2})(\eta)\cdot\mathbb{L}_{\mathfrak{f}p^\infty}\left((\chi^{k-2}\psi_\lambda^\iota)^{-1}\eta\right)\otimes t^{2-k}e_{k-2}\otimes\iota\gamma. \end{aligned}$$

As above, we identify $t^{2-k}e_{k-2}\in\mathbb{D}_{\mathrm{cris}}(\mathbb{Q}_p(k-2))$ with $1\in\mathbb{Q}_p$; and if ω is a basis of $S_L(\psi)$, the image of $\iota\omega$ under the comparison isomorphism is a basis of $\mathbb{D}_{\mathrm{cris}}(\iota(V_{L_\lambda}(\psi)))$, so if we define $\Omega_p^\iota=(\iota\gamma)/(\iota\omega)$ this becomes

$$L_{p,2}^{\gamma}(\eta) = (-1)^{k-2}\eta(\delta)(\ell_0\dots\ell_{k-2})(\eta)\cdot\mathbb{L}_{\mathfrak{f}p^\infty}\left((\chi^{k-2}\psi_\lambda^\iota)^{-1}\eta\right)\cdot\Omega_p^\iota\iota\omega.$$

∎

Definition 3.3: *Let ω be a basis of $S_L(\psi)$ as above, let $g=\bar{f}$, and let $L_{p,\alpha}(g)$ and $L_{p,\beta}(g)$ be the elements of $\mathcal{H}_{L_\lambda}(\Gamma)$ defined by*

$$L_{p,1}^{\gamma} = L_{p,\alpha}(g)\cdot\omega$$

and

$$L_{p,2}^{\gamma} = L_{p,\beta}(g)\cdot\iota\omega.$$

Then $L_{p,\alpha}$ and $L_{p,\beta}$ are the p-adic L-functions attached to g, where α and β are respectively the unit and non-unit roots of the Hecke polynomial of g.

As shown in [9, §16], this is consistent with the classical Amice–Velu–Vishik construction of the ordinary p-adic L-function $L_{p,\alpha}(g)$, and thus it is natural to regard $L_{p,\beta}(g)$ as a candidate for a critical-slope p-adic L-function. This is the definition of the Kato critical-slope L-function used in [11].

Theorem 3.4: *Up to multiplication by two nonzero scalars, one for each sign, $L_{p,\beta}(g)$ coincides with the modular symbol critical-slope L-function $L_{p,\beta}^{\mathrm{MS}}(g)$ attached to the non-ordinary p-stabilization of g in* [3].

Proof. This follows by comparing the formulae of Theorem 3.2 with Theorem 2 of [3]. Note that Bellaïche shows that if ρ_1 and ρ_2 are the two characters by which $\mathrm{Gal}(\overline{K}/K)$ acts on V_g^*, then

$$\begin{cases} L_{p,\alpha}(g)(\eta) &= \mathbb{L}_{\mathfrak{f}p^\infty}(\rho_2\eta^{-1})\cdot(\text{constant}^\pm), \\ L_{p,\beta}^{\mathrm{MS}}(g)(\eta) &= (\ell_0\cdots\ell_{k-2})(\eta)\cdot\mathbb{L}_{\mathfrak{f}p^\infty}(\rho_1\eta^{-1})\cdot(\text{constant}^\pm). \end{cases}$$

Here constant$^{\pm}$ indicates an equality of distributions on Γ up to multiplication by two nonzero constants (one for each sign). On the other hand, since $V_g^* = V_f(k-1)$, we have $\{\rho_1, \rho_2\} = \{\chi^{k-1}\psi_\lambda, \chi^{k-1}\psi_\lambda^\iota\}$ and the result of Theorem 3.2 shows that

$$\begin{cases} L_{p,\alpha}(g)(\eta) & = \mathbb{L}_{\mathfrak{f}p^\infty}(\chi\rho_1^{-1}\eta)\cdot(\text{constant}),\\ L_{p,\beta}(g)(\eta) & = (\ell_0\cdots\ell_{k-2})(\eta)\cdot\mathbb{L}_{\mathfrak{f}p^\infty}(\chi\rho_2^{-1}\eta)\cdot(\text{constant}).\end{cases}$$

To reconcile these formulae, we note that the p-adic L-function $\mathbb{L}_{\mathfrak{f}p^\infty}$ satisfies a functional equation [5, §II.6]

$$\mathbb{L}_{\mathfrak{f}p^\infty}(\iota(\eta)) = C(\eta)\cdot\mathbb{L}_{\mathfrak{f}p^\infty}(\chi\eta^{-1}),$$

for a function $C(\eta)$ (involving a p-adic root number and various other correction terms) which depends only on the coset of η modulo characters factoring through $\mathrm{Gal}(\mathbb{Q}_\infty^+/\mathbb{Q})$. Since $\iota(\rho_1) = \rho_2$ and vice versa, we deduce that

$$L_{p,\beta}(g) = L_{p,\beta}^{\mathrm{MS}}(g)\cdot(\text{constant}^{\pm}).$$

Since the modular symbol L-function is only defined up to scalars, this completes the proof. ∎

Remark 3.5: Both Kato's and Bellaïche's critical-slope p-adic L-functions are only defined up to multiplication by a nonzero constant for characters of each sign; in Kato's construction these constants correspond to the choice of γ, whose projection to each of the $\pm$ eigenspaces of complex conjugation must be nonzero. It seems natural to ask whether one can choose normalizations for both in a compatible fashion so Theorem 3.4 holds exactly, but the present authors do not feel sufficiently familiar with the modular symbol construction to comment further.

Remark 3.6: Since the Hodge–Tate weights of ψ_λ at $\mathfrak{p}$ and $\bar{\mathfrak{p}}$ are $(1-k, 0)$, we see that if η is a character of Γ whose single Hodge–Tate weight is t, the Hodge–Tate weights of $\eta\left(\psi_\lambda\chi^{k-2}\right)^{-1}$ and $\eta\left(\psi_\lambda^\iota\chi^{k-2}\right)^{-1}$ are respectively $(t+1, t+2-k)$ and $(t+2-k, t+1)$. Since the range of interpolation for the Katz p-adic L-function consists of those characters whose Hodge–Tate weights are (r, s) with $r \ge 1$ and $s \le 0$ ([5, Corollary II.6.7]), the line $(t+1, t+2-k)$ contains $k-1$ lattice points inside this range, but the line $(t+2-k, t+1)$ misses the range of interpolation entirely. The first statement corresponds to the well-known fact that $L_{p,\alpha}(g)(\eta)$ corresponds to a complex L-value for $0 \le t \le k-2$; but the second shows that, sadly, none of the values of $L_{p,\beta}(g)(\eta)$, nor its derivatives at the points where it is forced to vanish, correspond to a classical L-value for any value of η. In particular, we cannot rule out the possibility that $L_{p,\beta}(g)$ is zero.

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