The Characterisation of Chaos in Low Dimensional Spaces

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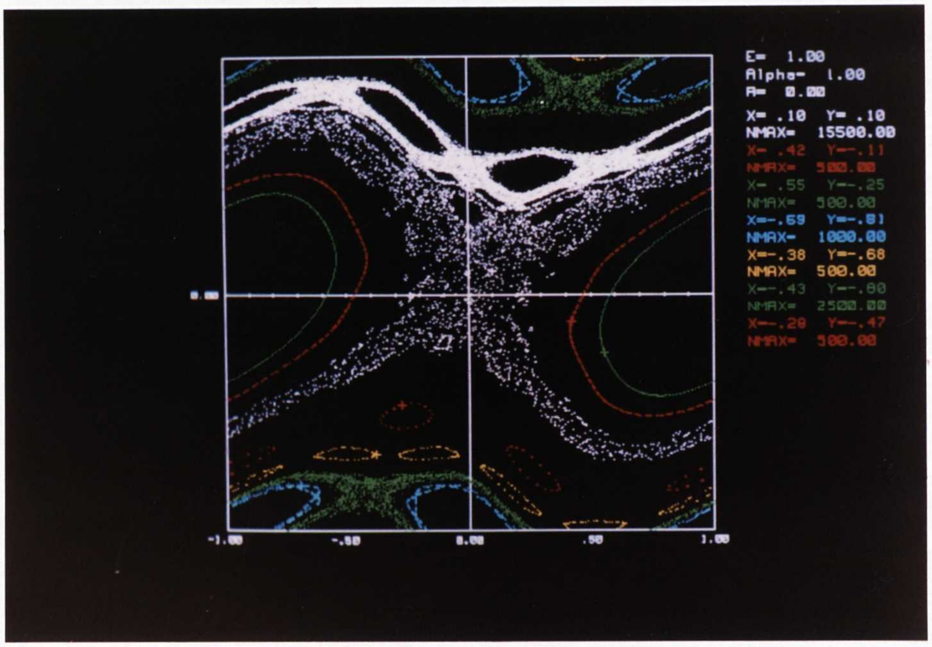
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for

my parents

Summary

This work attempts to characterise some of the complicated behaviour that is observed in many non-linear systems. For example, the frontispiece was generated by iterations of a two dimensional area-preserving mapping (the Chirikov map) that is typical of the systems studied herein. It will be shown that area-preserving deterministic mappings can be accurately characterised by a diffusion constant i.e. a quantity associated with random systems. In addition it shows how perturbation theory has a greater range of validity than might be expected.

The first three chapters introduce a number of physical systems that exhibit this chaotic behaviour and describe useful analytical techniques. Chapter 3 derives a general expression for a diffusion constant for 2D maps of the torus and shows the very good agreement between theory and numerical simulation for two example maps. Chapter 4 shows analytically how this type of deterministic system can be equivalent to a random system without the addition of external noise. Chapters 5 and 6 extend the theory to parameter values where chaos and order coexist and where the dynamics modulate an imposed noise. Chapter 7 calculates the Lyapunov exponent for the one dimensional logistic map. Chapter 8 examines the accuracy of computer models and perturbation schemes via the shadowing property of hyperbolic systems.

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Acknowledgements

I would like to thank all the members of the Physics and Mathematics Departments at Warwick with whom I have had helpful discussions and to the Environmental Science Department for much need distractions. Thanks go to Professor J.P. Eckmann who suggested looking at Wojtkowski's paper and to the other participants at the Les Houches Summer School for chaotic stimulation. When a chemist, a physicist and an applied mathematician come together, chaos might be expected. However for Dr. David Broomhead, Dr. George Rowlands and myself a synergetic reaction seemed to occur. I thank Dave for sharing an office with me and helping to sort out the relevance of various mathematical ideas. George was both a supervisor and a friend : I thank him for his gentle guidance and for supplying transport to Sainsbury's.

Special thanks are due several friends : Peter Gowland with whom I shared many a bottle of whiskey ; Sheila Chapman who was my cotutor in Cryfield ; Jeff Ralph who has a similar appreciation of mathematics and physics to myself ; and especially Denise Oakley with whom I shared many happy times. My parents have been very helpful, supplying transport and storage room whenever needed.

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Declaration

Except where otherwise indicated, this thesis contains an account of my own independent research undertaken in the Department of Physics, University of Warwick between October, 1980 and October, 1983 under the supervision of Dr. G. Rowlands. Some of this work has appeared in the scientific literature in the following joint publications :

1. D.S. BROOMHEAD, G.A. MCCREADIE, and G.ROWLANDS, "On the Analytic Derivation of Poincaré Maps - the Forced Brusselator Problem," PHYS.LETT. Vol. **84A** p. 229 (1981).

2. G.A. MCCREADIE and G. ROWLANDS, "An Analytic Approximation to the Lyapunov Number for 1D Maps," PHYS.LETT. Vol. **91A** p.146 (1982)

In addition the following paper is in preparation :

G.A. MCCREADIE, "Physics, Computers and Chaos"

A man said to the Universe,

"Sir, I exist."

"However", replied the Universe,

"That creats in me no obligation."

Preface

Ever since the dawn of mankind, the human mind has attempted to impose an order, a rational form on the behaviour that it observes around it. These attempts stretch from the witch-doctor who claims that mysterious omnipotent beings control the elements and require apeasement with strange gifts; through the philosophers of the nineteenth century who claimed that, given sufficient information and time, they could predict the complete behaviour of the universe ; to the elementary particle physicists who describe matter as being composed of a small regular family of basic quantities. This methodology arises partially because it is man's nature to organise but also because he has few tools with which he can understand disorder or chaos. The devil is the lord of chaos - evil incarnate!

At the end of the nineteenth centuary, scientists believed that all of nature was within their grasp. The equations describing the motion of any system of particles could be written down and solutions attempted by algebraic means. These ideas began to crumble when solutions to the equations of astronomy were attempted using perturbation theory. The great philosopher/mathematician H.Poincaré found solution curves which were 'too complicated to draw or ennumerate'. The death knell was truely sounded when Einstein, Heisenberg and many others introduced Quantum Mechanics and the wave-particle duality of matter. Though even then it was believed that order must reign: " God does not play dice "

For a number of years the arguements concerning deterministic yet random behaviour was confined to quantum mechanics and the statistical mechanics of systems of very many particles, but in the past decade this dichotomy has reached into the heart of classical mechanics and is changing the way in which we view the world.

The Road goes ever on and on
Down from the door where it began.
Now far ahead the Road has gone,
And I must follow, if I can,
Pursuing it with eager feet,
Until it joins some larger way
Where many paths and errands meet.
And wither then ? I cannot say.

Bilbo's walking song : J.R.R. Tolkien

CHAPTER 0

Introduction

0.1. Background

The idea that highly disordered behaviour could arise from quite simple deterministic equations has slowly grown over the past two decades. This complicated motion, which appears random is seen in many nonlinear systems and in systems which are small perturbations of integrable equations. Many people noticed the possibility of complicated motion but ignored it, either because there were not available suitable techniques for studying it or because they thought it non-physical. An example of this is Poincaré's treatise[99, 100] on perturbation theory which later led to the KAM theory and many works on nonlinear perturbation theory. Poincaré obtained the complicated heteroclinic structure that surrounds periodic points in perturbations of Hamiltonian systems but stated that "The complexity of this figure will be striking, and I shall not even try to draw it" (vol 3, p382[100]). In recent years this structure has been completely elucidated both analytically and by computer plots. One of the first attempts to use computers to study the ergodic behaviour of a nonlinear system was Fermi, Pasta and Ulam's[40] numerical study of a one-dimensional nonlinear lattice. This study showed that equipartition of energy was not always true as long-time recurrences between modes were observed. Another approach was stimulated by Lorenz's[78] computer study of a 3-mode approximation to the Navier-Stokes equations which showed a strange attracting region, appearing to fill a volume in 3D space, which was nothing like the fixed points or limit cycles studied up to that time.

One of the major contributions to the observation of chaotic behaviour was the development of high speed computers which were available to many researchers. This was most noticeable from the late seventies when many research groups had their own minicomputer with a high-resolution graphical display. With these tools, the solutions of many equations could be seen immediately and large ranges of parameters rapidly investigated. No longer was it necessary to wait several days while cards were punched, sent to a special computing department and then returned with strange error messages on reams of computer output. This use of computers, often produced a synergetic effect where the numerics and the theory greatly aided the development of each other[129].

Most of the pictures in this thesis were produced on desk-top microcomputers, where the ease of program development and the subsequent rapid computations enabled many interesting features of many systems to be quickly evaluated. One such example is the observation of correlations in the maps studied in chapter 3 (Chirikov map) and the partial trapping of a trajectory near a heteroclinic orbit.

0.2. What is Chaos ?

There are many answers to this question in the literature, most are obtained heuristically but a few authors have attempted rigour. This question will be discussed in greater detail in chapter 8 but for the present a working definition is required.

> Chaos is the complicated bounded motion that is not a periodic orbit nor quasi-periodic motion on a torus.

Refering to the frontispiece, the trajectory originating at (0.1,0.1) and filling a two dimensional region of the picture is called chaotic while the central fixed point (1.0,0.0) is surrounded by one dimensional closed loops on which quasi-periodic motion occurs. The pictures in §0.4 are for an area-preserving map and show that single trajectories can 'fill' the whole phase space, as opposed to the simple closed orbits usually observed in such Hamiltonian-like systems. This occupation of a two dimensional region by a one dimensional curve will be made precise later.

In systems where the volume can change, chaos can be observed when attracting fixed points or limit cycles become locally repelling but there remains some mechanism of global attraction. In this case an object known as a ***strange attractor*** may be created, composed of one dimensional trajectories but occupying a region of space that has nonintegral dimension. These regions are like the ***fractals*** introduced by Mandelbrot[84].

0.3. Examples

This section details several physical systems which can be modelled by nonlinear difference equations. Often nonlinear equations are obtained in a heuristic fashion by choosing the simplest nonlinear addition to soluble linear equations. The examples here are obtained via justifiable approximations to the full model equations.

0.3.1. Particle in a Magnetic Mirror Device

The problem is to confine energetic charged particles using magnetic fields. A simple mirror device consists of a cylindrical field with increased field strength at the ends to provide a constriction that reflects particles back into the cylinder.

The following derivation is given in Chirikov[25] §4.3 and considers the simplest case when the magnetic field is axisymmetric and slowly varying . Similar results can be obtained for current experimental devices e.g. quadrupole stoppers or the Tandem Mirror Solenoid[26].

It is well known that a charged particle in a uniform magnetic field moves in a spiral about a line of magnetic force. The frequency of rotation, the Larmour frequency, ω is given by $\omega = \dfrac{eB}{mc}$. Choose units such that $e = c = m = 1$, then ω

= B. Let v_ϑ denote the angular velocity about a line of force and v_z the velocity along the line of force. The particle's motion can be characterised by the orbital magnetic moment $\mu = \frac{v_\vartheta^2}{2B}$ which is constant for a uniform field. If now the field varies slowly along a line of force e.g. $B(s) = B_0(1+b^2s^2)$ where s is distance along the line of force, then μ is approximately constant and particles spiral about the curved line of force. Since B(s) has a minimum the particle is confined in some region and executes periodic motion as if it were in a potential $U = \mu B(s)$. This is still a Hamiltonian system so energy is conserved and v_ϑ decreases as v_z increases. As μ decreases the potential barrier decreases and v_z increases until the particle is able to escape. Thus the confinement problem can be solved by obtaining bounds on the variation of μ.

The time evolution of μ is given by[57]

$$\frac{d\mu}{dt} = \frac{\rho v_\vartheta}{B}(v^2 - \frac{v_\vartheta^2}{2}) \sin\Phi - \frac{v_z v_\vartheta^2}{2B^2}\frac{\partial B}{\partial s}\cos(2\Phi) \qquad (0.3.1.1)$$

Φ is a perturbation phase, ρ - magnetic line curvature

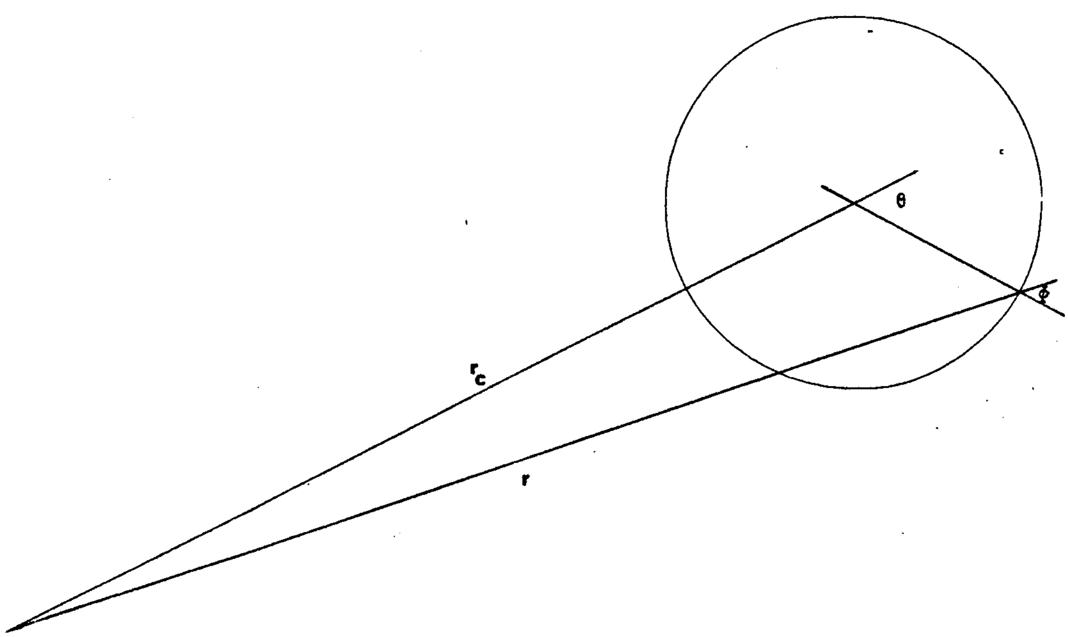


fig. 0.3.1 Coordinates for a particle in a magnetic mirror device

The closed curve is the projection of the trajectory onto a plane perpendicular to the symmetry axis. It is reasonable to assume that this projection is circular so that $r\sin\Phi = r_c\sin\vartheta$.

Rather than consider the differential equations it is usual to reduce the dimensionality of the problem and use a mapping obtained through use of a Poincaré section i.e. relate the value of (μ,ϑ) as the trajectory intersects some plane (e.g. s=0) to the values at the next intersection. For this problem, it is found by numerical simulation that μ is approximately constant apart from a jump near the mid-plane, s=0 between each bounce. Thus it is natural to use the nearly constant value of μ as a dynamical variable. The dynamical variables are μ for the motion before crossing s=0 with phase ϑ_0, becoming $\bar{\mu}$ after the crossing and before the next one with phase $\bar{\vartheta}_0$. In the following a subscript 0 will denote the value at s = 0.

To obtain the changes in (μ,ϑ) integrate the equations of motion over half the bounce period i.e. $\frac{\pi}{\Omega}$. When integrating (0.3.1.1), the second harmonic can be neglected as it gives an exponentially small contribution. Also, to lowest order, μ = const. and motion can be taken to be at r_c rather than r.

$$\therefore \quad \frac{d\mu}{dt} \approx \frac{v_{\vartheta 0}}{B_0}(v^2-\mu B)\sin\vartheta\,\rho(s)$$

This equation can be solved (see Chirikov[25] p296) and gives $\Delta\mu=\kappa(\mu)\sin\vartheta_0$ with $\kappa(\mu) \approx -\frac{3\pi r_{c0}v^2}{4\varepsilon a B_0}e^{-\frac{2}{3\varepsilon}}$. ε is the small parameter characterising the system and is given by $\varepsilon = ab\frac{\Omega}{\omega_0}$.

Since any trapped particles will execute harmonic oscillations (to a good approximation) let $s = a\sin(\Omega t)$ where Ω is the bounce frequency and $\Omega = bv_{\vartheta 0}$. Thus the average Larmour frequency is given by

$$\begin{aligned} \langle\omega\rangle &= \omega_0(1 + b^2\langle s^2\rangle) \\ &= \omega_0\left(1 + \frac{b^2a^2}{2}\right) \\ &= \omega_0\left(1 + \frac{1}{2}\left(\frac{B(a)}{B_0} - 1\right)\right) \end{aligned}$$

$$= \frac{\omega_0}{2}\left(1 + \frac{v^2}{2\mu B_0}\right)$$

since $\mu = v_{\vartheta 0}^2/2B_0 = v^2/2B(a)$. Thus the change in the phase angle ϑ_0 is given by

$$\Delta\vartheta_0 \doteqdot D(\bar{\mu}) \approx \frac{\pi}{\Omega}\langle\omega\rangle.$$

This gives a mapping :

$$\begin{aligned} \bar{\mu} &= \mu + \kappa(\mu)\sin\vartheta_0 \\ \bar{\vartheta}_0 &= \vartheta_0 + D(\bar{\mu}) \end{aligned} \tag{0.3.1.2}$$

Due to the various approximations made, (0.3.1.2) is not area-preserving (i.e. (0.3.1.2) does not represent a Hamiltonian system). This could be corrected by adding-in extra terms but it is simpler to use the smallness of $\Delta\mu$. Choose a resonant value of μ ,μ_r s.t. $D(\mu_r) = 2\pi n$ and then linearise about μ_r. Thus

$$\begin{aligned} D(\mu) &\approx D(\mu_r) + \frac{\partial D}{\partial \mu}(\mu_r)(\mu-\mu_r) \\ &= 2\pi n + I \end{aligned}$$

The new variable $I \equiv D'(\mu_r)(\mu-\mu_r)$ satisfies the area-preserving equation :

$$\begin{aligned} \bar{I} &= I + K\sin\vartheta_0 \\ \bar{\vartheta}_0 &= \vartheta_0 + \bar{I} \quad \mathrm{mod} 2\pi \end{aligned} \tag{0.3.1.3}$$

with all quantities evaluated at s=0 and $\mu=\mu_r$, and $K = \kappa(\mu_r)D'(\mu_r)$.

The map (0.3.1.3) is called the standard or Chirikov map[131] and describes motion in a mirror device about a resonance in μ . Similar equations have been derived for confinement in toral devices[109] and the interaction of particles in storage rings[60,119].

0.3.2. Atomic Lattices

An example that is naturally discrete but which has been treated for many years in a continuum limit is that of an atomic lattice interacting with a periodic potential. The model originally introduced by Frenkel and Kontorowa[42] consists of a 1D chain of atoms connected by linear springs sitting in a periodic cosine potential. If x_n is the position of the n^{th} atom, then the Hamiltonian for the system can be written:

$$H = \sum_n \frac{1}{2b^2}(x_{n+1}-x_n-a_0)^2 + V(1-\cos\frac{2\pi x_n}{b}) \tag{0.3.2.1}$$

Here a_0 is the natural lattice constant which in general will be incommensurate with the period, b of the potential V. The states $\{x_n\}$ of interest are those which minimise the energy. The condition for this is obtained by differentiating (0.3.2.1) giving $(x_{n+1}-x_n)-(x_n-x_{n-1}) = -4\pi^2 V\sin x_n$ where b has been set equal to 2π. Setting $w_n = x_n - x_{n-1}$ gives :

$$w_{n+1} = w_n - V'\sin x_n$$
$$x_{n+1} = x_n + w_{n+1}$$

namely the standard map. This approach has been used by Bak[20] to study metal-insulator transitions in Peierls systems and adsorbed monolayers (e.g. Krypton on graphite).

In general, the periodic potential attempts to 'lock' the system into a commensurate configuration with wavevector q ($= \frac{2\pi}{a}$) rationally related to b. As some parameter (e.g. natural frequency a_0) is changed the wavevector will change and tend to lock-in to a number of different commensurate phases. If an infinite number of different phases occurs, the plot is called ' the devil's staircase ' [6]. Referring to the frontispiece, the physics corresponding to different trajectories are :

i) the hyperbolic fixed point at (0,0) is a commensurate phase. The elliptic fixed point is unphysical as it maximises energy .

ii) the smooth invariant curves are the incommensurate phases. The smooth ellipses are energetically unfavourable.

iii) the chaotic trajectories are a random combination of pinned solitons and antisolitons.

Bruce[19] has compared the discrete model with the continuum approximation and has found similar results for both models except that the discrete case allows the formation of a devil's staircase. Other potentials than the cosine have been investigated : the quartic double well by Janssen and Tjon[65] and the piecewise quadratic by Allroth and Muller-Krumbhaar[2] who also considered more than nearest-neighbour interactions.

This type of model has also been used to study DNA molecules[121] since each base pair exists in a periodic potential. Thus solitons and chaotic phases exist at the very basis of life.

0.3.3. Bouncing Balls

The previous examples are both derived from Hamiltonians and are thus area-preserving. Rather than immediately considering a general dissipative system, it is useful to study a perturbation of the area-preserving case :

$$\begin{aligned} v_{n+1} &= \lambda v_n + K\sin x_n \\ x_{n+1} &= x_n + v_{n+1} \mod 2 \end{aligned} \tag{0.3.3.1}$$

For $\lambda < 1$ this map is dissipative and motion is not expected to fill the cylinder $[-1,1)x(-\infty,\infty)$, however it can fill a 'large' subset, in fact a set of non-integer dimension between one and two. This system can also be written in the form :

$$\begin{aligned} x_{n+1} &= (1+\lambda)x_n + \varepsilon f(x_n) - \lambda u_n \\ u_{n+1} &= x_n \end{aligned} \tag{0.3.3.2}$$

These equations are reminiscent of the Henon[58] or Lozi[80] maps.

Equations (0.3.3.1) have been obtained by Holmes[62] from the equations describing a ball bouncing on an oscillating table (which is similar to sheet metal moving over rollers[127]). Holmes was able to show that for λ near 1.0 and K sufficiently large there exists a 'horseshoe' in phase space and thus motion is topologically complicated. A similar problem has been studied by Zaslavskii and Chirikov[130] who obtained the exact discrete equations describing a ball bouncing between a stationary and oscillating wall. The 2D approximation to their equations is :

$$\begin{aligned} v_{n+1} &= v_n + K\sin x_n \\ \varphi_{n+1} &= \varphi_n + \frac{M}{v_{n+1}} \mod 2 \end{aligned} \tag{0.3.3.3}$$

The reciprocal dependence on v_{n+1} arises because in this case the ball moves linearly between impacts while in the Holmes case it moves quadratically under gravity.

Brahic[14] has made a numerical study of several different wall motions in the 2 wall case and finds an upper bound on the motion even when area-preserving (i.e perfectly elastic collisions). For the 1 wall case Pustylnikov[102] has shown that for $\lambda = 1$ and any analytic wall motion, the velocity is unbounded (i.e. accelerator modes §3.3.3)

Equations (0.3.3.3) are similar to those obtained in a model for cosmic-ray acceleration proposed by Fermi and studied by Lieberman and Lichtenberg[75] and more recently Lichtenberg, Lieberman and Cohen [74]. Equations (0.3.3.1) have been extended by Howard, Lichtenberg and Lieberman[63] to include a second forcing frequency which greatly increases the velocity at which the first adiabatic barrier occurs : these equations model electron cyclotron resonance heating.

0.3.4. Lorenz model

When modelling any system using differential equations it is very common to arrive at systems of first order autonomous (, no explicit time dependence) differential equations that are coupled in some complicated nonlinear way. In order to solve these equations it has been standard to find a fixed point and then linearise the equations. This can only produce a local non-chaotic solution. It is now feasible to study the qualitative features of such systems using high-speed computers. While it is often necessary to approximate the non-linearities by simple forms e.g. xy^2 , it is possible to observe very complicated behaviour and construct approximations to this behaviour [104].

One of the first systems to be studied was the Lorenz model[118] :

$$\dot{X} = -\sigma X + \sigma Y \qquad (0.3.4.1)$$
$$\dot{Y} = -Y + rX - XZ$$
$$\dot{Z} = -bZ \qquad + XY$$

These equations were obtained as an approximation to certain aspects of atmosphere dynamics [78]. Similar equations can be derived exactly for the Rikitake 2-disc dynamo[30] which may model the reversals of the Earth's magnetic field. The solutions of these equations are described in detail by Sparrow[118]. Essentially an orbit spirals out from one of two unstable fixed points until a saddle point causes an irregular transition into the domain of the other unstable fixed point, whereupon the motion continues in a like fashion.

Another example is the Oregonator model of the Belousov-Zhabotinski chemical reaction [41]. When well-stirred, this reaction shows erratic oscillations between products which are easily observed as colour changes. In addition, if the chemicals are poured into a flat dish, regular spatial structure (spirals) can be observed which can be partially explained in terms of nonlinear reaction-diffusion equations.

Even with the use of powerful computers it is difficult to solve or visualise systems of equations in many dimensions, thus it is necessary to reduce large systems to only a few dimensions (but at least 3 to observe chaos). One powerful method is the 'slaving principle' proposed by Haken[55] to study synergetic systems. Here many simple systems couple to produce a solution more complicated than any of its subcomponents but which may be described in a low dimensional subspace. The slaving principle says that most of the dynamical variables converge to an attracting solution, however this solution will in general be described by 3 or more variables and will thus tend to exhibit complicated behaviour.

As the above systems are described by limiting motion in a bounded region, a result of Poincaré says that any orbit is recurrent. This leads naturally to a reduction in the dimensionality of the problem by choosing some characteristic surface and studying the intersections of an orbit with that surface. The discrete mapping so generated contains all the information about the full equations (given certain smoothness conditions, see Channell[24]). This Lorenz section generalises the idea of a Poincaré section which relies on an explicit periodicity in the continuous equations, sampling at some multiple of this periodicity to produce a discrete map. (e.g. the forced Brusselator[120]) .

0.3.5. One Dimensional Maps

The simplest equations exhibiting chaos are ' maps of the interval'. Here a one dimensional nonlinear equation describes how the unit interval is mapped onto itself. These equations have been extensively studied[28, 38, 89] with conditions for being obtained for large classes of functional forms and many types of behaviour being characterised before being observed in higher dimensions (e.g. period-doubling occurs in 2D[9, 51] but has characteristic exponents different from those in 1D[37]).

The simplest chaotic equation $x' = \lambda x(1-x)$ is a discrete time approximation to the logistic equation used in biology to describe a population with saturation, but recently this equation has been used to describe many other processes e.g. RC circuits [54].

Work has also been done on smooth 1D equations with nonlinearities as cubic polynomials[117] , quartic polynomials[23] and the sine function[130] and on the many forms of piecewise linear maps[44, 52, 92].

0.4. Numerical Observations

The previous section detailed several systems which could be modelled by nonlinear difference equations. It is the purpose of this section to show, via plots of the phase space, how the solutions of these equations differ from the more usual continuous case and why the motion might be called random. This thesis is largely concerned with the area-preserving case, so the illustrations will be of

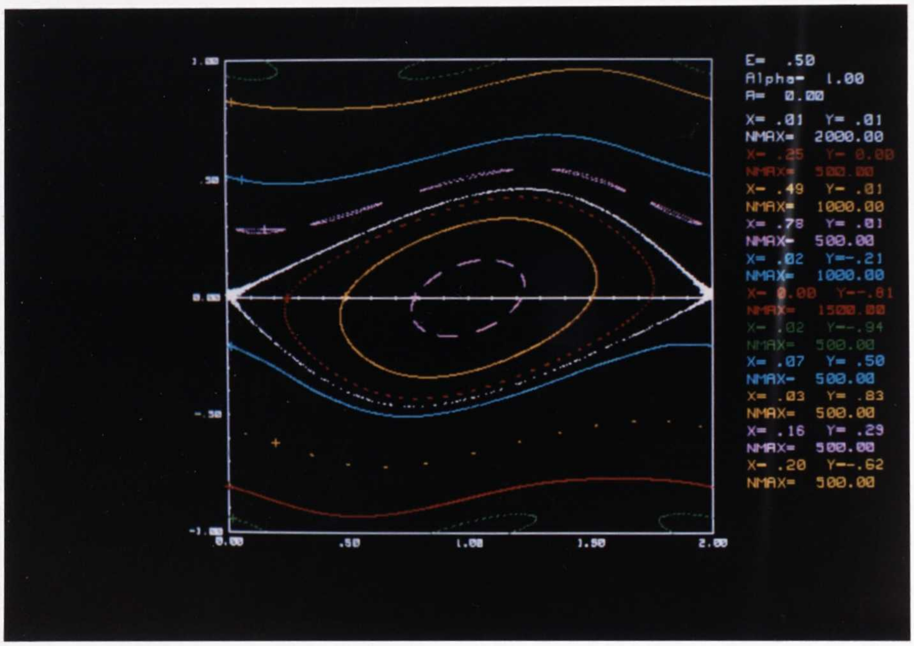


fig. 0.4.1 The Chirikov map: $E = \frac{0.5}{\pi}$

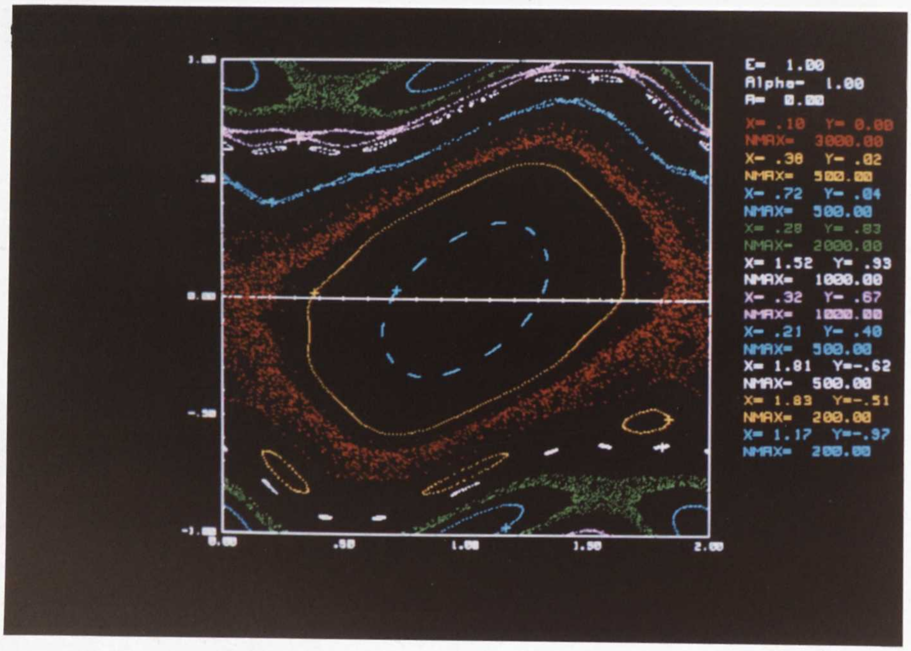


fig. 0.4.2 The Chirikov map: $E = \frac{1.0}{\pi}$

the Chirikov map (0.3.1.3) and a piecewise linear map :

$$v_{n+1} = v_n + EF(x_n;a) \qquad (0.4.1)$$
$$x_{n+1} = x_n + v_{n+1} \quad \mod 2$$

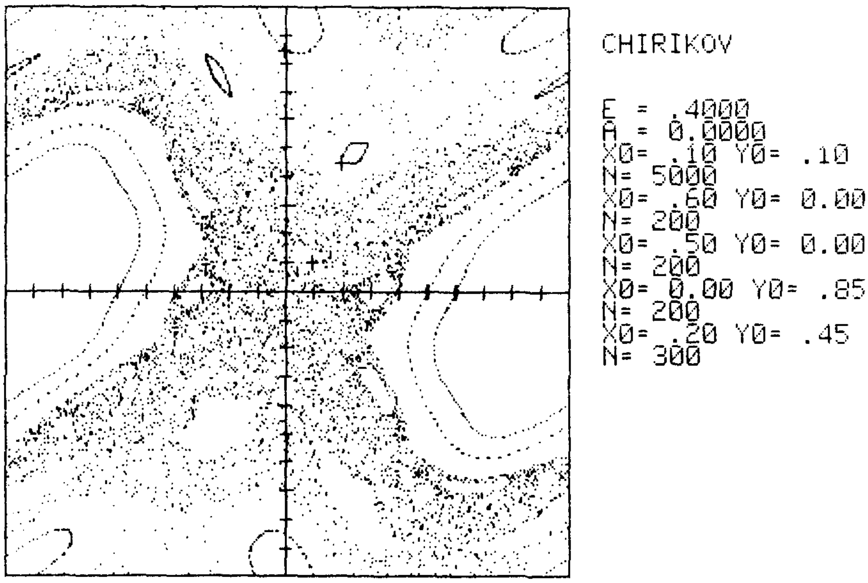
where F(x;a) is defined for $x \in [-1,1]$ and given by ,

$$F(x;a) = \begin{cases} \frac{1+x}{a-1} & x < -a \\ \frac{x}{a} & |x| < a \\ \frac{1-x}{1-a} & x > a \end{cases} \qquad (0.4.2)$$

For small values of the parameter E, these equations can be considered as a discrete approximation to the differential equations $\ddot{x} = -KF(x;a)$. $F(x) = \sin(\pi x)$, gives the equations for a pendulum with the standard phase plane plot : closed loops about x = 1 corresponding to simple oscillatory motion ; a separatrix from 0 to 2 ; lines unbounded in x which give rotations about the point of suspension. This type of phase plane plot is retained for the discrete map as can be seen from fig. 0.4.1.

[Aside : the parameter E in the photographs is π times the E of the figures. The parameter Alpha can be used to give non-area-preserving maps but for most of this thesis Alpha will be 1.0. Further details of the computations are in appendix B.]

Poincaré and Birkhoff proved that arbitrary small perturbations of Hamiltonian systems would cause the breakup of certain of the invariant curves into periodic points. For example, (0,1) is a point of period 2 which is not observed in the unperturbed differential form of the equations. The other new feature observed in discrete systems is the occurrence of a broad 'chaotic' orbit along the separatrix. The special properties of the separatrix will generate chaos in any map : this property - hyperbolicity , will be discussed in chapter 1. Thus even for small perturbations, there are definite differences between the differential and difference phase plots.



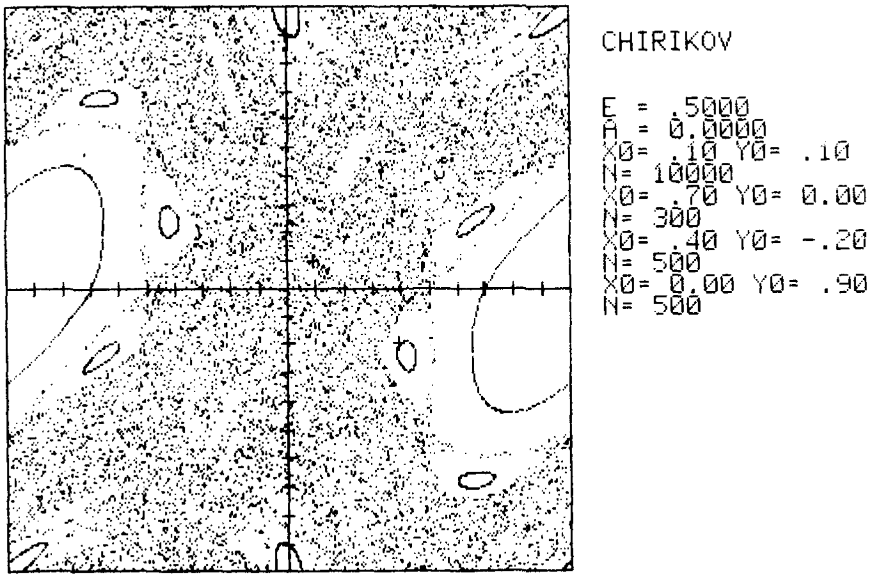
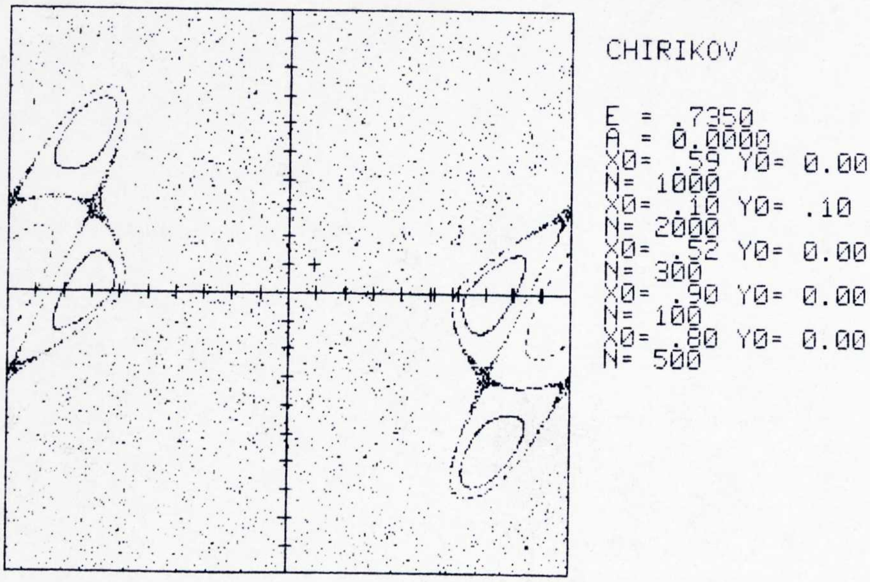


fig. 0.4.3 The Chirikov map:

a) E=0.4 b) E=0.5

As the perturbation is increased, the periodic orbits become more noticeable and the chaotic orbit about the main separatrix joins other chaotic orbits about the periodic points (fig. 0.4.2). For many years the existence of the curves that prevent motion in the vertical direction could not be shown. Their existence is connected with the problem of small divisors that affects many perturbation schemes and was proven by Kolmogorov, Arnold[4] and Moser[93] who gave a method of resummation for the perturbation schemes. Originally the proof required the existence of a large number of derivatives of the perturbation and E to be very small. It was soon possible to reduce the number of derivatives to three, but only recently has the proof been able to give realistic values for a critical value of E[59, 77].

The calculation of the critical value E_c at which the last invariant K.A.M. curve breaks up and allows unbounded motion in the v-direction has been refined to a high degree. The two methods commonly used are : resonance overlap[25, 36] and the change in stability of periodic orbits[50]. For the Chirikov map, the standard value for E_c is approximately $\frac{1}{\pi}$. Thus both the frontispiece and fig. 0.4.2 should show connected motion in the v-direction. The frontispiece shows an orbit of 15000 points which has drifted away from the origin but has become trapped around a period 3 resonance, thus numerical estimates for E_c put its value several percent higher than the analytic estimate. This may be due to the way in which the K.A.M. curve breaks up : an infinite number of very small gaps appear in the curve, thus it may still appear continuous to the orbits - the ghost of the curve remains[81].



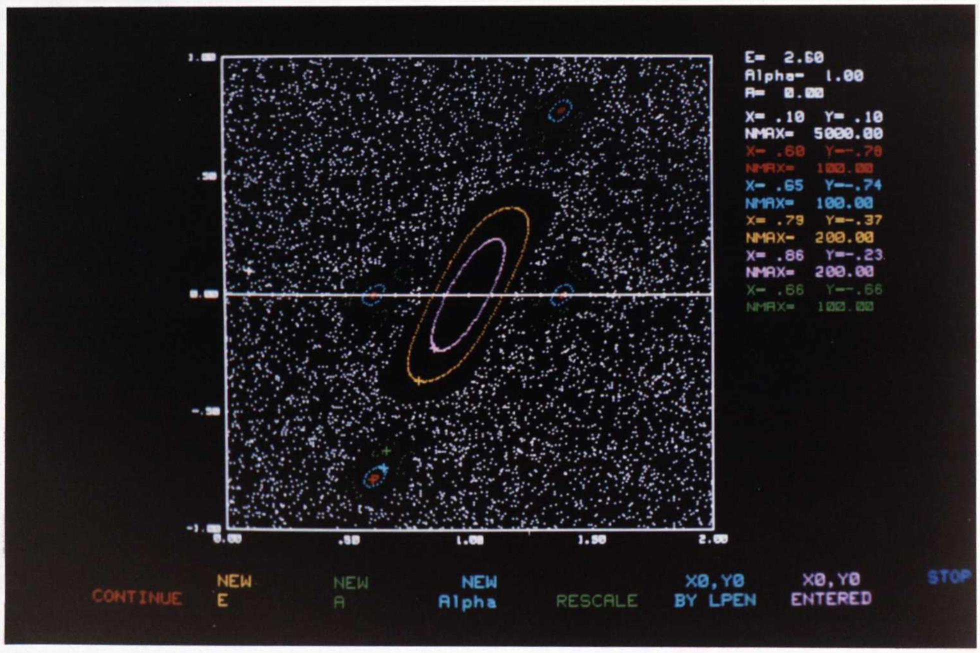


fig. 0.4.4 **The Chirikov map:**

a) E=0.73 **b)** $\mathbf{E=\frac{2.6}{\pi}}$

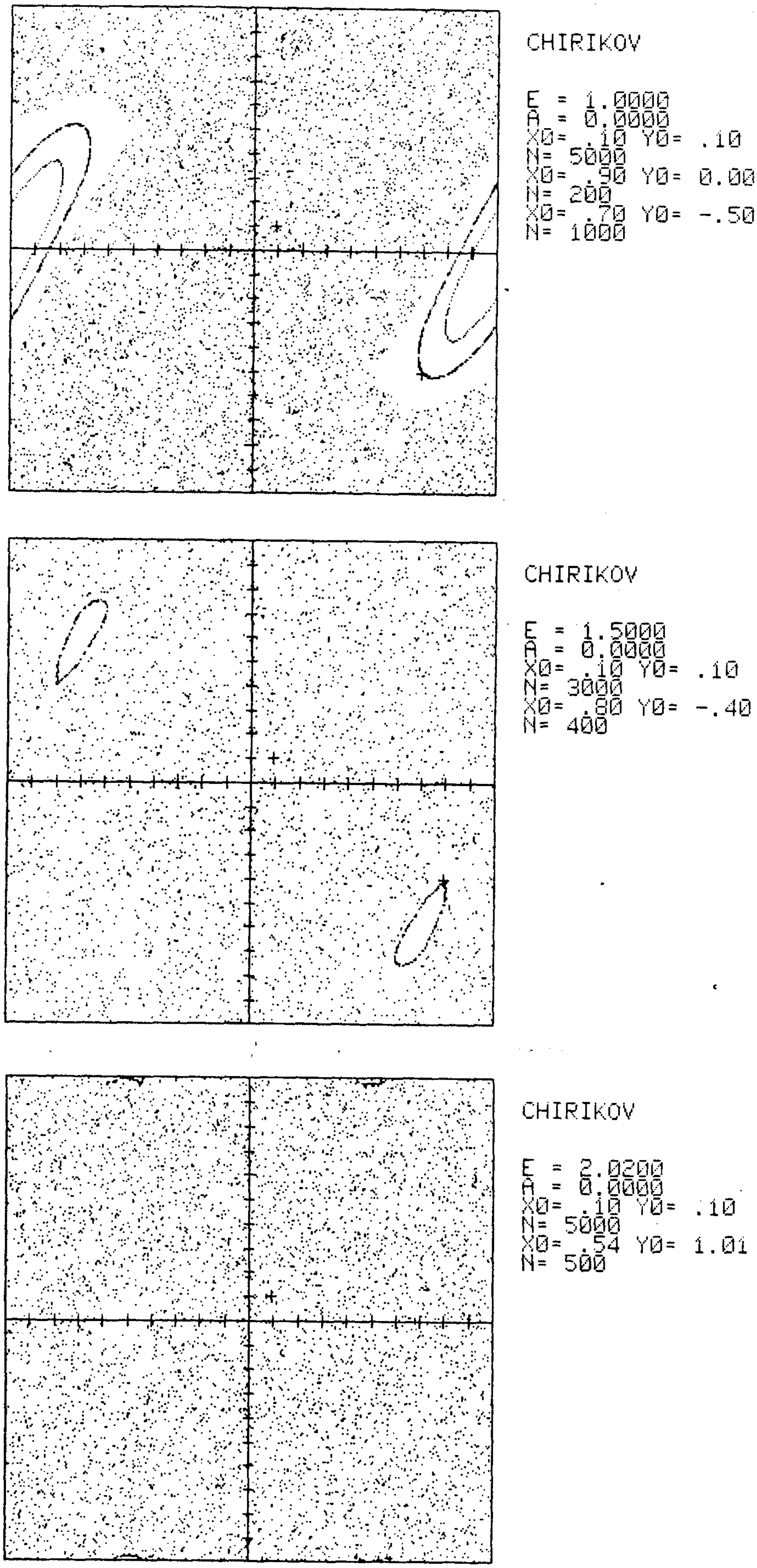


fig. 0.4.5 **The Chirikov map:** **a) E = 1.0** **b) E = 1.5**

c) E = 2.02

As E is increased, the chaotic bands produced by the heteroclinic orbits about the elliptic points grow and coalesce. Figures 0.4.3, 0.4.4 and 0.4.5 show the gradual change in structure. In fig. 0.4.3, while the period one elliptic island remains dominant, many higher periodic orbits appear and disappear e.g. a chain of elliptic islands of period six encircling the period one island. In fig. 0.4.4 the period two islands have disappeared and while the central island is dominant, there is a range of parameters for which the period four chain about the main island greatly increases the region of phase space over which ordered motion is dominant. Thus there are sharp changes in the volume of phase space dominated by ordered motion : this will be seen in the results of chapter 5. Fig. 0.4.5 shows the filling of the complete phase space by chaotic motion. Here the period one fixed point bifurcates giving two islands which have orbits alternating between each island. When this period two orbit bifurcates, rather than producing an orbit of period four, two orbits of period two are formed. This is due to the special structure of the Chirikov map (see MacKay[82]). This period doubling sequence now continues in the standard fashion and universal constants may be obtained[9,51] though these constants are different from those observed in 1D[37].

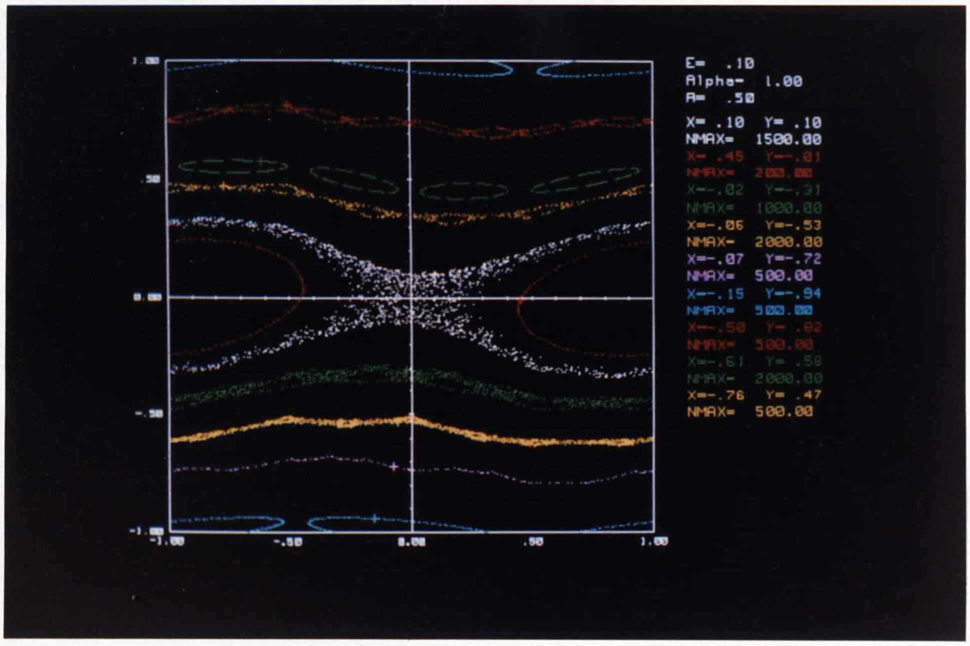
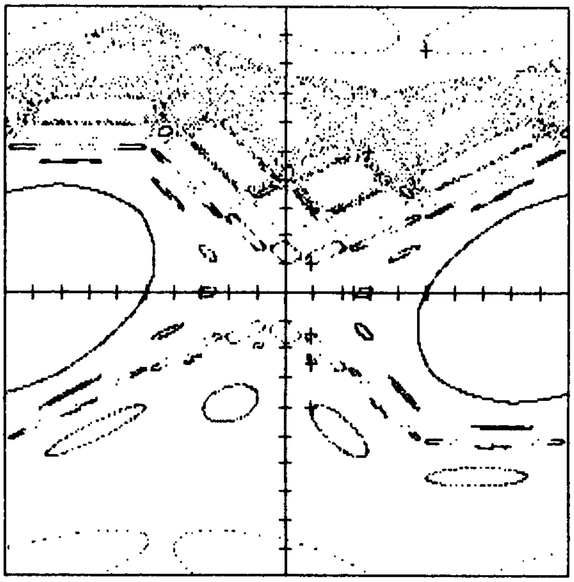


fig. 0.4.6 The piecewise linear map: E=0.1 a=0.5

Fig. 0.4.6 shows the piecewise linear map for a small parameter value. Comparing this figure with the Chirikov map in fig. 0.4.1 demonstrates the effect that reducing the number of derivatives produces, namely chaos is more prevalent. However the global properties are very similar, as might be expected from the similar form of the forcing function. The piecewise linear map does not undergo as many period-doublings as the Chirikov map, rather elliptic islands decrease in size and then disappear. This simplifies the description of the motion around the islands which will be given in chapter 5. Little work has been done when only a few islands are present since the perturbation theories which work for small and large E are not expected to work in this region : chapter 5 suggests that the standard techniques may be successfully extrapolated into this region.

Fig 0.4.7a shows the highly regular structure that is present for small E values in the piecewise linear case. The existence of high order periodic points is given by Poincaré and Birkhoff's theorem, the structure of the islands about these points is dictated by the linearity of the perturbation. Under a slightly larger perturbation, much of the linear structure disappears and the phase space is similar to that observed for the smooth Chirikov map. The form of the map allows highly singular behaviour. For example fig. 0.4.8b shows a chaotic region bounded by straight lines. Motion in the ordered region is singular, every point being a point of low period. Perturbing either of the parameters, E or a produces a more usual picture. Figures 0.4.9a and b show the reappearance of highly regular structure due to a resonance in the parameters which vanishes in fig. 0.4.9c. After E = 1.0 only the ordered region associated with period one remains. At this value, a singular structure is again observed which soon becomes an elliptic island (fig. 0.4.10a). The change in size of this single elliptic island with parameter is given in chapter 5. This is a smoothly varying function of parameter and does not include the singular behaviour observed at E = 1.0, 1.5, etc... . These values form a set of measure zero which is given in Wojtkowski[125]. This set is important because for these values the motion outside the polygon can be shown to be Bernoulli and to fill the chaotic region. As E is increased further still, all the periodic points become unstable and most orbits will wander throughout phase space - motion is ergodic.



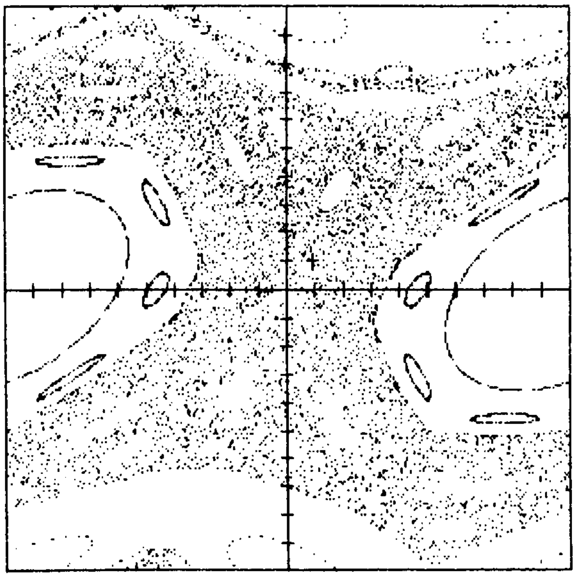


fig. 0.4.7 The piecewise linear map:

a) E=0.25 a=0.5

b) E=0.33 a=0.5

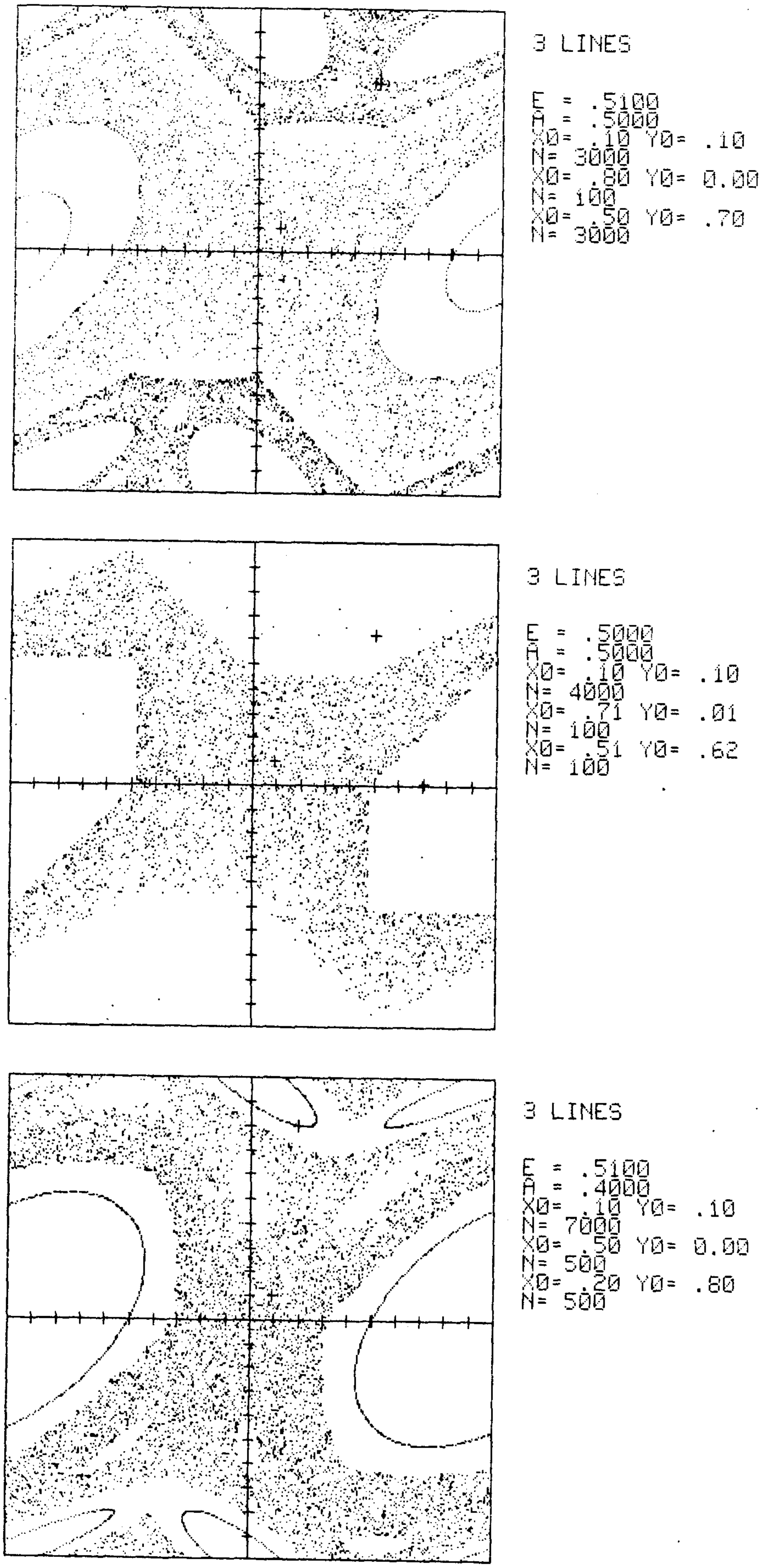


fig. 0.4.8 The piecewise linear map: a) E=0.51 a=0.5

b) E=0.50 a=0.5 c) E=0.51 a=0.4

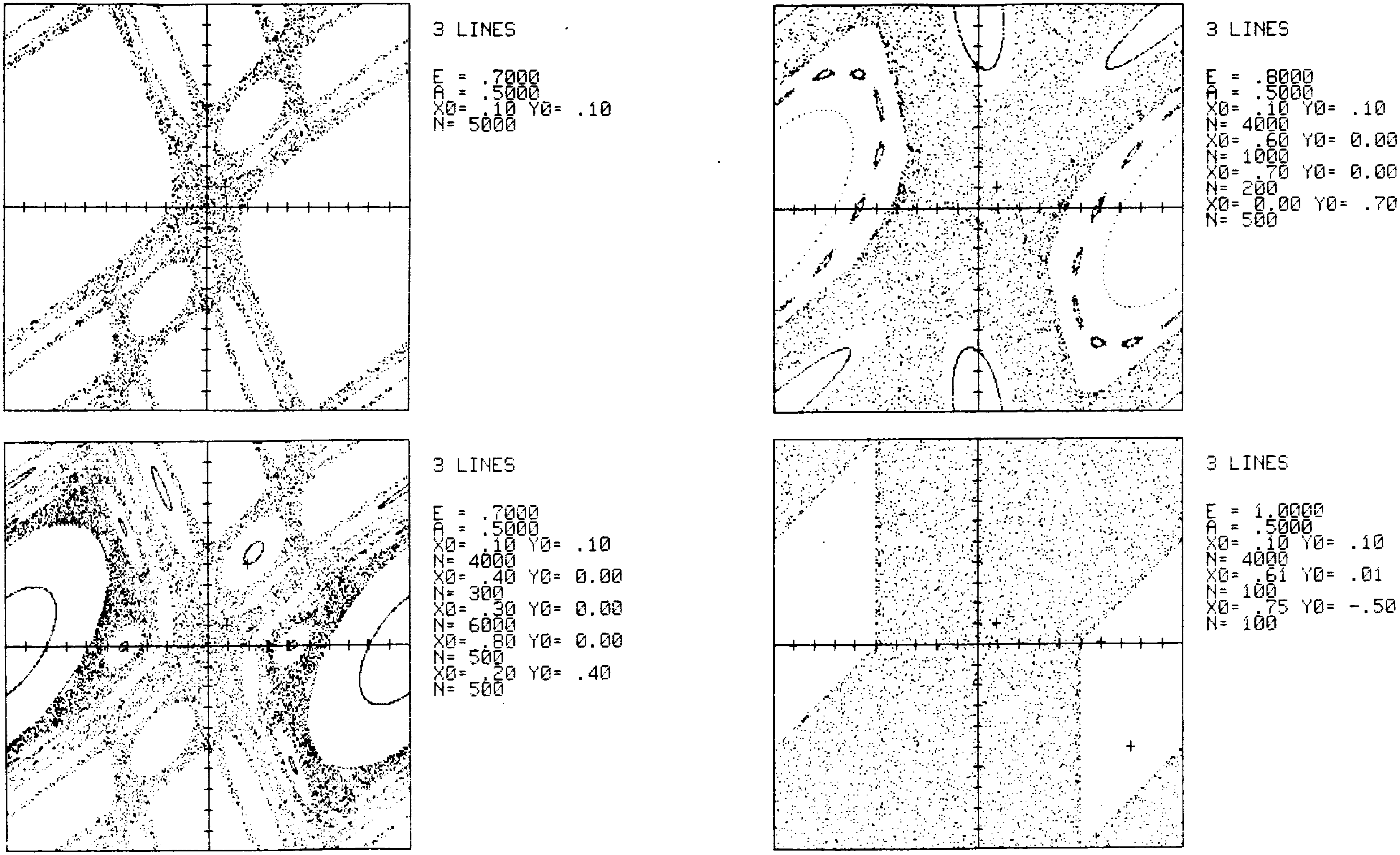


fig. 0.4.9 The piecewise linear map: a=0.5 a) E=0.7 1 orbit b) E=0.7 c) E=0.8 d) E=1.0

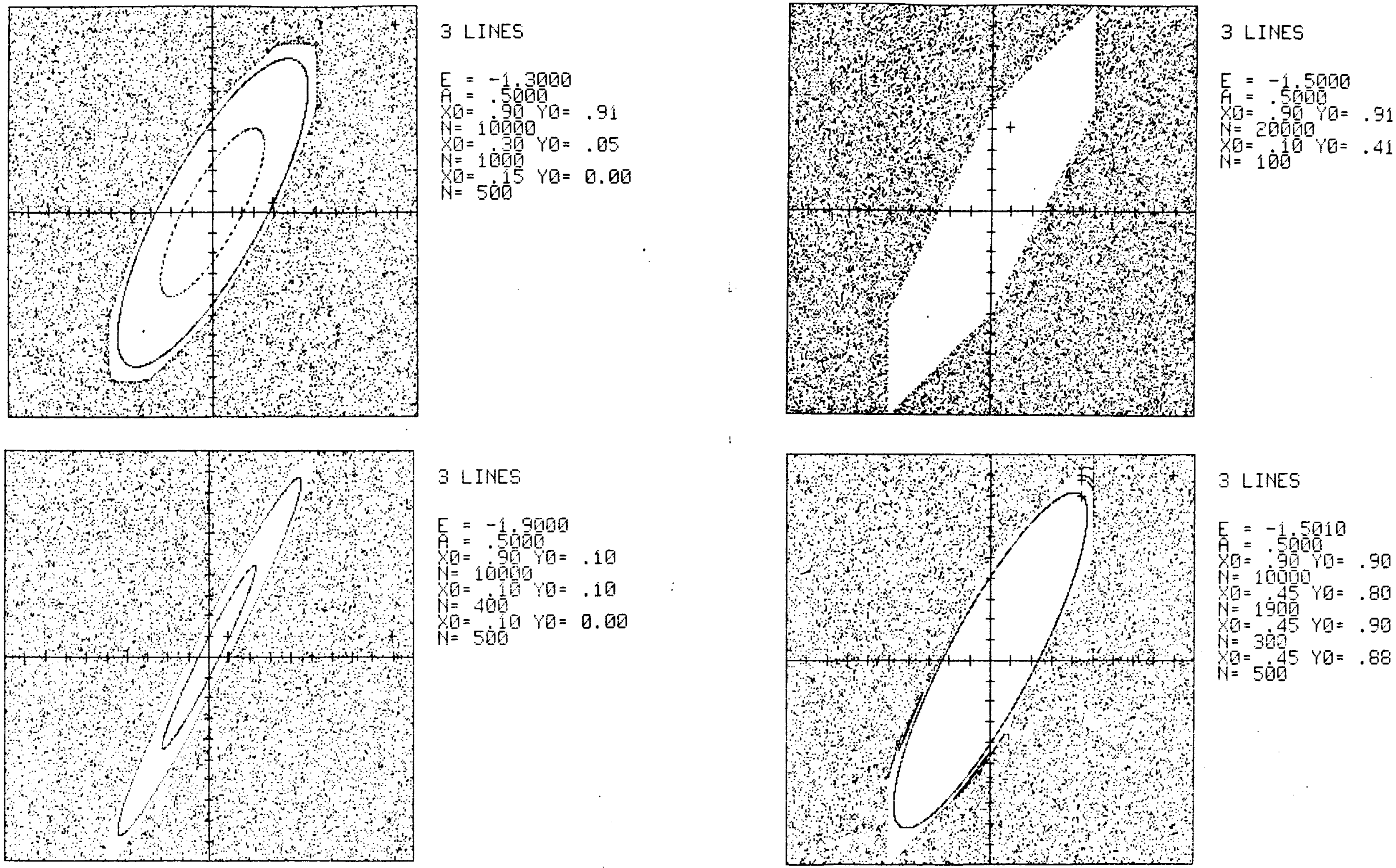


fig. 0.4.10 **The piecewise linear map: a=0.5 a) E=−1.3 b) E=−1.5 c) E=−1.9 d) E=−1.501**

0.5. Overview

Chapters 1 and 2 introduce necessary background material : chapter 1 describes certain mathematical techniques including the idea of hyperbolicity that will be needed in the proofs of chapter 4 ; chapter 2 describes some aspects of diffusion theory and also some results about distribution functions that suggest that the numerical calculations performed in chapter 3 are reasonably accurate.

Chapter 3 successfully calculates the average square spread in the v-direction as a function of E and advocates the description of the dynamics as a diffusion process. The good agreement observed between the theoretical results and numerical simulation encourages the application of rigorous methods to show that the deterministic equations can be explained in terms of stochastic variables. Thus chapter 4 proves that the piecewise linear map is a Bernoulli process and hence there exists a one-to-one transformation of the dynamics to a Markov process. It is possible that this process may satisfy a diffusion equation. It is still not possible to give any proofs for the Chirikov map, indeed there are numerous parameter ranges in which such properties are certainly not globally true. These parameter ranges are for E slightly larger than a multiple of 2. Here stable island structures are observed on the torus (fig. 3.3.3.1) which correspond to orbits whose v-coordinate increases by a multiple of 2 at each iteration : these islands are called accelerator modes[25,70].

Chapter 5 continues to model the dynamics as a diffusion but considers the situation when islands are present. An analogy is drawn between the map dynamics and the physical problem of conduction through a porous medium. In this way a simple model is obtained which explains some of the observed behaviour. In addition it is shown that the methods of chapter 3 can, in some instances, still be applied in regions where the perturbation expansion in $\frac{1}{E}$ might be expected to break-down.

Chapter 6 examines the small E region as a perturbation expansion in E. Results to order E^2 can easily be obtained but it is necessary to go to order E^4 to explain numerical observations[106]. This extension requires the use of a computer algebra package - REDUCE - to calculate various expansions and summations. Unfortunately this is a large problem and proved too large for the local mainframe. The method of attack for this problem when using REDUCE is given so that the calculations may be performed once a larger machine is available.

A one dimensional map - the logistic map - is considered in chapter 7 and an analytic expression obtained for its Lyapunov exponent in terms of periodic orbits. Chapter 8 returns to the problem of defining and numerically observing chaos. There are mathematical results and computations[7,8] that suggest that chaotic motion is stable (in a special sense) to perturbations and hence computer simulations are valid. Also discussed are several attempts at applying standard perturbation theory to chaotic systems to obtain dimensions or Poincaré sections. These methods cannot be rigorously justified but agree remarkably well with numerical results. This work is concluded with a summary of results and some comments on further problems that might be tractable.

CHAPTER 1

Some Mathematical Ideas

1.1. Purpose

The study of chaos is one of the areas in which there is great possibility for interaction between mathematics and the practical sciences. However much of the mathematics required is at a very abstract level and not readily available to the non-mathematician. Thus one of the goals of this thesis is to document some of the new methods together with examples arising in the sciences. The main target for the contents of this work are the practical scientist who has obtained a discrete model for some real system which exhibits possible chaotic behaviour and requires some tools with which to study the model.

The main problem is the characterisation of asymptotically complicated behaviour that appears to occupy large regions of phase space. Rather than study individual trajectories it is more useful to study certain averages. This is one aspect of Ergodic theory and can be stated as 'do time averages equal space averages ?'.

This chapter introduces many of the precise mathematical definitions used in the remainder of the thesis. Obviously much has been omitted and it is assumed that the reader has a familiarity with measure theory and differentiable dynamics. Most mathematical results are for very precise classes of equations which rarely include the models of practical interest, however the results are highly suggestive of extensions that may be proved in future. There is also the feedback from the numerical results that can suggest the type of theorems to be attempted. This feedback has been of great use in the study of one dimensional maps as many new techniques have been developed to study the logistic map and prove the existence of invariant measures with the special properties observed numerically. This interaction is starting to produce useful results in 2D. Chapter 3 presents numerical results agreeing with an approximate theory. Chapter 4 will give rigorous results to explain this agreement.

General reviews are : ergodic theory - Walters[123], Cornfeld, Fomin and Sinai[31], Halmos[56] ; symbolic dynamics and partitions - Bowen[12], Alekseev and Yakobsen[1] ; dynamical systems and ergodic theory - Arnold and Avez[5], Lanford[72].

1.2. Ergodicity and Mixing

Let X be a compact metric space and $\mathbf{B}(X)$ the Borel algebra of X. If $\mathbf{M}(X)$ is the set of normalised measures on X, then a continuous transformation $T{:}X \to X$ is called *measure-preserving* if $m(A) = m(T^{-1}A)\ \forall\, A \in \mathbf{B}(X), m \in \mathbf{M}(X)$. For example, by Liouville's theorem any Hamiltonian system preserves the natural smooth measure. A measure-preserving transformation is called *ergodic* if the only members $B \in \mathbf{B}$ with $T^{-1}B = B$ satisfy $m(B) = 0$ or 1. Thus an ergodic transformation cannot be decomposed into simpler transformations on subspaces of X. This property is related to averages via **Birkhoff's Ergodic Theorem.** If T is measure-preserving and $f \in L^1(m)$ then

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T_i x) \to_{\text{a.e.}} f^* \in L^1(m)$$

such that $f^*.T = f^*$ a.e. and $\int f^*\,dm = \int f\,dm$ if $m(X) < \infty$.

If T is ergodic then f^* is constant a.e. and $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{i=0}^{n-1} f(T^i x) = \int f\,dm$ a.e.

Another statement is :

$$T \text{ is ergodic} \Leftrightarrow \forall\, A,B \in \mathbf{B}\ \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} m(T^{-i}A \cap B) = m(A)\,m(B)$$

Thus, on average $T^{-n}A$ becomes independent of B. A stronger result would be to require asymptotic independence i.e.

$$T \text{ is } \textit{mixing} \Leftrightarrow \lim_{n\to\infty} m(T^{-n}A \cap B) = m(A)\,m(B)$$

Rather than look for measure-preserving transformations T, it is more usual to start with a continuous map T and find *invariant measures* $\mu \in \mathbf{M}(X)$ so that T is a measure-preserving transformation of $(X, \mathbf{B}(X), \mu)$. Define the set of measures invariant with respect to T, $\mathbf{M}(X,T) = \{\mu \in \mathbf{M}(X) : \mu(T^{-1}B) = \mu(B) \ \forall\ B \in \mathbf{B}(X)\}$. It can be shown that $\mathbf{M}(X,T)$ is non-empty, in fact if $\{\sigma_n\}_1^\infty$ is a sequence in $\mathbf{M}(X)$, form a new sequence $\{\mu_n\}_1^\infty$ where $\mu_n = \frac{1}{n}\sum_{i=1}^{n-1} \tilde{T}^i\sigma_n$ then any limit point μ of $\{\mu_n\}$ is a member of $\mathbf{M}(X, T)$ $[\tilde{T} : \mathbf{M}(X) \to \mathbf{M}(X)$ is given by $(\tilde{T}\mu)(B) = \mu(T^{-1}B)]$.

For T continuous, a measure $\mu \in \mathbf{M}(X,T)$ is ergodic or mixing if T has the corresponding property on $(X,\mathbf{B}(X),\mu)$. Thus there are the following equivalent definitions.

A measure $\mu \in \mathbf{M}(X,T)$ is called ergodic if :

a) $\forall\ A,B \in \mathbf{B}(x) : \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \mu(T^{-i}A \cap B) = \mu(A)\mu(B)$

b) $\forall\ f,g \in L^1(\mu) : \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \int f(T^i x)g(x)\, d\mu(x) = \int f d\mu \int g d\mu$

c) $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x} = \mu$ μ–a.e. where δ_y is the Dirac δ-measure concentrated at the point y.

In general, there will be several ergodic measures for a given map T, each singular with respect to the others (i.e. if $\mu_1(E) = 1$ then $\mu_2(E) = 0$). Each measure will be supported on a dense orbit $\{T^i x\}_0^\infty$ but a further condition must be imposed to obtain the relevant orbit. Physical measurements are related to the Lebesgue measure, m so it is useful to obtain measures, μ that are absolutely continuous with respect to m ($\mu \ll m$) i.e. μ has the same sets of measure zero as m. If μ is ergodic and $\mu \ll m$ then μ is unique.

As examples consider first a toral translation : $T(x,y) = (x+p, y+q) \bmod 1$ with measure dxdy. An orbit of T is dense and the measure ergodic if p/q is irrational. Consider an initial point (x_0,y_0) and a small rectangle about it. Under iterations of T, the rectangle will move over the whole torus covering it infinitely often and infinitely often returning to a neighbourhood of (x_0, y_0) but all the time remaining as a rectangle of constant shape and size. Thus on average, but not asymptotically, a region of the torus will be visited proportional to its measure.

Now consider the toral automorphism : $T(x,y) = (x+y, x+2y) \bmod 1$ with measure dxdy. This induces a linear map of the plane, $\tilde{T}$ with eigenvalues $0 < \lambda^{-1} < 1 < \lambda$. For large n, the set $\tilde{T}^n A$, $\forall A$ will appear as a long ribbon with gradient λ, this folds onto a dense helix on the torus. Thus any initial set follows a dense orbit on the torus but also is stretched in such a fashion that it densely fills the torus. This map is mixing.

1.3. Application to dynamical systems

The problem of rigorously describing the behaviour of a dynamical system is very difficult : only recently have methods become available that partially describe systems exhibiting chaos. Two approaches have been developed :

1. *topological dynamics* studies the evolution of individual points and obtains complicated behaviour on sets that are dense in the space but which may be small in measure ;
2. *measure theoretic* methods obtain the existence of limiting distributions and describe some of their properties. These distributions allow the calculation of quantities that might be measured in an experiment and thus are the methods most suited to describing chaos as regards this thesis.

To show chaos a typical method of attack might be :

a) show the existence of complicated behaviour on a subset of the space (often topological methods are used for this),

b) show the existence of a 'smooth' measure which gives equality of space and time averages, and which is defined on the topologically complicated set

c) extend (a) and (b) to a large (in the measure sense) subset of the space or to an attracting subset.

1.3.1. Hyperbolicity and Stable Manifolds

The local behaviour of a transformation, f is determined by the eigenvalues, λ_i of the derivative matrix at a point. This is adequate to describe sources ($|\lambda_i| < 1$), sinks ($|\lambda_i| > 1$) or elliptic motion ($|\lambda_i| = 1$) but for the hyperbolic case ($|\lambda_i| - 1$ positive and negative) more information is needed. Construct local linear approximations to the motion by decomposing the space into stable ($|\lambda_i| < 1$) and unstable ($|\lambda_i| > 1$) subspaces. Assume that there are no eigendirections corresponding to $|\lambda_i| = 1$ i.e. no centre manifold. Associate with each point in the space a vector space given by the eigenvectors of the derivative at that point, this is called the ***tangent space***, T_x at the point $x \in X$.

Definition : an orbit $\{f^n(x)\}_{-\infty}^{\infty}$ is called ***subexponentially hyperbolic*** if,

a) the space of tangents $T_{f^n(x)}$ splits as a direct sum $T_{f^n(x)} = E^s_{f^n(x)} \oplus E^u_{f^n(x)}$ such that the derivative Df respects this splitting, i.e. $Df(E^s_{f^n(x)}) = E^s_{f^{n+1}(x)}$.

b) $\exists\ 0 < \lambda < 1$ such that given $\varepsilon > 0$, $\exists$ constants $A_\varepsilon(f^n(x))$, $\tilde{A}_\varepsilon(f^n(x))$ that satisfy

1. $A_\varepsilon(f^{n+m}(x)) \le A_\varepsilon(f^n(x))e^{\varepsilon|m|}$: $\tilde{A}_\varepsilon(f^{n+m}(x)) \le \tilde{A}_\varepsilon(f^n(x))e^{\varepsilon|m|}$

2. $\forall$ n, $\forall\ v \in E^s_{f^n(x)}$, $\forall\ u \in E^u_{f^n(x)}$ and m>0 then

$$|| Df^{m}_{f^n(x)} (v) || \le A_\varepsilon(f^n(x))\, \lambda^m\, || v ||$$

$$|| Df^{-m}_{f^n(x)} (u) || \le \tilde{A}_\varepsilon(f^n(x))\, \lambda^m\, || u ||$$

Here $||\ .\ ||$ is a norm on the tangent space. If A_ε and $\tilde{A}_\varepsilon$ are independent of x then the orbit is *uniformly hyperbolic*.

It is now possible to give a theorem that allows the extension of the properties of an orbit (which may be dense and chaotic) to a region about that orbit which will have positive measure.

Theorem (Pesin[98] , Ruelle[115]) : Let f be a diffeomorphism with Holder derivative (f C^2 is sufficient) and let $\{f^n(x)\}_{-\infty}^{\infty}$ be a subexponentially hyperbolic orbit.

Then there exist nonlinear disks tangent to the subspaces $E^u_{f^n(x)}$ and $E^s_{f^n(x)}$ such that f expands the unstable disk at $f^n(x)$ onto the unstable disk at $f^{n+1}(x)$ and contracts the stable disk into.

This theorem says that there are measurable regions about the orbit which are continuously dependent on the orbit. For uniform hyperbolicity, this is the Stable Manifold Theorem that is used to obtain the global properties of Anosov and Axiom A systems (see Lanford[72]). The extension to subexponentially hyperbolic allows the description of systems more complicated than piecewise linear since the expansion constants are allowed to slowly vary along the orbit[71]. Unfortunately it is still not sufficiently general to enable the description of transformations like the Chirikov map as these have regions of zero expansion. Chapter 4 uses special properties of hyperbolic orbits when the subspaces E^u and E^s are linear segments. Then it is possible to state results about large subsets of the space.

1.3.2. Lyapunov Exponents, Invariant Measures and Ergodic Attractors

Given a tangent vector v at the point x, define the *Lyapunov characteristic exponent* as:

$$\chi(v) \equiv \overline{\lim_{k\to\infty}} \frac{1}{k} \log \| Df^k_x(v) \|$$

Since f is a diffeomorphism, the derivatives of f are bounded so $\frac{1}{n}\log \| Df^n \|$ is a bounded sequence of numbers. There are at most dim(X) distinct Lyapunov exponents, each corresponding to a different choice of eigenvector v. Chapter 6 will calculate the Lyapunov exponent for a one dimensional map. General numerical methods for obtaining $\chi(\nu)$ are given by Benettin, Galgani, Giorgilli and Strelcyn[10, 11]

If μ is an f-invariant measure, then μ has *no zero exponents* if for almost all (with respect to μ) x, $\chi(v) \neq 0$, $\forall\, v \in T_x$, $v \neq 0$. This allows the statement of a corollary of Oseledec : let μ be an invariant probability measure for f with no zero exponents.

Then for almost all x, the orbit through x is subexponentially hyperbolic. Moreover the spaces E_x^s and E_x^u depend measurably on x.

If now f has Holder derivatives then the theorem of §1.3.4 gives expanding and contracting leaves (disks) near the support of μ. However the measure of the intersections along the leaves may be meagre in a Lebesgue sense. Thus it necessary to require that μ induces an absolutely continuous conditional measure on the stable and unstable disks (the induced measures will be different on the unstable and stable disks, in fact singular). It can now be shown that forward time averages are constant on expanding leaves and backwards ones constant on contracting leaves, using Birkhoff's theorem and the existence of an invariant measure gives the same constant on all leaves. Thus $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{i=0}^{n-1} \varphi(f^n(x))$ exists and is constant on the leaves.

Definition : a measurable set $A \subset X$ and an f-invariant measure μ on A is called an *ergodic attractor* if $\exists$ a set $V \subset X$ of positive Lebesgue measure such that,

1) $f^n(x) \to A$ as $n \to \infty$, $\forall\, x \in V$

2) $\forall\, \varphi : X \to \mathbf{R}$ continuous, $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=0}^{n-1} \varphi(f^k(x)) = \int_A \varphi\, d\mu$ for almost all $x \in V$.

Theorem (Pugh and Shub[101] p 322) : Suppose f is a diffeomorphism and f' is Holder. If μ is an invariant probability measure for f such that :

a) μ has no zero exponents

b) μ induces absolutely continuous measures on unstable disks,

then $\exists$ an ergodic attractor in the support of μ and the ergodic measure is μ.

If f(X)=X, i.e. area-preserving, then the conditional measure is always smooth on expanding disks. Thus no zero exponents implies there exists an ergodic attractor for Lebesgue measure.

1.4. Bernoulli and Markov Processes

1.4.1. The Bernoulli shift

In order to show that a general dynamical system has random behaviour, one approach is to identify the system with some mathematically well-defined process that has strong random properties. This section describes two such processes which are obtained as shifts on symbol spaces.

Let **X** be the set of all sequences, infinite in both directions, $x = (\ldots, y_{-1}, y_0, y_1, \cdots)$ whose coordinates y_i are points of a fixed measurable space $(\mathbf{Y}, \mathbf{U})$ (usually $\mathbf{Y} = \{0,1,\ldots,k\text{-}1\}$ and $\mathbf{U} = 2^{\mathbf{Y}}$). **X** possesses a natural σ-algebra **C** generated by cylinders $C_r = \{x=(\ldots,y_{-1},y_0,y_1,\ldots) \in \mathbf{X} : y_{i_1} \in U_1, \ldots, y_{i_r} \in U_r\}$ where $1 \leq r < \infty$; $i_1,\ldots,i_r$ are integers ; $U_1, \ldots, U_r \in \mathbf{U}$. Let μ be a normalised measure on **C**. Then $(\mathbf{X}, \mathbf{C}, \mu)$ is a *discrete time random process* with *state space* $(\mathbf{Y}, \mathbf{U})$. A process is *stationary* if $\mu(\{x \in \mathbf{X} : y_{i_1+n} \in U_1, \ldots, y_{i_r+n} \in U_r\})$ is independent of n. (In statistical terms, any trial is an element of **Y** and the complete experiment which has run for infinite time in the past and will run for infinite time in the future is a point in **X**.)

Define the shift, T on **X** such that $x^* = Tx$ with $y_i^* = y_{i+1}$. For each cylinder set C_r , $T^{-n}C_r = \{x \in \mathbf{X} : y_{i_1-n} \in U_1, \ldots, y_{i_r-n} \in U_r\}$, thus stationarity gives $\mu(T^{-n}C_r) = \mu(C_r)\ \forall n$. Thus T is a measure-preserving automorphism of $(\mathbf{X}, \mathbf{C}, \mu)$. Suppose that a measure, σ is defined on **Y** then it is possible to construct the measure space $(\mathbf{X}, \mathbf{C}, \mu) = \prod_{n=-\infty}^{\infty} (\mathbf{Y}^{(n)}, \mathbf{U}^{(n)}, \sigma^{(n)})$ with $\mu = \bigotimes_{-\infty}^{\infty} \sigma$, the product measure generated from σ. The shift T with invariant measure μ is a *Bernoulli automorphism* with state space $(\mathbf{Y}, \mathbf{U}, \sigma)$. (Statistically, a Bernoulli shift is an independent, indentically distributed stochastic process.) Bernoulli-shifts are the most random processes and can be shown to be ergodic and mixing.

Example : two-sided $(p_0,\ldots,p_{k-1})$-shift

Let $k \geq 2$ be a fixed integer and let $(p_0,\ldots,p_{k-1})$ be a probability vector (i.e. $p_i \geq 0 \ \forall i$ and $\sum_i p_i = 1$). Let $\mathbf{Y}$ be the finite space $\{0,1,\ldots,k-1\}$ and associate with the point i measure p_i. Then the measure of the cylinder $C_r = \{ x \in \mathbf{X} : y_{i_1} \in U_1,\ldots,y_{i_r} \in U_r\}$ is given by $\mu^{(r)}(C_r) = p_{i_1}p_{i_2}\cdots p_{i_r}$ (the open sets U_i are just the elements of $\mathbf{Y}$ in the discrete topology).

1.4.2. The Markov Process

A process which is often used in physical models is the Markov process : here the system at time t 'remembers' the state at time t-1 but not earlier times. These processes could be introduced using the probability of transition between different states. However it will be of greater use to consider just topological properties which require only a statement of whether a transition between two states occurs or not. Let $\mathbf{Y}$ be the set $\{0,1,\ldots,k-1\}$ with the discrete topology. A neighbourhood of a point $x=\{x_n\}$ consists of the cylinder set $C_N = \{y \in \mathbf{X} : y_n = x_n \text{ for } |n| \leq N\}$ with $N \geq 1$. Defining the shift, T as before produces the *two-sided shift* on $\mathbf{X}$ Let $A = (a_{ij})$ be a kxk matrix with $a_{ij} \in \{0,1\}$. Restrict $\mathbf{X}$ to the set $\mathbf{X}_A = \left\{x = \{x_n\} : a_{x_n x_{n+1}} = 1 \ \forall\, n \in \mathbf{Z}\right\}$. $\mathbf{X}_A$ is closed and T restricted to $\mathbf{X}_A$ is a homeomorphism called the *2-sided topological Markov chain* (or subshift of finite type) determined by A. The matrix A determines which points x_{n+1} are images of other points x_n.

When studying a map f that exhibits complicated behaviour, it is found that the complicated motion occurs on hyperbolic sets, Ω_i which have an expanding and a contracting direction. It is easy to set-up a continuous surjection, π between the shift T and the map f, unfortunately π need not be 1-1. To obtain useful results it is necessary to produce a special partition of the hyperbolic set. Let $W^s_\varepsilon(x) \equiv E^s_x \cap B_\varepsilon(x)$ where $B_\varepsilon(x)$ is the ball of radius ε about x, similarly $W^u_\varepsilon(x)$.

A closed set $R \subset \Omega_i$ of small diameter is called a *rectangle* if :

1) $x,y \in R \Rightarrow [x,y] \in R$ where $[x,y]$ denotes the single point $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$.

2) the interior of R (as a subset of Ω_i) is dense in R.

Let $W^u(x,R) = R \cap W^u_\varepsilon(x)$ and $W^s(x,R) = R \cap W^s_\varepsilon(x)$ for $x \in R$. The boundary of R consists of two parts :

$$\partial^s R = \{x \in R : \text{for no } \gamma>0 \text{ does } W^u(x,R) \supset W^u_\gamma(x) \cap \Omega_i\}$$

$$\partial^u R = \{x \in R : \text{for no } \gamma>0 \text{ does } W^s(x,R) \supset W^s_\gamma(x) \cap \Omega_i\}$$

A *Markov Partition* for Ω_i is a finite cover $\mathbf{C}=\{R_1,\dots,R_n\}$ of Ω_i by rectangles such that:

1) $R_j \cap R_k$ has empty interior as a subset of Ω_i for $j \neq k$.

2) if $x \in \mathrm{int}(R_j \cap f^{-1}R_k)$ then $fW^s(x,R_j) \subset W^s(fx,R_k)$ and $fW^u(x,R_j) \supset W^u(fx,R^k)$.

Theorem : There exist Markov partitions $\mathbf{C}$ for Ω_i. Fixing $\mathbf{C}$ (with rectangles of small diameter) define the transition matrix A by $a_{ij} = 1 \Leftrightarrow \mathrm{int}(R_j \cap f^{-1}R_k)$ is non-empty. Then there exists a continuous surjection $\pi : \mathbf{X}_A \to \Omega_i$ with $\pi T_A = f\pi$ defined by $\pi(\bar{x})$ equalling the unique point $x \in \bigcap_{k=-\infty}^{\infty} f^{-k}R_{x_k}$.

Thus the dynamics of the invariant hyperbolic sets Ω_i under some map f may be shown to be equivalent to a topological Markov chain and thus the motion is random.

CHAPTER 2

Diffusion

2.1. Introduction

The previous chapter set out the framework for describing chaotic systems by obtaining a transformation from the given system to a mathematically well-defined stochastic process. This chapter will obtain in a more heuristic manner the framework that might be used by a scientist, namely the use of a Fokker-Planck equation and the calculation of a diffusion constant, D. D is readily obtained from the mean square spread of a phase-space variable which can easily be calculated in a computer simulation or a real experiment (e.g. plasma confinement).

It will be shown here that diffusion is a good concept for describing random systems, given certain assumptions. Chapters 3-5 will obtain analytic expressions for a characteristic diffusion quantity which agrees well with simulations in situations where the mathematical concepts of chapter 1 are not readily applied.

2.2. The kinetic equation

Let the object of study be a stochastic process X(t) which is sampled at discrete intervals $t = t_i$ $0 < i \leq n$. X(t) is completely determined if the joint probability density functions (p.d.f.) $f_n(x_n, t_n; x_{n-1}, t_{n-1}; \ldots; x_1, t_1)$ $\forall n > 0$ are known [f_n corresponds to $\mu(C_n)$ in §1.4.1] Using Bayes theorem, the conditional probability density can be defined as :

$$f(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1) = \frac{f_n(x_n, t_n; \ldots; x_1, t_1)}{f_{n-1}(x_{n-1}, t_{n-1}; \ldots; x_1, t_1)}$$

To obtain an usable description, make the Markov assumption,

$$f(x_n, t_n | x_{n-1}, t_{n-1}; \ldots; x_1, t_1) = f(x_n, t_n | x_{n-1}, t_{n-1})$$

then

$$f(x_n, t_n; \ldots; x_1, t_1) = f(x_n, t_n | x_{n-1}, t_{n-1}) f(x_{n-1}, t_{n-1} | x_{n-2}, t_{n-2}) \ldots f_1(x_1, t_1)$$

Thus the process $X(t)$ is completely determined by the probability density function $f_1(.)$ and the transition probability density $f(.|.)$.

The problem is now to obtain the transition p.d.f $f(.|.)$. Using the compatibility relation $f_2(x_1, t; x_0, t_0) = \int dy\, f_3(x_1, t; y, \tau; x_0, t_0)$ and the Markov property allows the derivation of the Smolukowski or Chapman-Kolmogorov equation,

$$f(x_1, t | x_0, t_0) = \int dy f(x_1, t | y, \tau) f(y, \tau | x_0, t_0) \tag{2.2.1}$$

These equations assume that the process $X(t)$ originates at some initial state with certainty i.e. $f(x, t_0 | x_0, t_0) = \delta(x - x_0)$. (2.2.1) can be converted into a differential form, giving the ***kinetic equation*** :

$$\frac{\partial f}{\partial t}(x, t | x_0, t_0) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n}\left[A_n(x,t) f(x, t | x_0, t_0)\right] \tag{2.2.2}$$

where $A_n(x,t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \int dy\, (y - x)^n f(y, t+\Delta t | x, t) \quad n=1,2,\ldots$

This equation is a ***forward equation*** as the evolution varies as the current state (x, t) while the initial state (x_0, t_0) is a parameter.

If the process is stationary then $f(.|.)$ depends only on time differences i.e. $f(y, t+\Delta t | x, t) = f(y, \Delta t | x, 0)$, whence the ***infinitesimal moments***, $A_n(x,t)$ are independent of t. Also, setting $y-x = \Delta x$, for sufficiently short times the Expectation is given by:

$$E\{(\Delta x)^n | x, t\} = A_n(x)\, \Delta t \tag{2.2.3}$$

2.3. The diffusion equation

Obviously it is impossible to solve (2.2.2) due to the presence of arbitrarily high derivatives. To proceed, (2.2.2) must be truncated at some n. Choosing n = 1 gives a deterministic process

$$f(x, t | x_0, t_0) = \delta(x - \int_{t_0}^{t} A_1(x_0, \tau) d\tau - x_0)$$

The simplest non-trivial truncation is n = 2, the ***diffusion equation***

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x}\left[A_1(x,t) f\right] + \frac{1}{2} \frac{\partial^2}{\partial x^2}\left[A_2(x,t) f\right] \tag{2.3.1}$$

A_1 and A_2 were originally defined in terms of f however they can usually be obtained from the probability of an increment over a short time interval (see (2.2.3) and later chapters). Thus (2.3.1) is well-defined and a solution may be attempted (Ricciardi[110] p 44).

The simplest diffusion equation models the Wiener process where $A_1 = 0$ and $A_2 = \sigma^2$. (2.3.1) becomes

$$\frac{\partial f}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 f}{\partial x^2} \tag{2.3.2}$$

The solution to this equation is the Gaussian distribution :

$$f(x, t \mid x_0, t_0) = \frac{1}{\sqrt{2\pi(t-t_0)\sigma^2}} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2(t-t_0)}\right] \tag{2.3.3}$$

2.4. Truncation schemes

In (2.2.2) it is possible to choose $n \geq 3$ as the truncation point and attempt to solve the resulting n^{th} order differential equations. This approach will be followed in chapter 5 with $n = 4$. However this truncation can never be exact by a theorem of Pawula[97].

Theorem : if A_n exists for all n and is zero for some even n then the conditional probability density $f(x, t \mid y, t_0)$ satisfies (2.3.1)

In other words, if A_n vanishes for some even n then it vanishes for all $n > 3$. Thus a probability density is either Gaussian or has moments of all orders. This theorem is related to the Marcinkiewicz theorem[76] and its corollaries which have been used by Lowe[79] to criticise hierarchical methods in quantum-field theory. Obviously the results of truncation schemes are sufficiently accurate that they will remain of great use, however Marcinkiewicz type theorems show that these schemes can never give the complete picture.

2.5. Random Walks

The maps considered in the body of this thesis appear very similar to random walks which suggests an approximation by a diffusion process. This section describes some results from the statistical theory of random walks on lattices that may prove relevant to the computer simulation of maps.

As a first example consider the process, X_n which takes values ± 1 with equal probability and examine the paths $S_n = \sum_{i=1}^{n} X_i$. Intuitively, it might be expected that for large n, S_n will have been positive for an equal time to it being negative : this is false !

Feller[39] (ch.3 p82) gives the result : if $0 < x < 1$ then the probability that $\leq$ xn time units are spent on the positive side and $\geq$ (1-x)n on the negative side tends to $\frac{2}{\pi}\arcsin\sqrt{x}$ as $n \to \infty$. This means that it is more probable that S_n remains of one sign, e.g. for n large, with probability 0.2, S_n is of the same sign for 97.6 per cent of the time.

The book by Ibragimov and Linnik[64] considers more general processes X_n and the resulting sums S_n. Let $(X_j)_1^\infty$ be a stationary sequence with $E(X_j) = 0$, $E(X_j^2) < \infty$ and $S_n = \sum_{j=1}^{n+1} X_j$, $\sigma_n^2 = \mathrm{Var}(S_n)$. Say that the sequence (X_j) satisfies a *central limit theorem* if $\lim_{n\to\infty} P(\frac{S_n}{\sigma_n} < z) = (2\pi)^{½} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} du = \Phi(z)$. Thus $\frac{S_n}{\sigma_n}$ is asymptotically distributed as a Gaussian. Among the conditions necessary for a sequence to satisfy a central limit theorem are :

1. the sequence (X_j) is a strongly mixing sequence ;
2. $\sigma_n^2 = nh(n)$ where h(n) is a slowly varying function that approaches 1 as $n \to \infty$.

This type of theorem is useful in calculating diffusion constants, which are related to σ_n^2, since the diffusion equation presupposes a solution that is Gaussian.

CHAPTER 3

Diffusion in 2D area-preserving maps

3.1. Introduction

Chapter 0 discussed in detail how the equations describing certain aspects of plasma confinement could be reduced to a simple area-preserving map of the plane - the Chirikov map. As was described there, this map exhibits very complicated behaviour - chaos. This chapter shows how to characterise such maps in terms of a diffusion constant or correlation coefficient and will attempt to demonstrate rigorously the equivalence between the heuristic ideas of chaos and the standard ideas of randomness and stochastic processes. This chapter will consider parameter values, $E > 6$ where motion appears chaotic over most of the plane.

Chapter 1 introduced the ideas of equality of time and space averages and hence the use of a probability distribution when studying long-time properties. Then chapter 2 defined a diffusion coefficient as a characteristic constant in the equation describing the evolution of an initial distribution function. §3.2 gives an expression for a probability distribution in terms of path-integrals. Little can be done with this expression directly but using the methods introduced by Rechester, Rosenbluth and White[106] an expression can be derived for a diffusion constant. This was originally used to describe the Chirikov map but will be extended here to any doubly-periodic map of the plane.

In §3.3 an expression for a diffusion constant is given for the Chirikov map, including the correction of Cohen and Rowlands[27] , and for a family of piecewise linear maps (special cases studied by Karney[68] and Cary and Meiss[21]). These analytic expressions are compared with the results of numerical iteration and good agreement is seen.

When watching the numerical plots of the maps, correlations between many successive iterations are often observed. §3.4 discusses the calculation of correlation coefficients using a recurrence relation for the characteristic functions[22]. There is a relation between the diffusion constant and correlation coefficients which shows that the oscillations with parameter observed in the diffusion constant are due to short-time correlations[68]. It has recently been shown by Meiss, Cary, Grebogi, Crawford, Kaufman and Abarbanel[88] that it is possible to sum certain of the correlations to all orders, obtaining terms of order $E^{-1/2}$, providing an improvement to the Fourier-path approximation. This resummation has greatest effect for small E and so is of importance for chapter 5 but it seems restricted to the Chirikov map as the summations involved do not simplify for more general cases.

There is a large set of parameter values for which the result for the Chirikov map breaks down, (the motion is dominated by small regions of phase space where particles are accelerated linearly rather than diffusing. Preliminary results for these accelerating islands have described much of the observed variation in the diffusion coefficient[51]). In order to prevent accelerator modes, a piecewise linear map is considered which also enables calculation of expansion coefficients which can be used in proofs of ergodicity and Bernoullicity. This will be described in chapter 4.

This chapter uses maps of the form :

$$\begin{aligned} v_{n+1} &= v_n + E\, f(x_n;a) \\ x_{n+1} &= x_n + v_{n+1} \qquad \mathrm{mod}2 \end{aligned} \tag{3.1.1}$$

where $f(x;a)$ has period 2.

Although this map is doubly-periodic (the periodicity in the x variable leads to periodicity in v) and can be restricted to a torus for plotting and topological descriptions, diffusive properties are obtained from motion on the cylinder $\{ (x,v) : x \in [-1,1] \, , \, v \in R \}$.

3.2. Diffusion

The method of path-integrals is used in several branches of physics : their use in analysising chaos by giving a description of the invariant measure has been developed by Rechester, Rosenbluth and White[106, 107]. for the Chirikov map and has since been used by several authors ([3, 27, 68]) to described periodic maps. This section obtains an expression for the diffusion constant for maps of the form (3.1.1) with arbitrary periodic function f.

The problem is to describe the evolution of a distribution function P(x,v,t) defined on the cylinder $\{ (x,v): x \in [-1,1],\ v \in \mathbf{R} \}$. Using the area-preservation property gives the relation $P(x,v,t)dxdv = P(x',v',t-1)dx'dv'$. Mapping (3.1.1) is invertible and so may be written :

$$\begin{aligned} v_n &= \mathbf{V}(v_{n+1}, x_{n+1}) \equiv v_{n+1} - Ef(x_{n+1} - v_{n+1}\ ;\ a) \\ x_n &= \mathbf{X}(v_{n+1}, x_{n+1}) \equiv x_{n+1} - v_{n+1} \end{aligned} \tag{3.2.1}$$

whence P(x,v,n+1) can be given as :

$$\begin{aligned} P(x,v,n+1) &= \int_{-1}^{1} dx' \int_{-\infty}^{\infty} dv'\ P(x',v',n)\,\delta(v' - \mathbf{V}(v,x))\,\delta(x' - \mathbf{X}(v,x)) \\ &= P(x - v, v - Ef(x-v), n) \end{aligned} \tag{3.2.2}$$

Expand P(x,v,t) as a Fourier series

$$P(x,v,t) = (½)^2 \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} dk\ e^{im\pi x}\, e^{ik\pi v}\, a_m^t(k) \tag{3.2.3}$$

then substituting in (3.2.1) and using the symmetry f(-z) = -f(z) leads to the relation (following Rechester et.al.) :

$$a_m^{n+1}(k) = \sum_{l=-\infty}^{\infty} g_l(\,|Ek'|\,)\, a_{m'}^{n}(k') \tag{3.2.4}$$

where $k' = k + m$, $m' = m + l\,\mathrm{sgn}(k')$ and

$$g_l(u) = ½\int_{-1}^{1} dz\, e^{-i\pi zl}\, e^{i\pi u f(z)} \quad \text{for } u \geq 0 \tag{3.2.5}$$

choosing an initial distribution $P(x,v,0) = ½\delta(v-v_0)$ i.e. a uniform distribution in x at some constant $v=v_0$ gives an initial term for the recurrence (3.2.4), $a_m^0(k) = \delta_{m,0}\, e^{-ikv_0\pi}$ where $\delta_{m,n}$ is the kronecker delta.

Defining a diffusion constant, D as

$$D = \lim_{T\to\infty} \frac{1}{2T} \int_{-1}^{1} dx \int_{-\infty}^{\infty} dv\, (v-v_0)^2 P(x,v,T) \tag{3.2.6}$$

and noting from (3.2.2) that

$$\int_{-1}^{1} dx \int_{-\infty}^{\infty} dv\, v^n P(x,v,t) = \lim_{k\to 0^+} \left\{ \frac{i}{\pi} \frac{\partial}{\partial k} \right\}^n a_0^t(k)$$

gives

$$D = \lim_{T\to\infty} \frac{1}{2T} \lim_{k\to 0^+} \left[\frac{-1}{\pi^2} \frac{\partial^2}{\partial k^2} a_0^T(k) - v_0^2 \right] \tag{3.2.7}$$

The contribution from the initial condition is asymptotically negligible and will be ignored in the analytical calculations but it does give a small correction to the finite time numerical simulations.

The problem is now to obtain the coefficient $a_0^T(k)$ in terms of the initial coefficient $a_0^0(k)$ and the transition coefficients $g_l(Ek)$ using relation (3.2.4). The simplest way of obtaining the T-fold products required is to examine paths in the integer space (k,m).

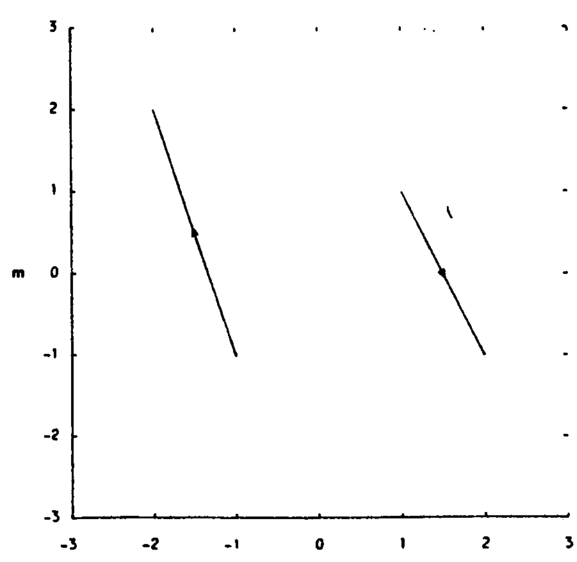


fig. 3.2.1 (k,m)–space

A point (k,m) is connected to any point (k',m') such that k' = k+m. To obtain $a_0^T(k\to 0^+)$ consider a path which starts at the origin at time T and iterate backwards along a path which ends at the origin at time 0. The simplest path is the one which remains at the origin $\forall t$, giving

$$a_0^T = g_0^T(|k|E)\, a_0^0(0) \tag{3.2.8}$$

Thus the main contribution to the diffusion, D is

$$D_{QL} = \frac{-1}{2\pi^2} \lim_{k \to 0^+} \frac{\partial^2}{\partial k^2} g_0(Ek) \tag{3.2.9}$$

D_{QL} is called the quasi-linear diffusion or the random phase approximation.

To obtain corrections to D_{QL} consider paths which leave the origin. These will have a contribution $g_l(0^+)$ on leaving and a similar term on returning to the origin. This term is zero $\forall l \neq 0, \forall$ functions f, so to obtain a contribution to the diffusion the derivative must act on these terms. Thus it is only necessary to consider paths that leave the origin once and the only derivative needed is $\lim_{k \to 0^+} \frac{\partial}{\partial k} g_l(Ek)$. Using (3.2.3) gives $\lim_{k \to 0^+} \frac{\partial}{\partial k} g_l(Ek) = i\pi E f_l$ where f_l is the Fourier coefficient of $f(x;a) = \sum_{-\infty}^{\infty} f_l e^{-i\pi x l}$. Notice that $g_l(Ek)$ decays as E increases, thus for large E the diffusion constant can be expanded in powers of E^{-1}. This essentially corresponds to expanding in the number of path steps away from the origin. Also (3.2.5) implies that paths must be traversed in a clockwise direction and that paths leave the origin via (0,l) for any l and return via (j,-j) for any j.

The first term in this expansion will be obtained by joining the leaving path to the returning path in one step.

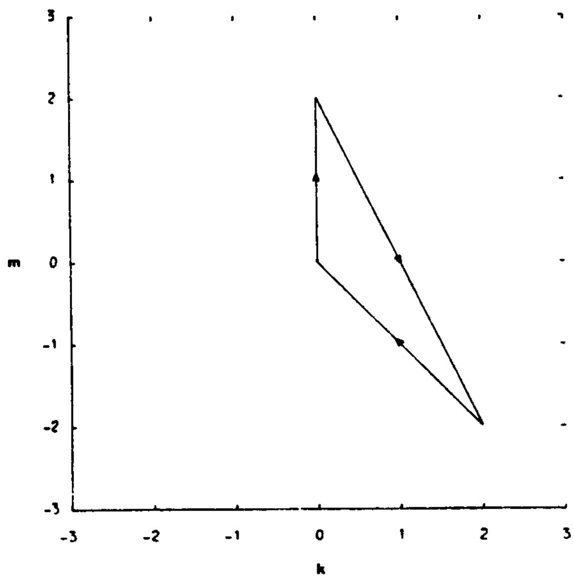


fig. 3.2.2 A path with 3 steps

This path is written :

$$(k,m):\ (0,0) \rightarrow^{l} (0,l) \rightarrow^{-a} (l,l-a) \rightarrow^{m} (0,0)$$

where the superscripts correspond to the step lengths. The equations (3.2.5) give relation between the step lengths :

$$l-a=-l \ \Rightarrow\ a=2l \quad \text{and} \quad m-l=0 \ \Rightarrow\ m=l$$

This path can be traversed in T steps in T-2 ways, each corresponding to remaining at the origin for 0,1,...,T-3 steps before moving. Also there is a symmetric path arising by leaving the origin through (0,-l). Thus the contribution to $a_0^T(k)$ has the form :

$$a_0^T(k) = 2\sum_{l=1}^{\infty}(T-2)\,a_0^0(k)\,g_0^{T-3}(kE)\left\{g_l^2(kE)\,g_{-2l}(|l+k|E) + g_{-l}^2(kE)\,g_{-2l}(|k-l|E)\right\}$$

This gives a contribution to the diffusion :

$$D_1 = E^2\,2\sum_{l=1}^{\infty} f_l^2\,g_{-2l}(lE) \tag{3.2.10}$$

Other paths of interest are:

$$(k,m):\ (0,0) \rightarrow^{l} (0,l) \rightarrow^{-b} (l,l-b) \rightarrow^{c} (2l-b,b-2l) \rightarrow^{2l-b} (0,0) \tag{3.2.11}$$

$$\text{with } c = \begin{cases} 2b-3l & b<2l \\ 3l-2b & b>2l \end{cases}$$

This gives a diffusion contribution :

$$D_2 = E^2 \sum_{l=1}^{\infty} \left[2 \sum_{b>2l} f_l f_{2l-b}\, g_{-b}(lE)\, g_{3l-2b}(|b-2l|E) \right.$$
$$\left. +2 \sum_{b<2l} f_l f_{2l-b}\, g_{-b}(lE)\, g_{2b-3l}(|2l-b|E) \right] \qquad (3.2.12)$$

Paths consisting of 5 steps are formed from the paths given in fig. 3.2.3.

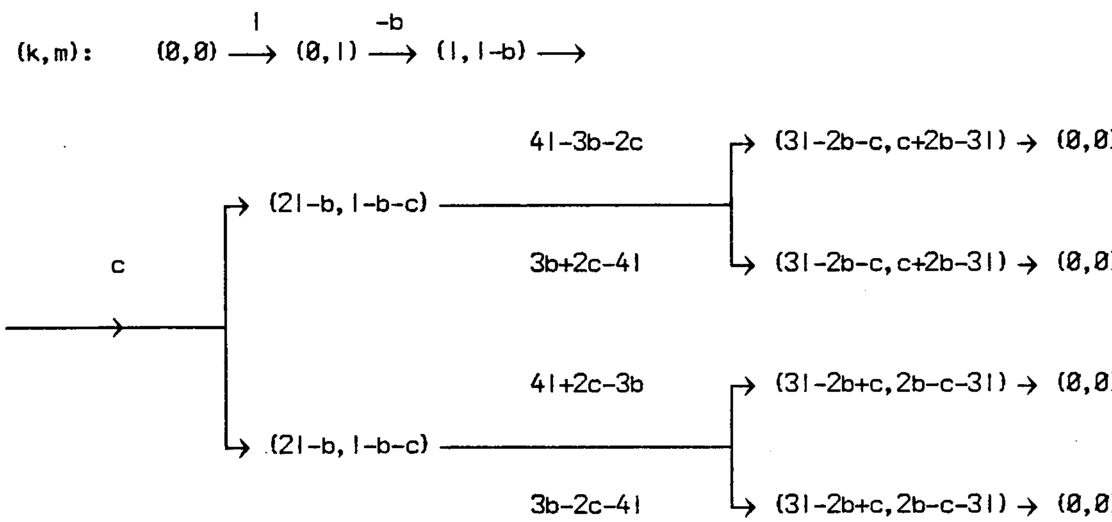


fig. 3.2.3 Construction of paths with 5 steps

In fig. 3.2.3 the upper branches are taken if the k variable is negative. Choosing c=0 and 2l=b gives a term $g_0(0)$ and hence a contribution of the same order as (3.2.12) namely,

$$D_3 = E^2\, 2 \sum_{l=1}^{\infty} f_l f_{-l}\, g_{-2l}(lE) g_{2l}(lE) \qquad (3.2.13)$$

When studying the Chirikov map this contribution was omitted by Rechester and White[107] but was pointed out by Cohen and Rowlands[27].

Thus the diffusion coefficient is given by :

$$D = \frac{E^2}{4} \int_{-1}^{1} dz\, f^2(z) \qquad (3.2.14)$$
$$+E^2\, 2 \sum_{l=1}^{\infty} f_l^2\, g_{-2l}(lE)$$
$$+E^2\, 2 \sum_{l=1}^{\infty} \left[\sum_{b>2l} f_l f_{2l-b}\, g_{-b}(lE) g_{3l-2b}((b-2l)E) + \sum_{b<2l} f_l f_{2l-b}\, g_{-b}(lE) g_{2b-3l}((2l-b)E) \right]$$
$$+E^2\, 2 \sum_{l=1}^{\infty} f_l f_{-l}\, g_{-2l}(lE) g_{2l}(lE)$$

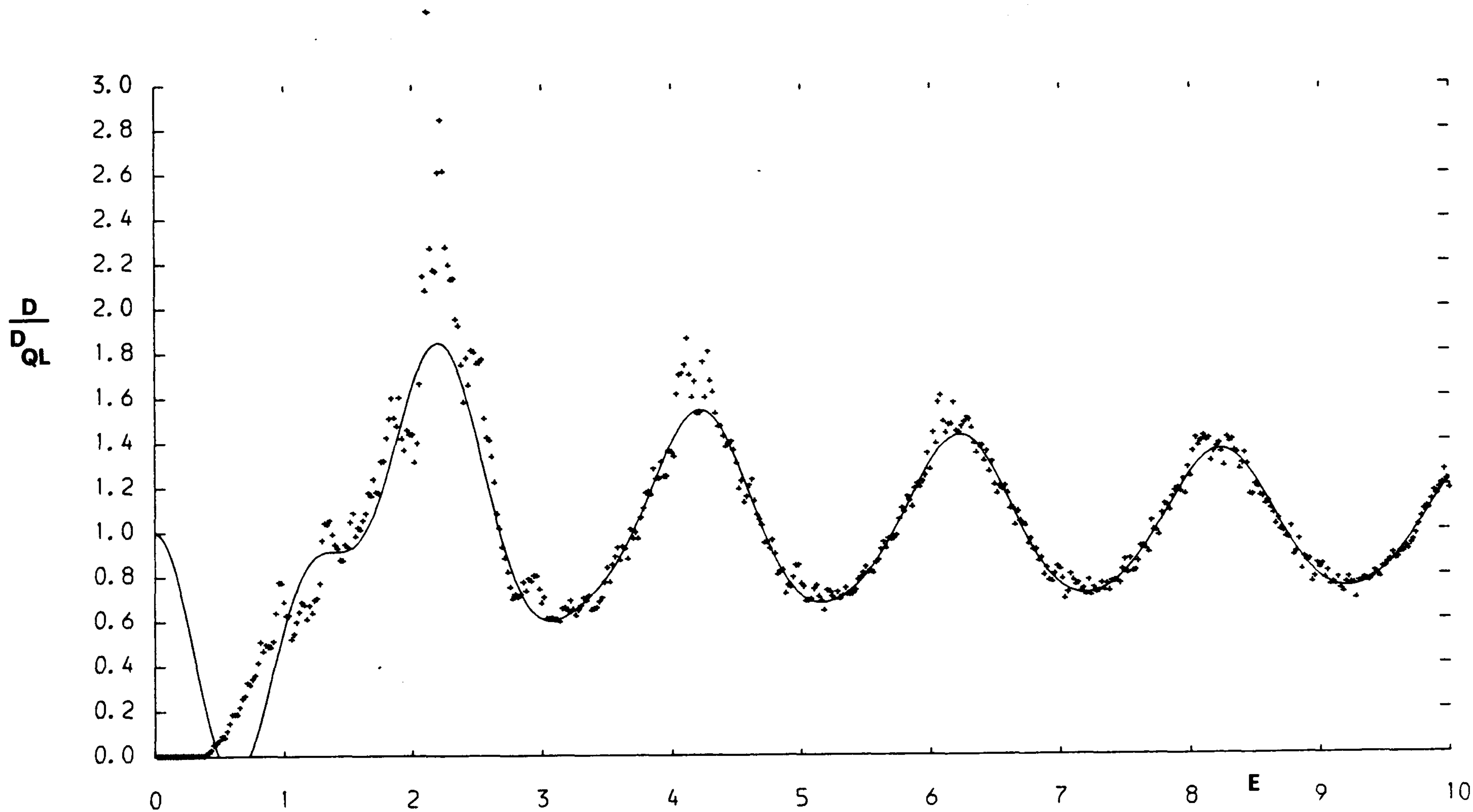


fig. 3.3.2.1 Comparison of numerics and the Fourier-path theory

The Chirikov map

This expression will be evaluated in the next section for two maps, giving results that agree well with numerics.

3.3. Comparison with numerical experiments

3.3.1. Computer experiments

The numerical results are from the approximation to the diffusion constant given by :

$$D_{expt} = \frac{1}{2T}\sum_{i=1}^{T}\int dx_0\,(p_i(x_0)-v_0)^2 \tag{3.3.1.1}$$

The integral over the initial conditions (δ-function at v_0 , uniform in x) is replaced by an average over 2000 points uniformly spaced in x. T is chosen to be 120 and an average taken over the final 20 values of D with respect to time to remove some of the long-time correlations. The computations where performed on a Burroughs B6700 in FORTRAN taking a few minutes for each E value. Due to the difficulty in estimating errors no error bars are drawn on the plots. After experimenting with larger numbers of points in the spatial average and using greater times T, the above values were taken as giving sufficiently good convergence without going to excessive lengths. The major problem arises because the diffusion constant is slowly varying with T (see §2.5). Numerically this appears as an oscillation of approximate period 100. It was decided that it was not feasible to increase T sufficiently to take account of this and so is ignored by the values chosen above. Further details of computational inaccuracies are discussed in §8.5.

3.3.2. The Chirikov map

The Chirikov map is the simplest map to which the method of §3.2 can be applied because of the form of the required Fourier transforms. Taking f(x) = $\sin(\pi x)$ gives :

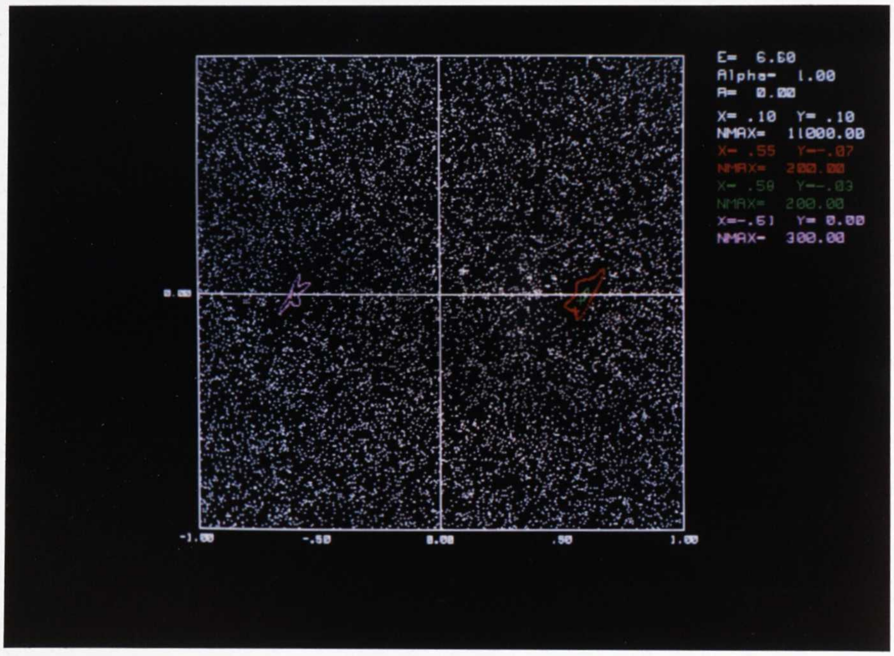


fig. 3.3.3.1 The Chirikov map showing small accelerator modes: $E = \frac{6.6}{\pi}$

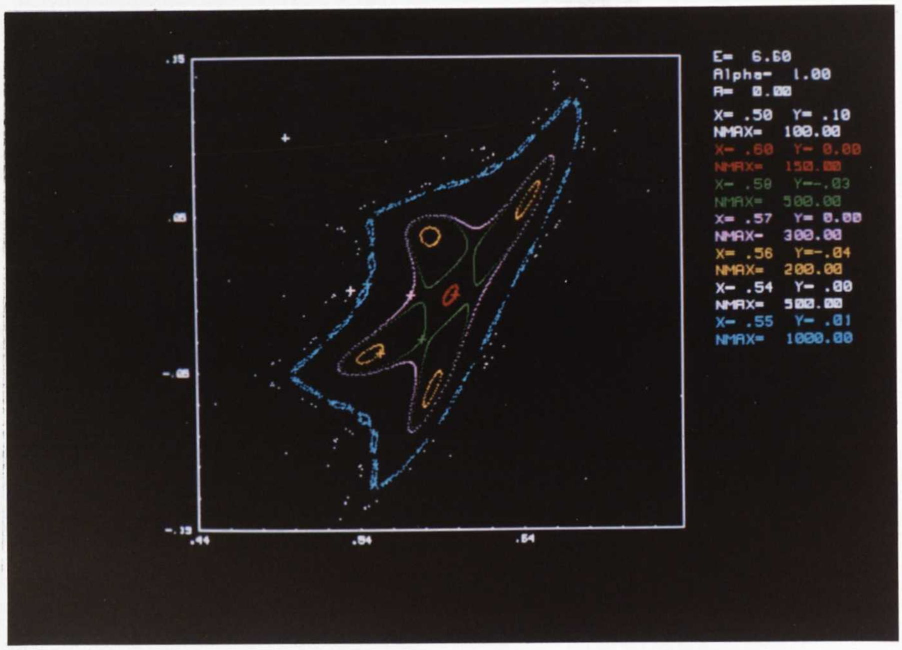


fig. 3.3.3.2 Enlargement of fig. 3.3.3.1

$$f_1 = \frac{-i}{2} \;:\; f_{-1} = \frac{i}{2} \;:\; f_l = 0 \;\forall\, l \neq \mp 1 \tag{3.3.2.1}$$

$g_l(u) = J_l(u\pi)$ – the Bessel function of integer order

Thus the terms in the expansion of the diffusion constant are :

$$D_{QL} = \frac{E^2}{4} \tag{3.3.2.2}$$

$$D_1 = \frac{-E^2}{2} J_2(\pi E)$$

$$D_2 = \frac{E^2}{2}\left\{J_3^2(\pi E) - J_1^2(\pi E)\right\}$$

$$D_3 = \frac{E^2}{2} J_2^2(\pi E)$$

Thus for large E the diffusion constant is

$$D = \frac{E^2}{4}\left\{1 - 2J_2(\pi E) - 2J_1^2(\pi E) + 2J_2^2(\pi E) + 2J_3^2(\pi E)\right\} \tag{3.3.2.3}$$

Equation (3.3.1.3) together with numerical results (marked by crosses) are plotted in fig. 3.3.2.1 There is very good agreement between the theory and experiment for $E > 2$ except when E is an even integer (see § 3.3.3). The inclusion of the D_3 term improves the agreement considerably compared with Rechester et.al[106].

3.3.3. Accelerator modes

When $E \geq 2n$ a phenomenon known as acceleration occurs. As can be seen in figure 3.3.3.1 and the enlargement in 3.3.3.2, two small islands come into existence at $E = 2n$ and persists for a range of E (decreasing with E). Although this structure appears to be a stable island (i.e. a fixed point surrounded by closed KAM curves) this is because the plots are for motion on a torus. On the cylinder, points in these islands are seen to move rapidly in v-space (accelerate) changing their v-coordinate by a multiple of 2 at each iteration. This gives a contribution proportional to T^2 in (3.3.1.1) which dominates the linear diffusion. This problem can be partially alleviated by choosing v_0 so that no initial points lie in an accelerating region. Unfortunately there is no well-defined boundary to these regions and any point can wander arbitrarily close to an accelerating region and be 'dragged along' for a number of iterations. These modes where originally discussed by Chirikov[25] and then in greater detail by Greene, MacKay, Vivaldi and Feigenbaum[51] who showed that in a moving coordinate frame the accelerating region was given locally by the area-preserving Henon map. This helps explain the structure observed in the peak near E = 2 (see fig 3.3.2.1) and may give a method for analytically obtaining the contribution to a 'diffusion constant'. These islands are not important in the original plasma physics model from which the Chirikov equations were derived since $E \equiv E(v)$ and thus the acceleration is soon damped. However for systems which are truly modelled by doubly-periodic maps, Pustylnikov[103] has shown that any smooth forcing function f will generate accelerating islands. Thus one of the reasons for considering the piecewise linear map is that it is not sufficiently smooth to show strong accelerator modes.

3.3.4. The Piecewise Linear map

Let the forcing function be given by

$$f(x;a) = \begin{cases} \dfrac{1+x}{a-1} & x<-a \\ \dfrac{x}{a} & |x|<a \\ \dfrac{1-x}{1-a} & x>a \end{cases} \tag{3.3.4.1}$$

with $x \in [-1,1]$ and $a \in (0,1)$. The required Fourier transforms are

$$f_l = \frac{-i\sin(\pi l a)}{a(1-a)\, l^2\pi^2} \tag{3.3.4.2}$$

$$g_l(u{:}a) = \frac{u}{\pi}\,\frac{\sin(\pi(u-la))}{(u-la)(u+l(1-a))}$$

This gives an expansion for the diffusion constant consisting of the terms:

$$D_{QL} = \frac{E^2}{6} \tag{3.3.4.3}$$

$$D_1 = \frac{-2E^2}{a^2(1-a)^2(2a+E)(E-2(1-a))\pi^5}\sum_{l=1}^{\infty}\frac{\sin^2(\pi l a)\sin[\pi l(2a+E)]}{l^5}$$

Writing the product of sines as a sum gives :

$$D_1 = \frac{-2E^2}{a^2(1-a)^2(2a+E)(E-2(1-a))\pi^5} \times \frac{1}{4}\sum_{l=1}^{\infty}\frac{1}{l^5}\left\{2\sin[\pi l(2a+E)]-\sin[\pi l(4a+E)]-\sin[\pi E l]\right\}$$

The summations may be performed using the result (from Wheelon[124])

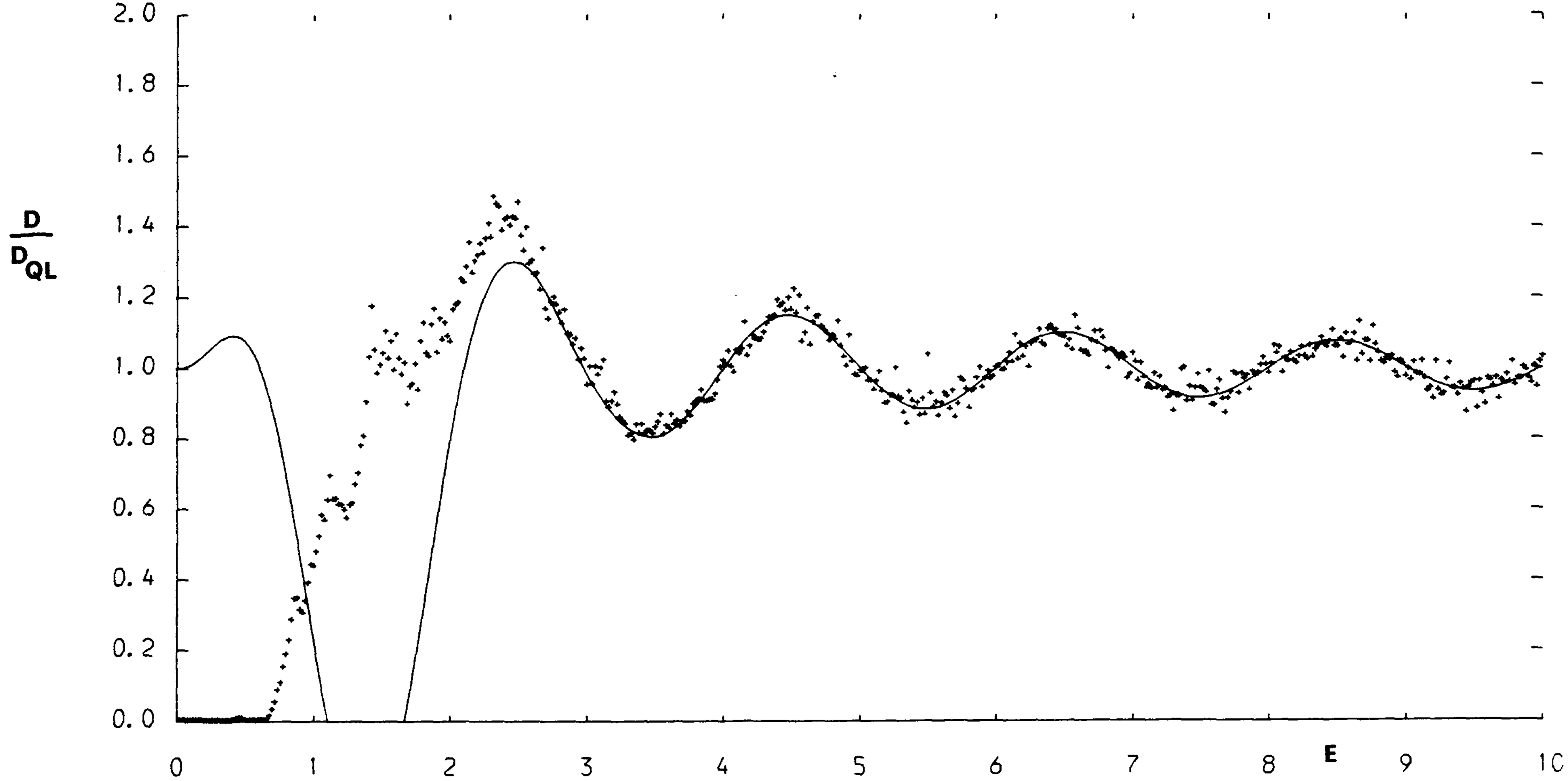


fig. 3.3.4.1 Comparison of numerics and the Fourier-path theory

The piecewise linear map: a=0.5

$$\sum_{n=1}^{\infty}\frac{\sin(nx)}{n^5}=\frac{\pi^4 x}{90}-\frac{\pi^2 x^3}{36}+\frac{\pi x^4}{48}-\frac{x^5}{240}$$

for $x \in (0,2\pi)$. Let $y = \frac{x}{\pi}$ then

$$\sum_{n=1}^{\infty}\frac{\sin(\pi l y)}{l^5}=\frac{\pi^5}{6!}\bar{y}(\bar{y}-1)(\bar{y}-2)(4+6\bar{y}-3\bar{y}^2)$$

where $\bar{y} = y \bmod 2$. Thus

$$D_1=\frac{-E^3}{2.6!a^2(1-a)^2(2a+E)(E-2(1-a))}\begin{Bmatrix}2v(v-1)(v-2)(4+6v-3v^2)\\-u(u-1)(u-2)(4+6u-3u^2)\\-w(w-1)(w-2)(4+6w-3w^{2)}\end{Bmatrix}\quad(3.3.4.4)$$

$$\text{where}\quad \begin{aligned} u &= E && \bmod 2, \\ v &= E+2a && \bmod 2, \\ w &= E+4a && \bmod 2. \end{aligned}$$

Making the change of variable m = b - 2l in (3.2.12) gives

$$D_2=2E^2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left\{f_l f_{-m}\,g_{-m-2l}(lE)\,g_{-2m-l}(mE)+f_l f_m\,g_{m-2l}(lE)\,g_{l-2m}(mE)\right\}$$

Thus

$$D_2=2E^2\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin(\pi la)\sin(\pi ma)}{a^2(1-a)^2m^2l^2\pi^6}\times\quad(3.3.4.5)$$
$$\begin{Bmatrix}\dfrac{lE\sin(\pi a(m+2l)+\pi lE)}{[lE+a(m+2l)][lE-(1-a)(m+2l)]}\dfrac{mE\sin(\pi mE+\pi a(2m+l))}{[mE+a(2m+l)][mE-(1-a)(2m+l)]}\\-\dfrac{lE\sin(\pi a(l-2m)+\pi lE)}{[lE-a(m-2l)][lE+(1-a)(m-2l)]}\dfrac{mE\sin(\pi mE+\pi a(2m-l))}{[mE+a(2m-l)][mE-(1-a)(2m-l)]}\end{Bmatrix}$$

Figure 3.3.4.1 and 3.3.4.2 are plots of numerics and theory for a = 0.5 and 0.4. The equation for D_2 is evaluated by taking the limit of both summations to be finite and equal to 20.

3.4. Calculation of Correlations

The previous method readily gave the velocity-moments of the probability distribution and hence a diffusion constant for maps of a cylinder. Another quantity that is often used to characterise random processes is the correlation function. This function is important because it is directly related to the correlations that are observed as the computer plots of the motion. In addition it is possible to say something about the convergence of the expansions in E for certain parameter values. The correlation functions are obtained from the

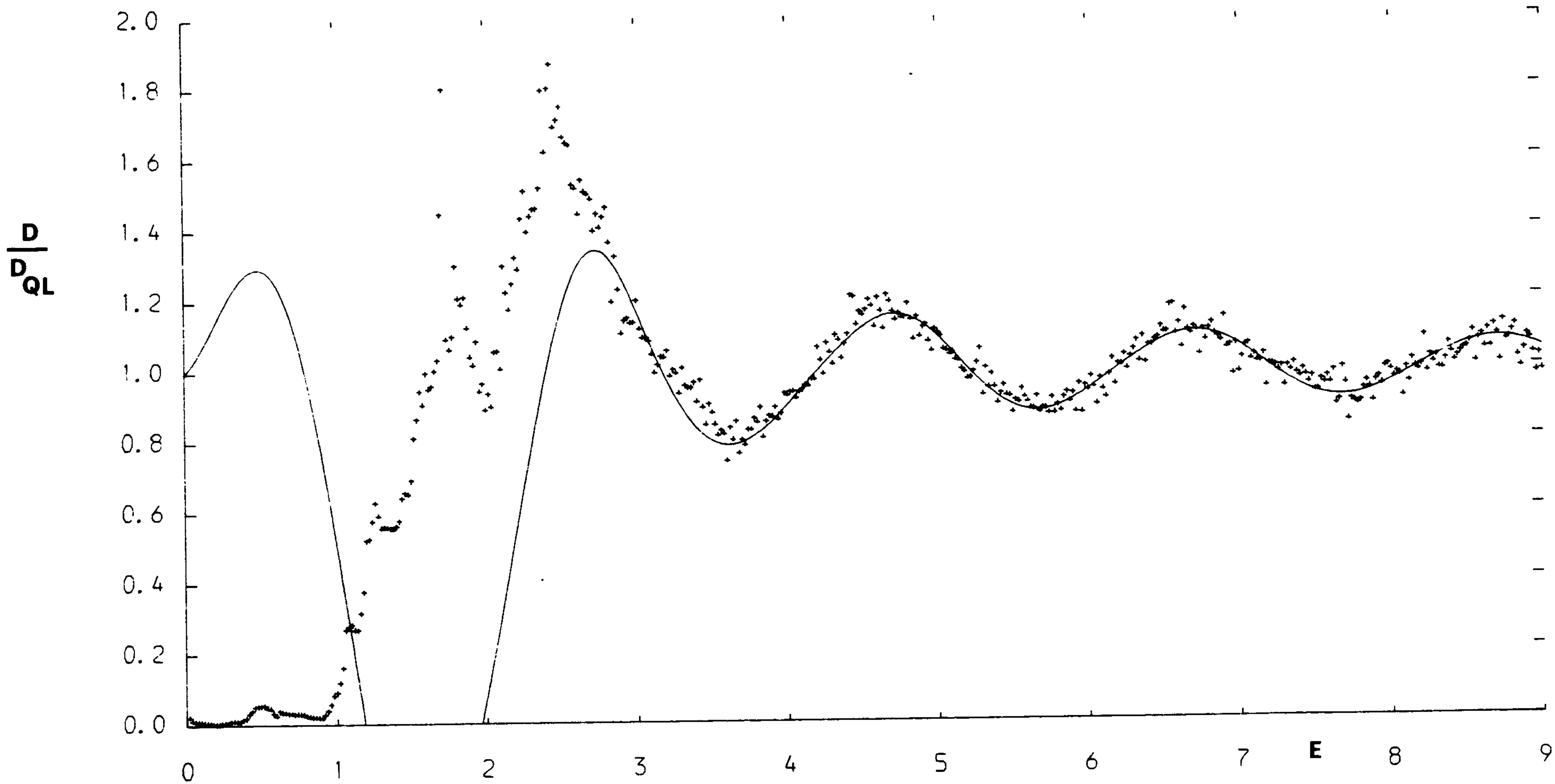


fig. 3.3.4.2 Comparison of numerics and the Fourier-path theory

The piecewise linear map: a = 0.4

characteristic function as described by Cary and Meiss[22].

The characteristic functions are defined as :

$$\chi_j(m_0,\ldots,m_j) \equiv \left\langle \exp(i\pi \sum_{k=0}^{j} m_k x_{n+k}(x_0,p_0)) \right\rangle_R \tag{3.4.1}$$

for $j=0,1,2,\ldots$; $m_j \in \mathbf{N}$; and the average given by

$$\left\langle F(x_0,p_0) \right\rangle_R \equiv \frac{\int_R dx_0 dp_0 F(x_0,p_0)}{\int_R dx_0 dp_0} \tag{3.4.2}$$

The region of integration R is chosen to be an invariant region (possibly the torus) so that the average in (3.4.1) is independent of n i.e. time-translation invariant. Writing the map equations (3.1.1) as,

$$x_{n+1} = 2x_n + Ef(x_n;a) - x_{n-1} \tag{3.4.3}$$

allows the expression of χ_{n+1} in terms of χ_n, namely

$$\chi_k^R = \sum_{l=-\infty}^{\infty} g_l(m_k E)\,\chi_{k-1}^R(m_0,\ldots,m_{k-3},m_{k-2}-m_k,m_{k-1}+2m_k+l) \tag{3.4.4}$$

Following Cary et.al. it is necessary to consider only the class of χ_k^R with the first and the last two indices non-zero. This gives the equation for χ_k^R as :

$$\begin{aligned}\chi_k^R(m_0,0,\ldots,0,m_{k-1},m_k) = \sum_{l_1=-\infty}^{\infty}\sum_{l_2=-\infty}^{\infty} \cdots \sum_{l_{k-1}=-\infty}^{\infty} & g_{l_1-2m_k-m_{k-1}}(m_k E) \\ & \times g_{l_2-2l_1+m_k}(l_1 E) \times g_{l_3-2l_2+l_1}(l_2 E) \times \cdots \\ & \times g_{l_{k-1}-2l_{k-2}+l_{k-3}}(l_{k-2}E)\chi_1^R(m_0-l_{k-2},l_{k-1})\end{aligned} \tag{3.4.5}$$

Choosing R=[-1,1]x[-1,1] gives $\chi_1^R(m_0,m_1) = \delta_{m_0,0}\,\delta_{m_1,0}$

Expanding f(x) as a Fourier series, $f(x) = \sum_{l=-\infty}^{\infty} f_l e^{i\pi l x}$ allows the expression of the force correlation in terms of χ_i :

$$\begin{aligned} C_j^R &\equiv \left\langle f(x_i) f(x_{i+j}) \right\rangle_R \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \left\langle f_m f_n e^{i\pi m x_i} e^{i\pi n x_{i+j}} \right\rangle \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \left\langle f_m f_n\, \chi_j^R(m,0,\ldots,0,n) \right\rangle \end{aligned} \tag{3.4.6}$$

with $C_0^R = \left\langle f^2(x_0) \right\rangle_R$

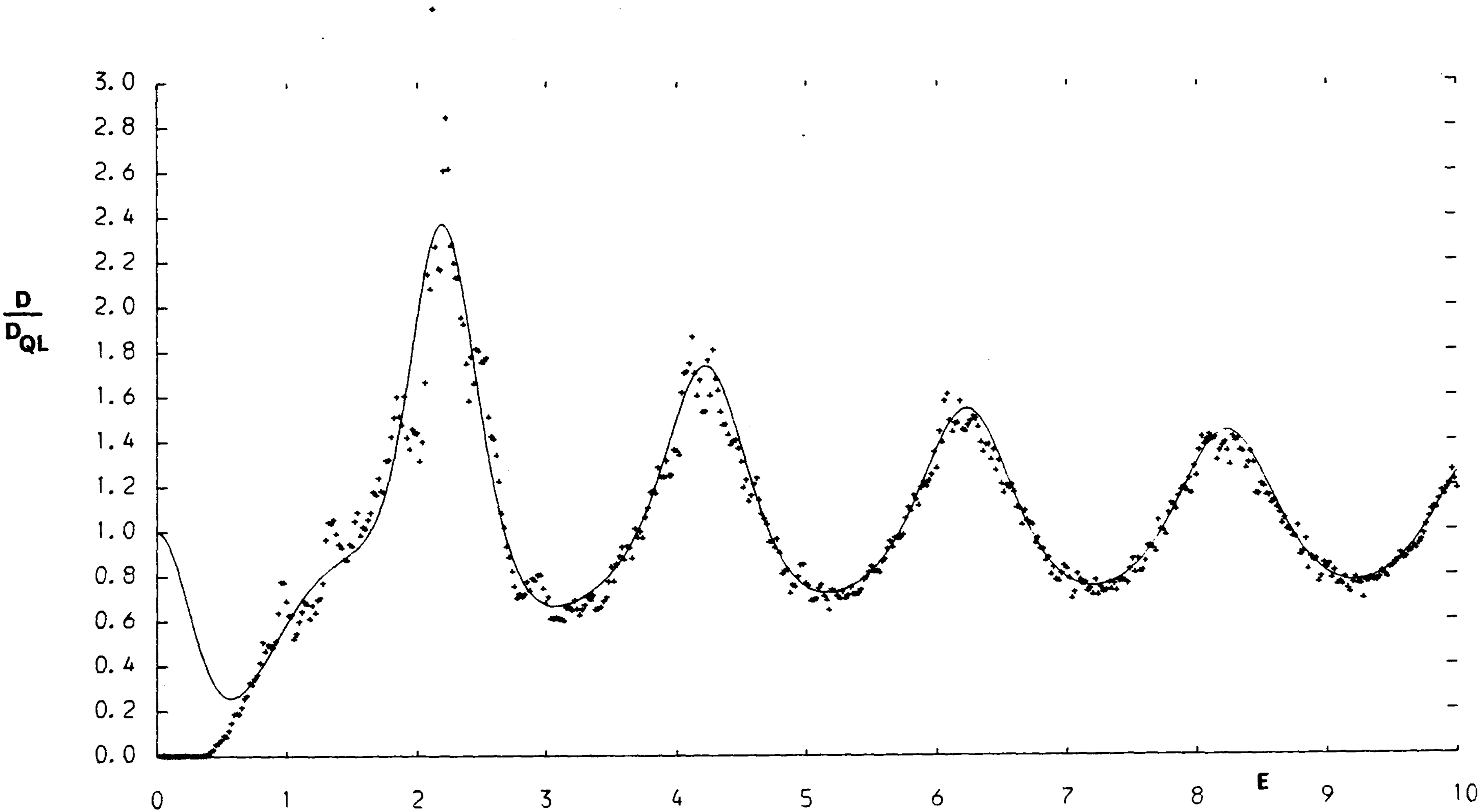


fig. 3.4.1.1 Comparison of numerics and Principal Terms

The Chirikov map

This method has been applied to the Chirikov map (Cary et.al.) and the diffusion constant obtained using the relationship :

$$D^R = \frac{E^2}{2}\left\{C_0^R + 2\sum_{k=1}^{\infty} C_k^R\right\} - E^2 \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} kC_k^R \tag{3.4.7}$$

Provided the correlation functions decay sufficiently rapidly, the diffusion constant is the sum of the correlations. Thus the oscillations observed in the diffusion constant for the Chirikov map are due to short-time correlations not the accelerator modes.

This method allows the writing of a general expression for the correlation coefficients. For the piecewise linear case this is

$$C_{j+1} = \sum_{\substack{l_1 \\ l_1\neq 0}} \cdots \sum_{\substack{l_j \\ l_j\neq 0}} \frac{-1}{\pi^2 a^2(1-a)^2}\left(\frac{1}{\pi}\right)^j \frac{\sin(\pi l_1 a)\sin(\pi l_j a)}{l_1^2 l_j^2} \tag{3.4.8}$$

$$\times \prod_{k=1}^{j} \frac{l_k E \sin(\pi(l_k E - a\tilde{l}_k))}{(l_k E - a\tilde{l}_k)(l_k E + (1-a)\tilde{l}_k)}$$

where $\tilde{l}_k = l_{k-1} - 2l_k + l_{k+1}$; $l_0 = 0$, $l_{j+1} = 0$. The invariant region R is precisely the torus when $E\alpha > 4$ where $\alpha = \min(\frac{1}{a}, \frac{1}{1-a})$. Thus C_{j+1} vanishes when $l_k E - a\tilde{l}_k$ is a non-zero integer. An example of this is the choice a = 1.0 and E a positive integer which has been considered by Cary and Meiss[21]. For this sawtooth map (also known as a hyperbolic toral automorphism[125]) all C_j vanish for $j > 0$. Thus the diffusion constant exists and is well-defined.

3.4.1. Principal terms

Meiss et.al[88]. have studied the correlation expression for the Chirikov map and obtained an improvement to the Fourier-path approximation (3.3.2.3). By choosing the l_k in (3.4.5) they obtain the Principal Terms (i.e. those with the maximum number of factors, $g_m(lE)$ equal to unity) for the characteristic functions of all orders. These expressions can then be used to given the force correlation functions and hence a diffusion constant. Their expression is

$$D^P = \frac{E^2}{4}\,\frac{1-2J_1^2(E)-J_2^2(E)+2J_3^2(E)}{[1+J_2(E)]^2} \qquad (3.4.1.1)$$

This expression is compared with the numerics in fig. 3.4.1.1 . This expression improves the agreement between theory and experiment, especially near the peaks where the accelerating modes are dominant.

Meiss et.al. make no comment about the surprisingly good agreement for E ~ 1 where islands are present and the invariant region R is certainly not the torus. This point will be examined further in chapter 5 where an attempt is made to describe the behaviour in this region in a phenomenological way.

CHAPTER 4

Rigorous Results

4.1. Aim

The previous chapter demonstrated that the second velocity moment for the two example maps converged to an analytically obtained expression. Thus the quantity called 'a diffusion constant' appears to be a good characterising quantity that allows the maps to be understood as a classical random process, namely diffusion. Thus a purely deterministic system can be accurately understood as a random process - diffusion.

This chapter will attempt to formalise this equivalence. It brings together several mathematical methods of attack and discusses their consequences in terms of the examples considered previously. While not being complete, this chapter presents an attempt at starting with realistic map, showing that it is ergodic and Bernoulli (Wojtkowski) then introducing a transformation to an irreversible Markov process (Prigogine et.al.) which has a limiting distribution and finally obtaining a diffusion constant (Rechester et.al.) which is of use in physical experiments. This path is the main theme of this thesis and it is hoped that this attempt for the piecewise linear case will encourage similar results for the more general smooth systems, in the same fashion that results for the tent map in one dimension led to results for the logistic map.

The first step will be to give a mathematical proof, using the methods of chapter 1, of the equivalence of the piecewise linear map on the torus and a Bernoulli system ! This technique was given by Wojtkowski[126] for the piecewise linear case with a=0.5. In addition, he presented a methodology for obtaining the same results for any piecewise differentiable system with the derivative bounded away from 0[125]. At present it is not possible to extend these methods to the Chirikov map because there are points at which the forcing function is flat and so it is not possible to obtain hyperbolicity everywhere.

Given the Bernoullicity of the map, it is possible to use the methods of Misra, Prigogine and Courbage to obtain equivalence with a Markov process that converges to an equilibrium distribution i.e. the motion becomes irreversible and intrinsically random. Their method introduces a new formulation of the second law of thermodynamics which breaks the time-reversal symmetry by choice of initial conditions. There is much in their method that is similar to Quantum Mechanics with the introduction of complementary variables that do not commute. In this way, accurate determination of an entropy excludes accurate determination of trajectories in phase space. Given the isomorphism with this Markov process it should be possible to show that the original map can be described by a diffusion equation as discussed in chapter 2.

4.2. Proof that a map is Bernoulli

Essentially the method is to show that the stable and unstable manifolds intersect sufficiently to produce subsets of positive measure on which the motion is Bernoulli. Then show that there is only one such ergodic subset, i.e. the torus.

4.2.1. Almost Hyperbolicity

Definition :- a piecewise linear measure-preserving transformation on the torus $T : \mathbf{T}^2 \to \mathbf{T}^2$ is *almost hyperbolic* in an invariant domain $K \subset \mathbf{T}^2$ iff there are 2 families of disjoint segments passing through almost all points, $p \in K$, called local contracting, $\gamma^s(p)$ and expanding, $\gamma^u(p)$ fibres satisfying :

1) $\gamma^s(p)$ and $\gamma^u(p)$ intersect transversally

2) $T(\gamma^s(p)) \subset \gamma^s(T(p))$ and $T^{-1}(\gamma^u(p)) \subset \gamma^u(T^{-1}(p))$

3) if $p_1, p_2 \in \gamma^s(p)$ [or $\gamma^u(p)$] then $\mathrm{dist}(T^n p_1, T^n p_2) < \mathrm{const}.\lambda^n$ for $n > 0$ [or $n < 0$] where $0 < \lambda < 1$ and the constant depend on the fibre.

Theorem 4.1 : Let K be some domain of the torus such that for almost all points $p \in K$ there is a sector $U(p)$ in the tangent space satisfying :

1) $DL_K(U(p)) \subset U(L_K(p))$ where $L_K : K \to K$ is the first return map for K

2) for vectors, $v \in U(p)$ with $\| DT^{n+1}v \| \geq \| DT^n v \|$ for n>0 and for almost all $p \in K$ there exists $0<\lambda<1$, $k>0$ such that $\| DT^k v \| \geq \lambda^{-1} \| v \|$;

then T is almost hyperbolic in $K_1 = \bigcup_{i=-\infty}^{\infty} T^i(K)$.

§4.2.2 details the explicit construction of sectors U and invariant domains K with the expansion constant λ for the piecewise linear map. Thus the piecewise linear map T_a is almost hyperbolic on the torus for $E \geq 4(1-a)$ $(a<0.5)$. The fibres $\gamma^u(p)$ and $\gamma^s(p)$ provide a partition of the space with conditional measures equal to arc length i.e. they are absolutely continuous with respect to Lebesgue measure. Classical theory gives the existence of a family of invariant subsets K_i of positive measure such that $\bigcup_{i=1}^{\infty} K_i = K$ and $T|_{K_i}$ is Bernoulli.

Theorem 4.2 : Let $T : \mathbf{T}^2 \to \mathbf{T}^2$ be almost hyperbolic in an invariant domain $K \subset \mathbf{T}^2$ and suppose that for any local expanding and contracting fibres $\exists$ $N \geq 0$ such that $T^k \gamma^u(p)$ intersects $T^{-l} \gamma^s(q)$ whenever $k,l \geq N$.

Then T and all its powers are ergodic in $\mathbf{T}^2$(and thus Bernoulli).

Using this theorem and constructing the fibres it can be shown that the number of ergodic components is one (i.e. the torus) for $E > E_0$ where E_0 is slightly greater than E_T. This is done in §4.2.3.

4.2.2. Obtaining stable and unstable fibres

To simplify the problem, consider a translation of the forcing function (0.4.2) so that it becomes two linear pieces i.e. $f(x,a) = \left\{ \frac{a+x}{a-1}, -1 \leq x \leq 1-2a ; \frac{x-1+a}{a}, 1-2a \leq x \leq 1 \right\}$ $0<a<0.5$. In addition consider the map in the form $T_a(x,y) = (y, 2x-y+Ef(y;a))$: this simplifies the study of the invariant regions. If S denotes the transformation $S(x,y) = (y,x)$ then $S.T_a.S = T_a^{-1}$. This property is called S-reversibility.

Define fundamental domains :

$$\begin{aligned} B_1 &= \{(x,y) : -1 \le y \le 1-2a\} \\ B_2 &= \{(x,y) : 1-2a \le y \le 1\} \\ C_1 &= \{(x,y) : -1 \le x \le 1-2a\} \\ C_2 &= \{(x,y) : 1-2a \le x \le 1\} \end{aligned} \qquad (4.2.2.1)$$

Then T_a is linear in B_1 and B_2 and $T_a(B_i) = C_i$, for $i=1,2$. The differential of T_a in each region is given by the matrices :

$$DT_a|_{B_1} = \begin{bmatrix} 0 & 1 \\ -1 & 2-\dfrac{E}{1-a} \end{bmatrix} \qquad (4.2.2.2)$$

$$DT_a|_{B_2} = \begin{bmatrix} 0 & 1 \\ -1 & 2+\dfrac{E}{a} \end{bmatrix}$$

These matrices have eigenvalues given by :

$$\begin{aligned} \lambda^1_\pm &= 1 - b_1 \pm [b_1(b_1-2)]^{1/2} \text{ in } B_1 \\ \lambda^2_\pm &= 1 - b_2 \pm [b_2(b_1+2)]^{1/2} \text{ in } B_2 \end{aligned} \qquad (4.2.2.3)$$

where $b_1 = \dfrac{E}{2(1-a)}$ and $b_2 = \dfrac{E}{2a}$. Let ξ=dx and η=dy then a suitable choice of sector in the tangent space is :

$$U(p) = \begin{cases} \left\{(\xi,\eta) : 1-b_1+[b_1(b_1-2)]^{1/2} \le \dfrac{\xi}{\eta} \le 0\right\} & p\in C_1 \\ \left\{(\xi,\eta) : 0 \le \dfrac{\xi}{\eta} \le 1+b_2-[b_2(b_2+2)]^{1/2}\right\} & p\in C_2 \end{cases} \qquad (4.2.2.4)$$

Define a norm on the tangent sector as $\|(\xi,\eta)_i\| = \max(|\xi|,|\eta|)$ Let $\bar{u}_1$ be the eigenvector corresponding to λ^1_- normalised such that $\bar{u}_1 = (d,1)$ with $d<1$; let $\bar{u}_2$ be a similarly normalised eigenvector corresponding to λ^2_+ and $\bar{v} = (0,1)$ a unit vector.

The expansion constants can now be obtained by considering the effect of the map in the four regions $B_i \cap C_j$. Let $\bar{w} = \alpha\bar{u}_1 + (1-\alpha)\bar{v}$ be a vector in the tangent sector of a point, $p \in B_1 \cap C_1$, where α ranging from 0 to 1 gives all vectors in the sector U(p). Then $DT_a\bar{w} = \alpha\lambda^1_-\bar{u}_1 + (1-\alpha)[1, 2-2b_1]$. Since $|2(1-b_1)| \ge |1-b_1-[b_1(b_1-2)]^{1/2}$, this vector is an element of the sector at the image point i.e.

$$DT_a(U(p)) \subset U(T_a p) \tag{4.2.2.5}$$

In addition $||DT_a\overline{w}|| = \max\left\{|\alpha + (1-\alpha)|, |\alpha\lambda^1_- + (1-\alpha)(2-2b_1)|\right\}$. Thus $||DT_a\overline{w}|| \geq (b_1 - 1 + [b_1(b_1-2)]^{1/2})\, ||\overline{w}||$. Similar arguments for the other three regions give (4.2.2.5) and expansion constants ≥ 1. Thus the four relationships are :

$$\begin{aligned}
&DT_a\overline{w} \geq (b_1 - 1 + [b_1(b_1-2)]^{1/2})\, ||\overline{w}|| && B_1 \cap C_1\\
&DT_a\overline{w} \geq 2(b_1 - 1)\, ||\overline{w}|| && B_1 \cap C_2\\
&DT_a\overline{w} \geq (b_2 + 1 + [b_2(b_2+2)]^{1/2})\, ||\overline{w}|| && B_2 \cap C_2\\
&DT_a\overline{w} \geq 2(1 + b_2)\, ||\overline{w}|| && B_2 \cap C_1
\end{aligned} \tag{4.2.2.6}$$

Theorem 4.1 can now be applied to say that T_a is almost hyperbolic in a region K_1 which is the torus for $b_1 > 2.0$ as then the regions $B_i \cap C_j$ map onto each other.

4.2.3. Construction of the fibres and the proof of ergodicity

The expanding fibre γ^u is constructed as follows. Let $U_n(p) = DT^n(U(T^{-n}(p))$ for $p \in K$. Obviously $U_{n+1}(p) \subset U_n(p)$ thus $U_\infty(p) = \bigcap_{n=0}^{\infty} U_n(p)$ is a single line. Consider a segment through p with the direction of $U_\infty(p)$. Under T^{-1} this segment is contracted exponentially while p approaches a singularity line ($\partial C_1 \cup \partial C_2$)at a rate not faster than const.n^{-2} where n is the number of iterations (see Wojtkowski[125]). Taking a sufficiently short segment, it will never cut a singularity line. Define $\gamma^u(p)$ as the maximal segment with this property. Thus $\gamma^u(p)$ is constructed for almost all points in K and $\gamma^s(p) = S(\gamma^u(Sp))$.

$\gamma^u(p)$ is contained in either C_1 or C_2 by construction. Suppose that γ^u intersects the singularity lines ∂B_1 , ∂B_2 at most once then :

$$\begin{aligned}
&l(\gamma^u(p) \cap B_1) = \mu\, l(\gamma^u(p))\\
&l(\gamma^u(p) \cap B_2) = (1-\mu)\, l(\gamma^u(p))
\end{aligned} \tag{4.2.3.7}$$

where $0 \leq \mu \leq 1$ and $l(.)$ is the length of an arc defined by the norm $||.||$.

In order to apply theorem 4.2 it is necessary to find an E_0 such that for $E > E_0$ there exists an $h>0$ so that if $\gamma^u(p)$ crosses $\partial B_1 \cap \partial B_2$ at most once then

$T_a(\gamma^u(p))$ contains a segment $\gamma^u(q), q \in T_a(\gamma^u(p))$ such that $l(\gamma^u(q)) \geq (1+h)\, l(\gamma^u(p))$. Refering to (4.2.2.3) E_0 is the smallest E satisfying

$$\inf_{0 \leq \mu \leq 1} \left[\max \left\{ (b_1 - 1 + (b_1[b_1-2])^{½})\mu \, , \, 2(b_2+1)(1-\mu) \right\} \right] > 1 \qquad (4.2.3.8)$$

$$\inf_{0 \leq \mu \leq 1} \left[\max \left\{ (2(b_1-1)(1-\mu)) \, , \, \mu(b_2 + 1 + (b_2[b_2+2])^{½}) \right\} \right] > 1$$

The first condition is for initial segments $\gamma^u(p) \subset C_1$,the second for $\gamma^u(p) \subset C_2$. While μ depends on p in (4.2.3.6) , the expansivity factors in (4.2.3.6) are independent of p so the p variation is contained in the infimum over μ.

Using the result : for $r,s \geq 0$, $\inf_{0 \leq \mu \leq 1} \{ \max(r\mu, (1-\mu)s) \} = \frac{1}{r^{-1} + s^{-1}}$, (4.2.3.8) becomes

$$\left[\{b_1 - 1 + (b_1[b_1-2])^{½}\}^{-1} + \frac{1}{2(b_2+1)} \right] < 1 \qquad (4.2.3.9)$$

$$\left[\{b_2 + 1 + (b_2[b_2+2])^{½}\}^{-1} + \frac{1}{2(b_1-1)} \right] < 1$$

For $a \in (0,½]$ then $b_1 \in (\frac{E}{2}, E)$, $b_2 \in (\infty, E)$. These two relations give cubic equations in E, namely

$$E^3 + E^2(7a-4) + E(14a^2 - 12a) - 9a^2(1-a) \geq 0 \qquad (4.2.3.10)$$

$$E^3 - 5E^2(1-a) + 6E(1-a)^2 - a(1-a)^2 > 0 \qquad (4.2.3.11)$$

Numerically it is easily seen that (4.2.3.11) is positive when (4.2.3.10) is positive and the required E_0 is the smallest positive value satisfying (4.2.3.10).

Thus sufficiently distant images of an arbitrary local expanding fibre $\gamma^u(p)$ contain segments crossing $B_2 \cap C_2$ from bottom to top. S-reversibility gives sufficiently distant preimages of $\gamma^s(p)$ contain segments crossing $B_2 \cap C_2$ from left to right. Thus theorem 4.2 gives T_a ergodic on the torus for all $E > E_0$.

Thus the map T_a is ergodic on the torus and is a Bernoulli system there. This implies that the integrals in the previous chapter are well-defined and as the motion is mixing, it is very likely that correlations decay exponentially and the various summations exist !

4.3. Passage to an Irreversible Markov Process

There are two representations for dynamical systems. The first, due to Newton and Hamilton considers point transforms, S_t and trajectories on phase space. The second, due to Gibbs and Koopman studies the group of unitary operators, U_t acting on square integrable distribution functions. If ρ is a distribution function and ω an initial point then

$$(U_t\rho)(\omega) = \rho(S_{-t}\omega) \tag{4.3.1}$$

Misra, Prigogine and Courbage[47,91], have shown that for the *intrinsically random* Bernoulli and K processes, the unitary group U_t can be transformed by a bounded nonunitary linear operator Λ into a strict semigroup of contractive operators of a Markov process W_t^* that satisfy $W_t^* = \Lambda U_t \Lambda^{-1}$ and provide a monotonic entropy for the distribution $\tilde{\rho}_t = W_t^* \tilde{\rho}$ where $\tilde{\rho} = \Lambda\rho$. Thus there is a change of representation, involving no loss of information which allows the transformation of deterministic dynamics into a Markovian process.

In addition these intrinsically random systems satisfy uncertainty relationships[90]. There is a theorem of Poincaré which states that there can exist no function of phase space that has definite sign and increases monotonically to a maximum. Thus in order to obtain a nonequilibrium entropy, it is necessary to consider objects more general than functions on phase space. Misra chooses the functional $-\ln\left\langle \psi_t \,,\, M\psi_t \right\rangle$ where M is a positive linear operator on L^2 restricted to the energy surface and ψ_t is a function in this space evolving under the Hamiltonian. In the usual Hilbert space formulation, physical observables correspond to multiplication by functions of phase space and hence commute. The above entropy functional can be shown to be valid only if physical observables no longer commute. Thus classical random systems should possess complementary variables. This implies that a description in terms of phase space trajectories is only valid when an entropy is not defined. This then leads to a definition of chaotic as those systems and regions of phase space where entropy is positive and hence where the motion can be shown to be intrinsically random. Non-chaotic systems are those where the dynamics can be described by following a trajectory e.g. an elliptic orbit or an attracting fixed point.

4.3.1. Irreversibility

The above arguments give a transformation to a Markov process but do not show how to break the time-reversal symmetry. In equilibrium thermodynamics this is done by using a very large number of particles subject to a heat bath which supplies a randomising influence. For intrinsically random systems, no source of external randomness is required and the methods work for simple, low-dimensional systems. Proceed by studying those fibres that are physically realisable, i.e. the expanding fibres[32,86]. Martinez and Tirapegui[86] calculate explicitly the time evolution of the measures concentrated on the contracting and expanding fibres under the Markov process given by the simple Bernoulli map, the baker transform. They show that an initial distribution concentrated on an expanding fibre evolves in positive time to an equilibrium distribution with finite entropy while the stable fibres do not converge to equilibrium and have infinite entropy. This can be understood since it requires an infinite amount of information to determine a stable fibre, if a point is not on the stable fibre of interest then expansivity causes iterations to take it further away. Thus by choosing measures that are absolutely continuous when restricted to expanding fibres (thus representing physically realisable states) breaks time-reversal symmetry.

The original deterministic, reversible map has been shown equivalent to an irreversible Markov process with continuous distributions provided that the map was Bernoulli or K. It can be shown that a map must be at least mixing for this equivalence to hold but the necessary and sufficient point between mixing and K is not yet known.

4.4. Diffusion

There are two problems in obtaining a diffusion equation. The first concerns the existence of a limit to the summations involved in obtaining D. Since the systems considered are strongly mixing, the correlations used in §3.4 should decay sufficiently rapidly that the summation (3.4.7) converges to a finite D. This is provided that the infinite summations involved in defining C_j converge to finite values. The results of chapter 3 suggest that these summations are well-defined however in is possible that the series involved may be asymptotic in the sense of Poincaré (i.e. they provide an good approximation if truncated but diverge for a large number of terms)[68].

The second problem involves the definition of an extended Fokker-Planck equation. It is unlikely that the time-evolution of the Markov distribution will satisfy a second order diffusion equation, so all terms must be retained in (2.2.2) and the relative magnitudes of the terms obtained. Further, the mathematical results presented above are for processes on a bounded phase space e.g. the torus, thus the relationship between diffusion on an infinite cylinder and the Markov process on the torus must be made clear.

CHAPTER 5

Coexistence of chaos and ordered motion

5.1. Introduction

Chapter 3 presented a methodology for describing the complicated behaviour observed in model systems for large parameter values. There it was shown that a diffusion constant provided a good characterisation of the chaotic behaviour. However, from figure 3.3.4.1, it is clear that the large E approximation to the diffusion constant becomes inaccurate as E decreases below 2.0. This is not surprising as the Fourier-path method is an expansion in 1/E. In addition, structure appears in the form of islands centred on stable periodic points and around which particles tend to circulate before wandering elsewhere, thus appearing to give appreciable correlations. This chapter shows that the idea of diffusion still has relevance in a situation where there is little hope of proving ergodicity, let alone any stronger property.

The essential question is : "how does the presence of an ordered region affect the chaotic behaviour in the remainder of phase space?" From the figures in chapter 0 it can be seen that as the parameter is decreased from large values, elliptic islands appear in phase space. At their centres are stable periodic points, surrounded by elliptic K.A.M. curves. As the parameter decreases, these islands change their shape and other islands appear or disappear until finally a K.A.M. surface appears connecting opposite sides of the square. This cuts the cylinder into short lengths and prevents free motion along it - this is the end of any possibility of diffusion.

The proposed method of attack is to consider this problem using the standard techniques of diffusion theory. That is, given a specific material whose diffusion constant (or conductivity) is known, how does that diffusion constant change when the material is filled with holes or random regions of another substance. This problem has been tackled by a number of authors, in particular, by

Fricke[43] in 1924 with application to measuring the concentration of red blood cells (insulators) in serum (conductor). Fricke obtained an expression for the change in conductivity due to randomly placed spheroids in a medium. The next section discusses how his ideas may be applied to chaos with islands and gives an expression for the diffusion that makes use of the property that the islands are on a regular lattice. §5.3 details how the theory may be applied to the two mappings and presents results showing reasonable agreement between the simple theory and the numerical calculations.

5.2. Diffusion in a medium with holes

In order to obtain the diffusion in a medium with holes Fricke[43] suggested making the ansatz,

$$D_{eff} = F(\nu, s)\, D \qquad (5.2.1)$$

where D is the expected diffusion for the medium with no holes, F is a scaling function depending on the volume fraction, ν of the holes and a shape factor, s. Fricke obtained an expression for F when the holes were all equal spheroids and observed that F was insensitive to changes in the shape factor, s for $s \approx 2$ - the spherical case.

To obtain an expression for F in two dimensions, Rowlands[112] considered the specific problem of diffusion through a lattice of circular holes. Diffusion through a doubly-periodic plane can be represented as diffusion through the square of figure 5.2.1 The boundary conditions are :

1) no flow through the lines $y = \pm b$

2) no flow across the boundary, C i.e. there is a hole in the medium

3) constant equal fluxes into the square at $x = -b$ and out off the square at $x = b$.

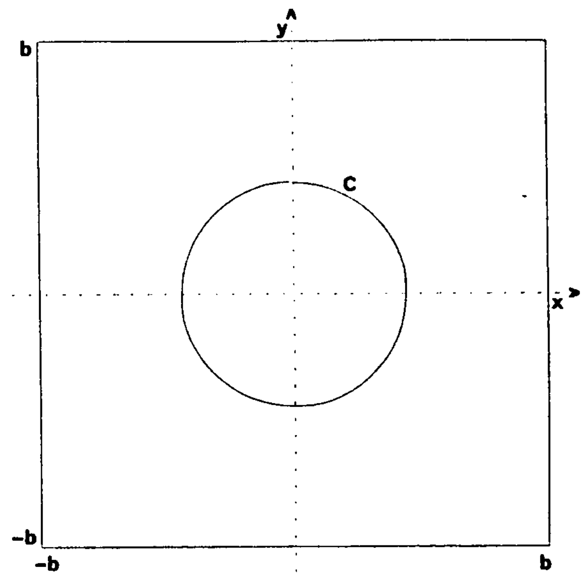


fig. 5.2.1 Diffusion in a medium with holes

It is then possible to obtain a variational expression for D_{eff} as :

$$D_{eff} = D\,b \left[\frac{\int \psi \frac{\partial \psi}{\partial x} dy}{\left(\int \psi dy \right)^2} \right]_{x=-b}$$

where ψ is a distribution function satisfying Laplace's equation $\nabla^2\psi = 0$ (see appendix A). A possible trial function is obtained by considering an infinite array of dipoles. As trial function consider :

$$\psi(x,y) = x + \frac{\pi\rho^2}{b} \frac{\sinh(\frac{2\pi x}{b})}{\left[\cosh(\frac{2\pi x}{b}) - \cos(\frac{2\pi y}{b})\right]}$$

here ρ is a measure of the size of the hole. It is not quite the radius of a circle since the trial function ψ gives boundary condition (2) over a non-circular shape. Setting the volume fraction, ν to be $\frac{\pi\rho^2}{4b^2}$ gives the equation for F as :

$$F(\nu,s) \equiv \frac{1}{1+2\nu} - \frac{C(s)\nu^2}{(1+2\nu)^2} \qquad (5.2.2)$$

with

$$C(s) = \frac{2\pi}{\sinh^2(\pi)}$$

This expression has a similar form to that obtained by Fricke for diffusion in three dimensions.

5.2.1. The growth of the islands

The variation of ν with E is difficult to obtain analytically for general maps but the piecewise linear map allows it to be found for a range of parameter. To obtain the variation for the piecewise linear map, notice that for $|x_n| < a$ the map is given by :

$$\begin{aligned} p_{n+1} &= p_n - \alpha x_n \\ x_{n+1} &= x_n + p_{n+1} \end{aligned} \tag{5.2.1.1}$$

with $\alpha = E/a$.

If all orbits are assumed to lie on the ellipses,

$$\frac{p^2}{A^2} + \frac{x^2}{B^2} + \frac{xp}{C^2} = 1 \tag{5.2.1.2}$$

then requiring (p_{n+1}, x_{n+1}) to be on the same ellipse as (p_n, x_n) gives relationships between the coefficients :

$$\begin{aligned} B^2 &= -C^2 \\ -\alpha B^2 &= A^2 \end{aligned} \tag{5.2.1.3}$$

Thus motion occurs on the one-parameter family of ellipses $p^2 - \alpha x^2 + \alpha xp = A^2$. The largest ellipse in the family is the one tangent to the singularity line x = a , whence $A^2 = a^2(\alpha - \frac{\alpha^2}{4})$. It can be shown that the area of an ellipse given by (5.2.1.2) is $\pi A/2$. Thus the area of the stable island is $\frac{\pi}{4}\sqrt{4Ea - E^2}$. Since the area of the torus is 4 in these units the volume faction, ν is given by :

$$\nu = \frac{\pi}{16}\sqrt{4Ea - E^2} \tag{5.2.1.4}$$

This expression is only valid when there is a single elliptic island, but this case covers a reasonable range for this map.

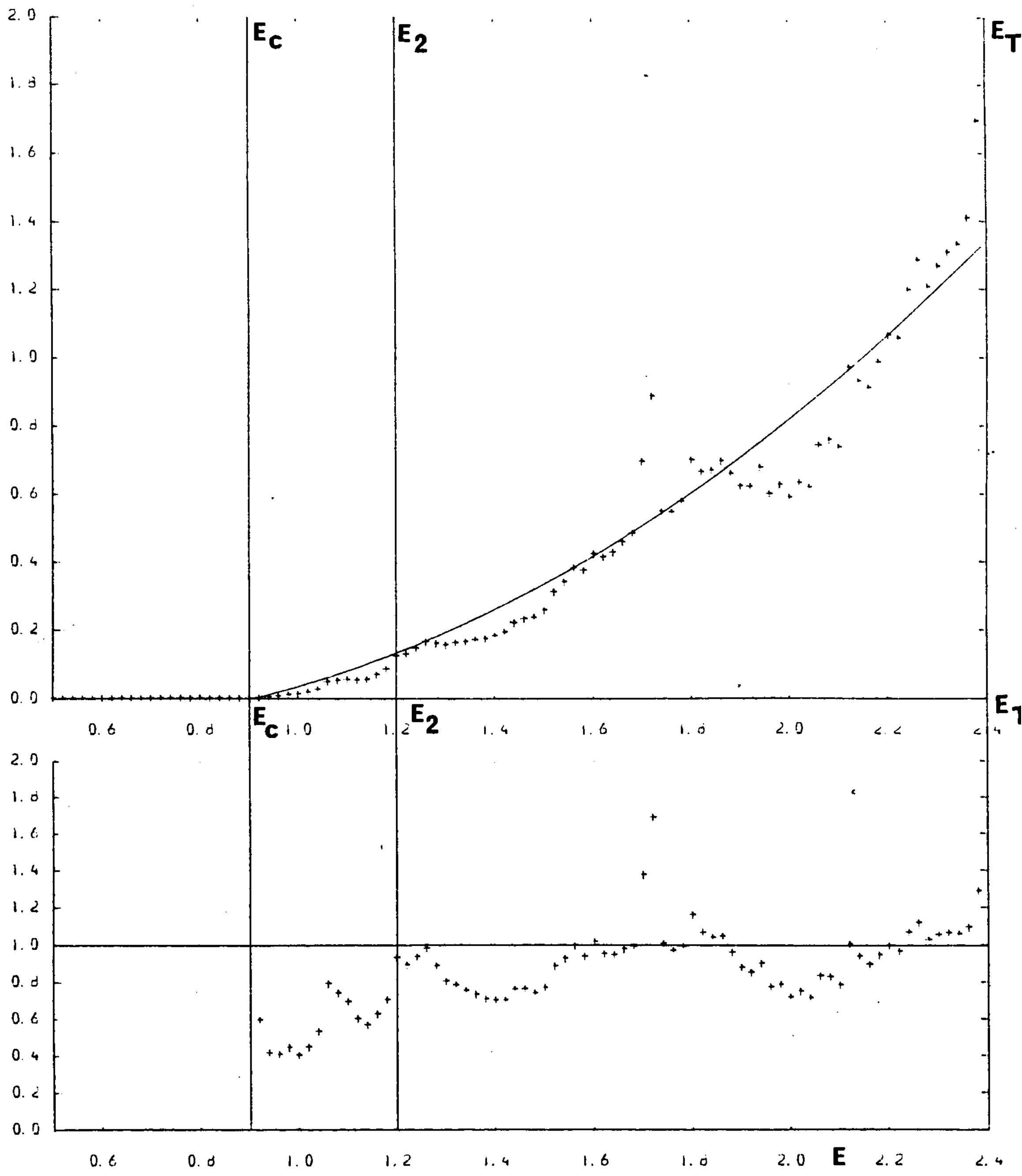


fig. 5.3.1.2 As fig. 5.3.1.1 with a=0.4

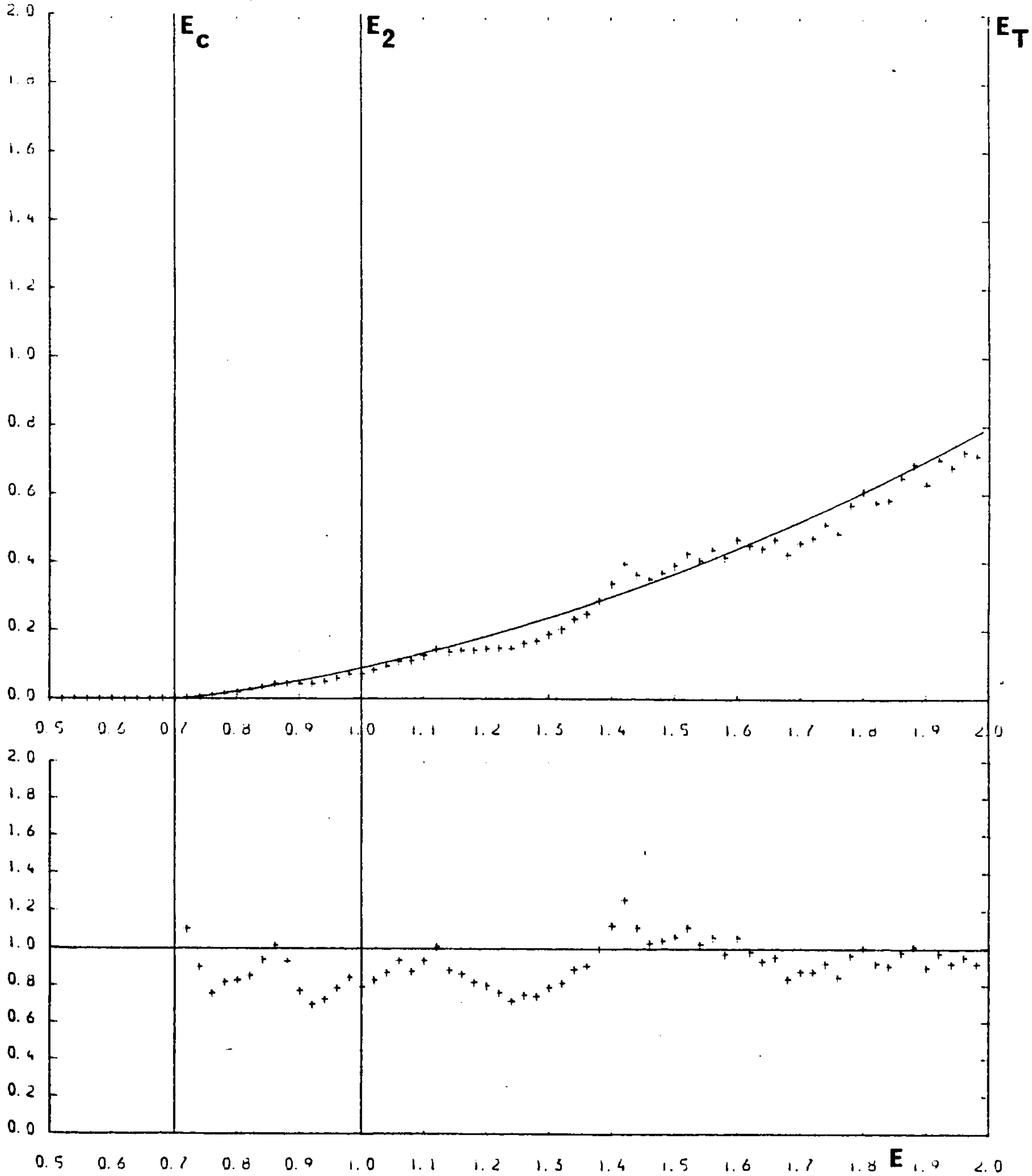


fig. 5.3.1.1 Comparison of (5.3.1) with numerics for the piecewise linear map: a=0.5

a) diffusion vs. E b) $\dfrac{\text{numerical diffusion}}{\text{theoretical diffusion}}$ vs. E

5.3. Numerical results

Before using the above results it is necessary to discuss their range of validity and certain other required assumptions. Let E_T denote the largest E at which a central stable island exists, let E_2 denote the largest E at which period 2 coexists with this central island and let E_c denote the smallest value of E at which connected motion occurs. It is now necessary to make a choice as to the form for D. Close to E_T choosing D to be the random phase approximation (e.g. $\frac{E^2}{6}$) gives good results but to obtain a reasonable fit over the complete range $E_T \to E_c$ use the form :

$$D = E(E-E_c)\,\alpha\,\frac{E_T}{(E_T-E_c)} \qquad (5.3.1)$$

where α is the coefficient of the random-phase approximation or a higher approximation so that theory and numerics matches at E_T.

5.3.1. The piecewise linear map

Here the Fourier-path method of chapter 3 is exact for $E \gtrsim E_T$. At $E = E_T = 4(1\text{-}a)$ a single central island appears and ν is given exactly by (5.2.1.4). This remains valid until $E = E_2 = 2a$. In the range $E_2 \to E_c$ no attempt is made to obtain ν as a function of E, rather D_{eff} goes to zero at E_c because of the form of D.

Figure 5.3.1.1a shows a plot of numerics and theory for a = 0.5 and D_{eff} given by :

$$D_{eff} = D\,\frac{1}{(1+2\nu)}$$

where D is given by (5.3.1) with $\alpha = \frac{1}{6}\frac{1.2}{1.3}$, i.e. random phase plus first order correction. The second term in (5.2.2) is sufficiently small to be neglected. Figure 5.3.1.1b gives the ratio, experiment over theory. Similarly fig. 5.3.1.2 for a=0.4

The figures show reasonable agreement between the diffusion approxima-

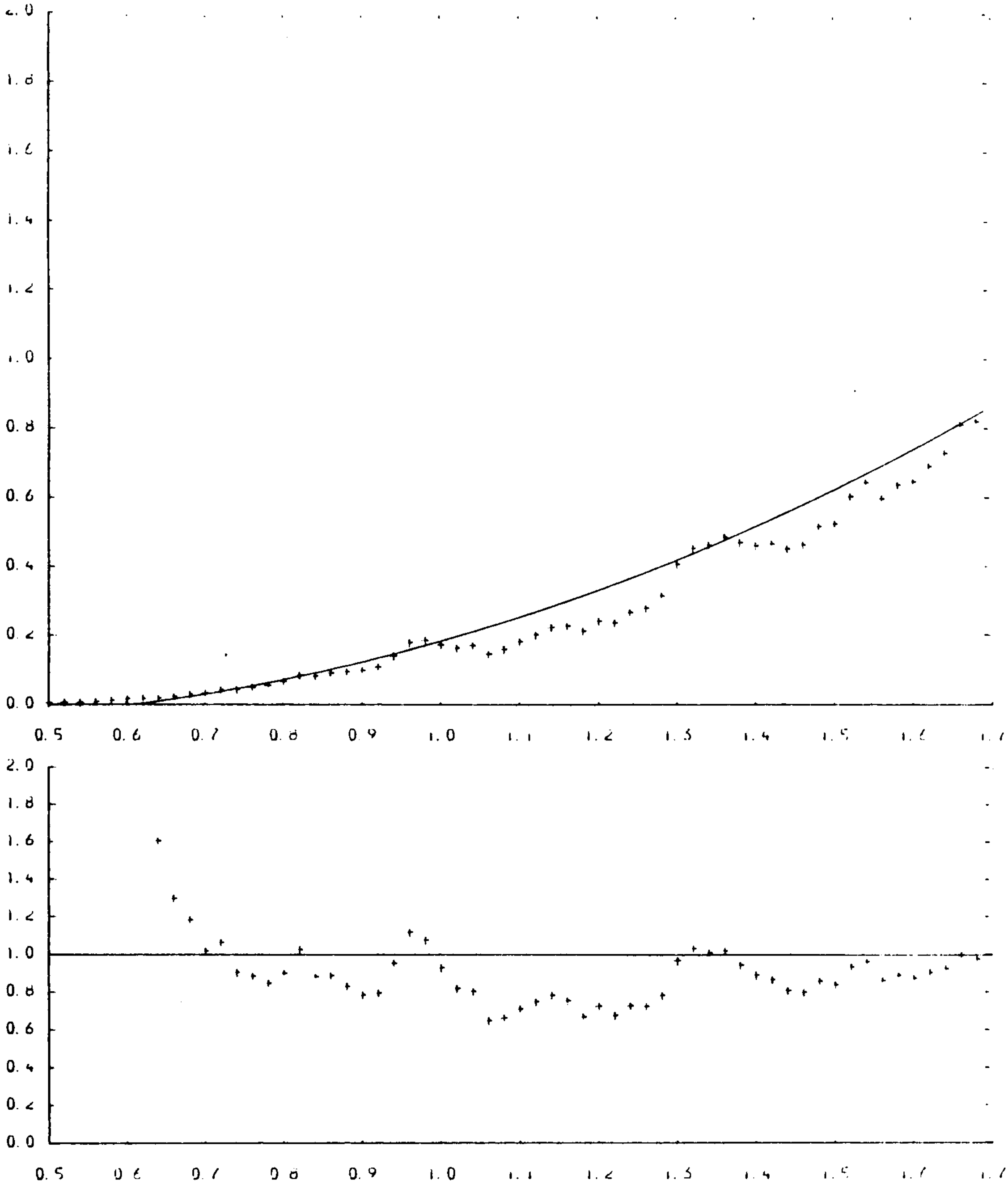


fig. 5.3.2.1 As fig. 5.3.1.1 for the Chirikov map

tion and the calculated average v^2. The disagreement near 1.7 for a = 0.4 is due to accelerating regions. Refering back to figures 3.3.4.1 and 3.3.4.2, it can be seen the above approximation improves greatly on the Fourier-path result.

5.3.2. The Chirikov map

Again the Fourier-path method gives good agreement for $E \gtrsim E_T$, though for this map the fixed point disappears by bifurcating into two points of period 2 each surrounded by a region of stable elliptic curves.

Figure 5.3.2.1a shows good agreement between the analytic approximation and the numerical results. The analytic curve is given by the same model as for the piecewise linear case but with the following choice of constants :

$E_T = 1.7$

$E_c = 0.6$

$\alpha = \frac{1}{4}1.2$

The problem of finding E_c has been studied extensively[50] and it is now possible to obtain theoretical estimates with great accuracy. Unfortunately, the theoretical estimates are several percent smaller than those observed numerically. This is due to the finite run-time of computer simulations and possible residual shadows of the K.A.M. curve [83]. Thus the value used for E_c above is chosen from the numerics. Additional details for the structure near E_c can be obtained using renormalisation group theory[29,51] and so may lead to better approximations to the scaling of ν and D_{eff} in this region.

While this approximation gives a better fit than the method of Fourier-paths, the Principal Terms expression of Meiss et.al. is even better (see fig. 5.3.2.2). Since the Principal Terms expression assumes motion over the whole torus, this good agreement shows that the change in diffusion is due to correlations. This might be expected as the presence of an island increases correlations as orbits are forced to circulate around the island.

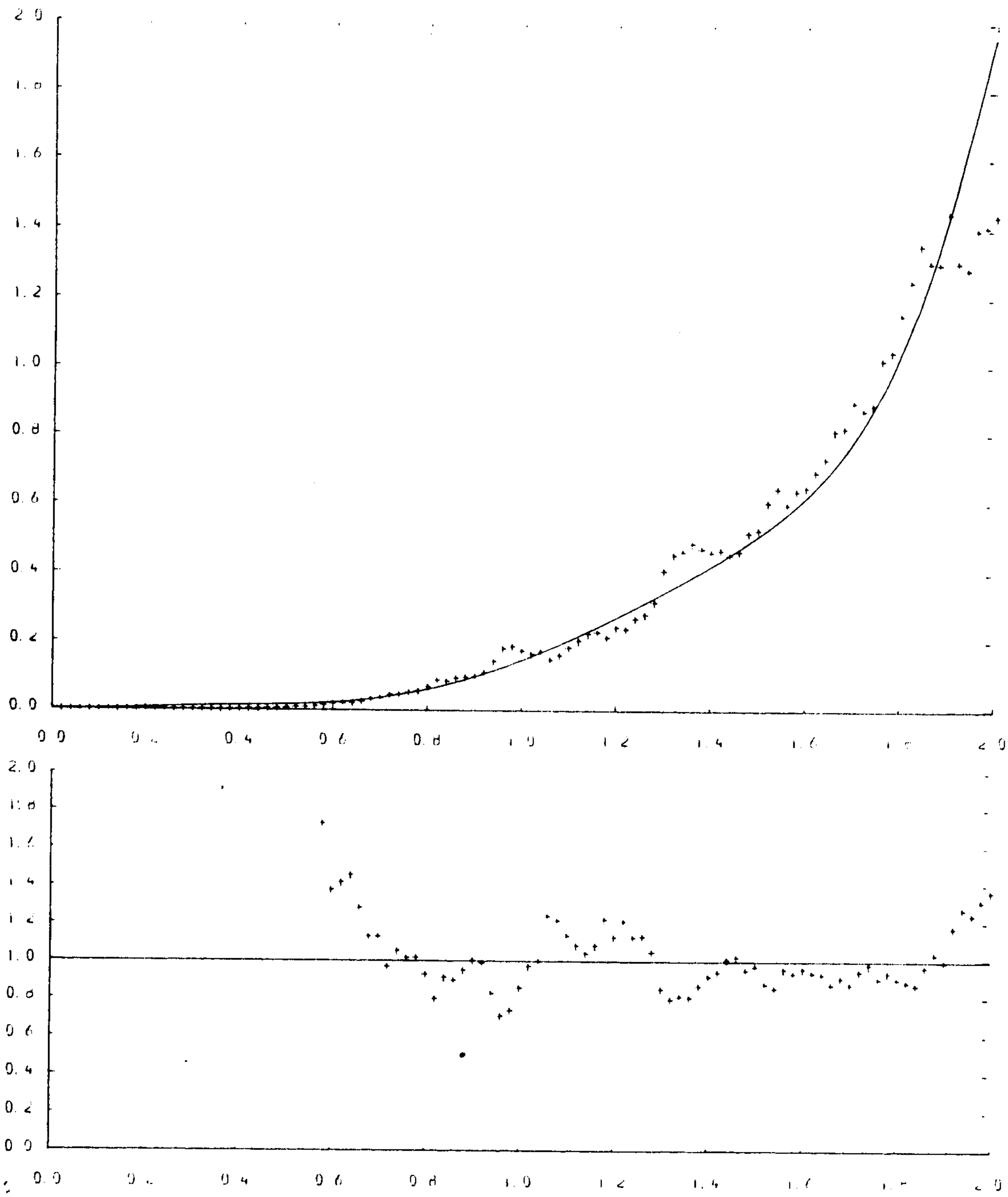


fig. 5.3.2.2 Comparison of numerics with the Principal Terms expression for the Chirikov map

Thus the model of Fricke can be applied to the study of discrete maps when numerics suggests that the motion is diffusion about a hole. This model gives good results for the piecewise linear case where a volume fraction can be precisely calculated and it may be extended to the Chirikov map (though this case is covered by the method of Meiss et.al[88].).

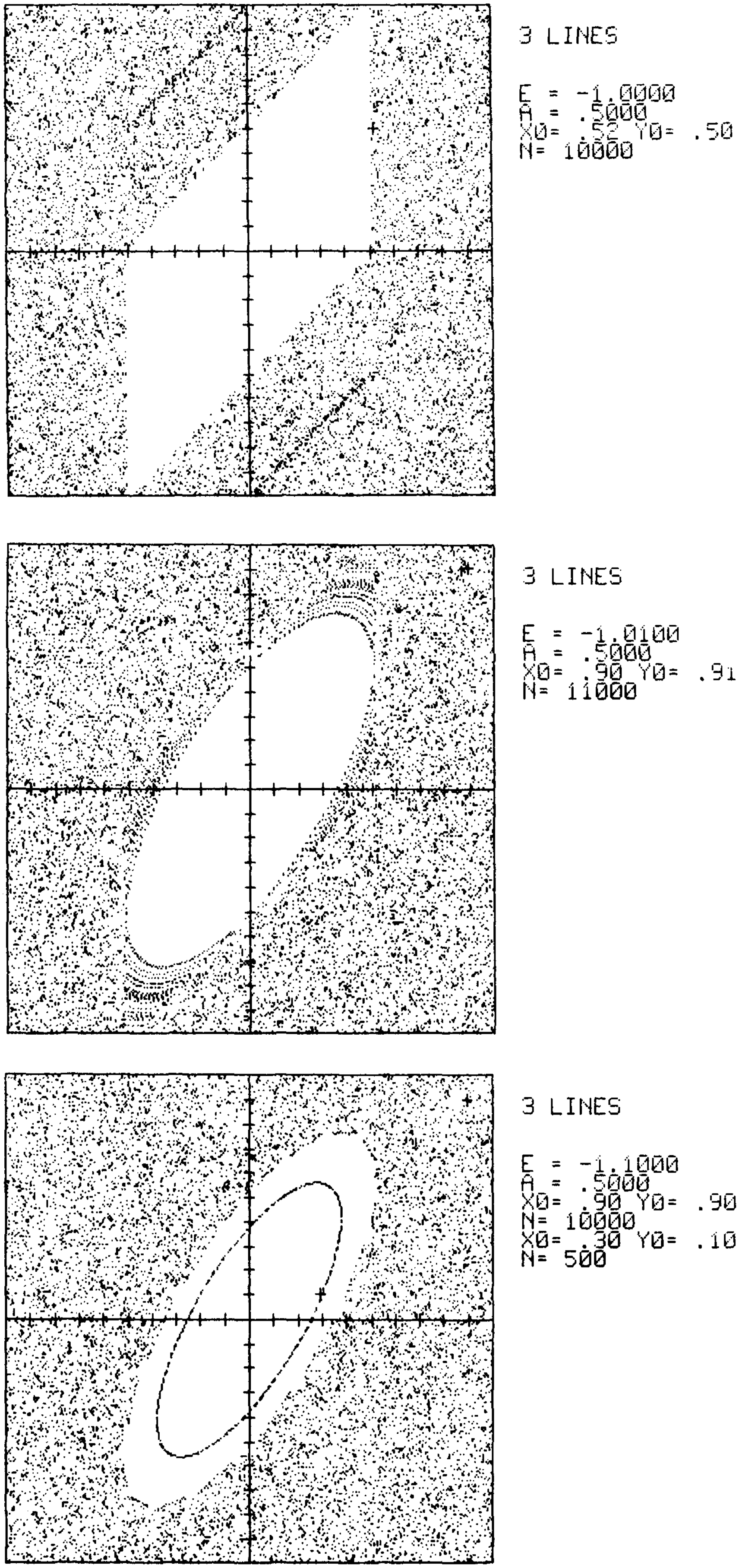


fig. 5.4.1 Correlations in the piecewise linear map: a = 0.5

a) E=−1.0 b) E=−1.01 c) E=−1.1

5.4. Rigorous results

For the piecewise linear case with a=0.5, Wojtkowski has shown that there is a set of parameter values $\{E_n\}$ satisfying $E_n = (1 + \cos\frac{\pi}{n})$ where the stable island is a 2n-sided convex polygon. (Indeed for each n there is a line in (a,E)-space along which the stable island is a 2n-sided polygon.) In addition the motion is almost hyperbolic outside this island. As these polygons are structurally unstable, the stable islands observed in the neighbourhoods of E_n are ellipses but with highly correlated motion in the regions that would have been inside the polygon if they changed continuously (see figure 5.4a-c). This highly correlated motion will give great difficulties in any rigorous proof of hyperbolicity or ergodicity, but from the above work it does not appear to affect the diffusion constant greatly.

CHAPTER 6

Diffusion for small perturbations

6.1. Introduction

The previous two chapters considered the characterisation of deterministic systems by a diffusion constant when those systems appeared random. Such systems can be viewed as large perturbations of ordered integrable systems, the size of the perturbation bringing about interactions and resonance between the natural frequencies of oscillation in the system. These are examples of intrinsic randomness.

This chapter considers very small perturbations of integrable systems where the KAM theorem[4, 93] or its extension[59] gives the existence of many closed curves which entrap the motion. In this case there is little intrinsic chaos, certainly not sufficient to generate a diffusion process. However, in analogy with the plasma confinement problem, it is interesting to examine how the system dynamics alter an imposed noise (usually gaussian) which can be used to model a collisional process[27]. Thus the system to be studied is chosen as :

$$p_{n+1} = p_n + Ef(x_n) \qquad (6.1.1)$$
$$x_{n+1} = x_n + p_{n+1} + \xi_n \mod 2$$

where f(x) is periodic of period 2 : ξ_n is a random variable with some distribution $P(\xi)$. It is convenient to choose $P(\xi) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{\xi^2}{2\sigma}}$. As before, motion is on the cylinder $(p,x) \in \mathbf{R} \times [-1,1]$ with the KAM curves hindering large-scale motion in the p-direction.

6.2. A Diffusion constant to $O(E^2)$

A diffusion constant might now be obtained using the path-summation procedure of Rechester et.al.[106], however this requires summation of an infinite number of diagrams even at the simplest level. A more profitable approach is to use perturbation theory (Cohen and Rowlands[27]), expanding in the small parameter E about some chosen orbit given by $p = p_0$. Equations (6.1.1) have the exact formal solution :

$$p_n = p_0 + Ef(x_0) + E\sum_{l=1}^{n-1} f(x_0 + \sum_{j=1}^{l}(p_j + \xi_{j-1})) \tag{6.2.1}$$

This expression can be iterated in E, stopping the procedure by setting $p_j = p_0 \equiv p$. To $O(E^2)$ the diffusion constant is defined as $D = \lim_{n\to\infty} \frac{\langle \Delta p_n^2 \rangle}{2n}$, where the average is over some ensemble of $\{\xi_j\}$ and $\Delta p_n = p_n - p_0$. Representing f as a Fourier expansion $f(x) = \sum_{\alpha=-\infty}^{\infty} f_\alpha e^{i\alpha\pi x}$ gives :

$$\begin{aligned}\Delta p_N^2 = E^2 \Bigg[& 2\sum_{l=1}^{N-1}\sum_{\sigma=-\infty}^{\infty} f_\sigma e^{i\sigma\pi x_0} \sum_{\tau=-\infty}^{\infty} f_\tau e^{i\tau\pi(x_0 + lp + \sum_{j=0}^{l-1}\xi_j)} \\ & + \sum_{l=1}^{N-1}\sum_{\sigma=-\infty}^{\infty} f_\sigma e^{i\sigma\pi(x_0 + lp + \sum_{j=0}^{l-1}\xi_j)} \sum_{\tau=-\infty}^{\infty} f_\tau e^{i\tau\pi(x_0 + lp + \sum_{j=0}^{l-1}\xi_j)} \\ & + 2\sum_{m=2}^{N-1}\sum_{l=1}^{m}\sum_{\sigma=-\infty}^{\infty} f_\sigma e^{i\sigma\pi(x_0 + mp + \sum_{j=0}^{m-1}\xi_j)} \sum_{\tau=-\infty}^{\infty} f_\tau e^{i\tau\pi(x_0 + lp + \sum_{j=0}^{l-1}\xi_j)} \Bigg]\end{aligned} \tag{6.2.2}$$

Here $f^2(x_0)$ has been neglected as it is $O(1)$. To obtain terms independent of x_0, set $\sigma = -\tau$. If the ξ_j are normally distributed then

$$\int_{-\infty}^{\infty} \Big(\prod_{j=0}^{n-1} \frac{e^{\frac{-\xi_j^2}{2\sigma}}}{\sqrt{2\pi\sigma}} d\xi_j\Big)\, e^{i\alpha\pi \sum_{l=m}^{n-1}\xi_l} = e^{\frac{-\alpha^2\pi^2\sigma(n-m)}{2}}$$

whence

$$\begin{aligned}\langle \Delta p_N^2 \rangle = E^2 \Bigg\{ & 2\sum_{l=1}^{N-1}\sum_{\tau=-\infty}^{\infty} f_{-\tau} f_\tau e^{i\tau\pi l p}\, e^{\frac{-\tau^2\pi^2\sigma l}{2}} \\ & + (N-1)\sum_{\tau=-\infty}^{\infty} f_{-\tau} f_\tau \\ & + 2\sum_{m=2}^{N-1}\sum_{l=1}^{m-1}\sum_{\tau=-\infty}^{\infty} f_{-\tau} f_\tau\, e^{i\tau\pi p[m-l]}\, e^{\frac{-\tau^2\pi^2\sigma[m-l]}{2}} \Bigg\}\end{aligned} \tag{6.2.3}$$

Setting $a = i\tau\pi p - \frac{\tau^2\pi^2\sigma}{2}$ gives an expression for the diffusion :

$$D = \frac{E^2}{2}\sum_{\tau=-\infty}^{\infty} f_{-\tau} f_{\tau}\left[\frac{1+e^{-a}}{e^{-a}-1}\right] + O(E^3) \tag{6.2.4}$$

All terms containing x_0 have been neglected: this can be justified either by integrating over an initial uniform distribution in the x-direction or by noticing that summations over terms containing x_0 leave an N dependence in the exponentials and thus are O(1).

Notice the implicit assumption that $f_0 \equiv 0$ else a term O(N) arises in (6.2.3) i.e. a non-symmetric forcing leads to a drift rather than diffusion.

For the standard map, $f(x) = \sin(\pi x)$, $f_1 = \frac{1}{2i}$, $f_{-1} = \frac{-1}{2i}$ and $f_i = 0\ \forall i \neq \pm 1$ whence

$$D = \frac{E^2}{4}\left\{\frac{1 - e^{-\pi^2\sigma}}{1 + e^{-2\pi^2\sigma} - 2\cos\pi p . e^{-\pi^2\sigma}}\right\} \tag{6.2.5}$$

This agrees with Cohen and Rowlands[27] and the result of Rechester et.al[106]. obtained by path-summation.

A detailed examination of this result for the piecewise linear map has not yet been performed.

This result is valid for 'short times', $n < E^{-2}$. However Cohen and Rowlands[27] obtain the more useful 'long time' diffusion, D_{eff} from this result using the expression : $D_{eff} = \left[\left\langle \frac{1}{D} \right\rangle_p\right]^{-1}$ where $\langle . \rangle_p$ denotes an average over the initial p. Thus the asymptotic diffusion can be derived from a short-time perturbation theory. Numerical results of Rechester et.al[106]. indicate that there is a significant contribution to D_{eff} from higher order contributions, thus §6.4 extends the above perturbation method to $O(E^4)$.

6.3. Expression for D to order E^4

Before obtaining the diffusion constant to $O(E^4)$ it is necessary to examine the transport equations describing the evolution of the phase-space distribution to obtain a definition of the diffusion-like quantity. The method follows

Chandrasekhar's derivation of the Fokker-Planck equation as given in Cohen and Rowlands[27].

Let $F(x,p,t)$ be the phase-space distribution function and the p-distribution given by the projection $N(p,t) \equiv \int dx F(x,p,t)$. Using the forward equation (2.2.2) gives :

$$N(p,t+\Delta t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\partial^k}{\partial p^k} \int dx\, F(x,p,t)\, [\Delta p(x,p,\xi)]^k \tag{6.3.1}$$

where Δt is a small time step and $\Delta p(x,p,\xi)$ is the change in p due to evolution of some deterministic equations in time Δt plus a random variable ξ. Expanding the L.H.S. to $O(\Delta t)$ and the R.H.S. to $O(\Delta p^4)$ gives :

$$\begin{aligned}\Delta t \frac{\partial N}{\partial t} = \int dx \frac{\partial}{\partial p}\Bigg\{ & F\left[-\Delta p + \frac{1}{2}\frac{\partial}{\partial p}(\Delta p^2) - \frac{1}{6}\frac{\partial^2}{\partial p^2}(\Delta p^3) + \frac{1}{24}\frac{\partial^3}{\partial p^3}(\Delta p^4)\right. \\ & + \frac{\partial F}{\partial p}\left[\frac{\Delta p^2}{2} - \frac{1}{3}\frac{\partial}{\partial p}(\Delta p^3) + \frac{1}{8}\frac{\partial^2}{\partial p^2}(\Delta p^4)\right] \\ & + \frac{\partial^2 F}{\partial p^2}\left[-\frac{\Delta p^3}{6} + \frac{1}{8}\frac{\partial}{\partial p}(\Delta p^4)\right] \\ & + \frac{\partial^3 F}{\partial p^3}\frac{\Delta p^4}{24}\Bigg\}\end{aligned} \tag{6.3.2}$$

Assuming F to be a smooth slowly varying function of p means that the terms $\frac{\partial^2 F}{\partial p^2}$ and $\frac{\partial^3 F}{\partial p^3}$ may be ordered out.

$$\therefore \quad \Delta t \frac{\partial N}{\partial t} = \int dx \frac{\partial}{\partial p}\left[AF + D' \frac{\partial F}{\partial p}\right] \tag{6.3.3}$$

Assuming that (6.3.3) models a diffusion equation and that the R.H.S. is a current, then F independent of p means there is no gradient to drive a flow and $\frac{\partial N}{\partial t}$ should be zero, thus require that $A \equiv 0$. This gives

$$\frac{\partial N}{\partial t} = \int dx \frac{\partial}{\partial p}\left\{\tilde{D} \frac{\partial F}{\partial p}\right\}$$

where

$$\tilde{D} = \frac{1}{n\tau}\left[\frac{\Delta p^2}{2} - \frac{\partial}{\partial p}\left\{\frac{\Delta p^3}{3}\right\} + \frac{\partial^2}{\partial p^2}\left\{\frac{\Delta p^4}{8}\right\}\right] \tag{6.3.4}$$

$\tau \equiv \frac{\Delta t}{n}$ is the time per iteration of the map and will be taken to be 1. Δp remains a function of the random variables ξ so define an averaging such that $\left\langle \psi \right\rangle = \int \left[\prod_{i=0}^{n-1} d\xi_i\, P(\xi_i)\right] \psi(\xi_0, \ldots, \xi_{n-1})$. The diffusion constant, D is defined to be $\left\langle \tilde{D} \right\rangle$.

To $O(\Delta p^2)$ this is the usual definition of a diffusion constant, as was used in §6.2. To study the E dependence expand as a formal power series in E, i.e.

$$\Delta p = a_1 E + a_2 E^2 + a_3 E^3 + a_4 E^4 + \cdots$$

where $a_i \equiv a_i(p,\xi)$. This gives the higher moments as : $\Delta p^2 = a_1^2 E^2 + 2a_1 a_2 E^3 + (a_2^2 + 2a_1 a_3)E^4 + \cdots$; $\Delta p^3 = a_1^3 E^3 + 3a_1^2 a_2 E^4 + \cdots$ and $\Delta p^4 = a_1^4 E^4 + \cdots$. Given that N(p,t) evolves under a symmetric system (like the Chirikov map) then the diffusion constant may be written

$$D = \frac{1}{n}\left\{ \frac{\left\langle a_1^2 \right\rangle}{2} E^2 + E^4 \left[\frac{\partial^2}{\partial p^2} \frac{\left\langle a_1^4 \right\rangle}{8} - \frac{1}{3}\frac{\partial}{\partial p}\left(\left\langle a_2^2 \right\rangle + 2\left\langle a_1 a_3 \right\rangle \right) \right] \right\} \qquad (6.3.5)$$

Thus to obtain the $O(E^4)$ correction to the diffusion it is only necessary to obtain the $O(E^3)$ terms, a_1, a_2 and a_3 in the expansion of Δp. (Notice that the requirement that A=0, can be used to give relationships between the a_i ,e.g. $a_2 = \frac{1}{2} \frac{\partial}{\partial p} a_1^2$ as is used by Cohen and Rowlands.)

6.4. The use of REDUCE to obtain the high order corrections

Having derived an expression for D, this section presents a method for obtaining the a_i's and the averages with respect to the noise. This calculation rapidly became algebraically complex and so a computer package for algebraic manipulation (REDUCE) was employed. Unfortunately the calculation could not be split into sufficiently small sections to be handled fully by the local mainframe. Thus no detailed results are presented, just the method by which the problem might be tackled.

Expanding (6.2.1) to third order gives :

$$\begin{aligned} p_n - p_0 = \; & E \sum_{l=1}^{n-1} \sin[l] \qquad (6.4.1) \\ & + E^2 \sum_{l=3}^{n-1} \cos[l] \sum_{j=1}^{l} \sum_{a=0}^{j-1} \sin[a] \end{aligned}$$

$$+ E^3 \left\{ \sum_{l=3}^{n-1} \cos[l] \sum_{j=1}^{l} \sum_{a=0}^{j-1} \cos[a] \sum_{b=1}^{a} \sum_{c=0}^{b-1} \sin[c] - \tfrac{1}{2} \sum_{l=3}^{n-1} \sin[l] \left(\sum_{j=1}^{l} \sum_{a=0}^{j-1} \sin[a] \right)^2 \right\}$$

where [l] denotes ($x_0 + lp + \sum_{m=0}^{l-1} \xi_m$). The double summation $\sum_{j=1}^{l} \sum_{a=0}^{j-1} \sin[a]$ may be replaced by $\sum_{j=0}^{l-1} (l-j)\sin[j]$. Combining the products of the four trigonometric functions and neglecting terms containing x_0 allows the terms to be written as $\frac{\pi}{4} \mathrm{Im}\tilde{S}(f,l,m,n)$ where $\tilde{S}$ is a sum or difference of the three terms : $S_1(l,m,n) = e^{i[f-l+m-n]}$; $S_2(l,m,n) = e^{i[f-l-m+n]}$; $S_3(l,m,n) = e^{i[f+l-m-n]}$, where [.] is as before. f is the summation variable with largest range, i.e. from 3 to n-1, and l,m,n are the other summation variables. Written in this form enables the use of relationships between the S_i to simplify the equations.

In order to perform the averaging over the noise, the summation variables must be ordered so that the summations of ξ_m are definite. If f > l > m > n then :

$$\left\langle S_1 \right\rangle \equiv T_1 = e^{i(f-l+m-n)p}\, e^{-\frac{\sigma}{2}(f-l+m-n)}$$
$$\left\langle S_2 \right\rangle \equiv T_2 = e^{i(f-l-m+n)p}\, e^{-\frac{\sigma}{2}(f-l-m+n)}$$
$$\left\langle S_3 \right\rangle \equiv T_3 = e^{i(f+l-m-n)p}\, e^{-\frac{\sigma}{2}(f+3l-3m-n)}$$

All the required expressions may now be written in the form :

$$\sum_{f=3}^{n-1} \sum_{l=2}^{f-1} \sum_{m=1}^{l-1} \sum_{n=0}^{m-1} P(f,l,m,n)\, T_i(f,l,m,n) \qquad (6.4.2)$$

where P is a polynomial.

The use of REDUCE (described in appendix B) allows checking of these expressions by choosing n=5,6,7 and comparing the terms generated by the summations with a direct expansion of $p_n - p_0$. Using the methods described in appendix B, an attempt has been made to calculate the summations for a general n and so obtain a correction to the expression for the diffusion. Unfortunately lack of core memory and process time prevented the completion of this program. The method proposed did provide a result as a function of n but contained many terms proportional to $e^{-\sigma n}$. Process time was excessive at each step and the attempts to simplify and collect the many expressions resulting from the summations had insufficient memory and so caused the program to fail. The current situation would require many hours of computer time to obtain the summations (~100 hours) followed by many hours of human intervention to obtain a concise result - thus it was judged infeasible to proceed.

CHAPTER 7

Lyapunov exponent for 1D maps

7.1. Introduction

In one dimension it is usual to consider maps confined to an interval since only bounded motion is of interest (however diffusion has recently been studied for unbounded motion[108]). 1D maps have been investigated in great detail, especially the logistic map $F_A(x) = A - x^2$; $A \in [0,2], x \in [-1,1]$. As there are only '2 degrees of freedom' this type of map can easily be examined numerically and any mathematical results are easily visualised. (General reviews are Shaw[116] and the book by Collet and Eckmann[28].)

For bounded motion a diffusion constant has no meaning, instead the Lyapunov exponent, λ is used. This measures the rate of separation of neighbouring orbits and is defined by,

$$\lambda_A = \frac{1}{\ln 2} \int \ln\left|\frac{dF_A(x)}{dx}\right| d\mu_A(x) \qquad (7.1.1)$$

Here $d\mu_A(x)$ is an invariant measure for F_A at a specific parameter value A. For any value of A there will be an infinity of invariant measures : it is usual to choose the unique measure that is absolutely continuous with respect to Lebesgue measure i.e. $d\mu_A(x) = \rho(x)dx$.

The form of λ_A can easily be obtained by numerical simulation see figure 7.4.2. Here the average in (7.1.1) is approximated by summing over a large number of iterations of the map, namely

$$\lambda_A = \frac{1}{\ln 2} \frac{1}{n} \sum_{i=1}^{n} \ln\left|\frac{dF_A^i(x_0)}{dx}\right| \qquad (7.1.2)$$

For fig. 7.4.2, x_0 took values from 1.25 to 2.0 at increments of 0.0025 with n=1000. This chapter describes a analytic approximation to this behaviour.

The behaviour of the map is completely described by the periodic orbits when these are stable. Assuming that these orbits still dominate the behaviour when they are unstable, McCreadie and Rowlands[87] obtain a good approximation to the Lyapunov exponent. A special case of this idea has been used by Geisel, Nierwetberg and Keller[46] to describe the scaling behaviour near the period-doubling onset of chaos for the logistic model. The idea of continuing the periodic orbits into regions where they are unstable has been discussed by Kai and Tomita[67], who also appeal to the rigorous limiting results of Bowen and Ruelle[13, 113]. The importance of the unstable orbits has also been discussed by Grebogi, Ott and Yorke[48] who show that when an unstable periodic orbit intersects a basin of attraction for a chaotic region, there is a discontinuous increase in the size of the basin, i.e. periodic orbits still have a great effect on the chaotic structure even when they are unstable.

7.2. An invariant measure

For the map F_A there are 2 distinct types of behaviour :

1) there exists a stable periodic orbit of period n ;

2) all periodic orbits are unstable and there exists a chaotic orbit .

In case (1) it is natural to use a measure consisting of δ-functions concentrated on the points of the stable orbit:

$$\mu_A(x) = \frac{1}{n}\sum_{i=1}^{n} \delta(x-p_i) \tag{7.2.1}$$

where p_i denotes the i^{th} point of the orbit with period n.

In case (2), following Tomita and Kai[120], consider all periodic orbits that have become unstable, then choose a measure

$$\mu_A(x) = \lim_{n\to\infty} \mu_A^n(x) \tag{7.2.2}$$

$$\mu_A^n(x) = \sum_{i=1}^{N(n)} w_i\,\delta(x-p_i)$$

where N(n) is the number of points of period n, w_i is a weighting factor.

(7.2.2) gives an invariant measure that is composed of an infinity of δ-functions, which would appear to be unobservable, however Ruelle[113] has shown that under certain conditions the limit $\mu_A(x)$ converges to the unique invariant measure absolutely continuous with respect to Lebesgue measure.

The weighting factor is obtained from a general variational principle[120] which is similar to the mathematical idea of pressure[13] for the case of Axiom A maps. Thus

$$w_i = \frac{C_n}{|(F^n)'(p_i)|} \tag{7.2.3}$$

with C_n chosen such that $\sum w_i = 1$. Let p_i^m denote the i^{th} element of the m^{th} orbit of period n, then $N(n) = kn$, $0 \le i \le n-1$, $1 \le m \le k$. Using the chain rule, for a given j shows that

$$(F^n)' = \prod_{i=1}^{n-1} F'(p_i^m)$$

Substituting in (6.2.2) gives

$$\mu_n(x) = \sum_{m=1}^{k} \frac{C_n}{\left|\prod_{i=0}^{n-1} F'(p_i^m)\right|} \sum_{i=0}^{n-1} \delta(x-p_i^m) \tag{7.2.4}$$

Thus (7.2.2) gives the Lyapunov exponent as :

$$\lambda_A = \lim_{n\to\infty} \lambda_n \tag{7.2.5}$$

$$\lambda_n = \sum_{i=1}^{N(n)} w_i \ln|F'(p_i)|$$

Thus both the invariant measure and the Lyapunov exponent can be obtained from the product of the gradients along a periodic orbit i.e. it is not necessary to know precise details about a periodic orbit, just a certain average along it.

7.3. The average gradient along an orbit .

The average gradient along an orbit has been obtained by Brown for the logistic map for $n \le 6$[17], and $n = 7$[18]. The method uses properties of polynomial equations and so is restricted in its application, however it gives good results for the quadratic nonlinearity and could be extended to cubic or quartic nonlinearities.

As an example, consider points of period 4. To directly obtain all these points it would be necessary to solve an equation of degree 2^4. However each orbit $\{b_i\}$ ($b_i \equiv p_i$ of (7.2.1)) must satisfy an equation of degree 4 ,

$$h_4(x) \equiv \prod_{i=1}^{4}(x-b_i) = 0 \tag{7.3.1}$$
$$= x^4 - \alpha x^3 + \beta x^2 - \gamma x + \delta$$

where $\alpha,\beta,\gamma,\delta$ are standard symmetric functions of the roots b_i. Using an additional equation, i.e. the equation for the map , it is possible to obtain β,γ,δ as functions of α. Thus $h_4(x)$ is characterised by the value of α.

Define $G_m(x) \equiv F^m(x) - x$. Thus $G_4(x) = 0$ is the polynomial of degree 16 that gives the points of period 4. Only non-degenerate orbits are required so factorise out period 2 and 1, giving $G_4(x) = G_2(x)H_4(x)$, where $H_4(x)$ is of degree 12 and has 3 factors of the form (7.3.1). $G_2(x)$ contains one orbit of period 2, and two of period 1. Thus α satisfies an equation of degree 3 which can be solved analytically.

7.3.1. The equation for α

Using the mapping in the form $b_{i+1} = b_i^2 - A$ for $i = 1,2,3,4$ and $b_5 = b_1$ gives the relationships

$$\sum_{i=1}^{4} b_i^2 = \sum_{i=1}^{4}(A+b_{i+1}) = 4A + \alpha \tag{7.3.1.1}$$
$$\sum_{i=1}^{4} b_i^3 = \sum_{i=1}^{4} b_i(A+b_{i+1}) = A\alpha + \beta_1$$
$$\sum_{i=1}^{4} b_i^4 = \sum_{i=1}^{4}(A+b_{i+1})^2 = 4A^2 + 2A\alpha + 4A + \alpha$$

where $\beta_1 = \sum b_i b_{i+1} = b_1b_2 + b_2b_3 + b_3b_4 + b_4b_1$. Considering the difference of squares of the roots gives the relation

$$(b_1+b_2)(b_2+b_3)(b_3+b_4)(b_4+b_1) = 1 \tag{7.3.1.2}$$

The general theory of equations gives[122]

$$\sum_{i=1}^{4} b_i^2 = \alpha^2 - 2\beta \tag{7.3.1.3}$$

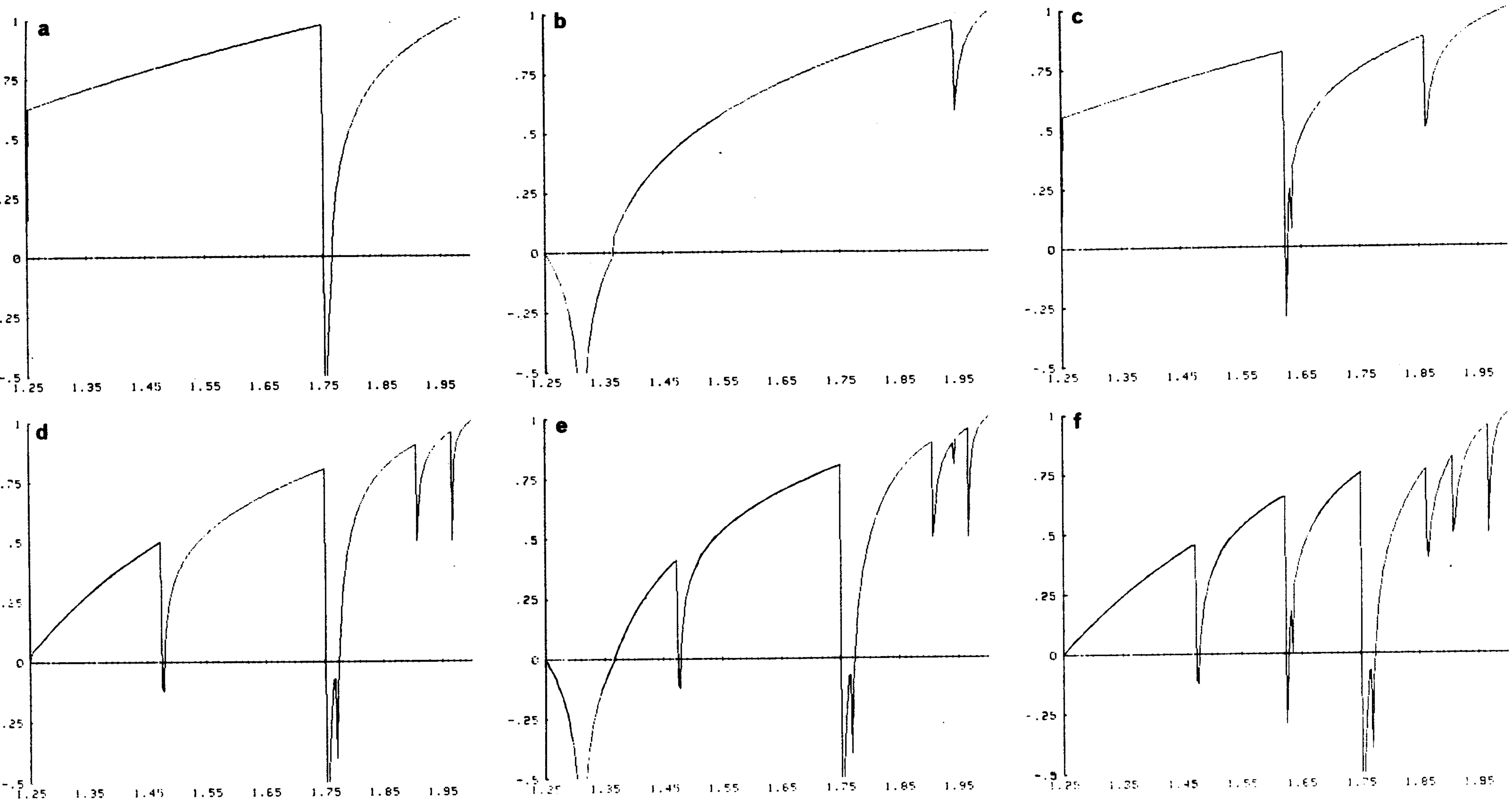


fig. 7.4.1 The approximation to the Lyapunov exponent for different n

a) n=3 b) n=4 c) n=5 d) n=6 e) n=12 f) n=30

$$\sum_{i=1}^{4} b_i^3 = \alpha^3 - 3\alpha\beta + 3\gamma$$

$$\sum_{i=1}^{4} b_i^4 = \alpha^4 - 4\alpha^2\beta + 2\beta^2 + 4\alpha\gamma - 4\delta$$

Combining (7.3.1.1) and (7.3.1.3) gives

$$2\beta = \alpha^2 - \alpha - 4\Lambda \qquad (7.3.1.4)$$

$$6\gamma = \alpha^3 - 3\alpha^2 - 10\alpha\Lambda - 2$$

$$24\delta = \alpha^4 - 6\alpha^3 + \alpha^2(3-16A) + \alpha(12A-14) + 24(A^2-A)$$

Using (7.3.1.2) gives an equation for α,

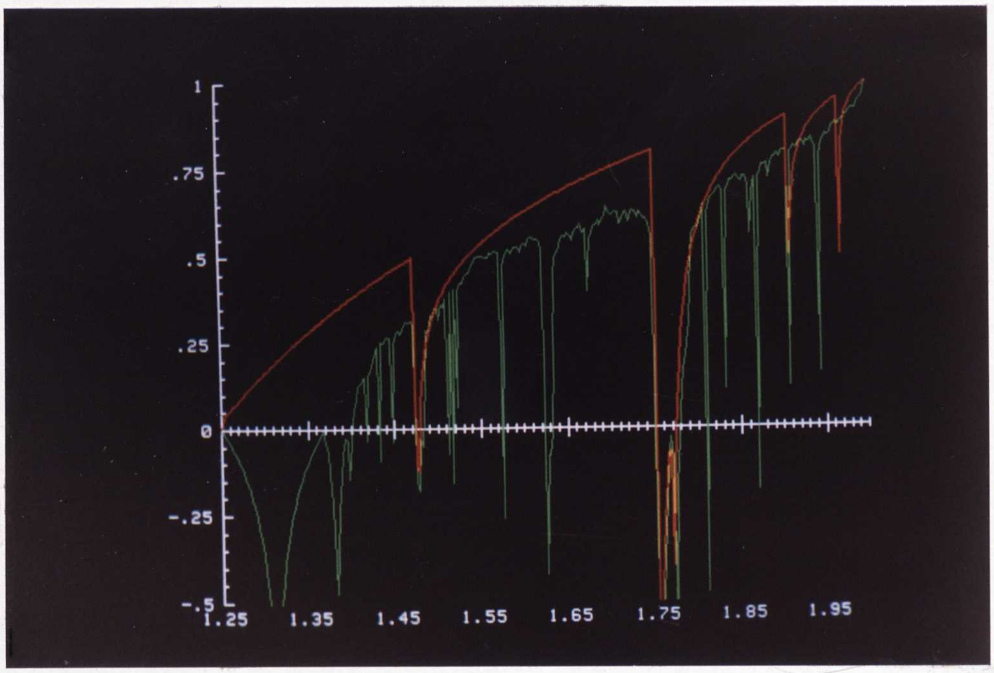
$$(\alpha-4)\,(\alpha^3 + \alpha(3-4A) + 4) = 0 \qquad (7.3.1.5)$$

The 3 solutions of this equation define the 3 nondegenerate orbits of period 4. Similar equations are obtained by Brown for periods up to 7.

The choice of $F_A(x)$ gives the derivative as 2x, whence the product of the derivatives along a periodic orbit is $2^n \prod_{i=1}^{n} b_i$. This is just the constant term in (7.3.1). Thus an equation of low degree defines values for the parameter α which then specifies the gradient product along an orbit and so the Lyapunov exponent using (7.2.5).

7.4. Numerical results

The equations for periods 1 to 4 can be solved analytically but period 5 and 6 require numerical solution of polynomials of degree 6 and 9. This was done on an HP9845C using a library routine. The resulting solutions were then used to construct the curves in figure 7.4.1 from equation (7.2.5) with (7.2.3). It can be seen that λ_6 (fig. 7.4.2) contains the solutions for period 2 and 3 but not 4 or 5. This arises from the limiting procedure in (7.2.5) : in order for the measures to converge to an absolutely continuous limit a sequence of n's must be chosen such that n' > n implies n divides n'. Thus a possible sequence is 1,2,3,6,12,60,... . To combine the first 6 periodic orbits, it is necessary to consider n=60, however this also requires inclusion of periods 10,12,15,20,30,60. As it is not feasible to calculate these high periods, figure 7.4.2b shows the approximation that



a) n=6

b) n=60

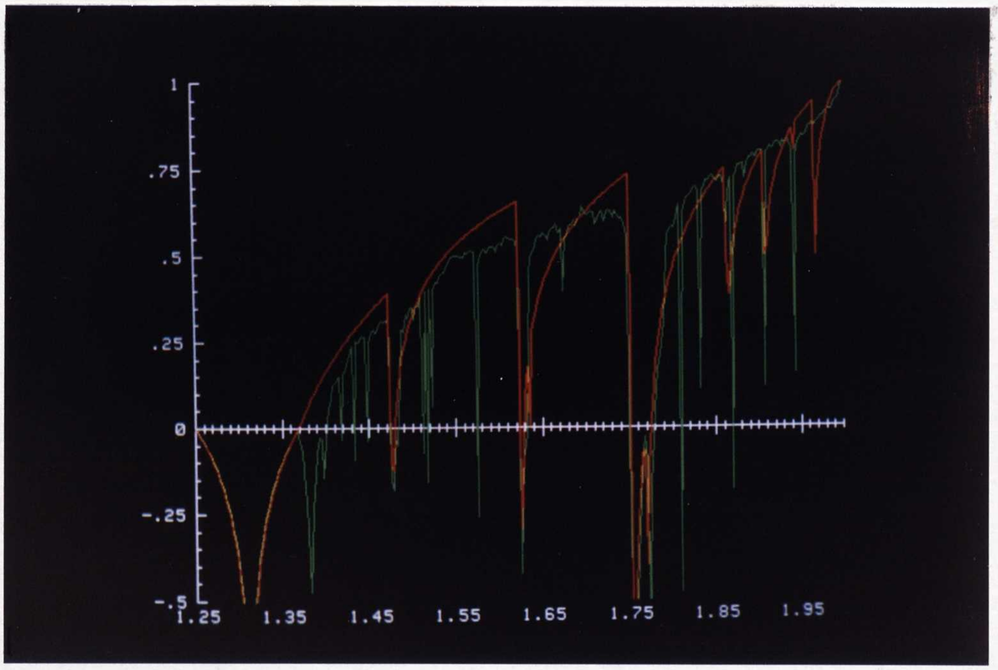


fig. 7.4.2 Comparison of the approximate Lyapunov exponent with numerics

results when n is taken to be 60 but only up to period 6 are included. This is qualitatively correct, only missing the small scale high period structure, a lowering of the local maxima by nearby stable orbits and a steepening of the positive parts of the local minima due to the increased background contribution. Thus consideration of a few periodic orbits is sufficient to give a good approximation to the Lyapunov exponent for the logistic map.

7.5. Discussion

7.5.1. Topological Entropy

Another quantity of great interest when studying chaotic maps is the topological entropy, $h_{top}(A)$. This forms an upper bound for the Lyapunov exponent (and measure-theoretic entropy h_μ) with equality only for absolutely continuous invariant measures. A method for obtaining this entropy has been given by Milnor and Thurston[89] , relating the entropy to the kneading determinant. This quantity must be calculated numerically and is essentially defined pointwise in parameter space. An approximation to the entropy using these ideas but with smooth variation in parameter space has been introduced by Dias de Deus, Dilao and Taborda Durate for the logistic map[35, 34]. Using a functional form for the change in entropy with parameter obtained from renormalisation ideas, they are able to obtain exact expressions for the scaling constants at certain critical points where the entropy is known exactly. In this way they are able to approximate the behaviour of $h_{top}(A)$ over the range of chaotic A.

7.5.2. Intermittency

This is the phenomenon of smooth oscillatory motion interspersed with bursts of chaotic behaviour. This is common in turbulent fluid dynamics but can also be seen in many model mappings including maps of the interval, where it occurs just before the onset of stable motion in a chaotic region e.g. onset of period 3. Renormalisation ideas have been used by Pomeau and Manneville[85] and by Hirsch,Nauenberg and Scalapino[61] to describe the average lengths of the laminar regions and to obtain scaling coefficients. Intermittency manifests itself as a smoothing of the Lyapunov exponent near transitions to stable periodic motion (see fig. 7.4.2 near period 3). It is possible that this behaviour could be taken into account in the approximation of §7.3 by considering the extension of the stable periodic orbit to parameter values before it exists. This corresponds to using the complex roots of equations like (7.3.1.5) and using either the modulus or a projection to obtain a real quantity for calculating the Lyapunov exponent. Preliminary investigations near period 3 do show the characteristic smoothing of the Lyapunov exponent at the transition to a stable orbit.

7.5.3. Path-integrals

The method of Fourier-space paths to obtain a characterisation of an invariant distribution function (chapter 3) can also be used in one dimension. Rechester and White[108] obtained expressions for the Lyapunov exponent and the invariant distribution for the map $x_{n+1} = -\alpha \sin x_n$ which agree very well with numerical calculation. A similar approach has been used by Jensen and Oberman[66] to study a dissipative 2D map which reduced to an essentially 1D map. As both these methods use Fourier transforms, they work best for periodic sinusoidal functions whereas the methods described in this chapter work best for simple polynomials.

7.5.4. Summary

This chapter has presented a method for obtaining characteristic quantities for 1D maps in terms of an invariant measure given by the periodic points. There is good agreement between the theory and a numerically obtained Lyapunov exponent[87]. There is the possibility that the method may be extended to describe intermittency effects by considering the complex roots of equations like (7.3.1.5) i.e. the effect of the periodic points before they actually exist. In addition the method may be used to give an upper bound to the topological entropy by choosing the least stable periodic orbits rather than the most stable when calculating the measure.

CHAPTER 8

Chaos

8.1. Introduction

This chapter reconsiders the question asked in §0.2 : 'what is chaos ?' §8.2 presents a number of equivalent definitions that have been proposed by Oono and Osikawa[96] to describe the chaos observed in one dimensional maps. In several dimensions, the possible behaviours are more difficult to enumerate but several characteristics do pass over from 1D. One important structure that occurs in several dimensions is the ***strange attractor***. §8.3 presents a definition of these structures as given by Ruelle[114]. This definition differs from the usual techniques in that it uses properties that are relevant to computer simulation of the structures. A perturbation technique that can be used to study attractors by analytic construction of the Poincaré map associated with the flow[15] is given in §8.4. Numerical simulations have been vital to this thesis, §8.5 considers the question : 'are the computer generated results a true reflection of the solutions of the equations ?'. This question was originally studied in depth by Benettin, Casartelli, Galgani, Giorgilli and Strelcyn[7,8] who suggested that techniques like those of chapter 1 gave an affirmative answer. While chapter 4 encourages this view, recent numerical studies by Karney[69] and Grebogi, Ott and Yorke[49] suggest that long-time correlations will present analytical difficulties for smooth maps.

8.2. Definition of Chaos

The physics literature contains examples of many apparently useful concepts for describing chaotic motion in particular systems which are then shown to be inaccurate for some other standard example. A summary of results in one dimension has been given by Oono and Osikawa[96]. For C^0-endomorphisms of an interval they are able to prove the equivalence of several concepts that have meaning both physically and mathematically.

1. There exists an invariant mixing measure ;

2. there exists a homoclinic point ;

2'. there are two closed intervals I_0, I_1 sharing at most one point such that $F^n(I_0) \cap F^m(I_1) \supset I_0 \cup I_1$ for some $m, n > 0$;

3. there are periodic orbits of arbitrarily long period and uncountably many nonperiodic orbits which itinerate amongst themselves and the periodic orbits ;

4. there is a natural correspondence with some Bernoulli system ;

5. the Kolmogorov entropy (a disorder parameter) is positive.

If F is not continuous or is defined in several dimensions then these properties are not equivalent. (1) is a consequence of (4) and as shown in chapter 4 there should be some distinct property between these two cases. (2') and (3) are not true for the horseshoe and so may be neglected in higher dimensions.

The arguments of chapter 4 concerning an uncertainty principle associated with entropy suggest that (5) is a good characterising quantity, however it is not clear which of several entropies is the relevant choice. Although positivity of Lyapunov exponents does not necessarily give mixing, it is possible that systems with positive measure entropy equal to the average of the positive Lyapunov exponents with respect to a smooth invariant measure are good chaotic systems.

8.3. Strange Attractors

When studying chaos in many dimensions there are two types of behaviour. One essentially corresponds to area-preserving maps where the chaos is defined on a set of the same dimension as the underlying phase space and with positive measure (though possibly with large ordered regions). The other situation occurs when the motion contracts onto a set with dimension (usually non-integer) less than that of the phase space. This gives rise to the structures called strange attractors, though no precise characterisation of these structures has yet been given. A possible approach is via Ruelle's[114] definition of attractor that takes account of the properties required for computer observations.

Call a closed set $\Lambda \subset M$ an ***attracting set*** if it has a neighbourhood U s.t.

a) for all neighbourhoods V of Λ, $f^n U \subset V$ for sufficiently large n

b) $f^n \Lambda \supset \Lambda$ for sufficiently large n.

The ***basin of attraction of*** Λ, is given by $W = \bigcup_n (f^{-n})(U)$. An attractor is a subset of an attracting set obtained by taking a minimal equivalence class of pseudoorbits. A sequence $X = \{ x_k \}_{k \geq 0}$ is called an *ε-pseudoorbit* for a map T if $d(Tx_k, x_{k+1}) \leq \varepsilon \ \forall k$. Then a point y, *β-shadows* the pseudoorbit X if $d(T^k y, x_k) \leq \beta$. Let $a,b \in M$ be points in a metric space. Introduce a preorder $\gg$ such that $a \gg b$ if, for arbitrarily small $\varepsilon > 0$, there is an ε-pseudoorbit going from a to b. Write $a \sim b$ if $a \gg b$ and $b \gg a$. $\gg$ induces an order on the equivalence classes given by $\sim$, given by $[a] \geq [b]$ if $a \gg b$. A minimal equivalence class is called an *attractor*. If Λ is a compact attracting set, f C^1 and a is in the basin of attraction of Λ, then there is at least one attractor $[b]$ such that $a \gg b$ and $[b] \subset \Lambda$. Define a strange attractor as an attractor which supports chaotic dynamics.

The above definition of an attractor is consistent with the standard definitions used in the study of Axiom A systems and automatically has the shadowing property that is important for computer studies. In addition this definition can be used to study the effect of small random perturbations (noise) on the dynamics. Following Ruelle, for $f: M \to M$ a continuous function and $\varepsilon > \delta > 0$, then an affine map F from the space of probability measures with compact support in M to itself is an *(ε,δ)-diffusion* associated with f if :

a) supp $F\delta_x \subset f\overline{B}_x(\varepsilon)$ where δ_x is the unit mass at x, $\overline{B}_x(\varepsilon)$ is the closure of the open ball centred at x, radius ε

b) $\operatorname{supp} F\delta_x \supset f\overline{B}_x(\delta)$

c) if $\varphi : M \to \mathbf{R}$ is continuous then $x \to (F\delta_x)\varphi$ is continuous and $(F\mu)\varphi = \int \mu(dx)(F\delta_x)\varphi$. In particular $\operatorname{supp} F\mu = \text{closure} \bigcup_{x \in \operatorname{supp}\mu} \operatorname{supp} F\delta_x$

d) if $\varphi : M \to \mathbf{R}$ is continuous then the set $\{(F\delta_x)\varphi : y \in \overline{B}_x(\delta)\}$ is a closed interval.

This formulation corresponds to thinking of F as f preceded by a small diffusion and it can be used to show that a probability measure ν with compact support in the basin of attraction of Λ has a vague limit under F with support in the union of all attractors in Λ. Thus topological properties of M (i.e. compactness) give details about the support of asymptotic measures. To actually find the measures requires details about the differentiable structure. For example, hyperbolicity conditions are needed to obtain asymptotic measures absolutely continuous on unstable manifolds (see chapters 2 and 4), while the stronger condition of Axiom A gives the exponential decay of the mass remaining in a neighbourhood of a basic set. The definition of (ε,δ)-diffusion can be extended to cases when $\operatorname{supp} F\delta_x$ is not small (e.g. a gaussian distribution which is not compact) but simple, general results become difficult to obtain.

Given the existence of an attractor with an asymptotic measure then the attractor might be called *strange* if the measure has positive characteristic exponent. This ties the strangeness of an attractor to the asymptotic measure and will vary with the measure.

The imposition of gaussian noise in model equations as a means of smoothing has been used by scientists for many years. Rechester, Rosenbluth and White[106] originally included a noise contribution in their Fourier-path summations and similar terms where used by Cohen and Rowlands[27] to model collisional processes, however the presentation of chapter 3 without noise converges sufficiently well to give excellent agreement with experiment. The effects of noise in one dimensional systems are reviewed by Crutchfield, Farmer and Huberman[33]. The preliminary results of Ruelle show how noise may be used in differentiable dynamics in a rigorous rather than a heuristic manner.

8.4. Poincaré Maps by Perturbation Theory

Chapter 0 introduced the idea of a surface of section plot (Poincaré section) as a means of reducing the dimensionality of a problem so that it might more easily be displayed on a computer screen. This idea can also be used in analytical work. The simplest approach is to generate a Poincaré section numerically and then choose a simple quadratic, piecewise linear, etc... approximation to it and study this approximation analytically. While not very satisfactory this approach has led to useful insights and appears all that can be done since obtaining the Poincaré section requires the solution of the original equations.

A way of using multiple-time perturbation theory to analytically obtain the Poincaré section from the equations of motion has been suggested by Broomhead, McCreadie and Rowlands[15] for the forced Brusselator and for the Lorenz equations by Rowlands[111].

Consider the equations :

$$\frac{dX}{dt} = A - X - BX + X^2Y + a\cos\omega t \qquad (8.4.1)$$
$$\frac{dY}{dt} = BX - X^2Y$$

When a = 0, there is a single steady state at X = A, Y = B/A. Using local coordinates x,y about this point and scaled by a small parameter ε, a multiple-time perturbation scheme gives :

$$x = \varepsilon Re^{iAt} + c.c. + O(\varepsilon^2) \qquad (8.4.2)$$
$$z \equiv x + y = \frac{\varepsilon i Re^{iAt}}{A} + c.c. + O(\varepsilon^2)$$

where c.c. denotes complex conjugate and $R = \rho e^{i\varphi}$, with ρ and φ given by the consistency conditions to remove secular terms :

$$\frac{d\rho}{dt_2} = \frac{\alpha\rho}{2} - \gamma_1\rho^3 - \bar{a}\rho\cos2(\varphi + \vartheta) \qquad (8.4.3)$$
$$\frac{d\varphi}{dt_2} = -\gamma_2\rho^2 + \bar{a}\sin2(\varphi + \vartheta)$$

Here $t_2 \equiv \varepsilon^2 t$ measures the slow time variation and the forcing frequency has been chosen to give chaos, $\omega = 2A$.

Equations (8.4.3) support at most a limit cycle, as will higher order consistency conditions, thus this perturbation scheme can generate at most quasi-periodic solutions - not chaos. This is to be expected as this perturbation scheme was originally developed to find just such hierarchies of periodic motion. In order to generate chaos a resummation of terms must occur which can take account of non-expandible terms like $e^{\frac{-1}{t}}$. Generate a Poincaré section by considering the flow at intervals $\tau = \frac{2\pi}{A}$ (the period of the driving force). Then (8.4.2) becomes :

$$x_n = 2\varepsilon\rho_n\cos\varphi_n + O(\varepsilon^2) \tag{8.4.4}$$
$$\tilde{z}_n \equiv Az_n = -2\rho_n\sin\varphi_n + O(\varepsilon^2)$$

where $\rho_n \equiv \rho(n\varepsilon^2\tau)$; $\varphi_n \equiv \varphi(n\varepsilon^2\tau)$. These equations provide an improvement since now the solutions ρ,φ are not required for long times, just the interval τ between iterations.

If a solution for (8.4.3) were known it could be used to relate (ρ_n,φ_n) to $(\rho_{n+1},\varphi_{n+1})$ and the Poincaré section would be obtained. For this example this is not possible, so a trial solution is used which is exact when $\bar{a} = 0$ and when the equations are linearised in ρ. This procedure is reasonable since : i) a is expected to be small $\approx$ 0.05 ; ii) the structure of the motion in the neighbourhood of the saddle point at $\rho = 0$ will be dominant. This method produces a Poincaré section that compares well with the numerically generated section as is reported in Broomhead et.al.[15].

While a rigorous justification of this method cannot yet be given (but see Guckenheimer and Holmes[53]), it should be noticed that the formation of the difference equation suggests that the definition of attractor due to Ruelle might be appropriate. A sufficiently accurate solution to the continuous equations (8.4.3) can be obtained so that the errors introduced in the interval τ are sufficiently small that a true orbit shadows the pseudoorbit and converges to the true attractor. Indeed it is the additional errors in the difference scheme that allow chaotic motion whereas the continuous equations are constrained to be at most quasiperiodic.

8.5. Computational Methods

Throughout this thesis it has been assumed that the computer simulations performed, accurately model the asymptotic properties of the systems of equations. This section will discuss these assumptions in more detail, describing some problems that arise and giving justifications for the numerical methods of chapters 3,5,6 and 7.

Due to the finite accuracy of computer numbers, the orbit that is studied is not the true orbit originating from a given initial point. Thus for any pseudoorbit generated by the computer it is necessary to find conditions such that there always exists a true orbit that shadows the computer orbit. Given these conditions it is possible to bound the errors between computer simulation and true results. This shadowing property was originally shown for Anosov maps but can be extended to almost hyperbolic maps and to neighbourhoods of hyperbolic sets (Bowen[12]). In order to study this problem Benettin et.al[7,8]. studied the long-time averages of various functions for several maps of the torus with an error introduced at each step e.g.truncation. For an Anosov map, the time averages with imposed errors up to 10^{-3} converged to the expected space averages. For non-Anosov maps, a similar convergence was observed , as was also seen in chapter 3, suggesting that a shadowing result holds for sufficiently hyperbolic systems.

Another study of discretisation effects was performed by Rannou[105]. Here a toral map was forced to be exact by only defining it on a lattice (up to 800x800). Obviously the map must repeat after a finite number of iterations. Often quite short periods were observed but a number of initial conditions gave similar area-filling results to those observed for continuous phase space. The problem of computer induced periodic orbits has been discussed by Levy[73] for the Henon attractor. Here periodic orbits of length 6000 were often observed when single precision was used. A more detailed study of aspects of this problem was made for certain piecewise linear two dimensional maps by Gambaudo and Tresser[45]. They calculated stable orbits of short period (e.g. 11) with very small basins of attraction. When these maps were simulated by computers, apparent strange attractors were observed until the number of iterations reached $\sim 10^6$ when the motion settled to a periodic orbit. Thus the presence of periodic orbits, of which there will be an infinite number[95], can cause doubts concerning the interpretation of numerical results.

Given that the presence of discretisation effects can be accounted for, it is still difficult to describe the limiting measure or the way in which this limit occurs. Benettin et.al[8]. studied the importance of limiting measures, μ_L that were absolutely continuous with respect to Lebesgue measure. They found that the time averages of various functions converged to the values of spatial averages using a smooth measure, μ_L rather than the measure of maximum entropy, μ_0 which could be found from Markov partition techniques. Thus computer simulations choose a smooth measure rather than the most random measure. (In the diffusion calculations of chapter 3, the unbounded phase space means that there is no limiting measure μ_L. Rather the distribution approaches a smooth spreading gaussian.)

The effects of correlations were seen in chapter 3 where they produced oscillations in the diffusion constant. If a map is mixing then correlations will decay, however a stronger condition must be imposed to obtain exponential decay e.g. an Anosov map where long-time correlations can be neglected. The decay of correlations has been investigated by many authors. For one dimensional maps Nagashima and Haken[94] found not only periodic modulation of the decay rate but also chaotic modulation. Similarly Mori, So and Ose[92] found many different rates of decay for piecewise linear maps of the interval. In two dimensions Grebogi et.al[49] found a decay of correlations that depended on the driving map but which seemed independent of the parameter E and possibly related to the Lyapunov exponent.

For non-mixing maps like the Chirikov map, long-time correlations will always be present. This has been investigated by Karney[69] who found that many correlations were caused by high period stable elliptic islands. As the map is analytic, these islands do not have well-defined boundaries which totally exclude orbits, rather an orbit can come arbitrarily close to the island and be trapped for a long time by the complicated hyperbolic structure surrounding the island. This problem is alleviated in the piecewise linear case since here the stable islands have well-defined boundaries and mixing results can be given for the motion outside the islands (see chapter 4).

The discussion of the arcsin distribution in §2.5 shows that any single orbit which is iterated many times will show correlation effects, thus the numerical results of chapter 3 were obtained by choosing a large number of orbits over which to take an ensemble average. The theorems given in chapter 2 show that this averaging still cannot be expected to give time independent results. For a random process, while the quantity S_n/n (where $S_n = \sum_{i=1}^{n} x_i$) converges to a normal distribution $N(0,\sigma^2)$, the variance of S_n, V is proportional to $\sigma^2 n h(n)$ where $h(n)$ is a slowly varying function converging to 1 as $n \to \infty$. This slowly varying function has been observed in the simulations generating the results of chapter 3 but was ignored there. This slow variation may explain the disagreement seen in fig.3.3.2.1 for $E \sim 2.5$.

Much work is still necessary to study the differences between ensemble averages and long-time averages both computationally and analytically. Computer simulations may well avoid many of the problems by introducing a small noise, thus aiding convergence. In addition the structure of higher moments than p_n^2 should be studied to make clear the relationship with true diffusion processes.

CHAPTER 9

Conclusions

The purpose of this thesis was to show that the complicated behaviour observed in many nonlinear deterministic systems can be well described by random processes like Bernoulli shifts or coin-tossing.

In order to demonstrate the equivalence between chaotic and random systems, two methods were employed. The first was the calculation of correlation coefficients and a diffusion constant using a path-integral formalism. Chapter 3 showed that there was excellent agreement between this theory and computer simulations for a class of area-preserving maps of the torus. Thus for maps obtained from a class of Hamiltonian systems it is possible to easily calculate a physically relevant diffusion constant.

The second method proved rigorously the equivalence of a piecewise linear map on the torus with a Bernoulli shift and hence with a general nondeterministic Markov process. Thus it is possible to show that some deterministic equations are precisely equivalent to a nondeterministic random process. This equivalence suggested the introduction of complementary variables that do not commute, namely position and entropy. Thus positivity of entropy, often used heuristically to characterise chaos, was shown to imply the impossibility of describing the system using trajectories related to an initial point.

The method of Fourier-paths has been used for one dimensional maps and very simple dissipative two dimensional maps. It should be possible to extend the method to the dissipative form of the Chirikov and the piecewise linear maps. The dissipative piecewise linear map is a special case of a generalised Lozi map (as introduced by Young[128]). Young has obtained absolutely continuous conditional measures for these maps using methods similar to Wojtkowski. This could help to explain the perturbative approach to obtaining the fractal dimension of such maps[16].

Diffusion is expected to describe chaotic motion when a parameter in the system was large and the motion occupied most of the torus. Chapter 5 demonstrated that the concept of diffusion was still able to characterise the motion when chaos only occupied a subset of the torus. Here a variational principle was proposed to obtain the diffusion through a medium which contained holes. These ideas might be refined to explain the observed variations about the simple result, caused by variations in the background diffusion or the appearance of uncounted islands.

Chapter 6 used a different perturbation expansion to describe the influence of the dynamics on an imposed gaussian noise. The method was able to reproduce previously obtained results but due to the complexity of the problem and limitations on computer resources it was not possible to carry through the calculations and obtain the corrections to the low order theory that were suggested by numerical results.

Having shown that two dimensional behaviour could be well described by a random process and a diffusion constant, the opposite view was taken in chapter 7 where a few low order periodic points were used to describe one dimensional chaos. By continuing a function of the periodic points into the regions where those points were unstable and not directly observable, it was possible to obtain a very good approximation to the Lyapunov exponent for the logistic map. This method could be extended to give a smooth approximation to the Lyapunov exponent for the cubic and quartic maps where there are several parameters, thus making full numerical investigation expensive. In addition the method might be of use in studying intermittency by using the complex roots of the equations involved.

Chapter 8 considered some of the problems associated with studying chaotic dynamics on a computer and appealed to the shadowing property of hyperbolic systems to explain the good results obtained by perturbation theories. Shadowing properties are needed to explain how computer simulations, which can never be exact, may accurately describe motion which is inherently unstable. Obviously further work must attempt to show rigorously that systems like the Brusselator do have a shadowing property and that the method of obtaining Poincaré sections is valid. In addition further numerical studies should be performed to elucidate the relevance of arcsin laws and other properties of true random systems to quasi-random systems like the Chirikov map.

Appendix A : Derivation of a Variational Principle

Refering to fig. 5.2.1 it is required to find a solution, φ of Laplace's equation, $\nabla^2\varphi = 0$ satisfying the boundary conditions : $\frac{\partial\varphi}{\partial y} = 0$ for $y = \pm b$; $\frac{\partial\varphi}{\partial n} = 0$ on the surface C ; constant flux $\frac{\partial\varphi}{\partial x} = J$ at $x = \pm b$. Consider the functional

$$K = \frac{1}{2}\int\int dxdy\, [\bar{\nabla}\varphi(x,y)]^2 - 2J\int dy\, \varphi(x,y)\Big|_{x=-b} \tag{A.1}$$

Then a variation in φ generates a variation in K given by

$$\begin{aligned}\delta K &= \int\int dxdy\, \bar{\nabla}\delta\varphi(x,y)\,.\,\bar{\nabla}\varphi(x,y) - 2J\int dy\delta\varphi(x,y)\Big|_{x=-b} \\ &= \int \delta\varphi(x,y)\bar{\nabla}\varphi(x,y)\circ \overline{dS} - \int\int dxdy\delta\varphi\nabla^2\varphi - 2J\int dy\delta\varphi(x,y)\Big|_{x=-b}\end{aligned}$$

The middle term vanishes since φ satisfies Laplace's equation. On performing the integral over the boundaries, the first term cancels the third. Thus $\delta K = 0$ and K is a variational function.

Now restrict the trial functions, φ to functions satisfying Laplace's equation and all the boundary conditions except at $x = \pm b$, then

$$K = \left\{\int dy\varphi\frac{\partial\varphi}{\partial x} - 2J\int dy\varphi\right\}_{x=-b} \tag{A.2}$$

with exact value $K_{exact} = -J\int dy\varphi\big|_{x=-b}$. To remove amplitude dependence set $\varphi(x,y) = A\psi(x,y)$ and vary A in (A.2) to give after resubstitution

$$K = \frac{-J^2\left(\int \psi dy\big|_{x=-b}\right)^2}{\left[\int\psi\frac{\partial\psi}{\partial x}dy\right]_{x=-b}}$$

with the exact value $K_{exact} = -J\int\psi dy\big|_{x=-b}$.

Define an effective diffusion constant D_{eff} in terms of the diffusion D when no holes are present and the medium is uniform by :

$$\text{Current} \equiv \frac{D[\psi(x=b) - \psi(x=-b)]}{2b} = \frac{D_{eff}\left[\int\psi dy\big|_{x=b} - \int\psi dy\big|_{x=-b}\right]}{2b}$$

In the uniform case $\frac{\partial\psi}{\partial x} = J$, $\therefore$ $\psi = \alpha + Jx$ and $\psi(x=b) - \psi(x=-b) = 2Jb$. Thus $D_{eff} = \frac{D2Jb}{2\int\psi dy\big|_{x=-b}} = \frac{-DJ^2b}{K_{exact}}$. Thus the effective diffusion satisfies the variational principle :

$$D_{eff} = \left[\frac{Db \int \psi \frac{\partial \psi}{\partial x} dy}{[\int \psi dy]^2} \right]_{x=-b}$$

Appendix B : The Use of Computers

Much of this thesis was the result of investigations performed on computers. This appendix describes some of the techniques used and some implications for the future.

One of the first problems studied was the calculation of a diffusion constant for the small parameter E as described in chapter 6. Initial calculations where performed by hand but it became apparent that extensive algebra was involved. At that time an implementation of A.C.Hearn's algebraic manipulation language, REDUCE became available on the local mainframe, a Burroughs B6700. Due to the lack of a general pattern matching algorithm, REDUCE was initially only used to check the ordered summations obtained from the various averaged products of a_i's. Although these were relatively simple evaluations, difficulties arose due to the extensive requirements for machine memory and process time.

A method was then developed in order to perform the summations as a function of n. Denote each summation as P(f,l,m,n)TI(f,l,m,n) where P is a polynomial and TI is a product of exponentials arising from the average over the noise. Since the arguments had been ordered $f > l > m > n$ so that the averaging could be performed, a simple pattern matching process can perform the summations. For example, let $TI = e^{fx}e^{-lx}e^{mx}e^{-nx}$ where $x = ip_0 - \frac{\sigma^2}{4}$; $P = flmn$ and $S\alpha(N,x) \equiv \sum_{k=0}^{N-1} k^\alpha e^{kx}$ and proceed as below :

1. replace expressions of the form ne^{nx} by $S1(x,m)$
2. substitute the explicit summation result for $S1(x,m)$
3. replace expressions of the form $m^2 e^{mx}$ by $S2(x,l)$; me^{mx} by $S1(x,l)$; e^{mx} by $S0(x,l)$ then the polynomial summations i.e. replace m^2 by $\frac{l(l-1)(2l-1)}{6}$ etc...

4. substitute the explicit summation results for S2, S1, etc... and repeat.

This procedure produces a rather large expression which if written to backup store cannot be reinput to REDUCE (this is due to the structure of REDUCE). Thus simplification of the expression for the summation must be performed directly - this gives process time problems (the job has already taken 4 hours) and because of an intermediate step in the simplification process overflows the available memory.

As the local implementation of REDUCE could not be recompiled to make use of a greater percentage of memory and as this program was very antisocial on the time-sharing system, this calculation has been halted. One problem with REDUCE is that it is not optimised to perform calculations on trigonometric summations, an extended implementation of a package like MuMath on a large microcomputer may be better suited to this problem.

Obviously the ability to view computer generated plots of phase space was of great importance. Initially this was done using the Burroughs mainframe and a Tektronix or Hewlett-Packard graphics display. This proved very unsatisfactory for several reasons : the mainframe was time-shared by up to thirty users, even ten users might produce a ratio of elapsed time to process time of 10 to 1 which is excessive for a plot that takes 4 minutes process time ; several people needed graphical output so the two terminals were oversubscribed ; attempts to produce pictures containing 10^4 points on the Calcomp plotter where not well-received and access to a digital plotter via the Hewlett-Packard was only available during office hours when the machine was busiest.

The solution to this problem was to obtain a SuperBrain microcomputer which initially produced plots on an Epson printer and later directly on the screen (via a Micronex graphics retrofit) with a direct dump to the printer. The map plots in this thesis were produced in this way using an interactive Fortran program. A further improvement was the availability of an HP9845C - a powerful microcomputer with high-resolution colour graphics. The photographs herein were taken directly from the screen of this machine. The ease of use of this machine enabled rapid program development so that numerous maps could be investigated and a powerful user-interface built. Using redefinable-keys a number of options could easily be provided e.g. the ability to concentrate interest on a small region of phase space. The possible options are shown in fig. 0.4.5. In addition the use of a light-pen enabled a flexible choice of initial points and of different colours.

It is the presence of this kind of powerful microcomputer that has enabled many of the developments in the study of chaotic dynamics.

During the latter part of the research, the use of the SuperBrain as an intelligent terminal to a mainframe via the Local Area Switching System proved of great value. Once the techniques of communication had been sorted out, the powerful screen editor on the SuperBrain could be used to produce programs that could be run on one of the local mainframes (if working) or on the SuperBrain itself. While Fortran programs are fairly portable, subroutines to enable the SuperBrain graphics system to be called like the mainframe GHOST package had to be written. Future research will rely heavily on powerful microcomputers with special features that are under ones own control with the occasional, very intensive jobs being submitted to some powerful mainframe over a transparent network.

This thesis was produced using the UNIX text-formatters, NROFF and TROFF on a VAX 11/780 with output to a Benson plotter. The increasing ability of computers to handle scientific text may reduce the quantity of scrawled, symbol-ridden text that secretaries must decipher, should researches write their papers directly on the machines which already contain all the results. However the machines cannot yet generate the papers unaided!

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