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# VARIATION IN WEIGHTED VOTING GAMES

*Haris Aziz and Mike Paterson*

THE UNIVERSITY OF
WARWICK

# Variations in weighted voting games

Haris Aziz and Mike Paterson
Department of Computer Science
University of Warwick
Coventry CV4 7AL, UK
Email: {haris.aziz, m.s.paterson}@warwick.ac.uk

**Abstract**

*Weighted voting games* are ubiquitous mathematical models which are used in economics, political science, neuroscience, threshold logic, reliability theory and distributed systems. They model situations where agents with variable voting weight vote in favour of or against a decision. A coalition of agents is winning if and only if the sum of weights of the coalition exceeds or equals a specified quota. *Tolerance* and *amplitude* of a weighted voting game signify the possible variations in a weighted voting game which still keep the game unchanged. We characterize the complexity of computing the tolerance and amplitude of weighted voting games. We give tighter bounds and results for the tolerance and amplitude of key weighted voting games. We then provide limits to how much the Banzhaf index of a player increases or decreases if it splits up into sub-players.

Keywords: weighted voting games, tolerance, amplitude, reliability theory, combinatorics, applied mathematics, algorithms and complexity.

## I. Introduction

### A. Motivation

Weighted voting games (WVGs) are mathematical models which are used to analyze voting bodies in which the voters have different number of votes. In WVGs, each voter is assigned a non-negative weight and makes a vote in favour of or against a bill. The bill is passed if and only if the total weight of those voting in favour of the bill is greater than or equal to some fixed quota. A power index attempts to measure the ability of a player in a WVG to determine the outcome of the vote. WVGs have been applied in various political and economic organizations ([9], [8], [1]). Voting power is used in joint stock companies where each shareholder gets votes in proportion to the ownership of a stock [5]. WVGs are also encountered in threshold logic, reliability theory, neuroscience and logical computing devices ([14], [15]). Nordmann et al. [11] deal with reliability and cost evaluation of weighted dynamic-threshold voting-systems. Systems of this type are used in various areas such as target and pattern recognition, safety monitoring and human organization systems. There are parallels between reliability theory and voting theory [13].

Tolerance and amplitude of WVGs signify the possible variances in a WVG which still keep the game unchanged. They are significant in mathematical models of reliability systems and shareholdings. Parhami [12] points out that voting has a long history in reliability systems dating back to von Neumann [16]. For reliability systems, the weights of a WVG can represent the significance of the components whereas the quota can represent the threshold for the overall system to fail. It is then a natural requirement to provide a framework which can help identify similar reliability systems. In shareholding scenarios, there is need to check the maximum changes in shares which still maintain the status quo. In political settings, the amplitude of a WVG signifies the maximum percentage change in various votes without changing the voting powers of the voters.

We give tight bounds for the tolerance of some key WVGs. Arcaini and Gambarelli [2] analyse the Shapley-Shubik index variations of a WVG which models share holdings. We look at the possible variances in the weights and quotas in a WVG which still keep the game unchanged. This kind of guarantee can be extremely useful in the modeling of shareholding, reliability systems and of course voting scenarios.

Splitting of a player into sub-players can be seen as a false name manipulation by an agent in which it splits itself into more agents so that the sum of the utilities of the split-up players is more than the utility of the original player [3]. We examine situations when a player splitting up into smaller players may be advantageous or disadvantageous in the context of WVGs and Banzhaf indices. We provide limits to how much the Banzhaf index of a player increases or decreases if it splits up into sub-players.

## II. PRELIMINARIES

In this section we give definitions and notations of key terms. The set of voters is $N = \{1, ..., n\}$.

**Definitions II.1.** *A* simple voting game *is a pair* $(N, v)$ *where the valuation function* $v : 2^N \rightarrow \{0, 1\}$ *has the properties that* $v(\emptyset) = 0$, $v(N) = 1$ *and* $v(S) \leq v(T)$ *whenever* $S \subseteq T$. *A* coalition $S \subseteq N$ *is* winning *if* $v(S) = 1$ *and* losing *if* $v(S) = 0$. *A simple voting game can alternatively be defined as* $(N, W)$ *where* $W$ *is the set of winning coalitions.*

**Definitions II.2.** *The simple voting game* $(N, v)$ *where* $W = \{X \subseteq N, \sum_{x \in X} w_x \geq q\}$ *is called a* weighted voting game *(WVG). A WVG is denoted by* $[q; w_1, w_2, ..., w_n]$ *where* $w_i \geq 0$ *is the voting weight of player* $i$. *Generally* $w_i \geq w_j$ *if* $i < j$.

Usually, $q > \frac{1}{2}\sum_{1 \leq i \leq n} w_i$ so that there are no two mutually exclusive winning coalitions at the same time. WVGs with this property are termed *proper*. Proper WVGs are also desirable because they satisfy the criterion of the majority getting preference. If the valuation function of a WVG $v$ is same as another WVG $v'$, then $v'$ is called a *representation* of $v$. If the quota $q'$ of $v'$ is such that for all $S \subseteq N$, $\sum_{i \in S} w_i' \neq q'$, then $v'$ is called a *strict representation* of $v$.

**Definitions II.3.** *A player $i$ is* critical in *a winning coalition $S$ when $S \in W$ and $S \setminus \{i\} \notin W$. For each $i \in N$, we denote the number of coalitions in which $i$ is critical in game $v$ by $\eta_i(v)$. The* Banzhaf Index *of player $i$ in WVG $v$ is $\beta_i = \frac{\eta_i(v)}{\sum_{i \in N} \eta_i(v)}$.*

**Definition II.4.** *A problem is in complexity class* P *if it can be solved in time which is polynomial in the size of the input. A problem is in the complexity class* NP *if its solution can be verified in time which is polynomial in the size of the input of the problem. A problem is in complexity class* co-NP *if and only if its complement is NP. A problem is in the complexity class* NP-hard *if any problem in* NP *is polynomial time reducible to that problem.* NP-hard *problems are as hard as the hardest problems in* NP.

## III. TOLERANCE & AMPLITUDE: BACKGROUND

### A. *Background*

The question we are interested in is to find the maximum possible variations in the weights and quotas of a WVG which still do not change the game. The two key references which address this question are [7] and [4]. Hu [7] developed on the theory of switching functions. He set forth the idea of *linearly separable switching functions* which are equivalent to each other. Freixas and Puente [4] extended the theory by framing it in the context of strict representations of WVG which are equivalent to linearly separable switching functions.

### B. *Tolerance*

The setting of the problem is that we look at a transformation, $f_{(\lambda_1,\ldots,\lambda_n),\Lambda}$ which maps a WVG, $v = [q; w_1, \ldots, w_n]$ to $v' = [q'; w_1', \ldots, w_n']$ such that $w_i' = (1 + \lambda_i)w_i$ and $q' = (1 + \Lambda)q$. Let $A$ be the maximum of $w(S)$ for all $\{S | v(S) = 0\}$. and let $B$ be the minimum of $w(S)$ for all $\{S | v(S) = 1\}$. Then $A < q \leq B$ (and $q < B$ if the representation is strict). Moreover, let $m = \mathrm{Min}(q - A, B - q)$ and $M = q + w(N)$. Hu [7] and then Freixas and Puente [4] showed that if for all $1 \leq i \leq n$, $|\lambda_i| < m/M$ and $|\Lambda| < m/M$, then $v'$ is just another representation of $v$. They defined $\tau[q; w_1, \ldots, w_n] = m/M$ as the *tolerance* of the system. Freixas and Puente [4] also showed that the tolerance is less than or equal to $1/3$ for strict representations of a WVG and less than or equal to $1/5$ for a non-monotonic[1] WVG.

### C. *Amplitude*

Freixas and Puente defined the *amplitude* as the maximum $\mu$ such that $f_{(\lambda_1,\ldots,\lambda_n),\Lambda}$ is a representation of $v$ whenever $\mathrm{Max}(|\lambda_1|, \ldots, |\lambda_n|, |\Lambda|) < \mu(v)$. For a strict representation of a WVG $[q; w_1, \ldots, w_n]$, for each coalition $S \subseteq N$, let $a(S) = |w(S) - q|$ and $b(S) = q + w(S)$.

[1]Freixas and Puente also consider WVGs where players' weights can be negative, giving perhaps "non-monotonic" WVGs.

Freixas and Puente [4] showed that the amplitude of a WVG is $\mu(v) = \inf_{S \subseteq N} \frac{a(S)}{b(S)}$. Although both tolerance and amplitude has been used in the WVG literature to signify the maximum possible variation in the weights and the quota without changing the game, the amplitude is a more precise and accurate indicator of the maximum possible variation than tolerance.

## IV. TOLERANCE & AMPLITUDE: SOME RESULTS

### A. *Complexity*

In all the complexity proofs in this section, we assume that the weights in a WVG are positive integers. We let WVG-STRICT be the problem of checking whether a WVG $v = [q; w_1, \dots, w_n]$ is strict or not, i.e., WVG-STRICT = $\{v$: $v$ is strict$\}$. Then we have the following proposition:

**Proposition IV.1.** *WVG-STRICT is co-NP-complete*

*Proof:* Let WVG-NOT-STRICT = $\{v$: $v$ is not strict$\}$. WVG-NOT-STRICT is in NP since a certificate of weights can be added in linear time to confirm that they sum up to $q$. Moreover $v$ is not strict if and only if there is a subset of weights which sum up to $q$. Therefore the NP-complete problem SUBSET-SUM (see Garey and Johnson [6]) reduces to WVG-NOT-STRICT. Hence WVG-NOT-STRICT is NP-complete and WVG-STRICT is co-NP-complete. ■

**Corollary IV.2.** *The problem of checking whether the amplitude of a WVG is* $0$ *is NP-hard.*

**Proposition IV.3.** *The problem of computing the amplitude of a WVG* $v$ *is NP-hard, even for integer WVGs.*

*Proof:* Let us assume that weights in $v$ are even integers whereas the quota $q$ is an odd integer $2k - 1$ where $k \in \mathbf{N}$. Then the minimum possible difference between $q$ and $A$, the weight of the maximal losing coalition, or $q$ and $B$, the weight of minimal winning coalition is 1. So $A \leq 2k - 2$ and $B \geq 2k$. We see that $\mu(v) \leq 1/2k$ if and only if there exists a coalition $C$ such that $w(C) = 2k$. Thus the problem of computing $\mu(v)$ of a WVG is NP-hard by a reduction from the SUBSET-SUM problem. ■

A similar proof can be used to prove the following proposition:

**Proposition IV.4.** *The problem of computing the tolerance of a strict WVG is NP-hard.*

### B. *Uniform and unanimity WVGs*

We show that the bound for the maximum possible tolerance can be improved when we restrict to strict representations of special cases of WVGs. We first look at uniform WVGs which are an important subclass of WVGs which model many multi-agent scenarios where each agent has the same voting power.

**Proposition IV.5.** *For a strict representation of a proper uniform WVG* $v = [q; \underbrace{w, \ldots, w}_{n}]$, $\tau(v) \leq \frac{2}{3n}$.

*Proof:* The tolerance reaches its maximum when $q$ is the arithmetic mean $\frac{A+B}{2}$. Since the minimum difference in the size of two coalitions can be $w$, $m = w$. We let the size of the maximal losing coalition be $r$ and the size of the minimal winning coalition be $r+1$. Then the weight of a maximal losing coalition is $rw$ and the weight of the minimal winning coalition is $(r+1)w$. Since $v$ is proper, $q = rw + \frac{1}{2}w \geq \frac{1}{2}(nw)$. Thus $r \geq \frac{n-1}{2}$. Then,

$$\tau(v) = m/M \leq \frac{w}{\frac{rw+(r+1)w}{2} + nw} = \frac{2}{2r+2n+1} \leq \frac{2}{\frac{2(n-1)}{2} + 2n + 1} = \frac{2}{3n}.$$

■

**Proposition IV.6.** *For a uniform WVG* $v = [q; \underbrace{w, \ldots, w}_{n}]$, *we have* $B = w\lceil \frac{q}{w} \rceil$ *and* $A = B - w$. *Then,*

$$\mu(v) = \begin{cases} \frac{q-A}{A+q}, & \text{if } q \leq \sqrt{AB} \\ \frac{B-q}{B+q}, & \text{otherwise.} \end{cases}$$

*Proof:*

It is clear that $B$, the weight of the minimal winning coalition is $w\lceil \frac{q}{w} \rceil$ and $A$, the weight of the maximal losing coalition is $B - w$. If $\frac{q-A}{q+a} \leq \frac{B-q}{q+B}$, then $q \leq \sqrt{AB}$. For losing coalitions with weight $w$, $\frac{q-w}{q+w}$ is a decreasing function for $w$. For winning coalitions with weight $w$, $\frac{w-q}{q+w} = 1 - \frac{2q}{q+w}$ is an increasing function for $w$. Thus if $q \leq \sqrt{AB}$, $\mu(v) = \frac{q-A}{A+q}$. Otherwise, $\mu(v) = \frac{B-q}{B+q}$. ■

**Corollary IV.7.** *The amplitude* $\mu(v)$ *of a uniform WVG* $v$ *can be found in* $O(1)$, *i.e., constant time.*

*Proof:* The corollary immediately follows from the previous theorem. ■

We now look at unanimity WVGs, which are another important subclass of WVGs in which a coalition is winning if and only if it is the grand coalition $N$.

**Proposition IV.8.** *For a unanimity WVG* $v = [q; w_1, \ldots, w_n]$, $\tau(v) \leq \frac{w_n}{4w(N)-w_n} \leq \frac{1}{4n-1}$.

*Proof:* We know that $B = w(N)$ and $A = w(N) - w_n$ which means that $w(N) - w_n < q \leq w(N)$. For maximum tolerance, $q = \frac{A+B}{2} = w(N) - \frac{w_n}{2}$. Therefore $m = w_n/2$ and $M = w(N) - \frac{w_n}{2} + w(N)$. Then the tolerance of $v$ satisfies:

$$\tau(v) \leq \frac{m}{M} = \frac{w_n}{4w(N) - w_n} \leq \frac{1}{4n-1},$$

since $w_n \leq w(N)/n$. ■

A multiple weighted voting game is composed of more than one weighted voting game and a coalition wins if and only if it is winning in each of the weighted voting games:

**Definition IV.9.** *An* $m$-multiple weighted voting game, $(N, v_1 \wedge ... \wedge v_m)$ *is the simple game* $(N, v)$*, where the games* $(N, v_t)$ *are the weighted voting games* $[q^t; w_1^t, ...w_n^t]$ *for* $1 \leq t \leq m$*. Then* $v = v_1 \wedge ... \wedge v_m$ *is defined as:* $v(S) = 1 \iff v_t(S) = 1$ *for* $1 \leq t \leq m$*. The game* $v$ *is called the* meet *of the* $v_i$*s.*

Let $(N, v) = (N, v_1 \wedge ... \wedge v_m)$ be a multiple weighted voting game. Then we can see that $\mu(v) \geq \mathsf{Inf}(\mu(v_1), \ldots, \mu(v_m))$. The reason is that for $v$ to change, at least one constituent game has to change. However it is not necessary that a change in any one game $v_i$ changes $v$. As a simple example, suppose $v_1 = [2; 2, 1]$ and $v_2 = [2; 1, 2]$. Then $\mu(v_1) = \mu(v_2) = \sqrt{3/2} - 1$, as witnessed by the coalition $\{1, 2\}$. However, $\mu(v_i) = 0$, witnessed by $\{i\}$, for $i = 1, 2$.

## V. Splitting

### A. *Background*

In the real world, WVGs may be dynamic. Players might have incentive to split up into smaller players or merge into voting blocks. Imputations of the players are acceptable distributions of the payoff of the grand coalition. Imputations of players in a coalitional games setting can be based on fairness, i.e., power indices or they can be based on the notion of stability which includes many cooperative game theoretic solutions such as core, nucleolus etc. We examine the situation when the Banzhaf indices of agents can be used as imputations in a cooperative game theoretic situation. Falsenthal and Machover [10] give this notion of voting power as P-power since the motivation of agents is prize-seeking as opposed to influence-seeking. Splitting of a player can be seen as a false name manipulation by an agent in which it splits itself into more agents so that the sum of the utilities of the split-up players is more than the utility of the original player [3].

Splitting does not always have to beneficial. We give examples where if we use Banzhaf index as an imputation for a WVG, then splitting can be advantageous, neutral or disadvantageous.

**Example V.1.** *Splitting can be advantageous, neutral or disadvantageous:*

- Disadvantageous splitting. *We take the WVG* $[5; 2, 2, 2]$ *in which each player has a Banzhaf index of* $1/3$*. If the last player splits up into two players, the new game is* $[5; 2, 2, 1, 1]$*. In that case, the split-up players have a Banzhaf index of* $1/8$ *each.*
- Neutral splitting. *We take the WVG* $[4; 2, 2, 2]$ *in which each player has a Banzhaf index of* $1/3$*. If the last player splits up into two players, the new game is* $[4; 2, 2, 1, 1]$*. In that case, the split-up players have a Banzhaf index of* $1/6$ *each.*

- Advantageous splitting. *We take the WVG* $[6; 2, 2, 2]$ *in which each player has a Banzhaf index of* $1/3$. *If the last player splits up into two players, the new game is* $[6; 2, 2, 1, 1]$. *In that case, the split-up players have a Banzhaf index of* $1/4$ *each.*

We analyse the splitting of players in the unaninimity WVG.

**Proposition V.2.** *In a unanimity WVG with* $q = w(N)$, *if Banzhaf or Shapley-Shubik indices are used as imputations of agents in a WVG, then it is beneficial for an agent to split up into agents.*

*Proof:* In a WVG with $q = w(N)$, the Banzhaf index of each player is $1/n$. Let player $i$ split up into $m + 1$ players. In that case there is a total of $n + m$ players and the Banzhaf index of each player is $1/(n + m)$. In that case the total Banzhaf index of the split up players is $\frac{m+1}{n+m}$, and for $n > 1$, $\frac{m+1}{n+m} > \frac{1}{n}$. An exactly similar analysis holds for Shapley-Shubik index. ■

However there is the same motivation for all agents to split up into smaller players which returns us to the status quo.

### B. General case

We recall that a player is critical in a winning coalition if the player's exclusion makes the coalition losing. We will also say that a player is *critical for* a losing coalition $C$ if the player's inclusion results in the coalition winning.

**Proposition V.3.** *Let WVG* $v$ *be* $[q; w_1, \ldots, w_n]$. *If* $v$ *transforms to* $v'$ *by the splitting of player* $i$ *into* $i'$ *and* $i''$, *then* $\frac{2\beta_i(v)}{3} \leq \beta_{i'}(v') + \beta_{i''}(v') \leq 2\beta_i(v)$.

*Proof:* We assume that a player $i$ splits up into $i'$ and $i''$ and that $w_{i'} \leq w_{i''}$. We consider a losing coalition $C$ for which $i$ is critical in $v$. Then $w(C) < q \leq w(C) + w_i = w(C) + w_{i'} + w_{i''}$.

- If $q - w(C) \leq w_{i'}$, then $i'$ and $i''$ are critical for $C$ in $v'$.
- If $w_{i'} < q - w(C) \leq w_{i''}$, then $i'$ is critical for $C \cup \{i''\}$ and $i''$ is critical for $C$ in $v'$.
- If $q - w(C) > w_{i''}$, then $i'$ is critical for $C \cup \{i''\}$ and $i''$ is critical for $C \cup \{i'\}$ in $v'$.

Therefore we have $\eta_{i'}(v') + \eta_{i''}(v') = 2\eta_i(v)$ in each case.

Now we consider a player $x$ in $v$ which is other than player $i$. If $x$ is not critical for a coalition $C$ or $C \cup \{i\}$ in $v$, then it is not critical for $C$, $C \cup \{i'\}$, $C \cup \{i''\}$ or $C \cup \{i', i''\}$ in $v'$. If is critical for both $C$ or $C \cup \{i\}$ in $v$, then it is critical for $C$, $C \cup \{i'\}$, $C \cup \{i''\}$ and $C \cup \{i', i''\}$ in $v'$.

If $x$ is only critical for $C$ in $v$, then the following cases are possible:

- $x$ is critical for $C$ in $v'$,
- $x$ is critical for $C$ and $C \cup \{i'\}$, or
- $x$ is critical for $C$, $C \cup \{i'\}$ and $C \cup \{i''\}$.

If $x$ is only critical for $C \cup \{i\}$ in $v$, then the following cases are possible:

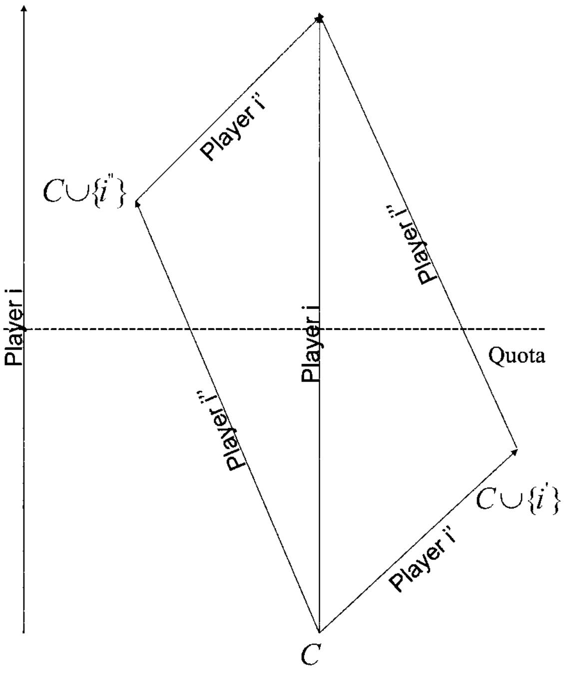


Fig. 1. Splitting of player $i$ into $i'$ and $i''$ .

- $x$ is critical for $C \cup \{i''\}$ in $v'$,
- $x$ is critical for $C \cup \{i''\}$ and $C \cup \{i'\}$ in $v'$, or
- $x$ is critical for $C$, $C \cup \{i'\}$ and $C \cup \{i''\}$ in $v'$.

Therefore $\eta_x(v) \leq \eta_x(v') \leq 3\eta_x(v)$. Hence

$$\begin{aligned}
\beta_{i'}(v') + \beta_{i''}(v') &= \frac{\eta_{i'}(v') + \eta_{i''}(v')}{\sum_{j\in N(v')} \eta_j(v')} = \frac{\eta_{i'}(v') + \eta_{i''}(v')}{\eta_{i'}(v') + \eta_{i''}(v') + \sum_{x\in N(v')\setminus\{i',i''\}} \eta_x(v')} \\
&= \frac{2\eta_i(v)}{2\eta_i(v) + \sum_{x\in N(v')\setminus\{i',i''\}} \eta_x(v')} \\
&\geq \frac{2\eta_i(v)}{2\eta_i(v) + 3\sum_{x\in N(v)\setminus\{i\}} \eta_x(v)} \\
&\geq \frac{2\eta_i(v)}{3\eta_i(v) + 3\sum_{x\in N(v)\setminus\{i\}} \eta_x(v)} = \frac{2\beta_i(v)}{3}
\end{aligned}$$

Moreover,

$$\begin{aligned}\beta_{i'}(v') + \beta_{i''}(v') &= \frac{2\eta_i(v)}{2\eta_i(v) + \sum_{x \in N(v') \setminus \{i',i''\}} \eta_x(v')} \\ &\leq \frac{2\eta_i(v)}{2\eta_i(v) + \sum_{x \in N(v) \setminus \{i\}} \eta_x(v)} \\ &\leq \frac{2\eta_i(v)}{\eta_i(v) + \sum_{x \in N(v) \setminus \{i\}} \eta_x(v)} = 2\beta_i(v)\end{aligned}$$

■

We can give examples which show that these bounds on the change in Banzhaf indices are almost tight:

**Example V.4.** Advantageous splitting. *We take a WVG* $[n; 2, 1, \ldots, 1]$ *with* $n+1$ *players. We find that* $\eta_1 = n + \binom{n}{2}$ *and for all other* $x$, $\eta_x = 1 + \binom{n-1}{2}$. *Therefore*

$$\beta_1 = \frac{n + \binom{n}{2}}{n + \binom{n}{2} + n(1 + \binom{n-1}{2})} = \frac{n+1}{(n-2)^2} \sim 1/n.$$

*In case player* 1 *splits up into* $1'$ *and* $1''$ *with weights* 1 *each, then for all players* $j$, $\beta_j = \frac{1}{n+2}$. *Thus for large* $n$, $\beta_{1'} + \beta_{1''} = \frac{2}{n+2} \sim 2\beta_1$.

**Example V.5.** Disadvantageous splitting. *We take a WVG* $[2n-2; 2, \ldots, 2]$ *with* $n+1$ *players. Therefore* $\beta_j = 1/(n+1)$ *for all* $j$. *Let us assume that the player* $i$ *splits up into players* $i'$ *and* $i''$ *with weights* 1 *each. In the new game* $v'$, $\eta_{i'} = \eta_{i''} = n$ *and for all* $x \neq i$, $\eta_x = 1 + (n-1) + 2(n-1) = 3n - 2$. *Then*

$$\beta_{i'} = \beta_{i''} = \frac{n}{n(3n-2) + 2n} = \frac{1}{3n}.$$

*Thus for large* $n$, $\beta_{i'} + \beta_{i''} \sim \frac{2}{3}\beta_i$.

## VI. Conclusion and Future Work

We have characterised the computational complexity of computing the tolerance and amplitude of WVGs. We have analysed the tolerance and amplitude of uniform and unanimity games. There is a need to devise approximate algorithms for computing the amplitude of a WVG. The analysis of amplitude and tolerance motivates the formulation of an overall framework to check the 'sensitivity' of voting games under fluctuations. It will be interesting to explore the limit of changes in WVGs in alternative representations of WVGs. We have also investigated the impact on the Banzhaf power distribution due to a player splitting into smaller players in a weighted voting game. There is more scope to analyse such situations with respect to cooperative game theoretic solutions.

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