The University of Warwick

THEORY OF COMPUTATION

REPORT NO. 55

IMPROVED UPPER BOUNDS ON THE AREA REQUIRED TO EMBED ARBITRARY GRAPHS

by

Paul E Dunne

Department of Computer Science
University of Warwick
Coventry
CV4 7AL
Great Britain

December 1983

Improved Upper Bounds On The Area Required To Embed Arbitrary Graphs

Paul E. Dunne

University Of Warwick
Coventry CV4 7AL
Great Britain

ABSTRACT

We prove that any n-vertex graph of maximum degree r (r=3 or 4) can be embedded in a square grid of area $A_r(n)$, where:

$$A_3(n) \le n^2/4 + O(n^{3/2})$$

$$A_4(n) \le n^2 + O(n^{3/2})$$

Improved Upper Bounds On The Area Required To Embed Arbitrary Graphs

Paul E. Dunne

University Of Warwick
Coventry CV4 7AL
Great Britain

1. Introduction

Valiant [5], Leiserson [4] and others have considered the problem of determining upper bounds on the grid areas required to embed certain classes of n-vertex graph, e.g trees and planar graphs.
Valiant also examined arbitrary graphs and proved;

$$A_r(n) \leq 9\, n^2$$

The construction used may be slightly sharpened to reduce the constant to 4.

In this paper we describe a different method and prove that;

$$A_3(n) \leq n^2/4 + O(\, n^{3/2}\,)$$

$$A_4(n) \leq n^2 + O(\, n^{3/2}\,)$$

In the remainder of this section we give definitions and the notation used and in the next section prove the main result.

We consider undirected graphs, with vertices **V** and edges **E**, which will be denoted by **G(V,E)**. For the graphs considered **E** will include no self-loops.

The *degree* of a graph **G(V,E)** is the maximum degree of any vertex in **G**.

We shall consider graphs with degree at most 4.

The vertex set of a graph **G** will be denoted by **V(G)**, the edge set by **E(G)**.

Definition 1

An *I,J grid* is the graph consisting of **IJ** vertices, formed by placing vertices at the Cartesian coordinates;

$$\{ \ (x,y) \ | \ 0 \leq x < I, \quad 0 \leq y < J \ \}$$

with edges between the pairs at unit distance apart.

Definition 2

An *embedding* of a graph **G(V,E)** into an I−J grid **X**, is a pair of mappings;

P : **V(G)** −> **V(X)**

Q : **E(G)** −> paths of edges in **X**

such that if (v,w) is in **E(G)** then Q((v,w)) is a path in **X** from P(v) to P(w).

Two paths, $Q(v_1,w_1)$ and $Q(v_2,w_2)$ may share grid vertices but not any grid edges.

Definition 3

$A_r(n)$ is defined to be the minimum **K** such that any degree r n-vertex graph may be embedded into a grid containing at most **K** vertices.

For further graph-theoretic definitions refer to Even [3] or Berge [1].

2. Upper Bounds

Consider the algorithm below which embeds an arbitrary n-vertex graph **G(V,E)**.

A1) Find a spanning tree **T** of **G** and embed **T** in a $(k_1n^{1/2}),(k_2n^{1/2})$-grid **X** (e.g following the methods of [4])

A2) For each edge (v,w) of **G** which is not an edge of **T**, embed (v,w)

In order to perform step (A2) additional rows and columns may have to be added to **X** in order to provide a path from P(v) to P(w) which is edge disjoint from existing paths in Q(**E**).
Let X_i denote the grid after the i'th edge has been embedded. (So that $X_0 = X$).
Let R_i and C_i denote the number of rows and columns in X_i.
The upper bound is obtained by proving;

$$R_i \leq R_{i-1}+1 \ ; \ C_i \leq C_{i-1}+1$$

for $1 \leq i \leq | E(G) - E(T) |$
Thus at most one row and one column need be added to X_{i-1} to route the i'th edge.

We prove this result in two stages. First we establish a sufficient condition allowing an edge to be embedded by adding at most one row and one column to the grid. We then prove that any graph having degree<=4 may be labelled in such a way, that when an edge is to be embedded this condition will hold.

Definition 4

Let v be a graph vertex, and let P(v)=z. An *exit-path* of v, is a grid edge (z,y), such that no graph edge incident to v has been embedded using (z,y) as a path component.

Initially an embedded vertex, with no incident edges added, has four exit-paths. We shall label these **N**, **S**, **E**, **W** in the obvious way.

Lemma 1

Let (v,w) be an edge of a graph **G**(**V**,**E**). Let each edge of **G** be embedded into a grid **X**, except for (v,w). If v has a **N** or **S** exit-path and w has a **E** or **W** exit-path (or vice-versa) the edge (v,w) may be embedded by adding at most one row and one column to **X**.

Proof

Wlog, suppose v has a **N** exit-path and w has a **E** exit-path. Let (x_v, y_v) be the Cartesian coordinates of P(v) in **X**, and let (x_w, y_w) be the Cartesian coordinates of P(w). We proceed as follows to embed the edge (v,w).

Insert a new row $\mathbf{R}^*$ between rows y_v and y_v+1. Similarly insert a new column $\mathbf{C}^*$ between columns x_w and x_w+1. The edge (v,w) can now be embedded by using a path consisting of:

> The **N** exit-path of v, followed by the edges in row $\mathbf{R}^*$, (between columns x_v and $\mathbf{C}^*$), followed by the edges in column $\mathbf{C}^*$, (between rows $\mathbf{R}^*$ and y_w), completing the path using the **E** exit-path of w.

Definition 5

A *routing scheme* RS for a graph **G**(**V**,**E**) is a pair of mappings $\{\mathbf{M_v}, \mathbf{M_w}\}$ satisfying (b1)-(b4) below;

b1) $\mathbf{M_v}(\mathbf{E}(\mathbf{G})) \text{ -> } \{\ \mathbf{N,S,E,W}\ \}$

b2) $\mathbf{M_w}(\mathbf{E}(\mathbf{G})) \text{ -> } \{\ \mathbf{N,S,E,W}\}$

b3) $\mathbf{M_v}(v,w) = \mathbf{M_v}(v,y) <=> y=w$

b4) $\mathbf{M_w}(v,w) = \mathbf{M_w}(x,w) <=> x=v$

Any routing scheme $\{\mathbf{M_v},\mathbf{M_w}\}$, for a graph **G(V,E)** defines a set of pairs (e_v,e_w) describing the exit-paths to be used when embedding the edge (v,w).

In order for the condition of Lemma(1) to hold when each edge is embedded, a scheme $\{\mathbf{M_v},\mathbf{M_w}\}$ must satisfy:

b5) $\mathbf{M_v}(v,w)=\mathbf{N}$ or **S** if and only if $\mathbf{M_w}(v,w)=\mathbf{E}$ or **W**.

Lemma 2

For any graph **G(V,E)** there exists a routing scheme RS satisfying (b5).

Proof

We distinguish two cases:

Case 1

Every vertex of **G** has even degree.
It is well known that **G** has an Eulerian circuit (i.e starting from any vertex of **G**, a path may be traced through **G** which visits each edge exacly once and ends at the starting vertex).
We proceed as follows.

Make **G** into a directed graph by tracing out an Eulerian circuit of **G** and marking each edge with the direction in which it is traversed, e.g if vertex w is visited from vertex v then the edge (v,w) is directed **from** v **to** w.

It is easy to see that in the directed graph which results, each vertex has at most 2 incoming edges and at most 2 outgoing edges incident to it. Furthermore every degree 2 vertex is incident to exactly one incoming edge and exacly one outgoing edge.

We can now define $\mathbf{M_v}$ and $\mathbf{M_w}$ as follows:

For each edge (v,w) in **E**(**G**)
  If (v,w) is directed **from** v **to** w
    then $\mathbf{M_v}$(v,w) = **N** or **S**
    and $\mathbf{M_w}$(v,w) = **E** or **W**
 else
   $\mathbf{M_v}$(v,w) = **E** or **W**
   and $\mathbf{M_w}$(v,w) = **N** or **S**

Clearly the resulting routing scheme for **G** satisfies (b5).

Case 2

G contains some odd degree vertices.

The number of odd degree vertices in any graph is even, since the sum of the vertex degrees is equal to twice the number of graph edges. We can thus reduce this case to Case(1) by "pairing" odd degree vertices and adding an edge between the vertices in a pair.

The resulting graph still has degree<=4 and the methods of Case(1) may be applied to find a routing scheme satisfying (b5). The extra edges can then be removed.

Theorem 1

$$A_3(n) \leq n^2/4 + O(n^{3/2})$$

$$A_4(n) \leq n^2 + O(n^{3/2})$$

Proof

Let $G(V,E)$ be a graph. Embed G into a grid as follows;

D1) Construct a routing scheme $\{M_v, M_w\}$ satisfying (b5), as in the proof of Lemma(2).

D2) Find a spanning tree T of G and embed T into a $(k_1 n^{1/2}),(k_2 n^{1/2})$-grid X, with each edge of T being embedded using the exit-paths specified by $\{M_v, M_w\}$

D3) For each edge (v,w) in $E(G)-E(T)$, embed (v,w) by adding one row and one column to the grid X_{i-1} to yield a new grid X_i.

The correctness of this algorithm follows from Lemma(2) and the fact that the tree embedding algorithms of Valiant and Leiserson may be amended to realise the requirements of Step(D2) ([2]).

Let K be the number of vertices in the final grid X_s, (where $s=|E(G)-E(T)|$). Then;

$$A_r(n) \leq K$$

$$\leq (k_1 n^{1/2}+s)(k_2 n^{1/2}+s)$$

For r=3

$$s \leq 3n/2 - (n-1) = n/2 + 1$$

For r=4

$$s \leq 2n - (n-1) = n + 1$$

Thus

$A_3(n) \leq n^2/4 + O(n^{3/2})$

$A_4(n) \leq n^2 + O(n^{3/2})$ as claimed.

The above result has also been independently derived by S.Skyum.

3. References

[1] C.Berge

Graphs and Hypergraphs

North Holland 1973

[2] P.E.Dunne

Embedding Graphs Into Grids

Unpublished manuscript, Univ. Of Edinburgh 1981

[3] S.Even

Graph Algorithms

Pitman 1979

[4] C.E.Leiserson

Area-Efficient Graph Layouts (For VLSI)

IEEE Proceedings on FOCS 1980

[5] L.G.Valiant

Universality Considerations In VLSI Circuits

Univ. Of Edinburgh, Computer Sc. Dept

Report CSR-54-80 1980