Research Report 212

Partial Metric Spaces

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Scott models are topological models of complete partial orders used for Tarskian fixed point semantics of the lambda calculus. As of yet there are no methods for deriving Scott models from specifications of the "complete" objects beyond an arbitrary choice. This paper introduces "partial metrics" for generalising a theory of complete objects into a Scott model including partial objects.

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Introduction

Metric space topology provides an excellent framework for studying the behaviour of continuous functions in many T_2 topologies. For example, Banach's contraction mapping theorem provides a foundation for much inductive proof theory for continuous functions. Metric spaces are of much interest to programming language designers as they provide the domain of totally defined or **complete** programmable data objects. For example, for the set of all flat finite & infinite lists L^S over a set S we can define a "Baire" style metric on L^S as follows.

$$\forall\, l \in L \;.\; d(\text{Nil}, l) = 1 \qquad \text{if } l \neq \text{Nil}$$

$$\begin{aligned} \forall\, s, s' \in S \;\; \forall\, l, l' \in L \;.\; & \\ d(s:l, s':l') &= 1/2 \times d(l, l') && \text{if } s = s' \\ &= 1 && \text{if } s \neq s' \end{aligned}$$

Unfortunately Godel's decidability results force us to include partial objects such as $\perp$ the totally undefined data object alongside the complete ones. This leads to Scott's partial order topological models as used in denotational semantics. However, by necessity these models are T_0 and so not describable by any metric as all metric spaces are T_2 (i.e. Hausdorff). This would seem to infer that metric space topology is not appropriate for denotational semantics. deBakker & Zucker [dB&Z82] have used metric spaces but without the usual Scott partial order topology. Smyth [Sm87] has generalised the metric axioms by dropping the symmetry axiom (see M2 in Definition A5 or [Su75]) in order to define partial orders. Also, Kopperman [Ko88] has shown that any topology can be generated by an appropriate generalised metric. This raises the possibility that there may be an appropriate notion of a generalised metric suitable for Scott topologies. In this paper we provide such a generalisation as an alternative to the approach taken by Smyth. In our approach the complete objects form a metric subspace of a **partial metric space**. Our generalisation keeps the symmetry axiom M2 while using a new generalised reflexive axiom to distinguish between partial and complete objects. This new approach promises an approach to denotational semantics which combines the elegance of Tarskian semantics and Scott topologies with conventiional metric space technology.

Definition 1

A **Partial Metric** (**Pmetric**) is a function $p : A \times A \rightarrow \mathbb{R}$ such that,

(P1) $\forall\ x, y \in A\ .\ x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$

(P2) $\forall\ x, y \in A\ .\ p(x,x) \leq p(x,y)$

(P3) $\forall\ x, y \in A\ .\ p(x,y) = p(y,x)$

(P4) $\forall\ x, y, z \in A\ .\ p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$

A metric is precisely a pmetric $p : A \times A \rightarrow \mathbb{R}$ in which,

$$\forall\ x \in A\ .\ p(x,x) = 0$$

Also note that for each pmetric $p : A \times A \rightarrow \mathbb{R}$,

$$\forall\ x, y \in A\ .\ p(x,y) = 0 \Rightarrow x = y$$

this being "half" of the metric reflexive axiom M1 . We can use pmetrics to make the following definition of when a data object is to be regarded as **complete**.

x is **complete** ::= $p(x,x) = 0$

Any object which is not complete is called **partial**, thus,

x is **partial** ::= $p(x,x) > 0$

Example 1

A **Flat Pmetric** is a pmetric $p_\perp : S_\perp \times S_\perp \rightarrow \{0,1\}$ where, $S_\perp ::= S \cup \{\perp\}$ for a set S, and $\perp \notin S$, and,

$$\forall\ x, y \in S_\perp\ .\ p_\perp(x,y) = 0 \Leftrightarrow x = y \in S$$

Later we will show how flat pmetrics relate to the usual partial order notion of a flat domain.

Example 2

A **Kahn Pmetric** is a pmetric $p^S : Ka^S \times Ka^S \to \{ 2^{-n} \mid n \in \omega \} \cup \{ 0 \}$ where Ka^S is defined to be the set of all finite & infinite sequences over the set S and,

$$\forall\ y \in Ka^S . \quad p^S(<> , y) = 1$$

$$\forall\ x , y \in {}^{\omega}S \quad \forall\ n , m > 0 . \quad p^S(< x_0 , \ldots , x_{n-1} > , < y_0 , \ldots , y_{m-1} >)$$

$$\begin{aligned} &= 1/2 \times p^S(< x_1 , \ldots , x_{n-1} > , < y_1 , \ldots , y_{m-1} >) && \text{if } x_0 = y_0 \\ &= 1 && \text{if } x_0 \neq y_0 \end{aligned}$$

$$\forall\ x \in {}^{\omega}S . \quad p^S(x , x) = 0$$

Kahn pmetrics are used for describing the partial order domain used by Kahn [Ka74] to give a denotational semantics to pipeline data flow networks. Later on we will show how Kahn pmetrics can be used to describe Kahn's partial ordering on Ka^S .

Definition 2

The **Open Balls** for a pmetric $p : A \times A \to \mathbb{R}$ are the sets of the form,

$$B_{\varepsilon}(x) \quad ::= \quad \{ \ y \in A \ \mid \ p(x , y) < \varepsilon \ \}$$

for each $\varepsilon > 0$ and $x \in A$.

Theorem 1

The set of all open balls of a pmetric $p : A \times A \to \mathbb{R}$ with $\varnothing$ form the basis of a topology on A .

Proof:

Proof using Definition A4 and Theorem A1

Suppose $p : A \times A \to \mathbb{R}$ is a pmetric.

Then, $A = \bigcup_{x \in A} B_{p(x,x) + 1}(x)$ and,

for any balls $B_{\varepsilon}(x)$ and $B_{\delta}(y)$,

$$B_{\varepsilon}(x) \cap B_{\delta}(y) =$$

$$\bigcup \{ \ B_{\eta}(z) \ \mid \ z \in B_{\varepsilon}(x) \cap B_{\delta}(y)$$

where, $\eta ::= p(z , z) + \min \{ \ \varepsilon - p(x , z) , \ \delta - p(y , z) \ \}$

❑

Theorem 2
For each pmetric $p : A \times A \rightarrow \mathbb{R}$, open ball $B_\varepsilon(a)$, and $x \in A$,

$$x \in B_\varepsilon(a) \quad \Rightarrow \quad \exists \; \delta > 0 \; . \; x \in B_\delta(x) \subseteq B_\varepsilon(a)$$

Proof:

Suppose $x \in B_\varepsilon(a)$
Then $p(x,a) < \varepsilon$
Let $\delta ::= \varepsilon - p(x,a) + p(x,x)$
Then $\delta > 0$ as $\varepsilon > p(x,a)$
Also, $p(x,x) < \delta$ as $\varepsilon > p(x,a)$
Thus $x \in B_\delta(x)$

Suppose now that $y \in B_\delta(x)$
$\therefore \; p(y,x) < \delta$
$\therefore \; p(y,x) < \varepsilon - p(x,a) + p(x,x)$
$\therefore \; p(y,x) + p(x,a) - p(x,x) < \varepsilon$
$\therefore \; p(y,a) < \varepsilon$ (by P4)
$\therefore \; y \in B_\varepsilon(a)$
Thus $B_\delta(x) \subseteq B_\varepsilon(a)$.

❑

Theorem 3
Pmetric topologies are T_0.

Proof:

Suppose $p : A \times A \rightarrow \mathbb{R}$ is a pmetric.
Suppose $x \neq y \in A$.

Then from P1 & P2 $p(x,x) < p(x,y)$ or $p(y,y) < p(x,y)$.

Wlog suppose $p(x,x) < p(x,y)$ then,
$x \in B_\varepsilon(x) \wedge y \notin B_\varepsilon(x)$ where, $\varepsilon ::= (p(x,x) + p(x,y)) / 2$

❑

Note that the open balls (with ∅) of the form,

$$B_\varepsilon(x) \quad ::= \quad \{ \; y \in A \mid p(x,y) < \varepsilon \; \}$$

form a basis for the same topoplogy as the balls (with ∅) of the form,

$$B'_\varepsilon(x) \quad ::= \quad \{ \; y \in A \mid p(x,y) < \varepsilon + p(x,x) \; \}$$

as, $\forall \; \varepsilon > 0 \; \forall \; x \in A \; . \; B'_\varepsilon(x) = B_{\varepsilon + p(x,x)}(x)$
and as, $\forall \; \varepsilon > p(x,x) \; . \; B_\varepsilon(x) = B'_{\varepsilon - p(x,x)}(x)$
and, $\forall \; 0 < \varepsilon \leq p(x,x) \; . \; B_\varepsilon(x) = \varnothing$

The next theorem gives us a pmetric analogue to the familiar metric condition for convergence. A sequence $X \in {}^{\omega}A$ in a metric space with metric $d : A \times A \rightarrow \mathbb{R}$ converges to $\lambda \in A$ iff,

$$\exists \ \lim_{n \to \infty} d(X_n, \lambda) = 0$$

Theorem 4

A sequence $X \in {}^{\omega}A$ in a pmetric space with pmetric $p : A \times A \rightarrow \mathbb{R}$ converges to $\lambda \in A$ iff $\exists \ \lim_{n \to \infty} p(X_n, \lambda) = p(\lambda, \lambda)$

Proof:

Suppose X converges to λ .
Then $\forall \ \varepsilon > 0 \ \exists \ k \geq 0 \ \forall \ n > k \ . \ X_n \in B_{\varepsilon + p(\lambda,\lambda)}(\lambda)$
Then $\forall \ \varepsilon > 0 \ \exists \ k \geq 0 \ \forall \ n > k \ . \ p(X_n, \lambda) - p(\lambda, \lambda) < \varepsilon$
Thus by P2 $\exists \ \lim_{n \to \infty} p(X_n, \lambda) = p(\lambda, \lambda)$

Suppose now that $\exists \ \lim_{n \to \infty} p(X_n, \lambda) = p(\lambda, \lambda)$.
Also, suppose that $\lambda \in B_{\varepsilon}(a)$.
We have to show that X is eventually in $B_{\varepsilon}(a)$.
As $\lim_{n \to \infty} p(X_n, \lambda) = p(\lambda, \lambda)$ we can choose $k \geq 0$ such that,

$$\forall \ n > k \ . \ p(X_n, \lambda) - p(\lambda, \lambda) < \varepsilon - p(\lambda, a)$$

$$\begin{aligned} \therefore \ \forall \ n > k \ . \ p(X_n, a) &\leq p(X_n, \lambda) - p(\lambda, \lambda) + p(\lambda, a) \\ &< \varepsilon - p(\lambda, a) + p(\lambda, a) \\ &= \varepsilon \end{aligned}$$

$$\therefore \ \forall \ n > k \ . \ X_n \in B_{\varepsilon}(a) .$$

❑

A primary motivation behind the development of generalising metrics to get pmetrics was that there should be a natural way of defining a partial order on a pmetric space, and so open up such spaces to applications in denotational semantics.

Definition 3

For each pmetric $p : A \times A \rightarrow \mathbb{R}$ $\bullet p \subseteq A \times A$ is the binary relation defined by,

$$\forall \ x, y \in A \ . \ x \bullet p \ y \Leftrightarrow p(x, x) = p(x, y)$$

Theorem 5

For each pmetric $p : A \times A \rightarrow \mathbb{R}$ $\bullet p$ is a partial order.

Proof:

$$\forall \ x \in A \ . \ x \bullet p \ x \quad \text{as} \quad p(x, x) = p(x, x)$$

$$\begin{aligned} \forall \ x, y \in A \ . \ & x \bullet p \ y \ \wedge \ y \bullet p \ x \\ & \Rightarrow \ p(x, x) = p(x, y) = p(y, y) \quad (\text{by P3}) \\ & \Rightarrow \ x = y \quad (\text{by P1}) \end{aligned}$$

$$\forall\ x, y, z \in A\ .\ x \bullet p\ y \ \wedge\ y \bullet p\ z$$
$$\Rightarrow\ p(x, x) = p(x, y) \ \wedge\ p(y, y) = p(y, z)$$
$$\text{but, } p(x, z) \ \leq\ p(x, y) + p(y, z) - p(y, y) \quad (\text{by P4})$$
$$\therefore\ p(x, z) \leq p(x, x)$$
$$\therefore\ p(x, z) = p(x, x) \quad (\text{by P2})$$
$$\therefore\ x \bullet p\ z \quad (\text{by definition of } \bullet p)$$

❑

For each flat pmetric p $\bullet p_{\perp}$ is the usual ordering on a flat domain, while for each Kahn pmetric p^S $\bullet p^S$ is the usual "initial segment" ordering on sequences. For a metric $d : A \times A \rightarrow \mathbb{R}$ $\bullet d$ is the equality relation. As we regard each metric space as a partial metric subspace of complete objects it is appropriate that this should be so as no totally defined object should be comparable with another distinct totally defined object.

The next theorem provides a warning of the dangers of working in T_0 spaces as not all sequences have unique limits, even if chains do.

Theorem 6

Suppose $X \in {}^{\omega}A$ converges to $\lambda \in A$ in a pmetric space with pmetric $p : A \times A \rightarrow \mathbb{R}$ and that $\lambda' \in A$ is such that $\lambda' \bullet p\ \lambda$. Then X converges to λ' as well.

Proof:

By Theorem 4 it is sufficent to show that ,

$$\exists\ \lim_{n \rightarrow \infty} p(X_n, \lambda') = p(\lambda', \lambda')$$

Suppose $\varepsilon > 0$, then as X converges to λ (by Theorem 4) we can choose $k \geq 0$ such that ,

$$\forall\ n > k\ .\ p(X_n, \lambda) - p(\lambda, \lambda) < \varepsilon$$
$$\therefore\ \forall\ n > k\ .\ p(X_n, \lambda') - p(\lambda', \lambda')$$
$$\leq p(X_n, \lambda) - p(\lambda, \lambda) + p(\lambda, \lambda') - p(\lambda', \lambda')$$
$$= p(X_n, \lambda) - p(\lambda, \lambda) \quad (\text{as } \lambda' \bullet p\ \lambda)$$
$$< \varepsilon$$

❑

In the above proof λ' is a "phoney" limit in the sense that it would not correspond to a chain limit if the sequence were a chain. The intention of the next definition is to overcome the problem of having sequences with more than one limit by introducing a restricted notion of convergence. This will ensure that the topological limit of a chain is also the least upper bound.

Definition 4
A sequence $X \in {}^{\omega}A$ in a pmetric space with pmetric $p : A \times A \to \mathbb{R}$ **Properly Converges** to $\lambda \in A$ if X converges to λ and,

$$\exists \lim_{n\to\infty} p(X_n, X_n) = p(\lambda, \lambda)$$

In other words $X \in {}^{\omega}A$ properly converges to $\lambda \in A$ if $\exists \lim_{n\to\infty} p(X_n, \lambda)$ and, $\exists \lim_{n\to\infty} p(X_n, X_n)$ and,

$$\lim_{n\to\infty} p(X_n, X_n) = \lim_{n\to\infty} p(X_n, \lambda) = p(\lambda, \lambda)$$

The next Theorem shows that proper convergence captures the limits we really want although notice, we have not used chains here to obtain unique limits in a T_0 space.

Theorem 7
Suppose $X \in {}^{\omega}A$ properly converges to both λ and λ' in a pmetric space with pmetric $p : A \times A \to \mathbb{R}$, then $\lambda' \bullet_p \lambda$.

Proof:

Suppose $X \in {}^{\omega}A$ properly converges to both λ and λ'.

Choose $\varepsilon > 0$, then we can choose $k \geq 0$ such that,

$$p(X_n, \lambda) - p(\lambda, \lambda) < \varepsilon/3$$
$$\wedge \quad p(X_n, \lambda') - p(\lambda', \lambda') < \varepsilon/3$$
$$\wedge \quad |\, p(X_n, X_n) - p(\lambda, \lambda)\,| < \varepsilon/3$$

$$\begin{aligned}
\therefore\ p(\lambda, \lambda') - p(\lambda', \lambda') \\
&\leq p(\lambda, X_n) - p(X_n, X_n) + p(X_n, \lambda') - p(\lambda', \lambda') \\
&\leq (p(\lambda, X_n) - p(\lambda, \lambda)) + (p(\lambda, \lambda) - p(X_n, X_n)) \\
&\qquad + (p(X_n, \lambda') - p(\lambda', \lambda')) \\
&< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \\
&= \varepsilon
\end{aligned}$$

Thus $p(\lambda', \lambda') = p(\lambda, \lambda')$ as ε was an arbitrary choice.

And so $\lambda' \bullet_p \lambda$

❑

The implication of the Theorem 7 is that limits to properly convergent sequences are unique. This is an interesting result for non-Hausdorff T_0 spaces.

Other standard metric constructions also generalise to pmetric spaces with both considerable & surprising ease.

Definition 5
A sequence $X \in {}^{\omega}A$ in a pmetric space with pmetric $p : A \times A \to \mathbb{R}$ is **Cauchy** if,

$$\forall\ \varepsilon > 0\ \ \exists\ k \geq 0\ \ \forall\ n, m > k\ .\ \ p(X_n, X_m) - p(X_m, X_m) < \varepsilon$$

Definition 6
A pmetric space is **Complete** if every Cauchy sequence properly converges.

Definition 7
A **Contraction** in a pmetric space with pmetric $p : A \times A \to \mathbb{R}$ is a function $f : A \to A$ such that,

$$\exists\ 0 \leq c < 1\ \ \forall\ x, y \in A\ .$$
$$p(f(y), f(x)) - p(f(x), f(x)) \leq c \times (p(y, x) - p(x, x))$$

Theorem 8
Each contraction in a complete partial metric space has a unique fixed point.

Proof:

Suppose $f : A \to A$ is a contraction in a complete partial metric space with pmetric $p : A \times A \to \mathbb{R}$, and that $0 \leq c < 1$ is such that,

$$\forall\ x, y \in A\ .\ p(f(y), f(x)) - p(f(x), f(x)) \leq c \times (p(y, x) - p(x, x))$$

Let $a \in A$, and let $X \in {}^{\omega}A$ be such that $\forall\ n \geq 0\ .\ X_n = f^n(a)$.

We will first show that X is a Cauchy sequence.

$$\forall\ n \geq 0\ .\ f(X_{n+2}, X_{n+1}) - f(X_{n+1}, X_{n+1}) \leq c \times (f(X_{n+1}, X_n) - f(X_n, X_n))$$

$$\therefore\ \forall\ n \geq 0\ .\ f(X_{n+2}, X_{n+1}) - f(X_{n+1}, X_{n+1}) \leq c^{n+1} \times (f(X_1, X_0) - f(X_0, X_0))$$

$$\therefore\ \forall\ n, k \geq 0\ .\ f(X_{n+k+1}, X_n) - f(X_n, X_n)$$
$$\leq f(X_{n+k+1}, X_{n+k}) - f(X_{n+k}, X_{n+k}) + f(X_{n+k}, X_n) - f(X_n, X_n)$$

$$\leq c^{n+k} \times (f(X_1, X_0) - f(X_0, X_0)) + f(X_{n+k}, X_n) - f(X_n, X_n)$$

$$\therefore\ \forall\ n, k \geq 0\ .\ f(X_{n+k+1}, X_n) - f(X_n, X_n)$$
$$\leq (c^{n+k} + \ldots + c^n) \times (f(X_1, X_0) - f(X_0, X_0))$$

$$= \frac{c^n \times (1 - c^{k+1})}{1 - c} \times (f(X_1, X_0) - f(X_0, X_0))$$

$$< (c^n / (1-c)) \times (f(X_1 , X_0) - f(X_0 , X_0))$$

Thus X is seen to be a Cauchy sequence.
Thus as our pmetric space is complete X properly converges to $\lambda \in A$ say.

We now show that λ is a fixed point of f.
Choose $\varepsilon > 0$, then as X properly converges to λ we can find $k \geq 0$ such that,

$$\forall n > k \ . \ p(\lambda , X_n) - p(X_n , X_n) < \varepsilon/(1+c)$$
$$\wedge \ p(X_n , \lambda) - p(\lambda , \lambda) < \varepsilon/(1+c)$$

Thus $\forall n > k \ . \ p(f(\lambda) , \lambda) - p(\lambda , \lambda)$

$$\leq p(f(\lambda) , X_{n+1}) - p(X_{n+1} , X_{n+1}) + p(X_{n+1} , \lambda) - p(\lambda , \lambda)$$

$$\leq c \times (p(\lambda , X_n) - p(X_n , X_n)) + p(X_{n+1} , \lambda) - p(\lambda , \lambda)$$

$$< c \times (\varepsilon/(1+c)) + \varepsilon/(1+c)$$

$$= \varepsilon$$

Thus, as ε is arbitrary, $p(f(\lambda) , \lambda) = p(\lambda , \lambda)$ (*)

Similarly, $\forall n > k \ . \ p(f(\lambda) , \lambda) - p(f(\lambda) , f(\lambda))$

$$\leq p(f(\lambda) , X_{n+1}) - p(X_{n+1} , X_{n+1}) + p(X_{n+1} , \lambda) - p(f(\lambda) , f(\lambda))$$

$$= (p(f(\lambda) , X_{n+1}) - p(f(\lambda) , f(\lambda))) + (p(X_{n+1} , \lambda) - p(X_{n+1} , X_{n+1}))$$

$$\leq c \times (p(\lambda , X_n) - p(\lambda , \lambda)) + \varepsilon/(1+c)$$

$$< c \times (\varepsilon/(1+c)) + \varepsilon/(1+c)$$

$$= \varepsilon$$

Thus, as ε is arbitrary, $p(f(\lambda) , \lambda) = p(f(\lambda) , f(\lambda))$
Thus from (*) and P1 $\lambda = f(\lambda)$, and so f has been shown to have a fixed point. It just reamins to show that λ is unique.

Suppose $\lambda' \in A$ and $\lambda' = f(\lambda')$, then,

$$\begin{aligned} p(\lambda, \lambda') - p(\lambda', \lambda') &= p(f(\lambda), f(\lambda')) - p(f(\lambda'), f(\lambda')) \\ &\leq c \times (p(\lambda, \lambda') - p(\lambda', \lambda')) \end{aligned}$$

$\therefore \quad p(\lambda, \lambda') - p(\lambda', \lambda') = 0 \quad$ as $0 \leq c < 1$

$\therefore \quad \lambda' \bullet_p \lambda$

Similarly we can show, $\lambda \bullet_p \lambda'$ and so $\lambda = \lambda'$

❑

Weighted Metrics

So far we have explained partial metrics in terms of a generalisation of the metric axioms M1 - M3 . However, there is another method for introducing partial metrics. This second approach sheds more light on the relationship between metrics and partial metrics. As has been clearly shown already partial metrics do allow discussion of Scott style partial objects in the spirit of metric spaces by introducing the idea that an object need not necessarily have zero distance from itself, i.e. $\forall x \in A \ . \ p(x, x) \geq 0$ instead of $\forall x \in A \ . \ d(x, x) = 0$. By concentrating on the idea that each object has a **weight** which in general is a non-negative real gives us an alternative way to define partial metrics. The result of this is the conclusion that the notion of **pmetric** is precisley the combination of the ideas of **metric** and **weight.**

Definition 8

A **Weighted Metric** over a set A is a pair $< d , |\,| >$ consisting of a metric $d : A \times A \rightarrow \mathbb{R}$ and a **Weight Function** $|\,| : A \rightarrow \mathbb{R}$ where,

$$\forall x, y \in A \ . \ d(x, y) \geq |x| - |y|$$

Theorem 9

Partial metrics and weighted metrics can be defined in terms of each other.

Proof :

Suppose $< d , |\,| >$ is a weighted metric over the set A .

Let $p : A \times A \rightarrow \mathbb{R}$ be the function such that,

$$\forall x, y \in A \ . \ p(x, y) = (d(x, y) + |x| + |y|) / 2$$

We will first show that p is a pmetric by proving P1 – P4 .

(P1$\Rightarrow$) Trivially $\forall x, y \in A \ . \ x = y \Rightarrow p(x, x) = p(x, y) = p(y, y)$

(P1⇐) $\forall\ x, y \in A\ .\ \ p(x,x) = p(x,y) = p(y,y)$

$$\Rightarrow \frac{d(x,x)+|x|+|x|}{2} = \frac{d(x,y)+|x|+|y|}{2} = \frac{d(y,y)+|y|+|y|}{2}$$

$$\Rightarrow\ 2 \times |x| = d(x,y) + |x| + |y| = 2 \times |y|$$

$$\Rightarrow\ d(x,y) = |x| - |y| = |y| - |x|$$

$$\Rightarrow\ d(x,y) = 0 \Rightarrow x = y \qquad (\text{by M1})$$

(P2) $\forall\ x, y \in A \quad p(x,y) = p(y,x)$ (by M2)

(P3) $\forall\ x, y \in A\ .\ p(x,x) = |x| \leq p(x,y)$ as $d(x,y) \geq |x| - |y|$

(P4) $\forall\ x, y, z \in A\ .\ d(x,z) \leq d(x,y) + d(y,z)$ (by M3)

$$\Rightarrow\ d(x,z)+|x|+|z| \leq (d(x,y)+|x|) + (d(y,z)+|z|)$$

$$\Rightarrow \frac{d(x,z)+|x|+|z|}{2} \leq \frac{d(x,y)+|x|+|y|}{2} + \frac{d(y,z)+|y|+|z|}{2} - |y|$$

$$\Rightarrow\ p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$$

Thus p has been shown to be a pmetric.

Suppose now that p is a pmetric. We will show that the pair $< d\ ,\ |\,| >$ defined by,

$$\forall\ x \in A\ .\ |x| ::= p(x,x)$$

$$\forall\ x, y \in A\ .\ d(x,y) := 2 \times p(x,y) - |x| - |y|$$

is a weighted metric by proving M1 - M3 .

(M1⇒) $\forall\ x, y \in A\ .\ x = y \Rightarrow d(x,y) = 0$ (by definition of $<d, |\,|>$)

(M1⇐) $\forall\ x, y \in A\ .\ d(x,y) = 0$

$$\Rightarrow\ 2 \times p(x,y) - |x| - |y| = 0$$

$$\Rightarrow\ (p(x,y) - p(x,x)) + (p(y,x) - p(y,y)) = 0 \quad (\text{by P3})$$

$$\Rightarrow\ p(x,x) = p(x,y) = p(y,y) \qquad (\text{by P2})$$

$$\Rightarrow\ x = y \qquad (\text{by P1})$$

(M2) $\forall\ x, y \in A\ .\ d(x,y) = d(y,x)$ (by P3)

(M3) $\forall\ x,y,z \in A\ .\ p(x,z) \leq p(x,y) + p(y,z) - p(y,y)$ (by P4)

$$\Rightarrow \frac{d(x,z)+|x|+|z|}{2} \leq \frac{d(x,y)+|x|+|y|}{2} + \frac{d(y,z)+|y|+|z|}{2} - |y|$$

$$\Rightarrow d(x,z) \leq d(x,y) + d(y,z)$$

❑

Using the one to one relationship between partial and weighted metrics used in the last proof we can define the equivalent of the pmetric ordering on weighted metrics by,

$$\forall\ x,y \in A\ .\ x \leq y \Leftrightarrow d(x,y) = |x| - |y|$$

Now we move on to the problems of how to build larger pmetric spaces from smaller pmetric spaces. For pmetrics to be of much use in denotational semantics we must have (at least) product, sum, and function space constructions to build useful spaces. The remainder of this paper demonstrates that such constructions do exist. First we need to apply to pmetrics a standard construction used for turning an unbounded metric into a bounded metric.

Definition 9
For each pmetric $p : A \times A \rightarrow \mathbb{R}$, $p^\wedge : A \times A \rightarrow [0,1)$ is the pmetric such that $\forall\ x,y \in A\ .\ p^\wedge(x,y) = p(x,y) / (1 + p(x,y))$

Using Theorem A3 to check P4 it can easily be verified that $p^\wedge$ is indeed a pmetric .

Theorem 10
For each pmetric p the topology induced by $p^\wedge$ is the same as p .

Proof :

Suppose $p : A \times A \rightarrow \mathbb{R}$ is a pmetric , then ,

$\forall\ x \in A\ \forall\ \varepsilon > 0\ .\ B_\varepsilon(x) = B^\wedge_{\varepsilon/(1+\varepsilon)}(x)$ and ,

$\forall\ x \in A\ \forall\ 0 < \varepsilon < 1\ .\ B^\wedge_\varepsilon(x) = B_{\varepsilon/(1-\varepsilon)}(x)$ and ,

$\forall\ x \in A\ \forall\ \varepsilon \geq 1\ .\ B^\wedge_\varepsilon(x) = \cup\{\ B_{p(x,x)+1}(y)\ |\ y \in B^\wedge_\varepsilon(x)\ \}$

❑

Also note that for each pmetric p, $\bullet p = \bullet(p^\wedge)$.

Definition 10

The **Countable Product** of the pmetrics $p_n : A_n \times A_n \rightarrow \mathbb{R}$ $(n \geq 0)$ is the function $p^\times : (\times_{n\geq 0} A_n)^2 \rightarrow \mathbb{R}$ where,

$$\forall\ x, y \in \times_{n\geq 0} A_n \ .\ p^\times(x,y) = \Sigma_{n\geq 0}\ (p_n)^\wedge(x_n, y_n) \times 2^{-n-1}$$

Theorem 10

The countable product of pmetrics is a pmetric.

Proof:

Suppose $p^\times : (\times_{n\geq 0} A_n)^2 \rightarrow \mathbb{R}$ is the countable product of the pmetrics

$$p_n : A_n \times A_n \rightarrow \mathbb{R}$$

We will show the countable product to be a pmetric by proving P1 - P4.

(P1$\Rightarrow$) trivial.

(P1$\Leftarrow$) $\forall\ x, y \in \times_{n\geq 0} A_n$.

$$
\begin{aligned}
& p^\times(x,x) = p^\times(x,y) = p^\times(y,y) \\
& \quad \Rightarrow \Sigma_{n\geq 0}\ (p_n)^\wedge(x_n, x_n) \times 2^{-n-1} \\
& \qquad = \Sigma_{n\geq 0}\ (p_n)^\wedge(x_n, y_n) \times 2^{-n-1} \\
& \qquad = \Sigma_{n\geq 0}\ (p_n)^\wedge(y_n, y_n) \times 2^{-n-1}
\end{aligned}
$$

$$
\begin{aligned}
& \Rightarrow \Sigma_{n\geq 0}\ (\ (p_n)^\wedge(x_n, y_n) - (p_n)^\wedge(x_n, x_n)\ ) \times 2^{-n-1} \\
& \ = \Sigma_{n\geq 0}\ (\ (p_n)^\wedge(x_n, y_n) - (p_n)^\wedge(y_n, y_n)\ ) \times 2^{-n-1} \\
& \ = 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \forall\ n \geq 0\ .\ (p_n)^\wedge(x_n, x_n) &= (p_n)^\wedge(x_n, y_n) \\
&= (p_n)^\wedge(y_n, y_n)
\end{aligned}
$$

$$\Rightarrow \quad \forall\ n \geq 0\ .\ x_n = y_n \qquad \Rightarrow \quad x = y$$

(P2) by P2 for each p_n.

(P3) by P3 for each p_n.

(P4) $\forall\ x, y, z \in \times_{n \geq 0} A_n$.

$$p^{\times}(x, z) \leq p^{\times}(x, y) + p^{\times}(y, z) - p^{\times}(y, y) \text{ as}$$
$$p^{\times}(x, z) = \Sigma_{n \geq 0}\ (p_n)^{\wedge}(x_n, z_n) \times 2^{-n-1} \text{ and as}$$
$$p^{\times}(x, y) + p^{\times}(y, z) - p^{\times}(y, y) =$$
$$\Sigma_{n \geq 0}\ ((p_n)^{\wedge}(x_n, y_n) + (p_n)^{\wedge}(y_n, z_n) - (p_n)^{\wedge}(y_n, y_n)) \times 2^{-n-1}$$

❑

The countable product has the "pointwise" ordering, i.e.

$$\forall\ x, y \in \times_{n \geq 0} A_n \ .\ x \bullet(p^{\wedge})\ y \Leftrightarrow \forall\ n \geq 0\ .\ x_n \bullet((p_n)^{\wedge})\ y_n$$

Definition 11
The **Disjoint Sum** of a family of pmetrics $p_i : A_i \times A_i \rightarrow \mathbb{R}\ (i \in I)$ is the pmetric $p^{+} : \cup_{i \in I} \{ <i, x> \mid x \in A_i \} \rightarrow [0, 1]$ where,

$$\forall\ <i,x>\ ,\ <j,y> \in \cup_{i \in I} \{ <i, x> \mid x \in A_i \} \ .$$
$$p^{+}(<i,x>\ ,\ <j,y>)$$
$$= (p_i)^{\wedge}(x, y) \quad \text{if } i = j$$
$$= 1 \quad \text{if } i \neq j$$

The topology and partial ordering for a disjoint sums are the expected ones.

We now come to the more involved problem of how to construct a pmetric function space. As with function space constructions of others we are forced to make certain assumptions on the type of functions allowed in such a function space.

Definition 12
For each pmetric $p : A \times A \rightarrow \mathbb{R}$ a set $A^* \subseteq A$ is **Properly Dense** in A if each member $x \in A$ is the limit of a sequence in A^* properly converging to x.

Definition 13
A set A with pmetric $p : A \times A \rightarrow \mathbb{R}$ is **Sufferable** if there there exists a countable properly dense set $A^* \subseteq A$ and function $p^* : A \rightarrow \mathbb{R} - \{ 0 \}$ such that for any $X, Y \in {}^{A^*}[0,2]$,

$$\Sigma_{a \in A^*}\ p^*(a) \times X_a = \Sigma_{a \in A^*}\ p^*(a) \times Y_a \quad \Leftrightarrow \quad X = Y$$

Sufferability is not an unreasonable assumption as any space of interest to a programming language designer is likely to have some kind of countable dense subset as the universe will probably be the

Erratum (RR212 Partial Metric Spaces)

Definition 13 is unnecessarily strong and should be replaced by,

Definition 13

A set A with pmetric $p : A \times A \to \mathbb{R}$ is **Sufferable** if,

$$\exists\ r \geq 0 \in \mathbb{R}\quad \forall\ x, y \in A \ .\quad p(x, y) \leq r$$

and there exists a countable properly dense set $A^* \subseteq A$ with function $p^* : A^* \to \mathbb{R} - \{0\}$ such that

$$\Sigma_{a \in A^*}\ p^*(a) \quad < \quad \infty$$

In Definition 14 read,

The set of all such properly continuous functions over sufferable A is denoted by $A \mapsto A'$.

closure of a recursively enumerable set. Although not proved here we can show that the countable product of sufferable spaces is sufferable.

Definition 14

A continuous function $f : A \rightarrow A'$ over pmetric spaces is **Properly Continuous** if for each sequence $X \in {}^{\omega}A$ properly converging to $x \in A$ the sequence $Y \in A'$ where,

$$\forall\ n \geq 0\ .\ Y_n = f(X_n)$$

properly converges to $f(x)$. The set of all such properly continuous functions is denoted by $A \mapsto A'$.

Definition 15

For pmetrics $p : A \times A \rightarrow \mathbb{R}$ and $p' : A' \times A' \rightarrow \mathbb{R}$, $d^{\rightarrow} : (A \mapsto A') \times (A \mapsto A') \rightarrow \mathbb{R}$ and $|\,| : (A \mapsto A') \rightarrow \mathbb{R}$ are the functions such that,

$$\forall\ f \in A \mapsto A'\ .\ |f| = \Sigma_{a \in A^*}\ p^*(a) \times |f(a)|'$$

$$\forall\ f, g \in A \mapsto A'\ .\ d^{\rightarrow}(f, g) = \Sigma_{a \in A^*}\ p^*(a) \times d'(f(a), g(a))$$

where $< d', |\,|' >$ is the weighted metric equivalent for $(p')^{\wedge}$ as constructed in the proof of Theorem 9.

Theorem 11

$< d^{\rightarrow}, |\,| >$ is a weighted metric.

Proof :

(M1⇒) $\forall\ f \in A \mapsto A'\ .\ d^{\rightarrow}(f, f)$

$$= \Sigma_{a \in A^*}\ p^*(a) \times d'(f(a), f(a))$$
$$= \Sigma_{a \in A^*}\ p^*(a) \times 0$$
$$= 0$$

(M2⇐) $\forall\ f, g \in A \mapsto A'\ .\ d^{\rightarrow}(f, g) = 0$

$\Rightarrow \forall\ a \in A^*\ .\ d'(f(a), g(a)) = 0$

$\Rightarrow \forall\ a \in A^*\ .\ f(a) = g(a)$

$\Rightarrow f|A^* = g|A^*$

$\Rightarrow f = g$ as f & g are continuous and A^* is dense in A.

(M2) $d^{\rightarrow}$ is symmetric as d' is symmetric.

(M3) $\forall\ f, g, h \in A \mapsto A' \quad . \quad d^{\rightarrow}(f, h)$

$$= \Sigma_{a \in A^*}\ p^*(a) \times d'(\ f(a)\ ,\ h(a)\ )$$
$$\leq \Sigma_{a \in A^*}\ p^*(a) \times (\ d'(\ f(a)\ ,\ g(a)\ ) + d'(\ g(a)\ ,\ h(a)\ )\ )$$
$$= \Sigma_{a \in A^*}\ p^*(a) \times d'(\ f(a)\ ,\ g(a))$$
$$+ \Sigma_{a \in A^*}\ p^*(a) \times d'(\ g(a)\ ,\ h(a)\ )$$
$$= d^{\rightarrow}(f, g) + d^{\rightarrow}(g, h)$$

Thus $d^{\rightarrow}$ is proven to be a metric. It just reamains to show that $|\,|$ is a weight.

$\forall\ f, g \in A \mapsto A' \quad . \quad |f| - |g|$

$$= \Sigma_{a \in A^*}\ p^*(a) \times (\ |f(a)|' - |g(a)|'\ )$$
$$\leq \Sigma_{a \in A^*}\ p^*(a) \times d'(\ f(a)\ ,\ g(a)\ ) \quad \text{as } < d'\ ,\ |\,|' > \text{ is a weighted metric}$$
$$= d^{\rightarrow}(f, g)$$

Thus $|\,|$ is a weight for $A \mapsto A'$.

Thus $< d^{\rightarrow}\ ,\ |\,| >$ is a weighted metric for $A \mapsto A'$.

❑

As in the proof of Theorem 9 we can construct a pmetric $p^{\rightarrow}$ for $A \mapsto A'$. An important result for a potential function space is the following.

Theorem 12

$\forall\ f, g \in A \mapsto A' \quad . \quad f \bullet p^{\rightarrow} g \quad \Leftrightarrow \quad \forall\ a \in A \quad . \quad f(a) \bullet p'\ g(a)$

Proof :

$\forall\ f, g \in A \mapsto A' \quad . \quad f \bullet p^{\rightarrow} g$

$$\Leftrightarrow d^{\rightarrow}(f, g) = |f| - |g|$$

$$\Leftrightarrow \Sigma_{a \in A^*}\ p^*(a) \times d'(\ f(a)\ ,\ g(a)\ ) = \Sigma_{a \in A^*}\ p^*(a) \times (\ |f(a)|' - |g(a)|'\ )$$

$$\Leftrightarrow \forall\ a \in A^* \quad . \quad d'(\ f(a)\ ,\ g(a)\ ) = |f(a)|' - |g(a)|' \quad \text{(by the definition of } p^*)$$

$$\Leftrightarrow \forall\ a \in A^* \quad . \quad f(a) \bullet p'\ g(a)$$

$$\Leftrightarrow \forall\ a \in A \quad . \quad f(a) \bullet p'\ g(a)$$ as A^* is properly dense in A and f & g are properly continuous and Theorem A4

❑

Conclusions

Pmetrics (= weighted metrics) allow the application of metric Hausdorff methods to the non Hausdorff T_0 topologies required for denotational semantics based upon partial orders. For Computer Scientists this approach promises a fresh approach to denotational semantics using well understood metric mathematics. For Mathematicians this approach suggests that the standard theory of metric spaces can be generalised to non-Hausdorff spaces without losing too many Hausdorff properties such as limits of sequences being unique. Perhaps more importantly there is a lesson to be learnt by both Computer Scientists & Mathematicians here. Too often the former have had to invent their own mathematics because the latter have not found computing problems mathematically interesting. The coincidence between the late David Park's work on bisimulation for process calculi and Peter Aczel's theory on non well founded sets was perhaps an earlier example of the same lesson. The challenge is to extend familiar mathemaical methods for reasoning about "total" well founded objects to include "partial" non well founded ones and so apply these methods for reasoning about programs.

The topology $\mathfrak{T}$ of a pmetric space with pmetric $p : A \times A \rightarrow \mathbb{R}$ always has the first of the two definitive properties,

$$\forall\, x, y \in A \;.\; x \bullet_p y \;\wedge\; x \in \mathfrak{T} \;\Rightarrow\; y \in \mathfrak{T}$$

which characterise a Scott topology. The second definitive property is that the least upper bound of a chain must be a topological limit of that chain. In the context of pmetric spaces this is equivalent to saying that all chains must be properly convergent. Thus if a **Scott pmetric** is defined to be one in which every chain is properly convergent and for which there exists a special element $\perp \in A$ such that

$$p(\perp, \perp) = \sup\{\, p(x, x) \mid x \in A \,\}$$

then the topology of a Scott pmetric is always a Scott topology. The conclusion from this is that pmetrics can be used to define Scott topologies, and so must be relevant t o denotational semantics. The open question is how many Scott topologies cannot be defined using pmetrics.

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Appendix

Definition A1

A **Topology** on a set A is a set $\mathfrak{T} \subseteq 2^A$ such that,

(T1) $\emptyset \in \mathfrak{T}$

(T2) $A \in \mathfrak{T}$

(T3) $\forall\ S \subseteq \mathfrak{T}\ .\ \ \bigcup S \in \mathfrak{T}$

(T4) $\forall\ S \subseteq \mathfrak{T}\ .\ \ |S| < \infty \ \Rightarrow\ \bigcap S \in \mathfrak{T}$

(Members of $\mathfrak{T}$ are called **open** sets)

Definition A2

A topology $\mathfrak{T}$ on a set A is $\mathbf{T_0}$ if,

$$\forall\ x \neq y \in A\ \ \exists\ O \in \mathfrak{T}\ .$$
$$(\ x \in O \wedge y \notin O\ )\ \vee\ (\ y \in O \wedge x \notin O\ )$$

Definition A3

A topology $\mathfrak{T}$ on A is $\mathbf{T_2}$ (i.e. **Hausdorff**) if,

$$\forall\ x \neq y \in A\ \ \exists\ O, O' \in \mathfrak{T}\ .$$
$$x \in O\ \wedge\ y \in O'\ \wedge\ O \cap O' = \emptyset$$

Definition A4

A **basis** for a set A is a set $\mathfrak{B} \subseteq 2^A$ such that,

(B1) $A = \bigcup \mathfrak{B}$

(B2) $\forall\ B_1 , B_2 \in \mathfrak{B}\ \ \exists\ \mathcal{B} \subseteq \mathfrak{B}\ .\ \ B_1 \cap B_2 = \bigcup \mathcal{B}$

Theorem A1
For each basis $\mathcal{B}$ for a non-empty set A, $\bigcup \mathcal{B}$ is a topology on A.

Definition A5
A **Metric** is a function $d : A \times A \rightarrow \mathbb{R}$ such that,

$$
\begin{array}{lll}
\text{(M1)} & \forall\, x, y \in A \; . & x = y \;\Leftrightarrow\; d(x,y) = 0 \\
\text{(M2)} & \forall\, x, y \in A \; . & d(x,y) = d(y,x) \\
\text{(M3)} & \forall\, x, y, z \in A \; . & d(x,z) \leq d(x,y) + d(y,z)
\end{array}
$$

Definition A6
An **Open Ball** for a metric $d : A \times A \rightarrow \mathbb{R}$ is a set of the form,

$$B_{\varepsilon}(x) \quad ::= \quad \{ \; y \in A \;\mid\; d(x,y) < \varepsilon \; \}$$

for any $x \in A$ and $\varepsilon > 0$.

Theorem A2
The open balls of a metric $d : A \times A \rightarrow \mathbb{R}$ with $\varnothing$ form a base for a T_2 topology on A.

Theorem A3

$$\forall\, a, b, c, d \geq 0 \; . \quad a \leq b + c - d \;\wedge\; d \leq b \;\wedge\; d \leq c$$

$$\Rightarrow \quad \frac{a}{1 + a} \leq \frac{b}{1 + b} + \frac{c}{1 + c} - \frac{d}{1 + d}$$

Theorem A4

Suppose $p : A \times A \rightarrow \mathbb{R}$ is a pmetric, and $X, Y \in {}^{\omega}A$, and $x, y \in A$ are such that X properly converges to x and Y properly converges to y, and,

$$\forall\, n \geq 0 \; . \; X_n \bullet p \; Y_n .$$

Then $x \bullet p \; y$.