The Topology of Partial Metric Spaces

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The $T_0$ world of Scott's topological models used in the denotational semantics of programming languages may at first sight appear to have nothing whatever in common with the Hausdorff world of metric space theory. Can this be true though when the notion of "distance" is so important in the application of inductive proof theory to recursive definitions? This paper shows that existing work on the application of quasi metrics to denotational semantics can be taken much further than just describing Scott topologies. Using our "partial metric" we introduce a new approach by constructing each semantic domain as an Alexandrov topology "sandwiched" between two metric topologies.

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The Topology of Partial Metric Spaces

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1. Introduction

In the study of the denotational semantics of programming languages topological models are created for programming languages represented by systems of logic. More often than not this means a $T_0$ model for the lambda calculus in the spirit of Scott. However, the inherent assumption in this approach that all suitable models must be $T_0$ appears to remove any possibility that the theory of metric spaces (which are all $T_2$) can be applied in any way to domain theory in Computer Science. Rare exceptions to this rule are the use of quasi metrics in [Sm87] and metrics in [La87]. If metrics are to be used at all then the more conventional wisdom in Computer Science would dismiss Scott's $T_0$ approach in favour of a purely metric approach [dB&Z82]. Unfortunately the latter $T_2$ approach pays the price of losing any notion of partial order, a concept of fundamental importance in any Tarskian approach to fixed point semantics in denotational semantics. The distinct advantage of using quasi metrics is that such generalised metrics can be used to define $T_0$ topologies with partial orders, and so allowing Tarskian semantics. Quasi metrics are not without their problems though. Being non-symmetric a quasi metric is arguably an "unnatural" notion of distance. Perhaps a more important criticism is that they shed little light on how to develop proof theory for programs based upon metric reasoning. In [Sm87] the phrase Reconciling Domains with Metric Spaces is used to suggest that conventional wisdom from the theory of metric spaces can be applied to programming language semantics.

[Ko88] contains a proof that All Topologies come from Generalised Metrics. This may or may not be of interest to topologists in general as many of the more pleasant $T_2$ properties usually associated with metric spaces may be lost in a process of generalisation. However, the point that metrics can, if only in principle, be generalised to explore non- $T_2$ topologies is established. Combining this with Smyth's work on quasi metrics we are naturally lead to the following question.

To what extent can the $T_2$ topological methods of metric space theory be applied to programming language semantics?
This report aims to demonstrate that a positive answer to this question can be found by showing that the existing work on quasi metrics can be taken much further than has previously been suggested. Not only can a generalised metric such as a quasi metric be used to describe Scott $T_0$ topologies but we can also use metrics as well. We introduce here the \textit{weighted metric}, a version of a metric algebraically equivalent to a restricted form of quasi metric.

\textit{Defined using partial metrics} we propose here a method of constructing domains as Alexandrov sub-topologies of weighted metric topologies.

The purpose of this work is to bring the theory of metric spaces ever closer to Scott's theory of domains, and so make the reconciliation of domains with metric spaces a reality.

2. Background Definitions & Results

\textbf{Definition 2.1}

A \textit{Topology} over a set $U$ is a set $\mathcal{T} \subseteq 2^U$ such that,

\begin{align*}
(T1) \quad & \emptyset \in \mathcal{T} \\
(T2) \quad & U \in \mathcal{T} \\
(T3) \quad & \forall S \subseteq \mathcal{T} . \ U S \in \mathcal{T} \\
(T4) \quad & \forall O , O' \in \mathcal{T} . \ O \cap O' \in \mathcal{T}
\end{align*}

Members of $\mathcal{T}$ are called \textit{open sets} and their complements \textit{closed sets}.

\textbf{Definition 2.2}

The $T_0$, $T_1$, and $T_2$ (i.e. Hausdorff) \textit{Separation Axioms} for a topology $\mathcal{T}$ are,

\begin{align*}
T_0 & = \forall x \neq x' \in U . \ \exists O \in \mathcal{T} . \ (x \in O \land x' \notin O) \\
& \lor (x' \in O \land x \notin O) \\
T_1 & = \forall x \neq x' \in U . \ (\exists O , O' \in \mathcal{T} . \ x \in O - O' \\
& \land x' \in O' - O) \\
T_2 & = \forall x \neq x' \in U . \ (\exists O , O' \in \mathcal{T} . \ x \in O \land x' \in O' \\
& \land O \cap O' = \emptyset)
\end{align*}

If $\mathcal{T}$ is $T_2$ then $\mathcal{T}$ is $T_1$, and if $\mathcal{T}$ is $T_1$ then $\mathcal{T}$ is $T_0$. 

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Definition 2.3
A Basis for a topology \( \mathcal{T} \) is a set \( \mathcal{B} \subseteq \mathcal{T} \) such that,
\[
\mathcal{T} = \{ US \mid S \subseteq \mathcal{B} \}
\]

Lemma 2.1
\( \mathcal{B} \subseteq 2^U \) is a basis for a topology \( \mathcal{T} \) over \( U \) if,
\[
U = U \mathcal{B} \text{ and } \forall O, O' \in \mathcal{B} \exists S \subseteq \mathcal{B} . \ O \cap O' = US
\]

Lemma 2.2
\( \mathcal{B}, \mathcal{B}' \subseteq 2^U \) are each a basis for the same topology on \( U \) if,
\[
\forall O \in \mathcal{B} \exists S' \subseteq \mathcal{B}' . \ O = US' \text{ and }, \ \forall O' \in \mathcal{B}' \exists S \subseteq \mathcal{B} . \ O' = US
\]

Definition 2.4
A Partial Order (poset) is a pair \( \langle U, \ll \rangle \subseteq U \times U \rangle \) such that,

\begin{align*}
(PO1) & \quad \forall x \in U . \ x \ll x \\
(PO2) & \quad \forall x, y \in U . \ x \ll y \land y \ll x \Rightarrow x = y \\
(PO3) & \quad \forall x, y, z \in U . \ x \ll y \land y \ll z \Rightarrow x \ll z
\end{align*}

Definition 2.5
An Alexandrov Topology over a poset \( \langle U, \ll \rangle \) is a topology \( \mathcal{T} \) over \( U \) such that,
\[
\forall O \in \mathcal{T} \forall x, y \in U . \ x \in O \land x \ll y \Rightarrow y \in O
\]
The topology, \( \mathcal{T} = \{ \{ n , n+1 , \ldots , \infty \} \mid n \in \omega \} \) over \( \langle \omega \cup \{ \infty \}, \leq \rangle \) is an example of an Alexandrov topology. The Full Alexandrov Topology \( \mathcal{T}[\ll] \) over a poset \( \langle U, \ll \rangle \) is the topology
\[
\{ O \subseteq U \mid \forall x \in O , y \in U . \ x \ll y \Rightarrow y \in O \}
\]
of all upward closures [Vi89]. The topology
\[
\mathcal{T}[\leq] := \{ \{ n , n+1 , \ldots , \infty \} \mid n \in \omega \cup \{ \infty \} \}
\]
is the Full Alexandrov Topology over \( \langle \omega \cup \{ \infty \}, \leq \rangle \). The Full Alexandrov Topology is always \( T_0 \), however, not every Alexandrov Topology is \( T_0 \) as the indiscrete topology.
$T = \{ \emptyset, \{0,1\} \}$ shows.

**Definition 2.6**

A *Scott Topology* is an Alexandrov Topology $T$ over a chain complete poset (cpo) $\langle U, \ll \rangle$ such that for each chain $X \in \omega U$,

$$\forall O \in T . \ \lub X \in O \Rightarrow \exists k \geq 0 . \ \forall n > k . \ X_n \in O$$

The only $T_2$ Scott topologies are the "flat" trivial ones in which,

$$\forall x, y \in U . \ \ x \ll y \iff x = y$$

The least element $\bot$ (bottom) is not included here in our definition of a Scott topology as it is not essential to the development of the work presented in this report. Later work using our methods for fixed point semantics and reflexive domains will need to include $\bot$.

**Definition 2.7**

A *Metric* [Su75] is a function $d : U \times U \to \mathbb{R}$ such that,

(M1) $\forall x, y \in U . \ \ x = y \iff d(x, y) = 0$

(M2) $\forall x, y \in U . \ \ d(x, y) = d(y, x)$

(M3) $\forall x, y, z \in U . \ \ d(x, z) \leq d(x, y) + d(y, z)$

**Definition 2.8**

An *Open Ball* for a metric $d : U \times U \to \mathbb{R}$ is a set of the form,

$$B^d_\epsilon(x) := \{ y \in U \mid d(x, y) < \epsilon \}$$

for any $x \in U$ and $\epsilon > 0$.

**Lemma 2.3**

The set of all open balls of a metric $d : U \times U \to \mathbb{R}$ is a basis for a $T_2$ topology $T[d]$ on $U$.

**Definition 2.9**

A *Quasi Metric* is a function $q : U \times U \to \mathbb{R}$ such that,

(Q1) $\forall x, y \in U . \ \ x = y \iff q(x, y) = q(y, x) = 0$

(Q2) $\forall x, y, z \in U . \ \ q(x, z) \leq q(x, y) + q(y, z)$

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Lemma 2.4
For each quasi metric \( q : U \times U \to \mathbb{R} \) the relation \( \preceq_q \subseteq U \times U \) defined by,
\[
\forall x, y \in U . \quad x \preceq_q y \iff q(x, y) = 0
\]
is a partial ordering.

Lemma 2.5
For each quasi metric \( q : U \times U \to \mathbb{R} \) the set of all open balls of the form,
\[
B^q_\varepsilon(x) := \{ y \in U \mid q(x, y) < \varepsilon \}
\]
for any \( x \in U \) and \( \varepsilon > 0 \) is the basis for an Alexandrov \( T_0 \) topology \( \mathcal{T}\{q\} \) over \( \langle U, \preceq_q \rangle \).

Example 2.1
For each complete metric \( d : U \times U \to \mathbb{R} \) and for each collection of closed sets \( U^c \subseteq 2^U \) the function \( q : U^c \times U^c \to \mathbb{R} \) where,
\[
\forall X, Y \in 2^U . \quad q(X, Y) := \sup \{ \inf \{ d(x,y) / (1 + d(x,y)) \} \mid y \in Y \} \mid x \in X \}
\]
is a quasi metric such that, \( \forall X, Y \in U^c . \quad X \subseteq Y \iff X \preceq_q Y \)

Lemma 2.6
For each quasi metric \( q : U \times U \to \mathbb{R} \) the function \( q^m : U \times U \to \mathbb{R} \) where,
\[
\forall x, y \in U . \quad q^m(x, y) := q(x, y) + q(y, x)
\]
is a metric such that \( \mathcal{T}\{q\} \subseteq \mathcal{T}\{q^m\} \).

Lemma 2.7 (Quasi Metric Contraction Mapping Theorem)
For each quasi metric \( q : U \times U \to \mathbb{R} \) such that \( q^m \) is complete, and for each function \( f : U \to U \) such that,
\[
\exists 0 \leq c < 1 . \quad \forall x, y \in U . \quad q(f(x), f(y)) \leq c \times q(x, y)
\]
there exists a unique \( a \in U \) such that \( a = f(a) \).

Note that the unique fixed point of a quasi metric contraction mapping need not be maximal as the following simple example shows.
Thus this theorem cannot be used to prove that recursive definitions define total objects. In Section 5 we present a second theorem specifically designed for this purpose.

3. Partial Metrics

Definition 3.1

A Partial Metric (pmetric) is a function \( p : U \times U \rightarrow \mathbb{R} \) such that,

(P1) \( \forall x, y \in U . \ x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y) \)
(P2) \( \forall x, y \in U . \ p(x, x) \leq p(x, y) \)
(P3) \( \forall x, y \in U . \ p(x, y) = p(y, x) \)
(P4) \( \forall x, y, z \in U . \ p(x, z) \leq p(x, y) + p(y, z) - p(y, y) \)

The pmetric axioms P1 thru P4 are intended to be a "minimal" generalisation of the metric axioms M1 thru M3 such that objects do not necessarily have to have zero distance from themselves. In this generalisation we manage to preserve the symmetry axiom M2 to get P3, but have to "massage" the transitivity axiom M3 to produce the generalisation P4. Consequently a metric is precisely a pmetric \( p : U \times U \rightarrow \mathbb{R} \) in which,

\( \forall x \in U . \ p(x, x) = 0 \)

For each pmetric \( p : U \times U \rightarrow \mathbb{R} \) we preserve "half" of the metric reflexive axiom M1 giving,

\( \forall x, y \in U . \ p(x, y) = 0 \Rightarrow x = y \)

Example 3.1

The poset \( < \mathbb{R} - \{0\}, \leq, > \) of positive real numbers can be defined by the pmetric \( p : (\mathbb{R} - \{0\})^2 \rightarrow \mathbb{R} \) where, \( \forall x, y \in U . \ p(x, y) := \max \{1/x, 1/y\} \)
as \( \forall x, y \in U . \ x \leq y \Leftrightarrow p(x, x) = p(x, y) \).

Example 3.2

The well known Baire Metric \( d^B : \omega S \times \omega S \rightarrow \mathbb{R} \) of \( \omega \) - sequences over a set \( S \) is defined by,
$\forall x \neq y \in \omega S \; . \; d^B(x, y) = 2^{\min\{ n: \omega \mid x(n) \neq y(n) \}}$

This definition can be extended to the set $(\bigcup \{ \{0, \ldots, n-1\}S \mid n \in \omega \}) \cup \omega S$ of all finite and $\omega$-sequences over $S$ by the Baire $p$-metric,

$p^B : ( (\bigcup \{ \{0, \ldots, n-1\}S \mid n \in \omega \}) \cup \omega S )^2 \rightarrow \mathbb{R}$

where, $\forall n \in \omega$, $x \in \{0, \ldots, n-1\}S$. $p^B (x, x) := 2^{-n}$. In this example we can use the condition $p(x, x) = 0$ to distinguish between those $x$ which are in the Baire Space and those which are not.

Example 3.3

A Flat $p$-metric is a metric $p_{\bot} : S_{\bot} \times S_{\bot} \rightarrow [0, 1]$ where, $S_{\bot} := S \cup \{\bot\}$ for a set $S$, and $\bot \notin S$, and,

$\forall x, y \in S_{\bot} . \; p_{\bot} (x, y) = 0 \iff x = y \in S$

Here an object $x \in S_{\bot}$ is totally defined if and only if $p_{\bot} (x, x) = 0$.

Definition 3.2

The Open Balls for a metric $p : U \times U \rightarrow \mathbb{R}$ are the sets of the form,

$B^p_\varepsilon (x) := \{ y \in A \mid p(x, y) < \varepsilon \}$

for each $\varepsilon > 0$ and $x \in A$.

Theorem 3.1

The set of all open balls of a metric $p : U \times U \rightarrow \mathbb{R}$ is the basis of a topology $\mathcal{T}[p]$ on $U$.

Proof:

Proof using Lemma 2.1

Suppose $p : U \times U \rightarrow \mathbb{R}$ is a metric.

Then, $A = \bigcup_{x \in A} B^p_{p(x,x)+1} (x)$ and,

for any balls $B^p_\varepsilon (x)$ and $B^p_\delta (y)$,

$B^p_\varepsilon (x) \cap B^p_\delta (y) = U \{ B^p_{\eta} (z) \mid z \in B^p_\varepsilon (x) \cap B^p_\delta (y) \}$

where, $\eta := p(z, z) + \min \{ \varepsilon - p(x, z), \delta - p(y, z) \}$
Theorem 3.2
For each metric \( p : U \times U \to \mathbb{R} \), open ball \( B_p(\epsilon)(a) \), and \( x \in U \),
\[
x \in B_p(\epsilon)(a) \quad \Rightarrow \quad \exists \ \delta > 0 \ . \ x \in B_p(\delta)(x) \subseteq B_p(\epsilon)(a)
\]
Proof:
Suppose \( p : U \times U \to \mathbb{R} \) is a metric.
Suppose \( x \in B_p(\epsilon)(a) \)
Then \( p(x, a) < \epsilon \)
Let \( \delta := \epsilon - p(x, a) + p(x, x) \)
Then \( \delta > 0 \) as \( \epsilon > p(x, a) \)
Also, \( p(x, x) < \delta \) as \( \epsilon > p(x, a) \)
Thus \( x \in B_p(\delta)(x) \)

Suppose now that \( y \in B_p(\epsilon)(x) \)
\[
\Rightarrow \quad p(y, x) < \delta \\
\Rightarrow \quad p(y, x) < \epsilon - p(x, a) + p(x, x) \\
\Rightarrow \quad p(y, x) + p(x, a) - p(x, x) < \epsilon \\
\Rightarrow \quad p(y, a) < \epsilon \quad \text{(by P4)} \\
\Rightarrow \quad y \in B_p(\epsilon)(a)
\]
Thus \( B_p(\delta)(x) \subseteq B_p(\epsilon)(a) \).

\( \square \)

Theorem 3.3
Metric topologies are \( T_0 \).

Proof:
Suppose \( p : U \times U \to \mathbb{R} \) is a metric.
Suppose \( x \neq y \in U \).

Then from P1 & P2 \( p(x, x) < p(x, y) \) or \( p(y, y) < p(x, y) \).

Wlog suppose \( p(x, x) < p(x, y) \) then,
\[
x \in B_p(\epsilon)(x) \land y \notin B_p(\epsilon)(x) \text{ where, } \epsilon := (p(x, x) + p(x, y)) / 2
\]

\( \square \)

Definition 3.3
For each metric \( p : U \times U \to \mathbb{R} \), \( \ll_p \subseteq U \times U \) is the binary relation defined by,
\[
\forall \ x, y \in A \ . \ x \ll_p y \quad \Leftrightarrow \quad p(x, x) = p(x, y)
\]
Theorem 3.4
For each metric \( p : U \times U \to \mathbb{R} \), \( \ll_p \) is a partial order.

Proof:
Suppose \( p : U \times U \to \mathbb{R} \) is a metric.
We prove PO1 thru PO3

(PO1) \[ \forall x \in U . \quad x \ll_p x \text{ as } p(x, x) = p(x, x) \]

(PO2) \[ \forall x, y \in U . \quad x \ll_p y \land y \ll_p x \]
\[ \Rightarrow p(x, x) = p(x, y) = p(y, y) \quad \text{(by P3)} \]
\[ \Rightarrow x = y \quad \text{(by P1)} \]

(PO3) \[ \forall x, y, z \in A . \quad x \ll_p y \land y \ll_p z \]
\[ \Rightarrow p(x, x) = p(x, y) \land p(y, y) = p(y, z) \]
but, \( p(x, z) \leq p(x, y) + p(y, z) - p(y, y) \) \quad \text{(by P4)}
\[ \therefore p(x, z) \leq p(x, x) \]
\[ \therefore p(x, z) = p(x, x) \quad \text{(by P2)} \]
\[ \therefore x \ll_p z \quad \text{(by definition of } \ll_p \text{)} \]

\[ \square \]

Theorem 3.5
Every metric topology is an Alexandrov Topology.

Proof:
Suppose \( p : U \times U \to \mathbb{R} \) is a metric,
then we have to show that \( \mathcal{T} [p] \subseteq \mathcal{T} [\ll_p] \).

It is sufficient to show that,
\[ \forall x \in U , \quad \epsilon > 0 . \quad B^p_{\epsilon}(x) = \bigcup \{ \{ z / y \ll_p z \} / y \in B^p_{\epsilon}(x) \} \]

Suppose \( x, y, z \in U \) and \( \epsilon > 0 \) are such that \( y \in B^p_{\epsilon}(x) \) and \( y \ll_p z \).

Then, \( p(x, z) \leq p(x, y) + p(y, z) - p(y, y) \) \quad \text{(by P4)}
\[ = p(x, y) \quad \text{as } y \ll_p z \]
\[ < \epsilon \quad \text{as } y \in B^p_{\epsilon}(x) \]

Thus, \( z \in B^p_{\epsilon}(x) \).

\[ \square \]
Theorem 3.6
For each p metric \( p : U \times U \to \mathbb{R} \), \( \mathcal{T}[p] = \mathcal{T}[\ll p] \) if and only if,

\[
\forall x \in U \quad \exists \varepsilon > 0 \quad B^p_\varepsilon(x) = \{ y \mid x \ll_p y \}
\]

Proof:
Suppose first that, \( \forall x \in U \quad \exists \varepsilon > 0 \quad B^p_\varepsilon(x) = \{ y \mid x \ll_p y \} \)
Then, \( \forall O \in \mathcal{T}[\ll_p] \quad O = \bigcup_{x \in O} \{ y \mid x \ll_p y \} \)
\( = \bigcup_{x \in O} B^p_\varepsilon(x) \)
\( \in \mathcal{T}[p] \)

\( \therefore \quad \mathcal{T}[\ll_p] \subseteq \mathcal{T}[p] \)
\( \therefore \quad \mathcal{T}[p] = \mathcal{T}[\ll_p] \)
(by Theorem 3.5)

Suppose now that, \( \mathcal{T}[p] = \mathcal{T}[\ll_p] \)
Then, \( \forall x \in U \quad \{ y \mid x \ll_p y \} \in \mathcal{T}[p] \)
Thus, by Theorem 3.2,
\[
\forall x \in U \quad \exists \varepsilon > 0 \quad x \in B^p_\varepsilon(x) \subseteq \{ y \mid x \ll_p y \}
\]
But, if \( x \in B^p_\varepsilon(x) \) then \( \{ y \mid x \ll_p y \} \subseteq B^p_\varepsilon(x) \)
Thus, \( \forall x \in U \quad \exists \varepsilon > 0 \quad B^p_\varepsilon(x) = \{ y \mid x \ll_p y \} \)

\( \square \)

4. Partial & Quasi Metrics

Theorem 4.1
For each p metric \( p : U \times U \to \mathbb{R} \) the function \( q : U \times U \to \mathbb{R} \) where,

\[
\forall x, y \in U \quad q(x, y) := p(x, y) - p(x, x)
\]
is a quasi metric such that \( \mathcal{T}[p] = \mathcal{T}[q] \) and \( \ll_p = \ll_q \).

Proof:
Suppose \( p : U \times U \to \mathbb{R} \) is a metric.

Let \( q : U \times U \to \mathbb{R} \) be such that,

\[
\forall x, y \in U \quad q(x, y) = p(x, y) - p(x, x)
\]

We show first that \( q \) is a quasi metric by proving Q1 and Q2.
\((Q1 \Rightarrow)\) \quad \forall x, y \in U \quad x = y \Rightarrow q(x, y) = 0 \quad \text{(by definition of } q)\\

\((Q1 \Leftarrow)\) \quad \forall x, y \in U \quad q(x, y) = q(y, x) = 0 \\
\Rightarrow p(x, y) - p(x, x) = p(y, x) - p(y, y) = 0 \\
\Rightarrow p(x, x) = p(x, y) = p(y, y) \quad \text{(by P3)} \\
\Rightarrow x = y \quad \text{(by P1)} \\

\((Q2)\) \quad \forall x, y, z \in U \quad p(x, z) \leq p(x, y) + p(y, z) - p(y, y) \\
\Rightarrow q(x, z) \leq q(x, y) + q(y, z) \\

Thus \( q \) is a quasi metric.

Now, \( \mathcal{I}[p] = \mathcal{I}[q] \) as,

\( \forall x \in U, \quad \varepsilon > p(x, x) \Rightarrow B_{p, \varepsilon}(x) = B_{q, -p(x,x)}(x) \) \\

\( \forall x \in U, \quad 0 < \varepsilon \leq p(x, x) \Rightarrow B_{p, \varepsilon}(x) = \emptyset \) \\

\( \forall x \in U, \quad \varepsilon > 0 \Rightarrow B_{q, \varepsilon}(x) = B_{p, \varepsilon + p(x,x)}(x) \) \\

and by Lemma 2.2.

Finally, \( \ll_p = \ll_q \) as,

\( \forall x, y \in U \quad p(x, x) = p(x, y) \Leftrightarrow q(x, y) = 0 \)

\( \square \)

**Definition 4.1**

A **Weighted Quasi Metric** over a set \( U \) is a pair \( < q, \ll > \) consisting of a quasi metric \( q : U \times U \to \mathbb{R} \) and a **Weight Function** \( \ll : U \to \mathbb{R} \) where,

\[(WQ)\quad \forall x, y \in U \quad q(x, y) + |x| = q(y, x) + |y|\]

A quasi metric \( q \) is **Weightable** if there exists a weight function \( \ll \) such that \( < q, \ll > \) is a weighted quasi metric.

**Theorem 4.2**

For each weighted quasi metric \( < q, \ll > \) over a set \( U \) the function \( p : U \times U \to \mathbb{R} \) where,

\( \forall x, y \in U \quad p(x, y) := q(x, y) + |x| \)

is a pmetric such that \( \mathcal{I}[p] = \mathcal{I}[q] \) and \( \ll_p = \ll_q \).
Proof:
Suppose \(< q, l | \) is a weighted quasi metric over a set \( U \).

Let \( p : U \times U \to \mathbb{R} \) be such that,
\[
\forall x, y \in U . \quad p(x, y) := q(x, y) + |x|
\]

We show first that \( p \) is a pmetric by proving P1 thru P4.

(P1 \( \Rightarrow \)) Trivial

(P1 \( \Leftarrow \))
\[
\forall x, y \in U . \quad p(x, x) = p(x, y) = p(y, y) \\
\Rightarrow |x| = q(x, y) + |x| = |y| \quad \text{(by Q1)} \\
\Rightarrow |x| = q(x, y) + |x| = q(y, x) + |y| \quad \text{(by WQ)} \\
\Rightarrow q(x, y) = q(y, x) = 0 \\
\Rightarrow x = y \quad \text{(by Q1)}
\]

(P2)
\[
\forall x, y \in U . \quad 0 \leq q(x, y) \\
\therefore \forall x, y \in U . \quad |x| \leq q(x, y) + |x| \\
\therefore \forall x, y \in U . \quad p(x, x) \leq p(x, y) \quad \text{(by Q1)}
\]

(P3)
\[
\forall x, y \in U . \quad q(x, y) + |x| = q(y, x) + |y| \quad \text{(by WQ)} \\
\therefore \forall x, y \in U . \quad p(x, y) = p(y, x)
\]

(P4)
\[
\forall x, y, z \in U . \quad q(x, z) \leq q(x, y) + q(y, z) \quad \text{(by Q2)} \\
\therefore \forall x, y, z \in U . \quad q(x, z) + |x| \\
\leq (q(x, y) + |x|) + (q(y, z) + |y|) - |y| \\
\therefore \forall x, y, z \in U . \quad p(x, z) \\
\leq p(x, y) + p(y, z) - p(y, y) \quad \text{(by Q1)}
\]

Thus \( p \) is a pmetric.

Now, \( \mathcal{T}[p] = \mathcal{T}[q] \) as by Lemma 2.2,
\[
\forall x \in U , \; \epsilon > |x| \quad B^p_\epsilon(x) = B^q_\epsilon - |x| (x) \\
\forall x \in U , \; 0 < \epsilon \leq |x| \quad B^p_\epsilon(x) = \emptyset \\
\forall x \in U , \; \epsilon > 0 \quad B^{q}_\epsilon(x) = B^{p}_\epsilon + |x| (x)
\]

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Finally, \( \ll_p = \ll_q \) as,
\[
\forall x, y \in U . \quad p(x, x) = p(x, y) \iff q(x, y) = 0
\]

Note that Theorems 4.1 & 4.2 establish an algebraic equivalence between the partial metric and a restricted form of the quasi metric.

**Theorem 4.3**

Not every quasi metric is weightable

**Proof:**

Let \( q : \{a, b, c\}^2 \to \{0, 1, 2, 3\} \) be the unique quasi metric such that,

\[
\begin{align*}
q(a, b) &= 0 & q(b, a) &= 2 \\
q(a, c) &= 1 & q(c, a) &= 1 \\
q(b, c) &= 3 & q(c, b) &= 0
\end{align*}
\]

Suppose that there exists a weight function \( \ll : \{a, b, c\} \to \{0, 1, 2, 3\} \) for \( q \), then,

\[
\begin{align*}
|b| + q(b, c) &= (|b| + q(b, a)) + 1 \\
&= (|a| + q(a, b)) + 1 & (\text{by WQ}) \\
&= |a| + q(a, c) \\
&= |c| + q(c, a) & (\text{by WQ}) \\
&= (|c| + q(c, b)) + 1 \\
&= (|b| + q(b, c)) + 1 & (\text{by WQ})
\end{align*}
\]

But this is impossible.

Unfortunately Theorem 4.3 does not answer the question of whether or not every quasi metric topology can be defined using a partial metric. Theorem 4.3 only shows that the method of defining a p-metric for a quasi metric topology using a weight function will not always work. The next result does answer the question for finite quasi metric topologies.

**Theorem 4.4**

For each quasi metric \( q : U \times U \to R \) over a finite set \( U \) there exists a p-metric \( p : U \times U \to R \) such that \( \tau[p] = \tau[q] \) and \( \ll_p = \ll_q \).
Proof Outline:

Let \( P := \{ U, \ll_q \} \) be the partial order induced by a quasi metric \( q : U \times U \rightarrow \mathbb{R} \) over a finite set \( U \).

Let \( P^* \subseteq 2^U \) be the set of all chains in \( P \) (i.e. all non-empty totally ordered subsets of \( U \)).

Let \( \$ : U \rightarrow \omega \) be the function where,

\[ \forall x \in U. \quad \$ (x) := 2 \max \{ \lvert c \rvert \mid c \in P^* \land \text{lub}_c \ll_q x \} \]

Then it can be shown that the function \( p : U \times U \rightarrow \omega \) where,

\[ \forall x, y \in U. \quad p(x, y) := 2 \lvert U \rvert - 1 - \max \{ \sum_{c \in \$ (x)} \lvert c \rvert \mid c \in P^* \land \text{lub}_c \ll_q x \land \text{lub}_c \ll_q y \} \]

is a pmetric such that \( \mathcal{T}[p] = \mathcal{T}[q] \) and \( \ll_p = \ll_q \).

An important implication of Theorem 4.4 is that any finite partial order can be defined by a partial metric.

5. Metrics & Partial Metrics

Definition 5.1

A Weighted Metric over a set \( U \) is a pair \( \langle d, \| \| \rangle \) consisting of a metric \( d : U \times U \rightarrow \mathbb{R} \) and a Weight Function \( \| : \rightarrow \mathbb{R} \) where,

\[ \text{(WM)} \quad \forall x, y \in U . \quad d(x, y) \geq \lvert x \rvert - \lvert y \rvert \]

A metric \( d \) is Weightable if there exists a weight function \( \| \) such that \( \langle d, \| \rangle \) is a weighted metric.

The next two results show the algebraic equivalence between the partial metric and the weighted metric.

Theorem 5.1

For each pmetric \( p : U \times U \rightarrow \mathbb{R} \) the pair \( \langle p^m : U \times U \rightarrow \mathbb{R} , \| : U \rightarrow \mathbb{R} \rangle \) where,
\( \forall x, y \in U \ . \ p^m(x, y) := 2 \times p(x, y) - p(x, x) - p(y, y) \)

\( \forall x \in U \ . \ |x| := p(x, x) \)

is a weighted metric such that,

\[ \mathcal{T}[p] \subseteq \mathcal{T}[p^m] \] , and,

\( \forall x, y \in U \ . \ p(x, y) := \frac{(p^m(x, y) + |x| + |y|)}{2} \)

Proof:

Suppose \( p : U \times U \to \mathbb{R} \) is a pmetric.

Then, by Theorem 4.1, the function \( q : U \times U \to \mathbb{R} \) where,

\[ \forall x, y \in U \ . \ q(x, y) := p(x, y) - p(x, x) \]

is a quasi metric such that \( \mathcal{T}[p] = \mathcal{T}[q] \).

Thus, by Lemma 2.6, the function \( p^m : U \times U \to \mathbb{R} \) where,

\( \forall x, y \in U \ . \ p^m(x, y) := 2 \times p(x, y) - p(x, x) - p(y, y) \)

is a metric such that, \( \forall x, y \in U \ . \ p^m(x, y) = q(x, y) + q(y, x) \).

Thus, \( \mathcal{T}[p] \subseteq \mathcal{T}[p^m] \).

Finally, WM holds as by P2 \( \forall x, y \in U \ . \ p^m(x, y) \geq |x| - |y| \)

\( \square \)

**Theorem 5.2**

For each weighted metric \( <d, ||> \) over a set \( U \) the function \( p : U \times U \to \mathbb{R} \) where,

\[ \forall x, y \in U \ . \ p(x, y) := \frac{(|x| + |y| + d(x, y))}{2} \]

such that \( d = p^m \) and \( \forall x \in U \ . \ |x| = p(x, x) \).

Proof:

as \( M1 \Rightarrow P1 \) and,\n
WM \( \Rightarrow P2 \) and, \n
M2 \( \Rightarrow P3 \) and, \n
M1 & M3 \( \Rightarrow P4 \)

\( \square \)

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We now turn to constructing a contraction mapping theorem for partial metrics. The beauty of the result in Theorem 5.3 is that its formulation is virtually the same as Banach's original theorem for metric spaces.

**Definition 5.2**
For each pmetric \( p : U \times U \to \mathbb{R} \), and for each \( X \in \omega U \), \( X \) is a **Cauchy Sequence** if,

\[ \forall \varepsilon > 0 \ \exists k \in \omega \ \forall n, m > k \quad p(X_n, X_m) < \varepsilon \]

**Definition 5.3**
A pmetric \( p : U \times U \to \mathbb{R} \) is **Complete** if for each Cauchy sequence \( X \in \omega U \) there exists \( a \in U \) such that, \( \exists \lim_{n \to \infty} p(X_n, a) = 0 \).

Note that these definitions for **Cauchy sequence** and **complete pmetric** are consistent with the usual metric definitions.

**Theorem 5.3 (Partial Metric Contraction Mapping Theorem)**
For each complete pmetric \( p : U \times U \to \mathbb{R} \), and for each function \( f : U \to U \) such that,

\[ \exists 0 \leq c < 1 \ \forall x, y \in U \quad p(f(x), f(y)) \leq c \times p(x, y) \]

firstly, there exists a unique \( a \in U \) such that \( a = f(a) \), and secondly \( p(a, a) = 0 \).

**Proof:**
Suppose \( p \), \( f \), \( \& \ c \) are as in the statement of the Theorem.
Suppose \( u \in U \).
We begin by showing that \( \lambda n \in \omega \). f^n(u) \) is a Cauchy sequence.

\( \forall n, k \in \omega \quad p(f^{n+k+1}(u), f^n(u)) \)
\[ \leq p(f^{n+k+1}(u), f^{n+k}(u)) + p(f^{n+k}(u), f^n(u)) - p(f^{n+k}(u), f^{n+k}(u)) \]
\[ \leq c^{n+k} \times p(f(u), u) + p(f^{n+k}(u), f^n(u)) \]

Thus,
\( \forall n, k \in \omega \quad p(f^{n+k+1}(u), f^n(u)) \)
\[ \leq (c^{n+k} + \ldots + c^n) \times p(f(u), u) + p(f^n(u), f^n(u)) \]
\[ \leq c^n \times ((1 - c^{k+1})/(1 - c)) \times p(f(u), u) + c^n \times p(u, u) \]
\[ \leq c^n \times ((p(f(u), u))/(1 - c)) + p(u, u) \]
Thus as, \( \forall n \in \omega . \ p( f^n(u) , f^n(u) ) \leq c^n \times p(u,u) \)
we see that \( \lambda n \in \omega . f^n(u) \) is a Cauchy sequence.

Thus as \( p \) is complete we can choose \( a \in U \) such that,

\[
\lim_{n \to \infty} p( f^n(u) , a ) = 0
\]

But, \( p( f(a), a ) = 0 \) as,

\[
\forall n \in \omega . \ p( f(a), a ) \\
\leq p( f(a), f^{n+1}(u) ) + p( f^{n+1}(u), a ) \\
- p( f^{n+1}(u), f^{n-1}(u) ) \\
\leq c \times p( a, f^n(u) ) + p( f^{n+1}(u), a )
\]

Thus \( a = f(a) \).

Suppose \( b \in U \) is such that \( b = f(b) \), then,

\[
p(a,b) = p( f(a), f(b) ) \leq c \times p(a,b)
\]

Thus, \( p(a,b) = 0 \), and so \( a = b \).

Thus the fixed point of \( f \) is unique.

Finally, \( p(a,a) = 0 \) as, \( p(a,a) = p( f(a), f(a) ) \leq c \times p(a,a) \)

\[\Box\]

6. Conclusions & Further Work

The principle conclusion from the research in this report is that generalised metric topology has a largely unexplored potential in the field of partial order denotational semantics. This conclusion is justified for the following reasons. Firstly, this research supports the quasi metric approach used in [Sm87] and the metric approach in [La87] to model partial order topologies. Secondly, we have shown that such work can be conducted using a symmetric distance function. Thirdly, we have shown that such work can be conducted within metric topology itself by adding the concept of a weight to points in a metric space. Finally, we have shown that Banach’s Contraction Mapping Theorem can be generalised for proof theory applications in Scott topologies.

Whether or not an arbitrary quasi metric topology can be defined by a weighted metric is an open question which needs an answer in order to decide exactly which topologies can be defined using our approach. However, it is unlikely that there exists any simple algebraic method for converting an arbitrary quasi metric into a weighted quasi metric. Similarly, it is not yet known exactly which Scott topologies can be defined using this approach. If a favourable answer to this question can be established then perhaps weighted metric models for the lambda calculus could be feasible.
The Wadge Cycle Test [Wa81] for proving Kahn data flow networks free of deadlock has yet to receive a proper denotational proof within Kahn's fixed point semantics. By formulating the domain of all finite & infinite streams as a partial metric space we plan to use the Partial Metric Contraction Mapping Theorem to prove Wadge's test. Beyond this we hope to use partial metrics to provide a denotational semantics for intensional programming languages such as LUCID [Wa&A85] which have been inspired by Kahn's work [Ka74]. For more results on partial metrics such as partial metric products, partial metric sums, and partial metric function spaces see [Ma92].

7. References


