

Original citation:

Goldberg, Leslie Ann, Hart, W. E. and Wilson, D. B. (1996) Learning foraging thresholds for lizards. University of Warwick. Department of Computer Science. (Department of Computer Science Research Report). (Unpublished) CS-RR-296

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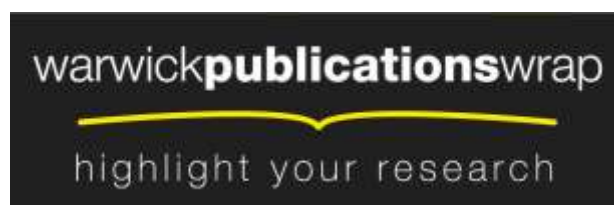
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Research Report 296

Learning Foraging Thresholds for Lizards

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This work gives a proof of convergence for a randomized learning algorithm that describes how anoles (lizards found in the Carribean) learn a foraging threshold distance. The model assumes that an anole will pursue a prey if and only if it is within this threshold of the anole's perch. This learning algorithm was proposed by the biologist Roughgarden and his colleagues. They experimentally confirmed that this algorithm quickly converges to the foraging threshold that is predicted by optimal foraging theory. Our analysis provides an analytic confirmation that the learning algorithm converges to this optimal foraging threshold with high probability.

Learning Foraging Thresholds for Lizards**

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17th of January, 1996

ABSTRACT This work gives a proof of convergence for a randomized learning algorithm that describes how anoles (lizards found in the Carribean) learn a *foraging threshold* distance. This model assumes that an anole will pursue a prey if and only if it is within this threshold of the anole's perch. This learning algorithm was proposed by the biologist Roughgarden and his colleagues. They experimentally confirmed that this algorithm quickly converges to the foraging threshold that is predicted by optimal foraging theory. Our analysis provides an analytic confirmation that the learning algorithm converges to this optimal foraging threshold with high probability.

1. Introduction

The model proposed by Roughgarden ([3], see also [1, 4]) models the behavior of anoles, which are lizards found in the Carribean. Anoles usually perch at a spot in their territory scanning the ground for prey. If an anole sees a prey that it considers worth pursuing, it leaves its perch and returns after the prey has been caught.

Optimal foraging theory assumes that it is good for a lizard to minimize the average time used to capture a prey. This is reasonable in circumstances of high predation, because the lizard is vulnerable to predators while chasing its prey [3]. Roughgarden [3] notes that this leads to a tradeoff for the lizard:

...if a lizard chases a very distant item, it is away from its perch and cannot see (or react) to prey that may appear while it is gone. Alternatively, a lizard may ignore a very distant item, and yet nothing may actually appear during the time it would have chased down that item. So, where should the lizard

** Part of this work was performed at Sandia National Laboratories and was supported by the U.S. Department of Energy under contract DE-AC04-94AL85000.

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draw the line? Clearly it should *not* chase extremely distant prey, for it would be away from its perch all day, and it *should* chase extremely close prey.

This tradeoff is captured in Roughgarden's analysis of the optimal foraging behavior that minimizes the average time used to capture a prey [3]. The basic model studied by Roughgarden assumes that the lizard captures all prey that it pursues, and that the lizard has a 180° field of view. The optimal foraging threshold that this analysis predicts is

$$X^* = \left(\frac{3v}{\pi a} \right)^{1/3},$$

where v represents the velocity of the lizard and a is the abundance of prey per square meter per second. If a lizard only pursues prey that fall within this distance from its perch, then the average time used to capture a prey is minimized.

There are a number of reasons why it is unlikely that lizards directly calculate the optimal foraging threshold. First, this rule is not flexible enough to account for variations in the abundance of prey in space and time. The model used in this analysis assumes a uniform, stochastic, stable environment, which is certainly not true in general. Second, the calculation of cubic roots is well beyond a lizard's cognitive abilities [3].

Roughgarden argues that a more realistic model of foraging is that a lizard learns an appropriate foraging threshold based on its past experiences pursuing prey. Specifically, Roughgarden proposes a learning algorithm that the lizard uses to refine its current foraging threshold. This algorithm assumes that the lizard can store two quantities: the time spent so far, T , and the total number of prey that have been captured, n .¹ Shamir and Roughgarden [4] note that experimental evidence suggests that many animals can conceptualize this type of information.

If a prey is seen at a distance d from the lizard, the learning algorithm specifies that the lizard will pursue if

$$\frac{T + 2d/v}{n + 1} < \frac{T}{n}.$$

This expression can be rewritten, to show that the lizard pursues if

$$d < \frac{vT}{2n}.$$

Consequently, we can view the right hand side of this expression as the lizard's current guess at the optimal foraging threshold.

Experimental results with this learning algorithm are reported by Roughgarden and Shamir [3, 4]. These results confirm that the foraging threshold is learned very quickly

¹ Roughgarden and Shamir [3, 4] divide T into two parts: the total time spent pursuing and the total time spent waiting for prey. However, the lizard really only needs to keep track of the total time to execute their learning algorithm.

by the learning algorithm. Menczer, Hart and Littman [2] formally argue that if the learning algorithm converges, then it converges to X^* . While this provides confirmation that the learning algorithm is reasonable, it does not constitute a proof of convergence.

Our analysis of the learning rule shows that it converges to X^* with probability one. We assume that appearance of prey is modeled by a stationary Poisson process, such that a is the expected number of prey that appear per unit area per second. Our proof uses a potential function argument to show that the probability that the foraging threshold remains far from X^* goes to zero. The fact that the learning algorithm converges provides analytic confirmation that the problem of learning optimal foraging thresholds can be solved using biologically plausible quantities. Thus, this work serves to strengthen the conclusions concerning the learning rule of Roughgarden [3].

2. Mathematical Analysis

Following Roughgarden, we assume that the visual area of the lizard is a perfect semicircle, and we assume that the lizard always catches prey that it pursues. Let v be the velocity of the lizard. We assume that the appearance of prey is a Poisson process with a being the expected number of prey that appear per unit area per second. The model assumes that if a prey arrives while the lizard is busy pursuing another prey, the new prey escapes instead of waiting to be caught. At time t , let n_t be the number of prey the lizard has captured. Let $X_t = vt/(2n_t)$ denote the lizard's foraging threshold at time t . The process starts with some initial value X_0 with $n_0 = 1$. (When the process is started, the first prey has just been caught.)

We can define the value of X_t as a random variable as follows. Let t_i denote the time at which the i th prey is seen, W_i denote the time spent waiting to see the i th prey, d_i denote the distance from the perch to the i th prey, and P_i denote the time spent pursuing the i th prey. We can describe the density functions of d_i, P_i and W_i as follows.

$$\Pr(d_i \leq y) = (y/X_{t_i})^2 = \Pr(r \leq (y/X_{t_i})^2) = \Pr(X_{t_i}\sqrt{r} \leq y),$$

where r is chosen uniformly from the range $[0, 1)$. $P_i = 2d_i/v$, so

$$\Pr(P_i \leq y) = \Pr\left(\frac{2X_{t_i}\sqrt{r}}{v} \leq y\right).$$

We now compute $\Pr(W_i > y)$. Let T_i be the time when the lizard starts looking for the i th prey (i.e., $T_i = t_{i-1} + P_{i-1}$.) Let the random variable C_y be the number of prey that appear within a radius of $X_{T_i} + vs/2(i-1)$ of the perch at time $T_i + s$, as s goes from 0 to y . Note that $\Pr(W_i > y) = \Pr(C_y = 0)$. But C_y is a Poisson random variable, whose parameter is a times the integral of the surveyed area over time. Thus,

$$\Pr(W_i > y) = \exp \left[- \int_0^y \alpha(X_{T_i} + vs/2(i-1))^2 ds \right],$$

where $\alpha = \pi a/2$.

Using these definitions, we have

$$\Pr(n_t \geq i) = \Pr \left(\sum_{j=2}^i (W_j + P_j) \leq t \right),$$

so

$$\Pr(X_t \leq y) = \Pr \left(\sum_{j=2}^{\lceil vt/(2y) \rceil} (W_j + P_j) \leq t \right).$$

Let $T(\tau)$ denote the last time at which the lizard has been waiting for τ seconds. Let $m_\tau = n_{T(\tau)}$ and $Y_\tau = X_{T(\tau)}$. Let τ_i be the total amount of time spent waiting before the i th prey is seen within the foraging threshold. The probability that a prey arrives exactly when the lizard returns to the perch after pursuing the i th prey is 0, so with probability 1, $\tau_i < \tau_{i+1}$. In this case, $T(\tau_i)$ is the time at which the lizard returns to its perch after catching the i th prey, so $m_{\tau_i} = i$. Let $X^* = (3v/(\pi a))^{1/3}$.

We will prove that the Y_τ 's converge to X^* and then show that this implies that the X_t 's converge to X^* . Our proof uses a potential function argument. Let the potential Φ be given by

$$\Phi_\tau = (Y_\tau - X^*)^4 m_\tau^{5/4};$$

the reason for these exponents will become evident. First we argue that the expected potential does not grow too rapidly (and probably even decreases) as a function of waiting time, then we argue that it does not grow too rapidly as a function of the number of prey caught so far. Since the potential gets large when the foraging threshold is far from X^* and many prey have been seen, this will imply that not much time is spent far away from the optimal threshold.

We will use the following technical lemma.

Lemma 1: If P is a Poisson random variable with parameter λ , then

$$\Pr[P \geq k] \leq \lambda^k / k!.$$

Proof:

$$\Pr(P \geq k) = e^{-\lambda} \sum_{i=k}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} \frac{\lambda^k}{k!} \sum_{i=0}^{\infty} \frac{\lambda^i k!}{(i+k)!} \leq e^{-\lambda} \frac{\lambda^k}{k!} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = \frac{\lambda^k}{k!}.$$

□

Lemma 2: For any fixed finite time interval, there is an upper bound Y such that for all τ in the interval, $Y_\tau \leq Y$. Furthermore, if τ_b and τ_e are drawn from the interval, and $\Delta\tau$ denotes $|\tau_e - \tau_b|$, then the probability that at least k prey arrive within the lizard's foraging threshold between time τ_b and τ_e is at most $O((\Delta\tau)^k)$, where the constant depends only upon the interval.

Proof: The number of prey that appear per unit area per second is a Poisson random variable with parameter a . Thus, the number of prey that appear within the lizard's foraging threshold is dominated by a Poisson random variable with parameter $\alpha Y^2 \Delta\tau$. The lemma follows from Lemma 1. \square

Lemma 3: There is a function $g(n)$ with $g(n) = O(n^{-7/4})$ such that the following is true for any fixed finite time interval: If τ_b and τ_e are drawn from the interval with $\tau_b \leq \tau_e$ and $m_{\tau_b} = n$, then

$$\mathbb{E}[\Phi_{\tau_e} | \text{state at time } \tau_b] - \Phi_{\tau_b} \leq (\tau_e - \tau_b)g(n) + O((\tau_e - \tau_b)^2),$$

where the constants in the $O((\tau_e - \tau_b)^2)$ term depend upon the interval (in particular upon its maximum foraging radius from Lemma 2) but not upon τ_b or τ_e .

Proof: We start by making a preliminary observation. Suppose that after waiting τ seconds, the lizard sees a prey at time T' when its foraging threshold is Y' . By the probability density function of d_{m_τ} , the distance of the prey from the perch is the random variable $Y'\sqrt{r}$, where r is uniformly distributed between 0 and 1. The lizard pursues the prey and returns to its perch. The pursuit time for this prey is $2Y'\sqrt{r}/v$. The new foraging threshold after the prey is caught is

$$Y_\tau = \frac{v(T' + 2Y'\sqrt{r}/v)}{2m_\tau} = \frac{v(2(m_\tau - 1)Y'/v) + 2Y'\sqrt{r}}{2m_\tau} = Y' + Y' \left(\frac{\sqrt{r} - 1}{m_\tau} \right).$$

(Note that the prey will be caught before the time that the lizard has spent waiting, τ , increases.)

Now consider the finite interval in the statement of Lemma 3 and let Y be the interval's foraging threshold bound from Lemma 2. Suppose that τ_b and τ_e are drawn from the interval with $\tau_b \leq \tau_e$ and that $m_{\tau_b} = n$. Let Δ denote $\mathbb{E}[\Phi_{\tau_e} | \text{state at time } \tau_b] - \Phi_{\tau_b}$. We wish to show that

$$\Delta \leq (\tau_e - \tau_b)g(n) + O((\tau_e - \tau_b)^2),$$

where the constant in the term $O((\tau_e - \tau_b)^2)$ depends upon the interval but not upon τ_b or τ_e and $g(n) = O(n^{-7/4})$ does not depend upon the interval (or upon τ_b or τ_e).

Consider the time period $(\tau_b, \tau_e]$ (the period of time between τ_b and τ_e), and let $\Delta\tau = \tau_e - \tau_b$. Either a prey appears during this period, or not. If no prey appears, the foraging threshold expands by $v\Delta\tau/(2n)$. If exactly one prey appears, then, (by the preliminary observation), the lizard pursues the prey and adjusts the foraging threshold by adding $Z_{\tau_b} = (Y_{\tau_b} + O(\Delta\tau))(\sqrt{r} - 1)/(n + 1)$; meanwhile the foraging threshold expands by $O(\Delta\tau)$. Let $f(y) = (y - X^*)^4$ and let W denote the probability that prey appears during the period. Putting all of this together, we have

$$\begin{aligned} \mathbb{E}[\Phi_{\tau_e} \mid \text{state at time } \tau_b, \text{ choice of } r] &\leq (1 - W)n^{5/4}f\left(Y_{\tau_b} + \frac{v\Delta\tau}{2n}\right) \\ &\quad + W(n + 1)^{5/4}f(Y_{\tau_b} + Z_{\tau_b} + O(\Delta\tau)) \\ &\quad + O((\Delta\tau)^2). \end{aligned}$$

So, conditioned on the choice of r ,

$$\begin{aligned} \Delta &\leq n^{5/4}f\left(Y_{\tau_b} + \frac{v\Delta\tau}{2n}\right) - n^{5/4}f(Y_{\tau_b}) + W \left[(n + 1)^{5/4}f(Y_{\tau_b} + Z_{\tau_b} + O(\Delta\tau)) \right. \\ &\quad \left. - n^{5/4}f\left(Y_{\tau_b} + \frac{v\Delta\tau}{2n}\right) \right] + O(\Delta\tau^2). \end{aligned}$$

We now derive an upper bound for W . Let Y^* denote $Y_{\tau_b} + v\Delta\tau/(2n)$ and note that the lizard's foraging threshold does not exceed Y^* during the period between τ_b and τ_e . The number of prey that appear within the lizard's foraging threshold during this period is dominated by a Poisson random variable with parameter $\alpha\Delta\tau Y^{*2}$. So by Lemma 1, $W \leq \alpha\Delta\tau Y^{*2}$, which is at most $\alpha\Delta\tau(Y_{\tau_b}^2 + Yv\Delta\tau/n + (v\Delta\tau/2n)^2)$. Thus, $W \leq \alpha Y_{\tau_b}^2 \Delta\tau + O((\Delta\tau)^2)$ where the constants in the $O((\Delta\tau)^2)$ depend upon the fixed interval (and therefore, on Y), but not upon τ_b and τ_e . In the interval the second derivative of f is bounded, so by Taylor's theorem², the value of $\Delta/\Delta\tau$ conditioned on the choice of r is as follows.

$$\begin{aligned} \Delta/\Delta\tau &\leq n^{5/4}f'(Y_{\tau_b})\left(\frac{v}{2n}\right) + \alpha Y_{\tau_b}^2 \left[(n + 1)^{5/4}f\left(Y_{\tau_b} + Y_{\tau_b} \frac{\sqrt{r} - 1}{n + 1}\right) \right. \\ &\quad \left. - n^{5/4}f(Y_{\tau_b}) \right] + O(\Delta\tau). \end{aligned} \tag{1}$$

² Equation 1 led us to choose the exponent 5/4 in the definition of Φ_{τ_b} . We want the derivative in Equation 1 to be small even when Y_{τ_b} is large. This makes it necessary to have the exponent less than 4/3.

We note that the constants in the $O(\Delta\tau)$ can be chosen uniformly for any τ_b and τ_e in the fixed interval.

Integrating with respect to r , and evaluating Equation 1 at $Y_{\tau_b} = X^*(1+z)$ yields a polynomial in z whose coefficients are functions of n . We need only concern ourselves with the behavior of these functions when n is large enough. Using Maple, we find that $\Delta/\Delta\tau$ evaluated at $Y_{\tau} = X^*(1+z)$ is equal to the following.

$$\begin{aligned} \Delta/\Delta\tau = \alpha X^{*6} & \left[-\frac{1+o(1)}{12}n^{1/4}z^6 - \frac{3+o(1)}{2}n^{1/4}z^5 - \frac{11+o(1)}{4}n^{1/4}z^4 + \frac{11+o(1)}{3}n^{-3/4}z^3 \right. \\ & \left. + \frac{1+o(1)}{1}n^{-3/4}z^2 - \frac{2+o(1)}{5}n^{-7/4}z + \frac{1+o(1)}{15}n^{-11/4} \right] + O(\Delta\tau). \end{aligned} \quad (2)$$

where the $o(1)$ terms are functions of n (independently of z).

Suppose that n is sufficiently large. Ignoring the $O(\Delta\tau)$ term for the moment, let β denote the first term. We consider the following cases.

Case 1: $z \geq 3n^{-1/2}$: In this case, the term $[(11+o(1))/4]n^{1/4}z^4$ is at least three times as large as each of the three positive terms, so $\beta \leq 0$.

Case 2: $-1 \leq z \leq -3n^{-1/2}$: In this case, the term $[(11+o(1))/4]n^{1/4}z^4$ is at least $3/2$ times as large as the term $[(3+o(1))/2]n^{1/4}z^5$ and at least nine times as large as each of the three other positive terms. Thus, $\beta \leq 0$.

Case 3: $|z| \leq 3n^{-1/2}$: In this case, each term is $O(n^{-7/4})$, so $\beta = O(n^{-7/4})$.

Note that $z \geq -1$. Thus we have that $\Delta/\Delta\tau = O(n^{-7/4}) + O(\Delta\tau)$ for all values of z , for all values of τ_b and τ_e contained in a given finite interval. \square

We will find it convenient to define

$$\Psi_{\tau}^n = \begin{cases} \Phi_{\tau_n}, & \text{if } \tau < \tau_n; \\ \Phi_{\tau}, & \text{if } \tau_n \leq \tau < \tau_{n+1}; \\ \Phi_{\tau_{n+1}}, & \text{if } \tau_{n+1} \leq \tau. \end{cases}$$

As a function of τ , Ψ_{τ}^n is discontinuous at τ_{n+1} but it is continuous elsewhere.

Lemma 4: For any fixed finite time interval, if τ_b and τ_e are drawn from that interval then

$$\mathbb{E} [\Psi_{\tau_e}^n - \Psi_{\tau_b}^n] = \mathbb{E} [\Phi_{\tau_e} - \Phi_{\tau_b} \mid m_{\tau_b} = n] \cdot \Pr(m_{\tau_b} = n) + O((\Delta\tau)^2),$$

where the constants in the $O((\Delta\tau)^2)$ term depend upon n and the fixed interval's maximum foraging radius from Lemma 2.

Proof: Let Y be the interval's maximum foraging radius from Lemma 2 and let $\gamma_r = \mathbb{E} [\Psi_{\tau_e}^n - \Psi_{\tau_b}^n \mid m_{\tau_b} = r]$. By the linearity of expectation,

$$\mathbb{E} [\Psi_{\tau_e}^n - \Psi_{\tau_b}^n] = \sum_{r=1}^{\infty} \gamma_r \Pr(m_{\tau_b} = r). \quad (3)$$

We now consider possible values of r .

Case $r > n$: $\gamma_r = 0$.

Case $r \leq n - 1$: If no prey are seen then $\Psi_{\tau_e}^n - \Psi_{\tau_b}^n = 0$. If one prey is seen (by Lemma 2, this happens with probability $O(\Delta\tau)$), then $|\Psi_{\tau_e}^n - \Psi_{\tau_b}^n| = O(\Delta\tau)$. Even if two or more prey are seen (by Lemma 2, this happens with probability $O((\Delta\tau)^2)$), $|\Psi_{\tau_e}^n - \Psi_{\tau_b}^n|$ is still $O(1)$. Hence $\gamma_r = O((\Delta\tau)^2)$.

Case $r = n$: Let $\Delta\Phi = \Phi_{\tau_e} - \Phi_{\tau_b}$ and $\Delta\Psi = \Psi_{\tau_e}^n - \Psi_{\tau_b}^n$. If k prey are found, then $|\Delta\Phi| \leq Y^4(n+k)^{5/4}$ and $|\Delta\Psi| \leq Y^4(n+1)^{5/4}$, so

$$|\Delta\Psi - \Delta\Phi| \leq 2Y^4(n+k)^{5/4}.$$

If $k = 0$, then $\Delta\Psi = \Delta\Phi$. If $k = 1$ (with probability $O(\Delta\tau)$, by Lemma 2) then $|\Delta\Psi - \Delta\Phi| = O(\Delta\tau)$. Using Lemmas 1 and 2,

$$\begin{aligned} |E[\Delta\Psi \mid m_{\tau_b} = n] - E[\Delta\Phi \mid m_{\tau_b} = n]| &\leq O((\Delta\tau)^2) + 2Y^4 \sum_{k=2}^{\infty} (n+k)^{5/4} (\alpha Y^2 \Delta\tau)^k / k! \\ &= O((\Delta\tau)^2). \end{aligned}$$

Since $\gamma_n = E[\Delta\Psi \mid m_{\tau_b} = n]$, $\gamma_n = E[\Phi_{\tau_e} - \Phi_{\tau_b} \mid m_{\tau_b} = n] + O((\Delta\tau)^2)$.

Summing over r according to Equation 3, we get

$$\mathbb{E} [\Psi_{\tau_e}^n - \Psi_{\tau_b}^n] = \mathbb{E} [\Phi_{\tau_e} - \Phi_{\tau_b} \mid m_{\tau_b} = n] \cdot \Pr(m_{\tau_b} = n) + O((\Delta\tau)^2),$$

where the constants in the $O((\Delta\tau)^2)$ term depend upon n and upon the fixed interval. \square

Lemma 5: For all n , $\mathbb{E}[\Phi_{\tau_n}] \leq \Phi_0 + O(1)$. (The $O(1)$ term does not depend on τ .)

Proof: We start by computing $\Phi_{\tau_{n+1}} - \Phi_{\tau_n} = \Psi_{\infty}^n - \Psi_0^n$. To do this, we will use Lemmas 3 and 4. Both lemmas may be applied to any finite interval, so we will apply them to the intervals $[0, 1], [1, 2], \dots$. Any interval $[i, i + 1]$ may be subdivided into subintervals of size $1/M$ (for any $M > 1$). Applying lemmas 3 and 4 to each subinterval and summing gives

$$\begin{aligned}
E[\Psi_{i+1}^n - \Psi_i^n] &= \sum_{j=1}^M E[\Psi_{i+j/M}^n - \Psi_{i+(j-1)/M}^n] \\
&\leq \sum_{j=1}^M [(g(n)/M + O(1/M^2)) \Pr(m_{i+(j-1)/M} = n) + O(1/M^2)]
\end{aligned}$$

where the constants in the $O(1/M^2)$ terms depend on the interval $[i, i+1]$ and on n but not the subintervals. Taking the limit of both sides as $M \rightarrow \infty$ gives

$$E[\Psi_{i+1}^n - \Psi_i^n] \leq g(n) \int_i^{i+1} \Pr(m_\tau = n) d\tau.$$

Summing over all i we get

$$E[\Phi_{\tau_{n+1}} - \Phi_{\tau_n}] \leq g(n) \int_0^\infty \Pr(m_\tau = n) d\tau. \quad (4)$$

The integral in Equation 4 is the expected waiting time after the n th prey is seen before the $(n+1)$ st prey is seen. We will show that this is $O(n^{2/3})$. To see this, note that if the lizard has been waiting $\Theta(n^{2/3})$ seconds then the foraging threshold is $\Omega(n^{-1/3})$ so the area of the region that the lizard surveys is $\Omega(n^{-2/3})$. After each additional $n^{2/3}$ seconds the probability that the lizard sees a prey is bounded away from 0. Thus the waiting time for the lizard is geometrically distributed. It follows that the expected time that the lizard waits is $O(n^{2/3})$.

Since $g(n) = O(n^{-7/4})$ (from Lemma 3), we conclude that $E[\Phi_{\tau_{n+1}} - \Phi_{\tau_n}] = O(n^{-13/12})$. Since $n^{-13/12}$ is a convergent series, the expected value of the potential just after the n th prey is caught is at most Φ_1 plus some constant independent of n . \square

Theorem 1: *With probability one $Y_\tau \rightarrow X^*$.*

Proof: First consider the sequence of foraging thresholds just after the lizard has caught a prey; suppose that these converge to X^* . If the threshold just before running after the prey is $vT/(2n)$ then the threshold just after is between $(vT/(2n))(1 - 1/(n+1))$ and $vT/(2n)$. Thus the sequence of foraging thresholds just before the catching of the prey converges too. Finally, since the foraging threshold is always bounded below by the threshold just after catching the previous prey, and above by the threshold just before catching the next prey, the foraging thresholds converge to X^* . Therefore we focus on the subsequence of foraging thresholds Y_{τ_n} just after the n th prey was caught.

Using Markov's inequality and recalling that, with probability 1, $m_{\tau_n} = n$, we get

$$\begin{aligned}
\Pr[|X^* - Y_{\tau_n}| > \varepsilon] &= \Pr[n^{5/4} |X^* - Y_{\tau_n}|^4 > n^{5/4} \varepsilon^4] \\
&= \Pr[\Phi_{\tau_n} > n^{5/4} \varepsilon^4] \leq E[\Phi_{\tau_n}] n^{-5/4} \varepsilon^{-4}.
\end{aligned}$$

If we take $\varepsilon = n^{-1/32}$ and note that, by Lemma 5, $\mathbf{E}[\Phi_{\tau_n}]$ is bounded from above by a constant, then we get

$$\Pr[|X^* - Y_{\tau_n}| > n^{-1/32}] = O(n^{-9/8}).$$

Thus, as the process runs, the expected number of integers n such that the Y_{τ_n} deviates from X^* by more than $n^{-1/32}$ is $O(1)$. Almost surely, there are a finite number of such integers n , so with probability one, the sequence of foraging thresholds Y_{τ_n} just after the catching of prey converges to X^* . \square

Corollary 1: *With probability one $X_t \rightarrow X^*$.*

Proof: Recall that for every i , with probability 1, the lizard waits some between pursuing the i th prey and pursuing the $(i + 1)$ st prey. By Theorem 1, the sequence of foraging thresholds during waiting times converge to X^* . During the pursuit of a prey, the threshold is bounded from below by the threshold during the waiting time before the pursuit of the prey and is bounded from above by $(1 + O(1/(n + 1)))$ times the threshold during the waiting time after the pursuit of the prey. Thus, the foraging thresholds converge to X^* . \square

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