CATEGORIES OF OPERATORS AND H-SPACES

by

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Abstract

By introducing categories of operators the concept of an associative H-space is generalized. Each such category gives rise to a structure on a space X if it can be made to act on it. To each category $\mathcal{C}$ of operators a category $\mathcal{W}_{\mathcal{C}}$ of operators is associated which gives rise to a $\mathcal{C}$-structure up to higher homotopies and all possible coherence conditions. After introducing the notion of a structure map and of homotopies of structure maps the category of $\mathcal{W}_{\mathcal{C}}$-spaces and homotopy classes of structure maps is set up and studied. The theory is applied to prove a classification theorem.
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Homotopy–Everything H–Spaces (a summary of joint work with Dr J.M. Boardman).
INTRODUCTION

The concept of an H-space, i.e. a space $X$ with base point $e$ and a multiplication map $m: X \times X \to X$ such that $e$ is a homotopy identity, arose as a generalization of that of a topological group. It turned out to be of great importance in homotopy theory, especially in the study of extraordinary cohomology theories.

Generally speaking most of the techniques which apply to topological groups cannot be applied to H-spaces because of their lack of structure. From the homotopy theoretic point of view the distinguishing feature is the associativity (and commutativity) of the multiplication rather than the existence of a continuous inverse. [for example see [1], Satz 8.2 and 8.3]. Since many spaces of interest have no natural monoid or commutative monoid structure, such as the loop space $\Omega X$ or the stable orthogonal group, this led to an extensive study of H-spaces which almost have the structure of a topological monoid or a commutative topological monoid such as homotopy associative, homotopy commutative, strongly homotopy commutative [7] H-spaces, and $A_\infty$-spaces [5]. In the last two cases, part of the structure consists of higher
homotopies and coherence conditions, and important constructions like the classifying space construction turn out to hold for them.

A problem in the theory of $H$-spaces with additional structure has been to find the right concept of maps between them. The notion of a homomorphism, i.e. a map that preserves the multiplication and the coherence conditions, turned out to be too restrictive, while a notion of a map that commutes with the multiplication up to homotopy was too weak for many applications. The complexity of structures with higher homotopies and coherence conditions made it so far impossible to find a satisfactory definition of maps between such spaces, while Sugawara [7] succeeded in doing this for monoids. A study of the category of topological monoids and homotopy classes of such maps can be found in [8].

The purpose of this thesis is to develop a satisfactory theory - from the view of homotopy theory - of spaces with homotopy-associative (and homotopy-commutative) multiplication and all possible higher homotopies and coherence conditions and of structure maps between such spaces. A suitable definition of homotopy between such maps makes these spaces and the homotopy classes of struc-
ture maps between them into a category. We adopt following as test propositions:

(A) If $X$ is a space in the category and $Y$ is homotopy equivalent to $X$ then $Y$ is in the category.

(B) A structure map over a homotopy equivalence is an isomorphism in the category.

To avoid the difficulties arising from the complexity of the topological models used to define the higher homotopies (for example of an $A_\infty$-space, such as the well known Stasheff pentagon), we approach the problem in a completely new way, which in addition provides us with results for a much wider range of "structure" spaces than just $A_\infty$-spaces or homotopy-commutative $A_\infty$-spaces with suitable higher coherence conditions. Rather than speaking of a particular space with a given structure we introduce categories of operators which "act" on spaces and thus induce a structure on them. Such a category $\mathcal{B}$ basically consists of objects $0, 1, 2, 3, \ldots$, a topological structure on each morphism set such that composition is continuous, a continuous bifunctor $\otimes: \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that $m \otimes n = m + n$. An action of $\mathcal{B}$ on a space $X$ associates with each morphism $f: m \to n$ a map $\alpha(f): X^m \to X^n$ continuously in $f$ and such
that $\alpha(f \oplus g) = \alpha(f) \times \alpha(g)$ and $\alpha$ is functorial. For example the category consisting of exactly one morphism $n \rightarrow 1$ for each $n$ gives rise to a topological monoid structure. With each category $B$ of operators we associate another category $\tilde{W}B$ of operators which gives rise to a structure that is a $\tilde{B}$-structure up to higher homotopies and all possible coherence conditions of which the morphism spaces keep track. $\tilde{W}B$ has a universal property such that a space with a $\tilde{W}(\tilde{W}B)^\sim$-structure can be given a $\tilde{W}B$-structure. This universal property is the key for the development of our theory.

A slight generalization of the concept of categories of operators gives rise to the definition of structure maps.

In order to avoid spurious difficulties in our topological constructions we work in the category $CG$ of compactly generated Hausdorff spaces. For details see [6]. Two of the properties of $CG$ which we frequently use without mentioning are full adjointness and the fact that the product of two quotient spaces is the quotient of the product.

This thesis is part of a joint work with my supervisor, Dr. J. M. Boardman, who applied the theory represented here to obtain results about the stable groups $O$, $SO$, $F$, $SF$,
u, SU, PL, SPL, Top, STop etc. and their classifying spaces. A summary of this joint work is included at the end of this thesis.

In the first chapter we give the definition of categories of operators and list a few examples. In the second chapter we construct the category $W\mathcal{B}$ for each category $\mathcal{B}$ of operators and discuss its basic properties. Chapter III deals with the concept of structure maps and we set up the category of $W\mathcal{B}$-spaces and homotopy classes of structure maps. It includes the proofs for the test theorems. In Chapter IV we study spaces with $W\mathcal{B}$-structures and state a classification theorem.

Example 4 in §1 and the results of the second section of Chapter IV are entirely due to Dr. J.M. Boardman and we restrict ourselves to sketching the proofs.
All our topological constructions will be in the category of compactly generated Hausdorff spaces. This means that we only need to check that the identification spaces constructed are Hausdorff. The rest is automatic.

Let $\mathcal{N}$ be the set of all sequences in $n$ generators $0, \ldots, n-1$ including the empty sequence. $\mathcal{N}$ is a free monoid under juxtaposition.

Define a left action of $S(k)$, the symmetric group in $k$ letters, on the sequences of length $k$ by

$$\xi(i_1, \ldots, i_k) = (i_{\xi^{-1}(1)}, \ldots, i_{\xi^{-1}(k)})$$

$\xi \in S(k)$.

We have two variants of categories of operators: with or without permutations.

**Definition 1.1**: In a category $\mathcal{B}$ of operators on $n$ object generators

(a) the objects are elements of $\mathcal{N}$

(b) the morphisms from $a$ to $b$ form a (compactly generated) topological space $\mathcal{B}(a,b)$ and composition is continuous

(c) we are given a strictly associative, continuous bifunctor $\oplus: \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ such that $a \oplus b = ab$
(d) if \( B \) has permutations we are also given a morphism \( \xi : a \rightarrow \xi a \) for each \( \xi \in S(k) \) and each sequence \( a \) of length \( k \) such that

(i) \( \xi \cdot \eta = \xi \circ \eta \)

(ii) if \( \xi \in S(k) \) is the identity, then \( \xi = 1_a \)

(iii) if \( \xi \in S(m) \) and \( \eta \in S(k) \) then \( \xi \circ \eta = \xi \circ \eta \),

where \( \xi \circ \eta \in S(m+k) \) is the usual sum permutation

(iv) given \( r \) morphisms \( a_i \) in \( B \) such that source \( a_i \) is a sequence of length \( m_i \) and target \( a_i \) one of length \( n_i \), and \( \xi \in S(r) \), then we have

\[
\xi(n_1, \ldots, n_r) \circ (a_1 \circ \cdots \circ a_r) = (a_\xi^{-1}(1) \circ \cdots \circ a_\xi^{-1}(r)) \circ \xi(m_1, \ldots, m_r)
\]

where \( \xi(n_1, \ldots, n_r) \in S(n_1 + \cdots + n_r) \) is defined as follows:

Let \( n_1 + \cdots + n_{k-1} + l = i < n_1 + \cdots + n_k \), \( l > 0 \).

Then \( \xi(n_1, \ldots, n_r)(i) = l + \sum_{j=1}^{\xi(k)-1} n_{\xi^{-1}(j)} \).

**Notation:** For \( \xi \in S(k) \) we denote the induced morphisms \( \xi \) simply by \( \xi \).

We call a category of operators on \( n \) object generators an \( M^T \)-category if it has no permutations and an \( M^T \)-category if it has.

Unless otherwise stated we denote sequences of length
1 by small Latin letters and general sequences by underlined small Latin letters. For morphisms we use the letters α, β, γ and for permutations the letters ξ, η, ζ. Categories are denoted by underlined capital Latin letters.

**Definition 1.2:** Let $\mathcal{C}$ be an $M^n_T(M^n_{TP})$ category and $\mathcal{D}$ an $M^m_T$(resp.$M^m_{TP}$)-category. An $M^n_T$(resp.$M^n_{TP}$)-functor $\delta: \mathcal{C} \rightarrow \mathcal{D}$ from $\mathcal{C}$ to $\mathcal{D}$ is a functor such that

(i) $\delta$ maps object generators into object generators, i.e. it maps sequences of length 1 into sequences of length 1

(ii) $\delta$ preserves sums, i.e. $\delta(\mathcal{C} \oplus \mathcal{D}) = \delta\mathcal{C} \oplus \delta\mathcal{D}$ and $\delta(\mathcal{C} \oplus \mathcal{D}) = \delta\mathcal{C} \oplus \delta\mathcal{D}$

(iii) $\delta: \mathcal{D}(\mathcal{C}, \mathcal{D}) \rightarrow \mathcal{D}(\mathcal{C}, \mathcal{D})$ is continuous

(iv) if $\mathcal{C}$ and $\mathcal{D}$ have permutations, then $\delta$ preserves permutations, i.e. $\delta(\xi) = \xi$

If $\mathcal{C}$ and $\mathcal{D}$ are categories on the same object generators and $\delta$ in addition preserves generators, i.e. $\delta(a) = a$, then $\delta$ is called an $M^n_T$(resp.$M^n_{TP}$)-functor.

Note that (iv) is equivalent to saying that $\delta$ is equivariant, i.e. $\delta(a \circ \xi) = \delta(a) \circ \xi$. 
Example 1: \( \text{End}(X_1, \ldots, X_n) \) for based spaces \( X_1, \ldots, X_n \) is an \( M^nT \)-category. \( \text{End}(X_1, \ldots, X_n)(a,b) \) is the space of all based maps \( X_a \to X_b \) (see [6], chapter 5), where \( X_a = X_{i_1} \times \ldots \times X_{i_k} \) if \( a = (i_1, \ldots, i_k) \). The functor \( \varepsilon \) is just \( x \). The permutations are the obvious shuffles.

Definition 1.3: An \( M^nT \) (resp. \( M^nTP \))-category \( \mathcal{B} \) is said to act on \( (X_1, \ldots, X_n) \) if we are given an \( M^nT \) (resp. \( M^nTP \))-functor \( \mathcal{B} \to \text{End}(X_1, \ldots, X_n) \).

If an \( M^1T \) (resp. \( M^1TP \))-category \( \mathcal{B} \) acts on \( X \) we call \( X \) a \( \mathcal{B} \)-space.

Example 2: \( A \), an \( M^1T \)-category: Denote the unique sequence of length \( m \) by \( m \). \( A(m,n) \) is the space of all order preserving functions \( (1, \ldots, m) \to (1, \ldots, n) \) with the discrete topology. There is exactly one function \( \lambda_n: n \to 1 \) for each \( n \). An \( A \)-space is a topological monoid (in CG, the category of compactly generated Hausdorff spaces).

Example 3: \( S \), an \( M^1TP \)-category: Again denote the unique sequence of length \( m \) by \( m \). \( S(m,n) \) is the set of all functions \( (1, \ldots, m) \to (1, \ldots, n) \) with the discrete topology. The permutations are the ordinary permutations \( (1, \ldots, m) \to (1, \ldots, m) \). An \( S \)-space is an abelian topolo-
gical monoid. Such a space $X$ is known to have the homotopy type of a product of Eilenberg–Mac Lane spaces, if $X$ is a connected CW-complex. (The proof of this is roughly as follows: Let $G_n$ denote the $n$th homotopy group of $X$, and $M(G,n)$ the Moore space with $G$ as $n$-th homology group. For each $n$ construct a map $f_n: M(G_n,n) \to X$ which induces an isomorphism of the $n$-th homotopy groups. The abelian monoid structure on $X$ enables one to construct maps of the infinite symmetric products $SP(M(G_n,n))$ into $X$ from the $f_n$'s. These give rise to a map of the restricted product $\Pi_n SP(M(G_n,n))$ into $X$. $SP(M(G_n,n))$ is of the same homotopy type as the Eilenberg–Mac Lane complex $K(G_n,n)$, and the constructed map gives the required homotopy equivalence. For details see [3]).

**Definition 1.4**: An $M^\text{TP}$-category $B$ is in normal form if

(a) each morphism is expressible as

$$\alpha = (\alpha_1 \oplus \ldots \oplus \alpha_k)^{\circ \xi}$$

where $\alpha_i$ is a morphism into an object generator for each $i$, $1 \leq i \leq k$, and $\xi$ a permutation

(b) this expression is unique up to following equivalence

$$(\alpha_1 \circ \eta_1 \oplus \ldots \oplus \alpha_k \circ \eta_k)^{\circ \xi} = (\alpha_1 \oplus \ldots \oplus \alpha_k)^{\circ [(\eta_1 \oplus \ldots \oplus \eta_k)^{\circ \xi}]}$$
where \( \eta_i \) and \( \xi \) are permutations

(c) The morphism spaces of \( B \) have the quotient topology of the appropriate disjoint unions of product spaces of morphisms into a generator under the relation (b).

Analogously for \( M^{n_T} \)-categories to be in normal form we demand that each morphism is uniquely expressible as a sum of morphisms into a generator and that the morphism spaces have the appropriate disjoint union topology of product spaces of morphisms into a generator.

The importance of categories in normal form is clear from

**Theorem 1.5:** Given an arbitrary \( M^{n_{TP}} \) (resp. \( M^{n_T} \))-category \( B \) there exists another \( M^{n_{TP}} \) (resp. \( M^{n_T} \))-category \( C \) in normal form and an \( M^{n_{TP}} \) (resp. \( M^{n_T} \))-functor \( \gamma : C \to B \) satisfying \( C(a, b) = B(a, b) \), and \( \gamma|_{C(a, b)} \) is the identity. \( C \) and \( \gamma \) are unique up to isomorphism. (Recall: \( b \) denotes an object generator)

**Proof:** Put \( C(a, b) = B(a, b) \) as required. Construct the spaces of morphisms into larger sequences according to condition 1.4 (c): For \( a = (i_1, \ldots, i_k) \) and \( b = (j_1, \ldots, j_l) \)
let \( V(\alpha, \beta) = \bigcup \mathcal{C}(\alpha_1, j_1) \times \mathcal{C}(\alpha_2, j_2) \times \cdots \times \mathcal{C}(\alpha_1, j_1) \times \{ \xi \} \) taken over all partitions of the sequence \( \alpha_1 \oplus \cdots \oplus \alpha_1 = (i_\xi^{-1}(1), \ldots, i_\xi^{-1}(k)) \) into connected subsequences and all permutations \( \xi \in S(k) \). \( V(\alpha, \beta) \) has the disjoint union topology of the products and hence is in \( \mathbb{C} \).

Introduce the relation 1.4 (b) into \( V(\alpha, \beta) \):

\[
(\alpha_1 \circ \eta_1, \ldots, \alpha_1 \circ \eta_1; \xi) \sim (\alpha_1, \ldots, \alpha_1; (\eta_1 \oplus \cdots \oplus \eta_1) \circ \xi)
\]

where \( \alpha_1 \) is the element \( \alpha_1 \) of \( \beta \) considered as element of \( \mathcal{C} \), and \( \eta_1 \) a permutation in \( S(\text{source} \alpha_1) \).

Let \( \mathcal{C}(\alpha, \beta) = V(\alpha, \beta)/\sim \). \( \mathcal{C}(\alpha, \beta) \) is Hausdorff and hence in \( \mathbb{C} \). Composition with permutations is forced on us by 1.4 (b) and 1.1 (d): Let \( \eta \in S(1) \) and \( \xi \in S(k) \) then

\[
\eta^\circ (\alpha_1, \ldots, \alpha_1; \xi) = (\alpha_{\eta^{-1}(1)}, \ldots, \alpha_{\eta^{-1}(1)}; \eta(r_1, \ldots, r_1) \circ \xi)
\]

\[
(\alpha_1, \ldots, \alpha_1; \xi) \circ \xi = (\alpha_1, \ldots, \alpha_1; \xi \circ \xi)
\]

where \( r_p \) is the length of source \( \alpha_p \).

Denoting \( (\alpha_1, \ldots, \alpha_1; \xi) \) by \( (\alpha_1 \oplus \cdots \oplus \alpha_1) \circ \xi \) we have defined a continuous associative sum in \( \mathcal{C} \).

To define composition, note that it suffices to define it for \( \bar{\alpha} \circ [\bar{\beta}_1 \oplus \cdots \oplus \bar{\beta}_n] \), where \( \alpha \) is a morphism into a generator, and to check associativity and the existence of a unit just for these elements since we have taken care of the permutations already. Denote composition in \( \mathcal{B} \) by \( * \) and in \( \mathcal{C} \) by \( \circ \). Define
\[ a^* [\beta_1 \oplus \ldots \oplus \beta_n] = a^* (\beta_1 \oplus \ldots \oplus \beta_n) \]

Since composition and sum in \( B \) are continuous and since this definition respects the identifications, this composition is well defined and continuous. It is associative and has \( \bigoplus_{j_1} \oplus \ldots \oplus \bigoplus_{j_n} \) as identity for \((j_1, \ldots, j_n)\).

By construction \( \oplus \) is a Bifunctor. The permutations are represented by an element of the form \((\text{sum of identities}) \circ \xi\). By construction \( C \) is in normal form.

Define the functor \( \gamma: C \to B \) by
\[ \gamma[(\alpha_1 \oplus \ldots \oplus \alpha_k) \circ \xi] = (\alpha_1 \oplus \ldots \oplus \alpha_k)^* \circ \xi \]

Since \( \oplus \) is a bifunctor the relation 1.4 (b) holds in any \( M_{TP} \)-category. Hence \( \gamma \) is well defined. It is continuous, and preserves sums, permutations and identities. From the definition of composition it is clear that \( \gamma \) is a functor. Hence it is an \( M_{TP} \)-functor.

The construction for \( M_{T} \)-categories is completely analogous, but simpler.

We refer to the construction of morphism spaces into longer sequences once the ones into generators and their compositions with permutations are given as the normal form construction.

**Corollary 1.6:** Let \( B \) be an \( M_{TP} \) (resp. \( M_{T} \)) -category and \( C \) the associated category in normal form. If \( B \) acts
on \((X_1, \ldots, X_n)\) then we can canonically make \(Q\) act on \((X_1, \ldots, X_n)\).

This effects a welcome simplification in the theory. Of our examples, 2 and 3 are in normal form, but 1 is not.

Example 4: (This example is due to J. M. Boardman)

\(Q_n\), an \(M^1 TP\)-category operating on the \(n\)-th loop space

\(\Omega^n Y = X\). The space \(\Omega^n Y\) is the space of all maps

\((I^n, \partial I^n) \to (Y, o)\), where \(o\) is the basepoint of \(Y\), \(I^n\) is the standard \(n\)-cube, and \(\partial I^n\) its boundary. A point \(\alpha \in Q_n(k, 1)\) ordered

where \(k\) is the unique sequence of length \(k\), is an collection \(\alpha\) of \(k\) \(n\)-cubes \(I^n_i\) linearly embedded in \(I^n\) with their axes parallel to those of \(I^n\), having disjoint interiors. It acts on \(\Omega^n Y\) as follows: Given \((f_1, \ldots, f_k) \in X^k\), i.e. maps \(f_i: I^n \to Y\), we construct the map

\(\alpha(f_1, \ldots, f_k): I^n \to Y\)

by using \(f_i\) on the little cube \(I^n_i\) and the zero map outside the little cubes. We topologize \(Q_n(k, 1)\) as a subspace of \(R^{2kn}\). The permutations permute the coordinates of \(X^k\). \(Q(k, 1)\) is now obtained by the normal form construction.

We observe that \(Q(k, 1)\) is \((n-2)\)-connected. We will make use of this fact in Chapter IV.
Example 5: Let $\mathcal{B}$ be an $M^1TP$ (resp. $M^1T$)-category in normal form and $\mathcal{C}$ a topological category with $n$ objects $0, \ldots, n-1$ (in a topological category the morphism sets are topological spaces and composition is continuous). Then $\mathcal{B}$ and $\mathcal{C}$ give rise to an $M^nTP$ (resp. $M^nT$)-category $B^*C$. Denote the unique sequence of length $m$ in $\mathcal{B}$ by $m$. A morphism from $a = (i_1, \ldots, i_k)$ to $j$ is a $(k+1)$-tuple $(\beta; f_1, \ldots, f_k)$ with $\beta \in B(k,1)$ and $f_p \in C(i_p, j)$. $B^*C(\emptyset, j) = B(\emptyset, 1)$ where $\emptyset$ denotes the empty sequence. Denote the morphisms of $B^*C(\emptyset, j)$ by $(\beta; \emptyset)$). Give $B^*C(a, j)$ the product topology of $B(k,1) \times C(i_1, j) \times \ldots \times C(i_k, j)$. Define composition with permutations on the right by

$$(\beta; f_1, \ldots, f_k) \circ \xi = (\beta \circ \xi; f_{\xi(1)}, \ldots, f_{\xi(k)}), \quad \xi \in S(k).$$

Define the morphisms into longer sequences by the normal form construction. Composition is given by

$$(\beta; f_1, \ldots, f_k) \circ (x_1 \oplus \ldots \oplus x_k) = (\beta \circ (x_1 \oplus \ldots \oplus x_k); f_1 \circ g_1, \ldots, f_1 \circ g_p, \ldots, f_k \circ g_k, \ldots, f_k \circ g_k)$$

where $x_i = (\beta_i; g_{i1}, \ldots, g_{ip})$, with the convention that $f_i \circ \emptyset$ drops out.

The composition is continuous and since it is induced by the compositions in $B$ and $C$ it is associative.

$(1_k; 1_{i_1}, \ldots, 1_{i_k})$ serves as identity and $\emptyset$ is a bifunctor.
by construction. Hence $B^*C$ is an $\mathcal{M}^\mathcal{M}_\text{TP}$-category.

Note intuitively that if we denote $(\beta;1_j,\ldots,1_j)$ by $(\beta;j)$ and $(1_j;f)$ by $f$, we have $(\beta;f_1,\ldots,f_k) = (\beta;j)^{o}(f_1 \circ \cdots \circ f_k)$.

Observe that if $(B(1,1),1_1)$ and $(C(k,k),1_k)$ are NDR-pairs, $1 \leq k \leq n$, then the $(B^*C(a,a),1_a)$ are NDR-pairs too for all sequences $a$. This follows from the fact that $(B^*C(j,j),1_j)$ is a NDR-pair for all $j$, $1 \leq j \leq n$ and $B^*C$ is in normal form. See also [6; Lemma 7.3].

We have $n$ canonical $\mathcal{M}^\mathcal{M}-\text{inclusion functors}$

\[ t_p : B \to B^*C \]

given by $t_p[(\beta_1 \circ \cdots \circ \beta_k)^{o} \xi] = [(\beta_1;P) \circ \cdots \circ (\beta_k;P)]^{o} \xi$ and a topological (i.e. continuous) inclusion functor

\[ \Lambda : C \to B^*C \]

given by $\Lambda(f) = f$.

All functors embed the respective categories as closed (in the topological sense) subcategories in $B^*C$. Hence their images have the relative topology in $B^*C$.

The construction in the $\mathcal{M}^\mathcal{M}_\text{TP}$-case is completely analogous.

For illustration: If $A$ is the category of Example 2 then an action of $A^*C$ induces a functor from $C$ into the
category of topological monoids. ]]

Let \( I_{\mathbb{S}} \) be the category with \( n \) objects \( 0, \ldots, n-1 \) and exactly one morphism between any two objects.

**Lemma 1.7:** Any \( M^n \mathbb{T} \)-category \( B \) in normal form is augmented over \( S^* I_{\mathbb{S}} \) (resp. \( A^* I_{\mathbb{S}} \)) by a (necessarily) unique \( M^n \mathbb{T} \)-functor \( \delta: B \rightarrow S^* I_{\mathbb{S}} \) (resp. \( A^* I_{\mathbb{S}} \)).

**Proof:** There exists exactly one morphism from \( a = (i_1, \ldots, i_k) \) to \( j \) in \( S^* I_{\mathbb{S}} \) uniquely represented by \((\lambda_k; (i_1, j), \ldots, (i_k, j))\) where \( \lambda_k \) is the unique function \( (1, \ldots, k) \rightarrow (1) \) in \( S \) and \((i, j)\) the unique morphism from \( i \) to \( j \) in \( I_{\mathbb{S}} \). This determines \( \delta \) uniquely on \( B(a, j) \). Using the normal form of \( B \) we get a necessarily unique extension of \( \delta \) to \( B \). That \( \delta \) is a functor follows again from the fact that there is exactly one morphism from \( a \) to \( j \) in \( S^* I_{\mathbb{S}} \). ]]}
CHAPTER II: THE UNIVERSAL CONSTRUCTION

Unless otherwise stated we only consider categories in normal form from now on.

The concept of monoid is not a good one from the point of view of homotopy theory, because the existence of a monoid structure on a space is not a homotopy invariant. For example, the loop space $\Omega X$ has no natural monoid structure, although it is a deformation retract of a natural monoid. Similarly for other categories of operators. For this reason we look for a "universal" structure.

Suppose given an $\mathcal{M}^{\mathcal{N}_{TP}} (\mathcal{M}^{\mathcal{N}_{T}})$-category $\mathcal{B}$. We want to construct a "universal" $\mathcal{M}^{\mathcal{N}_{TP}} (\mathcal{M}^{\mathcal{N}_{T}})$-category $\mathcal{U}$ with the following properties:

(U1) There exists an $\mathcal{M}^{\mathcal{N}_{TP}} (\mathcal{M}^{\mathcal{N}_{T}})$-functor $\varepsilon: \mathcal{U} \to \mathcal{B}$, the standard augmentation of $\mathcal{B}$, and a collection $\iota$ of equivariant maps (not functors) $\iota: \mathcal{B}(\mathbf{a}, \mathbf{b}) \to \mathcal{U}(\mathbf{a}, \mathbf{b})$ for all sequences $\mathbf{a}$ and all generators $\mathbf{b}$, the standard section of $\mathcal{B}$, such that

$\varepsilon \circ \iota \upharpoonright \mathcal{B}(\mathbf{a}, \mathbf{b}) = 1 \upharpoonright \mathcal{B}(\mathbf{a}, \mathbf{b})$

$\iota \circ \varepsilon \upharpoonright \mathcal{U}(\mathbf{a}, \mathbf{b}) \simeq 1 \upharpoonright \mathcal{U}(\mathbf{a}, \mathbf{b})$ equivariantly (if $\mathcal{B}$ has permutations) and fibrewise

(U2) $(\mathcal{U}, \varepsilon, \iota)$ is universal with respect to (U1), i.e.
given an $M^nTP (M^nT)$-category $C$, an $M^nTP (M^nT)$-functor $\delta: C \to B$ and a collection $\sigma$ of equivariant maps $\sigma: B(a,b) \to C(a,b)$ for all sequences $a$ and all generators $b$ such that

\[ \delta \circ \sigma |_{B(a,b)} = 1 |_{B(a,b)} \]
\[ \sigma \circ \delta |_{C(a,b)} = 1 |_{C(a,b)} \]

equivariantly and fibre-wise, then there exists an $M^nTP (M^nT)$-functor $v: U \to C$ such that $\delta \circ v = e$.

Notation: A collection of maps $\sigma$ as given in (U2) such that $\delta \circ \sigma |_{B(a,b)} = 1 |_{B(a,b)}$ is called a (equivariant)section of $\delta$.

A functor $\delta$ which has a section $\sigma$ satisfying the requirements of (U2) is called fibre homotopically trivial.

We are going to give a construction $W$ which associates with each $M^nTP (M^nT)$-category $B$ such that $(B(b,b),1_b)$ is a NDR-pair for all generators $b$, an $M^nTP (M^nT)$-category $WB$ together with an augmentation $e_B: WB \to B$ and a section $t_B: B \to WB$ such that the triple $(WB, e_B, t_B)$ satisfies (U1). Furthermore for any $M^nTP (M^nT)$-category $B$ we can find a triple $(B', e_B', t_B')$ such that $WB'$ exists and $(WB', e_B \circ e_B', t_B \circ t_B')$ satisfies (U1) and (U2).
§ 2 THE CONSTRUCTION \( W \)

Since the construction for the \( M^n_T \)-case is completely analogous to the one for the \( M^n_{TP} \)-case, rather more simple in fact, we restrict ourselves to the \( MTP \)-case.

Let \( B \) be an \( M^n_{TP} \)-category such that \( (B(b,b), 1_b) \) is a NDR-pair for all generators \( b \). To obtain the universal property we start off the free \( M^n_{TP} \)-category in the discrete topology over \( B \). We then topologize the morphism sets and attach "cells" to them to obtain the property \((U1)\).

We form a bar construction by considering words

\[ [a_0|\ldots|a_k] \]

where \( k \geq 0 \), each \( a_i \) is a morphism in \( B \), and the composite \( a_0 \circ \ldots \circ a_k \) exists in \( B \).

**Definition 2.1:** The category \( W^0B \) has as morphisms from \( a \) to \( b \) those words \( [a_0|\ldots|a_k] \) for which \( a_0 \circ \ldots \circ a_k \) is a morphism in \( B \) from \( a \) to \( b \) subject to the following relations and their consequences:

(a) \( [a \oplus \beta] = [a \oplus 1|1 \oplus \beta] = [1 \oplus \beta|a \oplus 1] \)

(b) \( [1] \) is an identity

(c) \( [\alpha|\xi] = [\alpha \circ \xi], [\xi|\beta] = [\xi \circ \beta] \) (if \( B \) has permutations).

Composition in \( W^0B \) is by juxtaposition.
Let us give an alternative pictorial description of $\mathbb{W}_1^0 B$. A morphism in $\mathbb{W}_1^0 B(a, b)$ is represented by a pair $(\theta, \xi)$ where $\xi \in S(r)$, $r$ being the length of $a$, and $\theta$ a finite tree in the plane with directed edges, labelled by $0, \ldots, n-1$, repetition is allowed, except that some edges do not join two vertices (see picture below). There is just one, called the root, labelled by $b$, that leaves a vertex and goes nowhere, and exactly $r$ twigs labelled by $i_{\xi-1}(1), \ldots, i_{\xi-1}(r)$ if $a = (i_1, \ldots, i_r)$, that come from nowhere. The other edges are called links and join two vertices. Each vertex has exactly one outgoing edge and the vertex is labelled by a morphism in $B(p, q)$ where $q$ is the label of the outgoing edge and $p = (j_1, \ldots, j_k)$ where $k$ is the number of incoming edges and $j_1, \ldots, j_k$ their labels from left to right.

Call the tree with no vertex consisting of a labelled edge only, a trivial tree.

The relation can now be described as follows:

(2.2) any vertex labelled by $1_b \in B(b, b)$ may be suppressed

(2.3) if we obtain the tree $\theta$ by substituting a vertex
\[ a \in \mathcal{B}(p, q) \] of the tree \( \varphi \) by the vertex \( a \circ \eta \), where \( \eta \) is a permutation, permute the incoming edges of \( a \) and the subtrees of \( \varphi \) sitting on them in such a fashion that the \( \eta(i) \)-th incoming edge of \( a \) is the \( i \)-th incoming edge of \( a \circ \eta \), then

\[
(\varphi, \xi) \sim (\theta, \eta^{-1}(r_1, \ldots, r_k) \circ \xi)
\]

where \( r_i \) is the number of twigs of the subtree of \( \varphi \) over the \( i \)-th incoming edge of \( a \).

Define composition with permutations on the right by

\[
(\theta, \xi) \circ \zeta = (\theta, \xi \circ \zeta)
\]

Now the sets \( \mathcal{B}^0(a, b) \) can be obtained by the normal form construction. A morphism of \( \mathcal{B}^0(a, b) \) with \( a = (i_1, \ldots, i_r) \) and \( b = (j_1, \ldots, j_s) \) is represented by a pair \((\theta, \xi)\) where \( \xi \in S(\{r\}) \) and \( \theta \) is an ordered collection of \( s \) such trees, called a **copse**, the twigs of this collection being labelled by \( i_{\xi^{-1}(1)}, \ldots, i_{\xi^{-1}(r)} \) in order (always from left to right) and the roots by \( j_1, \ldots, j_s \), again subject to the relation (2.2) and a generalized version of (2.3): Let \( \varphi \) be the copse obtained from the copse \( \theta \) by changing the tree the twigs of which are labelled by \( i_{\xi^{-1}(t+1)}, \ldots, i_{\xi^{-1}(t+q)} \) according to (2.3), and let \( e \in S(t) \) and \( e' \in S(r-t-q) \) be the identities, then
Composition $\theta \circ \phi$ of copses $\theta, \phi$ is obtained by attaching the roots of $\phi$ to the twigs of $\theta$ (see picture below). Since the roots of $\phi$ are labelled in the same way as the twigs of $\theta$, $\theta \circ \phi$ is a well defined copse. The sum $\theta \oplus \psi$ of the two copses $\theta$ and $\psi$ is obtained by putting them side by side, the trees of $\theta$ followed by the trees of $\psi$.

Let $\xi \in S(r)$ and let $\phi$ be a copse with $r$ trees. Let $\xi \circ \phi$ be the copse with the $j$-th tree being the $\xi^{-1}(j)$-th tree of $\phi$. Sum and composition in $\mathcal{W}_0^B$ are now given by

\[
(\theta, \xi) \oplus (\phi, \eta) = (\theta \oplus \phi, \xi \oplus \eta)
\]

\[
(\theta, \xi) \circ (\phi, \eta) = (\theta \circ (\xi \circ \phi), \xi(\eta_1, \ldots, \eta_q))
\]

where $\eta_q$ is the number of twigs of the $q$-th tree in $\phi$. 
If the edges of two copses are labelled in the same way they are said to have the same **type**. Give the set of all copses of one type the product topology of their vertex spaces, and give the set of all copses with a given source and target the union topology of the union of all types with the given source and target. The trivial copses, i.e. the copses consisting of trivial trees only, are their own open and closed components. Composition of copses is continuous, associative and the the trivial copses act as identities. θ is continuous, associative, and a bifunctor. Hence, disregarding all relations, the copses over B form an $M^n_T$-category, if we just consider the copses and leave the permutations out. Disregarding all relations, the spaces of all pairs (the topology is induced by the copse component) form an $M^n_{TP}$-category. Including the relations they give rise to the $M^n_{TP}$-category $\mathcal{W}^0B$.

In the $M^n_T$-case we would continue to work with the category of copses over B, while the $M^n_{TP}$-case requires the slightly more complicated category of pairs.

Let α and β be vertices of a copse $\theta$ joint by the j-th incoming edge of $\alpha$. Let $\alpha$ and $\beta$ have n resp. m incoming edges. Shrinking the link between $\alpha$ and $\beta$ means substituting the subtree of $\theta$ consisting of $\alpha$ and $\beta$ and their
and their edges by a vertex $\gamma = a^0(1 \otimes \beta \otimes 1)$ with $m+n-1$ incoming edges (see picture).

Let $\mathcal{T}_a(\alpha, \beta) = \{(\alpha, \xi) | \xi \in S(k), \text{ where } k = \text{length } \alpha, \theta \text{ a copse with target } \beta \text{ and source } \xi \alpha \text{ (the source of } \theta \text{ is given by the labels of its twigs)}\}$ topologized by the topology inherited from the copses. Each type of copses with target $\beta$ and source some $\eta \alpha, \eta \in S(k)$, determines an open and closed subset of $\mathcal{T}_a(\alpha, \beta)$, called a component.

Index the edges of a copse by $0, 1, \ldots, k, \ldots$ starting from the root of the first tree and going up the most left sequence of edges. Continue going upwards the next sequence of edges to the right, and continue (see picture). Call this the standard indexing.
Let $T_P(B(a,b)) \subset TB(a,b)$ be the subspace of those elements the copses of which have exactly $p$ links. Since $T_P(B(a,b))$ is a collection of components of $TB(a,b)$ it is closed in $TB(a,b)$ and hence has the relative topology.

If $i$ indexes the $i$-th link of the copse $\Theta$ in the standard indexing and $d^i\Theta$ is the copse obtained from $\Theta$ by shrinking the $i$-th link, the correspondence

$$(\Theta, \xi) \mapsto (d^i\Theta, \xi), \quad (\Theta, \xi) \in T_P(B(a,b)),$$

defines a continuous map, called a face map,

$$d^i : K \to T_{P-1}(B(a,b))$$

where $K$ is a component such that the $i$-th edge is a link.

Let $s^i\Theta$ be the copse obtained from the copse $\Theta$ by inserting the vertex $1_c$ in the $i$-th edge where $c$ is the label of this edge. The correspondence

$$(\Theta, \xi) \mapsto (s^i\Theta, \xi)$$

defines a continuous map, called a degeneracy map,

$$s^i : T_P(B(a,b)) \to T_{P+1}(B(a,b))$$

Call $x \in T_P(B(a,b))$ degenerate if it is in the image of some degeneracy map.

Following identities hold whenever the maps involved are defined:

$$(2.4) \quad d^j \circ d^i = d^i \circ d^{j+1}, \quad j \geq i$$

$$(2.5) \quad s^{j+1} \circ s^i = s^i \circ s^j, \quad j \geq i$$
(2.6) \( d^j \circ s^i = 1 \) \( j = i, i+1 \)
(2.7) \( s^i \circ d^j = d^{j+1} \circ s^i \) \( i < j \)
\[= d^j \circ s^{i+1} \] \( i \geq j \)

For the time being we restrict ourselves to the case where \( b = b \), i.e. \( b \) is a sequence of length 1.

**Lemma 2.8:** Each \( x \in TpR(a,b) \) can be written uniquely as
\[x = s^m \circ ... \circ s^1 y\]

where \( k_1 < ... < k_m \) and \( y \) is not degenerate.

**Proof:** \( y \) is uniquely determined by deleting all vertices labelled by an identity from the tree of \( x \). Hence \( x \) is obtained from \( y \) by inserting identities, i.e. by applying degeneracy maps. By (2.5) we can choose them uniquely in the required fashion. ]]

Let \( T = (R,M,Q) \) be a gadget consisting of a topological monoid \( M \) with multiplication \( \ast \) and unit \( e \), a closed right "ideal" \( R \) and a closed left "ideal" \( Q \), i.e. closed subspaces \( R \) and \( Q \) of \( M \) such that \( R \ast M \subset R \) and \( M \ast Q \subset Q \), satisfying

**Axiom M1:** (i) There exist no inverses in \( M \), i.e. if \( x \ast y = e \) then \( x = y = e \).
(ii) There exist \( r_0 \in R \) and \( t_0 \in Q \) such that 
\( r_0 \ast t_0 \) is a right identity in \( Q \) and a left identity in \( R \). And \( r_0 \ast t_0 \neq e \).

(iii) \((M, e)\) is a NDR-pair.

Put \( u = r_0 \ast t_0 \). Then \( u \ast u = u \). Hence without loss of generality we can assume that \( r_0 = t_0 = u \). Since \( R \ast M \subset R \) and \( M \ast Q \subset Q \), \( R \ast Q \subset R \cap Q \). Also if \( x \in R \cap Q \), then \( x = x \ast u \in R \ast Q \), and hence \( R \cap Q \subset R \ast Q \).

Example 2.9: Let \( M \) be the unit interval with the multiplication \( t_1 \ast t_2 = \max(t_1, t_2) \). Then \( e \) is \( 0 \in \mathbb{I} \). Take \( Q = R = \mathbb{I} \), and \( u = 1 \). Then \((M1)\) is satisfied.

More examples will come in some later section.

Define maps \( s^i: M^p \to M^{p-1} \) and \( d^i: M^p \to M^{p+1} \), called face and degeneracy maps, by

\[
s^i(t_0, \ldots, t_{p-1}) = (t_0, \ldots, t_{i-1}, t_i \ast t_{i+1}, t_{i+2}, \ldots, t_{p-1})
\]

\[
d^i(t_0, \ldots, t_{p-1}) = (t_0, \ldots, t_{i-1}, e, t_i, \ldots, t_{p-1}).
\]

The maps satisfy following identities:

\[
(2.4') \quad d^i \circ d^j = d^{j+1} \circ d^i \quad j > i
\]

\[
(2.5') \quad s^i \circ s^{j+1} = s^{j+1} \circ s^i \quad j > i
\]

\[
(2.6') \quad s^i \circ d^j = 1 \quad j = i, i+1
\]

\[
(2.7') \quad d^j \circ s^i = s^i \circ d^{j+1} \quad i < j
\]

\[
= s^{i+1} \circ d^j \quad i \geq j
\]
Call $\delta \in M^p$ degenerate if it is in the image of some degeneracy map.

**Lemma 2.10:** Each $\delta \in M^p$ can be expressed uniquely as

$$\delta = d^n \circ \ldots \circ d^1 \delta$$

where $1_1 < \ldots < 1_n$ and $\delta$ is not degenerate.

**Proof:** $\delta$ is uniquely determined by deleting the coordinates $e$ of $\delta = (t_1, \ldots, t_p)$. $\delta$ is then obtained from $\delta$ by applying degeneracy maps. (2.4') allows us to choose them in the required fashion.

Let $(\theta, \zeta) \in T_p(a, b)$. To each link of $\theta$ we assign an element of $M$, to each twig an element of $Q$, and to the root an element of $R$. In the case of a trivial tree root and twig coincide and we assign to it an element of $R \cap Q$. The elements of $T_p(a, b)$, together with all possible assignments of this form give rise to a topological space

$$C_p^B(a, b) = T_pB(a, b) \times (R \times M^p \times Q^k),$$

where $k = \text{length } a$, $p \geq 0$. Let $T^{-1}B(b, b)$ be the space consisting of the trivial tree with the edge labelled by $b$. Define $C_{-1}^B(a, b) = \emptyset$ if $a \neq b$ and $C_{-1}^B(b, b) = T^{-1}B(b, b) \times (R \cap Q) = R \cap \zeta$. For convenience we denote $C_p^B(a, b)$ simply by $C_p(a, b)$. 
Let \( V_p(a, b) = \bigcup_{p \in \mathbb{Z}} C_p(a, b) \) be the disjoint union and \( V(a, b) = \bigcup_{p \in \mathbb{Z}} C_p(a, b) \) the disjoint union of all \( C_p(a, b) \), \( p = -1, 0, 1, \ldots \). The \( C_p(a, b) \)'s are in \( CG \), and hence the \( V_p(a, b) \)'s and \( V(a, b) \) are in \( CG \) because they are Hausdorff [6; Lemma 9.2].

Introduce the following relations in \( V(a, b) \):

\[(2.11) \text{ Each } x \in C_p(a, b) \text{ is given by a pair } (\theta, \xi) \in T_p(a, b) \text{ with an element of } M \text{ assigned to each edge. Let } y \in C_p(a, b) \text{ be obtained from } x \text{ by changing } (\theta, \xi) \text{ to } (\varphi, \eta) \text{ according to relation } (2.3) \text{. The elements of } M \text{ assigned to each edge of } \varphi \text{ are given by carrying the elements of } M \text{ assigned to the edges of } \theta \text{ along during the permutation of edges which defines } \varphi. \text{ Then } x \sim y.\]

\[(2.12) (d^i_x, \delta) \sim (x, d^i\delta), \text{ i indexes a link in } x = (\theta, \xi)\]

\[(2.13) (s^i_x, \delta) \sim (x, s^i\delta)\]

where \( (x, d^i\delta), (x, s^i\delta) \in C_p(a, b) \).

Note that if \( i \) indexes a link then \( (d^i_x, \delta) \in C_{p-1}(a, b) \) iff \( (x, d^i\delta) \in C_p(a, b) \), and \( (s^i_x, \delta) \in C_{p+1}(a, b) \) iff \( (x, s^i\delta) \in C_p(a, b) \) since \( R \) is a right ideal and \( Q \) a left ideal of \( M \).
Call a point \((x, \delta) \in \mathcal{C}_p(a, b)\) **degenerate** if \(x\) or \(\delta\) are degenerate.

**Lemma 2.14:** Each \((x, \delta) \in V(a, b)\) is related under (2.12) and (2.13) to a unique non-degenerate point.

**Proof:** Let \(x = s^m \circ \ldots \circ s^1 y\) be the unique expression for \(x\) given in Lemma 2.8. Define a function \(\lambda\) by

\[
\lambda(x, \delta) = (y, s^1 \circ \ldots \circ s^m \delta)
\]

Define a function \(\rho\) by setting \(\delta = d^n \circ \ldots \circ d^1 \delta\) as uniquely given in Lemma 2.10. Define

\[
\rho(x, \delta) = (d^n \circ \ldots \circ d^1 x, \delta).
\]

By Axiom M1 (ii), \(e \in M\) cannot be assigned to a root or a twig since \(M\) does not have any inverses. Hence \(l_1, \ldots, l_n\) index links in \(x\) and \(d^1 \circ \ldots \circ d^n x\) is defined.

\(\lambda \rho(x, \delta)\) is not degenerate since \(\delta\) is not degenerate and hence since \(M\) does not have inverses \(s^i \delta\) is not degenerate for any \(i\).

It is easily seen that \(\lambda \rho(x_1, \delta_1) = \lambda \rho(x_2, \delta_2)\) if \((x_1, \delta_1)\) and \((x_2, \delta_2)\) are related under (2.12) and (2.13). Since \(\lambda \rho\) is the identity on non-degenerate points, \(\lambda \rho(x, \delta)\) is independent of the choice of \((x, \delta)\) in its equivalence class.
Let $W(a,b) = WFR(a,b)$ be obtained from $V(a,b)$ by factoring out the relations (2.11), (2.12), and (2.13), and $W(a,b)$ by factoring out the relations (2.12) and (2.13) only. Let $\bar{W}_p(a,b) = \pi(V_p(a,b))$, where 
\[ \pi: V(a,b) \to W(a,b) \] is the projection, and let $\pi_p = \pi|V_p$.

Lemma 2.15: (a) $\bar{W}_p(a,b)$ and $WFR(a,b)$ are in $CG$.  
(b) $WFR(a,b)$ has the limit topology from 
\[ W_{-1}FR(a,b) \subset \ldots \subset W_pFR(a,b) \subset \ldots \]  
(c) $(W_pFR(a,b), W_{p-1}FR(a,b))$ and 
\[ (WFR(a,b), W_pFR(a,b)) \] are NDR-pairs for all $p$.  
(d) $(WFR(b,b), 1_b)$ are NDR-pairs if $(RnQ, r_0 * t_0)$ is a NDR-pair.

Since we are required to prove similar statements to those of Lemma 2.15 in the further development of our theory we analyse the general problem before we prove 2.15.

We are given a space $X$ which is a disjoint union of spaces
\[ X_0 \cup X_1 \cup X_2 \cup \ldots \] 
and an equivalence relation $\sim$ on $X$. Let $Y = X/\sim$ and $Y_n = (X_0 \cup \ldots \cup X_n)/\sim$, and let
\[ \pi_n: (X_0 \cup \ldots \cup X_n) \to Y_n \] be the projection. Put
\[ DX_n = \{ x \in X_n \mid \text{There exists } y \in X_i, i < n, \text{ such that } y \sim x \}. \]

We suppose:
(1) satisfies: if \( x, y \in X_n - DX_n \) and \( x \sim y \) then \( x = y \).

(2) \( DX_n \) is a finite union of closed subspaces \( F_r \) and we are given continuous maps \( f_r: F_r \to X_{i_r} \) such that
\[
x \sim f_r(x) \quad \text{for all} \quad x \in F_r \quad \text{and} \quad i_r < n \quad \text{for all} \quad r.
\]

(3) \( f: DX_n \to Y_{n-1} \) given by \( f|_{F_r} = \pi_{n-1} \circ f_r \) is well defined.

Then \( Y_n \) is obtained from \( Y_{n-1} \) by attaching \( X_n \) to \( Y_{n-1} \) by the attaching map \( f: DX_n \to Y_{n-1} \), and \( Y \) is the direct limit of \( Y_0 \subset Y_1 \subset \ldots \).

Now we assume further:

(4) \((X_n, DX_n)\) are NDR-pairs for all \( n \) and \( Y_0, X_n \) are in CG for all \( n \).

Then by induction \((Y_n, Y_{n-1})\) are NDR-pairs for all \( n \) [6; Lemma 8.5] and hence \( Y \) is in CG and \((Y, Y_n)\) are NDR-pairs for all \( n \) [6; Theorem 9.4 and Lemma 9.2].

**Proof of Lemma 2.15:** Let \( X_p = C_p(a, b) \) and \( \sim \) the equivalence relation generated by (2.12) and (2.13). Hence \( Y_p = \overline{W_p}(a, b) \). \( DX_p \) is the space of all degenerate points of \( X_p \), i.e. of those points \((x, \delta) = (\theta, \xi, \delta)\) where a vertex of \( \theta \) is labelled by an identity or a coordinate of \( \delta \) has the value \( e \). By Lemma 2.14 two non-degenerate points cannot be related, and hence (1) holds.
Let $F_1$ and $G_1$ be the (closed) subspaces of $D_X$ consisting of those points $(x,\delta)$, $x = (\theta, \xi)$, where the vertex on top of the $i$-th link of $\theta$ is labelled by an identity resp. the $i$-th coordinate of $\xi$ has the value $e$. The maps $f_i: F_i \to X_{p-1}$ and $g_i: G_i \to X_{p-1}$, given by

$$f_i(x,\delta) = (y, s^i \delta) \quad \text{and} \quad g_i(x,\delta) = (d^i x, \delta)$$

where $y$ and $\delta$ are the unique elements such that $x = s^i y$ and $\delta = d^i \delta$, are continuous (since $s^i$ and $d^i$ are continuous and since $x \to y$ and $\delta \to \delta$ are given by projections). $f: D_X \to Y_{p-1}$, given by $f|_{F_i} = x_{p-1} f_i$ and $f|_{G_i} = x_{p-1} g_i$ is well defined by Lemma 2.14. Hence (2) and (3) hold.

Since $X_\alpha$ is the disjoint union of products arising from the different types of trees, and since $(M, e)$ and $(\mathbb{E}(b, b), 1_b)$ are NDR-pairs for all generators $b$, $(X_p, D_X)$ is an NDR-pair for all $p$ [6; Lemma 7.3]. Hence (4) holds.

Observe that $X_\alpha = Y_\alpha$ and we will show later that $r_\alpha s_\alpha \in R * Q = \mathbb{W}_{-1}(b, b) \subset \mathbb{W}(b, b)$ will serve as identity. Hence Lemma 2.15 follows from our general consideration.

For part (d) use [6; Lemma 7.2].

**Lemma 2.16:** If $(x, \delta) \sim (y, \delta)$ under (2.11), then

$$\lambda_p(x, \delta) \sim \lambda_p(y, \delta) \quad \text{under (2.11), where } \lambda_p \text{ is the function constructed in the proof of Lemma 2.14.}$$
Proof: Picturing each element as a pair \((\theta, \xi)\) with elements of \(M\) assigned to each edge, the proof is immediate since (2.11) commutes with the shrinking of links and the deleting of vertices labelled by an identity. ]]

Corollary 2.17: \(W_{\Phi}(a,b) = W_{\Phi}(a,b)/\sim\), where \(\sim\) is the equivalence relation generated by (2.11) applied to non-degenerate points only. ]]

Corollary 2.18: \(W_{\Phi}(a,b)\) has the limit topology from 
\[W_{\Phi}(a,b) = W_{\Phi}(a,b)/(2.11)\] and is in \(CG\).

Proof: \(W(a,b)\) is Hausdorff since \(W(a,b)\) is. Hence it is in \(CG\). If \(q: W(a,b) \to W(a,b)\) is the projection then 
\[q^{-1} \circ q(W_p(a,b)) = W_p(a,b)\]. The corollary now follows from 
[6;Theorem 9.5]. ]]

Let \((x,\delta) = (\theta, \xi, \delta) \in C_p(a,b)\) be a representative of an element in \(W(a,b)\). Define composition with permutations on the right by 
\[(\theta, \xi, \delta) \circ \zeta = (\theta, \xi \circ \zeta, \delta)\]
This defines a continuous composition with permutations in \(W(a,b)\).
Define the spaces of morphisms into longer sequences by the normal form construction. We can give an alternative description along the lines of copses. We have reduced our construction to trees because the trivial copses would have made the argument somewhat unclean. Let $a = (i_1, \ldots, i_k)$ and $b = (j_1, \ldots, j_l)$. Let $T^{-1}(a, a)$ denote the one-point space of the trivial copse from $a$ to $a$. To each link of $\theta$ in $(\theta, \xi) \in T \rho(a, b)$ assign an element of $M$, to each twig an element of $Q$, and to each root an element of $R$, thus constructing spaces $C_\rho(a, b)$. In case $\theta$ contains a trivial tree, assign to its only edge an element of $R \cap Q$.

Introduce in

$$V(a, b) = C_{-1}(a, b) \cup C_0(a, b) \cup C_1(a, b) \cup \ldots$$

the product relations from (2.12) and (2.13) and denote the quotient of $V(a, b)$ under these relations by $W(a, b)$. Applying our previous considerations to each tree individually we again find that each element of $W(a, b)$ is uniquely represented by a non-degenerate triple $(\theta, \xi, \delta)$. Here $(\theta, \xi)$ is called degenerate if $\theta$ contains a degenerate tree, while the definition for $\delta$ being degenerate is the old one. Let

$$W'(a, b) = W(a, b)/\sim$$

where the equivalence relation is generated as follows:
Let $\theta = \theta_1 \oplus \ldots \oplus \theta_1$ such that $\theta_i$ is a non-degenerate tree. An element of $M$ is assigned to each edge of $\theta_i$, thus giving rise to a representative $(\theta_i, \text{unit}, \delta_i)$ of $W(a, j_i)$. Let $(\varphi_i, \xi_i, \delta_i) \sim (\theta_i, \text{unit}, \delta_i)$ under (2.11). Then

$$(2.19) \ (\theta, \xi, \delta_1 \times \ldots \times \delta_1) \sim (\varphi_1 \oplus \ldots \oplus \varphi_1, (\xi_1 \oplus \ldots \oplus \xi_1)^o \xi, \delta_1 \times \ldots \times \delta_1)$$

Relation (2.19) can be formulated for any triple $(\theta, \xi, \delta) \in C_p(a, b)$ and as in the previous case $W'(a, b)$ is obtained from $V(a, b)$ by factoring out the relations (2.12), (2.13), and (2.19).

**Lemma 2.20:** $WRG(a, b) \cong W'R(a, b)$

**Proof:** Let $x_p = (\theta_p, \xi_p, \delta_p) \in V(a_p, j_p)$. Define

$h: W(a, b) \to W'(a, b)$ and $k: W'(a, b) \to W(a, b)$

by $h(x_1, \ldots, x_n; \xi) = \{\theta_1 \oplus \ldots \oplus \theta_n, (\xi_1 \oplus \ldots \oplus \xi_n)^o \xi, \delta_1 \times \ldots \times \delta_n\}$

$k(\theta_1 \oplus \ldots \oplus \theta_n, \xi, \delta_1 \times \ldots \times \delta_n)$

$= \{(\theta_1, \text{unit}, \delta_1), \ldots, (\theta_n, \text{unit}, \delta_n); \xi\}$

Where $\{\}$ denotes the equivalence class. $h$ and $k$ are well defined and since they are continuous on representatives they are continuous. They are inverse to each others.

As a consequence of the lemma we find that

$\oplus: W(a, b) \times W(c, d) \to W(a \oplus c, b \oplus d)$
is given by
\[ \{ \theta, \xi, \delta \} \circ \{ \varphi, \eta, \vartheta \} = \{ \theta \circ \varphi, \xi \circ \eta, \delta \times \vartheta \} . \]
\( \circ \) is continuous since \( W(a,b) \) is obtained by the normal form construction.

Because of the normal form construction it suffices to define composition for \( \{ \theta, \text{unit}, \delta \} \circ \{ \varphi, \eta, \vartheta \} \) and prove the associativity for this case. \( \{ \theta, \text{unit}, \delta \} \circ \{ \varphi, \eta, \vartheta \} \) is represented by the pair \( (\theta \circ \varphi, \eta) \) to each link, root, or twig of which coming from \( \theta \) or \( \varphi \) we assign the value of \( M \) which it had in \( \theta \) or \( \varphi \), and to each new link of which we assign the product in \( M \) of the elements assigned to the original twig in \( \theta \) with the element assigned to the original root in \( \varphi \). Since the multiplication in \( M \) is associative, composition factors through (2.12), (2.13), and from the intuitive idea of a tree it is clear that it factors through (2.19). Since the multiplication in \( M \) and the composition of copses are continuous and associative, the composition in \( W \) is continuous and associative. The triple consisting of the copse of trivial trees with labelled edges \( i_1, \ldots, i_k \), the unit permutation, and the element \( u = r_o \circ t_o \) assigned to each tree acts as identity. And from the intuitive idea of copses it follows that \( \circ \) is a bifunctor (this also follows from the fact that \( W \) is obtained by the normal
form construction and that the definition of composition is extended to the whole of $\mathcal{W}$ using the normal form). Hence

**Theorem 2.21:** $\mathcal{WBr}$ is an $\mathcal{M}^{\mathcal{TP}}$-category in normal form.

Suppose the links of $(\emptyset, \text{unit}) \in T_p(a, b)$ are indexed by $i_1 < \ldots < i_p$ in the standard indexing. Let

$$\varepsilon(\emptyset, \text{unit}) = d_{i_1} \circ \ldots \circ d_{i_p}$$

which determines a unique element in $\mathcal{B}$, namely the label of the unique vertex of $d_{i_1} \circ \ldots \circ d_{i_p}$. Putting $\varepsilon(\emptyset, \text{unit}) = 1_b \in B$ if $\emptyset$ is the trivial tree from $b$ to $b$, the correspondence

$$\varepsilon \{\emptyset_1 \oplus \ldots \oplus \emptyset_q, \xi, \delta\} = [\varepsilon(\emptyset_1, \text{unit}) \oplus \ldots \oplus \varepsilon(\emptyset_q, \text{unit})] \circ \xi$$

defines a continuous map from $\mathcal{W}(a, b)$ to $\mathcal{B}(a, b)$. Here as always in future $\{ \}$ denotes the equivalence class of the element in question. Since the shrinking process is basically composition in $\mathcal{B}$, it is associative. Hence, since trivial trees are mapped to the corresponding identities, $\varepsilon$ is an object preserving continuous functor. By definition it preserves sums and permutations. Hence it is an $\mathcal{M}^{\mathcal{TP}}$-functor. Since the definition of $\varepsilon$ is independent of $\Gamma$ we denote it by $\varepsilon_B$. We call it the **standard augmentation** of $\mathcal{B}$. 
Remark 2.22: Let $\mathcal{B}, \mathcal{C}, \mathcal{D}$ be an $M^ TP^-$, $M^ TP^-$, $M^ kTP^-$ category respectively, and let $\gamma: \mathcal{B} \to \mathcal{C}$ and $\delta: \mathcal{C} \to \mathcal{D}$ be $MTP^-$ functors. Then $\gamma$ and $\delta$ determine canonical $MTP^-$ functors $W\gamma$ and $W\delta$ such that $\epsilon_C \circ W\gamma = \gamma \circ \epsilon_B$, similarly for $\delta$, and $W(\delta \circ \gamma) = W\delta \circ W\gamma$. (You construct $W\gamma$ by applying $\gamma$ to the vertex labels of each cospe).

Definition 2.23: A $CW-M^nTP$-category $\mathcal{B}$ is an $M^nTP^-$ category such that the morphism spaces are $CW$-complexes, composition and sum are skeletal, and the identities are vertices.

Theorem 2.24: If $\mathcal{B}$ is a $CW-M^nTP^-$ category and $\Gamma = (R,M,Q)$ satisfies in addition to Axiom M1 following conditions: $M$ is a $CW$-complex, $R$ and $Q$ are subcomplexes of $M$, $r_0, t_0$ and $e$ are vertices, and the multiplication is skeletal. Then $W\mathcal{B}\Gamma$ is a $CW-M^nTP^-$ category.

Proof: Since the morphism spaces of $\mathcal{B}$ are $CW$-complexes, $W\mathcal{B}\Gamma$ exists. Since products in $CG$ of $CW$-complexes are $CW$-complexes, $G_p(a,b)$ is a $CW$-complex for each $p$. Hence $W_0(a,b)$ resp. $W_{-1}(a,b)$ are $CW$-complexes. $T_p(a,b)$ has the
product cell structure from \( U_q \left[ (\Pi_r B(a_r, b_r)) \times (\xi_q) \right] \),

the vertex labels of its trees. Since the identities in \( B \) are vertices the degenerate points of \( T_p(a, b) \) form a

subcomplex. Analogously the degenerate points of \( M^p \) form

a subcomplex. Hence \( DC_p(a, b) \) is a subcomplex of \( C_p(a, b) \).

Let \((x, \delta) \in DC_p(a, b), \delta = d^i s \circ \ldots \circ d^1 \delta, \)

\( y = d^1 \circ \ldots \circ d^i x = s_{t^o} \ldots \circ s_{t^1} z, \)

as given by the Lemmas 2.8 and 2.10, and suppose that \( x \) is in the \( q \)-skeleton of

\( T_p(a, b) \) and \( \delta \) in the \( r \)-skeleton of \( R \times M^p \times Q^k \). Then \( \delta \) is

in the \( r \)-skeleton of \( R \times M^p \times Q^k \) since \( e \) is a vertex and

\( y \) is in the \( q \)-skeleton of \( T_p-s(a, b) \) since composition in

\( B \) is skeletal. Since the identities of \( B \) are vertices, \( z \)

is in the \( q \)-skeleton of \( T_p-s-t(a, b) \), and since multipli-

cation in \( M \) is skeletal, \( s_{t^1} \circ \ldots \circ s_{t^j} \delta \) is in the \( r \)-skeleton

of \( R \times M^p \times Q^k \). Hence the attaching maps of Lemma 2.15

are skeletal and hence \( W(a, b) \) is a CW-complex.

Composition with permutations is cellular in \( B \). Hence

the relation (2.11) induces a cellular identification in

\( W(a, b) \), and therefore \( W(a, b) \) is a CW-complex. Composition

with permutations on the right is cellular since it is so

in \( V(a, b) \). Therefore \( W(a, b) \) is a CW-complex. \( \oplus \) is skeletal

because \( W(a, b) \) is obtained by the normal form construction.
Composition is skeletal since it is induced by inclusions of factors into a product. Since \( r_0 \ast t_0 \) is a vertex, the identities are vertices.
§ 3 THE CONTRACTABILITY OF WBT OVER B

Define the standard section \( \iota: B \to \text{WBT} \) of the standard augmentation (see p.39) by

\[
\iota(\beta) = \{\theta, \text{unit}, \delta\}, \quad \beta \text{ a morphism into a generator}
\]

where \((\theta, \delta)\) is the tree with exactly one vertex which is labelled by \(\beta\), and \(t_o\) assigned to each twig and \(r_o\) to the root (see picture).

\[\text{\begin{diagram}
t_o & \rightarrow & t_o \\
1 & \sim & r_o^*t_o \\
r_o & \rightarrow & \end{diagram}}\]

\(\iota\) is equivariant and since

\[
\iota |_{t_o} \sim \iota |_{r_o}
\]

it preserves identities.

Now suppose that \(\Gamma\) satisfies following additional axiom:

**Axiom M2:** There exists a homotopy \(m_t: M \to M\) such that

\[
m_t(v_1) * m_t(v_2) = m_t(v_1 * v_2)
\]

\[
m_t(e) = e \quad \text{for all } t \in I
\]

\[
m_o = \text{id}_M \quad \text{and} \quad m_t(v) = e \quad \text{for all } v \in M
\]
Putting \( u = r_0 = t_0 = r_0 \ast t_0 \) (see p.28), \( m_t \) induces homotopies \( l_t: R \to R \) and \( k_t: Q \to Q \) given by \( l_t(x) = u^*m_t(x) \), and \( k_t(x) = m_t(x) \ast u \). Then \( l_0 = \text{id}_R \), \( k_0 = \text{id}_Q \), 
\( l_t(x) = u = r_0 \), and \( k_t(y) = u = t_0 \) for all \( x \in R \) and \( y \in Q \).
Furthermore \( l_t(x) \ast m_t(v) = l_t(x \ast v) \) and \( m_t(v) \ast k_t(y) = k_t(v \ast y) \) for all \( t \in I \), \( v \in M \), \( x \in R \), and \( y \in Q \). In addition we have \( l_t(x) \ast k_t(y) = l_t(x \ast y) = k_t(x \ast y) \).

Note that the monoid of example 2.9 satisfies Axiom M2 with the homotopy \( m_t(v) = t \cdot v \), \( v \in M = I \), with the ordinary multiplication on the right of the equation.

**Theorem 3.1:** If \( \Gamma \) satisfies the Axioms M1 and M2, then 
\[ e_B: W\Gamma \to B \] is fibre homotopically trivial (see p.19 for the definition).

**Proof:** We have to construct equivariant fibrewise homotopies \( H_t: u \circ e| W(a,b) \simeq \text{id}| W(a,b) \).

Define \( h_t: R \times M^p \times Q^k \to R \times M^p \times Q^k \) by
\[
h_t(x, v_1, \ldots, v_p, y_1, \ldots, y_k) = (l_t(x), m_t(v_1), \ldots, m_t(v_p), k_t(y_1), \ldots, k_t(y_k))
\]
for each \( p \) and \( k \). The \( h_t \)'s induce homotopies
\[
H_t = (1 \times h_t): C_p(a,b) = T_p(a,b) \times (R \times M^p \times Q^k) \to C_p(a,b),
\]
for each \( p \geq 0 \). For \( p = -1 \) define \( H_t: C_{-1}(a,a) \to C_{-1}(a,a) \)
by \( H_t(x) = \frac{u_t(x)}{1_t(x)} \). (Recall that \( C_{-t}(a,a) = R \cap Q \)). The collection of the \( H_t \)'s induces a homotopy
\[
H_t: V(a,b) \to V(a,b).
\]
\( H_t \) factors automatically through relation (2.11). Since \( m_t(e) = e \) for all \( t \) it factors through (2.12) and because of \( M2 \) and the properties of \( l_t \) and \( k_t \) it factors through (2.13). Hence it induces an equivariant and fibrewise homotopy
\[
H_t: W(a,b) \to W(a,b)
\]
such that \( H_0 = \text{id}_{W(a,b)} \) and \( H_1 = \iota \circ \varepsilon \mid W(a,b) \) (by the properties of \( l_t \) and \( k_t \) and the conditions on \( m_0 \) and \( m_1 \)).]

**Lemma 3.2:** Under the assumptions of Theorem 2.24,
\( \varepsilon \mid W(a,b) \) and \( \iota \mid B(a,b) \) are skeletal.

**Proof:** \( \varepsilon \) is induced by the projection
\[
C_p(a,b) = T_p(a,b) \times (R \times M^P \times Q^k) \to T_p(a,b)
\]
followed by the shrinking of all links. Hence since composition in \( B \) is skeletal, \( \varepsilon \) is skeletal. Since \( \iota \) is induced by the identity \( B(a,b) \to T'_1(a,b) \), where \( T'_1(a,b) \) is the subspace of \( T_1(a,b) \) of all pairs of the form \((\emptyset, \text{unit})\), it is skeletal.]

We now give some further examples of systems \( \Gamma = (R, M, Q) \).
Example 3.3: Let $M$ be the unit interval with multiplication $t_1 \ast t_2 = t_1 \cdot t_2$. Then $e = 1$. Take $Q = R = (0)$ and $u = r_0 = t_0 = 0$. Then $M_1$ is satisfied.

$M_2$ cannot be satisfied since $m_t(0 \cdot v) \neq m_t(0) \cdot m_t(v)$ necessarily since $m_t = 1$ is required for all $t \in I$.

Example 3.4: Let $M$ be an arbitrary topological monoid with an idempotent $u \neq e$. Put $R = u \ast M$ and $Q = M \ast u$, $r_0 = t_0 = u$. Then this data satisfies $M_1$.

Example 3.5: Let $K$ be the free topological monoid over $I$, the unit interval, ($K$ is the reduced product space $I_\infty$ in the sense of James), i.e.

$$K = I^0 \cup I^1 \cup I^2 \cup \ldots /$$

where the equivalence relation is given by

$$(t_1, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \sim (t_1, \ldots, t_{i-1}, t_i, \ldots, t_n).$$

Hence $(0) = I^0$ is the identity in $K$.

Let $J$ be the monoid obtained from $K$ by introducing the relation

$$(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n) \sim (1, t_{i+1}, \ldots, t_n),$$

i.e. $1 \in I$ acts as a right zero. In particular it is an idempotent. Clearly $J$ is a monoid.
Let \( L_n = I^1 \cup I^2 \cup \ldots \cup I^n \) and \( J_n = \pi(L_n) \) where \( \pi: L = \lim_{\to n} L_n \rightarrow J \) is the projection. \( J_1 = L_1 \). Let
\[
I^n = \{(t_1, \ldots, t_n) \in I^n | t_i = 0 \text{ some } i, \text{ or } t_j = 1 \text{ some } j > 1\}
\]
\( J_{n+1} \) is obtained from \( J_n \) by attaching \( I^{n+1} \) by an attaching map \( f_{n+1}: I^{n+1} \rightarrow J_n \), and \( J_n \) and \( J \) are in CG (It is easy to verify that the conditions (1), \ldots, (4) of p. 33 hold with \( X_p = I^D \) and \( DX_p = I^D \)).

Since the attaching maps are skeletal, \( J \) is a CW-complex.

Claim: \( J_{n-1} \) is a strong deformation retract of \( J_n \).

Proof: \( J_n = J_{n-1} \cup f_n I^n \). All faces of \( I^n \) become attached to \( J_{n-1} \) with exception of the face \( t_i = 1 \). Hence the deformation retraction of \( I^n \) to the other faces induces a deformation retraction of \( J_n \) to \( J_{n-1} \).

The multiplication of \( K \) induces the monoid structure in \( J \). Since \( J \) is in CG and since it has an idempotent different from the identity it gives rise to a system satisfying M1 (see Example 3.4). Although \( J \) is contractible we cannot find a deformation satisfying M2, since any such deformation must be relative to \( u = 1 \in I^1 \) and to \( 0 \in I^1 \), the identity in \( J \).

Nevertheless the monoid \( J \) will be of some importance
later on. For note that if $A \subseteq X$ is a strong deformation retract and $p_t : X \to X$ is a deforming homotopy such that $p_0 = \text{id}_X$ and $p_1 = i \circ p$ where $i : A \to X$ is the inclusion and $p : X \to A$ the retraction, then the correspondence

$$(t_1, \ldots, t_n) \mapsto p_{t_1} \circ \cdots \circ p_{t_n}$$

defines a continuous map of $J$ into the space of maps from $X$ to $X$. 
§ 4 THE UNIVERSAL PROPERTY

From now on we restrict ourselves to the system \( \Gamma \) given in Example 2.9, and denote for this case \( \mathbb{W}B \) simply by \( \mathbb{W}B \). Since \( R \) and \( Q \) are just points in this case we neglect them and consider \( \delta \) in \( (\theta, \xi, \delta) \in C_p(a, b) \) simply as a \( p \)-tuple of points in \( I \).

Definition 4.1: \( x \in \mathbb{W}B(a, b) \) is called indecomposable if it cannot be written as a composition \( x = y \circ z \) such that \( y \) and \( z \) are not permutations. (Note that any identity is a permutation).

Lemma 4.2: (a) \( \{\theta, \xi, \delta\} \) with \( (\theta, \xi, \delta) \in C_p(a, b) \) non-degenerate and \( \delta = (t_1, \ldots, t_p) \) is decomposable iff 
\[ p > 1 \text{ and } t_i = 1 \text{ for some } i. \]

(b) \( \{\theta, \xi, \delta\} \) with \( (\theta, \xi, \delta) \in C_p(a, b) \) non-degenerate and \( \delta = (t_1, \ldots, t_p) \) is decomposable iff 
\[ p > 1 \text{ and } t_i = 1 \text{ for some } i. \]

Proof: (a) Suppose \( \{\theta, \xi, \delta\} \) is decomposable, 
\[ \{\theta, \xi, \delta\} = \{\psi_1, \xi_1, \delta_1\} \circ \{\psi_2, \xi_2, \delta_2\} \]
with \( (\psi_i, \xi_i, \delta_i), i = 1, 2 \), non-degenerate and not trivial.
Then \( \xi_1 \circ (\psi_2, \xi_2, \delta_2) \) is not degenerate and not trivial, and since \( \max(1,1) = 1 \), \( \lambda \rho [(\psi_1, \text{unit}, \delta_1) \circ (\xi_2, (\psi_2, \xi_2, \delta_2))] \) has at least one link to which \( 1 \in I \) is assigned. (\( \lambda \rho \) is the function defined in Lemma 2.14).

Conversely suppose that there is a link in \( \Theta \) to which \( 1 \in I \) has been assigned, the \( i \)-th link in the standard indexing, say. Let \( \psi' \) be the subtree of \( \Theta \) sitting on the \( i \)-th link, and suppose that \( \psi' \) has \( q \) twigs. Let \( \varphi \) be the tree obtained from \( \Theta \) by deleting \( \psi' \), and suppose the twigs of \( \varphi \) indexed by \( j < i \) are labelled by \( i_1^{-1}(1), \ldots, i_1^{-1}(s) \), then the twigs indexed by \( j > i \) are labelled by \( i_1^{-1}(s+q+1), \ldots, i_1^{-1}(k) \), if \( \mathbf{a} = (i_1, \ldots, i_k) \). Assign to the links of \( \varphi \) and \( \psi' \) the values in \( I \) inherited from \( \Theta \). Let

\[
\psi = 1_{\xi_1^{-1}(1)} \oplus \cdots \oplus 1_{\xi_1^{-1}(s)} \oplus \psi' \oplus 1_{\xi_1^{-1}(s+q+1)} \oplus \cdots \oplus 1_{\xi_1^{-1}(k)}
\]

where \( 1_b \) is the trivial tree with labelled edge \( b \). Then \( (\varphi, \text{unit}) \) and \( (\psi, \xi) \) with the values of \( I \) assigned to their links determine not trivial and not degenerate elements \( (\varphi, \text{unit}, \delta_1) \) and \( (\psi, \xi, \delta_2) \) such that

\[
\{\Theta, \xi, \delta\} = \{\varphi, \text{unit}, \delta_1\} \circ \{\psi, \xi, \delta_2\}.
\]

(b) follows from (a) by applying (a) to each tree.

We refer to the process of "cutting up" a tree into two composable ones by cutting off the \( i \)-th link as chopping the \( i \)-th link.
Lemma 4.3. Each element $x \in WB(a,b)$ which is not a permutation can be decomposed into indecomposable elements $x = x_1 \circ \ldots \circ x_p$. This decomposition is unique up to the equivalence generated by

(a) $x_1 \circ \ldots \circ (x_i' \oplus 1) \circ (1 \oplus x_{i+1}') \circ \ldots \circ x_p$

$$= x_1 \circ \ldots \circ (x_i' \oplus x_{i+1}') \circ \ldots \circ x_p$$

(b) $x_1 \circ \ldots \circ (x_i \circ \xi) \circ \ldots \circ x_p$

$$= x_1 \circ \ldots \circ x_i \circ \xi \circ x_{i+1} \circ \ldots \circ x_p$$

where $\xi$ is a permutation.

Proof: Represent $x$ by a non-degenerate triple $(\theta, \xi, \delta) = (\theta, \text{unit}, \delta) \circ \xi$. This representative is unique up to the relation (2.19). Chop each link of $\theta$ to which $1 \in I$ is assigned. This decomposes this representative into non-degenerate elements each of which represents an indecomposable element in $WB$. There are exactly three choices involved:

(1) the order in which we chop the links,
(2) the choice of the particular non-degenerate representative,
(3) in the chopping process the permutation $\xi$ can be broken
up into a block permutation (as defined on p. 7) and another one such that the block permutation can be associated with the copse on the left. Relation (a) takes care of (1), while relation (b) takes care of (2) and (3).

Let $W_{p,B}(a,b)$ be the subcategory of $WB$ generated by all $W_{p,B}(a,b)$, $p$ fixed. (Note that $W_{p,B}$ is not even a category). Let $V^D(a,b)$ be the subspace of $V(a,b)$ of all those elements $x$ such that $\{x\} = \{x_1\}^{\circ}\ldots^{\circ}\{x_m\}$, where $\{x\}$ denotes the equivalence class of $x$, i.e. its image in $WB$, where each tree in $x_k$ has at most $p$ links, $1 \leq k \leq m$. Observe that we do not require that $x$ is non-degenerate. $V^D(a,b)$ is closed in $V(a,b)$.

Let $\pi^D: V^D \to W_{p,B}$ be given by $\pi^D = \pi|_{V^D}$, where $\pi: V \to WB$ is the projection. Let $W_{p,B}$ be the inverse image of $W_{p,B}$ under the projection $\pi: WB \to WB$ induced by the relation (2.19), and $\pi^D$ its restriction to $W_{p,B}$. $W_{p,B}(a,b)$ is obtained from $W_B(a,b)$ by attaching $C_1(a,b), \ldots, C_p(a,b)$ in order. Consequently $W_{p,B}(a,b)$ is obtained from $W_B(a,b)$ by attaching $V^D(a,b) \cap C_1(a,b), \ldots, V^D(a,b) \cap C_q(a,b), \ldots$ in order with $q = 1, 2, 3, \ldots$

For each type $a$ of trees in $T_p(a,b)$ and each $\xi \in S(\text{length}a)$
we have a component $M_{a, p}^{\alpha, b} = \left[ \prod_k B(\alpha_k, j_k) \right] \times (\xi_\alpha)$. Denote the subspace of degenerate points of $T_p(\alpha, b)$ by $T'_p(\alpha, b)$ and let $C'_p(\alpha, b) = T'_p(\alpha, b) \times I^p \cup T_p(\alpha, b) \times \delta I^p$, where $\delta I^p$ denotes the boundary of the cube $I^p$. Set $Q_{a, p}(\alpha, b) = M_{a, p}(\alpha, b) \times I^p$, and $Q'_{a, p}(\alpha, b) = Q_{a, p}(\alpha, b) \cap C'_p(\alpha, b)$. $C'_p(\alpha, b)$ is the closed subspace of $C_p(\alpha, b)$ consisting of the degenerate or decomposable points.

We have characteristic maps

$\chi_{a, p} : (Q_{a, p}(\alpha, b), Q'_{a, p}(\alpha, b)) \to (W^p B(\alpha, b), W^{p-1} B(\alpha, b))$ which by Lemma 2.16 induce characteristic maps

$\chi_{a, p} : (Q_{a, p}(\alpha, b), Q'_{a, p}(\alpha, b)) \to (W^p B(\alpha, b), W^{p-1} B(\alpha, b))$.

Let $D$ be a subcategory of $\mathcal{W}_\mathcal{B}$, and let $D_{a, p} \subset Q_{a, p}$ be the subset of all those elements $x$ such that $\pi(x) \in D$. We assume that $D_{a, p}$ is closed in $Q_{a, p}$ (and hence has the relative topology) and that if $x \in D$, $x = y \circ z$, then $y$ and $z$ are in $D$.

**Definition 4.4:** Let $\mathcal{B}$ and $\mathcal{C}$ be topological categories and $\phi_0, \phi_1 : \mathcal{B} \to \mathcal{C}$ continuous functors such that $\phi_0(A) = \phi_1(A)$ for all objects $A$ in $\mathcal{B}$. Call $\phi_0$ and $\phi_1$ homotopic if there exist continuous functors $\Theta_t : \mathcal{B} \to \mathcal{C}$ for all $t \in I$ such that $\Theta_t(A) = \phi_0(A)$ for all $t \in I$ and for all objects $A$ in $\mathcal{B}$, $\Theta_0 = \phi_0$, $\Theta_1 = \phi_1$. 
$\Theta_1 = \varphi_1$, and $\Theta: \mathbb{B}(A_1, A_2) \times I \to \mathbb{C}(\varphi_0(A_1), \varphi_0(A_2))$ given by $\Theta(a, t) = \Theta_t(a)$ is continuous. $\Theta_t$ is called a \textit{homotopy of functors}. If $\varphi_0, \varphi_1$ are MTP-functors then $\Theta_t$ is called a \textit{homotopy of MTP-functors} if $\Theta_t$ is an MTP-functor for each $t \in I$.

Lemma 4.5: Let $\mathbb{C}$ be an $\mathbb{M}^{\text{TP}}$-category and $\mathbb{D}$ a subcategory of $\mathbb{W}_B$ as given above. Let $\delta_t: \mathbb{D} \to \mathbb{C}$ be a homotopy of functors preserving objects, sums and permutations ($\mathbb{D}$ need not be an $\mathbb{M}^{\text{TP}}$-category).

(1) Given a homotopy of $\mathbb{M}^{\text{TP}}$-functors $\gamma^{P-1}_t: \mathbb{W}_P \to \mathbb{C}$ and equivariant maps $f_{a, p}: Q_{a, p}(a, b) \times I \to \mathbb{C}(a, b)$ for all $a, b, \text{and } a$ such that

(i) $\gamma^{P-1}_t | \mathbb{W}_P \cap \mathbb{D} = \delta_t | \mathbb{W}_P \cap \mathbb{D}$

(ii) $f_{a, p} | D_{a, p}(a, b) \times (t) = \delta_t \circ (x_{a, p} | D_{a, p}(a, b))$

(iii) $f_{a, p}(x, t)$ factors through the relation (2.11) for each $t \in I$.

If $x$ is a trivial tree representing the identity of $b$, then $f_{a, t}(x, t) = 1_b$.

Then there exists a unique homotopy of $\mathbb{M}^{\text{TP}}$-functors $\gamma^P_t: \mathbb{W}_P \to \mathbb{C}$ extending $\gamma^{P-1}_t$ such that
\[ \gamma_t^D \mid W^D_B \cap D = \delta_t \mid W^D_B \cap D \] and
\[ \gamma_t^D(\chi_{a,p} | Q_{a,p}(a,b)) = f_{a,p} | Q_{a,p}(a,b)x(t). \]

(2) Given homotopies of \( M^1_{nTP} \)-functors \( \gamma_t^D : W^D_B \rightarrow C \) for all \( p \) such that \( \gamma_t^D \mid W^D_B = \gamma_t^{D-1} \) and
\[ \gamma_t^D \mid W^D_B \cap D = \delta_t \mid W^D_B \cap D \] then there exists a unique homotopy of \( M^1_{nTP} \)-functors \( \gamma_t : W_B \rightarrow C \) such that \( \gamma_t \mid W^D_B = \gamma_t^D \) and \( \gamma_t \mid D = \delta_t \).

**Proof:** Let \( \{e_1 \oplus \ldots \oplus e_n, \xi, \delta_1 \times \ldots \times \delta_n\} \in W^D_B(a,b) \) be indecomposable. Define
\[ \gamma_t^D[\theta_1 \oplus \ldots \oplus \theta_n, \xi, \delta_1 \times \ldots \times \delta_n] = \gamma_t^D[\theta_1, \text{unit}, \delta_1] \oplus \ldots \oplus \gamma_t^D[\theta_n, \text{unit}, \delta_n] \]
with \( \gamma_t^D[\theta_k, \text{unit}, \delta_k] = \gamma_t^{D-1}[\theta_k, \text{unit}, \delta_k] \) if \( (e_k, \text{unit}, \delta_k) \in V^{D-1}(a_k, b_k) \)
\[ = f_{a,p}(e_k, \text{unit}, \delta_k ; t) \] if \( (e_k, \text{unit}, \delta_k) \in Q_{a,p}(a,b) \).

This definition of \( \gamma_t^D \) on indecomposables is forced upon us by the condition that \( \gamma_t^D \) is an \( M^1_{nTP} \)-functor satisfying the extension conditions of the lemma. Because of (i), (ii), and (iii) \( \gamma_t^D \) is well defined and compatible with \( \delta \). It is continuous since sum and composition in \( C \) are continuous. Extend \( \gamma_t^D \) to the whole of \( W^D_B \) by
\[ \gamma^p_t(x_1 \circ \ldots \circ x_n) = \gamma^p_t(x_1) \circ \ldots \circ \gamma^p_t(x_n) \]
where the \( x_i \)'s are indecomposables. By definition \( \gamma^p_t \) preserves sums and permutations. Since indecomposables in \( D \) are indecomposables in \( \mathcal{W}_p \), \( \gamma^p_t \) extends \( \delta \). Since the \( f_{a,p} \)'s are equivariant, factor through (2.11), and preserve identities for \( p = -1 \), \( \gamma^p_t \) is a well defined functor by Lemma 4.3. Again this extension is forced upon us to make \( \gamma^p_t \) into a functor.

\( \gamma^p_t \) is continuous since the maps from \([\mathcal{V}_p(a,b) \cap C_q(a,b)] \times I\) to \( \mathbb{C} \) which induce \( \gamma^p_t \) are defined by projecting closed subspaces of \( \mathcal{V}_p(a,b) \cap C_q(a,b) \) to some product of such spaces of lower filtration \( q \) (factoring out vertices labelled by identities and links to which \( 1 \in I \) has been assigned) and following by product maps involving \( f_{a,p} \) and \( \gamma^p_t \circ \pi^p \). Different positions of identities in the copses and different assignments of elements \( 1 \in I \) require different projections, but since \( f_{a,p} \) extends \( \gamma^p_t \circ \chi_{a,p} \mid \mathcal{W}_a \) they coincide on their intersections.

Since \( \mathcal{W}_B \) has the limit topology from the \( \mathcal{W}_p \)'s the second part is immediate. 

\]

Remark: By taking the functor homotopies to be the trivial ones we obtain the same results for \( \mathcal{M}_1^{\mathcal{W}_{TP}} \) functors (delete \( t \) and \( I \) wherever they occur).
Theorem 4.6 (The universal property):

Given a commutative diagram

\[
\begin{array}{ccc}
D & \xleftarrow{\rho} & WB \\
\downarrow{\delta_t} & & \downarrow{\gamma} \\
\mathcal{D} & \xrightarrow{\varepsilon} & G \\
\downarrow{\mu} & & \downarrow{\varepsilon} \\
B & \xrightarrow{\gamma} & C
\end{array}
\]

of $\mathcal{M}^{nTP}$-categories $\mathcal{D}$, $\mathcal{C}$, $\mathcal{G}$ and a subcategory $\mathcal{D}$ of $\mathcal{WB}$, $\mathcal{M}^{nTP}$-functors $\gamma, \mu$, the standard augmentation $\varepsilon = \varepsilon_\mathcal{B}$, the inclusion functor $\rho$, and a homotopy of functors $\delta_t$ preserving objects, sums, and permutations for each $t \in \mathcal{I}$.

Assume:

1. If $x \in \mathcal{D}$ is a composition in $\mathcal{WB}$, $x = y \circ z$, then $y$ and $z$ are in $\mathcal{D}$.

2. For each generator $b$ there exists a closed neighbourhood $Z_b$ of $1_b$ in $\mathcal{B}(b,b)$ such that $(Z_b, 1_b \cup \text{fr } Z_b)$ is a NDR-pair ($\text{fr} = \text{frontier}$), and $\gamma(Z_b) = 1_b \in \mathcal{C}(b,b)$

3. $\mu$ is fibre homotopically trivial.
Then

I: There exists an $\mathcal{M}^{\text{TP}}$-functor $\nu_0 : \mathcal{W} \to \mathcal{G}$ such that $\mu \circ \nu_0 = \gamma \circ \epsilon$ and $\nu_0 \circ \rho = \delta_0$.

II: Given any two $\mathcal{M}^{\text{TP}}$-functors $\nu_0, \nu_1 : \mathcal{W} \to \mathcal{G}$ such that $\mu \circ \nu_0 = \mu \circ \nu_1 = \gamma \circ \epsilon$ and $\nu_0 \circ \rho = \delta_0, \nu_1 \circ \rho = \delta_1$, then there exists a homotopy $\mathcal{H}$ of $\mathcal{M}^{\text{TP}}$-functors between $\nu_0$ and $\nu_1$ extending $\delta_t$, and such that $\mu \circ \nu_t = \gamma \circ \epsilon$.

For the proof of Theorem 4.6 another filtration (really double filtration, and we induct over the sum of both) of $\mathcal{W}$ seems to be more suitable than the one used in Lemma 4.5.

$\mathcal{C}_o(b,b) \cong \mathcal{B}(b,b)$, and we can assume wlog that $Z_b \subset \mathcal{D}_a,b$ if the latter is not empty, since $\mathcal{D}_a,b$ is open and closed.

Let $Y_b = Z_b - (1_b \cup \text{fr } Z_b)$. Let $F_p \mathcal{V}(a,b)$ be the subspace of $\mathcal{V}(a,b)$ of those elements $x$ such that

$\{x\} = \{x_1\} \circ \ldots \circ \{x_q\}$ and $\{x_i\}$ is a sum $\{y_1\} \oplus \ldots \oplus \{y_k\}$ of morphisms into a generator for each $i$ such that $\{y_j\}$ is in $\mathcal{D}$ or $y_1$ has $s$ links and $t$ vertices labelled by elements in the $Y_b$'s with $s+t \leq p$. $F_p \mathcal{V}(a,b)$ is closed in $\mathcal{V}(a,b)$. $F_p \mathcal{W} = \pi(F_p \mathcal{V})$ is an $\mathcal{M}^{\text{TP}}$-subcategory of $\mathcal{W}$ containing $\mathcal{D}$ since it is closed under composition and sum and since it contains all permutations. We denote the $\mathcal{M}^{\text{TP}}$-subcategory
of $\mathcal{W}_B$ generated by $D$ and the identities by $F_{-1} \mathcal{W}_B$.

Let $D^\alpha, \rho$ be the union of all those connected components of $D^\alpha, \rho(a, b)$ which contain an element $x \in \mathcal{Q}^\alpha (a, b)$. Then by assumption (1) $D^\alpha, \rho(a, b)$ is a product $D^\alpha, \rho(a, b) \times I^\rho$.

Let $P^\alpha, \rho, k(a, b)$ be the subspace of $M^\alpha, \rho(a, b) - D^\alpha, \rho(a, b)$ of all those pairs $(\theta, \xi)$ such that at least $k - \rho$ vertices of $\theta$ are labelled by elements in the $Z_b$'s and at most $k - \rho$ ones by elements in the $(Y_b)^{\alpha}$'s. Denote the closed subspace of those points of $P^\alpha, \rho, k(a, b)$ with less than $k-\rho$ vertices labelled by elements in the $Y_b$'s by $P^\alpha, \rho, k(a, b)$. Note that for $k=0$, $P^\alpha, \rho, 0(a, b) = \emptyset$ unless $\rho=-1$ and $a = b$, when it contains the representative of $I_b$.

Let $R^\alpha, \rho, k(a, b) = P^\alpha, \rho, k(a, b) \times I^\rho$ and $R^\alpha, \rho, k(a, b) = P^\alpha, \rho, k(a, b) \times \partial I^\rho \cup P^\rho, \rho, k(a, b) \times I^\rho$.

$R^\rho, \rho, k(a, b)$ consists exactly of those points of $R^\rho, \rho, k(a, b)$ that are equivalent to a point in $F_{k-1} \mathcal{W}_B(a, b)$. We have characteristic maps $\chi^\alpha, \rho, k = \chi^\alpha, \rho, k \mid R^\alpha, \rho, k$

$\chi^\alpha, \rho, k : (R^\alpha, \rho, k(a, b), R^\rho, \rho, k(a, b)) \rightarrow (F_k \mathcal{W}_B(a, b), F_{k-1} \mathcal{W}_B(a, b))$

In a completely analogous way to Lemma 4.5 we can prove

**Lemma 4.7**: Let $\mathcal{C}$ be an $\mathcal{M}^{\mathcal{TP}}$-category, $\mathcal{D}$ a subcategory of $\mathcal{W}_B$ satisfying the requirements of Theorem 4.6, and
δ_t: D \rightarrow \mathcal{C} a homotopy of functors preserving objects sums and permutations. Then

(1) \(\delta_t\) determines a unique \(\mathcal{M}^{\text{TP}}\)-functor \(\gamma^{-1}_t:F_{-1}WB \rightarrow \mathcal{C}\) extending \(\delta_t\).

(2) Given a homotopy of \(\mathcal{M}^{\text{TP}}\)-functors \(\gamma^{k-1}_t:F_{k-1}WB \rightarrow \mathcal{C}\) \(k \geq 0\), and equivariant maps \(f_{a,p,k}^{}:R_{a,p,k}(a,b) \times I \rightarrow \mathcal{C}(a,b)\) for all \(a,p,k,a,b\) and such that

\[
\gamma^{k-1}_t \circ f_{a,p,k}^{}(x,t) = \gamma^{-1}_t \circ f_{a,p,k}^{}(x,t) \quad \text{factors through relation (2.11) for each } t.
\]

Then there exists a unique homotopy of \(\mathcal{M}^{\text{TP}}\)-functors \(\gamma^{k}_t:F_kWB \rightarrow \mathcal{C}\) extending \(\gamma^{k-1}_t\) and the maps \(f_{a,p,k}^{}\).

(3) Given a sequence of homotopies of \(\mathcal{M}^{\text{TP}}\)-functors \(\gamma^{k}_t:F_kWB \rightarrow \mathcal{C}\) such that \(\gamma^{k}_t|F_{k-1}WB = \gamma^{k-1}_t\) for all \(k\) and \(\gamma^{-1}_t\) extends \(\delta_t\), then there exists a unique \(\mathcal{M}^{\text{TP}}\)-functor \(\gamma^{k}_t:WB \rightarrow \mathcal{C}\) extending \(\delta_t\) and such that \(\gamma^{k}_t|F_{k}WB = \gamma^{k}_t\).

Proof of Theorem 4.6: We are going to prove the statements I and II simultaneously.

\(\delta_0\) resp. \(\delta_t\) determine \(\nu^{-1}_0(\text{resp. } \nu^{-1}_t): F_{-1}WB \rightarrow \mathcal{C}\).
Inductively suppose that we have defined
\[ \nu^{k-1}_o (\text{resp. } \nu^{k-1}_t): F_{k-1} \to G \] such that \( \mu \circ \nu^{k-1}_o, (\mu \circ \nu^{k-1}_t) = \gamma \circ \varepsilon | F_{k-1} \) and \( \nu^{k-1}_o, (\nu^{k-1}_t) | F_i \to G = \nu^{i}_o, (\nu^{i}_t) \) for all \( i < k \).

We have to define maps \( f_{a,p,k}: R_{a,p,k}(\alpha,\beta) \to G(a,b) \) (resp. \( R_{a,p,k}(\alpha,\beta) \times I \to G(a,b) \)) satisfying the requirements of Lemma 4.7 for \( t=0 \) (resp. for all \( t \in I \)), and such that
\[ \mu \circ f_{a,p,k} = \gamma \circ \varepsilon \circ \chi_{a,p,k} \]. The Theorem then follows from Lemma 4.7 (2) and (3).

Since we work with fixed \( a, p, k, \alpha, \beta, \) and \( b \) during the construction of any particular map \( f_{a,p,k} \) we denote \( R_{a,p,k}(\alpha,\beta), P_{a,p,k}(\alpha,\beta), G(a,b), f_{a,p,k}, \) and \( \chi_{a,p,k} \) simply by \( R, P, G, f, \) and \( \chi \).

Let \( \sigma \) be the equivariant section of \( \mu \) and \( H: \sigma \circ \mu | G = \text{id}_G \) the equivariant fibrewise homotopy given by assumption (3). Consider \( I^P \) as a cone on \( \partial I^P \), i.e. \( I^P = \{ (d,u) | d \in \partial I^P, u \in I, (d_1,0) \sim (d_2,0) \text{ for } d_1, d_2 \in \partial I^P \} \). Identify \( \partial I^P \subset I^P \) with \( \partial I^P \times I \).

**Case I:** The homotopy \( F': P \times \partial I^P \times I \to G \) given by
\[ F'(x,d,u) = H(\nu^{k-1}_o \circ \chi(x,d), u) \text{ for } p > 0 \text{ factors through the cone point since } F'(x,d,0) = \sigma \circ \mu \circ \nu^{k-1}_o \circ \chi(x,d) = \sigma \circ \gamma \circ \varepsilon \circ \chi(x,d) \text{ which is independent of } d \in \partial I^P. \] Hence \( F' \) induces a map
\[ F: R = P \times I^P \to G \]
such that \( F|_{P \times \partial \mathcal{I}^P = v_{\mathcal{O}}^{k-1} \circ \chi|_{P \times \partial \mathcal{I}^P} \). Since \( H \) is an equivariant fibre wise homotopy, \( F \) is equivariant and \( \mu F(x,d,u) \) is independent of \( u \). Hence \( \mu F(x,d,u) = \mu F(x,d,0) = \mu \circ \gamma \circ \epsilon \circ \chi(x,d) = \gamma \circ \epsilon \circ \chi(x,d) \).

Suppose \( (x',d') \sim (x,d) \) under \( (2.11) \), then \( F'(x,d,u) = F'(x',d',u) \). Hence \( F(x,d,u) = F(x',d',u) \) and since each permutation of coordinates of \( (d,u) \), now considered as a \( p \)-tuple, is induced by the same permutation of the \( p \)-tuple \( d \in \partial \mathcal{I}^P \), \( F \) factors through the relation \( (2.11) \).

For \( p=0 \) define \( F: R \rightarrow G \) by \( F(x) = \sigma \circ \gamma \circ \epsilon \circ \chi(x) \).

**Case II:** Define \( F': P \times (\partial \mathcal{I}^P \times I \cup \mathcal{I}^P \times \partial I) \times I \rightarrow G \) by

\[
F'(x,u,t) = H(g(x,u), t)
\]

where \( g(x,u) = v_t \circ \chi(x,u') \) if \( u = (u',t) \in \partial \mathcal{I}^P \times I \)

\[
= v_\epsilon \circ \chi(x,u') \text{ if } u = (u',\epsilon) \in \mathcal{I}^P \times \partial I, \epsilon = 0,1
\]

Using the same argument as above we obtain a map

\( F: R \times I \rightarrow G \)

which factors through the relation \( (2.11) \) and which satisfies:

\[
F|_{P \times \partial \mathcal{I}^P \times I = v_{t}^{k-1} \circ \chi|_{P \times \partial \mathcal{I}^P \times I}
\]

\[
F|_{P \times \mathcal{I}^P \times \epsilon = v_\epsilon \circ \chi|_{P \times \mathcal{I}^P}, \epsilon = 0,1
\]

\[
\mu F(r,t) = \gamma \circ \epsilon \circ \chi(r) \text{ for all } (r,t) \in R \times I
\]

If \( p' \neq p \), \( F \) serves as \( f \) (resp. \( f' \)) because then \( p' = p \times \partial \mathcal{I}^P \)

\( P \) is the union of the closed product spaces of \( M_{a,p}(a,b) - D_{a,p}(a,b) \) with exactly \( k-p \) factors being some of the neighbourhoods \( Z_\beta \) and the other factors being
$B(a_k, b_k)$ if $a_k \neq b_k$ or the closure of $B(b, b) - Z_b$. The intersection of two summands of $P$ is by definition in $P'$. After reshuffling the factors each summand is of the form $Z_{b_1} \times Z_{b_2} \times \ldots \times Z_{b_{k-p}} \times X$. We denote it by $Z \times X$. Let $Y \subset Z$ be the closed subspace of those points with at least one coordinate in some $(1_b \cup \text{fr } Z_b)$. $(Z, Y)$ is a NDR-pair by assumption (2) and [6; Lemma 7.3]. Note that $(Z \times X) \cap P' = Y \times X$.

$v_{k-1}^0$ (resp. $v_{k-1}^t$) determine $f$ on a subspace of $Z \times X \times I^p$ (resp. $Z \times X \times I^p \times I$) and to prove the theorem it now suffices to extend $f$ over each individual summand $Z \times X \times I^p$ of $R$ such that the required identities hold.

**Case I:** By induction $f$ is determined on $R'$ and hence on $Z \times X \times \partial I^p \cup Y \times X \times I^p$ in any reshuffled summand $Z \times X \times I^p$. 

**Case II:** $f$ is determined on $R' \times I \cup R \times \partial I$ and hence on $Z \times X \times \partial I^p \cup Y \times X \times I^p \times I \cup Z \times X \times I^p \times \partial I = Z \times X \times I^{p+1} \cup Y \times X \times I^{p+1}$ 

The maps $F$ (we delete the shuffling maps) satisfy for $(z, x) \in Z \times X$

\[ F | Z \times X \times \partial I^p = f | Z \times X \times \partial I^p \]

resp.

\[ F | Z \times X \times \partial I^{p+1} = f | Z \times X \times \partial I^{p+1} \]

since $\partial I^p$ (resp. $\partial I^{p+1}$) are identified with the level 1 in the cones $I^p$ (resp. $I^{p+1}$).

We restrict ourselves to case I for the rest of the proof since case II differs from it only in the number of
cube coordinates. Otherwise the proofs are from now on the same.

\[ K(y,x,s,t) = H(f(y,x,s), t) \] with \((y,x,s) \in Y \times X \times I^p\) defines a homotopy \(K : Y \times X \times I^p \times I \rightarrow G\) such that

\[ K(y,x,s,1) = f(y,x,s), \quad K(y,x,s,0) = \sigma \circ \mu \circ f(y,x,s) = \sigma \circ \gamma \circ \sigma \circ \chi(y,x,s) \] which is independent of \(s\). Since \(f\) factors through (2.11) by induction hypothesis and since \(f\) and \(H\) are equivariant, \(K\) is equivariant, factors through (2.11), and \(\mu \circ K(y,x,s,t)\) is independent of \(t\). \(K\) induces a homotopy

\[ L : Y \times X \times I^p \times I \rightarrow G \] such that \(L : F|_{Y \times X \times I^p} \sim f|_{Y \times X \times I^p} \) rel \(Y \times X \times \partial I^p\) by definition of \(F\) (see picture next page). \(L\) is equivariant and factors through (2.11) since \(K\) does. Furthermore

\[ \mu \circ L(y,x,s,t) = \mu \circ L(y,x,s,1) = \mu \circ f(y,x,s). \]
Define a map 

\[ N : Y \times X \times I^P \times I \cup Z \times X \times \partial I^P \times I \cup Z \times X \times I^P \times 0 \to G \]

by 

\[ N|_{Y \times X \times I^P \times I} = L \]

\[ N|_{Z \times X \times \partial I^P \times I} = \text{constant on } f|_{Z \times X \times \partial I^P} \]

\[ N|_{Z \times X \times I^P \times 0} = F|_{Z \times X \times I^P} \]

Then \( \mu \circ N(z, x, s, t) = \gamma \circ \circ x(z, x, s) \) which is independent of 

\( s \in I^P \). \( N \) is equivariant and factors through (2.11) since 

\( F, f, \) and \( L \) do.

\((Z \times I^P, Y \times I^P \cup Z \times \partial I^P)\) is a NDR-pair [6; Lemma 7.3]. Hence 

[6; Theorem 7.1] there exists a retraction 

\[ r' : Z \times I^P \times I \to Y \times I^P \times I \cup Z \times \partial I^P \times I \cup Z \times I^P \times 0 \]

which extends to a retraction 

\[ r : Z \times X \times I^P \times I \to Y \times X \times I^P \times I \cup Z \times X \times I^P \times I \cup Z \times X \times I^P \times 0 \]

given by \( r(z, x, s, t) = (z', x, s', t') \) where \( (z', s', t') = r'(z, s, t) \)

(Here we actually require that \( r' \) is symmetric in the coordinates of \( I^P \). Lemma 7.18 

p. 1217, this thesis shows the existence of such an \( r' \))
Define $f\mid Z \times X \times \mathbb{I}^P = N \circ r \mid Z \times X \times \mathbb{I}^P \times 1$. Then $f$ extends $f\mid Y \times X \times \mathbb{I}^P \cup Z \times X \times \partial \mathbb{I}^P$ and

$$\mu \circ f(z,x,s) = \mu \circ N \circ r(z,x,s,1) = \mu \circ N(z',x,s',1')$$

with $(z',s',1') = r'(z,s,1)$. Hence

$$\mu \circ f(z,x,s) = \gamma \circ \varepsilon \circ \chi(z',x,s')$$

which is independent of $s' \in \mathbb{I}^P$. $\varepsilon \circ \chi(z,x,s)$ is an expression in the coordinates of $z$ and $x$ involving composition and sum in $B$. $\varepsilon \circ \chi(z',x,s')$ can be obtained from this expression by substituting the coordinates of $z$ by the corresponding ones of $z'$, since only one type of tree is involved. Each coordinate of $z$ and its corresponding one of $z'$ are in the same neighbourhood $Z_b$. Since $\gamma$ is an $M^n_{TP}$-functor it preserves the expressions for $\varepsilon \circ \chi(z,x,s)$ and $\varepsilon \circ \chi(z',x,s')$, and since $\gamma(Z_b) = 1_b$ we obtain $\gamma \circ \varepsilon \circ \chi(z,x,s) = \gamma \circ \varepsilon \circ \chi(z',x,s')$. Hence $\mu \circ f(z,x,s) = \gamma \circ \varepsilon \circ \chi(z,x,s)$.

Since the retraction $r$ effects vertices of the trees involved which lie in some $B(b,b)$ on which the trivial permutation group operates and no others, $N \circ r$ is equivariant and factors through (2.11). Hence so does $f$. ]]

**Lemma 4.8:** Suppose that in addition to the assumptions of Theorem 4.6 we are given homotopies

$$\tau_{a,b} : B(a,b) \times I \to G(a,b)$$

for some $a$, $b$ such that
\[ \tau_{a,b}(x,t) = \delta_t(\nu_B x) \] if \( \nu_B x \in D \) (see p.43 for the definition of \( \nu_B \)).

\[ \tau_{b,b}(1_b,t) = 1_b \] if \( \tau_{b,b} \) is defined for all \( t \in I \).

\[ \mu \circ \tau_{a,b}(x,t) = \gamma(x) \] whenever it is defined.

Then there exists a homotopy of \( M^{nTP} \)-functors \( \nu_t : WB \to G \) such that \( \nu_t \circ \rho = \delta_t \), \( \mu \circ \nu_t = \gamma \circ e \), and

\[ \nu_t \circ \nu_B(x) = \tau_{a,b}(x,t) \] for \( x \in B(a,b) \) if \( \tau_{a,b} \) is defined.

**Proof:** The \( \tau_{a,b} \) determine some of the \( f_{a,p,k} \)'s for \( p=0 \) and \( k=0,1 \) compatibly with the boundary conditions. The Lemma now follows from Theorem 4.6. 

**Theorem 4.2:** Let \( D \) be a subcategory of \( WB \) satisfying 4.6 (1), \( G \) an \( M^{nTP} \)-category, \( \mu : G \to B \) a fibre homotopically trivial \( M^{nTP} \)-functor, and \( \delta_t : D \to G \) a homotopy of functors preserving objects sums and permutations and such that \( \mu \circ \delta_t = \varepsilon_B \circ \rho \) where \( \rho : D \to WB \) is the inclusion functor. Suppose the identities of \( B \) are isolated. Then there exists a homotopy of \( M^{nTP} \)-functors \( \nu_t : WB \to G \) such that \( \nu_t \circ \rho = \delta_t \) and \( \mu \circ \nu_t = \varepsilon_B \).

**Proof:** Since \( 1_b \in B(b,b) \) is isolated, \((B(b,b), 1_b)\) is a NDR-pair. Hence \( WB \) exists. Apply Theorem 4.6 with \( Z_b = (1_b), \gamma = \text{id}_B, \) and \( B = G \).
Lemma 4.10: Given any $\mathcal{M}^n\text{TP}$-category $\mathcal{B}$ (in normal form), then there exists an $\mathcal{M}^n\text{TP}$-category $\mathcal{B}^-$ (in normal form) such that

1. $(\mathcal{B}^-(b,b), 1_b)$ is a NDR-pair for all generators $b$. 
2. There exists a fibre homotopically trivial $\mathcal{M}^n\text{TP}$-functor $\varepsilon_B^- : \mathcal{B}^- \to \mathcal{B}$. 
3. Each $1_b \in \mathcal{B}^-(b,b)$ has a closed neighbourhood $Z_b$ such that $(Z_b, 1_b \cup \text{fr } Z_b)$ is a NDR-pair. 
4. $\varepsilon_B^-(Z_b) = 1_b \in \mathcal{B}(b,b)$. 

Proof: Let $\mathcal{B}^-(b,b) = \mathcal{B}(b,b) \cup I/\sim$ where $\mathcal{B}(b,b) \ni 1_b \sim 1 \in I$, and $\mathcal{B}^-(a,b) = \mathcal{B}(a,b)$ for $a \neq b$. Composition with permutations on the right is the one in $\mathcal{B}$. $\mathcal{B}^-(a,b)$ is now obtained by the normal form construction. Define composition as follows: Let $\beta$ be a morphism into a generator, $\beta$ not contained in one of the attached whiskers, let $\circ_B$ and $\oplus_B$ be the composition and sum in $\mathcal{B}$. Then

$$\beta \circ (\alpha_1 \oplus \cdots \oplus \alpha_k) = \beta \circ_B (\alpha'_1 \oplus_B \cdots \oplus_B \alpha'_k)$$

where $\alpha_i$ is a morphism into a generator, $\alpha'_i = \alpha_i$ if $\alpha_i$ is not contained in one of the attached whiskers, and $\alpha'_i = 1_b$, if $\alpha_i \in I \subset \mathcal{B}^-(b,b)$. 

For $\beta = t \in I \subset \mathcal{B}^-(b,b)$ define

$$\beta \circ \alpha = \alpha \text{ if } \alpha \text{ is not contained in one of the attached whiskers,}$$
and \( \beta \circ \alpha = \max(t, u) \) if \( \alpha = u \in I \subset B^-(b, b) \).

The composition is well defined, continuous and associative. \( 0 \in I \) serves as identity. By construction \( \circ \) is a bifunctor. Hence \( B^- \) is an \( M^{\text{TP}} \)-category.

Clearly (1) is satisfied and with \( Z_b = I \subset B^-(b, b) \) (3) holds. Define

\[
\epsilon'_B : B^- \to B
\]

by \( \epsilon'_B(a) = a \) if \( a \) is not contained in an attached whisker, and \( \epsilon'_B(a) = 1_b \) if \( a \in I \subset B^-(b, b) \). Extend \( \epsilon'_B \) to the whole of \( B^- \) using its normal form (this is possible since \( \epsilon'_B \) is equivariant where it is defined already).

The section \( t'_B : B \to B^- \) is given by \( t'_B(a) = a \). Then \( t'_B \circ \epsilon'_B \mid B^-(a, b) = \text{id} \mid B^-(a, b) \) equivariantly and fibrewise by shrinking the whiskers to \( 1 \in I \), and \( \epsilon'_B \circ t'_B \mid B(a, b) = \text{id} \mid B(a, b) \)

Condition (4) follows from the definition of \( \epsilon'_B \).

Remark 4.11: Of course, it would have sufficed to attach a whisker only to those morphism spaces \( B(b, b) \) for which \( 1_b \) is not isolated to obtain a category with the properties required in Lemma 4.10.

Notation: Denote \( \epsilon'_B \circ \epsilon_B^- \) and \( t_{B^-} \circ t_B^- \) by \( \epsilon_B^- \) resp. \( t_B^- \), where \( \epsilon_B^- \) and \( t_B^- \) are the standard augmentation and standard section of \( B^- \).
Theorem 4.12: Given any $M^{n\text{TP}}$-category (resp. $M^n_{\text{TP}}$-category) $\mathcal{B}$, then the triple $(\mathcal{B}^\sim, \varepsilon, \iota)$ satisfy the conditions $(U1)$ and $(U2)$ of p. 18.

**Proof:** $\varepsilon$ is fibre homotopically trivial with a section $\iota$. Hence $(U1)$ holds. $(U2)$ follows from Theorem 4.6 with $\mathcal{B} = \mathcal{B}^\sim$, $\mathcal{C} = \mathcal{B}$, and $\gamma = \varepsilon$.

Theorem 4.13: Let $\mathcal{B}$ be an $M^{n\text{TP}}$-category such that $\mathcal{B}^\sim$ exists. Then there exists an $M^{n\text{TP}}$-functor $\sigma: \mathcal{B} \to \mathcal{B}^\sim$ such that $\varepsilon = \varepsilon^\sigma \circ \sigma$ iff $\mathcal{B}$ has isolated identities.

**Proof:** If $\mathcal{B}$ has isolated identities then $\sigma$ exists by Theorem 4.9.

Suppose $\sigma$ exists. $\varepsilon^\sigma \circ \varepsilon = \sigma \circ \varepsilon = \varepsilon \circ \iota = \varepsilon \circ \iota = \text{id}_\mathcal{B}$

![Diagram](image)

Recall that $\iota$ preserves identities. $\varepsilon^\sigma \circ \iota$ defines a section of $\varepsilon^\sigma$, which preserves identities. Since $\varepsilon^\sigma |_{\mathcal{B}^\sim(b,b)}$ is given by the identity outside the attached whisker this section can only be continuous if the identity $\iota$ in $\mathcal{B}(b,b)$ is isolated.

[\]}
Remark 4.14: (1) In the case that $B$ has isolated identities it is easy to define $\sigma: WB \to WB^-$ without referring to Theorem 4.9 by constructing a functor $B \to B^-$ and using Remark 2.22.

(2) A similar theorem can be stated for the category obtained from $B$ by attaching a whisker to those morphism spaces $B(b,b)$ only for which $1_b$ is not isolated.

(3) Theorem 4.13 shows that Theorem 4.6 is false without some condition like 4.6 (2).

Lemma 4.15: Let $G$ be a discrete topological group, $X$ and $Y$ $G$-spaces with a free $G$-action, $Y$ a CW-complex and assume that $G$ acts freely on the cells of $Y$ (i.e. if $g \neq 1$, $g \in G$, then $x$ and $gx$ always lie in different cells). Let $p: X \to Y$ be an equivariant map and $s: Y \to X$ a section (not necessarily equivariant) of $p$ such that there exists a fibrewise homotopy $H: \text{id}_X \simeq s \circ p$. Then there exists an equivariant section $\tau: Y \ni X$ and an equivariant homotopy $T: \text{id}_X \simeq \tau \circ p$ which is fibrewise.

Proof: We construct a "regular" neighbourhood $V_n$ of the $n$-skeleton $Y^n$ of $Y$ which is invariant under the action of
G and such that the part $\mathcal{Q}$ of $V_n$ over an open $n$-cell $e$ does not intersect $g\mathcal{Q}$ over $g\hat{e}$ if $g \neq 1$, $g \in G$. We then construct a map $u: V_n \to I$ which is 1 outside $V_{n-1}, 0$ on $Y^n$, and satisfies $u(gx) = u(x)$. The section is then constructed by induction over the skeletons of $Y$. Assume we have constructed $\tau_{n-1}: V_{n-1} \to X$. We extend it to $V_n$ using $s$ on those points $x$ with $u(x) = 1$ and $H(\tau_{n-1}(x), u(x))$ on the others. The equivariant deformation is constructed analogously. Now the details:

Let $Z = Y/G$ and $\pi: Y \to Z$ the projection. Since the action of $G$ on $Y$ is free on cells $Z$ is a $\text{CW}$-complex, such that $\pi$ is cellular. Consider each cell $e^n$ as cone over its boundary, $e^n = \{(x, 1) \in \hat{e}^n \times [0, 2] | (x_1, 2) \sim (x_2, 2), x_1, x_2 \in \hat{e}^n\}$

Let $V$ be any subset of $Z$. We are going to construct a "regular" neighbourhood $N(V)$ of $V$. Let $Z^q$ be the $q$-skeleton of $Z$. Define

$$U_p, q(V) = Z^q \cap V \text{ for } q \leq p$$

Define $U_p, q(V) \subset Z^q$ for $q > p$ inductively by

$$U_p, q \cap \chi e^q_a = \chi (\chi^{-1} U_{p, q-1} \cap e^q_a) \times [0, 1]$$

where $\chi$ is the characteristic map, $e^q_a$ a $q$-cell and $e^q_a$ its boundary. Let

$$U_p(V) = U_q U_p, q(V) \text{ and } N(V) = U_p U_p(V).$$

Let $e$ be an $n$-cell of $Z$. Any two lifts of $N(\hat{e})$, where
\( \mathring{e} \) denotes the interior of \( e \), cannot intersect each other
for the two lifts of \( \mathring{e} \) itself are disjoint since the action
of \( G \) is free on cells.

**Claim:** \( N(U_p V_p) = U_p N(V_p) \)

This follows immediately from the definition.

**Claim:** \( N(V \cap W) = N(V) \cap N(W) \)

We first prove that \( U_{p,k}(V) \cap U_{q,k}(W) = U_{q,k}(V \cap W) \) for \( p \leq q \).

For \( k \geq p \):

\[
U_{p,k}(V) \cap U_{q,k}(W) = V \cap Z^k \cap W \cap Z^k
= U_{q,k}(V \cap W)
= U_{q,k}(V) \cap W
\]

For \( p < k \leq q \) we get inductively:

\[
U_{p,k}(V) \cap U_{q,k}(W) \cap \chi e_a^k = \chi \{(x^{-1} U_{p,k-1}(V) \ne_a^k) \times [0,1]\} \cap W \cap \chi e_a^k
= \chi \{(x^{-1} U_{p,k-1}(V) \ne_a^k) \times [0,1]\} \cap W \cap \chi e_a^k
= \chi \{(x^{-1} U_{q,k-1}(W) \ne_a^k) \times [0,1]\}
= \chi \{(x^{-1} U_{q,k-1}(W) \ne_a^k) \times [0,1]\}
= U_{q,k}(V \cap W) \cap \chi e_a^k
\]

Again by induction we obtain for \( k > q \):

\[
U_{p,k}(V) \cap U_{q,k}(W) \cap \chi e_a^k
= \chi \{(x^{-1} U_{p,k-1}(V) \ne_a^k) \times [0,1]\} \cap \chi \{(x^{-1} U_{q,k-1}(W) \ne_a^k) \times [0,1]\}
= \chi \{(x^{-1} U_{p,k-1}(V) \cap U_{q,k-1}(W)) \ne_a^k) \times [0,1]\}
= \chi \{(x^{-1} U_{q,k-1}(V \cap W) \ne_a^k) \times [0,1]\}
= U_{q,k}(V \cap W) \cap \chi e_a^k
\]
Now \( N(V) \cap N(W) = (U_p, k U_p, k(V)) \cap (U_q, l U_q, l(W)) \)
\[ = U_p, q, k, l U_p, k(V) \cap U_q, l(W) \]
\[ = U_p, q, k U_p, k(V) \cap U_q, l(W) \] since \( U_p, k \subset Z^k \), and \( U_q, l \subset Z^l \).
\[ = U_p, k U_p, k(V \cap W) \]
\[ = N(V \cap W). \]

Hence in particular \( N(e^\alpha_n) \cap N(e^\beta_n) = \emptyset \) for \( \alpha \neq \beta \).

We furthermore define a set \( M(Z^n) \) for each \( n \).
\( M(Z^n) = \bigcup_{q \geq n+1} M_q(Z^n) \), where \( M_q(Z^n) = M(Z^n) \cap Z^q \) is defined inductively by \( M_{n+1}(Z^n) = Z^n \), and given a \( k \)-cell \( e^k_\alpha \), \( k > n+1 \), then
\[ M_k(Z^n) \cap x e^k_\alpha = x \{(x^{-1} M_{k^{-1}}(Z^n) \cap e^k_\alpha) \times [0,1] \}. \]

The difference between \( M(Z^n) \) and \( N(Z^n) \) is that the "collar" part over the points of \( N(Z^n) \) which lie in the \((n+1)\)-skeleton has been omitted in \( M(Z^n) \). It follows from the construction that \( N(Z^n) = M(Z^n) \cup \bigcup_\alpha N(e^\alpha_n) \).

Let \( P X^n = p^{-1}(\pi^{-1}(N(Z^n))) \) and \( X^n = p^{-1}(Y^n) \), let \( P Y^n = \pi^{-1}(N(Z^n)) \). \( X^n \) and \( P X^n \) are in \( CG \) since they are closed. We are going to define equivariant sections \( \tau_n : P Y^n \to P X^n \) of \( p|P X^n \) and equivariant fibrewise homotopies \( T_n : \text{id}_{P X^n} \simeq \tau_n \circ (p|P X^n) \).

For each \( e^\alpha_0 \in Z^0 \) choose a lift \( l(N(e^\alpha_0)) \) in \( Y \). Define
\[ \tau_0 \mid l(N(e_\alpha^0)) = s \mid l(N(e_\alpha^0)) \]

Since \( N(e_\alpha^0) \cap N(e_\beta^0) = \emptyset \) for \( \alpha \neq \beta \) this is well defined. Now extend it to the whole of \( PY^0 \) by

\[ \tau_0(x) = \tau_0(x') \cdot \xi \quad \xi \in G \]

(we write the action of \( G \) on the right), where \( x = x' \cdot \xi \), \( x' \in l(N(e_\alpha^0)) \) some \( \alpha \). Since the action is free this is well defined. Define

\[ T_0(x, t) = H(x', t) \cdot \xi \]

if \( x = x' \cdot \xi \) and \( p(x') \in l(N(e_\alpha^0)) \), some \( \alpha \). \( T_0 \) is well defined equivariant and fibrewise. Since \( p \) is an equivariant map \( \tau_0 \) is a section, and \( T_0 : \text{id}_{PY^0} = \tau_0 \circ (p|PY^0) \).

Suppose inductively that \( \tau_{n-1} : PY^{n-1} \to PX^{n-1} \) and \( T_{n-1} : \text{id}_{PX^{n-1}} = \tau_{n-1} \circ (p|PX^{n-1}) \) have been defined. Define a map \( u : N(Z_n^{n-1}) \to I \) as follows: Let \( (x, t) \in \chi(e_n \times [0, 1]) \)

\[ \subset U_{n-1} \cap N(Z_n^{n-1}). \]

Set \( u(x, t) = t \). Extend \( u \) inductively by \( u(y, t) = u(y) \) for \( (y, t) \in \chi((x^{-1}U_{n-1}, k^{-1}(Z_n^{n-1}) \cap e^k_{\alpha}) \times [0, 1]) \)

\[ \subset U_{n-1}, k(Z_n^{n-1}), \quad k > n. \quad u \text{ is well defined and continuous.} \]

Extend \( u \) to \( u : N(Z_n^n) \to I \) by \( u(x) = 1 \) for \( x \in N(Z_n^n) - N(Z_{n-1}^{n-1}) \).

Since \( u(x) = 1 \) for \( x \in fr_N(Z_n^n) \cap N(Z_{n-1}^{n-1}) \) this is well defined.

Notice that \( u(x) = 0 \) iff \( x \in \text{M}(Z_n^{n-1}) \). For each \( n \)-cell \( e_\alpha^n \in Z_n \) choose a lift \( l(N(e_\alpha^n)) \). Define

\[ \tau_n(x) = \begin{cases} 
  s(x) & u \circ \pi(x) = 1 \\
  H(\tau_{n-1}(x), u \circ \pi(x)) & 0 \leq u \circ \pi(x) < 1
\end{cases} \]
x ∈ l(N(e^n_a)). \( \tau_n \) is well defined and continuous since it is independent of a on possible intersections

\[ l(N(e^n_a)) \cap l(N(e^n_b)). \]

Since furthermore \( \pi^{-1}M(Z^{n-1}) \cap \bigcup_a l(N(e^n_a)) \) = \( \emptyset \), any two lifts of \( N(e^n) \) are disjoint, the action of \( G \) is free on cells, and \( \tau_n \mid \pi^{-1}M(Z^{n-1}) \cap \bigcup_a l(N(e^n_a)) \) = \( \tau_{n-1} \mid \pi^{-1}M(Z^{n-1}) \cap \bigcup_a l(N(e^n_a)) \) we can extend \( \tau_n \) over the whole of \( \text{PY}^n \) by

\[ \tau_n(x) = \tau_n(x').\xi \]

if \( x = x'.\xi, x' \in l(N(e^n_a)) \), some \( a, \xi \in G \).

\[ H \text{ and } T_{n-1} \text{ define a product homotopy which is fibrewise} \]

\[ \begin{array}{c}
\tau_{n-1} \circ \text{op} \\
T_{n-1}(x,t_2) \\
\text{id} \\
\end{array} \]

\[ \begin{array}{c}
H(\tau_{n-1} \circ \text{op}(x),t_1) \\
H(T_{n-1}(x,t_2),t_1) \\
H(x,t_1) \\
\end{array} \]

\[ \begin{array}{c}
\text{op} = \text{op} \circ \tau_{n-1} \circ \text{op} \\
\text{op} T_{n-1}(x,t_2) = \text{op}(x) \\
\text{op} \end{array} \]

Hence there exists a homotopy \( K_{n-1} : \text{PX}^{n-1} \times I \times I \rightarrow \text{PX}^{n-1} \) which is fibrewise, such that

\[ K_{n-1}(x,0,t_2) = T_{n-1}(x,t_2) \]

\[ K_{n-1}(x,1,t_2) = H(x,t_2) \]

\[ K_{n-1}(x,t_1,0) = x \]

\[ K_{n-1}(x,t_1,1) = H(\tau_{n-1} \circ \text{op}(x),t_1) \].
Define
\[ T_n(x, t) = H(x, t) \]
\[ K_{n-1}(x, \mu_{\mathcal{P}}(x), t) \]
\[ 0 \leq \mu_{\mathcal{P}}(x) < 1 \]
\[ x \in p^{-1}l(N(e^n_\alpha)). \] Then \( T_n \) is well defined, continuous, and fibrewise, and since \( T_n(x, t) = T_{n-1}(x, t) \) for \( x \in p^{-1}o^{-1}M(Z^{n-1}) \) we can extend \( T_n \) over the whole of \( PX^n \times I \) by
\[ T_n(x', \xi, t) = T_n(x', t) \xi \]
for \( x' \in p^{-1}o^{-1}l(N(e^n_\alpha)), \) some \( \alpha, \xi \in \mathcal{G}. \) Then
\[ T_n: \text{id}_{P^X} = \tau_n \circ p | PX^n \] equivariantly and fibrewise.

Finally define \( \tau: Y \to X \) and \( T: X \times I \to X \) by \( \tau| Y^n = \tau_n | Y^n \) and \( T| X^n \times I = T_n | X^n \times I. \) Clearly \( \tau \) is a continuous equivariant section of \( p. \) Since we work in \( CG, \) \( T \) is continuous if it is continuous on each compact subset of \( X \times I. \) Each compact subset of \( X \times I \) is contained in a product \( C \times I \) where \( C \) is a compact subset of \( X. \) \( p(C) \) is compact and hence it is contained in some \( Y^n. \) Hence \( C \times I \) is contained in \( X^n \times I \) on which \( T \) is continuous.]

**Proposition 4.16:** Given a \( CW-M^n\text{TP} \)-category \( B \) such that composition with permutations is free on the cells of the morphism spaces of \( B. \) Let \( C \) be an \( M^n\text{TP} \)-category and \( \gamma: C \to B \) an \( M^n\text{TP} \)-functor such that there
exist maps \( s : B(a,b) \to C(a,b) \) satisfying
\[
\gamma \circ s | B(a,b) = \text{id}_{B(a,b)}
\]
and fibrewise homotopies
\[
H : s \circ \gamma | C(a,b) \simeq \text{id}_{C(a,b)}.
\]
Then \( \gamma \) is fibre homotopically trivial.

\textbf{Proof:} Put \( Y = \bigcup_{\xi \in S(k)} B(a,b) \) and \( X = \bigcup_{\xi \in S(k)} C(a,b) \) for each sequence \( a = (i_1, \ldots, i_K) \). Now apply Lemma 4.15 to \( X \) and \( Y \). We only have to make sure that the constructed new maps and homotopies map the morphism spaces of \( B \) resp. \( C \) into the corresponding morphism spaces of \( C \). Since \( s \) and \( H \) respect the morphism spaces a quick investigation of the proof of Lemma 4.15 shows that \( \tau \) and \( T \) do too. ]
]
CHAPTER III: STRUCTURE MAPS

§5 GENERALIZED HOMOTOPY B-MAPS

Suppose the category \( \mathcal{W} \) of operators acts on the spaces \( X \) and \( Y \), we want to give an appropriate definition of morphism between them. In fact there are various possibilities.

**Definition 5.1:** Let \( \mathcal{B} \) be an \( M^1 \)TP (resp. \( M^1 \)T)-category and \( (X, \alpha), (Y, \beta) \) be \( B \)-spaces, i.e. spaces in \( CG \) and we are given actions \( \alpha: B \to \text{End} X \) and \( \beta: B \to \text{End} Y \).

A map \( f: X \to Y \) is called a \( B \)-homomorphism if for each \( x \in \mathcal{B}(n, m) \), where \( k \) is the unique sequence of length \( k \), \( f^m \circ \alpha(x) = \beta(x) \circ f^n \).

We are more interested in a definition in which \( f \) merely commutes with the action up to coherent homotopies. This is more complicated and appears to be new.

Let \( L_n \) be the "linear" category with objects \( 0, \ldots, n \) and one morphism \( i \to j \) whenever \( i \leq j \).

**Definition 5.2:** Suppose \( (X, \gamma) \) and \( (Y, \delta) \) are \( \mathcal{W} \)-spaces.

A map \( f: X \to Y \) is a \( \text{generalized homotopy} \) \( B \)\( \text{-map} \) if we are given an action \( \rho: W(\mathcal{B} \ast L_1) \to \text{End} (X, Y) \) that
induces the given $WB$-actions on $X$ and $Y$ and the given map $f: X \to Y$ (for the definition of $B * L_1$ see p.15)

Later on we give a more precise definition of a generalized homotopy $B$-map.

If we attempt to construct the category of $WB$-spaces and generalized homotopy $B$-maps we find that it is not possible. The composite of two generalized homotopy $B$-maps is not defined, except up to a homotopy, which is itself defined only up to a homotopy, which is $\ldots \ldots$. Instead we form a semisimplicial complex $GMap_B$, whose $n$-simplexes are actions of $W(B * L_n)$ on $(n + 1)$-tuples of spaces.

**Lemma 5.3:** Let $B$, $C$ be $M^1TP$-categories in normal form and $\gamma: B \to C$ an $M^1TP$-functor. Let $D$ and $F$ be topological categories with objects $0, \ldots, n$ and $0, \ldots, m$ respectively, and $\delta: D \to F$ a continuous functor. Then there exists a unique $MTP$-functor

$$\nu = \gamma * \delta: B * D \to C * F$$

such that the following diagram commutes for all $p$, $0 \leq p \leq n$

$$
\begin{array}{ccccc}
B & \xrightarrow{\nu} & B * D & \xleftarrow{\Lambda} & D \\
\gamma & \downarrow & & \downarrow & \\
\gamma & & \nu & & \\
G & \xrightarrow{\nu * \delta(p)} & C * F & \xleftarrow{\Lambda} & F
\end{array}
$$
(For \( \mathcal{P} \) and \( \Lambda \) see p. 16)

**Proof:** On object generators \( \nu \) is given by \( \nu(i) = \delta(i) \).

Adopting the intuitive description of p. 16 the morphisms in \( \mathcal{B} \times \mathcal{D} \) from \( a = (i_1, \ldots, i_k) \) to \( b \) are given by a pair \( (\beta; b) \circ f \) where \( \beta \in \mathcal{B}(k, 1) \), \( (k \) is the unique sequence of length \( k \) in \( \mathcal{B} \) \) and \( f \) is a \( k \)-fold sum \( f_1 \circ \ldots \circ f_k \) of morphisms \( f_q \in \mathcal{D}(i_q, b) \). Define

\[
\nu[(\beta; b) \circ f] = (\gamma(\beta); \delta(b)) \circ \delta(f)
\]

where \( \delta(f_1 \circ \ldots \circ f_k) = \delta f_1 \circ \ldots \circ \delta f_k \). \( \nu \) is continuous and equivariant. Hence we can extend it to the whole of \( \mathcal{B} \times \mathcal{D} \) using the normal form. This automatically makes \( \nu \) commute with sums and permutations. Since \( \gamma \) and \( \delta \) are functors, \( \nu \) preserves identities, and it follows immediately from the definition that \( \nu \) preserves compositions. Hence it is an MTP-functor.

\[
\nu(\beta, b) = \nu \circ \mathcal{L}_b(\beta)
\]

\[
(\gamma(\beta); \delta(b)) = \nu \mathcal{L}_b(\beta) \circ \gamma(\beta)
\]

Hence \( \nu \circ \mathcal{L}_b(\beta) = \nu \mathcal{L}_b(\beta) \circ \gamma(\beta) \).

\[
\nu(f) = \nu \circ \Lambda(f) \quad f \in \mathcal{D}(i, j)
\]

\[
= \Lambda \circ \delta(f)
\]
Hence \( v((\beta; b) \circ f) = v(\beta; b) \circ v(f) \)

\[= (\gamma(\beta); \delta(b)) \circ \delta(f) \]

from which the commutativity of the diagram respectively

the uniqueness of \( v \) follow.

Each monotonically increasing map \( f: (0, \ldots, n) \rightarrow (0, \ldots, m) \)
gives rise to a unique functor \( f^i: L_n \rightarrow L_m \) such that

\( f(i) = fi \) for all objects \( i \in L_n \). Since \( f \) is monotonically

increasing, \( f(i, j) = (fi, fj) \) is defined, where \( i \leq j \) and

\( (i, j): i \rightarrow j \) is the unique map from \( i \) to \( j \).

Let \( f^i: (0, \ldots, n - 1) \rightarrow (0, \ldots, n) \) and \( g^i:(0, \ldots,n+1) \)

\( \rightarrow (0, \ldots, n) i = 0, \ldots, n, \) be given by

\[
f^i(j) = \begin{cases}  
  j & 0 \leq j < i \\
  j + 1 & i \leq j \leq n 
\end{cases}
\]

\[
g^i(j) = \begin{cases}  
  j - 1 & i < j \leq n + 1 \\
  j & 0 \leq j \leq i 
\end{cases}
\]

i.e. \( i \in (0, \ldots, n) \) is not in the image of \( f^i \) and its

counter image under \( g^i \) consists of two points

By Lemma 5.3 we have induced functors
\[ \bar{\rho}^i = 1 \ast \bar{\rho}^i : B \ast L_{n-1} \to B \ast L_n \]
\[ \bar{s}^i = 1 \ast \bar{s}^i : B \ast L_{n+1} \to B \ast L_n \]
satisfying following identities:
\[ \bar{\rho}^i \circ \bar{\rho}^{j-1} = \bar{\rho}^j \circ \bar{\rho}^i \quad i < j \]
\[ \bar{s}^{j-1} \circ \bar{s}^i = \bar{s}^i \circ \bar{s}^j \quad i < j \]
\[ \bar{s}^j \circ \bar{\rho}^i = \bar{\rho}^i \circ \bar{s}^{j-1} \quad i < j \]
\[ = 1 \quad i = j, j + 1 \]
\[ = \bar{\rho}^{i-1} \circ \bar{s}^j \quad i > j + 1 \]

By Remark 2.22 the same identities hold for \( \bar{W}(\bar{\rho}^i) = \bar{\rho}^i \)
and \( \bar{W}(\bar{s}^i) = \bar{s}^i \)

Let \( \rho^i : \text{End}(X_0, \ldots, \hat{X}_i, \ldots, X_n) \to \text{End}(X_0, \ldots, X_n) \),
where "\( \hat{\cdot} \)" means "delete", be the inclusion functor and
\( s^i : \text{End}(X_0, \ldots, X_{i-1}, X_i, X_i, X_{i+1}, \ldots, X_n) \to \text{End}(X_0, \ldots, X_n) \)
be the projection functor induced by the identity on the
mapping spaces. They are MTP-functors. Let
\[ \rho : \bar{W}(B \ast L_n) \to \text{End}(X_0, \ldots, X_n) \]
be an \( M^{n+1} \)TP-functor. Then \( \rho \) induces unique functors \( \rho_1 \)
and \( \rho_2 \) such that the following diagrams commute:
\[ W(B \ast L_{n+1}) \xrightarrow{s^i} \xrightarrow{s^i} W(B \ast L_n) \]
\[ \downarrow \rho_1 \downarrow \rho \]
\[ \text{End}(X_0, \ldots, X_i, X_i, \ldots, X_n) \xrightarrow{s^i} \text{End}(X_0, \ldots, X_n) \]

\[ W(B \ast L_{n-1}) \xrightarrow{\partial^i} \xrightarrow{\partial^i} W(B \ast L_n) \]
\[ \downarrow \rho_2 \downarrow \rho \]
\[ \text{End}(X_0, \ldots, \hat{X_i}, \ldots, X_n) \xrightarrow{\partial^i} \text{End}(X_0, \ldots, X_n) \]

\( \rho_1 \) and \( \rho_2 \) are understood to be the actions \( \rho \circ s^i \) and \( \rho \circ \partial^i \).

Hence \( G\text{Map}_B \) indeed is a semi simplicial complex, the \( n \)-simplexes of which are the actions of \( W(B \ast L_n) \) on \( (n+1) \)-tuples of spaces and the face and degeneracy operators are induced by composition with \( \partial^i \) respectively \( s^i \).

**Definition 5.2**: Let \((X, \gamma)\) and \((Y, \delta)\) be \( WB \)-spaces. A pair \((f, \rho)\), where \( f: X \to Y \) is a map and \( \rho: W(B \ast L_1) \to \text{End}(X, Y) \) an action, is called a **generalized homotopy \( B \)-map** if
\[ \begin{array}{cccccc}
W_B = W(B \ast L_0) & \xrightarrow{\partial^1} & W(B \ast L_1) & \xleftarrow{\partial^0} & W(B \ast L_0) = W_B \\
\gamma & \downarrow & \rho & \downarrow & \delta \\
End X & \xrightarrow{\partial^1} & End(X, Y) & \xleftarrow{\partial^0} & End Y
\end{array} \]

commutes and

\[ \rho \circ \iota \circ \Lambda(0, 1) = f \]

where \( \iota : B \ast L_1 \to W(B \ast L_1) \) is the standard section and \( \Lambda : L_1 \to B \ast L_1 \) the inclusion functor.

**Definition 5.4:** A semi simplicial complex \( K \) satisfies the **restricted Kan extension condition** if given \( n \)

\((n-1)\)-simplexes \( \sigma_i, i \in (0, \ldots, n), i \neq k \), where

\( 0 \leq k \leq n, k \neq 0, 1 \) fixed, such that

\[ \partial^{j-1} \sigma_i = \partial^i \sigma_j \quad 0 \leq i < j \leq n, i, j \neq k \]

then there exists an \( n \)-simplex \( \sigma \) such that

\[ \partial^i \sigma = \sigma_i, i \neq k. \] (i.e. it satisfies the Kan extension condition with the restriction that the omitted face is not the first or the last).
Theorem 5.5: The semi simplicial complex $G\text{Map } B^n$ satisfies the restricted Kan extension condition.

Before we start proving this theorem let us give a description of the trees representing the elements of $\mathcal{W}(B * L_n)$ which is simpler than the description in the general case. We make use of the fact that there is exactly one morphism from $i$ to $j$ in $L_n$ if $i \leq j$. In the general case we labelled the vertices by morphisms $(\beta, j) \circ (f_1 \oplus \ldots \oplus f_k)$ of $B * L_n$ into generator, the incoming edges by source $(f_1), \ldots$, source $(f_k)$, and the outgoing edge by target $(f_1) = \ldots = \text{target } (f_k) = j$. Since in $L_n$ the morphisms $f_i$ are uniquely determined by their source and target it suffices to label the vertices by a morphism of $B$ into a generator. A typical vertex now looks like

![Diagram](image)

\[\beta: r \to 1 \text{ in } B\]

\[i_1, \ldots, i_r \leq j\]

of course, we again have elements of $I$ assigned to each link. Note that in this representation a vertex labelled by 1 may only be suppressed if the incoming and outgoing edge are labelled by the same object generator.
Proof of Theorem 5.5: Given $k \neq 0$, $n$, $0 \leq k \leq n$, and for all $i \in (0,\ldots,n)$, an action $\rho_i : W(B^*L_{n-1}) \to \text{End}(X_0,\ldots,X_i,\ldots,X_n)$ such that

$$\rho_i \circ \delta^{j-1} = \rho_j \circ \delta^i \quad 0 \leq i < j \leq n, \; i, j \neq k$$

We have to construct an action $\rho : W(B^*L_n) \to \text{End}(X_0,\ldots,X_n)$ such that $\rho \circ \delta^i = \rho_i$.

For this we construct an action of a $M_{n+1}^{TP}$-subcategory of $W(B^*L_n)$ on $(X_0,\ldots,X_n)$ which extends the actions of the $\rho_i$'s and which is fibre homotopically trivial over $B^*L_n$. We then apply the Universal Theorem.

The elements of $\delta^i(W(B^*L_{n-1}))$ are represented by trees none of the edges of which has the label $i$. On those elements $\rho$ has to be given by $\rho_i$ for $i \neq k$ because of the condition that $\rho \circ \delta^i = \rho_i$. Since $\rho_i \circ \delta^{j-1} = \rho_j \circ \delta^i$, $0 \leq i < j \leq n$, $i, j \neq k$, $\rho_i$ and $\rho_j$ agree on the elements in $\delta^i(W(B^*L_{n-1})) \cap \delta^j(W(B^*L_{n-1}))$. Hence $\rho$ is well defined on all elements of $\delta^i(W(B^*L_{n-1}))$ for each $i \in (0,\ldots,n)$, $i \neq k$. This, of course, determines $\rho$ on all those elements of $W(B^*L_n)$ that are compositions of sums of elements in the $\delta^iW(B^*L_{n-1})$, $i \in (0,\ldots,n)$, $i \neq k$.

Let $C$ be the $M_{n+1}^{TP}$-subcategory of $W(B^*L_n)$ generated
by \( \delta^i \mathcal{W}(\mathcal{B}^-*L_{n-1}) \) \( i \in (0, \ldots, n) \), \( i \neq k \). By our consideration above the \( \rho_i \) define an action

\[
\eta: \mathcal{C} \rightarrow \text{End}(X_0, \ldots, X_n)
\]

by \( \eta(\delta^i \mathcal{W}(\mathcal{B}^-*L_{n-1})) = \rho_i \).

If a representing tree \( \Theta \) of a morphism of \( \mathcal{C} \) into a generator has all object generators 0, \ldots, n as labels for its edges, \( \Theta \) contains a collection of edges to which \( i \in I \) is assigned (twigs may be included) and which separate \( \Theta \) into a tree \( \varphi \) and a cone of trees \( \psi_q \) such that there exist \( i, j_q \neq k \) such that none of the edges of \( \varphi \) and \( \psi_q \) are labelled by \( i \) respectively \( j_q \).

Note that the subspace of the representing trees of the elements in \( \mathcal{C} \) is closed in the space of the representing trees of the elements of \( \mathcal{W}(\mathcal{B}^-*L_n) \). Furthermore if \( x \in \mathcal{C} \) is indecomposable in \( \mathcal{C} \) then it is indecomposable in \( \mathcal{W}(\mathcal{B}^-*L_n) \), for if \( y^o z \in \mathcal{C} \) is such that none of the edges of its representing tree is labelled by \( i \) then none of the edges of the representing trees of \( y \) and \( z \) is labelled by \( i \) and hence \( y \) and \( z \) are in \( \mathcal{C} \). Since with \{\( \Theta, \xi, \delta \)\} all elements \{\( \varphi, \xi, \delta \)\} are in \( \mathcal{C} \) where \( \varphi \) is a tree of the same type as \( \Theta, \mathcal{C} \) satisfies the requirements for the category \( \mathcal{D} \) in Theorem 4.6.
The standard augmentation \( \varepsilon = \varepsilon_{B \ast L_n} \) reduced to \( C \) augments \( C \) over \( B \ast L_n \). Define a section \( \sigma : B \ast L_n \rightarrow C \) of \( \varepsilon \mid C \) by \( \sigma|_C(a, b) = \) standard section if \( 0 \neq a = (i_1, \ldots, i_k) \) or \( b \neq n \). \( \sigma(\beta; (i_1, n), \ldots, (i_k, n)) = \{\emptyset, \text{unit}, \delta\} \) where \( \emptyset \) is the tree with the vertex at the root labelled by \( \beta \), the \( q \)-th incoming edge labelled by \( i_q \) if \( i_q \neq 0 \), and by \( i_q + 1 \) if \( i_q = 0 \), vertices labelled by \( 0 \in I \subset B^\sim(1, 1) \) on top of the \( q \)-th edge if it is labelled by \( i_q + 1 \) and their incoming edges labelled by \( 0 \). Assign \( 1 \in I \) to each link.

\[
\sigma(\beta; (0, n), (2, n), (1, n)) =
\]

The standard deformation (see p.144) gives the required deformation of \( C(a, b) \) with \( b \neq n \), into the section.

The equivariant fibrewise deformations of \( C(a, n) \) into the section are given in steps. We first shrink all links labelled by \( 0 \), we then introduce new vertices \( 0 \) (recall that \( 0 \in I \subset B^\sim(1, 1) \) is the unit) on top of each twig labelled by \( 0 \). Change the labels of the newly created links to \( 1 \) and label the new twigs \( 0 \). We get \( 1 \in I \) assigned to the new links by a deformation and then we shrink all links that are not
a new link. For each deformation we have to make sure that we stay in $C$.

Now the details:

$$H^1_t \{0, \xi, \delta\} = \{0, \xi, H^1_t(\delta)\} \text{ with } H^1_t(u_1, \ldots, u_p) = (t_1 \cdot u_1, \ldots, t_p \cdot u_p)$$

where $t_1 = t$ if $u_1$ is assigned to a link labelled by $0$ and $t_1 = 1$ otherwise. $H^1$ is well-defined, continuous, equivariant and fibrewise. If all $i, 0 \leq i \leq n$ occur as labels of links in $\delta$ then in the collection of edges to which $i \in I$ has been assigned and which decompose $\{0, \xi, \delta\}$ as mentioned at the beginning of the proof, none of the edges may be labelled by $0$. Hence this deformation stays in $C$. Denote $H^1_tC$ by $C^t$. Each element of $C^t$ can be represented by a tree none of the edges of which with exception, may be, of some twigs is labelled by $0$. The space of those trees is closed. If we stick a vertex labelled by $0 \in I \subset \mathbb{R}^+(1, 1)$ on top of each twig of those trees labelled by $0$, change the label of the newly created link from $0$ to $1$, and assign to it the value $0 \in I$, and label the twigs over the new vertices by $0$, we obtain a related representative (see picture).

\[\text{Diagram showing the transformation process.}\]
The next homotopy only affects the newly created links.

Define
\[ H_t^2(\theta, \xi, \delta) = \{ \theta, \xi, H_t^2(\delta) \} \]
with
\[ H_t^2(u_1, \ldots, u_p) = (\max(t_1, u_1), \ldots, \max(t_p, u_p)) \]
where \( t_i = t \) if \( u_i \) is assigned to an outgoing edge of a vertex whose incoming edge is labelled by 0 (such a vertex, of course, is labelled by 0). Since the multiplication \( t_1 \ast t_2 = \max(t_1, t_2) \) is associative, \( H_t^2 \) is well-defined. It clearly is continuous, equivariant, and fibrewise. By the same consideration as above, \( H_t^2 \) stays in \( C \). Denote \( H_t^2 C_1 \) by \( C_2 \).

Finally define \( H_t^3(\theta, \xi, \delta) = \{ \theta, \xi, H_t^3(\delta) \} \)
with
\[ H_t^3(u_1, \ldots, u_p) = (t_1u_1, \ldots, t_pu_p) \]
where \( t_i = 1 \) if \( u_i \) is assigned to an outgoing edge of a vertex whose incoming edge is labelled by 0 in the representation chosen above. \( t_i = t \) otherwise. \( H_t^3 \) is well-defined, continuous, equivariant, and fibrewise. Since \( H_t^3(\theta, \xi, \delta) \) is a composition of an element represented by a tree the edges of which are not labelled by 0, and an element which is a sum of elements \( t_B(0; (0, 1)) \), \((t_B^\ast \text{is the standard section})\), \( H_t^3 \) stays in \( C \). \( H_0^3(C_2(a, b)) = \sigma(\bar{B}^\ast L_n(a, b)) \).

Hence \( C \) is fibre homotopically trivially augmented over
\[ B^\sim L_n \text{ and hence over } B \ast L_n. \text{ Now apply Theorem 4.6 with } \]
\[ B = B^\sim L_n, \quad G = B \ast L_n, \quad G = D = G, \quad \delta = \text{id}_G \text{ and } \gamma = \varepsilon B \ast L_n, \]
which is possible since \( B^\sim L_n = (B \ast L_n)^\sim. \]

**Remark 5.6:** If \( B \) has isolated identities we get the same result for GMap \( B \) using the Theorem 4.9 instead of 4.6.

**Remark 5.7:** If \( n = 2 \) let \((f, \rho_2): (X, \alpha_0) \to (Y, \alpha_1)\) and \((g, \rho_0): (Y, \alpha_1) \to (z, \alpha_2)\) be generalized homotopy \( B^\sim - \) maps. Then there exists an extension \( \rho : W(B^\sim L_2) \to \text{End}(X, Y, Z) \) such that \( \rho \circ \delta^0 = \rho_0, \rho \circ \delta^2 = \rho_2 \) and \( \rho \circ \delta^1 = \tau_{B^\sim L_1} \circ L(0, 1) = g \circ f. \) (This follows from Lemma 4.8 choosing \( \tau: B^\sim L_2 (0, 2) \to G(0, 2) \) to be

\[
\tau(\beta; (0, 2)) = \begin{cases} 
0 & \text{if } \beta = 0 \\
1 & \beta = 0, 1 \\
\beta & \text{ assigned to the link} \\
2 & \text{assigned to the link}
\end{cases}
\]

The same holds for generalized homotopy \( B \)-maps if \( B \) has isolated identities.

For most purposes the concept of a generalized homotopy \( B \)-map has undesirable complications arising from the existence
of mixed maps, such as $X \times Y \rightarrow Y$. For this reason we discontinue to study them, although the Theorem 5.5 provides us with a good starting point for the development of the theory.

To be able to give some other definition for structure maps we have to introduce a new type of category of operators.
36 REDUCED CATEGORIES OF OPERATORS

Definition 6.1: A reduced $M^n_{\text{TP}}$-category $B$ has as objects finite sequences $a = (i_1, \ldots, i_k)$ of integers $0, \ldots, n-1$ such that $i_1 = \ldots = i_k$, the empty sequence is included. The morphisms between two objects form a topological space in $CG$ and composition is continuous. We are given a multiplicative structure $\oplus$ on $B$ such that

$$(i_1, \ldots, i_m) \oplus (j_1, \ldots, j_k) = (i_1, \ldots, i_m, j_1, \ldots, j_k)$$

whenever $i_1 = \ldots = i_m = j_1 = \ldots = j_k$. It induces a strictly associative map of the corresponding morphism spaces and behaves like a functor whenever it is defined, i.e.

$$(\beta \oplus \gamma) \circ (\beta' \oplus \gamma') = (\beta \circ \beta') \oplus (\gamma \circ \gamma')$$

$$1_a \oplus 1_b = 1_{a \oplus b}$$

Furthermore we are given permutations satisfying the conditions (d) of Definition 1.1.

Analogously we can define reduced $M^n_{\text{TP}}$-categories

Each $M^n_{\text{TP}}$-category $B$ gives rise to a reduced $M^n_{\text{TP}}$-category $RB$, the subcategory of $B$ consisting of all objects $(i_1, \ldots, i_k)$ of $B$ such that $i_1 = \ldots = i_k$ and all morphisms between such objects. Note that for $n = 1$ the definition of an $M^1_{\text{TP}}$-category and a reduced $M^1_{\text{TP}}$-category coincide.
Definition 6.2: A reduced MTP-functor between a reduced $M^\text{RTP}$-category $B$ and a reduced $M^\text{RTP}$-category $C$ is a continuous functor mapping object generators into object generators and preserving sums and permutations. If it in addition preserves object generators if $m = n$, it is called a reduced $M^\text{RTP}$-functor.

If $\gamma: B \to C$ is an MTP-functor then its restriction $\gamma: RB \to RC$ is a reduced MTP-functor.

We say that a reduced $M^\text{RTP}$-category $B$ acts on $(X_0, \ldots, X_{n-1})$ if we are given a reduced MTP-functor $\gamma: B \to \text{REmd}(X_0, \ldots, X_{n-1})$.

In order to develop a theory for actions of reduced $M^\text{RTP}$-categories we are going to prove a universal theorem equivalent to 4.6 for RWB. Clearly the notion of a fibre homotopically trivial augmentation holds for reduced $M^\text{RTP}$-categories too, as well as the notion of a section.

Lemma 6.3: Each element $x \in \text{RWB}(a, b)$ can be decomposed into indecomposable elements (in the sense of Definition 4.1), $x = x_1^o \circ \ldots \circ x_p$. This decomposition is unique up to the equivalence generated by
(a) \[ x_1 \circ \ldots \circ (x_i \circ 1) \circ (1 \circ x_{i+1}) \circ \ldots \circ x_p \]
\[ = x_1 \circ \ldots \circ (x_i' \circ x_{i+1}') \circ \ldots \circ x_p \]
\[ = x_1 \circ \ldots \circ (1 \circ x_{i+1}) \circ (x_i \circ 1) \circ \ldots \circ x_p \]

(b) \[ x_1 \circ \ldots \circ (x_i \circ \xi) \circ x_{i+1} \circ \ldots \circ x_p \]
\[ = x_1 \circ \ldots \circ x_i \circ \xi \circ x_{i+1} \circ \ldots \circ x_p \]
\[ = x_1 \circ \ldots \circ x_i \circ (\xi \circ x_{i+1}) \circ \ldots \circ x_p \]

where \( \xi \) is a permutation.

**Proof:** In view of Lemma 4.2 an element \( x \in \text{RWB}(a, b) \) is decomposable in \( \text{RWB}(a, b) \) iff there exists a collection of edges in a non-degenerate representing copse labelled by the same object generator and the values \( 1 \in \Gamma \) assigned to them which separate the copse into two copses (here we again suppose that \( 1 \) is assigned to the twigs and the roots. "Separate" means that each complete edge path runs through exactly one edge of this collection). Chop all edges of any such collection (chopping a twig or a root gives rise to a trivial tree) to obtain indecomposable elements. As in Lemma 4.3 there are three choices involved which are taken care of by the relations (a) and (b):

(1) the order in which we chop these collections

(2) the choice of the particular non-degenerate representative

(3) the choice of the position of permutations
Since a morphism in $\text{RWB}$ can be decomposable in $\text{WB}$ even if it is indecomposable in $\text{RWB}$, we have to refine the filtration $\text{RWP}^B$ of $\text{RWB}$: For any $\text{M}^{\text{TP}}$-category $B$, for which the construction $W$ is defined, let $\text{RWP}^p,q_B$ be the subcategory of $\text{RWP}^B$ generated by $\text{RWP}^{p-1}B$ and all those elements $x = \{\theta, \xi, \delta\}$ such that $(\theta, \xi) \in T_pB(a, b)$ and $\delta \in I^p$ has a collection $\beta$ of $p$-$q$ coordinates with value 1. Denote the (closed) subspace of $Q_{\alpha, p}(a, b)$ consisting of these representatives $(\theta, \xi, \delta)$ by $Q_{\alpha, p, \beta, q}(a, b)$. More precisely speaking, $Q_{\alpha, p, \beta, q}$ is the subspace of $Q_{\alpha, p}$ of those elements $(\theta, \xi)$ such that to a chosen collection $\beta$ of $p$-$q$ links of $\theta$ the value 1 has been assigned. Note that if the collection $\beta$ separates the tree $\theta$ into a tree and a copse representing elements in $\text{RWB}$ (we might have to add some twigs to the collection) then each element in $Q_{\alpha, p, \beta, q}$ represents a composition. Let $Q'_{\alpha, p, \beta, q} = Q_{\alpha, p, \beta, q}$ if $\beta$ is a collection that separates $\theta$ into a tree and a copse representing elements in $\text{RWB}$. Otherwise let $Q'_{\alpha, p, \beta, q_c} \subset Q_{\alpha, p, \beta, q}$ be the (closed) subspace of those representatives $(\theta, \xi, \delta)$ that are either degenerate or $1 \in I$ has been assigned to more links of $\theta$ than just to the ones in the collection $\beta$. $Q'_{\alpha, p, \beta, q}$ consists of all those elements of $Q_{\alpha, p, \beta, q}$ that are related to some element of lower filtration $p$ or $q$, or that represent
composites of elements, that can be represented by elements of some lower filtration $p$. If $(a,\xi,\beta) \in Q_{\alpha,\beta,0}(a,b)$ then $\beta = (1,\ldots,1)$. Hence $Q_{\alpha,\beta,0} = Q_{\alpha,\beta,0}^0$ and hence $RWP,0,B = RWP,0,B$. 

Let $D$ be a subcategory of $RWB$ such that $D_{\alpha,\beta}(a,b)$ is closed in $Q_{\alpha,\beta}(a,b)$ for all $\alpha,\beta,a,b$ (see p. 53) and such that if $x \in D$ is a composition $x = y \circ z$ with $y,z \in RWB$ then $y$ and $z$ are in $D$. Let $D_{\alpha,\beta,q}(a,b) = D_{\alpha,\beta}(a,b) \cap Q_{\alpha,\beta,q}(a,b)$.

Lemma 4.5 can now be stated for reduced $nTP$-categories and in view of Lemma 6.3 the proof goes over:

**Lemma 6.4:** Let $C$ be a reduced $nTP$-category and $D$ a subcategory of $RWB$ as given above. Let $\delta_t : D \to C$ be a homotopy of functors preserving objects, sums and permutations.

1. Given a homotopy of reduced $nTP$-functors $\gamma_{t,q}^p : RW_{t,q}B \to C$ and equivariant maps $f_{\alpha,\beta,q} : Q_{\alpha,\beta,q}(a,b) \times I \to C(a,b)$ for all $q,\beta, a,b$ such that
(a) \( \gamma_t^{p,q-1} \mid_{RW^p,q^{-1}B \cap D} = \delta_t \mid_{RW^p,q^{-1}B \cap D} \)

(b) \( f_{a,p,\beta,q} \mid_{D,a,p,\beta,q(a, b)x(t)} = \delta_t \circ (x_{a,p} \mid_{D,a,p,\beta,q(a, b)}) \)
\( f_{a,p,\beta,q} \mid_{Q',a,p,\beta,q(a, b)x(t)} = \gamma_t^{p,q-1} \circ (x_{a,b} \mid_{Q,a,p,\beta,q(a, b)}) \)

(c) \( f_{a,p,\beta,q}(x, t) \) factors through the relation (2.11) for each \( t \in I \).

If \( x \) is a trivial tree representing the identity of \( b \), then \( f_{a,-1,\beta,0}(x) = 1_b \).

Then there exists a unique homotopy of reduced \( M^n_{TP} \)-functors

\[ \gamma_t^{p,q} : RW^p,q_B \rightarrow C \]

extending \( \gamma_t^{p,q-1} \) and \( \delta_t \mid_{D} \cap RW^p,q_B \) such that
\[ \gamma_t^{p,q-1} \circ x_{a,p} \mid_{Q,a,p,\beta,q(a, b)} = f_{a,p,\beta,q} \mid_{Q,a,p,\beta,q(a, b)x(t)}. \]

If \( q-1=p \) we can substitute \( (p,q-1) \) by \( (p+1,0) \)

(2) Given homotopies of reduced \( M^n_{TP} \)-functors \( \gamma_t^{p,q} : RW^p,q_B \rightarrow C \) for all \( p \) and \( q \) such that
\[ \gamma_t^{p,q} \mid_{RW^p,q_B \cap RS^t_B} = \gamma_t^{S,t} \mid_{RW^p,q_B \cap RS^t_B} \]
and \( \gamma_t^{p,q} \mid_{RW^p,q_B \cap D} = \delta_t \mid_{RW^p,q_B \cap D} \), then there exists a unique homotopy of reduced \( M^n_{TP} \)-functors
\( \gamma_t : \text{RWB} \rightarrow \mathcal{C} \) extending \( \delta_t \) such that
\[
\gamma_t[R^{\mathcal{D}}]^B = \gamma_t^D^q.
\]

In the same manner we can state and prove the analogue of Lemma 4.7 with a refinement of the filtration \( F_p \) in the spaces \( R_{\alpha,p,k} \). Let \( R_{\alpha,p,k} \) be the (closed) subspace of \( R_{\alpha,p,k} \) of those elements \((\theta, \xi, \delta)\) such that \( 1 \in I \) has been assigned to a collection \( \beta \) of \( p-q \) links of \( \theta \). If the collection \( \beta \) separates the tree \( \theta \) then put \( R'_{\alpha,p,k,\beta} = R_{\alpha,p,k,\beta,q} \). Otherwise let \( R'_{\alpha,p,k,\beta,q} \) consist of those elements that are related to some element of lower filtration \( p,k \), (see p.59) or that have \( 1 \in I \) assigned to more than just the links of the collection \( \beta \). We refrain from stating the analogue of Lemma 4.7.

**Theorem 6.5** (The universal property):

Given a commutative diagram

\[
\begin{array}{ccc}
\text{RWB} & \xrightarrow{\rho} & D \\
\downarrow{\varepsilon} & & \downarrow{\delta_t} \\
\text{RB} & \xrightarrow{\gamma} & \mathcal{C} \\
\end{array}
\]

\[
\begin{array}{ccc}
D & \xrightarrow{\delta_t} & \mathcal{G} \\
\downarrow{\delta_t} & & \downarrow{\mu} \\
\mathcal{G} & \xrightarrow{\mu} & \mathcal{C} \\
\end{array}
\]
of reduced $M^nTP$-categories $RWB, RB, C, G$, where $B$ is an $M^nTP$-category, and a subcategory $D$ of $WB$, reduced $M^nTP$-functors $\gamma, \mu$, the standard augmentation $\varepsilon = \varepsilon_B$, the inclusion functor $\rho$ and a homotopy of functors $\delta_t$ preserving objects, sums and permutations for each $t \in I$.

Assume

(1) If $x \in D$ is a composition in $RWB$, $x = y \circ z$, then

$y$ and $z$ are in $D, D_{a,p}(a, b)$ is closed in $Q_{a,p}(a, b)$, and each connected component of $D_{a,p,b,q}(a, b)$ containing a point $x \notin Q'_{a,p,b,q}(a, b)$ is open and closed in $Q_{a,p,b,q}(a, b)$

(2) $B$ and $\gamma$ and $\mu$ satisfy the conditions (2) and (3) of Theorem 4.6

Then

I: There exists a reduced $M^nTP$-functor $\nu_0: RWB \to G$

such that $\mu \circ \nu_0 = \gamma \circ \varepsilon$ and $\nu_0 \circ \rho = \delta_0$

II: Given any two reduced $M^nTP$-functors $\nu_0, \nu_1: RWB \to G$

such that $\mu \circ \nu_0 = \mu \circ \nu_1 = \gamma \circ \varepsilon$ and $\nu_0 \circ \rho = \delta_0$, $\nu_1 \circ \rho = \delta_1$, then there exists a homotopy of reduced $M^nTP$-functors $\nu_t: RWB \to G$ between $\nu_0$ and $\nu_1$ such that $\nu_t \circ \rho = \delta_t$ and $\mu \circ \nu_t = \gamma \circ \varepsilon$. 
Proof: The proof proceeds on the same lines as the proof of Theorem 4.6. We again construct compatible functors

\( \nu_{0}^{k,q} : F_{k,q}^{\text{RWB}} \to G \) extending \( \delta_{0} \), respectively

\( \nu_{t}^{k,q} : F_{k,q}^{\text{RWB}} \to G \) extending \( \delta_{t} \). We restrict ourselves to proving I. The proof of II is similar. The differences have been described in the proof of Theorem 4.6.

\( \nu_{0}^{-1,q} \) and \( \nu_{t}^{-1,q} \) are uniquely determined by \( \delta_{0} \) respectively \( \delta_{t} \).

Suppose inductively that we have defined

\( \nu_{0}^{p,q-1} : F_{p,q-1}^{\text{RWB}} \to G \) such that

\[ \nu_{0}^{p,q-1} |_{F_{r,s}^{\text{RWB}}} = \nu_{0}^{r,s} \text{ for } r = p \text{ and } s < q - 1 \]

or \( r < p \) and

\[ \mu \circ \nu_{0}^{p,q-1} = \gamma \circ x_{p}^{\text{RWB}}. \]

Recall that \( \nu_{0}^{p,p} \) induces \( \nu_{0}^{p+1,0} \). We have to define equivariant maps

\[ f = f_{a,p,k,\beta} : R = R_{a,p,k,\beta} (a, b) \to G(a, b) \]

which factor through (2.11).

(We omit the indices whenever there is no danger of confusion)

satisfying

\[ f |_{R'} (a, b) = \nu_{0}^{p,q-1} \circ x_{a,p,k} |_{R'} (a, b) \]

and

\[ \mu \circ f = \gamma \circ x_{a,p,k} |_{R} \]

If \( \nu^{p,q-1} \) does not determine \( f \) on the whole of

\( R = P \times I_{\beta}^{q} \), i.e. if \( R \not\supset R' \), (recall \( R_{a,p,k} = P_{a,p,k} \times I_{\beta}^{q} \)). Hence

\( p = P_{a,p,k} \). \( \beta \in P_{a,p,k} \times I_{\beta}^{q} \), where \( I_{\beta}^{q} \subset I^{p} \) is the face with \( t_{i} = 1 \) for each \( i \) in the collection \( \beta \).
then it determines it exactly on \( P \times \partial I^q_\beta P' \times I^q_\beta \) 

Now we can proceed in exactly the same way as in the proof of Theorem 4.6 using \( I^q_\beta \) instead of \( I^P \).

**Remark 6.6:** The analogues of the Lemma 4.8, the Theorems 4.9, 4.12, and the Proposition 4.16 hold for reduced \( M^{nTP} \)-categories and reduced \( M^{nTP} \)-functors.
§7 HOMOTOPY B-MAPS

To simplify the notation we denote the sequences in $\text{RW}(B^*L_n)$ of length $m$ in the generators $0$, $1$, or $2$ by $m$, $m'$, $m''$ respectively. We hardly ever deal with $B^*L_n$ where $n>2$.

Definition 7.1: Let $B$ be a category of operators and $(X,\gamma)$, $(Y, \delta)$ $W_B$-spaces. A pair $(f, \rho)$, where $f: X \to Y$ is a map and $\rho: \text{RW}(B^*L_1) \to \text{REnd}(X,Y)$ a reduced $M^2\text{TP}$-functor, is called a homotopy $B$-map between $(X,\gamma)$ and $(Y, \delta)$ if

$$
\begin{array}{c}
\text{WB} = \text{RW}(B^*L_0) & \xrightarrow{\partial^1} & \text{RW}(B^*L_1) & \xleftarrow{\partial^0} & \text{RW}(B^*L_0) = \text{WB} \\
\downarrow \gamma & & \downarrow \rho & & \downarrow \text{End}Y \\
\text{End}X & \xrightarrow{\partial^1} & \text{REnd}(X,Y) & \xleftarrow{\partial^0} & \text{End}Y \\
\end{array}
$$

commutes (where $\partial^i$ is the restriction of the face operator $\partial^i$ to the restricted subcategories) and

$$
\rho \circ \frac{B^*L_1}{\Lambda(0,1)} = f
$$

where $\frac{B^*L_1}{\Lambda(0,1)}$ is the reduction of the standard section to $R(B^*L_1)$ and $\Lambda: L_1 \to R(B^*L_1)$ is the canonical inclusion functor.
Remark: Although we will distinguish between an $\mathcal{M}^n\mathcal{P}$-category $\mathcal{B}$ and its reduced subcategory $\mathcal{R}\mathcal{B}$ we use the same symbol for an $\mathcal{M}^n\mathcal{P}$-functor and its restriction to the reduced subcategory.

Definition 7.2: Let $(f, \rho), (g, \kappa): (X, \gamma) \to (Y, \delta)$ be homotopy $\mathcal{B}$-maps. We call $(f, \rho)$ and $(g, \kappa)$ homotopic and write $(f, \rho) \sim (g, \kappa)$ if there exists a homotopy of reduced $\mathcal{M}^2\mathcal{P}$-functors $\lambda_t: \text{RW}(\mathcal{B}*\mathcal{L}_1) \to \text{REnd}(X,Y)$ such that $\lambda_0 = \rho$ and $\lambda_1 = \kappa$, and $\lambda_t \circ \partial^2 = \partial^1 \circ \lambda_t$ for all $t \in I$.

Analogously define "homotopic" for generalized homotopy $\mathcal{B}$-maps.

A generalized homotopy $\mathcal{B}$-map $(f, \rho): (X, \gamma) \to (Y, \delta)$ canonically induces a homotopy $\mathcal{B}$-map $(f, \rho'): (X, \gamma) \to (Y, \delta)$ by restricting the functor $\rho: W(\mathcal{B}*\mathcal{L}_1) \to \text{End}(X,Y)$ to the reduced $\mathcal{M}^2\mathcal{P}$-subcategory $\text{RW}(\mathcal{B}*\mathcal{L}_1)$.

Theorem 7.3: Let $(f, \rho): (X, \gamma) \to (Y, \delta)$ be a homotopy $\mathcal{B}^\sim$-map. Then $\rho$ induces an action $\nu: W(\mathcal{B}^\sim*\mathcal{L}_1) \to \text{End}(X,Y)$ such that $(f, \nu): (X, \gamma) \to (Y, \delta)$ is a generalized homotopy $\mathcal{B}^\sim$-map. Furthermore if $(f, \rho)$ is the canonical homotopy $\mathcal{B}^\sim$-map obtained from a generalized homotopy
$B^\sim$-map $(f, \rho') : (X, \gamma) \to (Y, \delta)$, then $(f, \nu) \equiv (f, \rho')$.

**Proof:** $RW(B^\sim*L_1)$ generates an $M^2TP$-subcategory $Q$ of $W(B^\sim*L_1)$, i.e. each morphism of $Q$ is a composition of sums of elements in $RW(B^\sim*L_1)$ and of permutations. Let $y = x_1 \oplus \ldots \oplus x_p$ be an element in $Q$, $x_i \in RW(B^-*L_1)$. Define $\eta(x_1 \oplus \ldots \oplus x_p) = \rho(x_1) \times \ldots \times \rho(x_p)$ and $\eta(\xi) = \xi$, where $\xi$ is a permutation. Extend $\eta$ to an action of $Q$ by

$$\eta(y_1 \circ \ldots \circ y_n) = \eta(y_1) \circ \ldots \circ \eta(y_n)$$

where $y_i$ is a sum of morphisms in $RW(B^\sim*L_1)$.

$\varepsilon = \varepsilon_{B^\sim*L_1}|_Q$ augments $Q$ over $B^\sim*L_1$. We will show that $\varepsilon$ is fibre homotopically trivial and then we will apply the Universal Theorem.

Note that $Q(n,1) = W(B^\sim*L_1)(n,1)$, $Q(n,1') = W(B^\sim*L_1)(n,1')$ and $Q(n',1') = W(B^\sim*L_1)(n',1')$. Hence the standard section and the standard deformation guarantee that $\varepsilon$ is fibre homotopically trivial on these morphism spaces. So we can restrict our attention to $Q(a,1')$ where $a = (i_1, \ldots, i_k)$ with $0, 1 \in a$. As in §5 we use the simplified description for the trees.

Define a section $\sigma: B^\sim*L_1(a,1') \to C(a,1')$ by

$$\sigma(\beta; (i_1,1), \ldots, (i_k,1)) = \begin{cases} \end{cases}$$
The value 1 is assigned to each link. More precisely
\[ \sigma(\beta; (i_1, 1), \ldots, (i_k, 1)) = \{ \theta, \text{unit}, \delta \} \]
where \( \theta \) is the tree with the vertex at the root labelled by \( \beta \), all incoming and outgoing edges labelled by 1. If \( i_q = 0 \), then on top of the \( q \)-th incoming edge sits a vertex labelled by 0 (the identity of \( B^- \)), and its incoming edge is labelled by 0.

Each representing tree of \( C(a, 1') \) has a collection of edges to which 1 \( \in I \) is assigned, and which decompose the tree into a tree all twigs of which are labelled by 1, and a copse all the twigs of each individual tree of which are labelled by 0 or 1 only. Conversely each tree with such a collection of edges represents an element in \( C \).

Define the equivariant fibrewise deformation into the section in steps:

\[ H_t^1(\theta, \xi, \delta) = \{ \theta, \xi, H_t^1(\delta) \} \text{ with } H_t^1(u_1, \ldots, u_p) = (t_1 \cdot u_1, \ldots, t_p \cdot u_p) \]
where \( t_1 = t \) if \( u_1 \) is assigned to a link labelled by 0, and \( t_1 = 1 \) otherwise. Since each link in the separating collection of \( \theta \) is labelled by 1, this homotopy stays in \( C \). It certainly is well defined, continuous, equivariant, and fibrewise. Each element in \( H_0^1 C(a, 1') \) can be represented by a tree such that only its twigs are labelled by 0, and its vertices at the bottom of twigs labelled by 0 are labelled by 0 (the identity in \( B^- \)). Now define
\[ H_t^2(\theta, \xi, \delta) = \{ \theta, \xi, H_t^2(\delta) \} \] with \[ H_t^2(u_1, \ldots, u_p) = (\max(t_1, u_1), \ldots, \max(t_p, u_p)) \],

where \( t_i = t \) if \( u_i \) is assigned to a link that is preceded by a twig labelled by 0 in our chosen representation, and \( t_i = 0 \) otherwise. Since the multiplication map "\( \max \)" is associative, \( H_t^2 \) is well defined. It is continuous, equivariant, and fibrewise. Since links to which the value 1 is assigned are not affected, \( H_t^2 \) stays in \( \mathcal{C} \). Each element of \( H_1^2 \circ H_0^1(\mathcal{C}(a, 1')) \) is a composition \( y \circ z \), where \( y \in W(\mathcal{B}^*L_1)(k', 1') \) and \( z = x_1 \circ \ldots \circ x_k \) with \( x_q = 1 \) if \( i_q = 1 \), or

\[
\begin{align*}
x_q &= \{ \theta, \text{unit}, 1^0 \} \quad \text{with } \theta = \begin{cases} 0 & \text{if } i_q = 0, \\ 1 & \text{if } i_q = 1. \end{cases}
\end{align*}
\]

Hence \( z \) is uniquely determined by \( a \). If \( K_t \) is the standard deformation of \( W(\mathcal{B}^*L_1)(k', 1') \) into the standard section, then the deformation \( H_t^3 \), given by \( H_t^3(y \circ z) = K_t(y) \circ z \), deforms \( H_1^2 \circ H_0^1(\mathcal{C}(a, 1')) \) into the given section.

Let \( \mathcal{D} \) be the subcategory \( \partial^0WB^* \cup \partial^1WB^* \) of \( W(\mathcal{B}^*L_1) \). \( \mathcal{D} \) satisfies the requirements of the Universal Theorem. Define \( x : \mathcal{D} \to \mathcal{C} \) to be the inclusion. Define

\[ \tau = \sigma : \mathcal{B}^*L_1(1, 1') \to \mathcal{C}(1, 1'). \]

By the Universal Theorem and Lemma 4.8 we obtain an \( \mathbb{M}^2\text{TP}-\text{functor} \) \( \mu : W(\mathcal{B}^*L_1) \to \mathcal{C} \) extending \( x \), and such that
\[ \varepsilon \circ \omega = \varepsilon , \quad \text{and} \quad \eta \circ \omega = \Lambda (0,1) = f. \] (For \( \varepsilon \) and \( \omega \) see p. 69). Hence

\[(f, \eta \circ \omega) : (X, y) \to (Y, \delta) \]
gives the required generalized homotopy \( B^- \)-map.

Now suppose that \( \rho \) has been obtained by restricting \( \rho' : W(B^- L_1) \to \text{End}(X,Y) \). Let \( \lambda : C \to W(B^- L_1) \) be the inclusion functor. Then \( \eta = \rho' \circ \lambda \) since \( \rho' \) is a \( M^2 \)-TP-functor.

By definition \( \varepsilon = \varepsilon \circ \lambda \), and hence \( \varepsilon' \circ \omega = \varepsilon \). Let \( \delta_t : D \to W(B^- L_1) \) be the inclusion functor for all \( t \). Then by Theorem 4.6 II there exists a homotopy of \( M^2 \)-TP-functors \( F_t : W(B^- L_1) \to W(B^- L_1) \) extending \( \delta_t \), and such that \( F_0 = \lambda \circ \omega \) and \( F_1 = \text{id} \cdot \rho' \circ F_t \) gives the required homotopy \( (f, \eta \circ \omega) \cong (f, \rho') \).

Remark 7.4:

1. Since morphisms that are indecomposable in \( C \) can be decomposable in \( W(B^- L_1) \) we cannot expect that the reduced action \( \rho \) induces a canonical action \( \sigma : W(B^- L_1) \to \text{End}(X,Y) \).

2. Theorem 7.3 can be proved for actions \( \rho : RW(B^- L_n) \to R\text{End}(X_0, \ldots, X_n) \) with \( n \) arbitrary. But since the obtained \( M^{n+1} \)-TP-functor
\( v: W(B^*L_n) \to \text{End}(X_0, \ldots, X_n) \)
is not canonically induced by \( \rho \) it is not very interesting.

(3) An analogue theorem holds for homotopy \( B^- \)-maps if \( B \) has isolated identities. Just replace \( B^- \) by \( B \).

**Definition and Lemma 7.5:** Let \( f: (X, v) \to (Y, \mu) \) be a \( WB \)-homomorphism (see Definition 5.1). The induced homotopy \( B^- \)-map \( f_* = (f, f_*) : (X, v) \to (Y, \mu) \) is defined by

1. \( f_*|\delta^0WB = \mu \), \( f_*|\delta^1WB = v \)
2. \( f_*|\text{RW}(B^*L_1)(\alpha, 1') \) is given by the composite

\[
\text{RW}(B^*L_1)(\alpha, 1') \overset{\text{so}}{\to} \text{WB}(\alpha, 1) \to \text{End} X \to \text{REnd}(X, Y) \overset{f^0}{\to}
\]

where \( s^0 \) is the degeneracy functor, and \( (f_0)_q = f^q \phi \) for \( q: \alpha \to \alpha \) (we frequently shorten \( f^q = f_1 \cdots f_q, x \rightarrow f(x), \text{names} \rightarrow \text{df} \)).

Conversely each homotopy \( B^- \)-map \( (f, x): (X, v) \to (Y, \mu) \) such that \( x \) satisfies (2) is induced by a \( WB \)-homomorphism.

**Proof:** \( f_* \) is continuous, by the normality of \( \text{RW}(B^*L_1) \) well defined, and preserves sums, permutations, and identities. Since \( s^0 \) and \( v \) are functors we only have to show that composition of \( x \in \text{RW}(B^*L_1)(\alpha', 1') \) with \( y \in \text{RW}(B^*L_1)(\beta', \alpha') \) is preserved:
\[ f_*(x \circ y) = f \circ (v \circ s^0(x)) \circ (v \circ s^0(y)) \]
\[ = f \circ v(x) \circ (v \circ s^0(y)) \]
\[ = \mu(x) \circ f \circ (v \circ s^0(y)) \]
\[ = f_*(x) \circ f_*(y) . \]

Conversely given a homotopy \( B \)-map \( (f,x) \) such that \( x \) satisfies (2). Then \( x \circ \ell_{B^*L_1} \circ \Lambda(0,1) = f, \) and \( s^0 \circ \ell_{B^*L_1} \circ \Lambda(0,1) = 1 \) 
\( \in WB \). Hence for \( x \in WB \) considered embedded in \( W(B^*L_1) \) by \( \delta^0 \) or \( \delta^1 \):
\[ x(x) \circ (\ell_{B^*L_1} \circ \Lambda(0,1) \oplus \ldots \oplus \ell_{B^*L_1} \circ \Lambda(0,1))) = f \circ v(x) \circ 1_x \]
\[ = x(x) \circ f \circ 1_x \]

Hence \( f \circ v(x) = \mu(x) \circ f \).

This also follows from the tree representation. ]]

Clearly composites of \( B \)-homomorphisms are \( B \)-homomorphisms. Neither do we have any problems in defining composites of \( WB \)-homomorphisms with homotopy \( B \)-maps:

**Definition and Lemma 7.6:** Let \( (f,\rho): (X,\mu) \to (Y,\nu) \) be a homotopy \( B \)-map and \( g: (Y,\nu) \to (Z,\lambda) \) a \( WB \)-homomorphism. Then there exists a canonical composite homotopy \( B \)-map \( g^0(f,\rho) = (g^0f,\rho): (X,\mu) \to (Z,\lambda) \) defined by
\[ x| \delta^0_{WB} = \lambda , \quad x| \delta^1_{WB} = \mu \]
\[ x| RW(B^*L_1)(n,1') \) is defined by \( x(x) = g^0\rho(x) . \)
Proof: Again we have to show that $x$ is a functor. Since $\rho$ extends $\mu$ it suffices to show that $x$ preserves compositions of $x \in \delta^0_{WB}$ with $y \in \text{RW}(B^*L_1)(n,m')$:

$$x(x \circ y) = g \circ \rho(x) \circ \rho(y) = \lambda(x) \circ g \circ \rho(y) = x(x) \circ x(y).$$

Remark: Analogously we can define compositions $(f, \rho) \circ h$ where $h: (W, \sigma) \to (X, \mu)$ is a $WB$-homomorphism.

Again we run into trouble if we attempt to construct the category of $WB$-spaces and homotopy $B$-maps, for as in the case of the generalized homotopy $B$-maps the composite is only defined up to a homotopy, which is itself defined only up to a homotopy, which is ... . To get around this difficulty we again form a semi simplicial complex $\text{Map}_B$, the $n$-simplexes of which are actions of $\text{RW}(B^*L_n)$ on $(n+1)$-tuples of spaces. The face and degeneracy operators are induced by the compositions

$$\rho \circ \delta^1: \text{RW}(B^*L_{n-1}) \to \text{RW}(B^*L_n) \to \text{REnd}(X_0, \ldots, X_n)$$

$$\rho \circ s^1: \text{RW}(B^*L_{n+1}) \to \text{RW}(B^*L_n) \to \text{REnd}(X_0, \ldots, X_n)$$

(compare p. 84).

Theorem 7.7: The semi simplicial complex $\text{Map}_B$ satisfies the restricted Kan extension condition.
If $B$ has isolated identities, then $\text{Map}_B$ satisfies the restricted Kan extension condition.

The proof is exactly the same as the one of Theorem 5.5 with the exception that we use $RW(B^\sim L_n)$ instead of $W(B^\sim L_n)$ and Theorem 6.5 instead of Theorem 4.12.

The Remark 5.7 applies to the reduced case too.

**Definition 7.8:** Let $(f_i, \rho_i): (X, \mu) \rightarrow (Y, \nu)$, $i = 0, 1$, be homotopy $B$-maps. Then we call $(f_0, \rho_0)$ and $(f_1, \rho_1)$ s-homotopic and write $(f_0, \rho_0) \sim (f_1, \rho_1)$ if there exists a reduced $M^3TP$-functor $\sigma: RW(B^\sim L_2) \rightarrow REnd(X, Y, Y)$ such that $\sigma \circ \delta^0 = \rho_0$, $\sigma \circ \delta^1 = \rho_1$, and $\sigma \circ \delta^2 = 1_{Y^*}$.

The condition $\sigma \circ \delta^0 = (1_Y)_*$ is equivalent to saying that $\sigma \circ \delta^0$ is degenerate. It is easy to show that a homotopy $B$-map is degenerate iff it is the homotopy $B$-map induced by the identity.

**Lemma 7.9:** Let $(f, \rho): (X, \mu) \rightarrow (Y, \nu)$ be a homotopy $B$-map, $g: (Y, \nu) \rightarrow (Z, \lambda)$ a $WB$-homomorphism, $(g \circ f, x): (X, \mu) \rightarrow (Z, \lambda)$ their canonical composite. Then there exists an action $\sigma: RW(B^\sim L_2) \rightarrow REnd(X, Y, Z)$ such that $\sigma \circ \delta^1 = x$, $\sigma \circ \delta^0 = g_*$, and $\sigma \circ \delta^2 = \rho$. 
**Proof**: Define \( \sigma \) as follows:

\[
\begin{align*}
\sigma \mid \delta^2 &\text{RW}(B\ast L_1) = \rho \\
\sigma \mid \delta^0 &\text{RW}(B\ast L_1) = g_* \\
\sigma \mid &\text{RW}(B\ast L_2) \left( \eta, 1'' \right) = x \circ s^1.
\end{align*}
\]

\( \sigma \) is continuous, well defined, preserves sums, permutations, and identities. It satisfies the statement of the Lemma.

It remains to show that \( \sigma \) is a functor, and for this it suffices to show that \( \sigma \) preserves compositions of \( x \in \delta^0 \text{RW}(B\ast L_1) \) with \( y \in \delta^2 \text{RW}(B\ast L_1) \):

\[
\begin{align*}
\sigma(x \circ y) &= x(s^1(x)) \circ x(s^1(y)) \\
&= \lambda(s^0(x)) \circ g \circ \rho(y) \\
&= g \circ \nu(s^0(x)) \circ \sigma(y) \\
&= \sigma(x) \circ \sigma(y). \quad \text{]} \end{align*}
\]

**Remark**: We can prove an analogous lemma for compositions

\((f \circ h, \zeta) \circ (f, \rho)\) with a \( WB \)-homomorphism \( h: (W, \omega) \to (X, \mu) \).

Clearly "\( \approx \)" is an equivalence relation. From Theorem 7.7 and Lemma 7.9 we can immediately deduce that "\( \approx \)" is an equivalence relation. For reflexivity follows from Lemma 7.9, while symmetry and transitivity follow from Theorem 7.7 and a trivial version of Lemma 7.9 by con-
sidering the following 3-simplexes: All maps are supposed to be homotopy $B^n$-maps or homotopy $B$-maps and $B$ has isolated identities.

$\begin{array}{c}
\begin{array}{c}
\begin{array}{ccc}
0 & 1 & 2 \\
\uparrow & \uparrow & \uparrow \\
g & 1 & g \\
\end{array}
\end{array}
\end{array}$

I symmetry: The bottom is given by the homotopy $(f, \rho) \simeq (g, \alpha)$. The front and the right hand side are given by reflexivity (i.e. by Lemma 7.9). Since the second face is missing we can fill in the 3-simplex by Theorem 7.7. The resulting left hand side provides us with a homotopy $(g, \alpha) \simeq (f, \rho)$.

II transitivity: The homotopies $(f, \rho) \simeq (g, \alpha)$ and $(g, \alpha) \simeq (h, \lambda)$ give the bottom and the front. The right hand side is given by reflexivity. Since the second face is missing the 3-simplex can be filled by Theorem 7.7. The resulting left hand side provides us with a homotopy $(f, \rho) \simeq (h, \lambda)$.

**Definition 7.10:** Let $(f, \rho): (X, \mu) \to (Y, \nu)$ and $(g, \sigma): (Y, \nu) \to (Z, \lambda)$ be homotopy $B$-maps. Then the homotopy $B$-map $(h, \zeta): (X, \mu) \to (Z, \lambda)$ is called a **composite** of $(f, \rho)$ with $(g, \sigma)$ if there exists an action
Lemma 7.11: Let \((f, \rho): (X, \mu) \rightarrow (Y, \nu)\) and \((g, \sigma): (Y, \nu) \rightarrow (Z, \lambda)\) be homotopy \(B^\sim\)-maps. Then there exists a composite of \((f, \rho)\) with \((g, \sigma)\) and it is unique up to \(s\)-homotopy. If \(B\) is an \(M^\Lambda\)TP-category with isolated identities then the same holds if we substitute \(B^\sim\) by \(B\).

Proof: The first part follows from Theorem 7.7. Now suppose that \((h_i, \xi_i): (X, \mu) \rightarrow (Z, \lambda)\) are two composites of \((f, \rho)\) with \((g, \sigma)\), \(i = 0, 1\). Let \(\eta_i: RW(\overline{B^\sim} \ast L_2) \rightarrow REnd(X, Y, Z)\) be the actions defining them. Consider the following 3-simplex:

![3-simplex diagram]

The bottom and the left hand side are given by the actions \(\eta_1\) and \(\eta_2\). By Lemma 7.9 there exists an action determining the right hand side. Since the first face is missing we can apply Theorem 7.7 and fill in the 3-simplex. The resulting front face gives the required \(s\)-homotopy.
Lemma 7.12: Let \((f,\rho), (h,\chi)\): \((X,\mu) \to (Y,\nu)\) be homotopy \(B^\sim\)-maps. Then \((f,\rho) \simeq (h,\chi)\) iff there exists an action \(\sigma: \text{RW}(B^\sim * L_2) \to \text{REnd}(X,X,Y)\) such that \(\sigma \circ \sigma_0 = \rho\), \(\sigma \circ \sigma_1 = \chi\), \(\sigma \circ \sigma_2 = (1_X)_*\). (Recall that \(s\)-homotopy is defined by an action \(\text{RW}(B^\sim * L_2) \to \text{REnd}(X,Y,Y)\)). If \(B\) is an \(M^1\)TP-category with isolated identities then the same holds if we substitute \(B^\sim\) by \(B\).

Proof: The canonical composites \((f,\rho)_* (1_X)_*\) and \((1_Y)_* (f,\rho)\) are equal. From Lemma 7.9 and the uniqueness of composition of homotopy \(B^\sim\)-maps it follows that \((f,\rho) \simeq (h,\chi)\) iff \((h,\chi) \simeq (1_Y)_* (f,\rho)\), i.e. \((h,\chi)\) is a (not canonical) composite of \((1_Y)_*\) with \((f,\rho)\), and hence a composite of \((f,\rho)\) with \((1_X)_*\), which proves the Lemma one way. The converse follows in the same manner.

Lemma 7.13: Let \((f,\rho), (h,\chi)\): \((X,\mu) \to (Y,\nu)\) be \(s\)-homotopic homotopy \(B^\sim\)-maps and \((g,\zeta)\): \((Y,\nu) \to (Z,\lambda)\), \((k,\gamma)\): \((W,\omega) \to (X,\mu)\) homotopy \(B^\sim\)-maps. Then \((g,\zeta)_* (f,\rho) \simeq (g,\zeta)_* (h,\chi)\) and \((f,\rho)_* (k,\gamma) \simeq (h,\chi)_* (k,\gamma)\). If \(B\) is an \(M^1\)TP-category with isolated identities, then the same holds if we replace \(B^\sim\) by \(B\).

Proof: By Lemma 7.12 we have an action
σ: \( RW(B^\ast L_2) \to REnd(X,X,Y) \) such that \( σ^0 = ρ, \ σ^1 = κ, \) and \( σ^2 = (1_X)_\ast. \) In the following 3-simplexes

\[
\begin{array}{ccc}
\text{I} & & \text{II} \\
g^0 h & \downarrow g^0 f & g \\
1 & \downarrow 1 & g \\
0 & 1 & f \\
h & \downarrow 2 & 2 \\
& \ & \ \\
\end{array}
\]

in I the bottom is given by \( σ, \) the front and the right face by composition (Lemma 7.11), in II the bottom and the left face are given by composition, the right face by the given s-homotopy. Now apply Theorem 7.7. The resulting left face of I and front face of II give the required s-homotopies.

**Theorem 7.14:** The \( W_{B^\ast} \)-spaces and s-homotopy classes of homotopy \( B^\ast \)-maps form a category.

If \( B \) is an \( M^1 \text{-TP} \)-category with isolated identities then the \( W_{B^\ast} \)-spaces and s-homotopy classes of homotopy \( B^\ast \)-maps form a category.

**Proof:** By Lemma 7.11 and Lemma 7.13 we have a well defined composition. By Lemma 7.9 the \( W_{B^\ast} \)-homomorphisms

\( (1_X)_\ast: (X, μ) \to (X, μ) \) provide the identities. Associativity
is obtained from Theorem 7.7 by considering the following 3-simplex:

\[ \begin{array}{ccc}
3 & \xrightarrow{h \circ g} & 1 \\
\downarrow & & \downarrow \\
\circ (g \circ f) & \xrightarrow{f} & g \\
\downarrow & & \downarrow \\
0 & \xrightarrow{g \circ f} & 2
\end{array} \]

The bottom face defines \( g \circ f \), the front defines \( h \circ (g \circ f) \), the right face defines \( h \circ g \). Since the second face is missing we can fill in this 3-simplex. We find that the representative (composition is unique up to s-homotopy) for \( h \circ (g \circ f) \) represents \((h \circ g) \circ f\), too.

We next discuss the connection between the two definitions of homotopy between structure maps (see Definition 7.2 and 7.8). For this we first have to side track and study "equivariant" NDR-pairs of spaces of representing trees.

**Definition 7.15:** Call a subspace \( A \) of \( M = M_{a,b}(a,b) \), (see p. 53), an equivariant NDR, if the maps \( u:M \to I \) and \( h:M \times I \to M \) representing \( A \) as a NDR in \( M \) (see [6; Definition 6.2]) satisfy:
\[ u(x) = u(y) \text{ if } x \sim y \text{ under (2.3)} \]
\[ u(x^\circ \xi) = u(x) , \text{ where } \xi \text{ is a permutation.} \]
\[ h(x, t) \sim h(y, t) \text{ under (2.3) if } x \sim y \text{ under (2.3)} \]
\[ h(x^\circ \xi, t) = h(x, t)^\circ \xi \text{ , where } \xi \text{ is a permutation.} \]

By taking a radial map and a radial deformation
\[ v: I^\mathbb{R} \to I \text{ and } j: I^\mathbb{R} \times I \to I^\mathbb{R} \text{, (radial from the point } \left( \frac{1}{2}, \ldots, \frac{1}{2} \right) \text{), we can represent } \partial I^\mathbb{R} \text{ as a NDR in } I^\mathbb{R} \text{ in such a manner that } v \text{ and } j \text{ are symmetric in the } n \text{ coordinates of } I^\mathbb{R} \text{.} \]

Now suppose we are given an equivariant NDR \( A \) in
\[ M = M_{a, p}(a, b) \text{ represented by } u \text{ and } h. \]

**Lemma 7.16:** \( A \times I^\mathbb{R} \cup M \times \partial I^\mathbb{R} \) can be represented as NDR in \( M \times I^\mathbb{R} \) by maps \( w: M \times I^\mathbb{R} \to I \) and \( k: M \times I^\mathbb{R} \times I \to M \times I^\mathbb{R} \) such that
\[ (A) \ w(\theta, \xi, \delta) = w(\varphi, \eta, \delta) \text{ if } (\theta, \xi, \delta) \sim (\varphi, \eta, \delta) \text{ under } (2.11) \]
\[ w(\theta, \xi, \delta) = w(\theta, \text{unit}, \delta) \]
\[ (B) \ k(\theta, \xi, \delta, t) \sim k(\varphi, \eta, \delta, t) \text{ under } (2.11) \text{ if } (\theta, \xi, \delta) \sim (\varphi, \eta, \delta) \text{ under } (2.11) \]
\[ k(\theta, \xi, \delta, t) = k(\theta, \text{unit}, \delta, t)^\circ \xi . \]

**Proof:** Define \( w(\theta, \xi, \delta) = u(\theta, \xi).v(\delta) \). From the definition of \( u \) and \( v \) (A) follows immediately.
Define
\[ k(\theta, \xi, \delta, t) = (\theta, \xi, \delta) \quad \text{if} \quad (\theta, \xi) \in A, \quad \text{and} \quad \delta \in \partial I^n \]
\[ = [h(\theta, \xi, t), j[\delta, (u(\theta, \xi)/v(\delta)).t]] \]
\[ \quad \text{if} \quad v(\delta) \geq u(\theta, \xi) \quad \text{and} \quad v(\delta) > 0 \]
\[ = [h[\theta, \xi, (v(\delta)/u(\theta, \xi)).t], j(\delta, t)] \]
\[ \quad \text{if} \quad u(\theta, \xi) \geq v(\delta) \quad \text{and} \quad u(\theta, \xi) > 0 . \]

By [6; Theorem 6.3], \( k \) is continuous. It follows directly from the definition that it satisfies the condition (B).

Remark 7.17: Let \( K \subset I^n \) be a NDR such that the representing maps \( v': I^n \to I \) and \( j': I^n \times I \to I^n \) are symmetric in certain subsets \( U_i \) of the \( n \) coordinates of \( I^n \). Then by the same construction \( M \times K \cup A \times I^n \) can be represented as a NDR in \( M \times I^n \) by maps satisfying (A) and (B) of Lemma 7.16, if the coordinates of \( I^n \) in \( M \times I^n \) are only permuted inside the \( U_i \) under the relation (2.11).

Lemma 7.18: Let \( K \) and \( M \) and \( A \) be as in Remark 7.17, and suppose (2.11) permutes the coordinates of \( I^n \) in \( M \times I^n \) inside the subsets \( U_i \) of the coordinates of \( I^n \) only. Then there exists a retraction
\[ r: M \times I^n \times I \to M \times I^n \times 0 \cup (M \times K \cup A \times I^n) \times I \]
such that
(A) Let \( x = (\theta, \xi, \delta) \), \( y = (\varphi, \eta, \phi) \), \( r(x, t) = (x', t') \), \( r(y, t) = (y', t'') \). If \( x \sim y \) under (2.11), then \( t' = t'' \) and \( x' \sim y' \) under (2.11).

(B) \( r(\theta, \xi, \delta, t) = r(\theta, \text{unit}, \delta, t) \circ \xi \).

**Proof:** \( r \) is defined by

\[
r(\theta, \xi, \delta, t) = (\theta, \xi, \delta, t) \quad \text{if } t = 0 \text{ and } (\theta, \xi, \delta) \in \mathbb{M} \times K \cup \mathbb{A} \times \mathbb{I}^n
\]

\[
= [k(\theta, \xi, \delta, 1), 1[t, w(\theta, \xi, \delta)/s(t)]]
\]

if \( s(t) > w(\theta, \xi, \delta) \) and \( s(t) > 0 \)

\[
= [k[\theta, \xi, \delta, s(t)/w(\theta, \xi, \delta)], 1(t, 1)]
\]

if \( w(\theta, \xi, \delta) > s(t) \) and \( w(\theta, \xi, \delta) > 0 \),

where \( s: \mathbb{I} \to \mathbb{I} \) and \( l: \mathbb{I} \times \mathbb{I} \to \mathbb{I} \) are defined by \( s(t) = t/2 \) and \( l(t_1, t_2) = (1-t_2) \cdot t_1 \). \( w \) and \( k \) are the maps of Lemma 7.16. By [6; Theorem 6.3], \( r \) is continuous.

Let \( \mathcal{B} \) be an \( \mathbb{M}^{\text{TP}} \)-category such that \( (\mathcal{B}(b, b), 1_b) \) is a NDR-pair for all object generators \( b \). Let \( \mathcal{D} \) be a subcategory of \( \text{RWB} \) satisfying:

(1) If \( x \in \mathcal{D} \) is a composite in \( \text{RWB} \), \( x = y \circ z \), then \( y \) and \( z \) are in \( \mathcal{D} \).

(2) Suppose \( D_{a,p}(a, b) \) contains trees that do not represent decomposable elements of \( \mathcal{D} \), then \( D_{a,p}(a, b) \) is a product, \( D_{a,p}(a, b) = D' \times \mathbb{I}^p \), and \( D' \cup M_{a,p}(a, b) \subset M_{a,p}(a, b) \).
is an equivariant NDR, where $M_{a,p}^{'}(a,b) \subset M_{a,p}(a,b)$ is the subspace of those trees that contain a vertex labelled by an identity.

**Lemma 7.19:** Given an action $\rho_0 : \text{RWB} \rightarrow \text{REnd}(X_0, \ldots, X_{n-1})$ and a homotopy of functors $\delta_t : D \rightarrow \text{REnd}(X_0, \ldots, X_{n-1})$ preserving objects, sums, and permutations, such that $\rho_0|_D = \delta_0$. Then there exists a homotopy of reduced $M^n_{T,p}$-functors $\rho_t : \text{RWB} \rightarrow \text{REnd}(X_0, \ldots, X_{n-1})$ extending $\rho_0$ and $\delta_t$.

**Proof:** By Lemma 6.4 we have to construct homotopies of reduced $M^n_{T,p}$-functors $\gamma_t^{p,q_*} : \text{RWB}^{p,q} \rightarrow E$, where $E = \text{REnd}(X_0, \ldots, X_{n-1})$, such that $\gamma_t^{p+r,q+s}$ extends $\gamma_t^{p,q}$, $r,s > 0$, and such that $\gamma_t^{p,q}$ is compatible with $\delta_t$. For this we have to construct maps

$$f_{a,p,b,q} : Q_{a,p,b,q}(a,b) \rightarrow E(a,b)$$

satisfying the requirements of Lemma 6.4.

Since $(B(b,b), 1_b)$ is a NDR-pair for all object generators, and since the trivial group of permutations acts on it, $M_{a,p}^{'}(a,b) \subset M_{a,p}(a,b)$ is an equivariant NDR. (We know that it is a NDR. The representing maps are induced by those of the NDR-pairs $(B(b,b), 1_b)$. Hence it trivially
is an equivariant NDR-pair).

Induction start: For \( p = -1 \), \( \gamma_t^{-1}, q \) is uniquely determined since \( RW^{-1}, q \) consists of identities only.

Induction step from \((p, q-1)\) to \((p, q)\): We drop indices whenever there is no danger of confusion.

Suppose \( Q = Q_{a,p,b,q} \neq Q' \). Then \( f = f_{a,p,b,q} \) is determined on \((M \times \partial I_{P}^{q} \cup M' \times I_{P}^{q}) \times I\) by \( \gamma_{t}^{p,q-1} \), where \( I_{P}^{q} \subset I^{P} \) is the cube determined by the collection \( \beta \) of links in the trees of \( M \). \( f \) furthermore is given on \( Q \times 0 = M \times I_{P}^{q} \times 0 \) (compare p. 97) by \( \rho_{0} \). If \( D = D_{a,p} \cap Q \) contains an element which is not in \( Q' \), then \( D = D' \times I_{P}^{q} \), and \( D' \cup M' \) is an equivariant NDR in \( M \). \( f \) is determined on \( D' \times I_{P}^{q} \times I \) by \( \delta_{t} \). Denote \( f|\{[M \times \partial I_{P}^{q} \cup (M' \cup D') \times I_{P}^{q}] \times I \cup Q \times 0\} \) by \( g \). Define \( f: Q \times I \to \mathbb{E} \) by \( f = g \circ r \), where

\[
r: Q \times I = M \times I_{P}^{q} \times I \to [M \times \partial I_{P}^{q} \cup (M' \cup D') \times I_{P}^{q}] \times I \cup M \times I_{P}^{q} \times 0
\]

is the retraction of Lemma 7.18. Since \( r \) is equivariant and factors through (2.11) for each fixed \( t \), \( f = g \circ r \) satisfies the requirements of Lemma 6.4.

**Lemma 7.20:** Let \((g_i, \rho_i): (X, \mu) \to (Y, \nu), i = 0, 1\), be homotopy \( \mathcal{B} \)-maps such that \((g_0, \rho_0) \cong (g_1, \rho_1)\). If \((\mathcal{B}(1, 1), 1_1)\) is a NDR-pair, then \((g_0, \rho_0) \cong (g_1, \rho_1)\).
Proof: Let $x_t: (g_0, \rho_0) \cong (g_1, \rho_1)$ be the given homotopy of reduced $M^2TP$-functors. Let $\mathcal{D}$ be the reduced $M^3TP$-subcategory of $\text{RW}(B^*L_2)$ generated by $\delta^i \text{RW}(B^*L_1)$, $i = 0, 1, 2$. By Lemma 7.9 there exists an action $\sigma_0: \text{RW}(B^*L_2) \to \text{REnd}(X, Y, Y)$ such that $\sigma_0 \circ \delta^0 = (1_Y)^*_*$, $\sigma_0 \circ \delta^1 = \rho_0$, $\sigma_0 \circ \delta^2 = \rho_0$. Define $\delta_t: \mathcal{D} \to \text{REnd}(X, Y, Y)$ by

\[
\begin{align*}
\delta_t \mid \delta^0 \text{RW}(B^*L_1) &= (1_Y)^*_* \quad \text{for all } t \in I \\
\delta_t \mid \delta^1 \text{RW}(B^*L_1) &= x_t \\
\delta_t \mid \delta^2 \text{RW}(B^*L_1) &= \rho_0 \quad \text{for all } t \in I.
\end{align*}
\]

$\delta_t$ is a well defined homotopy of functors since $\rho_0$, $x_t$, and $(1_Y)^*_*$ extend the actions $\mu$ and $\nu$. $\mathcal{D}$ satisfies the requirements of Lemma 7.19. Hence there exists a homotopy $\sigma_t: \text{RW}(B^*L_2) \to \text{REnd}(X, Y, Y)$ of reduced $M^3TP$-functors extending $\delta_t$ and $\sigma_0$. $\sigma_1$ defines the required $s$-homotopy.]

Theorem 7.21: Let $(g_0, \rho_0): (X, \mu) \to (Y, v)$ be a homotopy $B$-map and $g_1: X \to Y$ a map homotopic to $g_0$. If $(B(1, 1), 1)$ is a NDR-pair, then $g_1$ can be made into
a homotopy \( \mathbb{B} \)-map \((g_1, \rho_1): (X, \mu) \to (Y, \nu)\) such that
\[
(g_0, \rho_0) \cong (g_1, \rho_1)
\)
and hence \((g_0, \rho_0) \cong (g_1, \rho_1)\).

**Proof:** Let \( g_t \) be the homotopy between \( g_0 \) and \( g_1 \). Let \( D \) be the subcategory of \( \text{RW}(\mathbb{B}^*L_1) \) consisting of the identities \( 1_1 \) and \( 1_1 \), and of the morphism \( j = \epsilon_{\mathbb{B}^*L_1}(1; (0,1)) \) only. Define a homotopy of functors \( \delta_t: D \to \text{REnd}(X, Y) \) by
\[
\delta_t(j) = g_t.
\]
Since \((\mathbb{B}(1,1), 1_1)\) is a NDR-pair, \( D \) satisfies the requirements of Lemma 7.19. Hence there exists a homotopy of reduced \( \mathbb{M}^2\text{TP} \)-functors \( \rho_t: \text{RW}(\mathbb{B}^*L_1) \to \text{REnd}(X, Y) \) extending \( \rho_0 \) and \( \delta_t \). Since \( \rho_1 \circ \epsilon_{\mathbb{B}^*L_1} \circ \Lambda(0,1) = g_1 \), \( \rho_t \) is a homotopy \((g_0, \rho_0) \cong (g_1, \rho_1)\).
§ 8 HOMOTOPY EQUIVALENCES AND HOMOTOPY TYPE

The aim of this chapter is to prove the following two theorems:

**Theorem 8.1:** Let \( B \) be an \( M^1 \)TP-category with isolated identities. Let \((f, \rho)\): \((X, \alpha) \to (Y, \beta)\) be a homotopy \( B \)-map and \( f: X \to Y \) a homotopy equivalence. Then \((f, \rho)\) is a \( s \)-homotopy equivalence, i.e. it is an isomorphism in the category of \( WB \)-spaces and \( s \)-homotopy classes of homotopy \( B \)-maps.

**Theorem 8.2:** Let \((X, \alpha)\) be a \( WB^\sim \)-space and \( f: X \to Y \) a homotopy equivalence. Then \( Y \) can be made into a \( WB^\sim \)-space \((Y, \beta)\) and \( f \) into a \( s \)-homotopy equivalence \((f, \rho)\): \((X, \alpha) \to (Y, \beta)\).
If \( B \) is an \( M^1 \)TP-category with isolated identities, the same holds if we replace \( B^\sim \) by \( B \).

By using the mapping cylinder these theorems reduce to proving the statements for strong deformation retracts, and this can be reduced in the case of Theorem 8.1 to proving that it holds if \( f \) is the identity.
In the proof that homotopy $B$-maps are $s$-homotopy equivalences it is often easier to work with the category $\text{RW}(B*\text{Is}_1)$ rather than the category $\text{RW}(B*L_2)$. Recall that $\text{Is}_1$ is the category with two objects and exactly one morphism between any two objects. We again can use the simplified description for the trees representing the elements of $\text{RW}(B*\text{Is}_1)$, (see p. 86).

The inclusion functors $d^i: L_0 \to \text{Is}_1$, $i = 0, 1$, given by $d^0(0) = 1$, and $d^1(0) = 0$, induce inclusion functors $\partial^i = W(1 \ast d^i): \text{WB} = \text{W}(B*L_0) \to \text{RW}(B*\text{Is}_1)$.

As in §5 each action $\rho: \text{RW}(B*\text{Is}_1) \to \text{REnd}(X_0, X_1)$ induces actions $\rho^i$ such that

$$
\begin{array}{cccccc}
\text{WB} & \xrightarrow{\partial^i} & \text{RW}(B*\text{Is}_1) \\
\rho^i & \downarrow & \downarrow \rho \\
\text{End} X_j & \xrightarrow{\partial^i} & \text{REnd}(X_0, X_1)
\end{array}
$$

commutes for $i \neq j$, $i, j = 0, 1$.

The inclusion functors $u, v: L_1 \to \text{Is}_1$ given by $u(0) = 1$, $u(1) = 0$, and $v(i) = i$, $i = 0, 1$, induce inclusion functors

$W(1 \ast u), W(1 \ast v): \text{RW}(B*L_1) \to \text{RW}(B*\text{Is}_1)$
Lemma 8.3: Any action $\rho: \text{RW}(B*\text{Is}_1) \to \text{REnd}(X,Y)$ induces actions

$$v: \text{RW}(B*\text{L}_2) \to \text{REnd}(X,Y,X)$$
$$\mu: \text{RW}(B*\text{L}_2) \to \text{REnd}(Y,X,Y)$$

such that

$$v \circ \delta^0 = \rho \circ \omega(1 * u) \quad \mu \circ \delta^0 = \rho \circ \omega(1 * v)$$
$$v \circ \delta^1 = \rho \circ \delta^1 \circ s^0 \quad \mu \circ \delta^1 = \rho \circ \delta^0 \circ s^0$$
$$v \circ \delta^2 = \rho \circ \omega(1 * v) \quad \mu \circ \delta^2 = \rho \circ \omega(1 * u).$$

In particular, $\lambda_1: \text{L}_2 \to B*\text{L}_1$ and $\lambda_2: \text{Is}_1 \to B*\text{Is}_1$ are the canonical inclusions,

$$v \circ \delta^0 \circ \lambda_1(0,1) = \rho \circ \omega \circ \lambda_2(1,0)$$
$$v \circ \delta^2 \circ \lambda_1(0,1) = \rho \circ \omega \circ \lambda_2(0,1).$$

Hence the actions $v \circ \delta^0$ and $v \circ \delta^2$ determine homotopy $B$-maps that are $s$-homotopy inverse to each others.

Proof: Define functors $k, l: \text{L}_2 \to \text{Is}_1$ by

$$k(i) = 0 \quad i = 0, 2$$
$$= 1 \quad i = 1$$
$$l(i) = 1 \quad i = 0, 2$$
$$= 0 \quad i = 1$$

$k$ and $l$ induce reduced MTP-functors $x = \omega(1 * k)$ and

$$\lambda = \omega(1 * l)$$

from $\text{RW}(B*\text{L}_2)$ to $\text{RW}(B*\text{Is}_1)$ which satisfy

$$x \circ \delta^0 = \omega(1 * (k \circ \delta^0)) = \omega(1 * u)$$
$$x \circ \delta^1 = \omega(1 * (k \circ \delta^1)) = \omega(1 * (\delta^1 \circ s^0)) = \delta^1 \circ s^0$$
$$x \circ \delta^2 = \omega(1 * (k \circ \delta^2)) = \omega(1 * v).$$

For $f$ and $g$ see p. 82. Similarly for $\lambda$ we obtain
\[ \lambda \circ \partial^0 = W(1 \ast v), \lambda \circ \partial^1 = \partial^0 \circ s^0, \lambda \circ \partial^2 = W(1 \ast u). \]

Now define \( v = \rho \circ x, \mu = \rho \circ \lambda \).]

**Lemma 8.4:** Suppose \( B \) has isolated identities. Let
\[
(1_X, \nu): (X, \mu) \to (X, \lambda)
\]
be a homotopy \( B \)-map. Then
\[
(1_X, \nu)
\]
is a \( s \)-homotopy equivalence.

**Proof:** Let \( G' \) be the reduced \( M^2 \text{TP} \)-subcategory of \( RW(\mathbb{B}^* \text{Is}_1) \)

generated under \( \ast \) and composition by all those elements
the representing trees \( \theta \) of which are either of the following forms:

(A) In each complete directed edge path of \( \theta \) the label of
the edges changes at most once, and then from 1 to 0.

(B) \( \theta \) is of the form
\[
\begin{array}{c}
0 \ast 1 \\
1
\end{array}
\]

(As in §5 and §6 the pictures give the labelling of the edges and not the value of \( I \) assigned to them).

The space of representing trees of \( G' \) is closed in
the space of the representing trees of \( RW(\mathbb{B}^* \text{Is}_1) \). Introduce a relation among the trees of \( G' \) by

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and its consequences (i.e. if any such sequence of edges occurs in a tree representing an element of $Q'$, and if $1 \in I$ is assigned to its incoming and outgoing edge, then this tree may be reduced under $(R)$). Let $\mathcal{G}$ be the reduced $M_2^{TP}$-quotient category of $Q'$ obtained by factoring out these relations. Using the general construction of p. 33 it is easy to show that the morphism spaces of $\mathcal{G}$ are in $CG$.

Define an action $\eta: \mathcal{G} \to R\text{End}(X,X)$ as follows: Each morphism of $\mathcal{G}$ can be represented as a composition of sums of elements which are represented by trees of the form (A) or (B), or which are permutations. Define

$$\eta[\theta, \xi, \delta] = 1_X \text{ if } \theta \text{ is of the form (B)}$$

$$= \nu[\theta, \xi, \delta] \text{ if } \theta \text{ is of the form (A)}.$$

This determines $\eta$ uniquely on $\mathcal{G}$. Since $\nu[\theta, \text{unit}, I^0] = 1_X$ if

$$\eta$$

$\nu[\theta, \xi, \delta] = 1_X$ if $\theta = I^1$. Then, $\eta$ is compatible with the relation $(R)$.

$\varepsilon_{B^* \text{Is}_1}|Q'$ induces an augmentation functor

$$\varepsilon: \mathcal{G} \to R(B^* \text{Is}_1)$$

Claim: $\varepsilon$ is fibre homotopically trivial.

Proof: Call the vertices with the labels

$$0 \quad 1$$

and

$$1 \quad 0$$

a g-vertex, resp. an f-vertex, and denote them by $g^\uparrow$ resp.
Then $(R)$ means $g \circ f = 1$, $f \circ g = 1$, where $f$ and $g$ are the elements of $G$ represented by a tree consisting of an $f$-vertex, resp. a $g$-vertex only.

The standard section $\sigma_{B^{*}I_{S_{1}}}^{\ast}$ induces a section

$$\sigma: R(B^{*}I_{S_{1}})(n',1) \rightarrow C(n',1).$$

(As usually $n$ and $n'$ denote the sequences of length $n$ in the generator 0 resp. 1).

For the other morphism spaces we construct a different section. For $(\beta;(0,1),\ldots,(0,1)) \in R(B^{*}I_{S_{1}})(n,1')$ define

$$\sigma(\beta;(0,1),\ldots,(0,1)) = \{\theta, \text{unit}, \delta\},$$

where $\theta$ is the tree with exactly three vertices on each (directed) edge path, labelled by 1, $\beta$, 1 in order, the edges change their label after each vertex, and the value 1 is assigned to each link:

Similarly define the section on $R(B^{*}I_{S_{1}})(n,1)$ by such a tree, deleting the $g$-vertex at the root:
and on $R(E*\text{Is}_1)(n',1')$ by deleting the $g$-vertices at the twigs:

![Diagram of tree with vertices numbered 0, 1, and 1']

We have four kinds of trees, namely those representing morphisms $n \to 1'$, $n' \to 1$, $n \to 1$, $n' \to 1'$. Using (R) we can choose the representatives such that each represents a composition of elements of the first two kinds. Replace

\[ a: n \to 1 \quad \text{by} \quad a \circ f^n \circ g^n: n \to n' \to n \to 1 \]
\[ \beta: n' \to 1' \quad \text{by} \quad \beta \circ f \circ g: n' \to 1' \to 1 \to 1' \]

We furthermore replace $a: n \to 1'$ by

\[ g \circ f \circ g \circ f^n \circ g^n: n \to n' \to n \to 1' \to 1 \to 1' \]

Let $Y$ be the space of those representing trees. Since the identities in $B$ are isolated, we can assume that none of the representatives in $Y$ can be reduced under the relation (2.13). In addition we can assume that sequences

\[ g \circ f \circ g \]

do not occur in any tree in $Y$, unless this tree consists of this sequence only.

We are now going to construct the equivariant, fibrewise deformation of $C = C(n,1')$ into the section. The deforma-
tion of the other morphism spaces is constructed analogously and therefore is omitted.

Filter $C$ as follows: $F_m C$ consists of all those elements that can be represented by a point in $Y$ which has at most $m$ $g$-vertices on any edge path. Then the lowest filtration is two, and each element of $F_2 C$ can be represented by a tree such that $B$ is a subtree that does not have any $g$-vertex. Deform $F_2 C$ into the section by mapping the values $t_1$ of the links in $B$ to $u t_1$ at the time $u$, $1 \geq u \geq 0$. At time 0 the tree represents an element in the section.

We now want to deform $F_n C$ strongly into $F_{n-1} C$. Consider a typical representative of $F_n C$ (in $Y$):
Here B is a tree which does not contain a g-vertex, and each $A_i$ does not have more than $n-1$ g-vertices in any edge path (we consider the g-vertices below the $A_i$ as a part of them). Let $N$ be the space of the trees of the form B, and $M_i$ the space of the trees of the form $A_i$, $i = 1, \ldots, r$. Index the $A_i$ by $1, \ldots, r$ in such a manner that $A_1, \ldots, A_k$ contain an edge path with $n-1$ g-vertices while $A_{k+1}, \ldots, A_r$ do not. Index the twig of B on which $A_i$ sits by $i$. Let $M = M_1 \times \ldots \times M_r$.

$\theta$ can only represent an element of lower filtration if an f-vertex is at a twig of B indexed (not labelled) by $i \in \{1, \ldots, k\}$, or if we have link combinations

$\begin{align*}
    \text{or } \\
    \begin{array}{c}
        t \\
        f \\
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    \begin{array}{c}
        \text{or } \\
        \begin{array}{c}
            g \\
            f \\
        \end{array}
    \end{array}
\end{align*}$

which do not include a twig, in $A_1, \ldots, A_k$. $t$ is the value to the particular incoming or outgoing edge of the f-vertex. Call such a link combination a critical sequence with value $t$, if it is part of a (directed) edge path through $n-1$ g-vertices in some $A_i$, and if it does not contain a twig.

Let $M_i'$, $i = 1, \ldots, k$, be the (closed) subspace of $M_i$ of those trees that contain a critical sequence in each
edge path that runs through \( n-1 \) g-vertices. Let \( s \) be a subsequence of \((1,\ldots,k)\). Let \( N_s \) be the (closed) subspace of those trees of \( N \) that have an \( f \)-vertex on the bottom of the \( i \)-th twig for all \( i \in s \), but not for \( i \in (1,\ldots,k) - s \).

If a tree \( \theta \) of \( Z = N \times M \) represents an element of lower filtration, then \( \theta \) is in some \( N_{s_1} \times R_1 \times \ldots \times R_k \times N_{k+1} \times \ldots \times N_r \), where \( R_i = M_i \) if \( i \in s \) and \( R_i = M_i' \) if \( i \notin s \).

We are now going to deform \( Z \) into the subspace of those trees representing an element of lower filtration. We do that by a triple induction: (A) on the number \( k \) of trees \( A_i \) that contain \( m-1 \) g-vertices in some edge path, (B) on the length of \( s \), and (C) on the total number of critical sequences in the \( A_i \). Notice that if \( s \) and \( r \) are subsequences of \((1,\ldots,k)\) such that none is contained in the other, and \( \theta \in N_s \) and \( \varphi \in N_r \) are related under (2.12), then both are related to a tree \( \psi \in N_{s \cap r} \). Hence they are dealt with in an earlier induction step. Similar arguments apply to the other induction stages.

Start (A): For \( k = 0 \), \( \theta \) (see two pages ago) represents an element of lower filtration.

Induction step (A): Suppose we have constructed the deformation for \( k-1 \).
We first construct a strong deformation retraction into the subspace of those elements that are either of lower filtration, or that can be represented by a tree in \( N_i \times M \), where \( \mathbf{i} = (1, \ldots, k) \), or by a tree such that less than \( k \) of the \( A_i \)'s contain a sequence through \( n-1 \) g-vertices.

Start (B): \( \mathcal{S} = \emptyset \). Let \( \mathbf{u} = (u_1, \ldots, u_p) \) be the collection of values assigned to the links of the tree \( B \) labelled by 1, which lie on an edge path starting in a tree \( A_i \), \( i \in (1, \ldots, k) \).

Start (C): Let \( q \) be the total number of critical sequences in \( A_1, \ldots, A_k \). Let \( q = 0 \). Define \( H \) to be the deformation which changes the value \( u_i \in \mathbf{u} \) to \( t \cdot u_i \) at the time \( t \), \( 1 \geq t \geq 0 \). Then \( H \) is well defined and compatible with the previous induction steps. \( H_1 \) is the identity.

Induction step (C): Suppose \( q > 0 \). Let \( \mathbf{t} = (t_1, \ldots, t_q) \) be the collection of values assigned to the incoming, resp. outgoing edges of the \( f \)-vertices of the \( q \) critical sequences in question, (i.e. the values of the critical sequences). Note that

- counts as two critical sequences and that
we neglect the values assigned to the links that start or end in a g-vertex (they are always 1).

$H$ is by induction defined on all trees $\theta$ for which $t \in \partial I^q$. For all the lower faces $\theta$ is related to a tree with $q-1$ critical sequences, and on the upper faces one of the critical sequences can be reduced by the relation (R). Since our aim is to get out of $N_{\emptyset} \times M$ keeping the elements of $N_{\emptyset} \times M$ fixed for $\emptyset \neq \emptyset$, we want to construct a strong deformation retraction

$$\mathbb{I}^{p+q} \rightarrow \mathbb{I}^p \times \partial I^q \cup \mathbb{I}^q$$

where $0 = (0, \ldots, 0) \in \mathbb{I}^p$. Since this deformation retraction has to be compatible with relation (2.11) we want it to be symmetric in the coordinates of $\mathbb{I}^p$ and $\mathbb{I}^q$ (recall $p$ is the number of coordinates of $u$). We construct such a deformation later on in the proof.

**Induction step (B): Length $s = m$.**

Again we induct on $q$. Let $\underline{v} = (v_1, \ldots, v_m)$ be the collection of values that are assigned to the outgoing edges of the $m$ f-vertices at the bottom of the twigs of the tree $B$ indexed by the elements of $s$. Let $t$ and $u$ be as above.

**Start (C): $q = 0$.** $H$ is defined exactly on all those trees $\theta$ for which $\underline{v} \in \partial I^m$. On the lower faces $\theta$ is related to a
tree on which $H$ has been defined by induction step $(B, m-1)$, and on the upper ones to a tree on which $H$ has been defined by induction step $(A, k-1)$. Again we want to define a strong deformation retraction, this time for

$$I^P + m \to I^P \times \partial I^m \cup 0 \times I^m$$

which is symmetric in the coordinates of $I^P$ and $I^m$.

Induction step (C): Suppose $q > 0$. Then $H$ has been defined exactly for $t \in \partial I^q$ or $v \in \partial I^m$, and hence for $(t, v) \in \partial I^{q+m}$. Hence again we want a strong deformation retraction

$$I^{P+q+m} \to I^P \times \partial I^{q+m} \cup 0 \times I^{q+m}$$

which is symmetric in the coordinates of $I^P$, $I^q$, and $I^m$.

This defines $H$ on the whole of $N \times M$. $H_0(N \times M)$ consists of trees that are related to a tree in $N_i \times M$, where $i = (1, \ldots, k)$. We are now going to construct a strong deformation retraction of $N_i \times M$ into the closed subspace of all those elements which represent an element in $C$ of lower filtration and such that this deformation extends the deformation given by induction step $(A, k-1)$.

Let $v$ and $t$ be as before.

Start (C): Denote the new deformation by $K$. By induction (A), $K$ has been defined on those trees for which $v$ is in an upper face of $I^m$. Hence we want a symmetric strong deformation retraction
\[ I^m \rightarrow UI^m \]

where UI^m is the collection of upper faces of I^m.

Induction step (C): Suppose \( q > 0 \). Then K is determined on those trees \( \theta \) for which \( t \in \partial I^q \) or \( v \in UI^m \). Hence we want a strong deformation retraction

\[ I^{m+q} \rightarrow I^m \times \partial I^q \cup UI^m \times I^q \]

which is symmetric in the coordinates of \( I^m \) and \( I^q \).

Since all deformations constructed are well defined, continuous, equivariant and fibrewise, the claim is proved if we can find the required deformations

\[ F_s : I^{p+r} \rightarrow I^p \times \partial I^r \cup I^p \times \partial I^r \]

\[ G_s : I^{p+r} \rightarrow I^p \times \partial I^r \cup UI^p \times I^r \]

Let \( \mathbf{u} = (u_1, \ldots, u_p), \mathbf{v} = (v_1, \ldots, v_r), 0 \leq u_i, v_j \leq 1, \]
\( i = 1, \ldots, p, j = 1, \ldots, r \). Let

\[ t(s) = t(s, \mathbf{u}, \mathbf{v}) = \min \{ s, \max (u_1/(2-u_1), \ldots, u_p/(2-u_p)), n(v_1), \ldots, n(v_r) \} \]

where \( s \in I \) and

\[ n(v_i) = \begin{cases} v_i/(1-v_i) & \text{if } v_i < \frac{1}{2} \\ 1 & \text{if } v_i = \frac{1}{2} \\ (1-v_i)/(v_i-\frac{1}{2}) & \text{if } v_i > \frac{1}{2} \end{cases} \]

Note that \( t(s, \mathbf{u}, \mathbf{v}) \) is continuous in \( s, \mathbf{u}, \) and \( \mathbf{v} \).
Now define:

\[ F_s(u, v) = \{ \max[0, u_1 + t(s)(u_1 - 2)], \ldots, \max[0, u_p + t(s)(u_p - 2)], \]

\[ m_s(v_i), \ldots, m_s(v_r) \}

where

\[ m_s(v_i) = \max[0, v_i + t(s)(v_i - \frac{1}{2})] \]

\[ = \min[1, v_i + t(s)(v_i - \frac{1}{2})] \]

if \( v_i \leq \frac{1}{2} \)

Define

\[ g(s) = q(s, u, v) = \min[s, (1 - u_1)/(2 - u_1), \ldots, (1 - u_p)/(2 - u_p), n(v_1), \]

\[ \ldots, n(v_r)] \]

where \( s \in I \) and \( n(v) \) as above. Note that \( g \) is continuous in \( s, u, v \). Define

\[ G_s(u, v) = \{ \min[1, q(s)(2 - u_1) + u_1], \ldots, \min[1, q(s)(2 - u_p) + u_p], \]

\[ m_s(v_i), \ldots, m_s(v_r) \}

where \( m_s(v) \) is defined as above, substitute only \( t(s) \) by \( q(s) \).

\( F_s \) and \( G_s \) satisfy our requirements on the deformations.

Let \( D \) be the subcategory of \( RW(B^* I_s_1) \) generated by all those elements that can be represented by a tree of form (A). Since the projection \( C' \to C \) is one-one on the trees of this form, there exists an inclusion functor \( \delta: D \to C \). Now apply the reduced version of Theorem 4.9.
We obtain a reduced $M^2TP$-functor

$$\rho: RW(B*Is_1) \to C$$

extending $\delta$. By the choice of the section we have

$$\eta^o\rho^o B*Is_1 A(i,j) = 1_X$$

for all $i,j = 0,1$. Hence $\eta^o\rho$ provides us with an action which in view of Lemma 8.3 gives the required result.

Lemma 8.5: Let $B, C$ be $M^1TP$-categories and $D$ a topological category with $n$ objects. Let $\gamma: B \to C$ be a fibre homotopically trivial $M^1TP$-functor. Then

$$\gamma^*1: B*D \to C*D$$

is fibre homotopically trivial.

The proof is immediate.

Lemma 8.6: Let $A$ be a strong deformation retract of $X$.

$$i \quad p$$

$$A \rightarrow X \rightarrow A$$

$p \circ i = 1_A$, and $H_t: 1_X = i \circ p \quad \text{rel} \quad A$. If $(A,\alpha)$ is a $WB^\sim$-space, then $X$ can be made into a $WB^\sim$-space $(X,\beta)$ and $p$ and $i$ into s-homotopy equivalences $(p,\pi): (X,\beta) \rightarrow (A,\alpha)$ , $(i,\gamma): (A,\alpha) \rightarrow (X,\beta)$ which are inverse to each other.

If $B$ is an $M^1TP$-category with isolated identities the same holds if we replace $B^\sim$ by $B$. 
Proof: For the time being put $M = WB^*$. For each $b \in M(m,n)$ the action $a$ induces a map $\overline{b} = i \circ a(b) \circ p :$

\[
\begin{array}{cccc}
X^m & p & \rightarrow & A^m \\
\downarrow & & & \downarrow a(b) \\
& A^n & i & \rightarrow & X^n \\
\end{array}
\]

(We write $p$ instead of $p^m$ as long as it is clear what we mean). $\overline{b} \circ a = \overline{b} \circ \overline{a}$, because $p \circ i = 1_A$, but unfortunately $1 \neq i \circ p$. This can be corrected by bringing in the homotopy $H_t$. Using this data we define a category $G$ which acts on $R\text{End}(X,A)$.

Let $J$ be the monoid of Example 3.5, denote the multiplication in $J$ by $*$ and $1 \in J_1$ by $u$. Using $J$ we are going to construct an $M^1\text{TP}$-category $G$, which in addition with $M$ gives rise to the category $G$.

Let $G(n,1) = M(n,1) \times J^n$, where $J^n$ is the $n$-fold product of $J$, $n \neq 1$. Define $G(1,1) = (M(1,1) \times J \cup J)/\sim$ where the equivalence relation is generated by

$$(1,u^*w) \sim (u^*w)$$

with $1 \in M(1,1)$ being the identity of $M$. Since the attaching map $f: u^*J \rightarrow M(1,1) \times J$ given by $f(u^*w) = (1,u^*w)$ is continuous and since $(J, u^*J)$ is a NDR-pair, $G(1,1)$ is in $CG$. (Recall: $u^*J$ is the image of the upper faces of the cubes $I^n$ under the attaching maps $I^n \rightarrow J_{n-1}$ and hence is a subcomplex of the CW-complex $J$).
Define an action of \( S(n) \) on \( C(n,1) \) as follows: Let \( a \in M(n,1) \) and \( (v_1, \ldots, v_n) \in J^n, \xi \in S(n), n > 1 \). Then

\[
(a; v_1, \ldots, v_n) \circ \xi = (a \circ \xi; v_{\xi(1)}, \ldots, v_{\xi(n)})
\]

Define \( C \) now by the normal form construction.

Composition in \( C \) is given as follows, motivated by the action on \( R \text{End}(X,A) \), see below:

Let \( (b; v_1, \ldots, v_n) \in M(n,1) \times J^n, (x_1 \oplus \cdots \oplus x_n) \circ \xi \in C(m,n) \) with \( x_i = (b_i; w_i) \in M(m_i,1) \times J_{m_i} \) or \( x_i = u_i \in J \subset C(1,1) \), \( i = 1, \ldots, n \). Then

\[
(C1) \quad (b; v_1, \ldots, v_n) \circ [(x_1 \oplus \cdots \oplus x_n) \circ \xi] = [b \circ (b_1' \oplus \cdots \oplus b_n'); w_1' \times \cdots w_n'] \circ \xi
\]

where \( (b_i', w_i') = (b_i, w_i) \) if \( x_i = (b_i, w_i) \) and \( (b_i', w_i') = (1; v_i \ast u_i) \) if \( x_i = u_i \).

If \( v \in J \subset C(1,1) \) and \( w \in J \), then define

\[
(C2) \quad v \circ (b; v_1, \ldots, v_n) = (b; v_1, \ldots, v_n) \quad v \circ w = v \ast w
\]

This definition factors through the relation imposed on \( M(1,1) \times J \cup J \) and hence is well defined. Since it is induced by the compositions in \( M \) and in \( J \) it is continuous and associative. \( 0 \in J_1 \subset J \) serves as identity. Hence by the normal form construction, \( C \) is an \( M^1 \text{TP} \)-category.

Let \( G \) be the reduced \( M^2 \text{TP} \)-category given by

\[
G(m,n) = C(m,n), \quad G(m',n') = M(m,n), \quad G(m',n) = M(m,n),
\]
and \( G(m,1') = C(m,1) \) for \( m \neq 1 \), and \( G(1,1') = M(1,1) \times J \subset C(1,1) \). Define the remaining morphism spaces by a reduced version of the normal form construction.

To define composition in \( G \) we embed \( M \) into \( C \) by

\[
b \rightarrow (b;0,\ldots,0),
\]

\( b \in M(a,1) \). Since

\[
(b;0,\ldots,0) \circ [(c_1;0,\ldots,0) \circ \ldots \circ (c_n;0,\ldots,0)] \circ \xi = (b \circ (c_1 \circ \ldots \circ c_n) \circ \xi;0,\ldots,0)
\]

the composition in \( M \) is "induced" by the one in \( C \). Hence composition in \( G \) can now be defined to be the one in \( C \), and hence is associative and continuous and has identities. Note that \((1;0,\ldots,0)\) serves as identity in \( G(1',1') \). It remains to check that for \( a \in G(n,1') \), \( b \in G(m',m) \), and \( c \in G(n,1) \), \( a \circ b \) and \( c \circ b \) are in the subcategory \( M \) of \( C \). But this follows immediately from (C1) and (C2).

Define an action \( \eta : G \rightarrow \text{REnd}(X,A) \) as follows:

\[
\eta(b;v_1,\ldots,v_n) = i \circ a(b) \circ \rho(H_{v_1} \times \ldots \times H_{v_n}) \quad \text{for} \quad (b;v_1,\ldots,v_n) \in G(n,1)
\]

\[
\eta(v) = H_v \quad \text{for} \quad v \in J \subset C(1,1)
\]

\[
\eta(b;v_1,\ldots,v_n) = a(b) \circ \rho(H_{v_1} \times \ldots \times H_{v_n}) \quad \text{for} \quad (b;v_1,\ldots,v_n) \in G(n,1')
\]

\[
\eta(b;0,\ldots,0) = a(b) \quad \text{for} \quad (b;0,\ldots,0) \in G(n',1')
\]

\[
= i \circ a(b) \quad \text{for} \quad (b;0,\ldots,0) \in G(n',1)
\]
where $H_v = H_{t_1} \circ \ldots \circ H_{t_n}$ if $v = (t_1, \ldots, t_n) \in J$.

Since $G$ is in normal form as reduced $M^2TP$-category, $\eta$ is uniquely determined on the whole of $G$. It is continuous and by definition preserves sums and permutations. Since $H_0 = 1_x$ and $H_v \circ i = i$, and $p \circ i = 1_A$ it preserves identities and compositions.

Define an augmentation $x: G \to R(M^*I)$ by
$$x(b; v_1, \ldots, v_n) = (b; (i, j), \ldots, (i, j))$$
if $(b; v_1, \ldots, v_n) \in G(a, b)$ with $a = (i, \ldots, i)$ and $b = j$, and
$$x(v) = (1; (0, 0))$$
if $v \in J \subset G(1, 1)$.

$x$ is well defined because of the normal form of $G$. It is continuous, and from the definition of composition in $G$ it follows immediately that $x$ is a reduced $M^2TP$-functor.

Define a section $\sigma$ of $x$ by
$$(b; (i, j), \ldots, (i, j)) = (b; u, \ldots, u) \in G(a, j), a = (i, \ldots, i),$$
$$(i, j) \neq (1, 0), (1, 1).$$

We have shown (p. 47) that $I = J_1$ is a strong deformation retract of $J$. Hence $u \in J_1$ is a strong deformation retract of $J$. Applying the product of the deformation of $J$ to $G(a, 1)$ and the identity deformation to $M(n, 1)$, we obtain an equivariant fibrewise deformation of $G$ into
\( \sigma(R(M^*IS_1)) \). Hence \( x \) is fibre homotopically trivial.

Now resubstitute \( M \) by \( WB^- \). Since 
\[ \varepsilon = \varepsilon_B * 1 : R(WB^-*IS_1) \rightarrow R(B^*IS_1) \] is fibre homotopically trivial by Lemma 8.5, \( \varepsilon \) is fibre homotopically trivial.

Let \( r : WB^- \rightarrow G \) be the embedding given by 
\[ WB^-((n,m)) \rightarrow G(n',m') \]. Let \( D \) be the subcategory of \( RW(B^-*IS_1) \) given by \( \delta^0WB^- \). Define \( \delta : D \rightarrow G \) to be the embedding \( r \). 
\( \delta \) and \( D \) satisfy the requirements of Theorem 6.5. Define 
\[ \tau_1 : R(B^-*IS_1)(1,1') \rightarrow G(1,1') \] and 
\[ \tau_2 : R(B^-*IS_1)(1',1) \rightarrow G(1',1) \] by 
\[ \tau_1(b;(0,1)) = \sigma(\iota_B^-b;(0,1)) \]
\[ \tau_2(b;(1,0)) = \sigma(\iota_B^-b;(1,0)) \]
where \( \iota_B^- \) is the standard section. By the Theorem 6.5 there exists a reduced \( M^2TP \)-functor \( \rho : RW(B^-*IS_1) \rightarrow G \) extending \( \delta \) and such that 
\[ \eta \circ \rho \circ \iota_{B^-*IS_1} \circ \Lambda(0,1) = p \circ i \circ p = p \quad \text{and} \]
\[ \eta \circ \rho \circ \iota_{B^-*IS_1} \circ \Lambda(1,0) = 1 \]
where \( \iota_{B^-*IS_1} \) is the standard section and \( \Lambda : IS_1 \rightarrow R(B^-*IS_1) \) the canonical inclusion.

By lemma 8.3 the lemma is proved putting \( \beta = \eta \circ \rho \circ \iota_{B^-} \)
\[ \gamma = \eta \circ \rho \circ W(1 * u), \quad \pi = \eta \circ \rho \circ W(1 * v). \]
Lemmas 8.7: Let $A$ be a strong deformation retract of $X$. $A \xrightarrow{i} X \xrightarrow{p} A$, $p \circ i = 1_A$, and suppose $H_t: 1_X \simeq 1_p$ rel $A$ satisfies $H_t \circ H_{t_2} = H_{\max(t_1, t_2)}$. If $(X, \zeta)$ is a $WB^-$-space then $A$ can be made into a $WB^-$-space $(A, \alpha)$ and $p$ and $i$ into $s$-homotopy equivalences $(p, \pi): (X, \zeta) \to (A, \alpha)$ and $(i, \gamma): (A, \alpha) \to (X, \zeta)$ inverse to each other.

Proof: Put $M = WB^-$. The action $\zeta$ induces a map

$$\overline{x}: A^n \xrightarrow{i} X^n \xrightarrow{\zeta(x)} X^m \xrightarrow{p} A^m$$

for each $x \in M(n, m)$. Although this time $1 = 1_A$, we have $\overline{x \circ y} \neq \overline{x \circ y}$. We again correct this by bringing in the homotopy $H_t$. The condition on the homotopy $H_t$ provides us with the condition we need for the degenerate trees.

Let $L(a, b)$ be the subspace of all those representing trees $(\theta, \xi, \delta)$ of $RW(M^*Is_1)(a, b)$ such that all links of $\theta$ are labelled by 1, and $\theta$ is not a trivial tree. If any edge of $\theta$ is labelled by 0, then it is either the root or a twig. $L(a, b)$ is closed in the space of the representatives of $RW(M^*Is_1)(a, b)$. Hence introducing the relations (2.11), (2.12), (2.13) in $L(a, b)$ we obtain a space $Q(a, b)$ in $CG$. The composition with permutations on the right is the one induced from the composition of the representing
trees with permutations. By applying the reduced version of the normal form construction we obtain morphism spaces into longer sequences. Define composition in $\mathfrak{G}$ as follows: Use the ordinary tree composition but assign the value $1$ to the newly created links iff these are labelled by $1$. If they are labelled by $0$, shrink the new links (see p. 25) to obtain a representative in $L$. This composition is well defined, continuous, and associative since the composition in $\mathfrak{M}$ is. Again the trees

\[
\begin{array}{c}
0 \\
1
\end{array}
\quad \text{and} \quad
\begin{array}{c}
0 \\
1
\end{array}
\]

serve as identities. It follows from the tree representation that $\mathfrak{G}$ is bifunctorial whenever it is defined (see also Lemma 2.20). Hence $\mathfrak{G}$ is a reduced $\mathfrak{M}^2$TP-category.

Define an action $\nu: \mathfrak{G} \to \text{REnd}(X,A)$ as follows: Given a representative $(\theta, \xi, \delta)$ of an element in $\mathfrak{G}$. Replace each vertex $v$ labelled by $b \in \mathfrak{M}$ by $H_t \circ \zeta(b)$, where $t$ is the value of the link below $v$. If $v$ is at the root simply replace it by $\zeta(b)$. Shrinking all links as defined on p. 25, using $x$ instead of $\oplus$, we obtain maps

\[
m(\theta, \xi, \delta): X^n \xrightarrow{\xi} X^n \xrightarrow{\xi} X^m
\]

where $n$ is the number of twigs and $m$ the number of roots in $\theta$.\]
For example,

\[ \begin{array}{ccc}
  & b & \\
  a & \downarrow & c \\
  & d & \\
\end{array} \]

with the values

\[ \begin{array}{ccc}
  & t_1 & \\
  t_2 & \downarrow & t_3 \\
\end{array} \]

gives rise to the map \( \xi(d) \circ H_{t_3} \circ \xi(a) \circ [H_{t_1} \circ \xi(b) \times H_{t_2} \circ \xi(c)] \)

Define

\[
\nu\{\theta, \xi, \delta\} = \begin{cases} 
  p^m \cdot m(\theta, \xi, \delta) & \text{if } \{\theta, \xi, \delta\} \in G(n, m') \\
  p^m \cdot m(\theta, \xi, \delta) \circ i^n & \text{if } \{\theta, \xi, \delta\} \in G(n', m') \\
  m(\theta, \xi, \delta) & \text{if } \{\theta, \xi, \delta\} \in G(n, m) \\
\end{cases}
\]

Since \( \xi \) and \( H \) are continuous, \( m \) is continuous. \( m \) factors through the relations since \( \xi \) is an \( M^1 \)TP-functor, \( H_0 = 1_X \), and \( H_{t_1} \circ H_{t_2} = \overline{H}_{\max(t_1, t_2)} \). Hence \( \nu \) is well defined and continuous. Since \( \xi \) is an \( M^1 \)TP-functor \( \nu \) preserves sums and permutations, and since \( p \circ i = 1_A \) it preserves identities. From the definition of composition in \( G \) it follows immediately that \( \nu \) is a functor because \( H_1 = 1^o p \).

The standard augmentation \( \varepsilon_{M*Is_1} \) induces an augmentation functor \( x: G \to R(M*Is_1) \). The standard section \( t_{M*Is_1} \) induces a section of \( x \) and the standard deformation of \( RW(M*Is_1) \) induces a deformation of \( G \) into the section. Hence \( x \) is fibre homotopically trivial.
Resubstitute $WB^-$ for $M$. Since
\[ \varepsilon = \varepsilon_B : R(WB^-*1) \to R(B*1) \]
is fibre homotopically trivial (Lemma 8.5), so is $\varepsilon\circ x$.

Let $D = \partial WB^- \subset RW(B^-*1)$. Define $\delta: D \to G$ as follows: For $x \in \partial WB^-((n,1))$ let $\delta(x) = t_{WB^-}(x) \in G(n,1)$. Extend $\delta$ over $\partial WB^-$ using the normal form. It follows immediately that $\delta$ is an MTP-functor. $D$ and $\delta$ satisfy the requirements of the Universal Theorem. Now define
\[ \tau_1: R(B^-*1)((1,1')) \to G(1,1') \]
and
\[ \tau_2: R(B^-*1)((1',1) \to G(1',1) \]
by
\[ \tau_1(b;(0,1)) = \{(t_B^-b;(0,1)), \text{unit}, I^0\} \]
\[ \tau_2(b;(1,0)) = \{(t_B^-b;(1,0)), \text{unit}, I^0\} \]
$\tau_1$ and $\tau_2$ satisfy the requirements of Lemma 4.8. Hence there exists a functor $\rho: RW(B^-*1) \to G$ extending $\delta$ and such that
\[ \nu^o \rho \circ t_{B^-*1} \circ \Lambda(0,1) = \rho \quad \text{and} \quad \nu^o \rho \circ t_{B^-*1} \circ \Lambda(1,0) = 1. \]
From the tree representation and the choice of $\delta|\partial WB^-$ it follows that $\nu^o \rho \circ \delta^1 = \zeta$. Let $\alpha = \nu^o \rho \circ \delta^0$. Then by Lemma 8.3, putting $\gamma = \nu^o \rho \circ W(1 \ast u)$ and $\pi = \nu^o \rho \circ W(1 \ast v)$, we obtain homotopy $B^-$-maps $(\rho, \pi): (X, \zeta) \to (A, \alpha)$ and $(i, \gamma): (A, \alpha) \to (X, \zeta)$, which are $s$-homotopy equivalences inverse to each other.
Remark: If $B$ is an $\mathcal{M}^1TP$-category with isolated identities then Lemma 8.7 holds also if we replace $B^-$ by $B$.

Proof of the Theorems 8.1 and 8.2:
Let $(X,\alpha)$ be a $WB^-$-space (or a $WB$-space and $B$ has isolated identities), and $f: X \to Y$ be a homotopy equivalence. Let $M$ be the mapping cylinder of $f$, $M = (X \times I \cup Y)/[(x,1) \sim fx]$. Let $i: X \to M$ and $j: Y \to M$ be the natural inclusions, and $p: M \to Y$ the natural projection. Define $H: i_!M = j_!p$ by

$$H_u(x,t) = (x, \max(u,t))$$
$$H_u(y) = y$$

Then $H_{u_1} \circ H_{u_2} = H_{\max(u_1,u_2)}$. Since $f$ is a homotopy equivalence, $i(X)$ is a strong deformation retract of $M$ (see Appendix). In the following diagram

\[
\begin{array}{ccc}
(X,\alpha) & \xrightarrow{(f,p)} & (Y,\beta) \\
\downarrow{(k,\delta)} & & \downarrow{(1_Y,\mu)} \\
(M,\alpha^*) & \xleftarrow{(p,x)} & (Y,\alpha^{**}) \\
\end{array}
\]

let $k: M \to X$ be the retraction, $\alpha^*$ the $WB^-$-structure induced on $M$ by $\alpha$ and $(k,\delta)$, $(i,\gamma)$ the $s$-homotopy equivalences given by Lemma 8.6. Let $\alpha^{**}$ be the $WB^-$-structure
induced on $Y$ by $a^*$ and $(p,x)$, $(j,v)$ the s-homotopy equivalences given by Lemma 8.7. The composite

$$(p,x) \circ (i,y) \simeq (p \circ i , \chi) = (f, \chi)$$

is a s-homotopy equivalence

$$(f, \chi) : (X, \alpha) \to (Y, \alpha^{**}) ,$$

which proves Theorem 8.2.

Now suppose $f$ is given as a homotopy $B$-map $(f, \rho)$ and the identities of $B$ are isolated. Since $p \simeq f \circ k$, and since $(f, \rho) \circ (k, \delta)$ is a homotopy $B$-map, there exists an action $\lambda$ such that $(p, \lambda) \simeq (f, \rho) \circ (k, \delta)$ by Theorem 7.21. Define $(1_Y, \mu)$ to be the composite $(p, \lambda) \circ (j, v)$. By Lemma 8.4 there exists a s-homotopy inverse $(1_Y, \omega)$ of $(1_Y, \mu)$. Now

$$(f, \rho) \circ (k, \delta) \circ (j, v) \circ (1_Y, \omega) \simeq (p, \lambda) \circ (j, v) \circ (1_Y, \omega) \simeq (1_Y, \mu) \circ (1_Y, \omega) \simeq (1_Y)_*$$

$$(k, \delta) \circ (j, v) \circ (1_Y, \omega) \circ (f, \rho)$$

$$(k, \delta) \circ (j, v) \circ (1_Y, \omega) \circ (f, \rho) \circ (k, \delta) \circ (i, \gamma)$$

$$(k, \delta) \circ (j, v) \circ (1_Y, \omega) \circ (p, \lambda) \circ (i, \gamma)$$

$$(k, \delta) \circ (j, v) \circ (1_Y, \omega) \circ (p, \lambda) \circ (j, v) \circ (p, x) \circ (i, \gamma)$$

$$(k, \delta) \circ (j, v) \circ (p, x) \circ (i, \gamma)$$

$$(k, \delta) \circ (i, \gamma)$$

$$(1_A)_* .$$

Hence $(k, \delta) \circ (j, v) \circ (1_Y, \omega)$ is a s-homotopy inverse of $(f, \rho)$.
Throughout this chapter we assume that \( R \) is an \( M^1T \)-category (so without permutations), such that \( (R(1,1), 1_1) \) is a NDR-pair. We choose an \( M^1T \)-category instead of an \( M^1TP \)-category, because the proofs are then slightly simpler. A refinement of the methods used in this chapter and the use of equivariant NDR's as studied in \( \S 7 \) should give the same results for categories with permutations.

Again we denote the sequences of length \( n \) in the object generators 0 or 1 of \( RW(R^*L^1) \) by \( n \), resp. \( n' \).

Denote \( RW(R^*L^1)(m, 1') \) by \( C_m \), and regard \( WB \) embedded in \( RW(R^*L^1) \) by \( \delta^0 \) and \( \delta^1 \), so that composition of elements of \( RW(R^*L^1) \) with elements of \( WB \) makes sense.

For each \( WB \)-space \( (X, \gamma) \) we are going to construct a \( WB \)-space \( UX \), a \( B \)-space \( MX \), which is a quotient of \( UX \), and a \( B \)-space \( NX \), which is a subspace of both \( UX \) and \( MX \). All three spaces have the same homotopy type as \( X \). In addition \( UX \) and \( MX \) satisfy certain universal properties with regard to homotopy \( B \)-maps.

Let \( \beta \in WB(n, m) \), and \( (X, \gamma) \) be a \( WB \)-space. We denote \( \gamma(\beta)(x_1, \ldots, x_n) \) by \( \beta.(x_1, \ldots, x_n) \) or simply by \( \beta.x \).
Let \( K_{p,q} = T_p(q,1') \times \mathbb{R}_p \times \mathbb{R}_q \), where \( T_p = T_p(B^*L_1) \), (see p. 26). Let \( K_p \) be the disjoint union \( K_p = \bigcup_{q=0}^{\infty} K_p,q \), and let \( K \) be the disjoint union \( K = \bigcup_{p=-1}^{\infty} K_p \).

Introduce an equivalence relation on \( K \) by

\[
\left( \theta, \delta, x \right) \sim \left( \phi, \delta, x \right) \text{ if } \{ \theta, \delta \} = \{ \phi, \delta \}, \text{ where } \{ x \} \text{ denotes the equivalence class of } x \text{ in } RW(B^*L_1), \text{ i.e. if }
\]

\[
(\theta, \delta) \sim (\phi, \delta) \text{ under (2.12) and (2.13). ((2.11) does not apply)}.
\]

(9.2) Suppose \( \theta \) in \( (\theta, \delta) \) has the value 1 assigned to a link labelled by 0. Let \( (\phi_1, \delta_1) \) and \( (\phi_2, \delta_2) \) be obtained by chopping this link (see p. 50). Then

\[
(\theta, \delta, x) \sim (\phi_1, \delta_1, \{ \phi_2, \delta_2 \} \cdot x).
\]

In (9.2) it suffices to restrict our attention to non-degenerate elements \( (\theta, \delta) \). Hence after having factored out (9.1), the relation (9.2) reads:

\[
(9.2)^* \quad (c \cdot \beta, x) \sim (c, \beta \cdot x),
\]

where \( c \in C_\mathbb{W} \), and \( \beta \in W_B(\mathbb{D}, \mathbb{D}) \).

Let \( UX = K/\sim \), \( \pi: K \to UX \) the projection, \( U_p X = \pi(U^p_{-1} K_1) \) and \( \pi_p = \pi| U^p_{-1} K_1 \). Note that \( U_0 X = K_0 \).

Call \( (\theta, \delta, x) \) degenerate if

(A1) \( (\theta, \delta) \) is degenerate (see p. 31)

(A2) the value \( 1 \in I \) is assigned to a link of \( \theta \) labelled by 0.
Denote the closed subspace of the degenerate points of $K_p, q$ by $DK_p, q$ and of $K_p$ by $DK_p$. $DK_p$ consists of exactly those points of $K_p$ that are related to a point in some $K_r$ with $r < p$. Note that if $x, y \in K_p - DK_p$, $x \sim y$, then $x = y$.

**Claim**: Each $(\theta, \delta, x) \in K_p$ is related to a unique non-degenerate point.

**Proof**: Let $\lambda$ be the function associating with $(\theta, \delta)$ a unique non-degenerate related point (Lemma 2.14). From the tree representation it follows that $(\theta, \delta)$ can be decomposed uniquely into $(\varphi_1, \delta_1) \circ (\varphi_2, \delta_2)$, such that the value 1 is not assigned to any link in $\varphi_1$ labelled by 0. Define $\rho(\theta, \delta, x) = (\varphi_1, \delta_1, \{\varphi_2, \delta_2\} \cdot x)$. The correspondence $(\theta, \delta, x) \rightarrow \rho(\lambda(\theta, \delta), x)$ associates with each element of $K_p, q$ a related non-degenerate one. It preserves non-degenerate elements and factors through the relations (9.1) and (9.2), which proves the claim.

Let $Y_p, q \subset K_p, q$ be the (closed) subspace of all those points $(\theta, \delta, x)$, for which $(\theta, \delta)$ is degenerate. Let $Z_{\alpha, p, q} \subset T_p(q, 1') \times IP$ be the subspace of all those trees of one type for which the value 1 is assigned to a particular link labelled by 0. Chopping this link induces a projection $\pi: Z_{\alpha, p, q} \rightarrow Z'_{\alpha, p, q} \times Z''_{\alpha, p, q}$. $DK_p, q$ is a
finite union of spaces $Y_{p,q}$ and $Z_{a,p,q}$. By Lemma 2.15 we have continuous maps $f: Y_{p,q} \to U_{p-1}X$ for all $p$ and $q$.

Since

$$Z_{a,p,q} \times X^q \xrightarrow{\times 1} Z'_a,p,q \times Z_a,p,q \times X^q \xrightarrow{1\times \text{action}} Z'_a,p,q \times \{Z_a,p,q\} . X^q \xrightarrow{1\times p-1} U_p X$$

is continuous, the conditions (1), (2), (3) of p. 33 are satisfied. Since $(I, \partial I)$, $(I, 0)$, and $(R(1,1), 1_1)$ are NDR-pairs, $(K_p,q, DK_p,q)$ and hence $(K_p, DK_p)$ are NDR-pairs.

Hence by the construction of p. 33 we obtain

**Lemma 9.3:**

(a) $UX, U_p X$ are in CG, $p=0,1,2,\ldots$

(b) $UX$ is the direct limit of $U_0 X \subset U_1 X \subset \ldots$

(c) $(UX, U_p X)$, $(U_{p+1} X, U_p X)$ are NDR-pairs for all $p \geq 0$.

To construct $MX$, we introduce a further relation in $K$, which is independent of (9.1) and (9.2). Hence $MX$ is a quotient of $UX$.

(9.4) Suppose $\theta$ in $(\theta, \delta)$ has the value 1 assigned to a collection of links labelled by 1, which separates the tree $\theta$ into a tree $\varphi$ the edges of which are labelled by 1 only, and into a copse $\psi$. Let $(\theta', \delta')$ be the pair
obtained from $(\theta, \delta)$ by shrinking (see p. 25) all links in $\varphi$. Then $(\theta, \delta, \underline{x}) \sim (\theta', \delta', \underline{x})$.

For example, suppose that in the following picture the links on the separating line have the value 1 assigned to them, while the values of the links above the line are the same. Then the elements $(\theta, \delta, \underline{x})$ and $(\varphi, \delta, \underline{x})$ are related under (9.4).

Let $MX = K/\sim$, $\omega: K \to MX$ be the projection, $M_p X = \omega(U_P^{-1} K_i)$ and $\omega_p = \omega|U_P^{-1} K_i$. Note that $M_0 X = K_0$.

Call $(\theta, \delta, \underline{x})$ degenerate, if it satisfies (A1) or (A2) of p. 155, or if (A3) the value 1 is assigned to a separating collection of links labelled by 1, and chopping these links decomposes $(\theta, \delta)$ into a tree at least one link and a copse.

Let $RK_{p,q}$ be the subspace of the degenerate points of $K_{p,q}$, and $RK_p$ of those of $K_p$. $RK_p$ consists of exactly those points of $K_p$, that are related to a point in some
Notice that if \( x, y \in K_p - RK_p \) and \( x \sim y \), then \( x = y \).

Suppose \((\theta, \delta)\) is degenerate under \((A3)\). From the tree representation it follows, that we can decompose \((\theta, \delta)\) into \((\phi_1, \delta_1)^{(\phi_2, \delta_2)}\) such that the edges of \( \phi_1 \) are labelled by 1 only, and \((\phi_2, \delta_2)\) is not degenerate under \((A3)\). Let \((\theta, \delta)^*\) be the pair obtained by substituting the values of the links in \( \theta \) which come from \( \phi_1 \) by 0. Then \((\theta, \delta)^*\) is not degenerate under \((A3)\). The correspondence 
\((\theta, \delta, x) \rightarrow \rho(\lambda[(\theta, \delta)^*], x)\) associates with each element of \(K_p\) a unique non-degenerate related one. Let \(C_{a,p,q}\) be the subspace of \(T_p(q, 1') \times I^p\) consisting of one type of trees, such that the value 1 is assigned to a separating collection of links labelled by 1 such that the tree below this separating collection has at least one link. Let \(x \in C_{a,p,q}\). The correspondence \(x \rightarrow x^*\) as defined above, induces a continuous map \(g\) of \(C\) into some \(T_r(q, 1') \times I^r\) with \(r < p\).

Hence the composite \(\omega_{p-1} \circ (gx1): C_{a,p,q} \times X^q \rightarrow M_{p-1}X\) is continuous. By the consideration of p. 157 we furthermore have continuous maps

\[
Y_{p,q} \rightarrow M_{p-1}X
\]
\[
Z_{a,p,q} \rightarrow M_{p-1}X.
\]

Since \(RK_{p,q}\) is a finite union of spaces \(Y_{p,q}\), \(Z_{a,p,q}\), and \(C_{a,p,q}\) the conditions (1), (2), (3) of p. 33 are satis-
fied. Using [6; Lemma 7.3] again, we find that \( R_{K_{p,q}} \) is a NDR in \( K_{p,q} \). Hence by the construction of p.33 we get:

**Lemma 9.5:**
(a) \( MX, M_pX \) are in CG, \( p = 0,1,2,... \)
(b) \( MX \) is the direct limit of \( M_0X \subset M_1X \subset ... \)
(c) \( (M_pX, M_pX), (M_{p+1}X, M_pX) \) are NDR-pairs for all \( p \geq 0 \).

\( NX \) is the subspace of \( MX \) and \( UX \) represented by all points \( (\theta, \delta, x) \) of \( K \), such that all edges of \( \theta \) with exception of the root are labelled by 0. On this set of representatives the relations defining \( MX \) and \( UX \) coincide. Hence, if \( \zeta: UX \to MX \) is the projection induced by the relation (9.4), \( \zeta|NX \) is the identity.

If \( (\theta, \delta, x) \in K_p \) represents an element of \( NX \), then so does \( \rho(\lambda(\theta, \delta), x) \). Furthermore if \( NK_p \) is the subspace of those elements in \( K_p \) that represent an element of \( NX \), then \( DK_p \cap NK_p \) is a NDR in \( NK_p \). Hence \( NX \) is in CG.

**Definition 9.6:** Let \((Z, \delta)\) be a \(B\)-space. Then the \(WB\)-structure on \( Z \) given by \( \delta \circ \rho_B: WB \to \text{End}Z \) is called the \(WB\)-structure on \( Z \) induced by \( \delta \).
Lemma 9.7: (a) UX is a WB-space $(UX, \chi)$ and there exists a homotopy $B$-map $(u, \mu): (X, \gamma) \rightarrow (UX, \chi)$.

(b) MX is a $B$-space $(MX, x)$ and there exists a homotopy $B$-map $(m, \nu): (X, \gamma) \rightarrow (MX, x^*)$, where $x^*$ is the WB-structure on $MX$ induced by $x$.

(c) NX is a $B$-space.

(d) $\zeta: (UX, \chi) \rightarrow (MX, x^*)$ is a WB-homomorphism.

Proof: We use the relation (9.2)*. Let $a \in WB(n, 1)$, and let $y_i \in U_{p_i} \ X$ be represented by $(c_i, x_i) \in C \times X_1$, $i = 1, \ldots, n$.

(Recall that $C_m = RW(B^*L_1)(m, 1')$). Define

$$a \cdot (y_1, \ldots, y_n) = \{a \circ (c_1 \otimes \ldots \otimes c_n), x_1 \times \ldots \times x_n\} \in U_{p_1 + \ldots + p_n} X$$

Extend this definition to actions of $a' \in WB(n, m)$ using the normal form of $WB$, and taking the $m$-fold product of the above definition. Since $(UX)^n$ is filtered by $(UX)^n_p = U_{p_1 + \ldots + p_n} (U_{p_1} X \times \ldots \times U_{p_n} X)$ and since the topology of $UX$ is the quotient topology from the disjoint union of the $C_q X^q$ under (9.2)*, this defines a continuous action $\chi$ of $WB$ on $UX$. To define the action $\mu: RW(B^*L_1) \rightarrow REnd(X, UX)$ it suffices to define the action of elements of $C_m$ compatibly with the action $\gamma$ on $X$ and $\chi$ on $UX$. Let $\beta \in C_q$. Define
\[ \beta(x_1, \ldots, x_q) = \{\beta; x_1, \ldots, x_q\} \]

Use the normal form of \( RW(B^*L_1) \) to extend this definition over the whole of \( RW(B^*L_1) \). It clearly is compatible with the action \( \chi \). For \( \alpha \in WB(\mathfrak{n}, \mathfrak{m}) \) and \( \beta \in C_{\mathfrak{m}} \) we have

\[ (\beta \circ \alpha)(x_1, \ldots, x_n) = \{\beta \circ \alpha; x_1, \ldots, x_n\} \]
\[ = \{\beta; \alpha.(x_1, \ldots, x_n)\} \quad \text{by (9.2)*} \]
\[ = \beta.(\alpha.(x_1, \ldots, x_n)) \]

Hence \( \mu \) is a continuous functor. By definition it preserves sums. Hence it is a reduced \( M^2T \)-functor, extending \( \gamma \) and \( \chi \).

\[ u = \mu \circ \mathfrak{B}^*L_1 \circ \Lambda(0,1) \]

is given by \( u(x) = \{t_{B^*L_1}(1;(0,1));x\} \). Note that \( u: X \to UX \) is an inclusion \((u(X) \) is closed in \( UX \).

(b) Since \( MX \) is a quotient of \( UX \), it is also a quotient of the disjoint union of the \( C_q \times X^q \). Let \( \beta \in B(\mathfrak{n}, \mathfrak{1}) \), and let \( y_1 \in M_{\mathfrak{1}}X \) be represented by \((c_1, x_1) \in C_{q_1} \times X_{q_1} \). Define the action \( \chi \) by

\[ \beta(y_1, \ldots, y_n) = \{(t_{B^*}(\beta) \circ (c_1 \oplus \cdots \oplus c_n); x_1 \times \cdots \times x_n\} \]

where \( \{x\} \) as usually denotes the equivalence class of \( x \).

Using the normal form of \( B \), we can extend this definition uniquely over the whole of \( B \). The relation (9.4) assures that \( \chi \) is a functor. For if \( \alpha \in B(\mathfrak{m}, \mathfrak{1}) \) and \( \beta \in B(\mathfrak{n}, \mathfrak{m}) \), \( \beta = \beta_1 \oplus \cdots \oplus \beta_m \), then \((a \circ \beta)(y_1, \ldots, y_n)\) is represented by \((\theta, \delta, x_1 \times \cdots \times x_n)\), where \( \theta \) is the tree with the vertex
at the root labelled by \((\alpha \circ \beta)\), the representing trees \(A_1, \ldots, A_n\) of \(c_1, \ldots, c_n\) on its incoming edges, and the value 1 assigned to each of these edges,

\[
\theta = \alpha \circ \beta
\]

while \(\alpha \cdot (\beta \cdot (y_1, \ldots, y_n))\) is represented by \((\phi, \delta, x_1 \times \cdots \times x_n)\), where \(\phi\) is the tree with the vertex \(\alpha\) at the root, the vertices on top of its incoming edges labelled by \(\beta_1, \ldots, \beta_m\), and the trees \(A_1, \ldots, A_n\) sitting on the incoming edges of the \(\beta_i\)'s.

\[
\phi = \beta_1 \circ \alpha \circ \beta_2
\]

The value 1 is assigned to the links ending in some \(\beta_i\) or in \(\alpha\). By relation (9.4) we can shrink the links below the vertices \(\beta_i\). But then we obtain the representative for 
\((\alpha \circ \beta) \cdot (y_1, \ldots, y_n)\).
Since the filtration of $MX$ induces the one of $(MX)^n$, $x$ is an $M^1T$-functor.

To construct the homotopy $B$-map $(x, y) : (X, \gamma) \to (MX, x^*)$, we have to extend the actions $\gamma$ and $x^*$ of $\partial^1_{WB} \subset RW(B*L_1)$, $i = 0, 1$, over the whole of $RW(B*L_1)$. Let $\beta \in C_n$. Define $\nu$ by

$$\beta \cdot (x_1, \ldots, x_n) = \{\beta ; x_1, \ldots, x_n\}$$

and extend this to an action of $RW(B*L_1)$ using the normal form. As in part (a) it follows that $\nu$ extends $\gamma$ and $x^*$, and is functorial. Hence $\nu$ is a reduced $M^2T$-functor.

$m = \nu \circ \alpha_{B*L_1}(0, 1)$ is given by $m(x) = \{i \circ B*L_1(1;(0,1)); x\}$.

Note that $m : X \to MX$ is an inclusion.

(c) The $B$-structure $\lambda$ of $NX$ is defined on representatives as follows: Let $\alpha \in B(n, 1)$, and let $(\theta_i, \delta_i, x_i) \in K_{P_i}$, $i = 1, \ldots, n$, be representatives of elements in $NX$. Each edge of $\theta_i$ with exception of the root is labelled by $0$. Let $\beta_i$ be the label of the vertex at the root of $\theta_i$, and let $A_{i1}, \ldots, A_{ik_i}$ be the subtrees of $\theta_i$ sitting on the incoming edges of $\beta_i$. Let $\varphi$ be the tree with the vertex at the root labelled by $\alpha(\beta_1 \oplus \ldots \oplus \beta_n)$, and the trees $A_{11}, \ldots, A_{1k_1}, \ldots, A_{n1}, \ldots, A_{nk_n}$ sitting on its incoming edges. Assign the values of the links of the $\theta_i$'s to the.
links of \( \varphi \) (the incoming edges of \( \alpha^\circ (\beta_1 \oplus \ldots \oplus \beta_n) \) have the values of the incoming edges of the \( \beta_i \) in \( \theta_i \)).

Let \( (\varphi, \delta) \) be the pair thus obtained. Define \( \lambda \) by

\[
\lambda(\{\theta_1, \delta_1, \xi_1\}, \ldots, \{\theta_n, \delta_n, \xi_n\}) = \{\varphi, \delta, \xi_1 \times \ldots \times \xi_n\}.
\]

From the tree representation it is clear that \( \lambda \) is an \( M^1T \)-functor.

(d) Let \( a \in \mathsf{WB}(n, 1) \), and let \( y_1 \in \mathcal{U}X \) be represented by \((c_1, \xi_1) \in \mathcal{C}_n \times X_1^1\). Then

\[
\lambda(y_1, \ldots, y_n) = \{\alpha^\circ (c_1 \oplus \ldots \oplus c_n), \xi_1 \times \ldots \times \xi_n\}
\]

Under relation (9.4) the representative

\[
[\alpha^\circ (c_1 \oplus \ldots \oplus c_n), \xi_1 \times \ldots \times \xi_n]
\]

of \( MX \) is related to

\[
[(\mathcal{B} \circ \mathbf{B}(a))^\circ (c_1 \oplus \ldots \oplus c_n); \xi_1 \times \ldots \times \xi_n],
\]

which represents \( \varepsilon_B(a).\{\{y_1\}, \ldots, \{y_n\}\} \), where \( \{y_i\} \) is the equivalence class of \( y_i \) in \( MX \). Hence, since \( WB \) is in normal form, \( \zeta \) is a \( WB \)-homomorphism.

\[ \]

Remark 9.8: (1) From the tree representation it follows immediately that the homotopy \( B \)-map
(m,v):(X,y)→(MX,x*) is the canonical composite of the $\mathcal{W}_B$-homomorphism $\zeta$ with the homotopy $B$-map $(u,\mu):(X,y)→(UX,\chi)$.

(2) $u(X)$ and $m(X)$ are subspaces of $NX$. Hence $u$ and $m$ factor into $X \subset NX \subset UX$ and $X \subset NX \subset MX$. The images of $u$ and $m$ in $NX$ agree.

(3) We can construct $\mathcal{W}_B$-homomorphisms $(UX,\chi)→(NX,\lambda^*)$, where $\lambda^*$ is the $\mathcal{W}_B$-structure on $NX$ induced by $\lambda$, and a $B$-homomorphism $(MX,x)→(NX,\lambda)$. Since we do not use them we refrain from giving the definitions.

**Theorem 9.9:**

(a) Each homotopy $B$-map $(f,\rho):(X,y)→(Y,\delta)$ factors uniquely as $(f,\rho) = Uf^\circ(u,\mu)$, where $Uf:(UX,\chi)→(Y,\delta)$ is a $\mathcal{W}_B$-homomorphism and $Uf^\circ(u,\mu)$ the canonical composite. Further, $Uf$ is a continuous function of $(f,\rho)$.

(b) Let $(Z,\eta)$ be a $B$-space and $\eta^*$ the $\mathcal{W}_B$-structure on $Z$ induced by $\eta$. Then each homotopy $B$-map $(f,\rho):(X,y)→(Z,\eta^*)$ factors uniquely as $(f,\rho) = Mf^\circ(m,v)$, where $Mf:(MX,x)→(Z,\eta)$ is a $B$-homomorphism (and hence a $\mathcal{W}_B$-homomorphism).
(MX,χ*) → (Z,η*) and Mf°(m,ν) the canonical composite. Furthermore, Mf is a continuous function of (f,ρ).

Proof: ρ induces maps \( f_n : C_n \times X^n \to Y \), which in turn determine maps \( h_{n,p} : Tp(n,1') \times IP \times X^n \to C_n \times X^n \xrightarrow{fn} Y \), where \( \chi \) is the characteristic map for \( Tp(n,1') \times IP \). \( \mu \) is induced by the identities \( Tp(n,1') \times IP \times X^n = Kp,n \). Hence if \( Uf \) with the required properties exists, then \( Uf \) must be induced by the collection of the \( h_{n,p} \)'s, \( h_{n,p} : Kp,n \to Y \). Since \( \rho \) is a functor, \( h_{n,p} \) respects the relations (9.1) and (9.2). Hence it indeed induces a map \( Uf : UX \to Y \), and the part (a) is proved if we can show, that \( Uf \) is a \( \mathcal{W} \)-homomorphism: Let \( \beta \in \mathcal{W}(n,1) \) be represented by \( (\theta,\delta) \) in \( Tp(n,1) \times IP \), and \( y_i \in UX \) by \( (\varphi_i,\delta_i,x_i) \in K_{p_i,q_i}, i=1,\ldots,n \). Then

\[
Uf[\beta.(y_1,\ldots,y_n)] = h_{m,r}[(\theta,\delta)^{\circ}(\varphi_1,\delta_1)\oplus\cdots\oplus(\varphi_n,\delta_n)];x_1\times\cdots\times x_n \]

some \( r,m \)

\[
= f_{m}([\theta,\delta]^{\circ}\{\varphi_1,\delta_1\}\oplus\cdots\oplus\{\varphi_n,\delta_n\};x_1\times\cdots\times x_n)
\]

\[
= \{\theta,\delta\}^{\circ}[f_{q_1}((\varphi_1,\delta_1);x_1)\times\cdots\times f_{q_n}((\varphi_n,\delta_n],x_n)] \]

since \( f_m \) is induced by an action

\[
= \beta.(Uf(y_1)\times\cdots\times Uf(y_n))
\]

From the definition it is obvious that \( Uf \) is a continuous
function of \((f, \rho)\).

(b) \(\rho\) induces maps \(f_n: C_n \times X^n \to Z\), which in turn induce maps \(h_{p,n}: K_{p,n} = Tp(n,1) \times T^n \times X^n \to C_n \times X^n \xrightarrow{f_n} Z\). If \(Mf\) exists it must be induced by the collection of maps \(h_{p,n}\).

Since \(Z\) is a \(B\)-space and \(\rho\) an action, \(h_{p,n}\) respects (9.2) and (9.4). By definition it respects (9.1). Hence the collection of the \(h_{p,n}\) indeed induces a map \(Mf: MX \to Z\).

It remains to show that \(Mf\) is a \(B\)-homomorphism. Let \(\beta \in B(n,1)\), and let \(y_i \in MX\) be represented by \((\varphi_i, \delta_i, x_i)\) in \(K_{p_i, q_i}, i = 1, \ldots, n\). Then

\[
Mf[\beta.(y_1, \ldots, y_n)] = h_{r,m}[t_B(\beta;(1,1)) \circ \{(\varphi_1, \delta_1) \oplus \ldots \oplus (\varphi_n, \delta_n)\}, x_1 \times \ldots \times x_n]
\]

\[
= f_m[t_B \circ \{(\varphi_1, \delta_1) \oplus \ldots \oplus (\varphi_n, \delta_n)\}, x_1 \times \ldots \times x_n]
\]

\[
= (e_B \circ t_B(\beta)) \cdot (f_{q_1} \{(\varphi_1, \delta_1, x_1)\} \times \ldots \times f_{q_n} \{(\varphi_n, \delta_n, x_n)\})
\]

since \(f_m\) is induced by an action and \(Z\) has the induced \(\mathbb{V}_{\mathbb{B}}\)-structure.

\[
= \beta.(Mf(y_1) \times \ldots \times Mf(y_n)).
\]

From the definition it is clear that \(Mf\) is a continuous function in \((f, \rho)\).
Proof: $X \subset NX$ is the subspace of $K_0 = U_0X = M_0X \subset NX$ of all triples $(\theta, 1^0, x)$, where

$$\theta = \begin{array}{ccc}
0 & \quad & 0 \\
\alpha & \quad & 1 \\
1 & \quad & 1
\end{array}$$

Recall that each element of $NX$ is represented by a triple $(\theta, \delta, x)$ such that each edge of $\theta$ is labelled by $0$ with exception of the root. Using the relation (9.1) we can choose the representatives such that the vertex at the root of $\theta$ is labelled by $1 \in B(1, 1)$. (Substitute the vertex at the root by the subtree)

and assign the value $\delta$ to the new link between the vertices labelled by $\alpha$ and $1$. Hence

The strong deformation retraction is induced by a deformation of the space of these representatives: Define

$$H_t\{\theta, \delta, x\} = \{\theta, H_t(\delta), x\}$$

with
\[ H_t(u_1, \ldots, u_p) = (\max(t_1, u_1), \ldots, \max(t_p, u_p)) \]

where \( t_i = t \) if \( u_i \) is assigned to a link ending in the vertex at the root, and \( t_i = 0 \) otherwise. Since links with the value 1 are not affected, \( H_t \) preserves the relation (9.2). Since the multiplication "\( \max \)" on \( I \) is associative, \( H_t \) also preserves the relation (9.1). \( H_t(\theta, \delta, \kappa) \) can be represented by a triple \((\varphi, \delta, \kappa)\) such that \((\varphi, \delta)\) represents a composition \( (\beta \ast \lambda_1(1; (0, 1))) \circ z \). Hence by the relation (9.2), \( H_t(\theta, \delta, \kappa) \in X \). Note that throughout the deformation the elements of \( X \) stay fixed.

Define the strong deformation retraction of \( UX \) into \( NX \) by

\[ H_t(\theta, \delta, \kappa) = (\theta, H_t(\delta), \kappa) \]

with \( H_t(u_1, \ldots, u_p) = (t_1 \cdot u_1, \ldots, t_p \cdot u_p) \)

where \( t_i = t \) if \( u_i \) is assigned to a link labelled by 1, and \( t_i = 1 \) otherwise. Since links labelled by 0 are not affected, (9.2) is preserved, and it follows immediately that (9.1) is preserved. Notice that \( H_t \) keeps the elements of \( NX \) fixed since only the roots of their representing trees are labelled by 1. \( H_0(\theta, \delta, \kappa) \in NX \).

The deformation retraction of \( MX \) into \( NX \) is more complicated. Filter \( MX \) as follows: \( F_n MX \) is the subspace of \( MX \) of those elements that can be represented by a triple \((\theta, \delta, \kappa)\) such that at most \( n \) links of \( \theta \) are labelled by 1.
Notice that the subspace of the representing elements of $F_n MX$ is closed in the space of the representing elements of $MX$. $F_0 MX = NX$. We are now going to define a strong deformation retraction of $F_n MX$ into $F_{n-1} MX$.

Consider a typical representative $(\theta, \delta, \underline{x})$ of $F_n MX$ with

![Diagram of a tree structure with nodes labeled $A_1$, $A_2$, $A_3$, $A_k$ and arrows indicating the root node $\theta$.]

The roots of $A_1, \ldots, A_k$ can be labelled by 0 or 1. Index $A_1, \ldots, A_k$ such that $A_1, \ldots, A_r$ have their roots labelled by 1 and $A_{r+1}, \ldots, A_k$ by 0. Index the incoming edges of $\alpha$ by the indices of the trees sitting on them. We consider one type of trees only.

Let $P_1$ be the space of the trees of the type of $A_1$. Let $P'_1$ be the subspace of those trees $A_1$ of $P_1$ such that the value 1 is assigned to a collection of links of $A_1$ labelled by 1, which separates $A_1$ into a tree each edge of which is labelled by 1 and a copse. Since $(I, 1)$ is a NDR-pair, $P'_1$ is a NDR in $P_1$. Let $Q \subset P_1 \times \ldots \times P_r = P$ be the subspace of all those copses $A_1 \Theta \ldots \Theta A_r$ such that the value 0 is assigned to a link labelled by 1, or a vertex
with label \( E \) has the incoming and outgoing edge labelled by \( E \). Since \((I, 0)\) and \((E(1,1), 1)\) are NDR-pairs, so is \((P, Q)\).

Let \( t = (t_1, \ldots, t_r) \) be the collection of values assigned to the incoming edges of \( \alpha \) indexed by \( 1, \ldots, r \).

**Case I:** \( r \neq k \). Then \((\theta, \delta, \kappa)\) represents an element of \( F_{n-1} MX \) iff \( (A_1 \oplus \cdots \oplus A_k) \in Q \) or \( t \in LI^R \) where \( LI^R \subset I^R \) is the collection of lower faces of \( I^R \). Hence we want a strong deformation retraction

\[
P_1 \times \cdots \times P_k \times I^R \rightarrow Q \times P_{r+1} \times \cdots \times P_k \times I^R \cup P_1 \times \cdots \times P_k \times LI^R
\]

Since \( Q \times P_{r+1} \times \cdots \times P_k \subset P_1 \times \cdots \times P_k \) is a NDR and since \( LI^R \) is a strong deformation retract of \( I^R \), such a deformation retraction exists by [6; Theorem 6.3].

**Case II:** \( r = k \). Then \((\theta, \delta, \kappa)\) represents an element of \( F_{n-1} MX \) iff one of the following conditions holds:

(1) \( A = A_1 \oplus \cdots \oplus A_k \subset Q \)

(2) \( t \in LI^k \)

(3) For each \( i \) either \( t_i = 1 \), \( t_i \in I \), or \( A_i \in P_i' \). But at least one \( A_i \) is in \( P_i' \) for some \( i \).

(4) \( \alpha = 1 \)

Construct the deformation \( H \) of \( F_n MX \) into \( F_{n-1} MX \) by induction on the number of trees in \( A = A_1 \oplus \cdots \oplus A_k \) that are in some \( P_i' \). Let \( P^q \) be the subspace of \( P \) of those
copses, such that at least $q$ of their trees are in some $P_i$. Then each element of $P_k$ represents an element of lower filtration.

Suppose inductively that $H$ has been defined on all elements $\{\emptyset, \delta, \mathbb{A}\}$, for which the subcops $A$ is contained in $P^{q+1}$.

Let $A \in P^q$; wlog $A \in P'_1 \times \ldots \times P'_q \times P_{q+1} \times \ldots \times P_k$, which we denote by $R$. $H$ has been defined on $\{\emptyset, \delta, \mathbb{A}\}$ iff

(5) $A_1 \oplus \ldots \oplus A_k \in Q$
(6) $A_i \in P'_i$ for some $i \neq 1, \ldots, q$
(7) $t \in L_1 I^k \cup \{(t_1, \ldots, t_k) \in I^k \mid t_{q+1} = \ldots = t_k = 1\}$, which we denote by $G_{II}^k$, for $q > 0$
(8) $a = 1$

Let $B = B(k, 1)$ and $B' = \emptyset$ if $k \neq 1$, $B' = (1, 1)$ if $k = 1$.

Let $R' \subset R$ be the subspace of all those copses $A$ satisfying (5) or (6). Since $(1, 0)$, $(1, \delta I)$, and $(B(1, 1), 1)$ are NDR-pairs, so are $(R, R')$ and $(B, B')$ and hence $(R \times B, R \times B' \cup R' \times B)$. We want a deformation retraction

$$R \times B \times I^k \to (R' \times B \cup R \times B') \times I^k \cup R \times B \times G_{II}^k.$$ 

By [6; Theorem 6.3] it suffices to show that there exists a deformation retraction

$$I^k \to G_{II}^k.$$ 

If $q \neq 0$, then $G_{II}^k = 0 \times I^{k-1} \cup I \times G_{II}^k$, where
\[ G^t I^{k-1} = L I^{k-1} \cup \{(t_2, \ldots, t_k) \in I^{k-1} \mid t_{q+1} = \ldots = t_k = 1\} \]

\((I^{k-1}, G^t I^{k-1})\) is a NDR-pair. Since 0 is a deformation retraction of \(I\), there exists a deformation retraction \(I^k \to GI^k\) for \(q \neq 0\).

In view of condition (3), \(GI^k\) reduces to \(LI^k\) if \(q = 0\), and \(LI^k\) is a deformation retraction of \(I^k\).}

\textbf{Corollary 9.11:} \(UX\) and \(MX\) have the same homotopy type as \(X\).

\textbf{Corollary 9.12:} If \(B\) is an \(M^1T\)-category with isolated identities, and \((X, \gamma)\) a \(WB\)-space, then

\[(u, \mu): (X, \gamma) \to (UX, \chi) \quad \text{and} \quad (m, \nu): (X, \gamma) \to (MX, \chi^*)\]

are \(s\)-homotopy equivalences.

This follows from Theorem 9.10 and Theorem 8.1.}

\textbf{Corollary 9.13:} Let \(A\) be the \(M^1T\)-category of Example 2, p.9. Then any \(WA\)-space is of the same homotopy type as a topological monoid.

The last result has been known to J.F. Adams and J.D. Stasheff (unpublished), but their topological monoid seems to be different from our monoid \(MX\).
The results of this chapter are entirely due to Dr J.M. Boardman. We enclose them to give some indication of applications of the theory we have developed.

Definition 10.1: A space $X$ is called an $E$-space if it is given an $E$-structure, which consists of an $M^1TP$-category $B$, acting on $X$, for which $B(n, 1)$ is contractible for all $n$.

Main Theorem 10.2: A CW-complex $X$ admits an $E$-structure with $\pi_0(X)$ a group, if and only if it is an infinite loop space.

Sketch proof. $X$ is an infinite loop space if and only if there is a sequence of spaces $X_n$ and homotopy equivalences $X_n \simeq \Omega X_{n+1}$ for $n \geq 0$, with $X = X_0$. Careful use of mapping cylinders and telescopes enables us to find a space $Y$ homotopy-equivalent to $X$, and spaces $Y_1, Y_2, \ldots$ such that

$$Y = \Omega Y_1, \quad Y_1 = \Omega Y_2, \quad Y_2 = \Omega Y_3, \ldots$$
Example 4 (p. 14) shows that the space $Y = Y^n$ admits a category of operators $Q_n$, which becomes highly connected for $n$ large. Moreover, we can include $Q_n$ in $Q_{n+1}$ as a subcategory of operators, so that the union $\bigcup_n Q_n$ acts on $Y$. This is an $E$-structure on $Y$. By Theorem 8.2, given a category $E$ acting on $Y$, we can make $WB^E$ act on $X$, and this is another $E$-structure on $X$.

Conversely, suppose we are given an $E$-structure on $X$. In this direction the theorem reduces to the following theorem, as induction step:

**Theorem 10.3:** Given an $E$-space $X$, where $X$ is a CW-complex, for which $\pi_0(X)$ is a group (by means of the $E$-structure) then there exists a "classifying space" $BX$ such that $X \simeq XB$, and $BX$ is an $E$-space, and $BX$ is a CW-complex.

The first step is the construction of a good category to act on $E$-spaces. What we need is a category $WB^E$, in which each space $B(n, 1)$ is a contractible CW-complex on which the symmetric group $S_n$ acts freely and cellularly.

We now return to the given $E$-structure on $X$, and deduce from it by Theorem 4.9 an action of $WA$ on $X$, where $A$ is the category of Example 2, p. 9. Then $WA$ also acts on $X^n$. We now use the relative universal property many times.
For each point $a \in \mathcal{B}(m, n)$ we construct a homotopy $A$-map $f_a : X^m \to X$, $n$ continuous in $a$. These must behave properly with respect to products $\otimes$ and permutations. However, we cannot compose homotopy $A$-maps. Whenever $a$ and $\beta$ are composable, we construct a 2-simplex of the semi simplicial complex $\text{Map}_A$ with faces $f_\alpha$, $f_\beta$ and $f_\beta \alpha$. This corresponds to an "edge" of $W^A$. Similarly for higher-dimensional simplexes, although the details become vastly more complicated. What we now have is a kind of $E$-space in the "category" of $WA$-spaces.

The next step is to reduce all the $WA$-actions to $A$-actions, the homotopy $A$-maps to $A$-homomorphisms, etc. The main tool for achieving this is Theorem 9.9 and Corollary 9.11 which first replaces $X$ by the universal monoid $MX$, and continues with the help of the restricted Kan extension condition. Much complication is caused by the fact that the natural map $M(X \times Y) \to MX \times MY$ is only a homotopy equivalence, so that homotopy inverses have to be chosen. Now monoid homomorphisms can be composed, which enables us to replace the semi-simplicial gadget by an $E$-space in the category of monoids, in which all the actions are monoid homomorphisms.
Finally we apply a suitable classifying space functor \( B \), and define \( BX = BMX \). The most convenient is Milgram's functor [4], because it has the property \( B(MX)^n = (BMX)^n \). Thus \( BX \) becomes an E-space, as required. Further, it is a CW-complex. Milgram [4] proves that \( \Omega BMX \simeq MX \) which with \( MX \simeq X \) (Corollary 9.13) shows that \( \Omega BX \simeq X \), provided that \( \pi_0(X) \) is a group.

Of course the theorem can be strengthened in all the obvious ways. The homotopy equivalence between the given E-space and the constructed infinite loop space can be made into an equivalence of E-spaces. Also we can consider higher-dimensional "simplexes" of E-actions, in the spirit of \( \text{Map}_B \), and prove results about these.
APPENDIX

The following lemma has been stated by A. Dold [1; Satz 3.6]. A proof of the dual situation can be found in [2; Theorem 6.1]. The lemma holds under slightly weaker conditions.

**Lemma (Dold):** Given cofibrations $i, i'$ and a homotopy equivalence $f$

\[
\begin{array}{c}
A \\
\downarrow i \phantom{0} \downarrow i' \\
X \phantom{0} \rightarrow \phantom{0} \rightarrow \phantom{0} Y \phantom{0} \rightarrow \phantom{0} \rightarrow \phantom{0} f
\end{array}
\]

such that $f \circ i = i'$. Then we can choose a homotopy inverse $f''$ and homotopies $D_t : X \to X, D_t' : Y \to Y$, such that $D_t : f'' \circ f \simeq \text{id}_X$ rel $iA$, and $D_t' : f \circ f'' \simeq \text{id}_Y$ rel $i'A$.

**Proof:** Let $f'$ be any homotopy inverse of $f$, and $F : X \times I \to X$ a homotopy between $f' \circ f$ and $\text{id}_X$. Since $F \circ (i \times 1) | A \times 0 = f' \circ f \circ i = f' \circ i'$, and since $i'$ has the HEP (homotopy extension property), there exists an extension of $F \circ (i \times 1)$ over $Y \times I$, i.e. a map $G : Y \times I \to X$ such that $G_0 = G | Y \times 0 = f'$, and $G \circ (i' \times 1) = F \circ (i \times 1)$. Let
\[ f'' = G_1 = G|_Yx1. \text{ Since } f'' \sim f', \text{ it is a homotopy equivalence. } f'' \circ i' = F_1 \circ i = i. \text{ Hence } f'' \text{ is a map "under" } A. \]

Let \( H: Xx[0,2] \to X \) be given by

\[
H(x,t) = \begin{cases} 
G(fx, 1-t) & 0 \leq t < 1 \\
F(x, t-1) & 1 \leq t \leq 2 
\end{cases}
\]

Since \( G(fix, 1-t) = G(i'x, 1-t) = F(ix, 1-t) \), we have \( H(i1x) = F_0(i1x) - F_0(i1x) \), (on the right side we have the addition of homotopies). Hence there exists a homotopy \( K': Ax[0,2]x[0,1] \to X \) such that \( K': H \circ i = (\text{constant on } 1) \text{ rel } ((0) \cup (2)), \) i.e.

\[
\begin{array}{c}
\text{const } 1 \\
\hline
\text{const } 1 \\
1 \\
\hline
F_0(i1x) \\
\hline
\text{const } 1 \\
\hline
i \\
\hline
t_1
\end{array}
\]

\[
H(i1x) \\
\hline
f' \circ i' \\
\hline
F_0(i1x)
\]

\[
K'(a,0,t_2) = ia \\
K'(a,2,t_2) = ia \\
K'(a,t_1,0) = H(ia,t_1) \\
K'(a,t_1,1) = ia 
\]

\[
Ax[0,2] \xrightarrow{ix1} Xx[0,2] \text{ has the HEP. Hence there exists a map } K: Xx[0,2]x[0,1] \to X, \text{ such that } K \circ (ix1x1) = K' \text{ and } K|_{Xx[0,2]x0} = H. \text{ Now define } D: Xx[0,4] \to X \text{ by}
\]
\[
D(x,t) = \begin{cases} 
K(x,0,t) & 0 \leq t \leq 1 \\
K(x,t-1,1) & 1 \leq t \leq 3 \\
K(x,2,4-t) & 3 \leq t \leq 4
\end{cases}
\]

Then \(D(ia,t) = ia\), since we move along the "boundary" parts of \(K'\) which are constant on \(i\).

\(D(x,0) = K(x,0,0) = H(x,0) = f'' \circ f\)

\(D(x,4) = K(x,2,0) = H(x,2) = \text{id}_X\).

Hence \(D: f'' \circ f \simeq \text{id}_X\) rel \(iA\).

Apply the procedure to \(f''\) to obtain a homotopy inverse \(g\) and a homotopy \(L: g \circ f'' \simeq \text{id}_Y\) rel \(i'A\). Let \(D'\) be following combined homotopy:

\(f \circ f'' \simeq (g \circ f'') \circ (f \circ f'') = g \circ (f'' \circ f) \circ f'' \simeq g \circ f'' \simeq \text{id}_Y\).

Since \(f \circ f'' \circ i' = f \circ i = i',\) and \(g \circ f'' \circ i' = g \circ i = i',\) this combined homotopy is a homotopy rel \(i'A\).  

\[
\]

Corollary: Let \(A \subset X\) be a cofibration which is a homotopy equivalence. Then there exists a retraction \(p: X \to A\) and a homotopy \(H_t: i \circ p \simeq \text{id}_X\) rel \(iA\), i.e. \(iA\) is a strong deformation retract of \(X\).

Proof: Use the previous Lemma with \(f = i, i = \text{id}_A',\) and \(i' = i\).
References


The following is a summary of joint work with my supervisor, Dr J.M. Boardman. We enclose it here to illustrate the position of the theory in a more general context.
An H-space is a space $X$ with basepoint $e$ and multiplication map $m: X^2 = X \times X \to X$, such that $e$ is a homotopy identity - the maps $x \mapsto m(x, e)$ and $x \mapsto m(e, x)$ are homotopic to the identity map $1$ of $X$. (We take all maps and homotopies in the based sense. We use the $k$-topologies (i.e. compactly generated) throughout in order to avoid spurious topological difficulties. Then function spaces have a canonical topology, obtained from the compact-open topology.) We call $X$ a monoid if $e$ is a strict identity and $m$ is associative.

In the literature there are various kinds of H-space: homotopy-associative, homotopy-commutative, strongly homotopy-commutative [4], and $A_\infty$-spaces [3]. In the last two cases, part of the structure consists of higher coherence conditions and homotopies. In this note we introduce in §2 the concept of homotopy-everything H-space, in which all possible coherence conditions hold; we abbreviate this to E-space. We also define E-maps, in §4. Our two main theorems are Theorem A, which is the structure theorem for E-spaces, and Theorem C, which shows that many familiar spaces such as BPL are in fact E-spaces. We sketch few of the proofs. Full details will appear elsewhere, in due course. Many of the results are folk theory.
A space \( X \) is called an **infinite loop space** if there is a sequence of spaces \( X_i \) and homotopy equivalences \( X_i \simeq \Omega X_{i+1} \) for \( i \geq 0 \), such that \( X = X_0 \).

**Theorem A**

A CW-complex \( X \) admits an E-space structure with \( \pi_0(X) \) a group, if and only if it is an infinite loop space. (Any multiplication on \( X \) induces a multiplication on \( \pi_0(X) \)). Every E-space \( X \) has a "classifying space" \( BX \) which is also an E-space.

**The machine**

Here we develop a machine for constructing numerous E-spaces.

We consider the category \( \mathcal{I} \) of real inner-product spaces of countable (algebraic) dimension, and linear isometric maps between them. As examples we have \( \mathbb{R}^\infty \), with orthonormal base \( \{e_1, e_2, e_3, \ldots\} \), and its subspaces \( \mathbb{R}^n \) with base \( \{e_1, e_2, \ldots, e_n\} \), for \( n \) finite. Every such space is isomorphic to one of these; in particular \( \mathbb{R}^\infty \oplus \mathbb{R}^\infty \cong \mathbb{R}^\infty \).

We topologize \( \mathcal{I}(A,B) \), the set of all isometric linear maps from \( A \) to \( B \), by first giving \( A \) and \( B \) the finite topology, which makes \( A \) the topological direct limit of its finite-dimensional subspaces. (The obvious metric topology on \( \mathcal{I}(A,B) \) is not acceptable.)

**Lemma** The space \( \mathcal{I}(A,\mathbb{R}^\infty) \) is contractible.

**Proof** This result is a consequence of two easy homotopies:

(a) \( i_1 \simeq i_2: A \rightarrow A \oplus A \)

(b) \( i_1 \simeq a: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty \oplus \mathbb{R}^\infty \), where \( a \) is an isomorphism.

To obtain (b), we first construct a homotopy \( 1 \simeq f: \mathbb{R}^\infty \rightarrow \mathbb{R}^\infty \), where \( f \) is defined by \( fe_n = e_{2n} \), by applying the Gram-Schmidt
orthogonalization process to the obvious linear homotopy $f_t$ given by $f_t e_n = (1 - t)e_n + te_{2n}$. Then we compose with an isomorphism $a: \mathbb{R}^\infty \cong \mathbb{R}^\infty \oplus \mathbb{R}^\infty$ chosen to make $a \circ f = i_1$.

Now fix $g: A \to \mathbb{R}^\infty$, and let $h: A \to \mathbb{R}^\infty$ be a typical linear isometric embedding. We construct a contraction homotopy $Q_t$ of $I(A, \mathbb{R}^\infty)$. For the first half, we take $Q_0h = h = a^{-1} \circ a \circ h$, $Q_{\frac{1}{2}}h = a^{-1} \circ i_1 \circ h$, and use homotopy (b). For the second half we rewrite $Q_1h = a^{-1} \circ i_1 \circ h = a^{-1} \circ (h \oplus g) \circ i_1$, take $Q_1h = a^{-1} \circ (h \oplus g) \circ i_2$, and use homotopy (a). But $Q_1h = a^{-1} \circ i_2 \circ g$, which is independent of $h$.

Thus $I(A, \mathbb{R}^\infty)$ is contractible.

Assume we have a functor $T$ defined on the category $I$, taking topological spaces as values, and a continuous natural transformation $\mu$ led 'Whitney sum $w: TA \times TB \to T(A \oplus B)$, such that

(a) $Tf$ is continuous in $f \in I(A, B)$,
(b) $T_{0}$ consists of one point (which will serve as basepoint of $TA$ for all $A$),
(c) $w$ preserves associativity, commutativity, and unit for $\times$ and $\oplus$,
(d) $T_{\infty}$ is the direct limit of the spaces $T_{n}$ for $n$ finite.

**Theorem B**

$T_{\infty}$ is an $E$-space. If $T$ is also monoid-valued (e.g. group-valued), the resulting classifying space $BT_{\infty}$ agrees with that given by Theorem A.
As a (non-canonical) multiplication on $T^\infty A$, we take
$$T^\infty A \times T^\infty A \xrightarrow{w} T(A^\infty \oplus A^\infty) \xrightarrow{Tf} T^\infty A,$$
where $f: A^\infty \oplus A^\infty \to A^\infty$ is some linear isometric embedding. It is homotopy-commutative, because if $s: A^\infty \oplus A^\infty \cong A^\infty \oplus A^\infty$ is the map interchanging factors, $f = f \circ s$ by the Lemma, and then $Tf = Tf \circ Ts$ by the axioms. Similarly, homotopy-associativity reduces to the existence of a homotopy
$$f \circ (f \oplus 1) = f \circ (1 \oplus f): A^\infty \oplus A^\infty \oplus A^\infty \to A^\infty.$$
It is fairly clear that the Lemma will provide all the coherence homotopies we could possibly desire.

In the examples we give below, we define $T^\infty A$ and $w$ explicitly only for finite-dimensional $A$, and note that axiom (d) allows us to extend the definition to $\mathbb{R}^\infty$ and hence to the whole of $I$. In each case the maps $Tf$ are obvious, in view of the inner-product structure.

**Examples**

1. $T^\infty A = O(A)$, the orthogonal group of $A$. Then $T^\infty n = O(n)$ and $T^\infty \mathbb{R} = O$.
2. $T^\infty A = U(A \otimes \mathbb{C})$, the unitary group of the complex vector space $A \otimes \mathbb{C}$. Then $T^\infty n = U(n)$ and $T^\infty \mathbb{C} = U$.
3. $T^\infty A = BQ(A)$, a suitable classifying space of the group $O(A)$. Then $T^\infty n = BQ(n)$ and $T^\infty \mathbb{R} = BQ$. Some care is needed in the choice of $BQ(A)$, if we are to obtain a Whitney sum map. We could take the Grassmannian of all $k$-planes in $A \otimes \mathbb{R}^\infty$, where $k = \dim A$. 

4. TA = F(A), the space of based homotopy equivalences of the sphere SA. Here, SA is the one-point compactification A ∪ ∞ of A, with ∞ as basepoint. The Whitney sum is the smash product, since s^A ∨ s^B ≅ s^{A ∪ B}. Then F(∞) = F.

There is also a semisimplicial analogue, in which T takes semisimplicial complexes as values, and I(Λ, B) is replaced by its singular complex.

5. TA = Top(A). A k-simplex of Top(A) is a fibre-preserving homeomorphism of A × Δ^k over Δ^k, where Δ^k is the standard k-simplex. Then TR^n = Top(n), and TR^∞ = Top.

6. The semisimplicial analogues of examples 1 - 4.

7. The orientation-preserving versions of the other examples, namely SQ, SU, BSQ, SF, STop.

8. TA = PL(A), defined as Top(A) but allowing only piecewise linear homeomorphisms of A × Δ^k. This fails, because the only singular simplexes of I(A, B) that map PL(A) into PL(B) are the constant ones! Thus the homotopies required for Theorem B are not allowed. Instead we must revise the machine, which turns out to be rather complicated. Suffice it to say that for a k-simplex of R(A, B) we take a pair (ξ, f), where ξ is a p.l. sub-bundle of the product bundle B × Δ^k over Δ^k, and f: ξ ⊕ (A × Δ^k) ≅ B × Δ^k is a p.l. fibre-homeomorphism that extends the inclusion of ξ.
Further, there are obvious natural transformations $Q(A) \rightarrow F(A)$, etc. The only one that causes difficulty is the construction of a suitable map $Q(A) \rightarrow PL(A)$, which is extremely awkward (compare §4).

**Theorem C** We have E-spaces

$Q$, $SQ$, $F$, $SF$, $U$, $SU$, $PL$, $SPL$, $Top$, $STop$, $\Gamma = "PL/Q"$, $F/PL$, etc., and all their iterated classifying spaces. The natural maps between these are all E-maps, including $Q \rightarrow PL$ and $PL \rightarrow \Gamma$.

2. **Categories of operators**

There are two variants: with or without permutations.

**Definition** In a category $B$ of operators

(a) the objects are $0, 1, 2, \ldots$;

(b) the morphisms from $m$ to $n$ form a topological space $B(m, n)$, and composition is continuous;

(c) we are given a strictly associative continuous functor $\Theta: B \times B \rightarrow B$ such that $m \Theta n = m + n$;

(d) if $B$ has permutations, we are also given for each $n$ a homomorphism $S(n) \rightarrow B(n, n)$, where $S(n)$ is the symmetric group on $n$ letters. We neglect any symbol for it.

In the case with permutations we demand two further conditions:

(i) if $\pi \in S(m)$ and $\rho \in S(n)$, then $\pi \Theta \rho$ lies in $S(m + n)$ and is the usual sum permutation;
(ii) given any $r$ morphisms $a_i: m_i \rightarrow n_i$ and $\pi \in S(r)$, we have

$$\pi(n) \circ (a_1 \oplus a_2 \oplus \ldots \oplus a_r) = \pi(a_1 \oplus a_2 \oplus \ldots \oplus a_r) \pi(m),$$

where $m = \Sigma m_i$, $n = \Sigma n_i$, $\pi$ acts on $a_1 \oplus a_2 \oplus \ldots \oplus a_r$ by permuting the factors, and the permutation $\pi(n) \in S(n)$ is obtained from $\pi$ by "thickening" - we replace $i \in \{1, 2, \ldots, r\}$ by a block of $n_i$ elements, and let $\pi$ permute these blocks.

All functors between such categories are required to preserve the objects, the functor $\oplus$, the topology, and the permutations (if any).

**Examples**

1. $\text{End}_X$, for a space $X$ with basepoint. $\text{End}_X(m,n)$ is the space of all (based) maps $X^m \rightarrow X^n$, where $X^n$ is the $n$th power of $X$. The functor $\oplus$ is just $\times$. This example has permutations.

**Definition** A category $B$ of operators acts on $X$ if we are given a functor $B \rightarrow \text{End}_X$. We then call $X$ a $B$-space.

2. $A$. $A(m,n)$ is the set of all order-preserving maps

$$\{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots, n\}.$$  

There is one map $\lambda_n: n \rightarrow 1$ for each $n$. Then an $A$-space is a monoid.

3. $S$. For $S(m,n)$ we take the set of all maps $\{1, 2, \ldots, m\} \rightarrow \{1, 2, \ldots, n\}$

This includes permutations. Then an $S$-space is an abelian monoid.

Such a space $X$ is known to have the homotopy type of a product of Eilenberg-MacLane spaces, if $X$ is a connected CW-complex.
Definition A space $X$ is called an $E$-space if it is given an $E$-structure, which consists of a category $B$ of operators with permutations, acting on $X$, for which $B(n,1)$ is contractible for all $n$. (We do not single out any canonical category $B$.)

4. Define $I(m,n) = (R^m, (R^n)^n)$ as in §1. By the Lemma in §1, $I(m,1)$ is contractible, so that any $I$-space, such as $T\infty$, is an $E$-space. Hence part of Theorem B.

5. $Q_n$, a category of operators on the $n$th loop space $\Omega^nY = X$. The space $\Omega^nY$ is the space of all maps $(I^n, \partial I^n) \rightarrow (Y, o)$, where $o$ is the basepoint of $Y$, $I^n$ is the standard $n$-cube, and $\partial I^n$ its boundary. A point $a \in Q_n(k,1)$ is a collection $a$ of $k$ $n$-cubes $I^n_i$ linearly embedded in $I^n$ with their axes parallel to those of $I^n$, having disjoint interiors. It acts on $\Omega^nY$ as follows: given $(f_1, f_2, \ldots, f_k) \in X^k$, i.e. maps $f_i: I^n \rightarrow Y$, we construct the map $a(f_1, f_2, \ldots, f_k): I^n \rightarrow Y$ by using $f_i$ on the little cube $I^n_i$ and the zero map outside the little cubes. We topologize $Q_n(k,1)$ as a subspace of $R^{2kn}$. To define $Q_n(k,r)$ for general $r$, we use $r$ range cubes instead of one. We observe that $Q_n(k,1)$ is $(n-2)$-connected, so that as $n$ tends to $\infty$, Theorem A becomes plausible.

We say that a category $B$ of operators is in standard form if there exists a (necessarily unique) augmentation functor $B \rightarrow A$ if $B$ is without permutations ($B \rightarrow S$ if $B$ has permutations), such
that every morphism \(a\) in \(B\) over \(\lambda m_1 \oplus \lambda m_2 \oplus \ldots \oplus \lambda r : m \to r\) is uniquely expressible in the form \(a_1 \oplus a_2 \oplus \ldots \oplus a_r\), where \(a_i : m_i \to 1\), and we have the appropriate product topology.

The importance of categories in standard form is that given an arbitrary category of operators \(B\) there is another category \(B'\) in standard form and a functor \(B' \to B\) satisfying \(B'(n,1) = B(n,1)\). Hence if \(B\) acts on \(X\), we can canonically make \(B'\) act on \(X\). This effects a welcome simplification in the theory. Of our examples, 2, 3, and 5 are in standard form, but 1 and 4 are not.

3. The bar construction

The concept of monoid is not a good one from the point of view of homotopy theory, because the existence of a monoid structure on a space is not a homotopy invariant. For example, the loop space \(\Omega X\) has no natural monoid structure, although it is a deformation retract of a natural monoid. Similarly for other categories of operators.

Suppose given a category \(B\) of operators, in standard form. We form a bar construction, by considering words \([a_0|a_1|\ldots|a_k]\), where \(k \geq 0\), each \(a_i\) is a morphism in \(B\), and the composite \(a_0 \circ a_1 \circ \ldots \circ a_k\) exists in \(B\).

**Definition** The category \(\mathbb{W}_B^0\) has as morphisms from \(m\) to \(n\) those words \([a_0|a_1|\ldots|a_k]\) for which the composite \(a_0 \circ a_1 \circ \ldots \circ a_k\) is a morphism in \(B\) from \(m\) to \(n\), subject to the following relations and their consequences:

\[
[a \oplus \beta] = [a \oplus 1|1 \oplus \beta] = [1 \oplus \beta|a \oplus 11],
\]

[1] is an identity,

\[
[a | \pi] = [a \circ \pi],\ [\pi | \beta] = [\pi \circ \beta]\ \text{if } B \text{ has permutations } \pi
\]
Composition in $W^0B$ is by juxtaposition.

To form the category $WB$, we take for each morphism $x$ in $W^0B$ a cube $C(x)$ of suitable dimension, having $x$ as one vertex, and identify the faces not containing $x$ with certain cubes $C(x_i)$ of lower dimension, where $x_i$ runs through the words formed from $x$ by one "amalgamation". (We give an alternative description below.) The categories $W^0B$ and $WB$ inherit obvious identification topologies. For composition we have $C(x) \circ C(y) \subseteq C(x \circ y)$ as a face, and $\varepsilon: C(x) \times C(y) \cong C(x \oplus y)$. The augmentation $\varepsilon: WB \longrightarrow B$ is defined by $\varepsilon C(x) = \varepsilon x$, and $\varepsilon [a_0|a_1|\ldots|a_k] = a_0 \circ a_1 \circ \ldots \circ a_k$.

In particular, the familiar pentagon in $WA(4,1)$ is now subdivided into 5 squares.

Let us give an alternative pictorial description of $W^0B$ and $WB$, in the case without permutations (for simplicity). A morphism in $W^0B(n,1)$ is represented by a finite tree with directed edges, except that some edges do not join two vertices (see pictures). There is just one, called the root, that leaves a vertex and goes nowhere; there are exactly $n$ twigs that come from nowhere; the other edges are called links and join two vertices. Each vertex has a label $a \in B(r,1)$, where $r$ is the number of incoming edges, and has exactly one outgoing edge. The only relation is that a vertex labelled with $1 \in B(1,1)$ may be suppressed.
A morphism in $WB(m,n)$ is an ordered collection of $n$ such trees, called a copse. Composition $x$ of $y$ is obtained by attaching the roots of $y$ in order to the twigs in $x$.

To describe a morphism in $C(x) \subseteq WB$ we simply assign a real number $t_i$ to each link of the copse $x$, ($0 \leq t_i \leq 1$), and add the relations:

(i) When we suppress a vertex labelled 1, if it separates links with values $t$ and $u$, we give the new link that appears the value $\max(t,u)$.

(ii) A link with value 0 joining $a$ to $\beta$ may be shrunk to form a new copse having one fewer vertex; the vertices $a$ and $\beta$ are amalgamated to form $\gamma$, which is obtained from $a$ and $\beta$ by using the composition in $B$.

When we compose copses, we assign the value 1 to each new link that appears. Consistency is assured by the tree differential calculus. Putting copses side by side describes the functor $\Theta$. 
To make the following theorems true, we need to replace $B$ by a slightly different category $\tilde{B}$ augmented over $B$, which is obtained from $B$ by growing a whisker on $B(1,1)$ rooted at $1$, and taking the outer end as a new identity morphism. However, we can replace $\tilde{B}$ by $B$ in all the results if we know that the identity $1 \in B(1,1)$ is isolated.

We call an augmentation functor $\theta: C \rightarrow B$ fibre-homotopically trivial if for each $n$ there exists a section $X: B(n,1) \rightarrow C(n,1)$ and a fibrewise homotopy $X \circ \theta \simeq 1$, $S(n)$-equivariantly if $B$ and $C$ have permutations.

Theorem D

(a) $\varepsilon: \tilde{W} \rightarrow B$ is fibre-homotopically trivial.

(b) Given any category of operators $C$ augmented over $B$ by a fibre-homotopically trivial augmentation, there exists a functor $F: \tilde{W} \rightarrow C$ that lifts $\varepsilon$ (not uniquely).

The superiority of our definition is clear from:

Theorem E

Suppose $X$ and $Y$ have the same homotopy type, and $\tilde{W}$ acts on $X$. Then we can make $\tilde{W}$ act on $Y$. 
4. Maps between $H$-spaces

Suppose the category of operators $\mathcal{W}\mathcal{B}$ acts on the spaces $X$ and $Y$. We need an appropriate definition of morphism between them. In fact there are two. If the map $f: X \rightarrow Y$ commutes strictly with the actions, we call $f$ a $\mathcal{W}\mathcal{B}$-homomorphism. We are more interested in the appropriate definition in which $f$ merely commutes with the actions up to coherent homotopies; this is more complicated and appears to be new.

Let $L_n$ be the "linear" category with objects $a_0, a_1, \ldots, a_n$ and one morphism $a_i \rightarrow a_j$ whenever $i \leq j$. We can generalize the bar construction in §3 to form $W(B \times L_n)$, a category which we make act on $(n+1)$-tuples of spaces, $(X_0, X_1, \ldots, X_n)$. (In $B \times L_n \otimes$ is no longer a functor, so that the first relation makes sense only inside each copy $B \times a_i$ of $B$.)

**Definition** We say the map $f: X \rightarrow Y$ is a homotopy $B$-map if we are given an action of $W(B \times L_n)$ on the pair $(X, Y)$ that induces the given $\mathcal{W}\mathcal{B}$-structures on $X$ and $Y$ and the given map $f: X \rightarrow Y$.

Similarly we say that a map $f: X \rightarrow Y$ between $H$-spaces is an $E$-map if there exists some suitable category of operators $\mathcal{C}$ on the pair $(X, Y)$ that induces the given $E$-structures on $X$ and $Y$, such that $f$ lies in $\mathcal{C}(X, Y)$, and each space $\mathcal{C}(X^n, Y)$ is contractible. We call
two $E$-structures on $X$ equivalent if the identity map between the two structures admits an $E$-structure.

**Theorem**

Let $X$ and $Y$ be $\mathcal{WB}$-spaces, and $f: X \rightarrow Y$ a homotopy $\tilde{B}$-map which is also a homotopy equivalence. Then any homotopy inverse $g: Y \rightarrow X$ can be made into a homotopy $\tilde{B}$-map.

**Example** Under suitable semisimplicial interpretations we have inclusions $i: Q(n) \subset PD(n)$ and $PL(n) \subset PD(n)$. As is well known, $PL(n)$ is a deformation retract of $PD(n)$, with a retraction $p: PD(n) \rightarrow PL(n)$, say. The only other fact we need is that $PD(n)$ admits an action of $Q(n)$ on the left and of $PL(n)$ on the right. Then it is obvious that $p \circ i: Q(n) \rightarrow PL(n)$ is a homotopy homomorphism (in the usual sense): take $x, y \in Q(n)$, then

$$p(x \cdot y) = p(x \cdot py) = p(px \cdot py) = px \cdot py.$$  

In fact it can be shown from the above information that $p \circ i$ admits the structure of homotopy $A$-map.

When we attempt to construct the category of $\mathcal{WB}$-spaces and homotopy $B$-maps, we find that it is not possible. The composite of two homotopy $B$-maps is not defined unless one of them is induced from a $\mathcal{WB}$-homomorphism, except up to a homotopy, which is itself defined only up to a homotopy, which is itself defined only up to a homotopy, which is ... Instead we form a semisimplicial complex
Theorem G

This complex $K$ satisfies the restricted Kan extension condition (in which the omitted face is not allowed to be the first or last).

This result provides all we need for composition up to homotopy, etc.

5. Structure theory

We consider $\text{WA}$-spaces, with $\text{A}$ as in §2. We first note that if $X$ and $Y$ are $\text{WA}$-spaces, so are $X \times Y$ and the powers $X^n$. The following theorem is essentially due to Adams.

Theorem H

Given a $\text{WA}$-space $X$, there is a universal monoid $\text{MX}$ with a homotopy $\text{A}$-map $i: X \rightarrow \text{MX}$, such that any $\text{WA}$-map $f: X \rightarrow Y$ to a monoid $Y$ factors uniquely as $g \circ i$, where $g: \text{MX} \rightarrow Y$ is a monoid homomorphism. Moreover, if $X$ is a $\text{CW}$-complex the map $i$ is a homotopy equivalence.

We know [2] that $\text{MX}$ has a classifying space $\text{BMX}$, which is functorial, connected, and satisfies $\text{MX} \simeq \Omega \text{BMX}$ provided $\pi_0(\text{MX})$ is a group. Further, we have $\text{B}(G \times H) \cong \text{BG} \times \text{BH}$. In one direction, the main theorem A follows from the more detailed theorem, by putting $\text{BX} = \text{BMX}$. 
Theorem J

Let $X$ be an $E$-space, so that in particular it supports a $WA$-structure by Theorem D. Then the classifying space $BMX$ is an $E$-space. If $Y$ is another $E$-space and $f: X \to Y$ an $E$-map, then $f$ admits a homotopy $A$-map structure, and we find an $E$-map $BMf: BMX \to BMY$ (not well defined).

Consider the $E$-spaces $X^n$. We can make each operator $a: X^n \to X$ into a homotopy $A$-map. This induces by Theorem H a monoid homomorphism $Ma: (MX)^n \to MX^n \to MX$, and hence $BMa: (BMX)^n \to BMX$. Along these lines we construct an $E$-structure on $BMX$, which makes it an $E$-space. The details are considerable.

6. Cohomology theories

Assume that the CW-complex $Y$ is an $E$-space such that $\pi_0(Y)$ is a group; then by Theorem A, $Y$ is an infinite loop space. Explicitly, put $Y_n = B^nY = B(B^{n-1}Y)$ by Theorem A and $Y_{-n} = \Omega^nY$, for $n \geq 0$; then we have homotopy equivalences $Y_n \simeq \Omega Y_{n+1}$ for all integers $n$, and we can define a graded cohomology theory $[1]$ by setting

$$t^n(X,A) = [X/A,Y_n],$$

the set of homotopy classes of based maps from $X/A$ to $Y_n$, for any CW-pair $(X,A)$. The coefficient groups are the groups $t^nP$, where $P$ is a point. Here they are zero for $n > 0$. Let us call such a theory connective.
Theorem K

Every connective graded additive cohomology theory t on CW-pairs arises from some E-space Y, which is uniquely defined up to homotopy equivalence of E-spaces.

In particular the E-space $\mathbb{Z} \times BU$ gives rise to the connective K-theory $cK$. This is more usually obtained by appealing to Bott periodicity and killing off the unwanted coefficient groups. In other cases we cannot appeal to Bott periodicity, for example

Definition We define connective p.l. K-theory by using the E-space $\mathbb{Z} \times BPL$: for $n > 0$ we put

$$cK^\mathbb{Z}_{PL}(X,A) = [X/A, B^n(\mathbb{Z} \times BPL)].$$

References:


