Coupled Oscillators With Internal Symmetries

David Andrew Wood

A thesis submitted for the degree of Doctor of Philosophy.

Nonlinear Systems Laboratory, Mathematics Institute, University of Warwick.

September 1995

Contents

List of Figures

List of Tables

Acknowledgements

So much has happened during the gestation and birth of this thesis, and I've met so many people, that I'm bound to forget someone, so I apologise now !

- First and foremost I would like to thank Ian, my supervisor, for his help towards the writing of this thesis and for believing in me when I started lacking confidence.
- A special thanks also to my old maths teacher, Graham Hoare, for encouraging me to realise my potential.
- Thanks to everyone at the Institute für Informationverarbeitung in Tübingen for the pleasant visits I have had there, and especially to Gerhard for suggesting the application for skew-equivariance in Chapter 9.
- I would also like to thank my parents for being there when I needed them, as well as for their generous financial support, and also to Paul for being my brother !
- A posthumous thankyou to my pet hamster, Harriet RIP, who nearly made it to the final stages of the writing of this thesis, for the years of enjoyment.
- Thanks also to my coach Colin, and training group at Coventry Godiva, it's amazing how much physical pain can stop you worrying about work !
- Thanks also to the Engineering and Physical Sciences Research Council for the grant, and the Mathematics Institute for the Teaching Assistants post.
- A special thanks to all the friends I've made during my time at Warwick, especially those who forced me to spend too much time drinking coffee - you know who you are !
- Last, but by no means least, a very special thankyou to Karen for her love, affection and support, especially during the final few months of the writing of this thesis - I love you !

Declaration

Unless stated otherwise, all material presented in this thesis is original work by the author under the supervision of Prof. Ian Stewart.

Summary

We investigate how the addition of an internal $\mathbf{Z}_2$ symmetry can affect the patterns seen in systems of coupled oscillators.

To do this we first consider the theoretical implications, and discover that we must investigate bifurcations in the presence of $\mathbf{S}_3$, $\mathbf{Z}_2 \wr \mathbf{S}_3$ (the wreath product) and $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetries. We do this for both steady-state and Hopf bifurcations. We also consider the general case of steady-state bifurcations with $\mathbf{S}_n$, $\mathbf{Z}_2 \wr \mathbf{S}_n$ and $\mathbf{Z}_2 \times \mathbf{S}_n$ symmetries.

We then try to answer the question of what it means for an oscillator to have an internal symmetry, and then how the form of coupling chosen between three oscillators with internal symmetries affects the global symmetry of the system. To this end we also introduce the notion of *skew-equivariance*, a generalisation of the notion of equivariance.

It turns out that the addition of an internal $\mathbf{Z}_2$ symmetry to a network of three coupled oscillators can have a substantial effect on the patterns of oscillation observed, which as well as showing theoretically we also show by some numerical experiments.

Finally we apply our results in two applications. The first is towards a model of how insects walk, their *gaits*, this problem being one of the main motivations for this work. The results found here show that the addition of an internal $\mathbf{Z}_2$ symmetry into the oscillators used to model locomotion, which can easily be justified by thinking of each leg as a pendulum, can be used to produce a much better model than those used in Wood [30] and [31].

The second application is considering a network of three clusters of three oscillators, where applying skew-equivariance to the coupling produces a new set of solutions to those calculated by Dangelmayer et al. [11], where they consider $\mathbf{D}_3 \times \mathbf{D}_3$ symmetry, by producing global symmetries of $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $\mathbf{Z}_2 \times \mathbf{S}_3$.

To Karen.

Chapter 1

Overview

The problem of networks of symmetrically coupled identical oscillators is a fairly extensively covered area of mathematics, and the results, and patterns observed, are by now reasonably familiar and predictable (see for example Golubitsky et al. [17]). Until very recently, however, there has been very little consideration of how these patterns are affected by each oscillator, in addition, having its own internal symmetries.

This problem arises, for example, in systems of coupled Van der Pol oscillators, a model which is of great importance in modeling to both mathematicians *and* biologists, and has been heavily relied upon to produce numerical simulations for many years. In its simplest from this oscillator has an internal $\mathbf{Z}_2$ symmetry, and we show some numerical results of how this symmetry can affect the patterns observed, depending on the form of coupling chosen, in Chapter 7.

Interest in this area of internal symmetries is beginning to gain momentum however. In 1991 Aronson et al. [3] considered $\mathbf{S}_n$ symmetry in a system of coupled Josephson Junctions, where an internal $\mathbf{Z}_2$ symmetry was introduced to detect period doubling bifurcations. The addition of this $\mathbf{Z}_2$ symmetry into their considerations produces global $\mathbf{Z}_2 \times \mathbf{S}_n$ symmetry, which introduces a whole new family of isotropy subgroups into the calculations corresponding to solutions with period-two points. We repeat the calculations of $\mathbf{Z}_2 \times \mathbf{S}_n$ steady-state bifurcations in this Thesis in our own notation for completeness, though our interpretation will be somewhat different since we are no longer considering $\mathbf{S}_n$ coupled maps.

In 1993 Dangelmayer et al. [10] considered a system of nine coupled oscillators arranged in three clusters of three oscillators. The coupling within each cluster there was chosen so as to be $\mathbf{D}_3$ symmetric, as was the coupling between the clusters, and then the full system exhibited $\mathbf{D}_3 \times \mathbf{D}_3$ symmetry, and the standard analysis was then applied. This was the first paper to be published that started to try and differentiate between local and global symmetries, although in this specific example we do not explicitly have 'internal' symmetries in a system of coupled oscillators, rather internal symmetries of clusters of oscillators. Therefore although the analysis becomes that of *direct product coupling* the interpretation will still be that of a system of nine coupled oscillators, but the effects of interactions between clusters *is* shown to be affected by the internal dynamics of each cluster. It was this paper that provided the inspiration for the work contained here, in particular the question arose as to how a different type of coupling (wreath product) would affect the patterns predicted in the direct product case.

We return to this system, from a different perspective, in Chapter 9.

In addition, in parallel to the writing of this thesis, Dionne et al. [12] and [13] have considered the general group-theoretic implications of coupled cells with internal symmetries with which there can be seen a certain amount of agreement with the results presented here.

In their paper the authors considered the general case of coupled cells where each cell had an internal $\mathcal{L}$ symmetry and the cells were coupled so that they were $\mathcal{G}$ symmetric. They then considered the irreducible representations that would be needed and the *axial isotropy subgroups* (essentially isotropy subgroups with one-dimensional fixed point subspaces) in terms of the representations and isotropies of the two cases of $\mathcal{L}$ and $\mathcal{G}$ symmetric bifurcations. Several of their results can be tied in with the results presented here, in particular they predict wreath-product coupled oscillator systems with quiescent oscillators, but they do not go as far as to calculate a general theory for stabilities, and it turns out that many of the new solutions that occur due to this internal $\mathcal{L}$ symmetry are in fact generically unstable.

For practical reasons, mainly to calculate these stabilities, we follow the standard methods of [17] to achieve our goals using the established techniques rather than try to apply the general theory of Dionne et al. to find the irreducible representations for our examples. It then remains to use a new interpretation of the results to realise the predictions as solutions of coupled cells. It is this interpretation that is only just beginning to be recognised as important, but as the results presented here show they can have some quite dramatic implications.

Our main goal is to find the possible patterns in networks of three symmetrically coupled oscillators where each oscillator has its own internal $\mathbf{Z}_2$ symmetry. We choose the internal symmetry to be $\mathbf{Z}_2$ since it is the simplest case to choose, and we choose a three oscillator system for both the simplicity (we could have chosen only two oscillators, but this did not seem intuitively to be a productive move) and so that we could apply the results to the problem already considered in Wood [30] and [31] of insect locomotion. This problem is considered further in Chapter 8. We now outline the rest of this thesis.

We begin, in Chapter 2, with a brief discussion of the background needed for the later work. We assume a familiarity with Golubitsky et al. [17], but re-cap the main techniques and results for reference, outline other techniques we shall use, and define the necessary group theory.

The next three chapters deal with the theoretical results of symmetry-breaking bifurcations, in the the presence of the appropriate symmetry, as well as discussion of how the symmetries used can arise in a modeling situation.

In Chapter 3, we consider the simplest case of steady state bifurcation in systems of three coupled cells, where each cell has an internal $\mathbf{Z}_2$ symmetry. We use this scenario to provide an example of how the coupling between cells can give rise to different symmetries, and so to different results.

In Chapter 4 we extend the investigation of steady-state bifurcation to the case of a general number of coupled cells with internal $\mathbf{Z}_2$ symmetries. It turns out that this extension is a reasonably straight forward process, which is not the case for Hopf bifurcation, which is why we do not consider the general Hopf bifurcation case here. There are however some lengthy and complicated calculations involved when we consider stabilities, for which extensive use was made of the computer program 'Maple'.

We do however carry out the calculations for Hopf bifurcations in three coupled cells with internal $\mathbf{Z}_2$ symmetries, and this is done in Chapter 5.

In Chapter 6 we consider how these symmetries, and the resulting theoretic implications, manifest themselves in systems of coupled oscillators to find some quite surprising results appearing. In this chapter we also introduce the notion of *skew-equivariance*, which provides a useful method for choosing the correct coupling between the oscillators to produce the required global symmetries with respect to the internal symmetries.

Various models are considered in Chapter 7 to numerically achieve the results predicted in the preceding chapter using 'Dstool' (©Center for Applied Mathematics, Cornell University), with considerable success, although the patterns observed here do not necessarily occur from a Hopf bifurcation in the required manner.

We then turn our attention to a couple of applications, of which return to the two problems which initially motivated the work of this thesis.

In Chapter 8 we consider how the addition of an internal $\mathbf{Z}_2$ symmetry in the oscillator networks already considered in Wood [31] can help to provide a more realistic model of insect locomotion, and in Chapter 9 we use our notion of skew-equivariance to produce some new patterns of oscillation in the network of three clusters of three oscillators mentioned earlier.

Finally we summarise the results of the thesis, and suggest directions for future research, in Chapter 10.

Chapter 2

Background and Preliminaries

Our aim is to consider systems of the form

$$\frac{dx}{dt} = f(x, \lambda) \tag{2.0.1}$$

where $x \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$ is a bifurcation parameter, and f is a suitably smooth function with an inherent symmetry. To do this we must recall the necessary theory, and we start by looking for solutions occurring from a steady-state bifurcation.

2.1 Group Theoretic Background

Consider a compact Lie Group Γ acting on a space $\mathbf{V} = \mathbf{R}^n$, and let $\mathbf{W}$ be a subspace of $\mathbf{V}$. Then we say that $\mathbf{W}$ is Γ-*invariant* if $\gamma w \in \mathbf{W}$ for all $\gamma \in \Gamma$ and for all $w \in \mathbf{W}$. A representation or action of Γ on $\mathbf{V}$ is *irreducible* if the only Γ invariant subspaces of $\mathbf{V}$ are $\{0\}$ and $\mathbf{V}$, and a subspace $\mathbf{W} \subset \mathbf{V}$ is Γ-*irreducible* if $\mathbf{W}$ is Γ-invariant and the action of Γ on $\mathbf{W}$ is irreducible.

We then have the following result

Lemma 2.1.1 (Theorem of Complete Reducibility) *Let Γ act on a space $\mathbf{V}$, where $\mathbf{V} = \mathbf{R}^n$, then we can decompose $\mathbf{V}$ as*

$$\mathbf{V} = \mathbf{V}_1 \oplus \ldots \oplus \mathbf{V}_s$$

where each $\mathbf{V}_i$ is Γ irreducible.

Proof: This is Golubitsky et al. [17] Corollary XII 2.2 ∎

In fact a stronger result can be proved

Theorem 2.1.2 *Let Γ be a compact Lie group acting on $\mathbf{V}$.*

a) *Up to Γ-isomorphism there are a finite number of distinct Γ-irreducible subspaces of $\mathbf{V}$. Call these $\mathbf{U}_1, \ldots, \mathbf{U}_t$.*

b) *Define $\mathbf{W}_k$ to be the sum of all Γ-irreducible subspaces $\mathbf{W}$ of $\mathbf{V}$ such that $\mathbf{W}$ is Γ-isomorphic to $\mathbf{U}_k$. Then*

$$\mathbf{V} = \mathbf{W}_1 \oplus \ldots \oplus \mathbf{W}_t.$$

Proof: This is Golubitsky et al. [17] Theorem XII 2.5. ∎

Here the $\mathbf{W}_i$ are called the *isotypic components*, and the decomposition the *isotypic decomposition*. We also recall that a representation of Γ on $\mathbf{V}$ is *absolutely irreducible* if the only linear mappings on $\mathbf{V}$ that commute with Γ are scalar multiples of the identity. Then Golubitsky et al. [17] Lemma XII 3.3 shows that absolute irreducibility implies irreducibility.

Define the *orbit* of the action of Γ on $x \in \mathbf{V}$ to be

$$\Gamma_x = \{\gamma x : \gamma \in \Gamma\}$$

and the isotropy subgroup of $x \in \mathbf{V}$ to be

$$\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

Note that points on the same group orbit have conjugate isotropy subgroups. We often shorten the phrase 'isotropy subgroup' to 'isotropy' where it is appropriate to do so, for example if a solution has isotropy subgroup Σ then we will quite often say that the solution has isotropy Σ.

Now consider the function $f(x) : \mathbf{R}^n \mapsto \mathbf{R}^n$. We say that f is Γ-*equivariant* if

$$f(\gamma x) = \gamma f(x)$$

for all $\gamma \in \Gamma$, and for all $x \in \mathbf{R}^n$, and Γ-*invariant* if

$$f(\gamma x) = f(x).$$

With this f, then if $x(t)$ is a solution to equation 2.0.1 then so is $\gamma x(t)$. We also note that when f vanishes, it vanishes on orbits of Γ, and if the *fixed-point subspace* of $\Sigma \subset \Gamma$ is

$$Fix(\Sigma) = \{x \in \mathbf{V} : \gamma x = x \; \forall \gamma \in \Sigma\}$$

then

$$f(Fix(\Sigma)) \subset Fix(\Sigma).$$

We now consider the theory of bifurcation problems with symmetry.

2.2 Bifurcations With Symmetry

A bifurcation problem with symmetry group Γ is an equivariant germ $g : \mathbf{V} \times \mathbf{R} \mapsto \mathbf{V}$ satisfying

$$g(\gamma x, \lambda) = \gamma g(x, \lambda)$$

for all $\gamma \in \Gamma$. By convention germs are based at the origin $(x, \lambda) = (0, 0)$, and require $g(0, 0) = 0$. We assume also that g has undergone Liapunov-Schmidt reduction (see Golubitsky et al. [16] for details) so that we also have $(dg)_{(0,0)} = 0$. Therefore by Golubitsky et al. [17] Proposition XIII 3.2 we may also assume that Γ acts on $\mathbf{V}$ absolutely irreducibly, by the following result.

Proposition 2.2.1 ([17] XIII 3.2) *Let $G : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}^n$ be a one parameter family of Γ-equivariant mappings with $G(0, 0) = 0$. Let $\mathbf{V} = ker(dG)_{0,0}$. Then generically the action of Γ on $\mathbf{V}$ is absolutely irreducible.*

Proof: See Golubitsky et al. [17], Proposition XIII 3.2. ∎

The stated result then follows since Liapunov-Schmidt reduction reduces our problem to precisely functions of the kernel of its linearisation.

This means, since we must have the identity

$$(dg)_{(0,\lambda)}\gamma = \gamma(dg)_{(0,\lambda)}$$

(from the chain rule on $g(\gamma x, \lambda) = \gamma g(x, \lambda)$), we have that $(dg)_{(0,\lambda)} = c(\lambda)\mathbf{I}$. Since $(dg)_{(0,0)} = 0$ we have $c(0) = 0$ and we assume that

$$c'(0) \neq 0. \tag{2.2.2}$$

It then follows that

Theorem 2.2.2 (Equivariant Branching Lemma) *Let Γ be a Lie group acting absolutely irreducibly on $\mathbf{V}$ and let g be a Γ-equivariant bifurcation problem satisfying 2.2.2. Let Σ be an isotropy subgroup satisfying*

$$dimFix(\Sigma) = 1.$$

Then there exists a unique smooth solution branch to $g = 0$ such that the isotropy subgroup of each solution is Σ.

Proof: See Golubitsky et al. [17] Theorem XIII 3.3. ∎

In fact, Golubitsky et al. [17] prove a more general result, which is one we state here also (note that the name given is our name, not that of the original authors).

Theorem 2.2.3 (A More General Equivariant Branching Lemma) *Let Γ be a Lie group acting on* $\mathbf{V}$. *Assume*

a) $Fix(\Gamma) = \{0\}$,

b) $\Sigma \subset \Gamma$ *is an isotropy subgroup satisfying* $dim Fix(\Sigma) = 1$,

c) $g : \mathbf{V} \times \mathbf{R} \mapsto \mathbf{V}$ *is a* Γ*-equivariant bifurcation problem satisfying*

$$(dg_\lambda)_{(0,0)}(v_0) \neq 0$$

where $v_0 \in Fix(\Sigma)$ *is nonzero.*

Then there exists a smooth branch of solutions $(tv_0, \lambda(t))$ *to the equation* $g(t, \lambda) = 0$.

Proof: See Golubitsky et al. [17] Theorem XIII 3.5. ∎

2.3 Stability Of The Branches

For the stability of the branches guaranteed by either the Equivariant Branching Lemma or its more general version we consider, for each symmetry Γ, the ring of invariant polynomials generated by the functions u_i say, and the module of equivariants, $\mathbf{X}_i$, generated over the invariants.

We can then write our equivariant germ g, up to any order, as

$$g = f_1\mathbf{X}_1 + \ldots + f_s\mathbf{X}_s$$

where each f_i is a function of the invariants. To reduce the amount of calculations needed further we note a result quoted in [28], a proof for which is given in [6]. Let Γ act on $\mathbf{V}$ irreducibly, and for the purposes of this result define the *degree of* $\Gamma \subset GL(\mathbf{V})$ to be $m = \dim \mathbf{V}$. Then

Theorem 2.3.1 *If Γ has degree m, then there exists m but not $m+1$, algebraically independent invariants.*

In fact we can also have a stronger result for some symmetry groups, again for details see [28]. Let $\mathbf{M} \in \Gamma$ above, and so in particular $\mathbf{M} \in GL(\mathbf{V})$, then $\mathbf{M}$ is a *pseudo-reflection* if precisely one eigenvalue of $\mathbf{M}$ is not equal to one. Then we have

Theorem 2.3.2 *Let Γ be a finite subgroup of $GL(\mathbf{V})$. There exist m algebraically independent (homogeneous) invariants $u_1, \ldots, u_m$ such that the algebra of invariants R^G is generated by $\{u_1, \ldots, u_m\}$ (over $\mathbf{C}$) if and only if Γ is generated by pseudo-reflections.*

A specific example is the case when Γ is the set of all $m \times m$ permutation matrices.

In particular, for the work presented here, we carry out all calculations in the proofs up to only third order, unless it becomes apparent that a higher order of approximation is needed at a later stage.

To find the invariant and equivariant polynomial mappings for a group Γ acting on $\mathbf{V}$ we use the following method, where we take $\mathbf{V} = \mathbf{R}^3$ for the purposes of illustration, but the methods are easily extendible to the case of $\mathbf{V} = \mathbf{R}^n$.

2.3.1 Invariants

Let Γ act on $\mathbf{R}^3$, and Λ_0^k be the set of generators for polynomials in $\mathbf{R}$ of order k over $\mathbf{R}^3$, $\Lambda_0^k = \{\alpha_1, \ldots, \alpha_l\}$. For example $\Lambda_0^2 = \{x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_1x_3\}$. Now extend Λ_0^k to a set Λ^k that contains all elements of the form $\gamma\alpha_i$ where $\gamma \in \Gamma$, where these elements are distinct from those already included unless $\gamma\alpha_i = \alpha_j$ some i, j (we assume $x_ix_j = x_jx_i$). Therefore we now have $\Lambda^k = \{\alpha_1, \ldots, \alpha_l, \alpha_{l+1}, \ldots, \alpha_{l+m}\}$.

The effect of every $\gamma \in \Gamma$ is now a permutation in $\mathbf{S}_{l+m}$ of the α_i's, and so Γ becomes a permutation group, and so there is a unique minimal partition of Λ^k into subsets Δ_i^k where each Δ_i^k is invariant under the action of Γ, $\gamma\Delta_i^k = \Delta_i^k$ for all $\gamma \in \Gamma$.

A set of invariants is then given by $\left\{u_j = \Sigma\{\alpha_i : \alpha_i \in \Delta_j^k\}\right\}$ and all invariants of order k are linear combinations of these functions. We can then use relations to extract those invariants necessary to form a basis of invariants up to order k.

2.3.2 Equivariants

If $h(x)$ is Γ equivariant then by definition $h(\gamma x) = \gamma h(x)$ and so $\gamma^{-1}h(\gamma x) = h(x)$, hence $h(x)$ is *invariant* under this latter action.

Define

$$\Omega_0^k = \left\{\epsilon_i(x) : \epsilon_i(x) \in \Lambda^k \times \Lambda^k \times \Lambda^k\right\} \cup \{0\} \times \{0\} \times \{0\}$$

and extend to $\Omega^k = \{\epsilon_1, \ldots, \epsilon_p, \epsilon_{p+1}, \ldots, \epsilon_{p+q}\}$ where we include elements of the form $\gamma\epsilon_i \notin \Omega_0^k$, where $\gamma \in \Gamma$.

Now let Γ act on Ω^k by $\gamma \cdot \epsilon_i = \gamma^{-1}\epsilon_i(\gamma x)$. The effect of this group action is, as in the invariants case, a permutation of the elements in Ω^k, and so with this action $\Gamma \subset \mathbf{S}_{p+q}$. Therefore there is a unique minimal partition of Ω^k into subsets Θ_i^k which are invariant under this action of Γ. The invariants of this action are then

$\left\{\mathbf{Y}_i = \{\Sigma\{\epsilon_j : \epsilon_j \in \Theta_j^k\}\}\right\}$ and these $\mathbf{Y}_i$ will generate the equivariants of the action we started with of Γ on $\mathbf{R}^3$.

We now define some notation to make things easier later, to do this we note that each Δ_i can be generated by one element of Δ_i, if not then we can find a partition of Δ_i with respect to Γ, contradicting our assumption that the partition was minimal. Define

$$< \epsilon >_\Gamma = \{\gamma \cdot \epsilon : \gamma \in \Gamma\}.$$

So that if $\epsilon \in \Delta_i^k$ then $< \epsilon >_\Gamma = \Delta_i^k$.

Now, each $\epsilon_i(x)$ can be written as

$$\begin{bmatrix} \epsilon_{i1} \\ \epsilon_{i2} \\ \epsilon_{i3} \end{bmatrix} = \begin{bmatrix} \epsilon_{i1} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_{i2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \epsilon_{i3} \end{bmatrix} = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$$

and so $< \epsilon_i > = < \mathcal{E}_1 > \cup < \mathcal{E}_1 > \cup < \mathcal{E}_1 >$ since

$$\gamma^{-1}\epsilon_i(\gamma x) = \gamma^{-1}\mathcal{E}_1(\gamma x) + \gamma^{-1}\mathcal{E}_2(\gamma x) + \gamma^{-1}\mathcal{E}_3(\gamma x).$$

This means that we need only consider elements of the form $\mathcal{E}_i$ to find a basis for the equivariants.

Define the *Generating Partitions* to be the smallest set of Θ_i to generate all possible Θ_i, then use relations to find a basis for the equivariant mappings. Note that since all the symmetry groups we will be considering contain $\mathbf{S}_3$, we need only consider generators for the Generating Partitions of the form $\begin{bmatrix} \epsilon_{i1} \\ 0 \\ 0 \end{bmatrix}$.

In the case when we consider an action of Γ on $\mathbf{R}_0^3$ where

$$\mathbf{R}_0^3 = \{x \in \mathbf{R}^3 : x_1 + x_2 + x_3 = 0\},$$

we first find the equivariants on $\mathbf{R}^3$ and then try to find relations using $x_1+x_2+x_3 = 0$, and finally take the orthogonal projection onto $\mathbf{R}_0^3$ giving us the required equivariants.

2.3.3 Linear Orbital Stability

Once we have solutions to the equation

$$\frac{dx}{dt} + g(x, \lambda) = 0, \tag{2.3.3}$$

we look for orbital stability. We say that the equilibrium x_0 is *orbitally stable* if x_0 is neutrally stable and whenever $x(t)$ is a trajectory beginning near x_0 then $\lim_{t\to\infty} x(t)$ exists and lies in Γx_0.

To find stability we use the notion of linear orbital stability. Let x_0 be an equilibrium of 2.3.3, where g commutes with the action of Γ. The steady-state x_0 is *linearly orbitally stable* if the eigenvalues of $(dg)_{x_0}$ other than those arising from $\mathbf{T}_{x_0}\Gamma_{x_0}$ have positive real part. Here $\mathbf{T}_{x_0}\Gamma_{x_0}$ is the tangent space of Γx_0, and $\mathbf{T}_{x_0}\Gamma_{x_0} \subset ker(dg)_{x_0}$ and so corresponds to zero eigenvalues. We then have

Theorem 2.3.3 *Linear orbital stability implies orbital (asymptotic) stability.*

Proof: This is Golubitsky et al. [17] Theorem XIII 4.3. ∎

Finally, we can use the isotropy of the solution to help us find $(dg)_{x_0}$ since we must have that $(dg)_{x_0}$ commutes with the action of the isotropy subgroup of the solution. In some cases it also makes calculations easier if we decompose $\mathbf{V}$ into it's irreducible subspaces, and then consider the eigenvalues restricted to each subspace in turn.

2.4 Hopf Bifurcations

When considering Hopf bifurcations we introduce an extension of the notion of absolute irreducibility. We say a representation $\mathbf{W}$ of Γ is Γ-*simple* if either

a) $\mathbf{W} \cong \mathbf{V} \oplus \mathbf{V}$ where $\mathbf{V}$ is absolutely irreducible for Γ or

b) $\mathbf{W}$ is non-absolutely irreducible for Γ.

We are then interested in solutions to

$$\frac{dv}{dt} + f(v, \lambda) = 0 \tag{2.4.4}$$

where $f : \mathbf{R}^n \to \mathbf{R}^n$ is smooth, commutes with Γ and $(df)_{(0,0)}$ has purely imaginary eigenvalues.

To find these solutions we extend our group action Γ to include temporal symmetries as phase shifts acting as elements of $\mathbf{S}^1$, and then basically apply our equivariant theory to this situation. We do this by, for a 2π-periodic solution $v(t)$ (if not 2π periodic then scale time), identifying $\mathbf{S}^1$ with $\mathbf{R}/2\pi\mathbf{Z}$. Then a symmetry of the periodic function $v(t)$ is an element $(\gamma, \theta) \in \Gamma \times \mathbf{S}^1$ such that

$$\gamma v(t) = v(t - \theta).$$

The collection of all symmetries for $v(t)$ forms a subgroup

$$\Sigma_{v(t)} = \{(\gamma, \theta) \in \Gamma \times \mathbf{S}^1 : \gamma v(t) = v(t - \theta)\}.$$

Note that here $\mathbf{S}^1$ acts on the space of all 2π-periodic mappings $v(t)$ (denoted $\mathcal{C}_{2\pi}$) *not* on $\mathbf{R}^n$. We take a natural action of $\Gamma \times \mathbf{S}^1$ on the space $\mathcal{C}_{2\pi}$ of 2π-periodic mappings from $\mathbf{R}$ to $\mathbf{R}^n$, that is we take the following action

$$(\gamma, \theta).v(t) = \gamma v(t + \theta).$$

where the Γ action is induced from its spatial action on $\mathbf{R}^n$, and the $\mathbf{S}^1$ action is by phase shift. Our original definition of a symmetry of a periodic function above then reduces to $(\gamma, \theta).v(t) = v(t)$ and so for this action $\Sigma_{v(t)}$ above is the isotropy subgroup for $v(t)$.

It is on this space and action ($\mathcal{C}_{2\pi}$) that we apply our Liapunov-Schmidt reduction, and not $\mathbf{R}^n$ as we did for the steady-state case. In more detail, we look for periodic solutions to 2.4.4 with period approximately 2π by rescaling time as $s = (1+\tau)t$ for a *period-scaling parameter* τ. This gives us a new system

$$(1+\tau)\frac{du}{ds} + f(u, \lambda) = 0 \tag{2.4.5}$$

where $u(s) = u((1+\tau)t)$. Then 2π-periodic solutions to this system correspond to $2\pi/(1+\tau)$-periodic solutions to 2.4.4.

To find these define the operator

$$\Phi : \mathcal{C}^1_{2\pi} \times \mathbf{R} \times \mathbf{R} \to \mathcal{C}_{2\pi}$$

by

$$\Phi(u, \lambda, \tau) = (1+\tau)\frac{du}{ds} + f(u, \lambda)$$

and then a solution (u, λ, τ) to $\Phi = 0$ corresponds to a $2\pi/(1+\tau)$-periodic solution to 2.4.4.

It is for this operator Φ that the Liapunov-Schmidt reduction lets us find solutions.

We now state the corresponding result to the Equivariant Branching Lemma for Hopf bifurcation.

Assume an equation of the form 2.4.4 and make the generic hypothesis that $\mathbf{R}^n$ is Γ-Simple and that the eigenvalues of $(df)_{(0,0)}$ cross the imaginary axis with non-zero speed. That is, if the eigenvalues of $(df)_{(0,0)}$ are $\sigma(\lambda) \pm i\rho(\lambda)$ then $\sigma(0) = 0$ and

$$\sigma'(0) \neq 0. \tag{2.4.6}$$

Temporal symmetries enter the following theorem through an isotropy subgroup $\Sigma \subset \Gamma \times \mathbf{S}^1$ acting on $\mathbf{R}^n$. Then we have

Theorem 2.4.1 (Equivariant Hopf Theorem) *Let the system of ODE's given by 2.4.4 satisfy 2.4.6. Suppose that*

$$dim Fix(\Sigma) = 2.$$

Then there exists a unique branch of small amplitude periodic solutions to 2.4.4 with period near 2π having Σ as their group of symmetries.

Proof: This is Golubitsky et al. [17] Theorem XVI 4.1. ∎

We can then examine the stability of these solutions in a similar manner to the steady state case.

2.5 Some Group Theory

We now re-cap some group theory that is necessary to understand the later work. We consider two ways of combining the symmetry groups $\mathcal{L}$ and $\mathcal{G}$, the direct product and the wreath product. With later work in mind we do this by considering how they act on the space $\mathbf{R}^{nk}$ where $\mathcal{G}$ acts on $(x_1, \ldots, x_n)$, where $x_i \in \mathbf{R}^k$, by permutation of indices, i.e. we assume that $\mathcal{G} \subset \mathbf{S}_n$, and where $\mathcal{L}$ acts on $\mathbf{R}^k$.

Direct Product

In this scenario, the direct product, $\mathcal{L} \times \mathcal{G}$, acts by $\mathcal{G}$ acting as usual by permutation of indices, and $\mathcal{L}$ acts on every x_i simultaneously. That is, if $(\gamma, \pi) \in \mathcal{L} \times \mathcal{G}$ then

$$(\gamma, \pi)(x_1, \ldots, x_n) = (\gamma x_{\pi(1)}, \ldots, \gamma x_{\pi(n)}).$$

Wreath Product

The wreath product $\mathcal{L} \wr \mathcal{G}$ however, has a more complicated action. The $\mathcal{G}$ part still acts as permutation of indices, but now the action of $\mathcal{L}$ acts as the action of $\mathcal{L} \times \ldots \times \mathcal{L}$, so that the action of the group on $(x_1, \ldots, x_n)$ is

$$(l_1 x_{\pi(1)}, \ldots, l_n x_{\pi(n)})$$

where $(l_1, \ldots, l_n) \in \mathcal{L} \times \ldots \times \mathcal{L}$ and $\pi \in \mathcal{G}$.

For a more formal definition see for example Hall [20] pp 81-82.

Chapter 3

Steady-State Bifurcations In 3 Coupled Cells

We begin our work on bifurcations of coupled ODE's with internal symmetries by considering the case of three coupled cells. This will give a good 'feel' for what is happening before embarking on the more complicated issue of a general number, n, of coupled cells. For the purposes of comparison, we finish this chapter by summarizing how the different symmetries could manifest their differences in a system of three coupled ODE's on $\mathbf{R}^3$.

3.1 Form Of The Equations

Here we consider how equations with internal symmetries can be coupled so as to produce wreath product and direct product symmetries. In particular we consider equations with $\mathbf{Z}_2$ symmetry, which can further simplify the problem to producing 'invariant coupling' and 'equivariant coupling'. Later, when considering Hopf-bifurcations, we shall prove that it is only $\mathbf{Z}_2$ symmetries that produce this possible simplification.

We consider three ODE's of the form

$$\dot{x}_i = \phi(x_i) \tag{3.1.1}$$

where $i \in \{1,2,3\}$, $x_i \in \mathbf{R}^k$, $\dot{x}$ denotes derivatives with respect to time, and where $\phi(x)$ is equivariant under an action of $\mathbf{Z}_2$. For the purposes of illustration we shall assume that $\kappa \in \mathbf{Z}_2$ acts as multiplication by -1, i.e.

$$\kappa x = -x$$

and so $\phi(x)$ satisfies

$$\phi(-x) = -\phi(x).$$

We now couple these three equations with a coupling term $K(x_i, x_j, x_k)$ so that the entire system possesses at least $\mathbf{S}_3$ symmetry, where the action of $\mathbf{S}_3$ is to permute the indices. Therefore the coupling term g must satisfy

$$K(u, v, w) = K(u, w, v).$$

This means that we now have a system of equations

$$\begin{aligned} \dot{x}_1 &= \phi(x_1) + K(x_1, x_2, x_3), \\ \dot{x}_2 &= \phi(x_2) + K(x_2, x_1, x_3), \\ \dot{x}_3 &= \phi(x_3) + K(x_3, x_2, x_1). \end{aligned} \tag{3.1.2}$$

The specific form of this coupling can now induce two very natural global symmetries, with respect to the internal $\mathbf{Z}_2$ Symmetry of the individual cells, if the necessary properties are present.

More specifically, the coupling can now lead to *invariant* coupling, where applying the $\mathbf{Z}_2$ action to one variable only, leaves the system of equations unchanged, or to *equivariant* coupling where if you apply the $\mathbf{Z}_2$ action to one variable, then to keep the equations consistent you must also apply the action to the remaining two variables. The coupling term need not strictly be $\mathbf{Z}_2$ equivariant to achieve the necessary properties, and so the same global symmetries, but this coupling is so named since the effect *can* be achieved by strict equivariance in all the variables. These in turn lead to overall $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetries. We shall also sometimes refer to these couplings as *wreath product* or *direct product* couplings, due to the global symmetries produced by them. We now show how these symmetries arise.

3.1.1 Invariant Coupling

The first case we consider is coupling where applying the $\mathbf{Z}_2$ action to any of the variables leaves the equations consistent with the original, i.e. the coupling ensures that applying a single $\mathbf{Z}_2$ action to any of the equations is an invariant action for the whole system.

We can denote this situation schematically as in Figure 3.1 where the $\mathbf{Z}_2$ action is represented by swapping the two elements x_i and $-x_i$. A symmetry of the whole system will leave the diagram topologically unchanged.

Therefore, for the case in question, it is easy to see that swapping any two of the elements within each cell will achieve this in the figure. This is equivalent to applying our $\mathbf{Z}_2$ action to one of the cells.

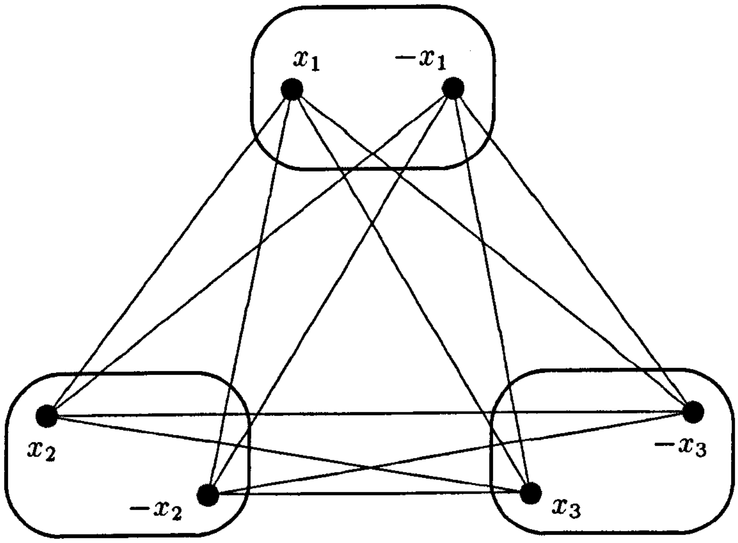


Figure 3.1: Schematic representation of invariant coupling of three cells with internal $\mathbf{Z}_2$ symmetry

In the equations this means that the coupling term $K(u, v, w)$ must be invariant under the $\mathbf{Z}_2$ action on the second two variables, but equivariant on the first variable otherwise we will not get the desired effect in equation 3.1.2.

Therefore, with our chosen action, we must have

$$K(u, -v, w) = K(u, v, -w) = K(u, v, w)$$

and

$$K(-u, v, w) = -K(u, v, w)$$

where $u, v, w \in \mathbf{R}^k$.

For a more realistic situation, we can place further constraints on the equations. For example, a usual restriction is that coupling between identical cells vanish, one way of achieving this is to consider the case when $K(u, v, w)$ can be decomposed as

$$K(u, v, w) = K^*(u, v) + K^*(u, w)$$

where

$$K^*(u, u) = 0,$$

and so the invariance under the $\mathbf{Z}_2$ action forces

$$K^*(u, \pm u) = 0,$$

and also

$$K^*(-u, v) = -K^*(u, v)$$

and

$$K^*(u, -v) = K^*(u, v).$$

Thus K^* could be of a form similar to

$$K^*(u, v) = u(u^2 - v^2)L(u^2, v^2).$$

However we choose to obtain the correct restrictions on K though, we eventually end up with a set of equations

$$\dot{x} = f(x)$$

where $x \in \mathbf{R}^{3k}$, which is equivariant under the action of the wreath product $\mathbf{Z}_2 \wr \mathbf{S}_3$.

3.1.2 Equivariant Coupling

Next we consider the case where if one of the variables in equation 3.1.2 has the $\mathbf{Z}_2$ action applied to it, then so must the other two variables. We shall then show that this can be equivalent to the coupling term K being equivariant in all its variables with respect to this action.

Firstly we consider this situation schematically, as shown in Figure 3.2. Again, we take the $\mathbf{Z}_2$ action as swapping over the elements x_i and $-x_i$ within each cell. The difference this time, when compared with Figure 3.1, is that if the $\mathbf{Z}_2$ action is applied to any cell, then to topologically preserve the diagram we *must* apply the $\mathbf{Z}_2$ action to *all* the cells.

One way to achieve this is to make the coupling term $K(u, v, w)$ equivariant under the $\mathbf{Z}_2$ action, in all its variables, and hence why this coupling is so named i.e.

$$K(-u, v, w) = K(u, -v, w) = K(u, v, -w) = -K(u, v, w)$$

which also forces

$$K(-u, -v, w) = K(-u, v, -w) = K(u, -v, -w) = K(u, v, w).$$

Note however, that a $\mathbf{Z}_2$ action is the only action that *can* be represented in this manner, for other symmetry groups other forms of equation must be used to achieve

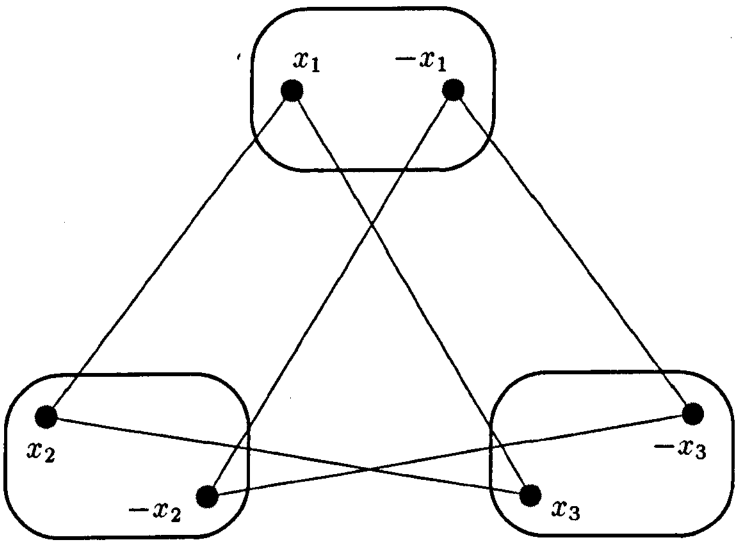


Figure 3.2: Schematic representation of equivariant coupling of three cells with internal $\mathbf{Z}_2$ symmetry

the same symmetries, for example skew-equivariance (see Chapter 6 for why only $\mathbf{Z}_2$-equivariant coupling terms can be used and for discussion of skew-equivariance).

It should be clear why coupling of this form, when used in equation 3.1.2, produces the required property. When considering more realistic situations, we now note that for the coupling to vanish between identical cells and for the whole coupling term to be $\mathbf{Z}_2$ equivariant becomes more complicated. We must for example, for the equivariance, start with a basic form something like

$$K(u,v,w) = uvwL(u^2,v^2,w^2).$$

From this start, for the coupling to vanish between identical cells, we must also, with this form, have the coupling vanish between cells that differ by a $\mathbf{Z}_2$ action. So L must decompose to

$$L(u^2,v^2,w^2) = L^*(u^2,v^2) + L^*(u^2,w^2)$$

where for example

$$L^*(u^2,v^2) = (u^2 - v^2)M(u^2,v^2).$$

If we don't necessarily want the equations to be equivariant under $\mathbf{Z}_2$ though, and only require that the final equations conform to our original specification that if the $\mathbf{Z}_2$ action is applied to one cell then it must be applied to all, then coupling between identical cells can be achieved by setting

$$K(u,v,w) = K^*(u,v) + K^*(u,w)$$

where for example

$$K^*(u,v) = (u-v)L((u^2,v^2).$$

However we achieve the required result though, the set of equations 3.1.2 with the appropriate coupling terms, will produce a set of equations

$$\dot{x} = f(x)$$

where $x \in \mathbf{R}^{3k}$ and f is equivariant under the action of the direct product $\mathbf{Z}_2 \times \mathbf{S}_3$.

3.1.3 Machinery

We now apply the methods of Golubitsky et al. [17] to systems of equations possessing the symmetries we have just discussed. In the following sections we consider the reduced map

$$\dot{x} = g(x,\lambda) \tag{3.1.3}$$

where x is in the appropriate irreducible subspace, λ is a bifurcation parameter and $g(x)$ is either $\mathbf{S}_3$, $\mathbf{Z}_2 \wr \mathbf{S}_3$ or $\mathbf{Z}_2 \times \mathbf{S}_3$ equivariant. That is, we consider the equation 3.1.2 in its most general form after it has undergone Liapunov-Schmidt reduction onto the irreducible subspace of the action. We seek solution branches satisfying

$$g(x,\lambda) = 0. \tag{3.1.4}$$

We shall then apply the results back to our original problem of considering three identical cells coupled so as to produce the appropriate symmetries.

3.2 Steady-State Bifurcation With $\mathbf{S}_3$ Symmetry

We begin with the simplest case of bifurcations with $\mathbf{S}_3$ symmetry. This is basically covered in Golubitsky et al. [17], Case Study 5, *The Traction Problem For Mooney-Rivlin Material.* There the authors consider a cube whose side lengths are given by the variables l_1, l_2 and l_3 subject to the constraint $l_1 l_2 l_3 = 1$. A uniform traction λ

is then placed on each face of the cube, and two solutions are expected, rod-like and plate-like.

This problem can be reduced to the problem of steady state bifurcation with $\mathbf{S}_3$ symmetry on $\mathbf{R}^3$ under the linear constraint $x_1 + x_2 + x_3 = 0$, where $\mathbf{S}_3$ acts by permutation of indices. This action is absolutely irreducible, and so we can apply the Equivariant Branching Lemma, the authors of [17] however go on to carry out the calculations on the space isomorphic to this, $\mathbf{C}^2$, where $\mathbf{D}_3$ acts by its standard action.

Here though, to stay consistent with the rest of this thesis, we carry out the calculations with the original coordinates in $\mathbf{R}^3$ and retrieve the results of [17] that the only branch of solutions guaranteed by the equivariant branching equation is generically unstable, and so to find a stable branch we must apply unfolding theory. We do not do that here, but instead try a simpler method of considering a non-generic form of equation.

For completeness with later work we also consider the irreducible action of $\mathbf{S}_3$ on $\mathbf{R}$ so that we have a complete analysis for $\mathbf{S}_3$ acting on $\mathbf{R}^3 \simeq \mathbf{R}^3_0 \oplus \mathbf{R}$.

Isotropy Subgroups

Proposition 3.2.1 *Up to conjugacy we have only three isotropy subgroups, given in the following table.*

Isotropy Subgroup (Σ)	*Fixed Point Subspace*	$dim Fix(\Sigma)$
$\mathbf{S}_3$	$(0,0,0)$	*0*
$\mathbf{S}_1 \times \mathbf{S}_2$	$(2x,-x,-x)$	*1*
$\mathbf{1}$	$(x,y,-(x+y))$	*2*

Proof:

It is clear these *are* isotropy subgroups, we now show that they are the only ones. We do this by considering all possible points in $\mathbf{V}$, and then computing their isotropy.

Any point in $\mathbf{R}^3_0$ can be written in the form $(x,y,-(x+y))$. If $x = y = 0$ then we have the point $(0,0,0)$ which has isotropy $\mathbf{S}_3$, so assume that this is not the case.

We cannot have points of the form $(x,0,0)$ (since this forces $x = 0$) and points of the form $(x,y,0)$ (and other points on the same orbit) must be in the form $(x,-x,0)$ which has only trivial isotropy, $\mathbf{1}$, as do points $(x,y,-(x+y))$ where x, y and $-(x+y)$ all distinct.

The only other possibility are points in the orbits given by representatives of the form $(2x,-x,-x)$. These have isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ (or conjugates).

∎

Therefore, up to conjugacy there is only one isotropy subgroup with a one-dimensional fixed point subspace, $\mathbf{S}_1 \times \mathbf{S}_2$, which guarantees a corresponding solution to our bifurcation problem by the Equivariant Branching Lemma.

Stabilities of Solutions

We now consider the stabilities of this solution branch by considering the equation restricted to the irreducible subspace, which we have seen is, in this case,

$$\mathbf{R}_0^3 = \{x \in \mathbf{R}^3 : x_1 + x_2 + x_3 = 0\}.$$

Proposition 3.2.2 *The details of the reduced equation are as follows*

i) every $\mathbf{S}_3$ *invariant germ* $f : \mathbf{R}_0^3 \to \mathbf{R}$ *has the form, up to any order,* $f(u, v)$ *where*

$$u = x_1^2 + x_2^2 + x_3^2, \; v = x_1^3 + x_2^3 + x_3^3;$$

and

ii) the module of $\mathbf{S}_3$ *equivariants is generated, up to any order order, by the mappings*

$$\mathbf{X}_1 = 3 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \; \mathbf{X}_2 = \begin{bmatrix} 2x_1^2 - x_2^2 - x_3^2 \\ 2x_2^2 - x_1^2 - x_3^2 \\ 2x_3^2 - x_1^2 - x_2^2 \end{bmatrix}.$$

Proof:

The action of $\mathbf{S}_3$ on $\mathbf{R}^3$ is generated by pseudo-reflections, as is, therefore, our action of $\mathbf{S}_3$ on $\mathbf{V}$, and so by Theorem 2.3.2 two invariants are sufficient to generate all the invariants to any order. By Field and Richardson [14] (page 80) we need the same number of equivariants as invariants to generate the module of equivariants over the invariants, and so again need only two.

Before getting in to the proof we note that for $\mathbf{S}_3$ symmetry we have $\Lambda_0^k = \Lambda^k$ for all k. We calculate invariants and equivariants up to third order.

Invariants We start with the linear invariants, and so start with $\Lambda^1 = \{x_1, x_2, x_3\}$. The only partition given by the symmetry is $\Delta_1 = \Lambda^1$ and so an invariant of $x_1 + x_2 + x_3$, but this is zero on $\mathbf{R}_0^3$, and so there are no non-zero invariants.

Now consider the quadratic invariants. There are now two possible subsets that partition Λ^2, namely $\Delta_1 = \{x_1^2, x_2^2, x_3^2\}$ and $\Delta_2 = \{x_1x_2, x_1x_3, x_2x_3\}$, but

$$0 = (x_1 + x_2 + x_3)^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_1x_3 + x_2x_3)$$

and so we only need an invariant generated by one of the Δ_i, say $u = x_1^2+x_2^2+x_3^2$.

Finally we consider the cubic invariants. Λ^3 partitions into three subsets, $\Delta_1 = \{x_1^3, x_2^3, x_3^3\}$, $\Delta_2 = \{x_1^2x_2, \ldots, x_3^2x_2\}$ and $\Delta_3 = \{x_1x_2x_3\}$, giving invariant functions $v_1 = \Sigma_i x_i^3$, $v_2 = \Sigma_{i\neq j}x_i^2x_j$ and $v_3 = x_1x_2x_3$. We can reduce this number though since $0 = (x_1 + x_2 + x_3)^3 = v_1 + 3v_2 + 3v_3$ and $0 = (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2) = v_1 + v_2$ and so again we need only one, say $v = x_1^3 + x_2^3 + x_3^3$.

Equivariants We begin with the linear equivariants. We need only two generating partitions, $\Theta_1^1 = \left\langle \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\Theta_2^1 = \left\langle \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_2 = \begin{bmatrix} x_2+x_3 \\ x_1+x_3 \\ x_1+x_2 \end{bmatrix}$. We now note that $\mathbf{Y}_1 + \mathbf{Y}_2$ is invariant under the group action, indeed we have that $\mathbf{Y}_1 + \mathbf{Y}_2 = 0$, and so therefore we need only one of them, say $\mathbf{X}_1 = \mathbf{Y}_1$.

Now consider the quadratic equivariants. We have the generating partitions $\Theta_1^2 = \left\langle \begin{bmatrix} x_1^2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ leading to $\mathbf{Y}_3 = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{bmatrix}$, $\Theta_2^2 = \left\langle \begin{bmatrix} x_2^2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ leading to $\mathbf{Y}_4 = \begin{bmatrix} x_2^2+x_3^2 \\ x_1^2+x_3^2 \\ x_1^2+x_2^2 \end{bmatrix}$, but $\mathbf{Y}_1 + \mathbf{Y}_2$ is strictly invariant under the group action so we need only one of these, say $\mathbf{X}_2 = \mathbf{Y}_3$. We also have $\Theta_3^2 = \left\langle \begin{bmatrix} x_1x_2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_5 = \begin{bmatrix} x_1x_2+x_1x_3 \\ x_2x_1+x_2x_3 \\ x_3x_1+x_3x_2 \end{bmatrix}$ but this is not needed since $0 = (x_1+x_2+x_3)\mathbf{X}_1 = \mathbf{X}_2 + \mathbf{Y}_5$. Finally $\Theta_4^2 = \left\langle \begin{bmatrix} x_2x_3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ gives us the equivariant $\mathbf{Y}_6 = \begin{bmatrix} x_2x_3 \\ x_1x_3 \\ x_1x_2 \end{bmatrix}$ but again this equivariant is not needed since $\mathbf{Y}_5 + \mathbf{Y}_6$ is strictly invariant under the group action. Once projected from $\mathbf{R}^3$ to $\mathbf{R}_0^3$, this gives us the two equivariants necessary for the proof.

Finally we consider the cubic equivariants. Now our generating partitions are contained among $\Theta_1^3 = \left\langle \begin{bmatrix} x_1^3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ leading to $\mathbf{Y}_7 = \begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{bmatrix}$, call $\mathbf{X}_3 = \mathbf{Y}_7$, $\Theta_2^3 = \left\langle \begin{bmatrix} x_2^3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ leading to $\mathbf{Y}_8 = \begin{bmatrix} x_2^3 + x_3^3 \\ x_1^3 + x_3^3 \\ x_1^3 + x_2^3 \end{bmatrix}$, but we can discount this one since $\mathbf{Y}_8 + \mathbf{X}_3$ is strictly invariant. We also have $\Theta_3^3 = \left\langle \begin{bmatrix} x_1^2 x_2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_9 = \begin{bmatrix} x_1^2(x_2 + x_3) \\ x_2^2(x_1 + x_3) \\ x_3^2(x_1 + x_2) \end{bmatrix}$, which is equal to $-\mathbf{X}_3$ since $(x_1 + x_2 + x_3)\mathbf{X}_2 = 0$. There is also $\Theta_4^3 = \left\langle \begin{bmatrix} x_1 x_2^2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_{10} = \begin{bmatrix} x_1(x_2^2 + x_3^2) \\ x_2(x_1^2 + x_3^2) \\ x_3(x_1^2 + x_2^2) \end{bmatrix}$ but this is equal to $u\mathbf{X}_1 - \mathbf{X}_3$. Next we have $\Theta_5^3 = \left\langle \begin{bmatrix} x_2 x_3^2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_{11} = \begin{bmatrix} x_2^2 x_3 + x_2 x_3^2 \\ x_1^2 x_3 + x_1 x_3^2 \\ x_1^2 x_2 + x_1 x_2^2 \end{bmatrix}$ but $\mathbf{Y}_{11} + \mathbf{Y}_{10} + \mathbf{Y}_9 + \mathbf{Y}_8 + \mathbf{Y}_7 = \begin{bmatrix} (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2) \\ (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2) \\ (x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2) \end{bmatrix} = 0$. Finally we have $\Theta_6^3 = \left\langle \begin{bmatrix} x_1 x_2 x_3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ which leads to $\mathbf{Y}_{12} = \begin{bmatrix} x_1 x_2 x_3 \\ x_1 x_2 x_3 \\ x_1 x_2 x_3 \end{bmatrix}$ which when projected onto $\mathbf{V}$ vanishes.

Therefore we are left with only one cubic equivariant, $\mathbf{X}_3$, but now note that

$$\begin{aligned} & x_1(u + x_1x_2 + x_1x_3 + x_2x_3) \\ &= x_1(u + x_1x_2 + x_1x_3 + x_2x_3) + (x_1 + x_2 + x_3)(x_2x_3 - u) \\ &= -(x_2^3 + x_3^3) + 2x_1x_2x_3. \end{aligned}$$

Therefore by symmetry we have that

$$(u + x_1x_2 + x_1x_3 + x_2x_3)\mathbf{X}_1 + \mathbf{Y}_8 = 2\mathbf{Y}_{12}.$$

Project onto $\mathbf{V}$ and we have the same equivariant as projecting $\mathbf{X}_3$ onto $\mathbf{V}$ and so this is not needed either and so we have *no* cubic equivariants.

∎

Therefore we can write our equivariant g up to any order as

$$g(x, \lambda) = P\mathbf{X}_1 + Q\mathbf{X}_2 \tag{3.2.5}$$

where P and Q are functions of u, v and λ.

Branching Equation

When we restrict ourselves to $Fix(\mathbf{S}_1 \times \mathbf{S}_2)$ and parametrise by t we have

$$Fix(\mathbf{S}_1 \times \mathbf{S}_2) = (2t, -t, -t),$$

$u = 6t^2$ and $v = 6t^3$ and the first component of g is given by

$$g_1 = 6Pt + 6Qt^2.$$

Setting this to zero gives us

$$P + Qt = 0$$

and so

$$\lambda(t) = -\left(6P_u t^2 + Q(0)t\right)/P_\lambda(0)$$

giving a solution branch that is transcritical, $\lambda'(0) = -Q(0)/P_\lambda(0) \neq 0$.

Now we use Theorem XIII 4.4 of Golubitsky et al. [17] to show that this branch is generically unstable because of the presence of the quadratic coefficient, Q, in our equation and we are done.

But we now ask the question, what if we set $Q(u, v, \lambda) \equiv 0$, that is if the equation is no longer a generic equivariant. This gives a revised branching equation of

$$\lambda(t) = -\left(6P_u(0)\right)t^2/P_\lambda(0).$$

Theorem 3.2.3 *Under the (non-generic) hypothesis that the quadratic coefficient is identically equal to zero, the solution branch corresponding to isotropy* $\mathbf{S}_1 \times \mathbf{S}_2$ *is stable iff* $P_u(0) > 0$.

Proof:

The general form for $(dg)_{x_0}$ which commutes with the action of $\mathbf{S}_1 \times \mathbf{S}_2$ is given by

$$\begin{pmatrix} a & d & d \\ c & b & e \\ c & e & b \end{pmatrix}.$$

which has eigenvalues $b-e$ and

$$\frac{1}{2}\left[(a+b+e)\pm\sqrt{(e+b-a)^2+8cd}\right]$$

where

$$\begin{aligned} a &= \tfrac{dg_1}{dx_1} = 30P_u(0)t^2, \\ b &= \tfrac{dg_2}{dx_2} = 12P_u(0)t^2, \\ c &= \tfrac{dg_2}{dx_1} = -12P_u(0)t^2, \\ d &= \tfrac{dg_1}{dx_2} = -12P_u(0)t^2 \end{aligned}$$

and

$$e = \tfrac{dg_2}{dx_3} = 6P_u(0)t^2.$$

This gives one eigenvalue, $b-e$, as $6P_u(0)t^2$, the other two being given by the above equation. So the final two eigenvalues are given as $42P_u(0)t^2$ and $6P_u(0)t^2$. Requiring all three eigenvalues to be positive gives us the result. ∎

As promised we now consider the other irreducible subspace of $\mathbf{R}^3$ for our action of $\mathbf{S}_3$

$\mathbf{S}_3$ Acting On R

We consider the action of $\mathbf{S}_3$ acting on $\mathbf{R}^* = \{(x,x,x) : x \in \mathbf{R}\}$ so that $\mathbf{R}^3 = \mathbf{R}_0^3 \oplus \mathbf{R}^*$. This action of $\mathbf{S}_3$ is trivial and so the only isotropy subgroup is $\mathbf{S}_3$ itself with fixed point subspace (t,t,t) with dimension one.

Theorem 3.2.4 *The branch of solutions with isotropy* $\mathbf{S}_3$ *guaranteed by the Equivariant Branching Lemma (2.2.2) is generically unstable.*

Proof: The invariants are generated by $u = x$ (we only need one) and the equivariants are generated over the invariants by $\mathbf{X}_1 = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$. Therefore $g = R\mathbf{X}_1$ where R is a function of u and λ (bifurcation parameter).

The branching equation satisfies $Rx = 0$ or $R(0)t + R_u(0)t^2 + R_\lambda(0)\lambda t = 0$ and so $\lambda(t) = -R_u(0)t/R_\lambda(0)$.

Therefore the branching equation $\lambda(t)$ is transcritical and so by Theorem XII 4.4 of Golubitsky et al. [17] this branch of solutions is generically unstable. ∎

We now consider the effect of adding an internal $\mathbf{Z}_2$ symmetry to each of the cells, beginning with the wreath product case.

3.3 Steady-State Bifurcation With $\mathbf{Z}_2 \wr \mathbf{S}_3$ Symmetry

We now consider the case of $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry, and so a system of the form 3.1.4 where g is equivariant under the action of $\mathbf{Z}_2 \wr \mathbf{S}_3$.

The Group Actions And Irreducible Representation

Since the standard action, $x \mapsto -x$, of $\mathbf{Z}_2$ on $\mathbf{R}$ is absolutely irreducible we start by considering the natural way for $\mathbf{Z}_2 \wr \mathbf{S}_3$ to act on $\mathbf{R}^3$, three copies of this. To do this let $\mathbf{S}_3$ act by permutation of indices, as in the case of $\mathbf{S}_3$ symmetric bifurcations, and then let $\mathbf{Z}_2$ act on each $x_i \in \mathbf{R}$ in the standard way

$$\kappa x_i = -x_i$$

where κ is the element that generates $\mathbf{Z}_2$.

Or, more precisely, let $\rho \in \mathbf{S}_3$, $\underline{\kappa} \in \mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$ where $\underline{\kappa} = (\kappa(1), \kappa(2), \kappa(3))$ and let $w \in \mathbf{Z}_2 \wr \mathbf{S}_3$ be the composition of ρ and $\underline{\kappa}$, then we can represent wx_i as

$$wx_i = \kappa(i)x_{\rho(i)}$$

where $\kappa(i) = \pm 1$. i.e.

$$(\underline{\kappa}, \pi).\underline{x} = \begin{bmatrix} \kappa(1)x_{\rho(1)} \\ \kappa(2)x_{\rho(2)} \\ \kappa(3)x_{\rho(3)} \end{bmatrix}.$$

This then generates an action of $\mathbf{Z}_2 \wr \mathbf{S}_3$ on $\mathbf{R}^3$ which is absolutely irreducible, as we shall now show.

Let the actions of $\mathbf{Z}_2 \wr \mathbf{S}_3$ act on the standard basis of $\mathbf{R}^3$ as 3×3 matrices, then by definition the action is absolutely irreducible if the only matrix that commutes with the group actions is a scalar multiple of the identity (see for example [17] XII 3.2, page 40).

The only matrices that commute with the matrix corresponding to the $\mathbf{S}_3$ action are of the form

$$\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}$$

and the only matrices that commute with all the $\mathbf{Z}_2$ actions are of the form

$$\begin{pmatrix} c & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix}.$$

Combining the two gives us the required result.

Remark 3.3.1 *This action is the the same as the symmetries of the cube considered in [17], exercises in chapter XII, 4.7 and 5.5.* $\mathbf{Z}_2 \wr \mathbf{S}_3$ *is also isomorphic to the octahedral group, steady state bifurcations of which have been considered by Melbourne [24],[25], but we carry out the calculations here for completeness.*

Remark 3.3.2 *We cannot restrict this action to the space*

$$V = \{x \in \mathbf{R}^3 : x_1 + x_2 + x_3 = 0\}$$

to compare with the results of Aronson et al. [3] since $\mathbf{V}$ *is not even invariant under the action; if* $x_1 + x_2 + x_3 = 0$ *then in general* $x_1 + x_2 - x_3 \neq 0$ *for instance.*

Therefore the representations are so different that there is no comparison that makes sense.

Isotropy Subgroups

Now that we have an absolutely irreducible action, our next step is to compute the associated isotropy subgroups, and their fixed point subspaces. This we do in the following proposition.

Proposition 3.3.3 *Up to conjugacy, the list of all the isotropy subgroups of* $\mathbf{Z}_2 \wr \mathbf{S}_3$ *is as given in the following table*

Isotropy Subgroup (Σ)	*Fixed Point Subspace*	$dimFix(\Sigma)$
$\mathbf{Z}_2 \wr \mathbf{S}_3$	$(0,0,0)$	*0*
$\mathbf{W}_1$	$(x,0,0)$	*1*
$\mathbf{W}_2$	$(x,x,0)$	*1*
$\mathbf{S}_3$	(x,x,x)	*1*
$[(1,1,-1),id]$	$(x,y,0)$	*2*
$\mathbf{S}_1 \times \mathbf{S}_2$	(x,y,y)	*2*
id	(x,y,z)	*3*

where $\mathbf{W_1}$ *is generated by the three elements of* $\mathbf{Z_2}^3 \times \mathbf{S_3}$

$$[(1,-1,1),id],[(1,1,-1),id],[id,(23)]$$

and $\mathbf{W_2}$ *is generated by the two elements*

$$[(1,1,-1),id],[id,(12)].$$

Proof: These are clearly isotropy subgroups, we now show they are the only ones.

If we have the zero vector $(0,0,0)$ then we have isotropy $\mathbf{Z_2} \wr \mathbf{S_3}$, so assume we have a non-zero vector (x_1, x_2, x_3). The group action can only permute the variables or multiply individual variables by ± 1. Therefore if two variables are scalar multiples of each other, other than by ± 1, then they are strictly contained in a fixed point subspace of higher dimension than otherwise. By conjugacy we may also assume that all the variables are of the same sign, and so positive.

First we consider the isotropies of vectors containing no zero elements. Now, if $x_1 \neq x_2 \neq x_3$ then we have isotropy $\mathbf{1}$. If $x_1 = x_2 = x_3$ then the vector is of form (u,u,u) which has isotropy $\mathbf{S_3}$. The only other possibility is that two are equal, by conjugacy say $x_2 = x_3$, when we have a vector of the form (u,v,v) which has isotropy $\mathbf{S_1} \times \mathbf{S_2}$.

Now let one component be zero, by conjugacy x_3, then either the other two variables are equal, so $(u,u,0)$ with isotropy $\mathbf{W_2}$ (generated by the transposition (12) and the $\mathbf{Z_2}$ action applied to only the third variable) or they are not equal, $(u,v,0)$ which is only fixed by the $\mathbf{Z_2}$ action applied to the third variable, $[(1,1,-1),id]$ and so this is its isotropy.

The final case to consider is vectors with two zero components, by conjugacy the second and third, giving vectors of the form $(u,0,0)$ which have isotropy $\mathbf{W_1}$. This completes the proof.

∎

Therefore, by the Equivariant Branching Lemma (Theorem 2.2.2), there exist three smooth branches of solutions to the equation $g(t,\lambda) = 0$ corresponding to the three isotropy subgroups with one-dimensional fixed point subspaces, namely $\mathbf{W_1}$,$\mathbf{W_2}$ and $\mathbf{S_3}$.

We know in addition, from Field and Richardson [14], that sub-maximal isotropy subgroups do not support solutions, and here a subgroup being maximal implies, and is implied by, its fixed point subspace being one-dimensional.

Stabilities of Solutions

We now consider the stabilities of these solution branches to the reduced equation by considering the equation as a function of the invariant and equivariant polynomials.

Proposition 3.3.4 *The details of the reduced equation are as follows*

i) every $\mathbf{Z}_2 \wr \mathbf{S}_3$ *invariant germ* $f : \mathbf{R}^3 \to \mathbf{R}$ *has the form,* $f(u,v,w)$ *where*

$$u = x_1^2 + x_2^2 + x_3^2, \quad v = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2, \quad w = x_1^2x_2^2x_3^2;$$

and

ii) the module of $\mathbf{Z}_2 \wr \mathbf{S}_3$ *equivariants is generated by the mappings*

$$\mathbf{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{X}_2 = \begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{bmatrix} \text{ and } \mathbf{X}_3 = \begin{bmatrix} x_1x_2^2x_3^2 \\ x_2x_1^2x_3^2 \\ x_3x_1^2x_2^2 \end{bmatrix}.$$

Proof:

By Theorem 2.3.1, and Theorem 2.3.2 we know that we need only three algebraically independent invariants to generate all possible invariant functions ($\mathbf{S}_3$ acting on $\mathbf{R}^3$ by permutations is generated by psuedo-reflections as are all the possible reflections from the combination of $\mathbf{S}_3$ and $\mathbf{Z}_2$), and so the above invariants will generate the invariants up to any order. By Field and Richardson [14] (page 80) we need the same number of equivariants as invariants to generate the module of equivariants over the invariants, and so again need only three.

As before, we compute all the invariants and equivariants up to only third order, higher orders follow by continuing the calculations further.

Invariants We begin with the linear invariants. Now $\Lambda_0^1 = \{x_1, x_2, x_3\}$ and the extension $\Lambda^1 = \{x_1, x_2, x_3, -x_1, -x_2, -x_3\}$. The only partition of Λ^1 gives the whole of Λ^1, and summing the elements gives zero, and so there are no non-zero linear invariants of $\mathbf{Z}_2 \wr \mathbf{S}_3$.

Next consider the quadratic invariants, where

$$\Lambda^2 = \{x_1^2, x_2^2, x_3^2, x_1x_2, x_2x_3, x_1x_3, -x_1x_2, -x_2x_3, -x_1x_3\}$$

which partitions into the two subsets $\Delta_1 = \{x_1^2, x_2^2, x_3^2\}$ giving the invariant $u = x_1^2 + x_2^2 + x_3^2$ and $\Delta_2 = \{x_1x_2, x_2x_3, x_1x_3, -x_1x_2, -x_2x_3, -x_1x_3\}$ giving the invariant $\Sigma_{i<j}(x_ix_j - x_ix_j)$, which sums to zero.

Finally consider the cubic invariants. We have

$$\Lambda^3 = \bigcup_{i \neq j} \left\{x_i^3, -x_i^3, x_i^2 x_j, -x_i^2 x_j\right\} \cup \{x_1x_2x_3, -x_1x_2x_3\},$$

which partitions into three subsets, $\Delta_1^3 = \bigcup\{x_i^3, -x_i^3\}$, $\Delta_2^3 = \bigcup_{i\neq j}\{x_i^2x_j, -x_i^2x_j\}$ and $\Delta_3^3 = \{x_1x_2x_3, -x_1x_2x_3\}$. All of these sum to zero, and so there are no cubic invariants.

Equivariants We now turn our attention to the $\mathbf{Z}_2 \wr \mathbf{S}_3$ equivariants, which will all have generating partitions of the form $\Theta_i \bigcup \gamma\Theta_i \bigcup \ldots$ where $\gamma \in \mathbf{Z}_2 \wr \mathbf{S}_3$ and Θ_i is a generating partition for $\mathbf{S}_3$ equivariance.

We begin with linear equivariants, and find we have $\Theta_1^1 = \left\langle \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$. Call this equivariant $\mathbf{X}_1 = \mathbf{Y}_1$. The only other equivariant is found from $\Theta_2^1 = \left\langle \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$, but for every $\mathcal{E}_i \in \Theta_2^1$, $-\mathcal{E}_i$ is also in Θ_2^1 and so when we sum, they cancel and we get zero.

The quadratic generating partitions all suffer the same fate as the latter of the two linear equivariants considered, that is if a $\mathcal{E}_i \in \Theta_j^2$ then so is $-\mathcal{E}_i$ and they cancel when summed. Therefore there are no quadratic equivariants.

Finally, consider the cubic equivariants. $\Theta_1^3 = \left\langle \begin{bmatrix} x_1^3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ gives the equivariant $\mathbf{Y}_3 = \begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{bmatrix}$ call $\mathbf{X}_2 = \mathbf{Y}_3$. $\Theta_2^3 = \left\langle \begin{bmatrix} x_2^3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ is the same as the subset $\left\langle \begin{bmatrix} -x_2^3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$, i.e. the elements cancel in pairs, summing to zero, so discard. Similarly, $\Theta_3^3 = \left\langle \begin{bmatrix} x_1^2 x_2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ is also generated by $\left\langle \begin{bmatrix} -x_1^2 x_2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ and terms cancel in pairs producing a zero sum. $\Theta_4^3 = \left\langle \begin{bmatrix} x_1 x_2^2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ leads to the equivariant

$\mathbf{Y}_4 = \begin{bmatrix} x_1(x_2^2+x_3^2) \\ x_2(x_1^2+x_3^2 \\ x_3(x_1^2+x_2^2 \end{bmatrix}$ but we have the relation $\mathbf{Y}_4 = u\mathbf{X}_1 - \mathbf{X}_2$ so discard.

$\Theta_5^3 = \left\langle \begin{bmatrix} x_2x_3^2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ is also generated by $\left\langle \begin{bmatrix} -x_2x_3^2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ so terms cancel in pairs,

as do the terms in $\Theta_6^3 = \left\langle \begin{bmatrix} x_1x_2x_3 \\ 0 \\ 0 \end{bmatrix} \right\rangle$.

∎

Therefore, due to Golubitsky et al. [17] XII §5, the general bifurcation problem on $\mathbf{R}^3$ with symmetries $\mathbf{Z}_2 \wr \mathbf{S}_3$ has the form,

$$g(x_1, x_2, x_3, \lambda) = P\mathbf{X}_1 + Q\mathbf{X}_2 + R\mathbf{X}_3 \tag{3.3.6}$$

where P, Q, R are functions of u, v, w, λ, where λ is the bifurcation parameter.

In what follows, however, we are concerned with the derivatives of g, and in particular we begin by considering the calculations up to only third order. It turns out that this level of accuracy is sufficient to classify the results, and so we therefore consider the simpler equation

$$g(x_1, x_2, x_3, \lambda) = P(u, \lambda)\mathbf{X}_1 + Q(u, \lambda)\mathbf{X}_2.$$

Branching Equations

The next step is to calculate the branching equations for each isotropy, which we do by first parametrising the solution branch by t.

For isotropy $\mathbf{W}_1$ we have, since $Fix(\mathbf{W}_1) = (t, 0, 0)$ means that $u = t^2$, the following

$$g|_{Fix(W_1)} = \begin{bmatrix} P(t^2, \lambda)t + Q(t^2, \lambda)t^3 + \dots \\ 0 \\ 0 \end{bmatrix}.$$

Setting this equation to zero yields, for the first component of g, up to third order,

$$P(0)t + P_\lambda(0)\lambda t + P_u(0)t^3 + Q(0)t^3 + \dots = 0$$

where subscripts denote derivatives. Linear terms must vanish at the origin since we assume that $g(0, 0) = 0$ (see [17]), and so $P(0) = 0$. Rearranging gives us the following branching equation for isotropy $\mathbf{W}_1$

$$\lambda(t) = -\left(P_u(0) + Q(0)\right) t^2 / P_\lambda(0) + \dots . \tag{3.3.7}$$

Similarly the branching equation for isotropy $\mathbf{W}_2$ is given by

$$\lambda(t) = -\left(2P_u(0) + Q(0)\right) t^2 / P_\lambda(0) + \dots \tag{3.3.8}$$

and that for $\mathbf{S}_3$ by

$$\lambda(t) = -\left(3P_u(0) + Q(0)\right) t^2 / P_\lambda(0) + \dots . \tag{3.3.9}$$

We can now state the necessary conditions for stability of these branches.

Theorem 3.3.5 *With the preceding forms of equation and labeling of isotropies, the stabilities of the branches of solutions in the presence of $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry guaranteed by the Equivariant Branching Lemma are as follows*

i) The trivial branch, corresponding to isotropy $\mathbf{Z}_2 \wr \mathbf{S}_3$, branching equation $x = 0$, is stable if $P_\lambda(0)\lambda > 0$. We assume $P_\lambda(0) < 0$.

ii) The branch corresponding to isotropy $\mathbf{W}_1$ is stable iff both

$$P_u(0) + Q(0) > 0 \text{ and } Q(0) < 0.$$

iii) The branch corresponding to isotropy $\mathbf{W}_2$ is generically unstable, with the genericity condition being $Q(0) \neq 0$. If $Q(0) = 0$ then this branch is stable iff $P_u(0) > 0$ and a condition involving fifth order terms holds.

iv) The branch corresponding to isotropy $\mathbf{S}_3$ is stable iff both

$$3P_u(0) + Q(0) > 0 \text{ and } Q(0) > 0.$$

In addition, if any branch is stable, then all three branches are sub-critical.

Remark 3.3.6 *It follows from this theorem that generically, and up to conjugacy, only one branch can be stable at any one time if $Q(0) \neq 0$.*

Proof:

i) Follows from definitions.

ii) We simplify the calculations involved in finding the eigenvalues for $(dg)_{x_0}$ by noting that $(dg)_{x_0}$ must commute with the action of the isotropy subgroup involved. This is a trick we shall extensively utilise throughout the work presented here. Therefore, for isotropy $\mathbf{W}_1$, $(dg)_{x_0}$ must be of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{pmatrix}.$$

where

$$a = \tfrac{dg_1}{dx_1}|_{Fix(W_1)} = 2t^2[P_u(0) + Q(0)]$$

and

$$c = \tfrac{dg_2}{dx_2}|_{Fix(W_1)} = -Q(0)t^2.$$

The signs of the eigenvalues of $(dg)_{x_0}$, which must have positive real part for stability, are given by the signs of a and c, hence the result.

iii) The form for $(dg)_{x_0}$, so that it commutes with the action of $\mathbf{W}_2$, is given by

$$\begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix}.$$

where

$$a = \tfrac{dg_1}{dx_1}|_{Fix(W_2)} = 2t^2[P_u(0) + Q(0)],$$
$$b = \tfrac{dg_2}{dx_1}|_{Fix(W_1)} = 2P_u(0)t^2$$

and

$$c = \tfrac{dg_3}{dx_3}|_{Fix(W_1)} = -Q(0)t^2.$$

Now $(dg)_{x_0}$ has eigenvalues $a+b$, $a-b$ and c, corresponding to

$$2t^2[2P_u(0) + Q(0)],\ 2t^2Q(0) \text{ and } -Q(0)t^2.$$

Therefore, for stability, we require both $Q(0) > 0$ and $Q(0) < 0$, hence the result.

iv) The general from for $(dg)_{x_0}$, for isotropy $\mathbf{S}_3$ is given by

$$\begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}.$$

where this time

$$a = \frac{dg_1}{dx_1}|_{Fix(S_3)} = 2t^2[P_u(0) + Q(0)]$$

and

$$b = \frac{dg_2}{dx_1}|_{Fix(S_3)} = 2P_u(0)t^2.$$

The eigenvalues of $(dg)_{x_0}$ are now $a - b$ (twice) and $a + 2b$, giving us eigenvalues of

$$2t^2[3P_u(0) + Q(0)] \text{ (twice) and } 2Q(0)t^2.$$

The result now follows.

The final result on criticality of the branches follows since, if the branch corresponding to isotropy $\mathbf{W}_1$ is stable, then $P_u(0) > 0$ so $2P_u(0) + Q(0)$ and $3P_u(0) + Q(0)$ must be positive. If the $\mathbf{S}_3$ branch is stable then $P_u(0)$ and $Q(0)$ are either both positive, or $|P_u(0)| < 1/3|Q(0)|$.

∎

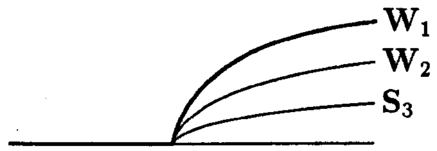
The results of the Theorem are summarised in Table 3.1 and are the same as those found by Melbourne [24]. Representative bifurcation diagrams are given in Figure 3.3.

Isotropy (Σ)	Branching Equation	Signs of Eigenvalues
$\mathbf{Z}_2 \wr \mathbf{S}_3$	$x = 0$	$P_\lambda(0)$
$\mathbf{W}_1$	$\lambda(t) = -(P_u(0) + Q(0))t^2/P_\lambda(0) + \dots$	$P_u(0) + Q(0), -Q(0)$
$\mathbf{W}_2$	$\lambda(t) = -(2P_u(0) + Q(0))t^2/P_\lambda(0) + \dots$	$2P_u(0) + Q(0), -Q(0), Q(0)$
$\mathbf{S}_3$	$\lambda(t) = -(3P_u(0) + Q(0))t^2/P_\lambda(0) + \dots$	$3P_u(0) + Q(0), Q(0)$

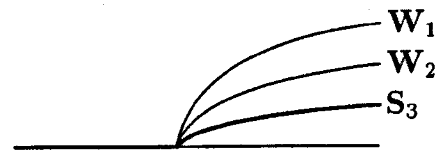
Table 3.1: Stability of branches of solutions in the presence of $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry.

3.4 Steady-State Bifurcation With $\mathbf{Z}_2 \times \mathbf{S}_3$ Symmetry

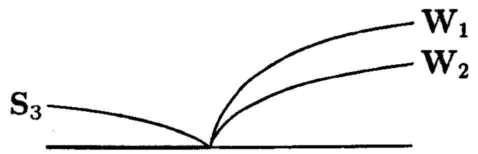
Next we consider the case of $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry. This is done by following the same steps as for $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry except that now we may find sub-maximal isotropy subgroups supporting solutions.



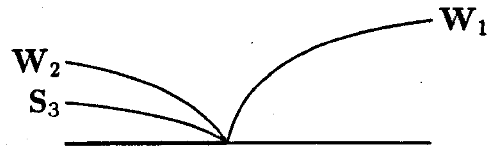
$Q(0) < 0, P_u(0) + Q(0) > 0$



$Q(0) > 0, 3P_u(0) + Q(0) > 0$



$Q(0) > 0, P_u(0) + Q(0) > 0$
$2P_u(0) + Q(0) > 0, 3P_u(0) + Q(0) < 0$



$Q(0) > 0, P_u(0) + Q(0) > 0$
$2P_u(0) + Q(0) < 0, 3P_u(0) + Q(0) < 0$

Figure 3.3: Representative bifurcation diagrams for $\mathbf{Z}_2 \wr \mathbf{S}_3$ steady-state bifurcations when $P_\lambda < 0$. Thick lines denote stability, thin lines instability.

The Group Actions And Irreducible Representation

Let $\rho \in \mathbf{S}_3$ act on $\mathbf{R}^3$ as before by permutation of indices, and let $\kappa \in \mathbf{Z}_2$ act on $x \in \mathbf{R}^3$ by $\kappa x = -x$ so that now, for $(\kappa, \rho) \in \mathbf{Z}_2 \times \mathbf{S}_3$,

$$(\kappa, \rho)\underline{x} = \begin{bmatrix} \kappa x_{\rho(1)} \\ \kappa x_{\rho(2)} \\ \kappa x_{\rho(3)} \end{bmatrix}.$$

This action of $\mathbf{Z}_2 \times \mathbf{S}_3$ on $\mathbf{R}^3$ is no longer absolutely irreducible. However, since $Fix(\mathbf{Z}_2 \times \mathbf{S}_3) = \{0\}$ we can still apply the more general version of the Equivariant Branching Lemma (see Theorem 2.2.3 for details), subject to the non-degeneracy condition that $dg_\lambda(v_0) \neq 0$ where $v_0 \in Fix(\Sigma)$, Σ the corresponding isotropy. Here though we consider the action of $\mathbf{Z}_2 \times \mathbf{S}_3$ as the two irreducible representations on $\mathbf{R}_0^3$ and $\mathbf{R}$ so that $\mathbf{R}^3 = \mathbf{R}_0^3 \oplus \mathbf{R}$. We begin with the action on $\mathbf{V} = \mathbf{R}_0^3$ and so we let

$$\mathbf{V} = \{x \in \mathbf{R}^3 : x_1 + x_2 + x_3 = 0\}$$

and on this space the action of $\mathbf{Z}_2 \times \mathbf{S}_3$ described above *is* absolutely irreducible.

Remark 3.4.1 *The action of $\mathbf{Z}_2 \times \mathbf{S}_3$ on $\mathbf{R}^2$ is isomorphic to the standard action of $\mathbf{D}_6$ on $\mathbf{C}$ which was considered by Golubitsky et al. [17] (actually they considered the general $\mathbf{D}_n$ case).*

The analysis for the action of $\mathbf{Z}_2 \times \mathbf{S}_3$ on $\mathbf{R}$ is carried out at the end of this section.

Isotropy Subgroups

We now list all the isotropy subgroups of $\mathbf{Z}_2 \times \mathbf{S}_3$, and compute the dimensions of their fixed point subspaces, as subspaces of $\mathbf{V} = \mathbf{R}_0^3$.

Proposition 3.4.2 *The list of all isotropy subgroups of $\mathbf{Z}_2 \times \mathbf{S}_3$, up to conjugacy, is as given in the following table.*

Isotropy Subgroup (Σ)	*Fixed Point Subspace*	$dimFix(\Sigma)$
$\mathbf{Z}_2 \times \mathbf{S}_3$	$(0,0,0)$	*0*
$\mathbf{T}_1$	$(x,-x,0)$	*1*
$\mathbf{S}_1 \times \mathbf{S}_2$	$(2x,-x,-x)$	*1*
id	$(x,y,-(x+y))$	*2*

where $\mathbf{T}_1$ is generated by the simultaneous application of $\kappa \in \mathbf{Z}_2$ and $(12) \in \mathbf{S}_3$.

Proof: It is clear these *are* isotropy subgroups, we now show that they are the only ones. We do this by considering all possible points in $\mathbf{V}$, and then computing their isotropy.

Any point in $\mathbf{R}_0^3$ can be written in the form $(x,y,-(x+y))$. If $x = y = 0$ then we have the point $(0,0,0)$ which has isotropy $\mathbf{S}_3$, so assume that this is not the case.

We cannot have points of the form $(x,0,0)$ (since this forces $x = 0$) and points of the form $(x,y,0)$ (and other points on the same orbit) must be in the form $(x,-x,0)$ which has isotropy, $\mathbf{T}_1$. Points $(x,y,-(x+y))$ where x, y and $-(x+y)$ all distinct have only trivial symmetry and so isotropy $\mathbf{1}$.

The only other possibility are points in the orbits given by representatives of the form $(2x,-x,-x)$. These have isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ (or conjugates).

∎

Therefore, by the General Equivariant Branching Lemma 2.2.3, we are, generically, guaranteed solution branches to equation 3.1.4 which possess symmetries corresponding to those isotropy subgroups with one-dimensional fixed point subspaces. This means we are guaranteed, up to conjugacy, two solutions having isotropy $\mathbf{T}_1$ and $\mathbf{S}_1 \times \mathbf{S}_2$.

Remark 3.4.3 *Since the fixed point subspaces must be two, one or zero dimensional we cannot have sub-maximal isotropy subgroups occurring.*

Stabilities of Solutions

As with the previous section, we begin by examining the form which the reduced equation must take to be $\mathbf{Z}_2 \times \mathbf{S}_3$ equivariant.

Proposition 3.4.4 *The details of the reduced equation, for $\mathbf{Z}_2 \times \mathbf{S}_3$-equivariance, are as follows*

i) every $\mathbf{Z}_2 \times \mathbf{S}_3$ invariant germ $f : \mathbf{R}_0^3 \to \mathbf{R}$ has the form, up to at least third order, $f(u)$ where

$$u = x_1^2 + x_2^2 + x_3^2.$$

ii) The module of $\mathbf{Z}_2 \times \mathbf{S}_3$ equivariants is generated, up to third order, by the mapping

$$\mathbf{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Proof:

As previously we calculate invariants and equivariants to third order. Higher orders follow by continuing the calculations.

By Theorem 2.3.1, Theorem 2.3.2 and Filed and Richardson [14] (page 80) we need only two invariants and two equivariants to generate the necessary invariants and equivariants to any order, but here we only consider to third order.

Invariants We begin with the linear invariants by noting that the partition of Λ^1 gives us $\Delta_1^1 = \Lambda^1 = \{x_1, x_2, x_3, -x_1, -x_2, -x_3\}$ which when summed gives us zero, and so there is no linear invariant.

The quadratics give us two subsets in the partition of Λ^2, $\Delta_1^2 = \{x_1^2, x_2^2, x_3^2\}$ and $\Delta_2^2 = \{x_1x_2, x_2x_3, x_1x_3\}$ which sum to the invariants $x_1^2 + x_2^2 + x_3^2$ and $x_1x_2 + x_1x_3 + x_2x_3$, but these are related since $(x_1 + x_2 + x_3)^2 = 0$.

The cubics all vanish since if $\alpha_i \in \Delta_j^3$ then so is $-\alpha_i$ and so the elements cancel in pairs when we sum.

Equivariants We now look at the linear equivariants, and find that we have the two subsets $\Theta_1^1 = \left\langle \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and

$\Theta_2^1 = \left\langle \begin{bmatrix} x_2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$ giving the equivariant $\mathbf{Y}_2 = \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_1 + x_2 \end{bmatrix}$. Take $\mathbf{X}_1 = \mathbf{Y}_1$, but $\mathbf{Y}_1 + \mathbf{Y}_2 = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = 0.$

For every $\mathcal{E}_i \in \Delta_j^2$ then we must have $-\mathcal{E}_i \in \Delta_j^2$ and so all the elements will cancel in pairs when we sum, and so there are no quadratic equivariants.

Finally, consider the cubics. This case is simplified since every $\mathcal{E}_i \in \Delta_j^3$ satisfies $\kappa\mathcal{E}_i = \mathcal{E}_i$ for $\kappa \in \mathbf{Z}_2 \times \mathbf{S}_3$ when we let the group act as permutations on Ω^3. Therefore we reduce our subsets to those partitioning the Ω^3, and the negatives of those partitioning Ω^3, for the case of $\mathbf{S}_3$ symmetry, but we may now not have the same relations. Therefore we have the equivariant mappings $\mathbf{Y}_3 = \begin{bmatrix} x_1^3 \\ x_2^3 \\ x_3^3 \end{bmatrix}$, call this equivariant $\mathbf{X}_2$, $\mathbf{Y}_4 = \begin{bmatrix} x_2^3 + x_3^3 \\ x_1^3 + x_3^3 \\ x_1^3 + x_2^3 \end{bmatrix}$, $\mathbf{Y}_5 = \begin{bmatrix} x_1^2(x_2 + x_3) \\ x_2^2(x_1 + x_3) \\ x_3^2(x_1 + x_2) \end{bmatrix}$, $\mathbf{Y}_6 = \begin{bmatrix} x_1(x_2^2 + x_3^2) \\ x_2(x_1^2 + x_3^2) \\ x_3(x_1^2 + x_2^2) \end{bmatrix}$, $\mathbf{Y}_7 = \begin{bmatrix} x_2^2x_3 + x_2x_3^2 \\ x_1^2x_3 + x_1x_3^2 \\ x_1^2x_2 + x_1x_2^2 \end{bmatrix}$ and $\mathbf{Y}_8 = \begin{bmatrix} x_1x_2x_3 \\ x_1x_2x_3 \\ x_1x_2x_3 \end{bmatrix}$. Now, $u\mathbf{X}_1 = \mathbf{Y}_6 + \mathbf{X}_2$ so we don't need $\mathbf{Y}_6$, $v\mathbf{X}_1 = \mathbf{Y}_5 + \mathbf{Y}_8$ so we don't need $\mathbf{Y}_5$, $v\mathbf{X}_0 = 3\mathbf{Y}_8 + \mathbf{Y}_5 + \mathbf{Y}_6 + \mathbf{Y}_7$ so we don't need $\mathbf{Y}_7$ and finally $u\mathbf{X}_0 = \mathbf{X}_2 + \mathbf{Y}_4 + \mathbf{Y}_5 + \mathbf{Y}_6 + \mathbf{Y}_7$ so we don't need $\mathbf{Y}_4$. This leaves us with just $\mathbf{X}_2$ and $\mathbf{Y}_8$.

When we orthogonally project onto $\mathbf{V}$ the equivariant $\mathbf{Y}_8$ vanishes and so we can discard. This leaves us with only $\mathbf{X}_2$, but now note that

$$\begin{aligned} &x_1(u + x_1x_2 + x_1x_3 + x_2x_3) \\ &= x_1(u + x_1x_2 + x_1x_3 + x_2x_3) + (x_1 + x_2 + x_3)(x_2x_3 - u) \\ &= -(x_2^3 + x_3^3) + 2x_1x_2x_3. \end{aligned}$$

Therefore by symmetry we have that

$$(u + x_1x_2 + x_1x_3 + x_2x_3)\mathbf{X}_1 + \mathbf{Y}_4 = \mathbf{X}_2 + 2\mathbf{Y}_8.$$

Project onto $\mathbf{V}$ and we have the same equivariant as projecting $\mathbf{X}_2$ onto $\mathbf{V}$ and so this is not needed either and so we have *no* cubic equivariants.

■

As with the $\mathbf{Z}_2 \wr \mathbf{S}_3$ case we are interested in derivatives up to third order when considering stability, therefore the general bifurcation problem on $\mathbf{R}^3$ with symmetry $\mathbf{Z}_2 \times \mathbf{S}_3$ that we will consider has the form

$$g(x_1, x_2, x_3, \lambda) = P\mathbf{X}_1 \tag{3.4.10}$$

where P is a function of u and λ.

Branching Equations

We now compute the branching equations for each isotropy, as before parametrising by t.

For isotropy $\mathbf{T}_1$ we have that $Fix(\mathbf{T}_1) = (t, -t, 0)$ and so $u = 2t^2$. This means that the first component of g gives us

$$\begin{aligned} g_1(t,\lambda) &= Pt \\ &= P(0)t + 2P_u(0)t^3 + P_\lambda(0)\lambda t + \ldots \end{aligned}$$

$P(0) = 0$ again since linear terms must vanish at 0 and so setting $g = 0$ yields the branching equation for isotropy $\mathbf{T}_1$

$$\lambda(t) = -2P_u(0)t^2/P_\lambda(0). \tag{3.4.11}$$

Similarly, for isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ we have that on $Fix(\mathbf{S}_1 \times \mathbf{S}_2)$ the invariant $u = 6t^2$ and so

$$g_1(t,\lambda) = 2Pt$$

which means that

$$g_1(t,\lambda) = 2P(0)t + 12P_u(0) + 2P_\lambda(0)\lambda t + \ldots$$

and so setting $g(t,\lambda) = 0$, and noting that $P(0)$ must vanish yields the branching equation corresponding to isotropy $\mathbf{S}_1 \times \mathbf{S}_2$

$$\lambda(t) = -6P_u(0)t^2/P_\lambda(0). \tag{3.4.12}$$

We are now ready to state the stability conditions for these branches of solutions to equation 3.1.4.

Theorem 3.4.5 *With the preceding form of equation we have the following* *i) The trivial branch corresponding to isotropy $\mathbf{Z}_2 \times \mathbf{S}_3$, with branching equation $x = 0$ is stable if $P_\lambda(0)\lambda > 0$. We assume $P_\lambda(0) < 0$.*

ii) Stability of the branch corresponding to isotropy $\mathbf{T}_1$ is undetermined at third order, but a necessary condition for stability is that

$$2P_u(0) > 0.$$

iii) The stability of the branch corresponding to isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ is undetermined to third order, but a necessary condition for stability is that

$$2P_u(0) > 0.$$

Remark 3.4.6 *In Golubitsky et al. [17] in the case of $\mathbf{D}_6$ symmetry they find that the two branches of solutions* cannot *be stable simultaneously when they work to higher orders.*

Remark 3.4.7 *So, as with $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry, there are only two possible stable solution branches with maximal isotropy, and both branches cannot be stable simultaneously.*

Proof: As it stands the equivariant $\mathbf{X}_1$ does not explicitly lie in $\mathbf{V}$, and so for the proof we use the equivalent mapping

$$\mathbf{X}_1 = \frac{1}{3}\begin{bmatrix} 2x_1 - x_2 - x_3 \\ 2x_2 - x_1 - x_3 \\ 2x_3 - x_1 - x_2 \end{bmatrix}.$$

i) Follows from the definition.

ii) In this case the only examples of $(dg)_{x_0}$ which commute with the subgroup action of $\mathbf{T}_1$ are of the form

$$\begin{pmatrix} a & b & c \\ b & a & c \\ g & g & i \end{pmatrix}$$

which has eigenvalues $a - b$ and

$$\frac{1}{2}\left[(i + b + a) \pm \sqrt{(a + b - i)^2 + 8gc}\right]$$

Where this time

$$a = \tfrac{dg_1}{dx_1} = 2P_u(0)t^2,$$

$$b = \frac{dg_1}{dx_2} = -2P_u(0)t^2,$$

$$c = \frac{dg_1}{dx_3} = 0,$$

$$g = \frac{dg_3}{dx_1} = 0,$$

and

$$i = \frac{dg_3}{dx_3} = 0.$$

Substituting into the above equations and rearranging yields the following eigenvalues for $(dg)_{x_0}$

$$4P_u(0)t^2$$

and 0 twice ($i + b + a = 0$, $a + b - i = 0$, $8gc = 0$).

For this branch to be stable we require that all the eigenvalues have non-zero real part, which occurs iff the conditions of the theorem hold (the second zero eigenvalue means that stability is undetermined to third order, but we do not attempt higher order calculations here).

iii) The general form for $(dg)_{x_0}$ which commutes with the action of $\mathbf{S}_1 \times \mathbf{S}_2$ is given by

$$\begin{pmatrix} a & d & d \\ c & b & e \\ c & e & b \end{pmatrix}.$$

which has eigenvalues $b - e$ and

$$\frac{1}{2}\left[(a + b + e) \pm \sqrt{(e + b - a)^2 + 8cd}\right]$$

where

$$a = \frac{dg_1}{dx_1} = 9P_u(0)t^2,$$
$$b = \frac{dg_2}{dx_2} = 2P_u(0)t^2,$$
$$c = \frac{dg_2}{dx_1} = -4P_u(0)t^2,$$

$$d = \frac{dg_1}{dx_2} = -4P_u(0)t^2,$$

and

$$e = \frac{dg_3}{dx_2} = 2P_u(0)t^2.$$

We immediately see that one of the eigenvalues, $b - e$, will be zero (as expected due to the orthogonal projection from $\mathbf{R}^3$ to $\mathbf{V}$).

The other two eigenvalues are given by the above formula which gives us 0 and $12P_u(0)t^2$.

Setting the real parts of the eigenvalues to being positive achieves the result stated in the theorem.

∎

Therefore, as in the case of $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry, maximal isotropy subgroups yield, up to conjugacy, only two possible stable branches of solutions. Here however, up to third order, both branches could be stable simultaneously. If one of the branches *is* stable then both branches bifurcate subcritically. Unlike the $\mathbf{Z}_2 \wr \mathbf{S}_3$ case however, the paper Field and Richardson [14] does not indicate whether or not sub-maximal isotropy subgroups support solutions to 3.1.4. This does not matter here however since there *are* no non-trivial sub-maximal isotropy subgroups, but *will* be important when we consider the general case of n coupled cells.

We now consider the case when $\mathbf{Z}_2 \times \mathbf{S}_3$ acts on the complementary irreducible subspace $\mathbf{R}$ of $\mathbf{R}^3$.

$\mathbf{Z}_2 \times \mathbf{S}_3$ acting on $\mathbf{R}$

To complete the analysis of $\mathbf{Z}_2 \times \mathbf{S}_3$ when acting on $\mathbf{R}^3$ we must consider the one-dimensional representation of $\mathbf{Z}_2 \times \mathbf{S}_3$ on the complementary subspace to $\mathbf{R}^3_0$, namely $\mathbf{R}^* = \{(x,x,x) : x \in \mathbf{R}\}$. On this space $\mathbf{S}_3 \subset \mathbf{Z}_2 \times \mathbf{S}_3$ acts trivially and $\mathbf{Z}_2$ by $\kappa(x,x,x) = -(x,x,x)$.

Proposition 3.4.8 *The isotropy subgroups of $\mathbf{Z}_2 \times \mathbf{S}_3$ acting on $\mathbf{R}^*$ are given by*

Isotropy Subgroup (Σ)	*Fixed Point Subspace*	$dim Fix(\Sigma)$
$\mathbf{Z}_2 \times \mathbf{S}_3$	$(0,0,0)$	*0*
$\mathbf{S}_3$	(x,x,x)	*1*

Proof:

This is trivial, but for completeness, $(0,0,0)$ must have symmetry and so isotropy $\mathbf{S}_3$. Any other point on $\mathbf{R}^*$ can be represented by (t,t,t) which has isotropy $\mathbf{S}_3$.

Therefore we are guaranteed a solution branch with isotropy $\mathbf{S}_3$ in $\mathbf{R}^*$.

Proposition 3.4.9 *The* $\mathbf{Z}_2 \times \mathbf{S}_3$ *invariants on* $\mathbf{R}^*$ *are generated by the one invariant* $u = x^2$ *and the equivariants are generated over the invariants by the one mapping* $\mathbf{X} = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$.

Proof: By Theorem 2.3.1 we need only one invariant and one one equivariant. It is clear that u is indeed invariant, and is the lowest order invariant, and it should also be clear that the same is true of $\mathbf{X}$. ∎

Therefore any bifurcation problem on $\mathbf{R}^*$ with symmetry $\mathbf{Z}_2 \times \mathbf{S}_3$ can be written $g(x, \lambda) = R\mathbf{X}$ where R is a function of u and λ.

Theorem 3.4.10 **i)** *The trivial branch, corresponding to isotropy* $\mathbf{Z}_2 \times \mathbf{S}_3$ *is stable iff* $R_\lambda(0)\lambda > 0$. *We assume* $R_\lambda(0) < 0$.

ii) *The solution branch corresponding to isotropy* $\mathbf{S}_3$ *is stable iff* $R_u(0) > 0$.

Proof:

i) Follows from definition.

ii) The branching equation is given by $R\mathbf{X}_1 = 0$ and so $Rx = 0$ giving $R(0)t + R_u(0)t^2 + R_\lambda(0)\lambda t = 0$ and so $\lambda(t) = -R_u(0)/R_\lambda(0)$.

Stability is given by

$$\begin{aligned} \frac{dg}{dx} &= R + 2R_u t^2 \\ &= R(0) + R_u(0)t^2 + R_\lambda(0)\lambda + 2R_u(0)t^2 \\ &= 2R_u(0)t^2. \end{aligned}$$

Requiring this to be positive yields the result.

∎

Remark 3.4.11 *The solution branch for* $\mathbf{S}_3$ *bifurcates subcritically if stable, supercritically if unstable.*

We summarize the stability results for $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry in table 3.2, and representative bifurcation diagrams are shown in Figure 3.4.

Isotropy (Σ)	Branching Equation	Signs of Eigenvalues
$\mathbf{Z}_2 \times \mathbf{S}_3$	$x = 0$	$P_\lambda(0)\lambda$ $(R_\lambda(0)\lambda)$
$\mathbf{T}_1$	$\lambda(t) = -(2P_u(0) + 3Q(0))t^2/P_\lambda(0)$	$2P_u(0) + 3Q(0)$, 0
$\mathbf{S}_1 \times \mathbf{S}_2$	$\lambda(t) = -3(2P_u(0) + 3Q(0))t^2/P_\lambda(0)$	0, $2P_u(0) + 3Q(0)$
$\mathbf{S}_3$	$\lambda(t) = -R_u(0)t^2/R_\lambda(0)$.	$R_u(0)$

Table 3.2: Stability of branches of solutions in the presence of $\mathbf{Z}_2 \times \mathbf{S}_3$ Symmetry

3.5 Comparisons Between Solutions of Equations with $\mathbf{S}_3$, $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $\mathbf{Z}_2 \times \mathbf{S}_3$ Symmetries

The first comparison that can be made between solutions to systems with the different symmetries is that for systems with $\mathbf{S}_3$ symmetry there are, generically, no possibly stable solution branches guaranteed by the Equivariant Branching Lemma, whereas when we add the internal $\mathbf{Z}_2$ symmetries, stable branches are now possible. In a modeling situation this could have a dramatic effect on the results if the existence of the $\mathbf{Z}_2$ symmetry is not intended.

We now compare the results obtained for the two different symmetries $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $\mathbf{Z}_2 \times \mathbf{S}_3$, and so see how different coupling with respect to the internal $\mathbf{Z}_2$ symmetry can affect the steady state solutions to the problem.

We start by observing that, apart from the trivial solutions (branching equation $x = 0$ for both cases), we can have, up to conjugacy, only two possible stable branches of solutions, with the $\mathbf{Z}_2 \wr \mathbf{S}_3$ case producing an additional, generically unstable, branch of solutions. In addition, in the $\mathbf{Z}_2 \wr \mathbf{S}_3$ case the branches cannot be stable simultaneously, whereas this *could* be possible in the $\mathbf{Z}_2 \times \mathbf{S}_3$ case when only working to third order (in fact they cannot both be stable, see Remark 3.4.6).

When considering the types of solutions actually seen in systems in$\mathbf{R}^3$, it is necessary to remember that until now we have been considering conjugacy classes of solutions. For example, both symmetries support solutions with corresponding isotropy

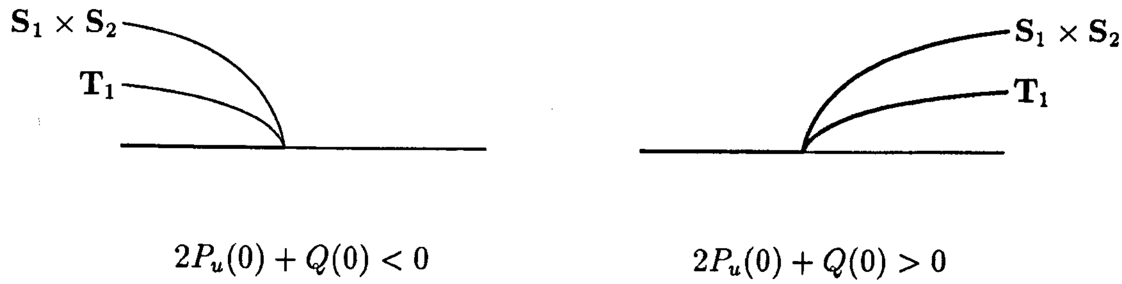


Figure 3.4: Representative bifurcation diagrams for $\mathbf{Z}_2 \times \mathbf{S}_3$ steady-state bifurcations when $P_\lambda < 0$. Thick lines denote stability undetermined to third order, thin lines instability.

$\mathbf{S}_3$, but in the $\mathbf{Z}_2 \times \mathbf{S}_3$ scenario these solutions correspond to only two half-branches of solutions, lying in the fixed point subspaces represented by (t,t,t) and $(-t,-t,-t)$ where the parameter t is taken to be positive. These solutions however only form a subset of the set of solutions conjugate to the $\mathbf{S}_3$ solution seen in the case of $\mathbf{Z}_2 \wr \mathbf{S}_3$. In this case, there are now eight half-branches of solutions with isotropy conjugate to $\mathbf{S}_3$.

Similarly, the six half-branches of solutions corresponding to conjugacies of $\mathbf{T}_1$ in $\mathbf{Z}_2 \times \mathbf{S}_3$ form a subset of the twelve solutions with isotropies conjugate to $\mathbf{W}_2$ in the $\mathbf{Z}_2 \wr \mathbf{S}_3$ case, but, unlike the $\mathbf{S}_3$ case detailed above, we also go from a subset of solutions where stability *is* possible ($\mathbf{Z}_2 \times \mathbf{S}_3$) to a set of solutions which are generically unstable ($\mathbf{Z}_2 \wr \mathbf{S}_3$).

In addition, we obtain six half-branches of solutions in the wreath product case ($\mathbf{Z}_2 \wr \mathbf{S}_3$), with symmetries conjugate to $\mathbf{W}_1$, which the cross product case ($\mathbf{Z}_2 \times \mathbf{S}_3$) does not generically support. In the $\mathbf{Z}_2 \times \mathbf{S}_3$ case, these solutions, as indeed some of the other 'extra' solutions seen in the $\mathbf{Z}_2 \wr \mathbf{S}_3$ case, *can* be seen in solutions with isotropy $\mathbf{S}_1 \times \mathbf{S}_2$.

If we think of each isotropy subgroup as being the conjugacy set of solutions corresponding to that isotropy, we can summarize the relationship between solutions obtained by different symmetries in Figure 3.5, where the superscripts '$\times$' and '$\wr$' denote that the isotropy comes from $\mathbf{Z}_2 \times \mathbf{S}_3$ or $\mathbf{Z}_2 \wr \mathbf{S}_3$ respectively, the subscripts 's' and 'u' denote possible stabilities, and an entry connected by a line in the lattice to the one above denotes that it is a subset of the one below.

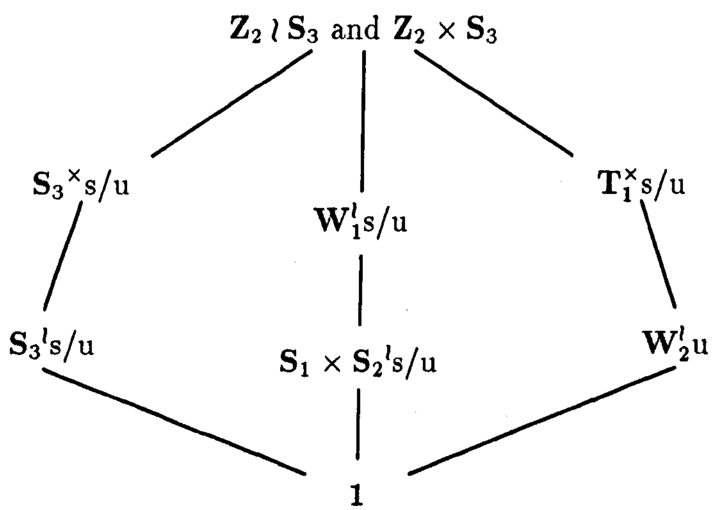


Figure 3.5: Relationships between solutions with isotropies in $\mathbf{Z}_2 \times \mathbf{S}_3$ and $\mathbf{Z}_2 \wr \mathbf{S}_3$ in $\mathbf{R}^3$

Chapter 4

Steady-State Bifurcations In n Coupled Cells

We now extend the work in the last Chapter to the case of n coupled ODEs, where each cell has an internal $\mathbf{Z}_2$ symmetry and these cells are coupled identically so as to produce a global symmetry of at least $\mathbf{S}_n$, i.e. we require 'all-to-all' coupling.

As in the case of three cells, there are again two very natural types of coupling with respect to the internal symmetries, and they can be formed by a direct extension of the method of coupling used for three cells, giving us coupling that is

- invariant under an individual application of an internal $\mathbf{Z}_2$, leading to $\mathbf{Z}_2 \wr \mathbf{S}_n$ symmetry, or
- equivariant under an application of an internal $\mathbf{Z}_2$ symmetry, so that the $\mathbf{Z}_2$ action must be applied to *all* the cells, leading to $\mathbf{Z}_2 \times \mathbf{S}_n$ symmetry.

As before we consider a bifurcation problem

$$g(x, \lambda) = 0$$

where g is Γ-equivariant, where now $\Gamma = \mathbf{S}_n,\ \mathbf{Z}_2 \wr \mathbf{S}_n$ or $\mathbf{Z}_2 \times \mathbf{S}_n$.

The Group Actions

Without loss of generality we let each dynamical system for an individual cell live on $\mathbf{R}$, so that the n copies will live on $\mathbf{R}^n$. As in the case of three maps, the natural $\mathbf{Z}_2$ action to take is multiplication by -1, i.e. if $\kappa \in \mathbf{Z}_2$ and $x_i \in \mathbf{R}$, then $\kappa x_i = -x_i$, and we also let $\mathbf{S}_n$ act by permutation of indices. So if $\rho \in \mathbf{S}_n$ then $\rho x_i = x_{\rho(i)}$.

4.1 Steady-State Bifurcations with $\mathbf{S}_n$ Symmetry

We first consider the case where there are no internal $\mathbf{Z}_2$ symmetries, so that we can compare to the cases where there is an internal symmetry present.

The Group Action and Irreducible Representation

Let $\mathbf{S}_n$ act on $\mathbf{V} = \{x \in \mathbf{R}^n : x_1 + \ldots + x_n = 0\}$ by permutation of indices, i.e. if $\rho \in \mathbf{S}_n$ then $\rho\underline{x} = (x_{\rho(1)}, \ldots, x_{\rho(n)})$. This action is absolutely irreducible and so we can again apply the Equivariant Branching Lemma. At the end of the section we also consider the irreducible action of $\mathbf{S}_n$ on $\mathbf{R} \simeq \{(x, \ldots, x) : x \in \mathbf{R}\}$ to compare results for coupled ODE's on $\mathbf{R}^n$ with the results where internal symmetries are also present.

Proposition 4.1.1 *Up to conjugacy, the isotropy subgroups of* $\mathbf{S}_n$ *are precisely those subgroups of the form* $\mathbf{S}_{k_1} \times \mathbf{S}_{k_2} \times \ldots \times \mathbf{S}_{k_l}$ *where* $k_1 + \ldots + k_l = n$, *and*

$$Fix(\mathbf{S}_{k_1} \times \ldots \times \mathbf{S}_{k_l}) = (\underbrace{x_1, \ldots, x_1}_{k_1}, \ldots, \underbrace{x_l, \ldots, x_l}_{k_l}).$$

By conjugacy we may also assume that $k_1 \leq \ldots \leq k_l$.

Proof: It is clear that these subgroups are isotropy subgroups, it now suffices to show that they are the only ones. By conjugacy we can take *any* vector $\underline{x}$ of $\mathbf{V}$ and rearrange the variables so that it becomes of the form $(\underbrace{x_1, \ldots, x_1}_{k_1}, \ldots, \underbrace{x_l, \ldots, x_l}_{k_l})$, where $k_1 \leq \ldots \leq k_l$, which has isotropy precisely $\mathbf{S}_{k_1} \times \ldots \times \mathbf{S}_{k_l}$. This proves the result. ∎

Corollary 4.1.2 *Those isotropy subgroups having one-dimensional fixed point subspaces are those of the form* $\mathbf{S}_k \times \mathbf{S}_{n-k}$ *which have fixed point subspaces of the form* $(\underbrace{(n-k)x, \ldots, (n-k)x}_{k}, \underbrace{-kx, \ldots, -kx}_{n-k})$.

Proof: Follows directly from the preceding Proposition ∎

Therefore, by the Equivariant Branching Lemma, there exist solution branches to our bifurcation problem having as their isotropy precisely those subgroups with one-dimensional fixed point subspaces, namely those subgroups of the form $\mathbf{S}_k \times \mathbf{S}_{n-k}$.

Further, we do not need to consider sub-maximal isotropy subgroups, due to Field and Richardson [14] Theorem 9.4.1. They show that sub-maximal isotropy subgroups of $\mathbf{S}_n$ do not support solutions, in their notation sub-maximal isotropy subgroups of $W(A_{k-1})$.

Stabilities of Solutions

We must now consider the stabilities of these solution branches, which we do in the same way as in the earlier work by considering our bifurcation problem to be written as a function of the invariant and equivariant polynomials.

Proposition 4.1.3 *The details of the reduced equation are as follows*

i) every $\mathbf{S}_n$ invariant germ $f : \mathbf{R}^n \to \mathbf{R}$ has the form, up to third order, $f(u,v)$ where

$$u = \sum_{i=1}^{n} x_i^2 \text{ and } v = \sum_{i=1}^{n} x_i^3;$$

and

ii) the module of $\mathbf{S}_n$ equivariants is generated, up to at least third order over the invariants, by the mappings (for $n > 3$)

$$\mathbf{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} (n-1)x_1^2 - x_2^2 - \ldots - x_n^2 \\ (n-1)x_2^2 - x_1^2 - \ldots - x_n^2 \\ \vdots \\ (n-1)x_n^2 - x_1^2 - \ldots - x_{n-1}^2 \end{bmatrix}$$

and

$$\mathbf{X}_3 = \begin{bmatrix} (n-1)x_1^3 - x_2^3 - \ldots - x_n^3 \\ (n-1)x_2^3 - x_1^3 - \ldots - x_n^3 \\ \vdots \\ (n-1)x_n^3 - x_1^3 - \ldots - x_{n-1}^3 \end{bmatrix}.$$

Proof: We show the list of invariants and equivariants by the methods outlined in Chapter 2, and calculate up to third order. To do this we take the natural extension of the method outlined for the case of $n = 3$ to the case of general n.

Invariants We begin with the linear invariants. As for the case of $\mathbf{S}_3$ symmetry we have that $\Lambda_0^k = \Lambda^k$ and so for the linear case we have that $\Lambda^1 = \{x_1, \ldots, x_n\}$ with only one partition $\Delta_1^1 = \Lambda^1$ which when summed gives us the only invariant as $x_1 + \ldots + x_n = 0$ and so there are no non-zero linear invariants.

For the quadratic invariants there are only two partitions $\Delta_1^2 = \{x_1^2, \dots, x_n^2\}$ and $\Delta_2^2 = \{x_1x_2, \dots, x_{n-1}x_n\}$ leading to the two invariants $u_1 = \Sigma_i x_i^2$ and $u_2 = \Sigma_{i<j} x_i x_j$. However, $0 = (x_1 + \dots + x_n)^2 = u_1 + 2u_2$ and so we need only one of these, say $u = u_1$.

The cubic partitions of Λ^3 are $\Delta_1^3 = \{x_1^3, \dots, x_n^3\}$, $\Delta_2^3 = \{x_1^2x_2, \dots, x_n^2x_{n-1}\}$ and $\Delta_3^3 = \{x_1x_2x_3, \dots, x_{n-1}x_{n-1}x_n\}$. These give us the three invariants

$$v_1 = \Sigma_i x_i^3, \; v_2 = \Sigma_{i \neq j} x_i^2 x_j$$

and

$$v_3 = \Sigma_{i<j<k} x_i x_j x_k.$$

But we have the relations $0 = (x_1 + \dots + x_n)u_1 = v_1 + v_2$ and so $v_1 = -v_2$ and also $0 = (x_1 + \dots + x_n)u_2 = 2v_2 + v_3$ and so $2v_2 = -v_3$. Therefore we need only one cubic invariant, say $v = v_1$.

Equivariants We now consider the equivariant mappings by, as before, considering the generating partitions. We do this first by considering the possible equivariants on the full space $\mathbf{R}^n$, we then use the condition $x_1 + \dots + x_n = 0$ to find relations, and then we project the equivariant mappings we have left onto the space $\mathbf{V}$.

We begin with the linear equivariants where we have only two generating partitions, $\Theta_1^1 = \left\langle \begin{bmatrix} x_1 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_1 = \mathbf{X}_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\Theta_2^1 = \left\langle \begin{bmatrix} x_2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_2 = \begin{bmatrix} \Sigma_{i\neq 1} x_i \\ \vdots \\ \Sigma_{i \neq n} x_i \end{bmatrix}$ with $\mathbf{Y}_1 + \mathbf{Y}_2 = 0$ and so we need only one.

For the quadratic equivariants we have generating partitions $\Theta_1^2 = \left\langle \begin{bmatrix} x_1^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ giving the equivariant $\mathbf{Y}_3 = \mathbf{X}_2 = \begin{bmatrix} x_1^2 \\ \vdots \\ x_n^2 \end{bmatrix}$, $\Theta_2^2 = \left\langle \begin{bmatrix} x_2^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ giving the equivariant $\mathbf{Y}_4 = \begin{bmatrix} \Sigma_{i \neq 1} x_i^2 \\ \vdots \\ \Sigma_{i\neq n} x_i^2 \end{bmatrix}$, but $\mathbf{Y}_3 + \mathbf{Y}_4$ is purely invariant under $\mathbf{S}_n$ and so we need only one of these. We also have $\Theta_3^2 = \left\langle \begin{bmatrix} x_1x_2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ giving $\mathbf{Y}_5 = \begin{bmatrix} x_1\Sigma_{i\neq 1}x_i \\ \vdots \\ x_n\Sigma_{i\neq n}x_i \end{bmatrix}$ and $\Theta_4^2 = \left\langle \begin{bmatrix} x_2x_3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ giving $\mathbf{Y}_6 = \begin{bmatrix} \Sigma_{1<i<j}x_ix_j \\ \vdots \\ \Sigma_{i<j<n}x_ix_j \end{bmatrix}$ but

$$0 = (x_1 + \ldots + x_n)\mathbf{X}_1 = \mathbf{X}_2 + \mathbf{Y}_5$$

and so we don't need $\mathbf{Y}_5$ and, furthermore, $\mathbf{Y}_5 + \mathbf{Y}_6$ is purely invariant. Finally consider the cubic equivariants.

$\Theta_1^3 = \left\langle \begin{bmatrix} x_1^3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ gives the equivariant $\mathbf{Y}_7 = \begin{bmatrix} x_1^3 \\ \vdots \\ x_n^3 \end{bmatrix}$ call $\mathbf{X}_3 = \mathbf{Y}_7$. $\Theta_2^3 = \left\langle \begin{bmatrix} x_2^3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ gives the mapping $\mathbf{Y}_8 = \begin{bmatrix} \Sigma_{i\neq 1}x_i^3 \\ \vdots \\ \Sigma_{i\neq n}x_i^3 \end{bmatrix}$ but $\mathbf{Y}_7 + \mathbf{Y}_8$ is purely invariant, and so we need only one of these. $\Theta_3^3 = \left\langle \begin{bmatrix} x_1^2x_2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ gives $\mathbf{Y}_9 = \begin{bmatrix} x_1^2\Sigma_{i\neq 1}x_i \\ \vdots \\ x_n^2\Sigma_{i\neq n}x_i \end{bmatrix}$ but $\mathbf{Y}_9 = -\mathbf{X}_3$ since $(x_1 + \ldots + x_n)\mathbf{X}_2 = 0$. $\Theta_4^3 = \left\langle \begin{bmatrix} x_1x_2^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ leads to the equivariant $\mathbf{Y}_{10} = \begin{bmatrix} x_1\Sigma_{i\neq 1}x_i^2 \\ \vdots \\ x_n\Sigma_{i\neq n}x_i^2 \end{bmatrix}$ but we have the relation $\mathbf{Y}_{10} = u\mathbf{X}_1 - \mathbf{X}_3$ so discard. $\Theta_5^3 = \left\langle \begin{bmatrix} x_2x_3^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ gives us $\mathbf{Y}_{11} = \begin{bmatrix} \Sigma_{i\neq j\neq 1}x_i^2x_j \\ \vdots \\ \Sigma_{i\neq j\neq n}x_i^2x_j \end{bmatrix}$ but $\mathbf{Y}_{11} + \mathbf{Y}_{10} + \mathbf{Y}_9 + \mathbf{Y}_8 + \mathbf{Y}_7 = 0$. Finally we have the two partitions $\Theta_6^3 = \left\langle \begin{bmatrix} x_1x_2x_3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ giving $\mathbf{Y}_{12} = \begin{bmatrix} x_1\Sigma_{1<i<j}x_ix_j \\ \vdots \\ x_n\Sigma_{i<j<n}x_ix_j \end{bmatrix}$ and $\Theta_7^3 = \left\langle \begin{bmatrix} x_2x_3x_4 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$. giving $\mathbf{Y}_{13} = \begin{bmatrix} \Sigma_{1<i<j<k}x_ix_jx_k \\ \vdots \\ \Sigma_{i<j<k<n}x_ix_jx_k \end{bmatrix}$. Now note that $\mathbf{Y}_{12} + \mathbf{Y}_{13}$ is $\mathbf{S}_n$ invariant

and also that $\mathbf{Y}_{12} = u_2\mathbf{X}_1 - \mathbf{Y}_9$, leaving us with only one cubic equivariant $\mathbf{X}_3$. Finally we apply a projection onto the space $\mathbf{V}$ by the mapping

$$\mathbf{X}_i \mapsto \mathbf{X}_i - (1, \ldots, 1) \cdot \mathbf{X}_i$$

and scaling where necessary to get rid of fractions. This then gives the results of the Proposition and we are done.

∎

Therefore we can now write g up to third order as

$$g = P\mathbf{X}_1 + Q\mathbf{X}_2 + R\mathbf{X}_3 + \ldots$$

where P, Q and R are functions of u, v and λ.

Branching Equations

When we restrict ourselves to the fixed point subspace of isotropy $\mathbf{S}_k \times \mathbf{S}_{n-k}$, which when parametrised by t is given by

$$Fix(\mathbf{S}_k \times \mathbf{S}_{n-k}) = (\underbrace{(n-k)t, \ldots, (n-k)t}_{k}, \underbrace{-kt, \ldots, -kt}_{n-k}),$$

we have that $u = nk(n-k)t^2$ and $v = (n-2k)nkt^3$. Setting the first component of g to zero gives us

$$(n-k)Pt + (n-k)(n^2 - 2nk)Qt^2 + (n-k)(n^3 + 3nk^2 - 3n^2k)Rt^3 = 0$$

or

$$P + (n^2 - 2nk)Qt + (n^3 + 3nk^2 - 3n^2k)Rt^2 = 0$$

and so

$$P_\lambda(0)\lambda + nk(n-k)P_u(0)t^2 + (n^2 - 2nk)Q(0)t + (n^3 + 3nk^2 - 3n^2k)R(0)t^2 = 0.$$

This gives us a branching equation of

$$\lambda(t) = -\left[nk(n-k)P_u(0)t^2 + (n^2 - 2nk)Q(0)t + (n^3 + 3nk^2 - 3n^2k)R(0)t^2\right] / P_\lambda(0). \tag{4.1.1}$$

If $n \neq 2k$ then $\lambda(t)$ is transcritical, $\lambda'(0) \neq 0$, and so by Golubitsky et al. [17] the corresponding solution branch is unstable. If however $n = 2k$ then we have isotropy $\mathbf{S}_k \times \mathbf{S}_k$, with branching equation

$$\lambda(t) = -\left[2k^3 P_u(0)t^2 + 2k^3 R(0)t^2\right]/P_\lambda(0)$$

or

$$\lambda(t) = -2k^3\left[P_u(0) + R(0)\right]t^2/P_\lambda(0). \tag{4.1.2}$$

We must now consider the stability of this branch of solutions.

Theorem 4.1.4 *The branches of solutions corresponding to the isotropy subgroups conjugate to $\mathbf{S}_k \times \mathbf{S}_k \subset \mathbf{S}_n$ are generically unstable.*

Proof: To prove this we first note that as it stands the equivariant $\mathbf{X}_1$ does not explicitly lie in $\mathbf{V}$ and so for the proof we use the equivalent mapping

$$\mathbf{X}_1 = \frac{1}{n}\begin{bmatrix} (n-1)x_1 - x_2 - \ldots - x_n \\ \vdots \\ \vdots \\ (n-1)x_n - x_1 - \ldots - x_{n-1} \end{bmatrix}$$

(add $x_1 + \ldots + x_n = 0$ to each row to recover original form).

We now apply a change of variables. Let $y_1 = x_1$, $y_2 = x_{k+1}$, $y_3 = x_2 - x_1, \ldots, y_{k+1} = x_k - x_1$, $y_{k+2} = x_{k+2} - x_{k+1}, \ldots, y_n = x_n - x_{k+1}$. This has decomposed $\mathbf{V}$ into it's isotypic components and in our new coordinates, after parametrising by t, we have

$$Fix(\mathbf{S}_k \times \mathbf{S}_k) = (kt, -kt, 0, \ldots, 0).$$

This change of variables also gives us the invariants as

$$u = ky_1^2 + ky_2^2 + y_3^2 + \ldots + y_n^2 + 2y_1(y_3 + \ldots + y_{k+1}) + 2y_2(y_{k+2} + \ldots + y_n)$$

and

$$\begin{aligned} v \;=\; & ky_1^3 + ky_2^3 + y_3^3 + \ldots + y_n^3 + 3y_1(y_3^2 + \ldots + y_{k+1}^2) \\ & + 3y_2(y_{k+2}^2 + \ldots + y_n^2) + 3y_1^2(y_3 + \ldots + y_{k+1}) + 3y_2^2(y_{k+2} + \ldots + y_n) \end{aligned}$$

and the equivariants

$$\mathbf{Y}_1 = \begin{bmatrix} k(y_1 - y_2) - (y_3 + \ldots + y_n) \\ k(y_2 - y_1) - (y_3 + \ldots + y_n) \\ 2ky_3 \\ \vdots \\ 2ky_n \end{bmatrix}$$

$$\mathbf{Y}_2 = \begin{bmatrix} ky_1^2 - ky_2^2 - (y_3^2 + \ldots + y_n^2) - 2y_1(y_3 + \ldots + y_{k+1}) - 2y_2(y_{k+2} + \ldots + y_n) \\ ky_2^2 - ky_1^2 - (y_3^2 + \ldots + y_n^2) - 2y_1(y_3 + \ldots + y_{k+1}) - 2y_2(y_{k+2} + \ldots + y_n) \\ 2ky_3^2 + 4ky_1y_3 \\ \vdots \\ 2ky_{k+1}^2 + 4ky_1y_{k+1} \\ 2ky_{k+2}^2 + 4ky_2y_{k+2} \\ \vdots \\ 2ky_n^2 + 4ky_2y_n \end{bmatrix}$$

and

$$\mathbf{Y}_3 = \begin{bmatrix} ky_1^3 - ky_2^3 - (y_3^3 + \ldots + y_n^3) - 3y_1(y_3^2 + \ldots + y_{k+1}^2) \\ -3y_2(y_{k+2}^2 + \ldots + y_n^2) - 3y_1^2(y_3 + \ldots + y_{k+1}) - 3y_2^2(y_{k+2} + \ldots + y_n) \\ ky_2^3 - ky_1^3 - (y_3^3 + \ldots + y_n^3) - 3y_1(y_3^2 + \ldots + y_{k+1}^2) \\ -3y_2(y_{k+2}^2 + \ldots + y_n^2) - 3y_1^2(y_3 + \ldots + y_{k+1}) - 3y_2^2(y_{k+2} + \ldots + y_n) \\ ky_3^3 + 6ky_1y_3(y_1 + y_3) \\ \vdots \\ ky_{k+1}^3 + 6ky_1y_{k+1}(y_1 + y_{k+1}) \\ ky_{k+2}^3 + 6ky_2y_{k+2}(y_2 + y_{k+2}) \\ \vdots \\ ky_n^3 + 6ky_2y_n(y_2 + y_n) \end{bmatrix}$$

and we now have

$$g = P\mathbf{Y}_1 + Q\mathbf{Y}_2 + R\mathbf{Y}_3 + \ldots.$$

Since we have effectively decomposed $\mathbf{V}$ into it's isotypic components, $(dg)_{x_0}$ will have block form

$$\begin{pmatrix} \frac{dg_1}{dy_1} & \cdots & \frac{dg_1}{dy_n} \\ \vdots & & \vdots \\ \frac{dg_n}{dy_1} & \cdots & \frac{dg_n}{dy_n} \end{pmatrix} = \begin{array}{cc} & \begin{array}{ccc} 2 & k-1 & n-k-1 \end{array} \\ \begin{array}{c} 2 \\ k-1 \\ n-k-1 \end{array} & \left(\begin{array}{c|c|c} A & F & G \\ \hline D & B & J \\ \hline E & H & C \end{array}\right) \end{array}$$

where D, E, F and G are all zero matrices (this is easy to see by direct computation also), A is of the form

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and B and C are both diagonal matrices with diagonal entries e and f where

$$a = 2k^3P_u(0)t^2 + 2k^2Q(0)t + 2k^3R(0)t^2,$$

$$b = -2k^3P_u(0)t^2 + 2k^2Q(0)t - 2k^3R(0)t^2,$$

$$c = -2k^3P_u(0) - 2k^2Q(0)t - 2k^3R(0)t^2,$$

$$d = 2k^3P_u(0) - 2k^2Q(0)t + 2k^3R(0)t^2,$$

$$e = 4k^2Q(0)t + 4k^3R(0)t^2$$

and

$$f = -4k^2Q(0) + 4k^3R(0)t^2.$$

The eigenvalues of $(dg)_{x_0}$ are given by $\frac{1}{2}\left[(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}\right]$, e and f. We have $ad - bc = 0$ and so we have eigenvalues of

$$a + d = 4k^3P_u(0)t^2 + 4k^3R(0)t^2,$$

$$0,$$

$$e = 4k^2Q(0)t + 4k^3R(0)t^2$$

and

$$f = -4k^2Q(0)t + 4k^3R(0)t^2.$$

Therefore to the lowest necessary order, first order, we have eigenvalues of 0 (as expected due to our orthogonal projection onto $\mathbf{V}$), $4k^2Q(0)t$ and $-4k^2Q(0)t$ and so the branch must be generically unstable, since we require both $Q(0) > 0$ and $Q(0) < 0$ for stability. If we apply the degeneracy condition $Q(0) = 0$ then to second order we have eigenvalues of $4k^3P_u(0)t^2 + 4k^3R(0)t^2$ and $4k^3R(0)t^2$ and so the branch is stable if both these are positive, ∎

$\mathbf{S}_n$ acting irreducibly on R

The action of $\mathbf{S}_n$ on $\mathbf{R}^* = \{(x, \ldots, x) : x \in \mathbf{R}\}$ reduces to the same problem that was considered in the $\mathbf{S}_3$ case (Theorem 3.2.4) and so generically we will have no stable branches of solutions lying in this subspace.

Therefore we have that generically, at the point of bifurcation, problems with $\mathbf{S}_n$ symmetry do not have any stable solution branches guaranteed by the Equivariant Branching Lemma. There may however be stable branches produced by secondary bifurcations, but we do not consider them here.

4.2 Steady-State Bifurcations with $\mathbf{Z}_2 \wr \mathbf{S}_n$ Symmetry

The first case to consider once we have the additional $\mathbf{Z}_2$ symmetry is the wreath product one, which, as in the three cell case, turns out to be the easier case of the two.

The Group Action and Irreducible Representation

The irreducible representation of $\mathbf{Z}_2 \wr \mathbf{S}_n$ that we take is a direct extension of the representation used for the case of $n = 3$. That is, let $\rho \in \mathbf{S}_n$, $\underline{\kappa} \in \mathbf{Z}_2 \times \ldots \times \mathbf{Z}_2$ and $w \in \mathbf{Z}_2 \wr \mathbf{S}_n$ then we can represent the action of $\mathbf{Z}_2 \wr \mathbf{S}_n$ on $\mathbf{R}^n$ by

$$wx_i = \kappa(i)x_{\rho(i)}$$

where $x_i \in \mathbf{R}$, $\kappa(i) = \pm 1$, so that we have

$$(\underline{\kappa}, \rho).\underline{x} = \begin{bmatrix} \kappa(1)x_{\rho(1)} \\ \kappa(2)x_{\rho(2)} \\ \vdots \\ \kappa(n)x_{\rho(n)} \end{bmatrix}$$

This action of $\mathbf{Z}_2 \wr \mathbf{S}_n$ on $\mathbf{R}^n$ is absolutely irreducible, and so we can apply the Equivariant Branching Lemma directly to find possible branches of solutions to our bifurcation problem.

Proposition 4.2.1 *Up to conjugacy, the list of all isotropy subgroups of* $\mathbf{Z}_2 \wr \mathbf{S}_n$ *is given by* $\mathbf{Z}_2 \wr \mathbf{S}_n$, $\mathbf{S}_n$, $\mathbf{1}$, *and subgroups of the form* $\mathbf{S}_{l_1} \times \ldots \times \mathbf{S}_{l_{k-1}} \times (\mathbf{Z}_2 \wr \mathbf{S}_{l_k})$ *where* $l_1 + \ldots + l_k = n$ *and* $0 < l_1 \leq l_2 \leq \ldots \leq l_{k-1}$. *With the corresponding fixed-point subspaces having dimensions* $0, 1, n$ *and* $k-1$ *respectively, with forms* $(0, \ldots, 0)$, $(x, \ldots, x)$, $(x_1, \ldots, x_n)$ *and* $(x_1, \ldots, x_1, \ldots, x_{k-1}, \ldots, x_{k-1}, 0, \ldots, 0)$.

Proof:

It is clear that these are isotropy subgroups, we now show that they are the only ones.

Vectors of the form $(0, \ldots, 0)$ have isotropy $\mathbf{Z}_2 \wr \mathbf{S}_n$ so assume that we have a non-zero vector $(x_1, \ldots, x_n)$. By conjugacy we may assume that all the non-zero x_i are positive, if not multiply by -1.

Now, if all the x_i are non-zero, then by conjugacy we can put all the equal x_i into a block, and obtain a vector of form

$$(\underbrace{y_1, \ldots, y_1}_{l_1}, \underbrace{y_2, \ldots, y_2}_{l_2}, \ldots, \underbrace{y_k, \ldots, y_k}_{l_k})$$

which has isotropy $\mathbf{S}_{l_1} \times \ldots \times \mathbf{S}_{l_k}$. The other alternative is that there are some zero entries, in which case, by conjugacy, put them at the end in a block, giving a vector of form

$$(\underbrace{y_1, \ldots, y_1}_{l_1}, \ldots, \underbrace{y_k, \ldots, y_k}_{l_k}, \underbrace{0, \ldots, 0}_{l_{k+1}})$$

which has isotropy $\mathbf{S}_{l_1} \times \ldots \times \mathbf{S}_{l_k} \times \mathbf{Z}_2 \wr \mathbf{S}_{l_{k+1}}$. There are no other vectors to consider, so we are done.

∎

Corollary 4.2.2 *Those isotropy subgroups having one-dimensional fixed point subspaces are* $\mathbf{W}_k = \mathbf{S}_k \times (\mathbf{Z}_2 \wr \mathbf{S}_{n-k})$ *and* $\mathbf{S}_n$, *with corresponding fixed point subspaces* $(\underbrace{x, \ldots, x}_{k}, \underbrace{0, \ldots, 0}_{n-k})$ *and* $(x, \ldots, x)$ *respectively. To make calculations easier to manage we identify* $\mathbf{S}_n$ *with* $\mathbf{W}_n$.

Proof: Follows directly from proposition 4.2.1. ∎

Field and Richardson [14] have shown that, generically, sub-maximal isotropy subgroups of $\mathbf{Z}_2 \wr \mathbf{S}_n$ do not support solutions ([14] Theorem 9.5.1 where, in their notation, the group in question is the reflection group $W(B_k)$).Therefore the symmetry only gives the n smooth branches of solutions to $g(x, \lambda)$ guaranteed by the Equivariant Branching Lemma, corresponding to isotropies $\mathbf{W}_1, \mathbf{W}_2, \ldots, \mathbf{W}_{n-1}$ and $\mathbf{W}_n = \mathbf{S}_n$, which have one-dimensional fixed point subspaces. We now consider the stabilities of these solutions on our irreducible subspace.

Stabilities of Solutions

We consider the stabilities of the solution branches by considering the equations as functions of the invariant and equivariant polynomial germs of $\mathbf{Z}_2 \wr \mathbf{S}_n$ acting on $\mathbf{R}^n$.

Proposition 4.2.3 *The details of the reduced equation are as follows*

i) every $\mathbf{Z}_2 \wr \mathbf{S}_n$ invariant germ $f : \mathbf{R}^n \to \mathbf{R}$ has the form, up to sixth order, $f(u, v, w)$ where

$$u = \sum_{i=1}^{n} x_i^2, \quad v = \sum_{i<j} x_i^2 x_j^2, \quad w = \sum_{i<j<k} x_i^2 x_j^2 x_k^2;$$

and

ii) the module of $\mathbf{Z}_2 \wr \mathbf{S}_n$ equivariants is generated, up to at least fifth order over the invariants, by the mappings

$$\mathbf{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{X}_2 = \begin{bmatrix} x_1^3 \\ x_2^3 \\ \vdots \\ x_n^3 \end{bmatrix} \textit{ and } \mathbf{X}_3 = \begin{bmatrix} x_1^5 \\ x_2^5 \\ \vdots \\ x_n^5 \end{bmatrix} (\textit{when } n > 4).$$

Proof:

We show the list of invariants and equivariants by the methods outlined in Chapter 2, and calculate up to only third order.

Invariants We begin with the linear invariants. For symmetry $\mathbf{Z}_2 \wr \mathbf{S}_n$ on $\mathbf{R}^n$ we have that $\Lambda^1 = \{x_1, \ldots, x_n, -x_1, \ldots, -x_n\}$ which does not partition any further, and so the only linear invariant is given by $\Sigma_i(x_i - x_i) = 0$, and so we have no non-zero linear invariants.

Now consider the quadratic invariants. We have that

$$\Lambda^2 = \{x_1^2, \ldots, x_n^2, x_1x_2, \ldots, x_{n-1}x_n, -x_1x_2, \ldots, -x_{n-1}x_n\}$$

which partitions into $\Delta_1^2 = \{x_1^2, \ldots, x_n^2\}$ giving $u = x_1^2 + \ldots + x_n^2$ and $\Delta_2^2 = \{x_1x_2, \ldots, x_{n-1}x_n, -x_1x_2, \ldots, -x_{n-1}x_n\}$ which gives zero when summed, and so there is only one quadratic invariant.

All the partitions of Λ^3 will contain elements that cancel in pairs when summed, i.e. if $h(x) \in \Delta_i^3$ then so is $-h(x)$, and so we have no non-zero cubic invariants.

Equivariants We now consider the equivariant mappings. Before getting into the calculations though we note that, similarly to the $\mathbf{Z}_2 \wr \mathbf{S}_3$ case, generating partitions for $\mathbf{Z}_2 \wr \mathbf{S}_n$ equivariants must be of the form $\Theta_i \cup_i \gamma_j \Theta_i$ where $\gamma_j \in \Gamma$, and the Θ_i are generating partitions for $\mathbf{S}_n$ on $\mathbf{R}^n$.

We begin with the linear equivariants where we have $\Theta_1^1 = \left\langle \begin{bmatrix} x_1 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant $\mathbf{Y}_1 = \mathbf{X}_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\Theta_2^1 = \left\langle \begin{bmatrix} x_2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ which also contains all the elements of the form $\left\langle \begin{bmatrix} -x_2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ and so the elements cancel in pairs and we have only one linear equivariant.

For the quadratic equivariants we have, for example, generating partitions like $\Theta_1^2 = \left\langle \begin{bmatrix} x_1^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ which is the same partition as $\Theta_1^2 = \left\langle \begin{bmatrix} -x_1^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ and so elements cancel in pairs. In fact if $\epsilon_i(x)$ is a member of a partition Θ_i^2 then so is $-\epsilon_i(x)$ due to the element $[(-1, \ldots, -1), id] \in \mathbf{Z}_2 \wr \mathbf{S}_n$ and so there are no non-zero quadratic equivariants.

Finally consider the cubic equivariants.

$\Theta_1^3 = \left\langle \begin{bmatrix} x_1^3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ gives the equivariant $\mathbf{Y}_2 = \begin{bmatrix} x_1^3 \\ \vdots \\ x_n^3 \end{bmatrix}$ call $\mathbf{X}_2 = \mathbf{Y}_2$.

$\Theta_2^3 = \left\langle \begin{bmatrix} x_2^3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ is the same as the subset $\left\langle \begin{bmatrix} -x_2^3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$, i.e. the elements cancel in pairs, summing to zero, so discard. Similarly, $\Theta_3^3 = \left\langle \begin{bmatrix} x_1^2 x_2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ is also generated by $\left\langle \begin{bmatrix} -x_1^2 x_2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ and terms cancel in pairs producing a zero sum. $\Theta_4^3 = \left\langle \begin{bmatrix} x_1 x_2^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ leads to the equivariant $\mathbf{Y}_3 = \begin{bmatrix} x_1 \Sigma_{i \neq 1} x_i^2 \\ \vdots \\ x_n \Sigma_{i \neq n} x_i^2 \end{bmatrix}$ but we have the relation $\mathbf{Y}_3 = u\mathbf{X}_1 - \mathbf{X}_2$ so discard. $\Theta_5^3 = \left\langle \begin{bmatrix} x_2 x_3^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ is also generated by $\left\langle \begin{bmatrix} -x_2 x_3^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ so terms cancel in pairs, as do the terms in $\Theta_6^3 = \left\langle \begin{bmatrix} x_1 x_2 x_3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$. Finally we also get elements cancelling in pairs in $\Theta_7^3 = \left\langle \begin{bmatrix} x_2 x_3 x_4 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$.

We have now proved the Proposition.

∎

Therefore, the general bifurcation problem on $\mathbf{R}^n$ with symmetry $\mathbf{Z}_2 \wr \mathbf{S}_n$ has the form, up to sufficient order for derivatives of at least second order,

$$g(x_1, \ldots, x_n, \lambda) = P\mathbf{X}_1 + Q\mathbf{X}_2 + \ldots$$

where P and Q are functions of u and λ.

Branching Equations

We now consider the branching equations for our solution branches corresponding to the isotropy subgroups of $\mathbf{Z}_2 \wr \mathbf{S}_n$. It turns out they involve fairly simple calculations. We parametrise each fixed point subspace by t.

For isotropy $\mathbf{W}_k$ we have that $u = kt^2$ and $Fix(\mathbf{W}_k) = (\overbrace{t,\ldots,t}^{k},\overbrace{0,\ldots,0}^{n-k})$ so that

$g|_{FixW_k} = P(kt^2,\lambda)t + Q(kt^2,\lambda)t^3 + \ldots = P(0)t + P_\lambda(0)\lambda t + kP_u(0)t^3 + Q(0)t^3 + \ldots$

Linear terms must vanish at zero (see Golubitsky et al. [17]), and so $P(0) = 0$, and we obtain

$$\lambda(t) = -(kP_u(0) + Q(0))t^2/P_\lambda(0).$$

This gives us all the necessary branching equations by setting $1 \le k \le n$ for isotropy subgroup $\mathbf{W}_k$ and we are now in a position to work out the stabilities of our solution branches.

Theorem 4.2.4 *i) The trivial branch, corresponding to isotropy $\mathbf{Z}_2 \wr \mathbf{S}_n$, branching equation $x = 0$, is stable if $P_\lambda(0)\lambda > 0$. We assume that $P_\lambda(0) < 0$.*

ii) The branch corresponding to isotropy $\mathbf{W}_1$ is stable iff both

$$P_u(0) + Q(0) > 0 \text{ and } Q(0) < 0.$$

iii) The branches corresponding to isotropies $\mathbf{W}_k$, where $2 \le k \le n-1$, are generically unstable, the genericity condition being $Q(0) \ne 0$. If $Q(0) = 0$ then all these branches are stable iff $P_u(0) > 0$.

iv) The branch of solutions corresponding to isotropy $\mathbf{W}_n \equiv \mathbf{S}_n$ is stable iff both

$$nP_u(0) + Q(0) > 0 \text{ and } Q(0) > 0.$$

Proof:

i) Follows from definitions and fact that the action of $\mathbf{Z}_2 \wr \mathbf{S}_n$ on $\mathbf{R}^n$ is absolutely irreducible.

Before continuing we note that the $k \times k$ matrix

$$\begin{pmatrix} a & b & b & \ldots & b \\ b & a & b & \ldots & b \\ b & b & a & \ldots & b \\ \vdots & \vdots & \vdots & \ddots & b \\ b & \ldots & \ldots & b & a \end{pmatrix}.$$

only has eigenvalues of $a - b$ and $a + (k-1)b$. This is easily shown by observing that $det(A - \lambda I) = 0$ means that the rows must be linearly dependent, so the eigenvalues must satisfy $(a-\lambda)+(k-2)b=tb$ and $(k-1)b=t(a-\lambda)$ for some t, which yields a quadratic in λ with roots $a-b$ and $a+(k-1)b$.

Next we note that the form of matrix that will commute with the action of the subgroup $\mathbf{W}_k$, and hence the form of $(dg)_{x_0}$, is given by

$$\left(\begin{array}{ccccc|ccc} a & b & b & \dots & b & & & \\ b & a & b & \dots & b & & & \\ b & b & a & \dots & b & & 0 & \\ \vdots & \vdots & \vdots & \ddots & b & & & \\ b & \dots & \dots & b & a & & & \\ \hline & & & & & c & & \\ & & 0 & & & & \ddots & \\ & & & & & & & c \end{array}\right).$$

Where the block in the top left of the matrix is $k \times k$, and the block in the lower right is $(n-k)\times(n-k)$. This has eigenvalues $a-b$, $(k-1)$ times, $a+(k-1)b$ and c, $(n-k)$ times.

We now continue with the proof.

ii) The branching equation for $\mathbf{W}_1$ is

$$\lambda(t) = -(P_u(0)+Q(0))t^2/P_\lambda(0)$$

and in the above matrix the top left block is 1×1, and so in effect $b=0$. Therefore the eigenvalues are given by a and c where, after substituting for λ,

$$a = \tfrac{dg_1}{dx_1} = 2t^2[P_u(0)+Q(0)]$$

and

$$c = \tfrac{dg_n}{dx_n} = -Q(0)t^2.$$

iii) The branches corresponding to isotropy $\mathbf{W}_k$, where $2 \le k \le n-1$, have eigenvalues for $(dg)_{x_0}$ of $a-b$, $a+(k-1)b$ and c where

$$a = \tfrac{dg_1}{dx_1} = 2[P_u(0)+Q(0)t^2],$$

$$b = \tfrac{dg_1}{dx_2} = 2P_u(0)t^2$$

and

$$c = \tfrac{dg_n}{dx_n} = -Q(0)t^2.$$

Therefore the eigenvalues of $(dg)_{x_0}$ are given by

$$2Q(0)t^2, \ -Q(0)t^2$$

and

$$2[kP_u(0) + Q(0)]t^2,$$

hence the result, since for stability we require both $Q(0) < 0$ and $Q(0) > 0$.

iv) The branching equation is now

$$\lambda(t) = -(nP_u(0) + Q(0))t^2/P_\lambda(0)$$

and the eigenvalues of $(dg)_{x_0}$ are given by $a - b$ and $a + (n-1)b$, where now

$$a = \tfrac{dg_1}{dx_1} = 2[P_u(0) + Q(0)]$$

and

$$b = \tfrac{dg_1}{dx_2} = 2P_u(0)t^2.$$

Therefore, we now have eigenvalues $2Q(0)t^2$ and $2[nP_u(0) + Q(0)]t^2$. Requiring these to be positive yields the result of the Theorem, and we are done.

∎

Therefore, surprisingly, of the n possible branches of solutions guaranteed by the Equivariant Branching Lemma to exist (up to conjugacy), only two can be generically stable at the point of bifurcation and, in addition, these two branches cannot both be stable.

We summarize the results in Table 4.1, and present some bifurcation diagrams in Figure 4.1 including the only two containing a stable branch, and the one diagram from which the two bifurcation diagrams in Figure 3.3 with all unstable branches can be deduced.

4.3 Steady-State Bifurcations with $\mathbf{Z}_2 \times \mathbf{S}_n$ Symmetry

Next we turn our attention to the case of direct product coupling of n cells where each cell has an internal $\mathbf{Z}_2$ symmetry, which leads to global $\mathbf{Z}_2 \times \mathbf{S}_n$ symmetry.

Fixed Point Subspace	Isotropy	Branching Equation	Signs of Eigenvalues
$(x, 0, \ldots, 0)$	$\mathbf{W}_1$	$\lambda(t) = -(P_u(0) + Q(0))t^2/P_\lambda(0)$	$P_u(0) + Q(0)$, $-Q(0)$ $(n-1)$times
$(\underbrace{x, \ldots, x}_{k}, \underbrace{0, \ldots, 0}_{n-k})$ $2 \leq k \leq n-1$	$\mathbf{W}_k$	$\lambda(t) = -(kP_u(0) + Q(0))t^2/P_\lambda(0)$	$kP_u(0) + Q(0)$, $-Q(0)$, $Q(0)$
$(x, \ldots, x)$	$\mathbf{S}_n$	$\lambda(t) = -(nP_u(0) + Q(0))t^2/P_\lambda(0)$	$nP_u(0) + Q(0)$, $Q(0)$

Table 4.1: Stability of branches of solutions in the presence of $\mathbf{Z}_2 \wr \mathbf{S}_n$ symmetry

The Group Action and Irreducible Representation

In a similar manner to the case of $n = 3$ already considered we take the natural action of $\mathbf{Z}_2 \times \mathbf{S}_n$ on $\mathbf{R}^n$. That is, we let $\rho \in \mathbf{S}_n$ act by permutation of indices, and $\kappa \in \mathbf{Z}_2$ act on $x \in \mathbf{R}^n$ by $\kappa x = -x$ so that now

$$(\kappa, \rho)\underline{x} = \begin{bmatrix} -x_{\rho(1)} \\ -x_{\rho(2)} \\ \vdots \\ -x_{\rho(n)} \end{bmatrix}$$

This action of $\mathbf{Z}_2 \times \mathbf{S}_n$ is irreducible but *not* absolutely irreducible on $\mathbf{R}^n$, but since $Fix(\mathbf{Z}_2 \times \mathbf{S}_n) = \{0\}$ we can still apply the more general version of the Equivariant Branching Lemma, subject to an extra constraint on the derivative of g (see Theorem 2.2.2), to obtain results that can be compared directly to those of $\mathbf{Z}_2 \wr \mathbf{S}_n$ on $\mathbf{R}^n$ where the action *is* absolutely irreducible. Here however we let, as earlier,

$$\mathbf{V}^n = \{x \in \mathbf{R}^n : x_1 + \ldots + x_n = 0\},$$

then on this space the action of $\mathbf{Z}_2 \times \mathbf{S}_n$ described above *is* now absolutely irreducible. Results on this space have been found by Aronson et al. [3] who considered period-doubling in $\mathbf{S}_n$ symmetrically coupled maps, resulting in $\mathbf{Z}_2 \times \mathbf{S}_n$ symmetry. We repeat the calculations here for coupled ODE's for completeness, and at the end of the section we also consider the irreducible action of $\mathbf{Z}_2 \times \mathbf{S}_n$ on

$$\mathbf{R}^{n*} \simeq \{(x, \ldots, x) : x \in \mathbf{R}\}$$

to consider the results of a $\mathbf{Z}_2 \times \mathbf{S}_n$ symmetric system in $\mathbf{R}^n$.

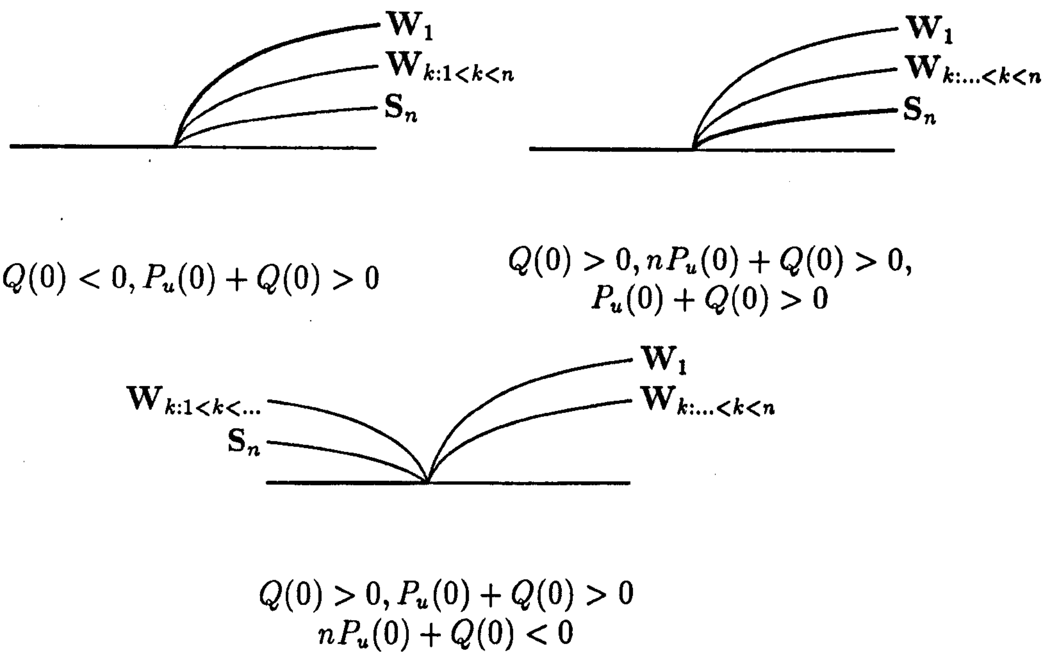


Figure 4.1: Representative bifurcation diagrams for $\mathbf{Z}_2 \wr \mathbf{S}_n$ steady-state bifurcations when $P_\lambda < 0$. Thick lines denote stability, thin lines instability.

Summary of Notation

For our definitions of the isotropy subgroups we use the notation of Aronson et al. [3], which will now be summarised almost directly from that paper with amendments due to our notation.

The isotropy subgroups for the action of $\mathbf{Z}_2 \times \mathbf{S}_n$ on $\mathbf{R}_0^n$ divide naturally into two classes: subgroups of $\mathbf{S}_n$ and subgroups of $\mathbf{Z}_2 \times \mathbf{S}_n$ not in $\mathbf{S}_n$. The first case leads to subgroups defined as follows. Partition n into s blocks with the j^{th} block having k_j elements. Thus

$$k_1 + \ldots + k_s = n.$$

Then the corresponding isotropy subgroup is

$$\Sigma_k = \mathbf{S}_{k_1} \times \mathbf{S}_{k_2} \times \ldots \times \mathbf{S}_{k_s}$$

where $dimFix(\Sigma_k) = s$.

To define the second class, partition n into $2r + 1$ blocks. Pair the blocks so that the $(2j-1)^{th}$ and the $(2j)^{th}$ blocks have l_j members, and the last block has l_{r+1} members. Thus

$$2l_1 + \ldots + 2l_r + l_{r+1} = n.$$

Define

$$\mathbf{T}_{\underline{l}} \equiv \mathbf{S}_{l_1} \times \mathbf{S}_{l_1} \times \ldots \times \mathbf{S}_{l_r} \times \mathbf{S}_{l_r} \times \mathbf{Z}_2(\rho_{\underline{l}})$$

where $dimFix(\mathbf{T}_l) = r$ (in $Fix(\mathbf{T}_l)$ the last block consists of zeros since from the following definition of the $\mathbf{Z}_2$ action and 'x' in the last block implies that $x = -x$) and ρ_i is an order-two group element defined as follows. Consider the j^{th} pair of blocks consisting of members $a_1, \ldots, a_{l_j}$ and $b_1, \ldots, b_{l_j}$ respectively. Define the permutation

$$\rho^j = (a_1 b_1) \ldots (a_{l_j} b_{l_j})$$

and

$$\rho_{\underline{l}} = (\rho^1 \ldots \rho^r, \kappa) \in \mathbf{Z}_2 \times \mathbf{S}_n$$

i.e. swap blocks in each pair and multiply by $-id$. Then we have that

Proposition 4.3.1 *Up to conjugacy, the isotropy subgroups of $\mathbf{Z}_2 \times \mathbf{S}_n$ acting on $\mathbf{R}^n$ are given by*

i) $\Sigma_{\underline{k}}$ *where* $\underline{k} = (k_1, \ldots, k_s)$ *and* $s \geq 1$ *with* $k_1 + \ldots + k_s = n$. *We may also assume* $k_1 \leq k_2 \leq \ldots \leq k_s$;

ii) $\mathbf{T}_{\underline{l}}$ *where* $\underline{l} = (l_1, \ldots, l_{r+1})$ *and* $2l_1 + \ldots + 2l_r + l_{r+1} = n$ *and* $s \geq 1$. *We may also assume* $l_1 \leq l_2 \leq \ldots \leq l_r$.

Proof:

It is clear that these are isotropy subgroups, we now show that they are the only ones.

Vectors of the form $(0, \ldots, 0)$ have isotropy $\mathbf{Z}_2 \times \mathbf{S}_3$ so assume we have a non zero vector $(x_1, \ldots, x_n)$. Now *any* vector in $\mathbf{R}^n$ is on the same group orbit (and so has conjugate isotropy) to a vector of the form

$$(\underbrace{y_1, \ldots, y_1}_{a_1}, \underbrace{-y_1, \ldots, -y_1}_{b_1}, \ldots, \underbrace{y_k, \ldots, y_k}_{a_k}, \underbrace{-y_k, \ldots, -y_1}_{b_k}, \underbrace{0, \ldots, 0}_{a_{k+1}})$$

where all the y_i are distinct, up to the condition that the entries must sum to zero, and so $\Sigma_{i=1}^{k+1} a_i + \Sigma_{i=1}^{k} b_i = n$. If all the b_i^s are zero then we have isotropy $\mathbf{S}_{a_1} \times \ldots \times \mathbf{S}_{a_{k+1}}$. So assume not all the b_i are zero and consider a block $\mathbf{J}_i$ where $\mathbf{J}_i = (\underbrace{y_i, \ldots, y_i}_{a_i}, \underbrace{-y_i, \ldots, -y_i}_{b_i})$. This block has at least isotropy $\mathbf{S}_{a_i} \times \mathbf{S}_{b_i}$. If $a_i \neq b_i$ then applying the $\mathbf{Z}_2$ action cannot be countered by a permutation, and so we have only isotropy $\mathbf{S}_{a_i} \times \mathbf{S}_{b_i}$. If however $a_i = b_i$ then applying any bijection that swaps all the y_{a_i} with the y_{b_i} and then applying the $\mathbf{Z}_2$ action also fixes the block, as well as any element of $\mathbf{S}_{a_i} \times \mathbf{S}_{b_i}$. So we can only apply the $\mathbf{Z}_2$ action as part of an isotropy if $a_i = b_i$ and so can only apply the $\mathbf{Z}_2$ action as part of an isotropy for the whole vector if $a_i = b_i$ for all $1 \leq i \leq k$, when we have isotropy $\mathbf{T}_l$. If any $a_i \neq b_i$ then we have isotropy $\mathbf{S}_{a_1} \times \ldots \times \mathbf{S}_{a_{k+1}}$. Note that if $n = 2k$ then the point $(\underbrace{x, \ldots, x}_{k}, \underbrace{-x, \ldots, -x}_{k})$ has symmetry Σ_k but has isotropy $T_{\underline{l}}$ where $\underline{l} = (k, k)$.

∎

Corollary 4.3.2 *Those isotropy subgroups having one-dimensional fixed point subspaces are* $\Sigma_k = \mathbf{S}_k \times \mathbf{S}_{k-1}$ *where* $k < \frac{n}{2}$ *and* $\mathbf{T}_l = \mathbf{S}_l \times \mathbf{S}_l \times \mathbf{S}_{n-2l} \times \mathbf{Z}_2(\rho_l)$ *where* $1 \leq l \leq n/2$, *with corresponding fixed point subspaces given by*

$$(\underbrace{(n-k)x, \ldots, (n-k)x}_{k}, \underbrace{-kx, \ldots, -kx}_{n-k})$$

and

$$(\underbrace{x, \ldots, x}_{l}, \underbrace{-x, \ldots, -x}_{l}, \underbrace{0, \ldots, 0}_{n-2l})$$

respectively.

Proof: Follows directly from proposition 4.3.1 and the definitions of the isotropy subgroups. ∎

Therefore, by the Equivariant Branching Lemma (Theorem 2.2.2), we are generically guaranteed solution branches at the point of bifurcation corresponding to the isotropy subgroups with one-dimensional fixed point subspaces, namely Σ_k and $\mathbf{T}_l$ where $1 \leq l \leq n/2$. Unlike the case of $\mathbf{Z}_2 \wr \mathbf{S}_n$ however we must also consider solution branches corresponding to sub-maximal isotropies. This will be dealt with later.

Stabilities of Solutions with Maximal Isotropy Subgroups

As previously, we consider the most general form of the bifurcation problem on the irreducible subspace and then calculate the possible stabilities.

Proposition 4.3.3 *The details of the equations are as follows* *i) Every $\mathbf{Z}_2 \times \mathbf{S}_n$ invariant germ f has the form, up to at least third order, $f(u)$ where*

$$u = \sum_{i=1}^{n} x_i^2;$$

ii) the module of $\mathbf{Z}_2 \times \mathbf{S}_n$ equivariants is generated, up to at least third order over the invariants, by the mappings

$$\mathbf{X}_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \ \mathbf{X}_2 = \begin{bmatrix} (n-1)x_1^3 - x_2^3 - \ldots - x_n^3 \\ (n-1)x_2^3 - x_1^3 - \ldots - x_n^3 \\ \vdots \\ (n-1)x_n^3 - x_1^3 - \ldots - x_{n-1}^3 \end{bmatrix}.$$

Proof:

We show the list of invariants and equivariants by the methods outlined in Chapter 2, and calculate up to only third order.

Invariants We begin with the linear invariants. For symmetry $\mathbf{Z}_2 \times \mathbf{S}_n$ on $\mathbf{R}^n$ we have that $\Lambda^1 = \{x_1, \ldots, x_n, -x_1, \ldots, -x_n\}$ which does not partition any further, and so the only linear invariant is given by $\Sigma_i(x_i - x_i) = 0$, and so we have no linear invariants.

Now consider the quadratic invariants. We have that

$$\Lambda^2 = \{x_1^2, \ldots, x_n^2, x_1x_2, \ldots, x_{n-1}x_n\}$$

which partitions into $\Delta_1^2 = \{x_1^2, \ldots, x_n^2\}$ giving $u = x_1^2 + \ldots + x_n^2$ and $\Delta_2^2 = \{x_1x_2, \ldots, x_{n-1}x_n\}$ which gives $v = x_1x_2 + \ldots + x_{n-1}x_n$ when summed. However, $(x_1 + \ldots + x_n)^2 = 0 = u + 2v$ and so one of these is not needed.

All the partitions of Λ^3 will contain elements that cancel in pairs when summed, i.e. if $h(x) \in \Delta_i^3$ then so is $-h(x)$, and so we have no non-zero cubic invariants.

Equivariants We now consider the equivariant mappings. We do these calculations in the same way as for the $\mathbf{Z}_2 \wr \mathbf{S}_n$ case by taking the $\mathbf{S}_n$ partitions Θ_i and finding the $\mathbf{Z}_2 \times \mathbf{S}_n$ partitions by considering the sets $\Theta_i \cup \kappa\Theta_i$, where $\kappa \in \mathbf{Z}_2 \times \mathbf{S}_n$.

For the linear case this gives us $\Theta_1^1 = \left\langle \begin{bmatrix} x_1 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ which leads to the equivariant

$\mathbf{Y}_1 = \mathbf{X}_1 = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\Theta_2^1 = \left\langle \begin{bmatrix} x_2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ which gives us $\mathbf{Y}_2 = \begin{bmatrix} \Sigma_{i \neq 1} x_i \\ \vdots \\ \Sigma_{i \neq n} x_i \end{bmatrix}$, but if we in fact take the equivariant $\mathbf{X}_0 = \mathbf{X}_1 + \mathbf{Y}_2 = \begin{bmatrix} \Sigma x_i \\ \vdots \\ \Sigma x_i \end{bmatrix} = 0$ so is not needed but will be used in this proof.

For the quadratic equivariants we have that every $\epsilon_i(x)$ satisfies

$$\kappa^{-1}\epsilon_i(\kappa x) = -\epsilon_i(x)$$

and so elements in every partition cancel in pairs, and so there are no quadratic equivariants.

Finally consider the cubic equivariants, which must satisfy $\kappa \cdot \Theta_i = \Theta_i$ for $\kappa \in \mathbf{Z}_2$.

$\Theta_1^3 = \left\langle \begin{bmatrix} x_1^3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ gives the equivariant $\mathbf{Y}_3 = \begin{bmatrix} x_1^3 \\ \vdots \\ x_n^3 \end{bmatrix}$ call $\mathbf{X}_2 = \mathbf{Y}_3$.

$\Theta_2^3 = \left\langle \begin{bmatrix} x_2^3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ giving the equivariant $\mathbf{Y}_4 = \begin{bmatrix} \Sigma_{i\neq 1} x_i^3 \\ \vdots \\ \Sigma_{i\neq n} x_i^3 \end{bmatrix}$.

Similarly, $\Theta_3^3 = \left\langle \begin{bmatrix} x_1^2 x_2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ gives us $\mathbf{Y}_5 = \begin{bmatrix} x_1^2 \Sigma_{i\neq 1} x_i \\ \vdots \\ x_n^2 \Sigma_{i\neq n} x_i \end{bmatrix}$, but this is just $-\mathbf{X}_2$, since $\sum_{i\neq 1} x_i = -x_1$ and so we can discard. $\Theta_4^3 = \left\langle \begin{bmatrix} x_1 x_2^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ leads to the equivariant

$\mathbf{Y}_6 = \begin{bmatrix} x_1 \Sigma_{i\neq 1} x_i^2 \\ \vdots \\ x_n \Sigma_{i\neq n} x_i^2 \end{bmatrix}$ but we have the relation $\mathbf{Y}_6 = u\mathbf{X}_1 - \mathbf{X}_2$ so again discard. $\Theta_5^3 = \left\langle \begin{bmatrix} x_2 x_3^2 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ giving $\mathbf{Y}_7 = \begin{bmatrix} \Sigma_{i,j\neq 1} x_i^2 x_j \\ \vdots \\ \Sigma_{i,j\neq n} x_i^2 x_j \end{bmatrix}$, but $\sum_{i,j\neq 1} x_i^2 x_j = -x_1 \sum x_i^2 - \sum x_i^3$ and so $\mathbf{Y}_7 = -\mathbf{Y}_6 - \mathbf{Y}_4$. $\Theta_6^3 = \left\langle \begin{bmatrix} x_1 x_2 x_3 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$ giving $\mathbf{Y}_8 =$

$\begin{bmatrix} x_1\Sigma_{1<i<j}x_ix_j \\ \vdots \\ x_n\Sigma_{i<j<n}x_ix_j \end{bmatrix}$. Call $\mathbf{Y}_8 = \mathbf{X}_3$, but $v\mathbf{X}_1 = \mathbf{Y}_5 + \mathbf{Y}_8$. Finally we get $\Theta_7^3 = \left\langle \begin{bmatrix} x_2x_3x_4 \\ \vdots \\ 0 \end{bmatrix} \right\rangle$, giving $\mathbf{Y}_9 = \begin{bmatrix} \Sigma_{1<i<j<k}x_ix_jx_k \\ \vdots \\ \Sigma_{i<j<k<n}x_ix_jx_k \end{bmatrix}$. Now we note that

$$0 = v\mathbf{X}_0 = \mathbf{X}_3 + \mathbf{Y}_5 + \mathbf{Y}_6 + \mathbf{Y}_7 + \mathbf{Y}_9,$$

so we don't need $\mathbf{Y}_9$, and

$$u\mathbf{X}_0 = \mathbf{X}_2 + \mathbf{Y}_4 + \mathbf{Y}_5 + \mathbf{Y}_6 + \mathbf{Y}_7,$$

so we don't need $\mathbf{Y}_4$. Now project these equivariants onto $\mathbf{R}_0^n$ and we are done.

We have now proved the Proposition.

∎

Again, for stability purposes, we are only interested in derivatives of up to second order, we therefore need to consider our general bifurcation problem up to only third order. Therefore the general bifurcation problem on $\mathbf{R}_0^n$ with $\mathbf{Z}_2 \times \mathbf{S}_n$ symmetry has the form, up to at least third order

$$g(x, \lambda) = P\mathbf{X}_1 + Q\mathbf{X}_2 + \ldots$$

where P and Q are functions of u and λ.

Branching Equations

We now work out the branching equations for our solution branches, after parametrising the fixed point subspaces, and so the equations, by t.

For isotropy Σ_k we have that $Fix(\Sigma_k) = ((n-k)t, \ldots, (n-k)t, -kt, \ldots, -kt)$ and so $u = nk(n-k)t^2$ and so by considering the first component of g

$$g_1(x, \lambda)|_{Fix(\Sigma_k)} = (n-k)P(u, \lambda)t + (n-k)[(n-k)^3 + k^3]Q(u, \lambda)t^3$$

Therefore we have

$$\begin{aligned} g_1(x, \lambda)|_{Fix(\Sigma_k)} = &(n-k)P(0)t + nk(n-k)^2P_u(0)t^3 + \\ &(n-k)P_\lambda(0)\lambda t + (n-k)[(n-k)^3 + k^3]Q(0)t^3. \end{aligned}$$

Linear terms vanish at the origin, so $P(0) = 0$, and so setting $g = 0$ yields the branching equation

$$\lambda(t) = -[nk(n-k)P_u(0) + [(n-k)^3 + k^3]Q(0)]t^2/P_\lambda(0) \qquad (4.3.3)$$

We now do the same calculations for Isotropy $\mathbf{T}_k$ where we have $Fix(\mathbf{T}_k) = (\overbrace{t,\ldots,t}^{k}, \overbrace{-t,\ldots,t}^{k}, \overbrace{0,\ldots,0}^{n-2k})$ and so now $u = 2kt^2$. This time we have

$$g_1(x,\lambda)|_{Fix(T_k)} = P(u,\lambda)t + nQ(u,\lambda)t^3 + \ldots$$

or

$$g_1(x,\lambda)|_{Fix(T_k)} = 2kP_u(0)t^3 + nQ(0)t^3 + P_\lambda(0)\lambda t + \ldots$$

so setting to zero yields the branching equation

$$\lambda(t) = -[2kP_u(0) + nQ(0)]t^2/P_\lambda(0).$$

We are now ready to state the necessary conditions for the stability of our solution branches

Theorem 4.3.4 *The stabilities of the solution branches corresponding to isotropy subgroups of $\mathbf{Z}_2 \times \mathbf{S}_n$ of form $\mathbf{T}_k$ (having one dimensional fixed point subspaces) are as follows*

i) The trivial branch corresponding to isotropy $\mathbf{Z}_2 \times \mathbf{S}_n$, branching equation $x = 0$, is stable if $P_\lambda(0)\lambda > 0$. We assume that $P_\lambda(0) < 0$.

ii) The solution branch corresponding to isotropy $\mathbf{T}_1$ is generically unstable, the generic condition being $Q(0) \neq 0$. If $Q(0) = 0$ then stability is undetermined to third order, but a necessary condition for stability is that $P_u(0) > 0$.

iii) The branches of solutions corresponding to the isotropy subgroups $\mathbf{T}_k$ where $2 \leq k \leq [n/2]$, where $[n/2]$ means the 'integer part of $n/2$', are generically unstable, the generic condition being $Q(0) \neq 0$. If $Q(0) = 0$ then stability is undetermined at third order, a necessary condition for stability is that $P_u(0) > 0$.

iv) If n is odd, $n = 2k+1$, then the branch corresponding to isotropy $\mathbf{T}_{[(n-1)/2]}$ is generically unstable, the generic condition being that $Q(0) \neq 0$. As with the previous two cases if $Q(0) = 0$ then stability is undetermined to third order but a necessary condition for stability is that $P_u(0) > 0$.

v)If n is even, $n = 2k$, then the branch corresponding to isotropy $\mathbf{T}_{n/2} = \mathbf{T}_k$ is stable iff

$$Q(0) > 0,$$
$$P_u(0) + Q(0) > 0.$$

Proof:

i) Our action is absolutely irreducible,so $(dg)_{x_0}$ is a scalar multiple of the identity with eigenvalues $P_\lambda(0)\lambda$. By construction we want this branch to be stable for $\lambda < 0$, hence the result.

For the purposes of the proof we use an equivalent equivariant to $\mathbf{X}_1$ given by

$$\mathbf{X}_1 = \frac{1}{n}\begin{bmatrix} (n-1)x_1 - x_2 - \ldots - x_n \\ \vdots \\ \vdots \\ (n-1)x_n - x_1 - \ldots - x_{n-1} \end{bmatrix}$$

(to regain the original add $0 = x_1 + x_2 + \ldots + x_n$ to each row). We do this since the original version does not explicitly include the condition $x_1 + x_2 + \ldots + x_n = 0$.

To prove parts ii), iii) and iv) we introduce a change of variables for each $\mathbf{T}_k$, where $1 \leq k \leq n/2$. Let $y_1 = x_1$, $y_2 = x_{k+1}$, $y_3 = x_{2k+1}$, $y_4 = x_2 - x_1$, $\ldots$, $y_{k+2} = x_k - x_1$, $y_{k+3} = x_{k+2} - x_{k+1}$, $\ldots$, $y_{2k+1} = x_{2k} - x_{k+1}$, $y_{2k+2} = x_{2k+2} - x_{2k+1}$, $\ldots$, $y_n = x_n - x_{2k+1}$
so that the x_i are the old co-ordinates, the y_i the new. This is equivalent to reducing $\mathbf{R}^n$ into isotypic components.

Thus (rearranging), $x_1 = y_1$, $x_2 = y_1 + y_4$, $x_3 = y_1 + y_5$, $\ldots$, $x_n = y_3 + y_n$, and $Fix(\mathbf{T}_k)$ in our new coordinates can be written, after parametrising by t,

$$Fix(\mathbf{T}_k) = (t, -t, 0, \ldots, 0).$$

which will make the calculations a lot easier to handle. We now have for our invariant germ

$$\begin{aligned} u = {} & ky_1^2 + ky_2^2 + (n-2k)y_3^2 + y_4^2 + \ldots + y_n^2 + 2y_1(y_4 + \ldots + y_{k+2}) \\ & +2y_2(y_{k+3} + \ldots + y_{2k+1}) + 2y_3(y_{2k+2} + \ldots + y_n). \end{aligned}$$

Note that when restricted to $Fix(\mathbf{T}_k)$ this still equates to $u|_{Fix(\mathbf{T}_k)} = 2kt^2$.

The equivariants in the new coordinates are now (making obvious allowances if k is small)

$$\mathbf{Y}_1 = \frac{1}{n}\begin{bmatrix} (n-k)y_1 - ky_2 - (n-2k)y_3 - (y_4 + \ldots + y_n) \\ (n-k)y_2 - ky_1 - (n-2k)y_3 - (y_4 + \ldots + y_n) \\ 2ky_3 - ky_1 - ky_2 - (y_4 + \ldots + y_n) \\ ny_4 \\ \vdots \\ ny_n \end{bmatrix},$$

$$\mathbf{Y}_2 = \begin{bmatrix} (n-k)y_1^3 - ky_2^3 - (n-2k)y_3^3 - A(y_1,\ldots,y_n) \\ (n-k)y_2^3 - ky_{1^3} - (n-2k)y_3^3 - A(y_1,\ldots,y_n) \\ 2ky_3^3 - ky_1^3 - ky_2^3 - A(y_1,\ldots,y_n) \\ n(y_4^3 + 3y_1y_4^2 + 3y_1^2y_4) \\ \vdots \\ n(y_n^3 + 3y_3y_n^2 + 3y_3^2y_n) \end{bmatrix},$$

where $A(y_1,\ldots,y_n)$ is equal to

$$3\begin{pmatrix} y_1^2(y_4 + \ldots + y_{k+2}) + y_2^2(y_{k+3} + \ldots + y_{2k+1}) \\ +y_3^2(y_{2k+2} + \ldots + y_n) + y_1(y_4 + \ldots + y_{k+2})^2 \\ +y_2(y_{k+3} + \ldots + y_{2k+1})^2 + y_3(y_{2k+2} + \ldots + y_n)^2 \end{pmatrix}$$

For later calculations we also note that on $Fix(\mathbf{T}_k)$

$$\begin{aligned} &\tfrac{\partial u}{\partial y_1} = 2kt, \quad \tfrac{\partial u}{\partial y_2} = -2kt, \quad \tfrac{\partial u}{\partial y_3} = 0, \quad \tfrac{\partial v}{\partial y_3} = 0, \\ &\tfrac{\partial u}{\partial y_s} = 2t, \quad (4 \le s \le k+2), \\ &\tfrac{\partial u}{\partial y_s} = -2t, \quad (k+3 \le s \le 2k+1), \\ &\tfrac{\partial u}{\partial y_s} = 0, \quad (2k+2 \le s \le n). \end{aligned}$$

So that now, in our new coordinates, we are working with an equation of form

$$g(y,\lambda) = P\mathbf{Y}_1 + Q\mathbf{Y}_2$$

where P and Q are functions of u and λ. In block form $(dg)_{x_0}$ will now look like

$$\begin{pmatrix} \frac{dg_1}{dy_1} & \cdots & \frac{dg_1}{dy_n} \\ \vdots & & \vdots \\ \frac{dg_n}{dy_1} & \cdots & \frac{dg_n}{dy_n} \end{pmatrix} = \begin{array}{c|c} & \begin{array}{cccc} 3 & k-1 & k-1 & n-2k-1 \end{array} \\ \begin{array}{c} 3 \\ k-1 \\ k-1 \\ n-2k-1 \end{array} & \left(\begin{array}{c|c|c|c} \mathrm{A} & \mathrm{E} & \mathrm{F} & \mathrm{G} \\ \hline \mathrm{H} & \mathrm{B} & \mathrm{W} & \mathrm{K} \\ \hline \mathrm{I} & \mathrm{R} & \mathrm{C} & \mathrm{L} \\ \hline \mathrm{J} & \mathrm{M} & \mathrm{N} & \mathrm{D} \end{array}\right) \end{array}.$$

Since we have reduced to isotypic components we would expect all the blocks H, I, J, M, N and R to consist of zero matrices. To verify this we can carry out direct computation. Note that all the entries of the above blocks involve calculations of the form

$$\tfrac{d}{dy_i}[Py_j + Q(y_j^3 + 3y_k y_j^2 + 3y_k^2 y_j)]$$

where $k \in \{1,2,3\}$ and $y_j = 0$ on the fixed point subspace, and so all the derivatives in these blocks involve a factor of a y_j which is zero, and so all the entries will be zero.

Therefore the eigenvalues of the whole matrix $(dg)_{x_0}$ are the eigenvalues of the matrices given by the blocks A, B, C and D. Of these, the eigenvalues corresponding to matrix A are the most complicated, and so we consider these first.
Matrix A is of the form

$$\begin{pmatrix} a & b & c \\ b & a & c \\ h & h & i \end{pmatrix}.$$

which has eigenvalues $a - b$ and $\frac{1}{2}\left[(i + b + a) \pm \sqrt{(a + b - i)^2 + 8hc}\,\right]$. Where in this case

$$\begin{aligned} a &= \tfrac{d}{dy_1}[P(\ldots) + Q(\ldots)] \\ &= \frac{1}{n}[(n-k)P + 2nkP_u(0)t^2 + 3(n-k)Q(0)t^2] \\ &= [2kP_u(0) + 2(n-k)Q(0)]\,t^2 \end{aligned}$$

$$b = -[2kP_u(0) + 2kQ(0)],$$

$$c = (n - 2k)Q(0)t^2,$$

$$h = -2kQ(0)t^2,$$

and

$$i = -2kQ(0)t^2.$$

Substituting, and rearranging gives us eigenvalues of

$$2[2kP_u(0) + nQ(0)]t^2,$$

$$2[(n - 3k)Q(0)]t^2$$

and 0 (from the direction of the projection from $\mathbf{R}^3$ to $\mathbf{R}^3_0$).

Which means that the eigenvalues of A are all positive iff

$$2kP_u(0) + nQ(0) > 0$$

and

$$(n-3k)Q(0) > 0.$$

The diagonal elements of B and C are given by

$$\tfrac{d}{dy_s}[Py_s + nQy_sy_1^2] = P = 3nQ(0)t^2 = 2nQ(0)t^2$$

and the off diagonals are again all equal to zero since they have a factor of y_s in them where $y_s = 0$. Note also that blocks B and C only exist if $k > 1$.

Similarly, diagonal elements of D are given by

$$\tfrac{d}{dy_s}[Py_s + nQ(\ldots)] = P = -nQ(0)t^2$$

and again off diagonal elements vanish. In our calculations block D only exists if $2k < n-1$.

We now know all the possible eigenvalues of dg and must consider the final three cases of the Theorem separately

ii) If $k = 1$ then blocks B and C do not exist and so the eigenvalues of $(dg)_{x_0}$ are

$$-nQ(0)t^2,$$

$$2[2P_u(0) + nQ(0)]t^2$$

and

$$2(n-3)Q(0)t^2$$

(remember that $n > 3$ here).

Requiring these to be positive yields the result.

iii) If $1 < k < [n/2]$ then all the blocks exist, and so we have eigenvalues of $2nQ(0)t^2$ (from block B) and $-nQ(0)t^2$ (from block D) and so the solution branch with the corresponding isotropy must be generically unstable since generically $Q(0) \neq 0$. If $Q(0) = 0$ then we have, up to third order, $n-1$ zero eigenvalues, and so stability is undetermined. However, the remaining eigenvalue becomes

$$4kP_u(0)t^2$$

which must be positive for a stable solution branch.

iv) If $n = 2k+1$, i.e. n is odd, then block D no longer exists and so the eigenvalues of $(dg)_{x_0}$ are going to be

$$2nQ(0)t^2,$$

$$2[2kP_u(0) + (2K+1)Q(0)]t^2$$

and

$$2(1-k)Q(0)t^2.$$

Requiring these to be positive for stability yields the result.

v) If n is even, $n = 2k$, then we can no longer use the above change of variables used for the fixed point subspaces of $\mathbf{T}_k$, instead we must use a different, though similar, change of variables, which is carried out again, more generally, in the later section on isotropies of type $\mathbf{S}_k \times \mathbf{S}_{n-k}$. The coordinate change we use is given by $y_1 = x_1$, $y_2 = x_{k+1}$, $y_3 = x_2 - x_1, \ldots, y_{k+1} = x_k - x_1$, $y_{k+2} = x_{k+2} - x_{k+1}, \ldots, y_n = x_n - x_{k+1}$. This again means that the old fixed point subspace parametrised by t

$$Fix(\mathbf{T}_{n/2}) = (t, \ldots, t, -t, \ldots, -t)$$

becomes, in our new coordinates,

$$Fix(\mathbf{T}_{n/2}) = (t, -t, 0, \ldots, 0),$$

as before. It works out that it is clearer to use k than n in what follows, where $n = 2k$, and we will then substitute for k once we have done the necessary calculations where considered necessary.

We now have

$$u = ky_1^2 + ky_2^2 + y_3^2 + \ldots + y_n^2 + 2y_1(y_3 + \ldots + y_{k+1}) + 2y_2(y_{k+2} + \ldots + y_n)$$

and the equivariants

$$\mathbf{Y}_1 = \frac{1}{2k}\begin{bmatrix} k(y_1 - y_2) - (y_3 + \ldots + y_n) \\ k(y_2 - y_1) - (y_3 + \ldots + y_n) \\ 2ky_3\vdots \\ 2ky_n \end{bmatrix},$$

$$\mathbf{Y}_2 = \begin{bmatrix} k(y_1^3 - y_2^3) - B(y_1, \ldots, y_n) \\ k(y_2^3 - y_1^3) - B(y_1, \ldots, y_n) \\ 2k(y_3^3 + 3y_1y_3^2 + 3y_1^2y_3) \\ \vdots \\ 2k(y_n^3 + 3y_2y_n^2 + 3y_2^2y_n) \end{bmatrix},$$

where on $Fix(\mathbf{T}_{n/2})$ we have $B(y_1, \ldots, y_n) = 0$ and $\frac{dB}{dy_1} = \frac{dB}{dy_2} = o$, and where again, allowances must be made for small n, in particular for $n < 4$.

We also note for the calculations later that

$$\begin{array}{ll} \frac{\partial u}{\partial y_1} = 2kt, & \frac{\partial u}{\partial y_2} = -2kt, \\ \frac{\partial u}{\partial y_s} = 2t, & (3 \le s \le k+1), \\ \frac{\partial u}{\partial y_s} = -2t, & (k+2 \le s \le n). \end{array}$$

So that once again we have g in the form

$$g(y,\lambda) = P\mathbf{Y}_1 + Q\mathbf{Y}_2$$

where P and Q are functions of u and λ, but this time $(dg)_{x_0}$ will be in block form

$$\begin{pmatrix} \frac{dg_1}{dy_1} & \cdots & \frac{dg_1}{dy_n} \\ \vdots & & \vdots \\ \frac{dg_n}{dy_1} & \cdots & \frac{dg_n}{dy_n} \end{pmatrix} = \begin{array}{c c} & \begin{array}{ccc} 2 & k-1 & k-1 \end{array} \\ \begin{array}{c} 3 \\ k-1 \\ k-1 \end{array} & \left(\begin{array}{c|c|c} A & F & G \\ \hline D & B & J \\ \hline E & H & C \end{array}\right) \end{array}$$

where as before blocks D, E, H and J must all be identically equal to zero matrices since they will all contain a factor of a y_s where $y_s = 0$ (we have again reduced $\mathbf{R}^n$ into it's isotypic components).

The diagonal elements of B and C are given by

$$\frac{d}{dy_3}[Py_3 + 2kQ(y_3^3 + 3y_3^2y_1 + 3y_3y_1^2)] = 4kQ(0)t^2,$$

and the off-diagonal elements are all zero for the same reason that block D and others are zero matrices.

Block A is a matrix of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

which has eigenvalues $a \pm b$ where

$$\begin{aligned} a &= \tfrac{1}{2}P + 2kP_u(0)t^2 + 3kQ(0)t^2 \\ &= 2k[P_u(0) + Q(0)]t^2. \end{aligned}$$

and

$$b = -\tfrac{1}{2}P - 2kP_u(0)t^2 - 3kQ(0)t^2 = -a.$$

Therefore the eigenvalues for matrix A are

$$0$$

and

$$a - b = 4k[P_u(0) + Q(0)]t^2,$$

so that the sign of the non-zero eigenvalue is given by the sign of

$$P_u(0) + Q(0).$$

This means that for stability we require both these values to be positive along with the additional condition $Q(0) > 0$ from blocks B and C, finishing the proof of the Theorem.

∎

Now consider the other solution branches with maximum isotropy.

Theorem 4.3.5 *Let* $\alpha_k = [2nk(n-k)P_u(0)+2n(n^2+3k^2-3nk)Q(0)]t^2$ *then solution branches with isotropy* $\mathbf{S}_k \times \mathbf{S}_{k-1}$ *have stabilities as follows*

i)Solutions corresponding to $k = 1$, *i.e. isotropy* $\mathbf{S}_1 \times \mathbf{S}_{n-1}$, *are stable iff* $Q(0) < 0$ *and*

$$\alpha_1/2n = (n-1)P_u(0) + (n^2 - 3n + 3)Q(0) > 0.$$

ii) Solutions corresponding to isotropy $\mathbf{S}_k \times \mathbf{S}_{n-k}$ *where* $2 \leq k < n/3$ *are generically unstable, the genericity condition being* $Q(0) \neq 0$.

iii) If $k = n/3$ *then the stability of solutions with isotropy* $\mathbf{S}_{n/3} \times \mathbf{S}_{2n/3}$ *is not determined at third order. However, necessary conditions for stability are that* $Q(0) > 0$ *and* $\alpha_{n/3}/2n = [2k^2P_u(0) + 3k^2Q(0)]t^2 > 0$.

iv) Solutions corresponding to isotropy $\mathbf{S}_k \times \mathbf{S}_{n-k}$ *where* $n/3 < k < n/2$ *are stable iff* $Q(0) > 0$ *and* $\alpha_k > 0$.

Proof:

In the same way as the calculations for the isotropies T_k, we can make the work a lot easier by a change of variables, again equivalent to decomposing into isotypic components. To this end let $y_1 = x_1$, $y_2 = x_{k+1}$, $y_3 = x_2 - x_1$, $\ldots, y_{k+1} = x_k - x_1$, $y_{k+2} = x_{k+2} - x_{k+1}, \ldots, y_n = x_n - x_{k+1}$ so that $x_1 = y_1$, $x_2 = y_1 + y_3, \ldots, x_k = y_1 + y_{k+1}$, $x_{k+1} = y_2$, $x_{k+2} = y_2 + y_{k+2}, \ldots, x_n = y_2 + y_n$.

In these coordinates we now have that

$$Fix(\mathbf{S}_k \times \mathbf{S}_{n-k}) = ((n-k)t, -kt, 0, \ldots, 0)$$

and the equivariants become

$$\mathbf{Y}_1 = \frac{1}{n}\begin{bmatrix} (n-k)(y_1 - y_2) - (y_3 + \ldots + y_n) \\ k(y_2 - y_1) - (y_3 + \ldots + y_n) \\ ny_3 \vdots \\ ny_n \end{bmatrix},$$

$$\mathbf{Y}_2 = \begin{bmatrix} (n-k)(y_1^3 - y_2^3) - B(y_1, \ldots, y_n) \\ k(y_2^3 - y_1^3) - B(y_1, \ldots, y_n) \\ n(y_3^3 + 3y_1y_3^2 + 3y_1^2y_3) \\ \vdots \\ n(y_n^3 + 3y_2y_n^2 + 3y_2^2y_n) \end{bmatrix},$$

where on $Fix(\mathbf{S}_k \times \mathbf{S}_{k-1})$, $B(y_1, \ldots, y_n) = 0$ and $\frac{dB}{dy_1} = \frac{dB}{dy_2} = 0$. For later calculations note that, on $Fix(\mathbf{S}_k \times \mathbf{S}_{n-k}$, we have

$$\begin{array}{ll} \frac{\partial u}{\partial y_1} = 2k(n-k)t, & \frac{\partial u}{\partial y_2} = -2k(n-k)t, \\ \frac{\partial u}{\partial y_s} = 2(n-k)t, & (3 \leq s \leq k+1), \\ \frac{\partial u}{\partial y_s} = -2kt, & (k+2 \leq s \leq n), \end{array}.$$

We are now in a position to find the eigenvalues associated with the stabilities of the solutions corresponding to each isotropy subgroup $\mathbf{S}_k \times \mathbf{S}_{n-k}$. In our new coordinates the matrix $(dg)_{x_0}$ will be of the block form

$$\begin{pmatrix} \frac{dg_1}{dy_1} & \cdots & \frac{dg_1}{dy_n} \\ \vdots & & \vdots \\ \frac{dg_n}{dy_1} & \cdots & \frac{dg_n}{dy_n} \end{pmatrix} = \begin{array}{c|c} & \begin{array}{ccc} 2 & k-1 & n-k-1 \end{array} \\ \begin{array}{c} 2 \\ k-1 \\ n-k-1 \end{array} & \left(\begin{array}{c|c|c} A & F & G \\ \hline D & B & J \\ \hline E & H & C \end{array}\right) \end{array}$$

The off-diagonal blocks we would expect to be zero matrices since we have decomposed $\mathbf{R}^n$ into isotypic components. We can check this by direct computation since the elements of blocks D, E, H and J all contain a factor of y_s where $y_s = 0$ (for example, a block D representative is given by $\frac{d}{dy_1}[Py_3 + nQ(y_3^3 + 3y_3^2y_1 + 3y_3y_1^2)] = 0$).

Block B, which only exists if $k \geq 2$, has diagonal entries

$\frac{dg_3}{dy_3} = \frac{d}{y_3}\left[Py_3 + nQ(y_3^3 + 3y_3^2 y_1 + 3y_3 y_1^2)\right] = P + 3n(n-k)^2 t^2 Q = n^2[2n-3k]Q(0)t^2,$

off-diagonal elements all vanish.

Similarly block C has off diagonal entries

$$-n^2(n-3k)Q(0)t^2,$$

and zero off-diagonal entries.

Block A takes the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

whose eigenvalues ζ must, obviously, satisfy $\zeta^2 - (a+d)\zeta + (ad-bc) = 0$ where

$$a = [2k(n-k)^2 P_u(0) + n(n-k)(2n-3k)Q(0)]t^2,$$

$$b = [-2k(n-k)^2 P_u(0) + n(n-k)(n-3k)Q(0)]t^2,$$

$$c = [-2k^2(n-k)P_u(0) + nk(3k-2n)Q(0)]t^2$$

and

$$d = [2k^2(n-k)P_u(0) + nk(3k-n)Q(0)]t^2$$

.

This gives us

$$\alpha_k = a + d = [2nk(n-k)P_u(0) + 2n(n^2 + 3k^2 - 3nk)Q(0)]t^2$$

and

$$ad - bc = 0.$$

The eigenvalues of block A are given by

$$\tfrac{1}{2}\left[\alpha_k \pm \sqrt{\alpha_k^2 - 4(ad-bc)}\right] = \tfrac{1}{2}\left[\alpha_k \pm \alpha_k\right],$$

and so 0 and α_k. We now consider the four cases of the theorem separately

i) If $k = 1$ then block B does not exist and so, since $n > 3$, we have that $n - 3k > 0$ so for stability we require $Q(0) < 0$, $\alpha_1 > 0$.

ii) If $2 \leq k < n/3$ then $2n - 3k > 0$ and $n - 3k > 0$ giving the signs of the eigenvalues as the signs of $Q(0)$, $-Q(0)$ (from blocks B and C), and α_k. Therefore this branch is 'generically' unstable.

iii) If $k = n/3$ then we have a zero eigenvalue from block C, and so stability is undetermined at third order. For the other eigenvalues to be positive we require that $Q(0) > 0$, $\alpha_{n/3} > 0$.

iv) If $n/3 < k \leq n/2$ then $2n - 3k > 0$ and $n - 3k < 0$ so for stability we require that $Q(0) > 0$, $\alpha_k > 0$.

This completes the proof. ∎

We now consider sub-maximal isotropy subgroups.

Sub-maximal Isotropy Subgroups (when $n > 3$)

Unlike the wreath product case, we do now have the possibility of sub-maximal isotropy subgroups supporting solutions (since the paper by Field and Richardson [14] no longer excludes this event), and so we must consider the possibilities.

Theorem 4.3.6 *We also have the following*

i) Isotropy subgroups of the form $\mathbf{T}_{\underline{l}}$, *where* $r > 1$, *do not generically support solutions, where*

$$Fix(\mathbf{T}_{\underline{l}}) = (y_1, -y_1, y_2, -y_2, \ldots, y_r, -y_r, 0),$$

where $y_i \in \mathbf{R}^{l_i}$ *is a multiple of* $(1, \ldots, 1)$.

ii) Isotropy subgroups of the form $\Sigma_{\underline{k}}$ *where* $s \geq 4$ *do not generically support solutions, where*

$$Fix(\Sigma_{\underline{k}}) = (x_1, \ldots, x_1, x_2, \ldots, x_2, \ldots, x_s, \ldots, x_s).$$

Proof: We attempt to follow the proofs of Aronson et al. [3] as closely as possible, amending for our irreducible action if appropriate.

Let $y_i = (t_i, \ldots, t_i)$.

i) Take one of the components of g, which must satisfy

$$Pt_i + Qt_i^3 + \ldots = 0,$$

and compare to its corresponding 'partner'

$$-Pt_i - Qt_i^3 + \ldots = 0.$$

Subtracting the two gives us, up to third order

$$2Pt_i + 2Qt_i^3 + \ldots = 0.$$

We must have $t_i \neq 0$ otherwise we will have a larger isotropy than assumed, so we can divide through by $2t_i$ to obtain

$$P + Qt_i^2 + \ldots = 0.$$

Similarly we obtain

$$P + Qt_j^2 + \ldots = 0$$

for some $t_j \neq t_i$. We are now in a position to follow the proof of [3] by subtracting these last two equations to get

$$(t_i^2 - t_j^2)(Q + \ldots) = 0.$$

Generically $Q(0) \neq 0$ and so we must have that $y_i = \pm y_j$, but this cannot hold because in either case we would have a larger isotropy than assumed.

ii) We follow exactly the proof of [3], substituting our notation where necessary. Up to third order we know that g has the form

$$g = P\mathbf{X}_1 + Q\mathbf{X}_2 + \ldots$$

so that each coordinate of $g|_{Fix(\Sigma)}$ has the form

$$F_a = Pa + (n - l_a)Qa^3 + \mathbf{X}_a Q + \ldots \tag{4.3.4}$$

where l_a is the number of variables equal to a and $\mathbf{X}_a$ contains the non-'a' terms.

When $s \geq 4$, $g|_{Fix(\Sigma)}$ has at least four distinct components of this form, and so we may replace a in equation 4.3.4 by b, c and d. We have, by assumption, $a \neq b$, otherwise we will have larger isotropy than that assumed, and so we can now define

$$G_{ab} = \frac{F_a - F_b}{a - b} = P + Q\frac{a^3 - b_3}{a - b} + \ldots \tag{4.3.5}$$

Next we assume, for the same reasons as above that $a \neq b$ and $b \neq c$ and define

$$H_{abc} = \frac{G_{ab} - G_{ac}}{b - c} = Q\frac{\frac{a^3-b^3}{a-b} - \frac{a^3-c^3}{a-c}}{b-c} + \ldots \tag{4.3.6}$$

The coefficient of Q in 4.3.6 reduces to $a+b+c$ and finally, assuming $c \neq d$ we perform

$$\frac{H_{abc} - H_{abd}}{c-d} = Q + \ldots = 0 \tag{4.3.7}$$

Since generically $Q(0) \neq 0$ it follows that there are no solutions to our problem with isotropy $\Sigma_{\underline{k}}$ near the bifurcation point.

∎

Proposition 4.3.7 *The final cases have yet to be solved for our notation, but we note that Aronson et al. [3] proved the following results:*

i)Isotropy subgroups of the form $\Sigma_{\underline{k}}$ where $\underline{k} = (k, k, n-2k)$ do not, generically, support solutions.

ii) Isotropy subgroups of the form $\Sigma_{(k_1,k_2,k_3)}$ where $0 \leq k_1 < k_2 < k_3$ do generically support solutions.

Remark 4.3.8 *We do not attempt to prove these results here for our notation, as they are not an integral part of our considerations in this Thesis.*

Unlike the previous sections we do not try to draw any representative bifurcation diagrams for $\mathbf{Z}_2 \times \mathbf{S}_n$ steady-state bifurcations due to the complexity of the branching equations and conditions for stability as functions of n and k. We do note however that if a branch *is* stable then it bifurcates subcritically.

$\mathbf{Z}_2 \times \mathbf{S}_n$ acting on $\mathbf{R}$.

We finish this section by completing the analysis of $\mathbf{Z}_2 \times \mathbf{S}_n$ bifurcations on $\mathbf{R}^n$ by considering the other irreducible representation contained in this space, namely $\mathbf{Z}_2 \times \mathbf{S}_n$ acting on $\mathbf{R}$.

This case reduces to the earlier case of $\mathbf{Z}_2 \times \mathbf{S}_3$ acting on $\mathbf{R}$, except that now we replace $\mathbf{R}^* = \{(x,x,x) : x \in \mathbf{R}\}$ by $\mathbf{R}^* = \{(x,\ldots,x) : x \in \mathbf{R}\}$. Again we will have that $\mathbf{S}_n \subset \mathbf{Z}_2 \times \mathbf{S}_n$ acts trivially and that the Equivariant Branching Lemma guarantees a branch of solutions in this space corresponding to isotropy $\mathbf{S}_n$, which can be stable.

4.4 Comparisons Between $\mathbf{S}_n$, $\mathbf{Z}_2 \times \mathbf{S}_n$ and $\mathbf{Z}_2 \wr \mathbf{S}_n$ Symmetries

As in the case of three coupled cells we begin by noting that in adding the internal $\mathbf{Z}_2$ symmetries we go from the $\mathbf{S}_n$ case of, generically, no stable solution branches guaranteed by the Equivariant Branching Lemma to the cases where stable branches are now possible. Therefore care must be taken when considering models of coupled cells with $\mathbf{S}_n$ symmetry, since if an internal $\mathbf{Z}_2$ symmetry is present it could dramatically affect the results. We now consider the differences between systems with $\mathbf{Z}_2 \wr \mathbf{S}_n$ and $\mathbf{Z}_2 \times \mathbf{S}_n$ symmetries.

Comparison of Irreducible Representations

Surprisingly, up to conjugacy, of the n possible maximal isotropy branches of solutions for the $\mathbf{Z}_2 \wr \mathbf{S}_n$ case and the $2[n/2]-1$ for $\mathbf{Z}_2 \times \mathbf{S}_n$, it is only possible for two of the $\mathbf{Z}_2 \wr \mathbf{S}_n$ solutions, and at most four of the $\mathbf{Z}_2 \times \mathbf{S}_n$ branches to be generically stable, all the other branches being generically unstable. As in the case of $n = 3$ considered earlier the wreath product case again does not allow more than one branch to be stable at any one time, and unlike in the $n = 3$ case (see below) the cross product case doesn't either.

Comparisons as Coupled ODE's in $\mathbf{R}^n$

To discuss any further similarities we must consider the solutions outside of conjugacy classes, and we also consider when both systems live on the same space, $\mathbf{R}^n$. For example, the two half-branches of solutions corresponding to isotropy $\mathbf{S}_n$ in $\mathbf{Z}_2 \times \mathbf{S}_n$ lie in the same fixed point subspace as the 2^n half-branches with isotropy $\mathbf{S}_n$ in $\mathbf{Z}_2 \wr \mathbf{S}_n$. Indeed if n is even then the $n\mathbf{C}_{n/2}$ half-branches of solutions with isotropy $\mathbf{T}_{n/2}$ in $\mathbf{Z}_2 \times \mathbf{S}_n$ also fall into this category with the corresponding solutions in the $\mathbf{Z}_2 \wr \mathbf{S}_n$ case. If however n is odd, then those solutions with isotropy $\mathbf{T}_{[n/2]}$ share the same fixed point subspaces as those of $\mathbf{W}_{n-1}$ with the difference that the wreath product solutions are generically unstable.

Similarly, fixed point subspaces which contain the half branches of solutions with isotropy $\mathbf{T}_k \subset \mathbf{Z}_2 \times \mathbf{S}_n$, where $2 \leq k < [n/2]$, are all contained in the fixed point subspaces corresponding to isotropy $\mathbf{W}_{2k}$ in the wreath product case.

Those solutions corresponding to wreath product isotropy subgroups that share no fixed point subspaces with the cross product case are those half-branches with isotropy $\mathbf{W}_k$ where k is odd and also those solutions corresponding to isotropy $\mathbf{S}_k \times \mathbf{S}_{k-1}$ in $\mathbf{Z}_2 \times \mathbf{S}_n$

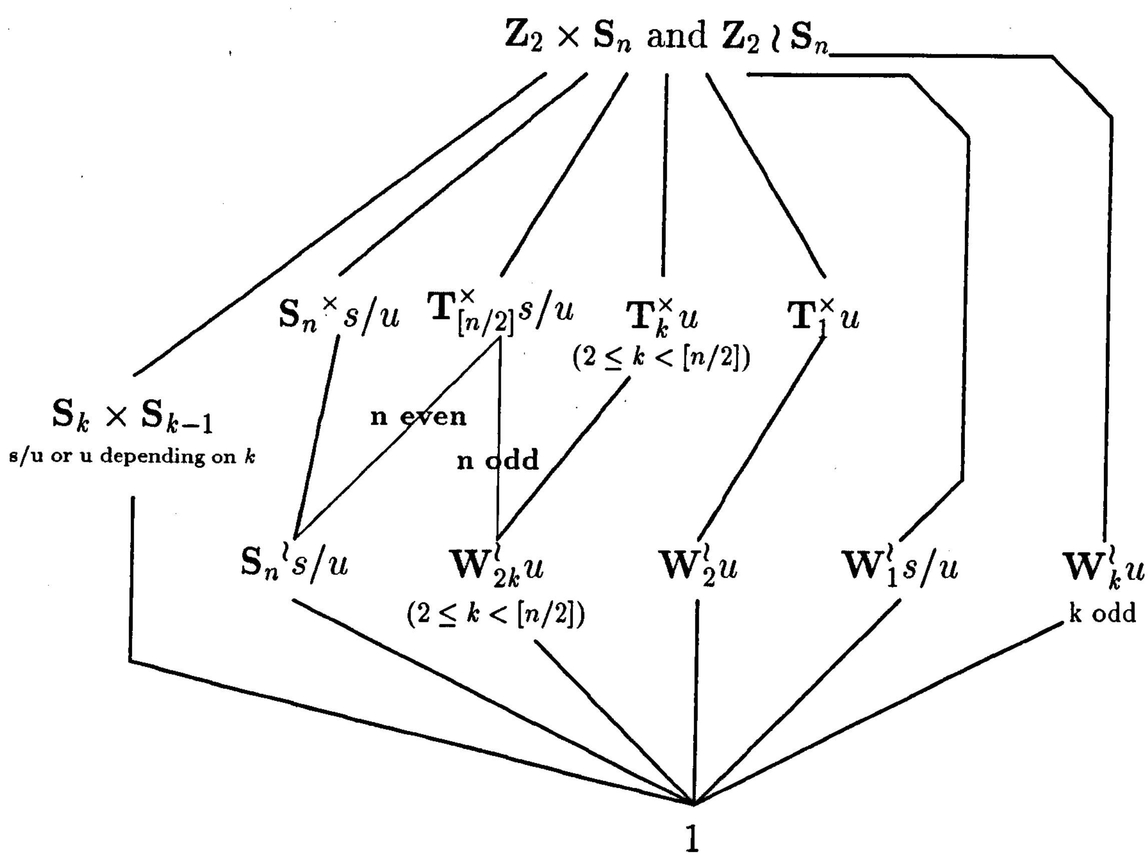


Figure 4.2: Relationships between solutions with isotropies in $\mathbf{Z}_2 \times \mathbf{S}_n$ and $\mathbf{Z}_2 \wr \mathbf{S}_n$.

Thinking of each isotropy as a set of representative fixed point subspaces, the above can be represented in Figure 4.2 where the superscripts '$\times$' and '$\wr$' denote isotropy subgroups of $\mathbf{Z}_2 \times \mathbf{S}_n$ and $\mathbf{Z}_2 \wr \mathbf{S}_n$ respectively, the symbols s/u and u denote possible stabilities and an entry above another, joined by a line, is a subspace of it.

4.5 Comments on Differences Between General n and $n = 3$

Direct Product Coupling

In the case of $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry both of the branches guaranteed by the Equivariant Branching Lemma have stability that is undetermined to third order, and so it is unclear if stability *is* possible. In going to the case of $\mathbf{Z}_2 \times \mathbf{S}_n$ however we find that

$\mathbf{T}_1$ is generically unstable for $n > 3$ (with one of the eigenvalues vanishing for $n = 3$), and $\mathbf{S}_1 \times \mathbf{S}_{n-1}$ can be stable. However, in going from $n = 3$ to a general number n it is not necessarily clear that the two solution branches in the $\mathbf{Z}_2 \times \mathbf{S}_3$ case *do* correspond to these two in the $\mathbf{Z}_2 \times \mathbf{S}_n$ case, and so we would not expect to be able to predict the general case from the results of $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry.

Wreath Product Coupling

Here in the $n = 3$ case we have the possibility of $\mathbf{W}_1$ or $\mathbf{S}_3$ branches being stable with the branch corresponding to $\mathbf{W}_2$ being generically unstable. From this we would not expect *all* the branches with isotropy $\mathbf{W}_k$ where $2 \leq k \leq n-1$ being generically unstable. Indeed a more intuitive result might be that the $\mathbf{W}_k$ with k odd might support stable solutions, instead of for the $n = 3$ case two out of the three solutions having the possibility of stability, but only two out of the n solutions in the general n case.

Chapter 5

Hopf Bifurcation In Three Coupled Cells

We now repeat the calculations already carried out for steady-state bifurcations in the case of Hopf bifurcations, which we can then apply to the study of coupled oscillators with internal symmetries.

As usual we consider the equation

$$\frac{dx}{dt} = g(x, \lambda) \tag{5.0.1}$$

where $x \in \mathbf{R}^n$ and λ is a bifurcation parameter. We assume that g undergoes a Hopf bifurcation at $x = 0$ when $\lambda = 0$ for λ increasing, and further assume that g is Γ equivariant for some Γ. i.e.

$$g(\gamma x) = \gamma g(x)$$

where $\gamma \in \Gamma$.

5.1 Hopf Bifurcations With $\mathbf{S}_3$ Symmetry

With a view to comparing the results for $\mathbf{S}_3$ symmetry with the cases of $\mathbf{Z}_2 \times \mathbf{S}_3$ and $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetries we repeat here the calculations carried out in Golubitsky et al. [17], where they consider the action of $\mathbf{D}_3$ on $\mathbf{C}^2$, except that here we consider the action of $\mathbf{S}_3$ on

$$\mathbf{C}_0^3 = \{z \in \mathbf{C}^3 : z_1 + z_2 + z_3 = 0\}$$

where $\mathbf{S}_3$ acts by permutation of indices. In this way we have avoided a Liapunov-Schmidt reduction step, which although in principal is routine, in practice is complicated and introduces unnecessary working.

Throughout this chapter we also assume that the original (before reduction) vector field is in *Birkhoff normal form* (commutes with $\Gamma \times \mathbf{S}^1$ where $\Gamma = \mathbf{S}_3, \mathbf{Z}_2 \wr \mathbf{S}_3$ or $\mathbf{Z}_2 \times \mathbf{S}_3$; see [17] XVI §5 for more detail). In this way it means that we can solve the branching equations in the following calculations for the period-scaling parameter τ (see Chapter 2 for more detail on τ).

Therefore we have decomposed $\mathbf{C}^3$ into its irreducible representations,

$$\mathbf{C}^3 \cong \mathbf{C}_0^3 \oplus \mathbf{C}$$

where $\mathbf{S}_3$ acts on $\mathbf{C}_0^3$ Γ-Simply, by acting on each component of $\mathbf{R}_0^3 \oplus \mathbf{R}_0^3$ absolutely irreducibly, and on the other component trivially.

To complete the extension of $\mathbf{S}_3 \times \mathbf{S}^1$ acting on $\mathbf{C}_0^3$, let $\theta \in \mathbf{S}^1$ act as $\theta.z = e^{i\theta}z$, and denote $\mathbf{V} = \mathbf{C}_0^3 \subset \mathbf{C}^3$. We are now ready to continue with the analysis.

Isotropy Subgroups

If we let $\zeta = e^{2\pi i/3}$ and $\eta = e^{i\pi}$ then the following proposition (Proposition 5.1.1) gives all the isotropy subgroups of $\mathbf{S}_3 \times \mathbf{S}^1$, the dimensions of their fixed point subspaces and isotypic decomposition.

Proposition 5.1.1 *The list of isotropy subgroups, up to conjugacy, of* $\mathbf{S}_3 \times \mathbf{S}^1$ *acting on* $\mathbf{V}$ *along with fixed point subspaces, and complementary isotypic components* $\mathbf{V}_1$ *are as given in the following table (up to conjugacy)*

Group Orbit	*Isotropy* Σ	$Fix(\Sigma) = \mathbf{V}_0$	$\mathbf{V}_1$	$dim Fix(\Sigma)$
(z,z,z)	$\mathbf{S}_3 \times \mathbf{S}^1$	$(0,0,0)$	$(w_1, w_2, -(w_1+w_2))$	*0*
$(z,\zeta z,\zeta^2 z)$	$\widetilde{\mathbf{Z}}_3$	$(w,\zeta w,\zeta^2 w)$	$(w,\zeta^2 w,\zeta w)$	*2*
$(z,\eta z,0)$	$\widetilde{\mathbf{Z}}_2$	$(w,-w,0)$	$(-w,-w,2w)$	*2*
$(2z,-z,-z)$	$\mathbf{S}_1 \times \mathbf{S}_2$	$(2w,-w,-w)$	$(0,w,-w)$	*2*
$(z_1,z_2,-(z_1+z_2))$	$\mathbf{1}$	$(w_1,w_2,-(w_1+w_2))$	$(0,0,0)$	*4*

Proof:

It should be clear that these are isotropy subgroups, we now show that they are the only ones. We do this by considering every possible vector, which will then span all possible fixed point subspaces.

Vector $(0,0,0)$ has isotropy $\mathbf{S}_3 \times \mathbf{S}^1$, so assume we have a non-zero vector (z_1, z_2, z_3) where $z_1 + z_2 + z_3$=0. The group action either permutes the variables, by permuting indices, or multiplies all the variables by $e^{i\theta}$ some $\theta \in \mathbf{S}^1$, so if two variables are scalar multiples of each other, but have different modulus, then they must be considered as two independent variables.

First assume we have $|z_1| \neq |z_2| \neq |z_3|$, then this vector must be contained in a four-dimensional fixed-point subspace, and so have isotropy $\mathbf{1}$.

Next assume that two of the variables have the same modulus, by conjugacy $|z_1| = |z_2|$ giving us a vector of the form (w, aw, z) where $a \in \mathbf{C}$. Multiplying this vector by an $e^{i\theta}$ leaves it irretrievibly changed with respect to our action of $\mathbf{S}_3 \times \mathbf{S}^1$ unless $z = 0$. So if $z \neq 0$ we have either (w, w, z) with isotropy conjugate to $\mathbf{S}_1 \times \mathbf{S}_2$, and $2w + z = 0$, or (w, aw, z) where $a \neq 1$ with isotropy $\mathbf{1}$ and $(a + 1)w + z = 0$. If $z = 0$ then we have $(w, aw, 0)$, and we are forced to have $a = -1$ so that the variables sum to zero. This has isotropy $\widetilde{\mathbf{Z}}_2$, generated by a permutation (12) followed by multiplication by $-1 = e^{\pi i}$.

The final case to consider is when $|z_1| = |z_2| = |z_3|$, giving a vector of form (w, aw, bw) where $a, b \in \mathbf{C}$ and $1 + a + b = 0$. If $a = b$ or $a = 1$ or $b = 1$ then we have $a + b = -1$ and have isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ or a conjugate, so assume $a \neq b \neq 1$. It should be clear that no subset of $\mathbf{S}_3$ will fix the vector now, so consider what happens when we multiply the vector by $e^{i\theta}$ by letting $a = e^{i\alpha}$ and $b = e^{i\beta}$. Multiplying by $e^{i\theta}$ gives us $(e^{i\theta}w, e^{i(\theta+\alpha)}w, e^{i(\theta+\beta)}w)$. We assume $e^{i\theta} \neq 1$. Therefore we need one of the other coefficients to be equal to unity, by conjugacy let $e^{i(\theta+\alpha)} = 1$, but either $e^{i\alpha}$ or $e^{i\beta}$ must equal $e^{i\theta}$, so either $\alpha = \theta$ or $\beta = \theta$. Therefore one of the original coefficients must be equal to $e^{i\theta}$, and so another to $e^{2i\theta}$ and a third to $e^{3i\theta}$. By assumption these must all be distinct, and so one of these being equal to unity forces $e^{3i\theta} = 1$ and so $\theta = 2\pi/3$. This means we have to have vectors of the form $(w, \zeta w, \zeta^2 w)$ where $\zeta = e^{2\pi/3}$, with isotropy $\widetilde{\mathbf{Z}}_3$. Not that this also satisfies $1 + \zeta + \zeta^2 = 0$. Any other choice of distinct a and b leaves us with isotropy $\mathbf{1}$.

∎

Remark 5.1.2 *When these isotropies are considered with respect to three coupled oscillators then the trivial solution, isotropy* $\mathbf{S}_3 \times \mathbf{S}^1$*, corresponds to the equilibrium of no oscillations,* $\widetilde{\mathbf{Z}_3}$ *to each oscillator* $2\pi/3$ *out of phase with each other,* $\widetilde{\mathbf{Z}_2}$ *to two oscillators of identical waveform but* π *out of phase, while the third oscillates with double the frequency, and* $\mathbf{S}_1 \times \mathbf{S}_2$ *to two oscillators identical and in phase, the third of a different waveform.*

Therefore by the Equivariant Hopf Theorem (Theorem 2.4.1) we are guaranteed solutions to equation 5.0.1 having as their symmetry one of those isotropy subgroups listed above with a two dimensional fixed point subspace.

We are now interested in the possible stabilities of these solutions.

Stabilities Of Solutions

The $\mathbf{S}^1$-invariants and equivariants are well known, and so we quote Lemma XVI 9.3 of Golubitsky et al. [17], where the invariants and equivariants are considered on $\mathbf{C}^m$.

Proposition 5.1.3 *On $\mathbf{C}^m$ we have that*

a) A Hilbert basis for the $\mathbf{S}^1$-invariants functions is given by the m^2 quadratics

$$u_j = z_j\overline{z_j},\ 1 \leq j \leq m,$$

$$Re\ v_{ij} \text{ and } Im\ v_{ij}\ 1 \leq,\ i < j \leq m$$

where $v_{ij} = z_i\overline{z_j}$.

Relations are given by $v_{ij}\overline{v_{ij}} = u_i u_j$.

b) Let $g = (g_1, \ldots, g_m) : \mathbf{C}^m \to \mathbf{C}^m$ be $\mathbf{S}^1$-equivariant. Then each g_j satisfies

$$g_j(\theta z) = e^{i\theta} g_j(z).$$

The module of each $g_j : \mathbf{C}^m \to \mathbf{C}$ is generated over the invariants by the $2m$ mappings $\mathbf{X}_j(z) = z_j$ and $\mathbf{Y}_j(z) = i\overline{z_j}$. Thus the modules of $\mathbf{S}^1$-equivariants $\mathbf{C}^m \to \mathbf{C}^m$ has $2m^2$ generators of the form

$$(0, \ldots, 0, \mathbf{X}_j, 0, \ldots, 0)$$

and

$$(0, \ldots, 0, \mathbf{Y}_j, 0, \ldots, 0).$$

Proof:
See [17], Lemma XV 9.3. ∎

Note that $Re\ v_{ij}$ and $Im\ v_{ij}$ span the same space as v_{ij} and $\overline{v_{ij}}$. Therefore for the case of $m = 3$ we have that, on $\mathbf{C}^3$

Corollary 5.1.4 *i) The $\mathbf{S}^1$ invariants on $\mathbf{C}^3$ are generated by the six functions*

$$u_1 = z_1\overline{z_1},\ u_2 = z_2\overline{z_2},\ u_3 = z_3\overline{z_3},$$

$$v_{12} = z_1\overline{z_2},\ v_{13} = z_1\overline{z_3},\ v_{23} = z_2\overline{z_3}.$$

ii) And the $\mathbf{S}^1$ equivariants are generated over the invariants by the mappings

$$(\mathbf{X}_1, 0, 0),\ (0, \mathbf{X}_1, 0),\ (0, 0, \mathbf{X}_1),\ (\mathbf{X}_2, 0, 0),\ etc.$$

Proof:

Follows directly from the previous lemma. ∎

It remains to calculate the invariants and equivariants for $\mathbf{S}_3 \times \mathbf{S}^1$ symmetric bifurcation problem. We summarize the results in the following lemma.

Lemma 5.1.5 *The details of the $\mathbf{S}_3 \times \mathbf{S}^1$ invariants and equivariants on the Γ-Simple subspace $\mathbf{V} = \{z \in \mathbf{C}^3 : z_1 + z_2 + z_3 = 0\}$ are as given below.*

i) Up to at least fifth order every $\mathbf{S}_3 \times \mathbf{S}^1$-invariant germ $f : \mathbf{C}_0^3 \to \mathbf{R}$ has the form $f(u, v)$ where

$$u = z_1\overline{z_1} + z_2\overline{z_2} + z_3\overline{z_3},$$

and

$$v = (z_1\overline{z_1})^2 + (z_2\overline{z_2})^2 + (z_3\overline{z_3})^2.$$

ii) Every smooth $\mathbf{S}_3 \times \mathbf{S}^1$-equivariant map germ $h : \mathbf{C}_0^3 \to \mathbf{C}_0^3$ has the form (up to at least fifth order)

$$h(z_1, z_2, z_3) = P\mathbf{X}_0 + Q\mathbf{X}_1 + R\mathbf{X}_2$$

where

$$\mathbf{X}_0 = 3\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \quad \mathbf{X}_1 = \begin{bmatrix} 2z_1^2\overline{z_1} - (z_2^2\overline{z_2} + z_3^2\overline{z_3}) \\ 2z_2^2\overline{z_2} - (z_1^2\overline{z_1} + z_3^2\overline{z_3}) \\ 2z_3^2\overline{z_3} - (z_1^2\overline{z_1} + z_3^2\overline{z_3}) \end{bmatrix}$$

and

$$\mathbf{X}_2 = \begin{bmatrix} 2z_1^3\overline{z_1}^2 - (z_2^3\overline{z_2}^2 + z_3^3\overline{z_3}^2) \\ 2z_2^3\overline{z_2}^2 - (z_1^3\overline{z_1}^2 + z_3^3\overline{z_3}^2) \\ 2z_3^3\overline{z_3}^2 - (z_1^3\overline{z_1}^2 + z_3^3\overline{z_3}^2) \end{bmatrix},$$

and where P, Q, R are complex-valued $\mathbf{S}_3 \times \mathbf{S}^1$-invariant functions. The equivariant mapping $\mathbf{X}_0$ contains a factor of three to simplify stability calculations later.

Proof:

i) We begin by calculating to fifth order, it turns out that this is sufficient for our calculations. We consider all the possible $\mathbf{S}_3 \times \mathbf{S}^1$ invariant functions on $\mathbf{C}^3$, and then consider relations that exist when we restrict ourselves to $\mathbf{V} = \mathbf{C}_0^3$ when we must have $z_1 + z_2 + z_3 = 0$.

Consider the transposition $(ij) \in \mathbf{S}_3$, this maps u_i to u_j, v_{ij} to $\overline{v_{ij}}$ and v_{ik} to v_{jk}. Therefore any invariant containing u_i must contain u_1, u_2 and u_3, and any invariant containing v_{ij} must contain all the v_{12},v_{13},v_{23}, $\overline{v_{12}}$,$\overline{v_{13}}$,$\overline{v_{23}}$. So, up to third order the only possible invariants are generated by

$$u_1 + u_2 + u_3, \text{ and } v_{12} + v_{13} + v_{23} + \overline{v_{12}} + \overline{v_{13}} + \overline{v_{23}}.$$

and up to fifth order, generated by these and the functions

$$u_1^2 + u_2^2 + u_3^2,\ u_1u_2 + u_2u_3 + u_1u_3,$$

$$v_{12}^2 + v_{13}^2 + v_{23}^2 + \overline{v_{12}}^2 + \overline{v_{13}}^2 + \overline{v_{23}}^2,$$

$$v_{12}\overline{v_{12}} + v_{13}\overline{v_{13}} + v_{23}\overline{v_{23}}$$

and all the other v_{ij}'s found by equating

$$\begin{aligned}(v_{12} + v_{13} + v_{23} + \overline{v_{12}} + \overline{v_{13}} + \overline{v_{23}})^2 \\ - \left(v_{12}^2 + v_{13}^2 + v_{23}^2 + \overline{v_{12}}^2 + \overline{v_{13}}^2 + \overline{v_{23}}^2\right) \\ - (v_{12}\overline{v_{12}} + v_{13}\overline{v_{13}} + v_{23}\overline{v_{23}})\end{aligned}$$

i.e.

$$\Sigma_{i,j,k,l} \left\{v_{ij}v_{kl} : \{i,j\} \cap \{k,l\} \neq \{i,j\}\right\}.$$

Now note that

$$u_1u_2 + u_2u_3 + u_1u_3 = \tfrac{1}{2}\left[(u_1 + u_2 + u_3)^2 - (u_1^2 + u_2^2 + u_3^2)\right]$$

and so is redundant. Note also that since $v_{12}\overline{v_{12}} = u_1u_2$ and so on, we also have that

$$v_{12}\overline{v_{12}} + v_{13}\overline{v_{13}} + v_{23}\overline{v_{23}} = u_1u_2 + u_2u_3 + u_1u_3$$

and so is also redundant.

Now restrict these invariants to the space $\mathbf{V}$ so that now $z_1 + z_2 + z_3 = 0$ and not that this also means that $\overline{z_1} + \overline{z_2} + \overline{z_3} = 0$. We can now make our necessary list of invariants even smaller. First note that

$$(z_1 + z_2 + z_3)(\overline{z_1} + \overline{z_2} + \overline{z_3}) = 0$$

but

$$\begin{aligned}&(z_1 + z_2 + z_3)(\overline{z_1} + \overline{z_2} + \overline{z_3}) = \\ &\quad z_1\overline{z_1} + z_2\overline{z_2} + z_3\overline{z_3} + z_1\overline{z_2} + z_1\overline{z_3} + z_2\overline{z_3} + z_2\overline{z_1} + z_3\overline{z_1} + z_3\overline{z_2} = \\ &\quad u_1 + u_2 + u_3 + v_{12} + v_{13} + v_{23} + \overline{v_{12}} + \overline{v_{13}} + \overline{v_{23}}\end{aligned}$$

which means that

$$u_1 + u_2 + u_3 = -(v_{12} + v_{13} + v_{23} + \overline{v_{12}} + \overline{v_{13}} + \overline{v_{23}})$$

and so we have only *one* second order invariant left.

On $\mathbf{V}$ we also have

$$0 = z_i(\overline{z_i} + \overline{z_j} + \overline{z_k}) = z_i\overline{z_i} + z_i\overline{z_j} + z_i\overline{z_k} = u_i + v_{ij} + v_{ik}$$

which means that

$$u_i = -(v_{ij} + v_{ik}) = -(\overline{v_{ij}} + \overline{v_{ik}}).$$

This in turn means that

$$v_{ij} + \overline{v_{ij}} = u_k - u_i - u_j$$

and so

$$v_{ij}^2 + \overline{v_{ij}}^2 = u_i^2 + u_j^2 + u_k^2 - 2u_iu_k - 2u_ju_k$$

which means that using symmetry to obtain the appropriate equations we find that $v_{12}^2 + v_{13}^2 + v_{23}^2 + \overline{v_{12}}^2 + \overline{v_{13}}^2 + \overline{v_{23}}^2$ also becomes redundant, as does the final invariant of the list. Thus we are left with only two invariants needed up to fifth order.

ii) We now consider the equivariants.

As before, we first consider the equivariants on $\mathbf{C}^3$ and then by projecting them onto $\mathbf{V}$ we prove the results of the Proposition. This works since the vectors perpendicular to V are acted on trivially by our action of $\mathbf{S}_3 \times \mathbf{S}^1$. As a minimum, we need the equivariants to be of the form

$$\begin{bmatrix} z_1h_1(\underline{z}) \\ z_2h_1(\underline{z}) \\ z_3h_1(\underline{z}) \end{bmatrix}$$

where each h_i is $\mathbf{S}^1$ invariant, and we must also have that $h_1 + h_2 + h_3$ is $\mathbf{S}_3$ invariant. Therefore the h_i must consist of components of the above invariants, and also they must be invariant under the transposition (jk). To third order this means that we can only have $h_i = u_i$ or its 'complement' $h_i = u_j + u_k$ or $h_i = v_{ij} + v_{ik}$. However, we know from above that $v_{ij} + v_{ik} = -u_i$ and so is redundant.

Similarly for fifth order equivariants we must have $h_i = u_i^2$ or its complement $h_i = u_j^2 + u_k^2$ and the other possibilities are catered for: $h_i = u_ku_k$ or its complement can be obtained from, for example, $u_1(u_1 + u_2 + u_3) = u_1^2 + u_1u_2 + u_1u_3$, in another words as an invariant times the third order equivariant . Similarly we obtain

$$v_{ij}^2 + \overline{v_{ij}}^2 = u_i^2 + u_j^2 + u_k^2 - 2(u_i u_k + u_j u_k)$$

and so on.

Finally we have the linear equivariants

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \text{ and } \begin{bmatrix} z_1 + z_2 + z_3 \\ z_1 + z_2 + z_3 \\ z_1 + z_2 + z_3 \end{bmatrix}$$

but is is easy to see that the latter of these two will vanish on $\mathbf{V}$, and so we disregard it.

Therefore we have three equivariants to consider

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \begin{bmatrix} z_1^2\overline{z_1} \\ z_2^2\overline{z_2} \\ z_3^2\overline{z_3} \end{bmatrix} \text{ and } \begin{bmatrix} z_1^3\overline{z_1}^2 \\ z_2^3\overline{z_2}^2 \\ z_3^3\overline{z_3}^2 \end{bmatrix}.$$

We shall now project these mappings, via the projection π_V, onto $\mathbf{V}$, given by

$$\pi_V(z_1, z_2, z_3) = (z_1, z_2, z_3) - \tfrac{1}{3}(z_1 + z_2 + z_3)(1, 1, 1).$$

After scaling to get rid of fractions, this gives us the equivariants of the Proposition, namely

$$\mathbf{X}_0 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \ \mathbf{X}_1 = \begin{bmatrix} 2z_1^2\overline{z_1} - (z_2^2\overline{z_2} + z_3^2\overline{z_3}) \\ 2z_2^2\overline{z_2} - (z_1^2\overline{z_1} + z_3^2\overline{z_3}) \\ 2z_3^2\overline{z_3} - (z_1^2\overline{z_1} + z_3^2\overline{z_3}) \end{bmatrix}$$

and

$$\mathbf{X}_2 = \begin{bmatrix} 2z_1^3\overline{z_1}^2 - (z_2^3\overline{z_2}^2 + z_3^3\overline{z_3}^2) \\ 2z_2^3\overline{z_2}^2 - (z_1^3\overline{z_1}^2 + z_3^3\overline{z_3}^2) \\ 2z_3^3\overline{z_3}^2 - (z_1^3\overline{z_1}^2 + z_3^3\overline{z_3}^2) \end{bmatrix}.$$

∎

Therefore, we have that up to fifth order we can write

$$g(z_1, z_2, z_3, \lambda, \tau) = P\mathbf{X}_0 + Q\mathbf{X}_1 + R\mathbf{X}_2$$

where P, Q and R are complex functions of u, v, the bifurcation parameter λ and the period-scaling parameter τ. In addition, since we have assumed that the original vector field was in Birkhoff normal form we can also say that the function P can be written

$$P = P' - (1+\tau)i$$

where $P'(0) = i$ (due to the way in which the Liapunov Schmidt reduction is applied) See for example Golubitsky et al. [17]. When we have the branching equations we can then solve the complex part of these for τ, we do not explicitly carry out these calculations here, but note that they are routine.

Branching Equations

After the initial calculations, we write, for clarity, $z = re^{i\theta}$ so that $z\bar{z} = r^2$. We now find the branching equation for the solution corresponding to each isotropy with a two dimensional fixed point subspace, up to fifth order.

Isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ has fixed point subspace $Fix(\mathbf{S}_1 \times \mathbf{S}_2) = (2z, -z, -z)$ and so setting $g(z, \lambda, \tau) = 0$ yields that

$$6Pz + 18Qz^2\bar{z} + 66Rz^3\bar{z}^2 = 0.$$

The branching equation is then the real part of this (the imaginary part can be solved for τ) so that we have a branching equation of form

$$Re(P + 3Qr^2 + 11Rr^4) = 0. \tag{5.1.2}$$

Similarly for isotropy $\tilde{\mathbf{Z}}_2$, $Fix(\tilde{\mathbf{Z}}_2) = (z, -z, 0)$ and setting $g(z, \lambda, \tau) = 0$ gives us, from the first component of g,

$$3Pz + 3Qz^2\bar{z} + 3Rz^3\bar{z}^2 = 0$$

giving us a branching equation of

$$Re(P + Qr^2 + Rr^4) = 0. \tag{5.1.3}$$

Finally, isotropy $\tilde{\mathbf{Z}}_3$ has fixed point subspace $Fix(\tilde{\mathbf{Z}}_3) = (z, \zeta z, \zeta^2 z)$ where we have $\zeta = e^{2\pi/3}$, and again setting $g(z, \lambda, \tau) = 0$ gives us

$$(2 - \zeta - \zeta^2)Pz + (2 - \zeta^2\bar{\zeta} - \zeta^4\bar{\zeta}^2)Qz^2\bar{z} + (2 - \zeta^3\bar{\zeta}^2 - \zeta^6\bar{\zeta}^4)Rz^3\bar{z}^2 = 0.$$

After noting that we must have that $\zeta\bar{\zeta} = 1$ and $1 + \zeta + \zeta^2 = 0$, this can be rearranged so that the real part gives the branching equation

$$Re(P + Qr^2 + Rr^4) = 0. \tag{5.1.4}$$

Before proceeding any further with our stability calculations, we note some useful details concerning the derivatives of certain elements of our equations.

Preliminaries

We first note that to explicitly have that all the equivariant mappings are from $\mathbf{V}$ to $\mathbf{V}$ we use the equivariant

$$\begin{bmatrix} 2z_1 - z_2 - z_3 \\ 2z_2 - z_1 - z_3 \\ 2z_3 - z_1 - z_2 \end{bmatrix}$$

instead of the form computed earlier. The two are equivalent however, just add $z_1 + z_2 + z_3 = 0$ to each row.

If we write $g(\underline{z}, \lambda, \tau) = (\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ then, up to fifth order we have that

$$\mathbf{Z}_i = P(2z_i - z_j - z_k) + Q(2z_i^2\overline{z_i} - z_j^2\overline{z_j} - z_k^2\overline{z_k}) + R(2z_i^3\overline{z_i}^2 - z_j^3\overline{z_j}^2 - z_k^3\overline{z_k}^2),$$

$$u = z_i\overline{z_i} + z_j\overline{z_j} + z_k\overline{z_k}, \ v = (z_i\overline{z_i})^2 + (z_j\overline{z_j})^2 + (z_k\overline{z_k})^2.$$

and so

$$u_{z_i} = \overline{z_i}, \ u_{\overline{z_i}} = z_i, \ u_{z_j} = \overline{z_j}, \ u_{\overline{z_j}} = z_j$$

$$v_{z_i} = 2z_i\overline{z_i}^2, \ v_{\overline{z_i}} = 2z_i^2\overline{z_i}, \ v_{z_j} = 2z_j\overline{z_j}^2, \ v_{\overline{z_j}} = 2z_j^2\overline{z_j}$$

where subscripts denote 'derivatives with respect to' and

$$\begin{aligned}
\mathbf{Z}_{i,z_i} = {} & 2P + P_u\overline{z_i}(2z_i - z_j - z_k) + 2P_v z_i\overline{z_i}^2(2z_i - z_j - z_k) \\
& +4Qz_i\overline{z_i} + Q_u\overline{z_i}(2z_i^2\overline{z_i} - z_j^2\overline{z_j} - z_k^2\overline{z_k}) + 6Rz_i^2\overline{z_i}^2,
\end{aligned}$$

$$\begin{aligned}
\mathbf{Z}_{i,\overline{z_i}} = {} & P_u z_i(2z_i - z_j - z_k) + 2P_v z_i^2\overline{z_i}(2z_i - z_j - z_k) \\
& +2Qz_i^2 + Q_u z_i(2z_i^2\overline{z_i} - z_j^2\overline{z_j} - z_k^2\overline{z_k}) + 4Rz_i^3\overline{z_i},
\end{aligned}$$

$$\begin{aligned}
\mathbf{Z}_{i,z_j} = {} & -P + P_u\overline{z_j}(2z_i - z_j - z_k) + 2P_v z_j\overline{z_j}^2(2z_i - z_j - z_k) \\
& -2Qz_j\overline{z_j} + Q_u\overline{z_j}(2z_i^2\overline{z_i} - z_j^2\overline{z_j} - z_k^2\overline{z_k}) - 3Rz_j^2\overline{z_j}^2,
\end{aligned}$$

$$\begin{aligned}
\mathbf{Z}_{i,\overline{z_j}} = {} & P_u z_j(2z_i - z_j - z_k) + 2P_v z_j^2\overline{z_j}(2z_i - z_j - z_k) \\
& -Qz_j^2 + Q_u z_j(2z_i^2\overline{z_i} - z_j^2\overline{z_j} - z_k^2\overline{z_k}) - 2Rz_j^3\overline{z_j}.
\end{aligned}$$

Where the second subscript of the $\mathbf{Z}_i$'s denotes derivatives.

Before stating the theorem for the stabilities of each solution branch, we explain why the above calculations will become useful in its proof. The reason is that in finding the eigenvalues of the corresponding $(dg)_{x_0}$, we use the methods of Golubitsky et al. [17] (Section XVIII 3) where they considered the coordinates of our problem to be z and $\overline{z}$, as opposed to x and y where $z = x + iy$, and then noted that an **R**-linear mapping on **C** has precisely the form

$$w \longrightarrow \alpha w + \beta \overline{w}$$

where α and β are both in **C**. The trace of this mapping is then given as $2Re(\alpha)$ and the determinant as $|\alpha|^2 - |\beta|^2$.

We then note that $(dg)_{x_0}|_{V_0}$ and $(dg)_{x_0}|_{V_1}$ are both **R**-linear maps on **C** where $\mathbf{V}_0$ is the fixed point subspace and $\mathbf{V}_1$ its complementary isotypic component, and

$$\alpha w_k = d_z g.w_k, \;\; \beta w_k = d_{\overline{z}} g.w_k$$

where w_k is a basis vector for $\mathbf{V}_k$.

We then recover the theorem of [17], with our notation and Γ-Simple action.

Theorem 5.1.6 *Assuming suitable non-degeneracy conditions, there exist precisely one branch of small amplitude, near-2π periodic solutions, for each of the isotropy subgroups $\mathbf{S}_1 \times \mathbf{S}_2$, $\tilde{\mathbf{Z}}_2$ and $\tilde{\mathbf{Z}}_3$. Assume that the trivial branch is stable sub-critically and loses stability as λ passes through 0, then*

i) The $\mathbf{S}_1 \times \mathbf{S}_2$ branch is super- or sub-critical according to whether

$$Re(2P_u(0) + Q(0))$$

is positive or negative. It is stable if $Re(2P_u(0) + Q(0)) > 0$, $Re(Q(0)) < 0$ and $Re(Q(0)\overline{R(0)}) > 0$.

ii) The $\widetilde{\mathbf{Z}_2}$ branch is super- or sub-critical according to whether $Re(2P_u(0) + Q(0))$ is positive or negative. It is stable if $Re(2P_u(0) + Q(0)) > 0$, $Re(Q(0)) < 0$ and $Re(Q(0)\overline{R(0)}) < 0$.

iii) The $\widetilde{\mathbf{Z}_3}$ branch is super- or sub-critical according to whether $Re(3P_u(0) + Q(0))$ is positive or negative. It is stable if $Re(3P_u(0) + Q(0)) > 0$, $Re(Q(0)) > 0$ and $3|P_u(0)|^2 + 2Re(P_u(0)\overline{Q(0)}) < 0$.

In particular note that only one branch may be stable at any one time, and for any branch to be stable, all three branches must be supercritical.

Proof:

i) For isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ we have that $\mathbf{V}_0 = Fix(\mathbf{S}_1 \times \mathbf{S}_2) = (2w, -w, -w)$, its other isotypic component is given by $(0, w, -w)$ and the branching equation is

$$P + 3Qr^2 + 11Rr^4 = 0.$$

$\mathbf{V}_0$

One of the eigenvalues corresponding to the fixed point subspace is forced to be zero, so therefore the other eigenvalue is given by the trace, $2Re(\alpha)$, where

$$2\alpha = 2\mathbf{Z}_{1,z_1} - \mathbf{Z}_{1,z_2} - \mathbf{Z}_{1,z_3}.$$

Now

$$\begin{aligned}
\mathbf{Z}_{1,z_1} &= 2P + 12P_u w\overline{w} + 96P_v(w\overline{w})^2 + 16Qw\overline{w} + 36Q_u(w\overline{w})^2 + 96R(w\overline{w})^2, \\
\mathbf{Z}_{1,z_2} &= -P - 6P_u w\overline{w} - 12P_v(w\overline{w})^2 - 2Qw\overline{w} - 18Q_u(w\overline{w})^2 - 3R(w\overline{w})^2, \\
\mathbf{Z}_{1,z_3} &= -P - 6P_u w\overline{w} - 12P_v(w\overline{w})^2 - 2Qw\overline{w} - 18Q_u(w\overline{w})^2 - 3R(w\overline{w})^2.
\end{aligned}$$

So that

$$2\alpha = 6P + 36P_u r^2 + 216P_v r^4 + 36Qr^2 + 108Q_u r^4 + 198Rr^4$$

which after substituting in the branching equation gives us

$$2\alpha = 36P_u r^2 + 216P_v r^4 + 18Qr^2 + 108Q_u r^4 + 132Rr^4 \qquad (5.1.5)$$

giving the eigenvalue

$$Re\left((36P_u + 18Q)r^2 + (216P_v + 108Q_u + 132R)r^4\right)$$

Hence, up to third order, which is all we need, the sign of the eigenvalue is given by the sign of $Re(2P_u + Q)$.

$\mathbf{V}_1$

The eigenvalues corresponding to the other isotypic component can be found from $\alpha = \mathbf{Z}_{2,z_2} - \mathbf{Z}_{2,z_3}$ and $\beta = \mathbf{Z}_{2,\overline{z_2}} - \mathbf{Z}_{2,\overline{z_3}}$ where

$$\begin{aligned}
\mathbf{Z}_{2,z_2} &= 2P + 3P_u w\overline{w} + 6P_v(w\overline{w})^2 + 4Qw\overline{w} + 9Q_u(w\overline{w})^2 + 6R(w\overline{w})^2, \\
\mathbf{Z}_{2,z_3} &= -P + 3P_u w\overline{w} + 6P_v(w\overline{w})^2 - 2Qw\overline{w} + 9Q_u(w\overline{w})^2 - 3R(w\overline{w})^2, \\
\mathbf{Z}_{2,\overline{z_2}} &= 3P_u w^2 + 6P_v w^2(w\overline{w}) + 2Qw^2 + 9Q_u w^2(w\overline{w}) + 4Rw^2(w\overline{w}), \\
\mathbf{Z}_{2,\overline{z_3}} &= 3P_u w^2 + 6P_v w^2(w\overline{w}) - Qw^2 + 9Q_u w^2(w\overline{w}) - 2Rw^2(w\overline{w}).
\end{aligned}$$

So that

$$\alpha = 3P + 6Qw\overline{w} + 9R(w\overline{w})^2 = -3Qr^2 - 24Rr^4 = -3r^2(Q + 8Rr^2)$$

and

$$\beta = 3Qw^2 + 6Rw^2(w\overline{w}) = 3Qw^2 + 6Rw^2r^2 = 3w^2(Q + 2Rr^2),$$

giving $trace = 2Re(\alpha) = -6r^2Re(Q + 8Rr^2)$, so that the sign of the trace is given by the sign of $-Re(Q + 8Rr^2)$. Now note that $|w\overline{w}| = |r^2| = r^2$ and $|w^2| = |r^2e^{2i\theta}| = r^2$ so that

$$|\alpha| = 3r^2|Q + 8Rr^2|$$

and

$$|\beta| = 3r^2|Q + 2Rr^2|.$$

Up to third order $det((dg)_{x_0}) = 0$ implying another zero eigenvalue other than that imposed by the symmetry, so we calculate up to fifth order.

$$det((dg)_{x0})|_{V_1} = |\alpha|^2 - |\beta|^2 = 9r^4\left(|Q + 8Rr^2|^2 - |Q + 2Rr^2|^2\right)$$

So, remembering that $|z|^2 = z\overline{z}$ and $z + \overline{z} = 2Re(z)$ we have

$$\begin{aligned} det((dg)_{x_0})|_{V_1} &= 9r^4\left[(Q + 8Rr^2)(\overline{Q} + 8\overline{R}r^2) - (Q + 2Rr^2)(\overline{Q} + 2\overline{R}r^2)\right] \\ &= 108r^6\left[12|R|^2r^2 + Re(Q\overline{R})\right]. \qquad (5.1.6) \end{aligned}$$

Taking the lowest necessary orders for each of our three calculations, trace for $\mathbf{V}_0$ and trace and determinant for $\mathbf{V}_1$ gives us the result of the Theorem.

ii) For isotropy $\widetilde{\mathbf{Z}}_2$ we have that $\mathbf{V}_0 = Fix(\widetilde{\mathbf{Z}}_2) = (w, -w, 0)$, the other isotypic component is given by $\mathbf{V}_1 = (-w, -w, 2w)$ and the branching equation by

$$P + Qr^2 + Rr^4 = 0.$$

$\mathbf{V}_0$

Again, one of the eigenvalues is forced to zero by the symmetry, and so the one non-zero eigenvalue corresponding to $\mathbf{V}_0$ is given by the trace, $2Re(\alpha)$, where this time $\alpha = \mathbf{Z}_{1,z_1} - \mathbf{Z}_{1,z_2}$ with

$$\begin{aligned} \mathbf{Z}_{1,z_1} &= 2P + 3P_uw\overline{w} + 6P_v(w\overline{w})^2 + 4Qw\overline{w} + 3Q_u(w\overline{w})^2 + 6R(w\overline{w})^2, \\ \mathbf{Z}_{1,z_2} &= -P - 3P_uw\overline{w} - 6P_v(w\overline{w})^2 - 2Qw\overline{w} - 3Q_u(w\overline{w})^2 - 3R(w\overline{w})^2 \end{aligned}$$

so that

$$\alpha = 3P + 6P_uw\overline{w} + 12P_v(w\overline{w})^2 + 6Qw\overline{w} + 6Q_u(w\overline{w})^2 + 9R(w\overline{w})^2$$

which, after substituting the branching equation, gives us

$$\alpha = 6P_u w\overline{w} + 12P_v(w\overline{w})^2 + 3Qw\overline{w} + 6Q_u(w\overline{w})^2 + 6R(w\overline{w})^2. \qquad (5.1.7)$$

This means that up to third order the trace is given by

$$2Re(\alpha) = 6r^2 Re(2P_u + Q).$$

$\mathbf{V}_1$

On the other isotypic component we have that $-\alpha = -\mathbf{Z}_{1,z_1} - \mathbf{Z}_{1,z_2} + 2\mathbf{Z}_{1,z_3}$ and $-\beta = -\mathbf{Z}_{1,\overline{z_1}} - \mathbf{Z}_{1,\overline{z_2}} + 2\mathbf{Z}_{1,\overline{z_3}}$ where

$$\begin{aligned}
\mathbf{Z}_{1,z_3} &= -P, \\
\mathbf{Z}_{1,\overline{z_1}} &= 3P_u w^2 + 6P_v w^2 w\overline{w} + 2Qw^2 + 3Q_u w^2(w\overline{w}) + 4Rw^2(w\overline{w}), \\
\mathbf{Z}_{1,\overline{z_2}} &= -3P_u w^2 - 6P_v w^2 w\overline{w} - Qw^2 - 3Q_u w^2(w\overline{w}) + 2Rw^2(w\overline{w}), \\
\mathbf{Z}_{1,\overline{z_3}} &= 0
\end{aligned}$$

so that

$$\alpha = 3P + 2Qw\overline{w} + 3R(w\overline{w})^2 = -Qw\overline{w} = -Qr^2$$

and

$$\beta = Qw^2 + 6Rw^2(w\overline{w}) = w^2(Q + 6Rr^2).$$

This means that up to third order the trace is given by

$$trace = 2Re(\alpha) = -2r^2 Re(Q)$$

and for the determinant we again work to fifth order and get

$$det = -12r^6\left[3|R|^2r^2 + Re(Q\overline{R})\right].$$

Taking each calculation to lowest necessary order, again gives us the result of the Theorem.

iii) For isotropy $\widetilde{\mathbf{Z}_3}$ we have that $\mathbf{V}_0 = Fix(\widetilde{\mathbf{Z}_3}) = (w, \zeta w, \zeta^2 w)$, the other isotypic component is given by $\mathbf{V}_1 = (w, \zeta^2 w, \zeta w)$, where $\zeta = e^{2\pi i/3}$, and the branching equation by $P + Qr^2 + Rr^4 = 0$.

$\mathbf{V}_0$

Yet again one of our eigenvalues corresponding to $\mathbf{V}_0$ is forced to be zero by the symmetry, and so the other is given by the trace, $2Re(\alpha)$, where $\alpha = \mathbf{Z}_{1,z_1} + \zeta\mathbf{Z}_{1,z_2} + \zeta^2\mathbf{Z}_{1,z_3}$. To calculate α we note that, using the facts that $\zeta\overline{\zeta} = 1$, $1+\zeta+\zeta^2 = 0$, $\overline{\zeta} = \zeta^2$ and $\zeta^3 = 1$, we have

$$\begin{aligned}
\mathbf{Z}_{1,z_1} &= 2P + 3P_u w\overline{w} + 6P_v(w\overline{w})^2 + 4Qw\overline{w} + 3Q_u(w\overline{w})^2 + 6R(w\overline{w})^2, \\
\mathbf{Z}_{1,z_2} &= -P + 3\zeta^2 P_u w\overline{w} + 6\zeta^2 P_v(w\overline{w})^2 - 2Qw\overline{w} + 3\zeta^2 Q_u(w\overline{w})^2 - 3R(w\overline{w})^2, \\
\mathbf{Z}_{1,z_3} &= -P + 3\zeta P_u w\overline{w} + 6\zeta P_v(w\overline{w})^2 - 2Qw\overline{w} + 3\zeta Q_u(w\overline{w})^2 - 3R(w\overline{w})^2.
\end{aligned}$$

This gives us that

$$\alpha = 3P + 9P_u r^2 + 18P_v r^4 + 6Qr^2 + 9Q_u r^4 + 9Rr^4$$

so that after substituting the branching equation we have

$$\alpha = 9P_u r^2 + 18P_v r^4 + 3Qr^2 + 9Q_u r^4 + 6Rr^4. \tag{5.1.8}$$

Therefore, up to third order the trace is given by $6r^2Re(3P_u + Q)$.

$\mathbf{V}_1$

When we restrict ourselves to the other isotypic component of $\widetilde{\mathbf{Z}_3}$ the trace and determinant are determined by $\alpha = \mathbf{Z}_{1,z_1} + \zeta^2\mathbf{Z}_{1,z_2} + \zeta\mathbf{Z}_{1,z_3}$ and also by $\beta = \mathbf{Z}_{1,\overline{z_1}} + \zeta^2\mathbf{Z}_{1,\overline{z_2}} + \zeta\mathbf{Z}_{1,\overline{z_3}}$ where for these values we need the following additional calculations.

$$\begin{aligned}
\mathbf{Z}_{1,\overline{z_1}} &= 3P_u w^2 + 6P_v w^2(w\overline{w}) + 2Qw^2 + 3Q_u w^2(w\overline{w}) + 4Rw^2(w\overline{w}), \\
\mathbf{Z}_{1,\overline{z_2}} &= 3\zeta P_u w^2 + 6\zeta P_v w^2(w\overline{w}) - \zeta^2 Qw^2 + 3\zeta Q_u w^2(w\overline{w}) - 2\zeta^2 Rw^2(w\overline{w}), \\
\mathbf{Z}_{1,\overline{z_3}} &= 3\zeta^2 P_u w^2 + 6\zeta^2 P_v w^2(w\overline{w}) - \zeta Qw^2 + 3\zeta^2 Q_u w^2(w\overline{w}) - 2\zeta Rw^2(w\overline{w}).
\end{aligned}$$

This gives us

$$\begin{aligned}
\alpha &= (2-\zeta^2-\zeta)P + (3+3\zeta+3\zeta^2)P_u r^2 + (6+6\zeta+6\zeta^2)P_v r^4 \\
&\quad +(4-2\zeta^2-2\zeta)Qr^2 + (3+3\zeta+3\zeta^2)Q_u r^4 + (6-3\zeta^2-3\zeta)Rr^4
\end{aligned}$$

which yields

$$\alpha = 3P + 6Qr^2 + 9Rr^4 = 3Qr^2 + 6Rr^4 \tag{5.1.9}$$

or

$$\alpha = 3r^2[Q + 2Rr^2],$$

and the trace is twice this. We will also have

$$\begin{aligned}\beta &= (3+3+3)P_u w^2 + (6+6+6)P_v w^2(w\overline{w}) + (2-\zeta-\zeta^2)Qw^2 \\ &\qquad +(3+3+3)Q_u w^2(w\overline{w}) + (4-2\zeta-2\zeta^2)Rw^2(w\overline{w}) \\ &= 9P_u w^2 + 18P_v w^2 r^2 + 3Qw^2 + 9Q_u w^2 r^2 + 6Rw^2 r^2,\end{aligned}$$

giving

$$\beta = 3w^2\left[3P_u + Q + (6P_v + 3Q_u + 2R)r^2\right].$$

For the determinant it turns out that we need only consider everything up to third order, but for completeness we do the calculations to fifth order. We find that

$$\begin{aligned}|\alpha|^2 &= 9r^4(Q+2Rr^2)(\overline{Q+2Rr^2}) \\ &= 9r^4\left[|Q|^2 + 4|R|^2r^4 + 4r^2Re(Q\overline{R})\right]\end{aligned}$$

and similarly

$$|\beta|^2 = 9r^4\left(3P_u + Q + (6P_v + 3Q_u + 2R)r^2\right)\left(\overline{3P_u + Q + (6P_v + 3Q_u + 2R)r^2}\right).$$

When multiplied out, and subtracted from the previous expression this gives us

$$\begin{aligned}&|\alpha|^2 - |\beta|^2 = \\ &\quad -27r^4\Big[3|P_u|^2 + 2Re(P_u\overline{Q}) \\ &\quad +2r^2[6Re(P_u\overline{P_v}) + 3Re(P_u\overline{Q}) + 2Re(P_u\overline{R}) + 2Re(Q\overline{P_v}) + Re(Q\overline{Q_u})] \\ &\quad +r^4[12|P_V|^2 + 3|Q_u|^2 + 12Re(P_v\overline{Q_u}) + 8Re(P_v\overline{R}) + 4Re(Q_u\overline{R})]\Big]\end{aligned}$$

so that taking the lowest necessary order for each calculation above yields the result of the Theorem, and we are done.

∎

The results of Theorem 5.1.6 are summarised in Table 5.1, and the bifurcation diagrams in the cases where there is a stable branch of solutions are shown in Figure 5.1.

This finishes our analysis of Hopf Bifurcation in the presence of $\mathbf{S}_3$ symmetry, and so we now consider what happens if we have three coupled cells, where each cell has an internal $\mathbf{Z}_2$ symmetry. In exactly the same way as for steady-state bifurcation the global symmetry group depends on whether the coupling is invariant or equivariant with respect to the internal symmetries. The exact form of the equations will be dealt with later when we consider the example of coupled oscillators, but, as before the coupling can lead to either $\mathbf{Z}_2 \wr \mathbf{S}_3$ or $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetries.

Isotropy (Σ)	Fix(Σ)	Branching Equation	Signs of Eigenvalues
$\mathbf{S}_1 \times \mathbf{S}_2$	$(2w, -w, -w)$	$P + 3Qr^2 + 11Rr^4 = 0$	$Re(2P_u + Q)$ $trace = -Re(Q)$ $det = Re(Q\overline{R})$
$\widetilde{\mathbf{Z}_2}$	$(-w, w, 0)$	$P + Qr^2 + Rr^4 = 0$	$Re(2P_u + Q)$ $trace = -Re(Q)$ $det = -Re(Q\overline{R})$
$\widetilde{\mathbf{Z}_3}$	$(w, \zeta w, \zeta^2 w)$	$P + Qr^2 + Rr^4 = 0$	$Re(3P_u + Q)$ $trace = Re(Q)$ $det =$ $-[3\|P_u\|^2 + 2Re(P_u\overline{Q})]$

Table 5.1: Stability of branches of solutions in the presence of $\mathbf{S}_3$ symmetry arising from Hopf bifurcations.

5.2 Hopf Bifurcation With $\mathbf{Z}_2 \wr \mathbf{S}_3$ Symmetry

We take the irreducible representation of $\mathbf{Z}_2 \wr \mathbf{S}_3$ on $\mathbf{R}^3$ and, in the manner of Golubitsky et al. [17], we extend it to an action of $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$ on $\mathbf{R}^3 \oplus \mathbf{R}^3 \cong \mathbf{C}^3$. That is, we let $\mathbf{S}_3$ act on $(z_1, z_2, z_3) \in \mathbf{C}^3$ by permutation of indices as in the previous section, $\mathbf{Z}_2$ by $\kappa z_i = -z_i$, where $\kappa \in \mathbf{Z}_2$, and we introduce the phase shift $\theta \in \mathbf{S}^1$ which acts by $\theta z = e^{i\theta} z$. As for the steady-state case the wreath product action is then obtained by

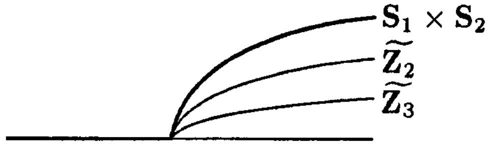
$$(\underline{\kappa}, \pi) \cdot z = \begin{bmatrix} \kappa(1) z_{\rho(1)} \\ \kappa(2) z_{\rho(2)} \\ \kappa(3) z_{\rho(3)} \end{bmatrix}$$

where $\rho \in \mathbf{S}_3$ and $\underline{\kappa} \in {\mathbf{Z}_2}^3$.

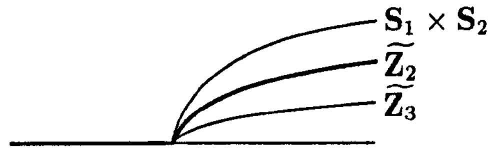
$\mathbf{Z}_2 \wr \mathbf{S}_3$ now acts on $\mathbf{C}^3$ Γ-Simply, and so we can apply the results of [17], the Equivariant Hopf Theorem, to find the solution branches corresponding to isotropy subgroups with two-dimensional fixed point subspaces.

Isotropy Subgroups

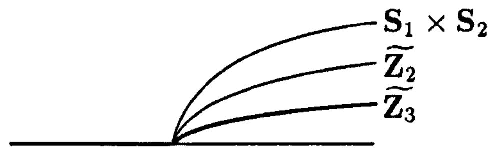
We let $\zeta = e^{2\pi i/3}$, and list the isotropy subgroups of $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$, together with the other necessary details in the following proposition



$Re(Q(0)) < 0, Re(2P_u(0) + Q(0)) > 0,$
$Re(Q\overline{R}) > 0$



$Re(Q(0)) < 0, Re(2P_u(0) + Q(0)) > 0,$
$Re(Q\overline{R}) < 0$



$Re(Q(0)) > 0, Re(3P_u(0) + Q(0)) > 0,$
$Re(2P_u(0) + Q(0)) > 0$

Figure 5.1: Representative bifurcation diagrams for $\mathbf{S}_3$ Hopf bifurcations when some solution branch is stable. Thick lines denote stability, thin lines instability.

Proposition 5.2.1 *The list of isotropy subgroups, up to conjugacy, of $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$ acting on $\mathbf{C}^3$ is given by the subgroups generated by each Σ in the following table and the element $[(\kappa, \kappa, \kappa), \pi]$ which acts as the identity in our representation. For clarity we denote the isotropy generated by Σ and $[(\kappa, \kappa, \kappa), \pi]$ by only the symbol Σ.*

Group Orbit	*Isotropy* Σ	$Fix(\Sigma) = \mathbf{V}_0$	$\mathbf{V}_1$	$dim Fix(\Sigma)$
$(0,0,0)$	$\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$	$(0,0,0)$	(u,v,w)	*0*
(z,z,z)	$\mathbf{S}_3$	(w,w,w)	$(u,v,-(u+v))$	*2*
$(z,z,0)$	$\mathbf{W}_2$	$(w,w,0)$	$(w,-w,0)$	*2*
$(z,0,0)$	$\mathbf{W}_1$	$(w,0,0)$	$(0,u,v)$	*2*
$(z,\zeta z,\zeta^2 z)$	$\widetilde{\mathbf{Z}}_3$	$(w,\zeta w,\zeta^2 w)$	$(u,v,-(\zeta u+\zeta^2 v)$	*2*
(z_1,z_2,z_2)	$\mathbf{S}_1 \times \mathbf{S}_2$	(u,v,v)	$(0,w,-w)$	*4*
$(z_1,z_2,0)$	$[(\kappa,\kappa,0),\pi]$	$(u,v,0)$	$(0,0,w)$	*4*
(z_1,z_2,z_3)	$\mathbf{1}$	(u,v,w)	$(0,0,0)$	*6*

We must, in addition, have a third isotypic component, $\mathbf{V}_2 = (0,0,w)$ for isotropy subgroup $\mathbf{W}_2$.

Proof:

It should be clear that these *are* isotropy subgroups, we now show that they are the only ones.

Assume we have a non-zero vector (z_1, z_2, z_3), since $(0,0,0)$ has isotropy $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$. If all the variables have different modulus, $|z_1| \neq |z_2| \neq |z_3|$ then we have isotropy $\mathbf{S}_3$, so now assume not all entries are distinct up to multiplication by $e^{i\theta}$, some θ. By conjugacy assume $|z_1| = |z_2|$, giving a vector of form (w, aw, z) where $a \in \mathbf{C}$ and $|w| \neq |z|$. If z is non-zero, then there is no permutation in $\mathbf{S}_3$ that will fix the original vector after a multiplication by $e^{i}\theta$, and so we have isotropy conjugate to $\mathbf{S}_1 \times \mathbf{S}_2$ if $a = 1$, or $\mathbf{1}$ if $a \neq 1$. If $z = 0$ then if $a = 1$ we have $(w, w, 0)$ which is fixed by both the permutation (12) and the element $[(1,1,-1), id] \in \mathbf{Z}_2 \wr \mathbf{S}_3$ and so has isotropy $\mathbf{W}_2$. The only other possibility is that we have a twisted subgroup, and so $(w, aw, 0) = (e^{i\theta}aw, e^{i\theta}w, 0)$, and so $a = e^{i\theta} = -1$ giving $\theta = \pi$, a vector of form $(w, -w, 0)$ and an isotropy of $\widetilde{\mathbf{Z}_2}$. But $(w, -w, 0)$ is on the same group orbit as $(w, w, 0)$ and so has isotropy conjugate to $\mathbf{W}_2$.

If $a \neq \pm 1$ then we must have isotropy $\mathbf{1}$.

The final possibility is $|z_1| = |z_2| = |z_3|$, and so a vector of form (w, aw, bw). By the proof of the isotropies of $\mathbf{S}_3 \times \mathbf{S}^1$ we know a twisted subgroup must satisfy $a = e^{2\pi i/3}$ and $b = e^{4\pi i/3}$ or another solution on the group orbit, or we can have $a = b \neq 1$ with isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ or $a = b = 1$ with isotropy $\mathbf{S}_3$.

∎

Therefore by the Equivariant Hopf Theorem we are guaranteed branches of solutions having as their isotropies those isotropy subgroups with two dimensional fixed point subspaces, namely $\mathbf{S}_3$, $\mathbf{W}_2$, $\mathbf{W}_1$ and $\widetilde{\mathbf{Z}_3}$ (up to conjugacy).

Remark 5.2.2 *The theory no longer guarantees a branch of solutions with isotropy* $\mathbf{S}_1 \times \mathbf{S}_2$ *since now* $Fix(\mathbf{S}_1 \times \mathbf{S}_2)$ *does not have a two dimensional fixed point subspace), but since the theory for* $\mathbf{S}_3$ *does do so, we would expect such a branch to exist with suitable non-generic conditions.*

We now consider the stabilities of the branches of solutions which correspond to the isotropies with two dimensional fixed point subspaces.

Stabilities Of Solutions

The invariants with $\mathbf{S}^1$ symmetry have been dealt with (see proposition 5.1.3), and so we now consider the invariants and equivariants with $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$ symmetry. Note that this becomes a little more complicated than the case of $\mathbf{S}_3$ symmetry because we no longer have the fact that $z_1 + z_2 + z_3 = 0$ which helped simplify the lists before.

In the proof we use the notation of Proposition 5.1.3 that $u_i = z_i\overline{z_i}$ and $v_{ij} = z_i\overline{z_j}$.

Lemma 5.2.3 *The details of the* $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$ *invariants and equivariants on the* Γ*-Simple subspace* $\mathbf{C}^3$ *are as given below.*

i) Up to fifth order every $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$ *invariant germ* $f : \mathbf{C}^3 \to \mathbf{R}$ *has the form* $f(u, v, w)$ *where*

$$u = z_1\overline{z_1} + z_2\overline{z_2} + z_3\overline{z_3},$$
$$v = (z_1\overline{z_1})^2 + (z_2\overline{z_2})^2 + (z_3\overline{z_3})^2$$

and

$$w = (z_1\overline{z_2})^2 + (z_1\overline{z_3})^2 + (z_2\overline{z_3})^2 + (z_2\overline{z_1})^2 + (z_3\overline{z_1})^2 + (z_3\overline{z_2})^2.$$

ii) Every smooth $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$ *equivariant map germ* $h : \mathbf{C}^3 \to \mathbf{C}^3$ *has the form (up to at least fifth order)*

$$h(z_1, z_2, z_3) = P\mathbf{X}_1 + Q\mathbf{X}_2 + R\mathbf{X}_3 + S\mathbf{X}_4$$

where

$$\mathbf{X}_1 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \ \mathbf{X}_2 = \begin{bmatrix} z_1^2\overline{z_1} \\ z_2^2\overline{z_2} \\ z_3^2\overline{z_3} \end{bmatrix}, \ \mathbf{X}_3 = \begin{bmatrix} z_1^3\overline{z_1}^2 \\ z_2^3\overline{z_2}^2 \\ z_3^3\overline{z_3}^2 \end{bmatrix}$$

and

$$\mathbf{X}_4 = \begin{bmatrix} z_1\left((z_2\overline{z_3})^2 + (\overline{z_2}z_3)^2\right) \\ z_2\left((z_1\overline{z_3})^2 + (\overline{z_1}z_3)^2\right) \\ z_3\left((z_1\overline{z_2})^2 + (\overline{z_1}z_2)^2\right) \end{bmatrix},$$

where P, Q, R *and* S *are complex-valued* $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$*-invariant functions.*

Proof:

i) To be invariant under the action of $\mathbf{S}^1$, the invariants here must be built using the $\mathbf{S}^1$ invariants u_i and v_{ij}, and to be $\mathbf{S}_3$ invariant, each invariant function must contain all the possible permutations of indices. We now consider how the action of ${\mathbf{Z}_2}^3$ acts on each of these mappings. Let $(\kappa_1, \kappa_2, \kappa_3) \in {\mathbf{Z}_2}^3$ then we have that $\kappa_i u_i = u_i$, $\kappa_j u_i = u_i$, $\kappa_i v_{ij} = \kappa_j v_{ij} = -v_{ij}$ and $\kappa_i v_{jk} = v_{jk}$. This means that all the invariants must be built up from all possible u_i or v_{ij}^2 to be invariant under the $\mathbf{Z}_2$ action. We then note that $2[u_1u_2 + u_1u_3 + u_2u_3] = (u_1 + u_2 + u_3)^2 - u_1^2 + u_2^2 + u_3^2$ and so is redundant.

This gives us, up to fifth order the result of the Lemma.

ii) The equivariants must be of the form

$$\begin{bmatrix} z_1 h_1(\underline{z}) \\ z_2 h_2(\underline{z}) \\ z_3 h_3(\underline{z}) \end{bmatrix}$$

where h_i is invariant under the action of $\mathbf{Z}_2{}^3$, but if $\rho \in \mathbf{S}_3$ then $\rho h_i(\underline{z}) = h_{\rho(i)}(\underline{z})$. So we need, up to fifth order, $h_i(\underline{z}) = id$, $h_i(\underline{z}) = u_i$ (or its complement $u - u_i$), u_i^2 (or its complement $v - u_i^2$) or $v_{jk}^2 + \overline{v_{jk}}^2$ (or its complement $w - v_{jk}^2 - \overline{v_{jk}}^2$). This means that the only possible equivariants, up to fifth order are

$$\mathbf{X}_1 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}, \; \mathbf{X}_2 = \begin{bmatrix} z_1 u_1 \\ z_2 u_2 \\ z_3 u_3 \end{bmatrix}, \; \mathbf{X}_3 = \begin{bmatrix} z_1 u_1^2 \\ z_2 u_2^2 \\ z_3 u_3^2 \end{bmatrix}$$

and

$$\mathbf{X}_1 = \begin{bmatrix} z_1(v_{23}^2 + \overline{v_{23}}^2) \\ z_2(v_{13}^2 + \overline{v_{13}}^2) \\ z_3(v_{12}^2 + \overline{v_{12}}^2) \end{bmatrix}.$$

Substituting back for z and $\overline{z}$ yields the required result, and we are done.

∎

Therefore, we can write our bifurcation problem g, up to at least fifth order, as

$$g(z, \lambda, \tau) = P\mathbf{X}_1 + Q\mathbf{X}_2 + R\mathbf{X}_3 + S\mathbf{X}_4 + \ldots \tag{5.2.10}$$

where P, Q, R and S are complex-valued functions of u, v, w, the bifurcation parameter λ and period-scaling parameter τ.

As with the $\mathbf{S}_3 \times \mathbf{S}^1$ case, since we have assumed the original (before Liapunov Schmidt reduction) vector field to be in Birkhoff normal form we again have that

$$P = P' - (1 + \tau)i$$

where $P'(0) = i$, and using this we can solve for τ the imaginary part of the branching equations. As before however we do not perform these routine calculations here.

Branching Equations

As before, we set $g(z, \lambda, \tau) = 0$, and restrict g to $Fix(\Sigma)$ for each isotropy subgroup Σ of $\mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$. Remember that $\zeta = e^{2\pi i/3}$.

Isotropy $\mathbf{S}_3$ has $Fix(\mathbf{S}_3) = (z, z, z)$ so that $g = Pz + Qz^2\overline{z} + Rz^3\overline{z}^2 + 2Sz^3\overline{z}^2$. Setting to zero and taking the real part (again the imaginary part can be solved for τ, the period-scaling parameter) yields the branching equation

$$Ra(P + Qr^2 + Rr^4 + 2Sr^4) = 0. \tag{5.2.11}$$

Isotropy $\mathbf{W}_2$ has $Fix(\mathbf{W}_2) = (z, z, 0)$ giving $g(\underline{z}, \lambda) = Pz + Qz^2\overline{z} + Rz^3\overline{z}^2$, and so the branching equation

$$Re(P + Qr^2 + Rr^4) = 0. \tag{5.2.12}$$

Isotropy $\mathbf{W}_1$ has $Fix(\mathbf{W}_1) = (z, 0, 0)$, but the same branching equation as isotropy $\mathbf{W}_2$, namely

$$Re(P + Qr^2 + Rr^4) = 0. \tag{5.2.13}$$

Isotropy $\widetilde{\mathbf{Z}}_3$ has $Fix(\widetilde{\mathbf{Z}}_3) = (z, \zeta z, \zeta^2 z)$, but the only term that needs any special care is the coefficient of S which is

$$\begin{aligned} & z\left[(\zeta z\overline{z}^2\overline{z})^2 + (\overline{\zeta}\overline{z}\zeta^2 z)^2\right] \\ = \;& z\left[(\zeta z\zeta\overline{z})^2 + (\zeta^2\overline{z}\zeta^2 z)^2\right] \\ = \;& z\left[\zeta z\overline{z} + \zeta^2 z\overline{z}\right] \\ = \;& -z^2\overline{z}. \end{aligned}$$

This gives us a branching equation of

$$Re(P + Qr^2 + Rr^4 - Sr^4) = 0. \tag{5.2.14}$$

As for the case of $\mathbf{S}_3$ Hopf bifurcations, before proceeding any further we carry out some preliminary calculations to use in later work.

We write $g(\underline{z}, \lambda, \tau) = (\mathbf{Z}_1, \mathbf{Z}_2, \mathbf{Z}_3)$ so that, up to fifth order,

$$\mathbf{Z}_i = Pz_i + Qz_i^2\overline{z_i} + Rz_i^3\overline{z_i}^2 + Sz_i(z_j^2\overline{z_k}^2 + \overline{z_j}^2 z_k^2),$$

$$u = z_i\overline{z_i} + z_j\overline{z_j} + z_k\overline{z_k},$$

$$v = (z_i\overline{z_i})^2 + (z_j\overline{z_j})^2 + (z_k\overline{z_k})^2$$

and

$$w = (z_i\overline{z_j})^2 + (z_i\overline{z_k})^2 + (z_j\overline{z_k})^2 + (z_j\overline{z_i})^2 + (z_k\overline{z_i})^2 + (z_k\overline{z_j})^2.$$

Therefore

$$u_{z_i} = \overline{z_i},\ u_{\overline{z_i}} = z_i,\ u_{z_j} = \overline{z_j},\ u_{\overline{z_j}} = z_j,$$

$$v_{z_i} = 2z_i\overline{z_i}^2,\ v_{\overline{z_i}} = 2z_i^2\overline{z_i},\ v_{z_j} = 2z_j\overline{z_j}^2,\ v_{\overline{z_j}} = 2z_j^2\overline{z_j},$$

$$w_{z_i} = 2z_i(\overline{z_j}^2 + \overline{z_k}^2),\ w_{\overline{z_i}} = 2\overline{z_i}(z_j^2 + z_k^2),\ w_{z_j} = 2z_j(\overline{z_i}^2 + \overline{z_k}^2),\ w_{\overline{z_j}} = 2\overline{z_j}(z_i^2 + z_k^2).$$

This then gives us

$$\begin{aligned}
\mathbf{Z}_{i,z_i} &= P + P_u z_i\overline{z_i} + 2P_v(z_i\overline{z_i})^2 + 2P_w z_i^2(\overline{z_j}^2 + \overline{z_k}^2) + \\
&\qquad 2Qz_i\overline{z_i} + Q_u(z_i\overline{z_i})^2 + 3R(z_i\overline{z_i})^2 + S(z_j^2\overline{z_k}^2 + \overline{z_j}^2 z_k^2),\\
\mathbf{Z}_{i,\overline{z_i}} &= P_u z_i^2 + 2P_v z_i^3\overline{z_i} + 2P_w z_i\overline{z_i}(z_j^2 + z_k^2) + Qz_i^2 + Q_u z_i^3\overline{z_i} + 2R\overline{z_i}z_i^3,\\
\mathbf{Z}_{i,z_j} &= P_u z_i\overline{z_j} + 2P_v z_i z_j\overline{z_j}^2 + 2P_w z_i z_j(\overline{z_i}^2 + \overline{z_k}^2) + Q_u z_i^2\overline{z_i}\overline{z_j} + 2Sz_i z_j\overline{z_k}^2,\\
\mathbf{Z}_{i,\overline{z_j}} &= P_u z_i z_j + 2P_v z_i z_j^2\overline{z_j} + 2P_w z_i\overline{z_j}(z_i^2 + z_k^2) + Q_u z_i^2 z_j\overline{z_j} + 2Sz_i z_k^2\overline{z_j}.
\end{aligned}$$

To find the eigenvalues of $(dg)_{x_0}$ we again use the method of Golubitsky et al. [17] by considering z and $\overline{z}$ to be our coordinate system. To recap, a $\mathbf{R}$-linear mapping of the form $w \longrightarrow \alpha w + \beta\overline{w}$ will have trace $2Re(\alpha)$ and determinant $|\alpha|^2 - |\beta|^2$. So for two dimensional fixed point subspaces and isotypic components we use that $\alpha w_k = d_z g.w_k$ and $\beta w_k = d_{\overline{z}} g.w_k$ where the w_k are the base vectors of the spaces in question.

However, unlike the case of $\mathbf{S}_3$ symmetry, it may now be possible to have *four* dimensional isotypic components, and so we must extend this method to cater for this eventuality.

We will have maps of the form

$$\begin{bmatrix} u \\ \overline{u} \\ v \\ \overline{v} \end{bmatrix} = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \overline{a_2} & \overline{a_1} & \overline{a_4} & \overline{a_3} \\ b_1 & b_2 & b_3 & b_4 \\ \overline{b_2} & \overline{b_1} & \overline{b_4} & \overline{b_3} \end{bmatrix} \begin{bmatrix} u \\ \overline{u} \\ v \\ \overline{v} \end{bmatrix} \tag{5.2.15}$$

which has trace $2Re(a_1 + b_3)$, and the a_i and b_i can be found as before. Unless the entries are simplified by duplication or nullification however we cannot find all the eigenvalues explicitly because of the complexity of the situation, but we can always find the trace.

We are now ready to state and prove the stability conditions for our branches of solutions

Theorem 5.2.4 *Assuming suitable non-degeneracy conditions, there exists, up to conjugacy, precisely one branch of small amplitude, near 2π periodic solutions, for each of the isotropy subgroups $\mathbf{S}_3$, $\mathbf{W}_2$, $\mathbf{W}_1$ and $\widetilde{\mathbf{Z}_3}$.*

Assume that the trivial branch is stable sub-critically and loses stability as λ passes through 0, then we have the following.

i) The $\mathbf{S}_3$ branch is super- or sub-critical according to whether $Re(3P_u(0) + Q(0))$ is positive or negative. Stability is undetermined to fifth order but necessary conditions for stability are that both $Re(3P_u(0) + Q(0)) > 0$ and $Re(Q(0)) > 0$.

ii) The $\mathbf{W}_2$ *branch is super- or sub-critical according to whether* $Re(2P_u(0)+Q(0))$ *is positive or negative. It is stable if* $Re(2P_u(0)+Q(0)) > 0$, $Re(Q(0)) > 0$ *and* $Re(Q(0)) < 0$. *i.e. this branch is generically unstable.*

iii) The $\mathbf{W}_1$ *branch is super- or sub-critical according to whether* $Re(P_u(0)+Q(0))$ *is positive or negative. It is stable if both* $Re(P_u(0)+Q(0)) > 0$ *and* $Re(Q(0)) < 0$.

iv) The $\tilde{\mathbf{Z}}_3$ *branch is super- or sub-critical according to whether* $Re(3P_u(0)+Q(0))$ *is positive or negative. It is stable if* $Re(3P_u(0)+Q(0)) > 0$, $Re(Q(0)) > 0$ *and*

$$0 \leq Q(0)\overline{Q(0)} < Im(3P_u(0)+Q(0))^2 + \tfrac{1}{4}Re(3P_u(0)+Q(0))^2.$$

Remark 5.2.5 *Note that generically, if the* $\mathbf{W}_1$ *branch is stable, it is the only stable branch, and if the* $\tilde{\mathbf{Z}}_3$ *branch is stable, it forces the non-zero (up to fifth order) eigenvalues of the* $\mathbf{S}_3$ *branch to be positive, but the reverse does not necessarily hold.*

Proof: As in the proof of the other stability theorems, most of this proof consists of systematically working through the calculations.

i) Isotropy $\mathbf{S}_3$ has fixed point subspace of the form $Fix(\mathbf{S}_3) = (w,w,w)$, and branching equation $P + Qr^2 + Rr^4 + 2Sr^4 = 0$. The only other isotypic component is four dimensional, and is given by $\mathbf{V}_1 = (u, v, -(u+v))$, where $u, v, w \in \mathbf{C}$.

We make all our calculations to fifth order, even though we actually only need them up to third, for completeness, and note, for later calculations that due to the symmetry $\mathbf{Z}_{1,z_1} = \mathbf{Z}_{2,z_2} = \mathbf{Z}_{3,z_3}$, $\mathbf{Z}_{1,z_2} = \mathbf{Z}_{2,z_1} = \ldots$ etc.

We have

$$\begin{aligned}
\mathbf{Z}_{1,z_1} &= P + P_u w\overline{w} + 2Qw\overline{w} + 2P_v(w\overline{w})^2 \\
&\qquad +4P_w(w\overline{w})^2 + Q_u(w\overline{w})^2 + 3R(w\overline{w})^2 + 2S(w\overline{w})^2, \\
\mathbf{Z}_{1,z_2} &= P_u w\overline{w} + 2P_v(w\overline{w})^2 + 4P_w(w\overline{w})^2 + Q_u(w\overline{w})^2 + 2S(w\overline{w})^2, \\
\mathbf{Z}_{1,z_3} &= P_u w\overline{w} + 2P_v(w\overline{w})^2 + 4P_w(w\overline{w})^2 + Q_u(w\overline{w})^2 + 2S(w\overline{w})^2, \\
\mathbf{Z}_{1,\overline{z_1}} &= P_u w^2 + Qw^2 + 2P_v w^2 w\overline{w} + 4P_w w^2 w\overline{w} + Q_u w^2 w\overline{w} + 2Rw^2 w\overline{w}, \\
\mathbf{Z}_{1,\overline{z_2}} &= P_u w^2 + 2P_v w^2 w\overline{w} + 4P_w w^2 w\overline{w} + Q_u w^2 w\overline{w} + 2Sw^2 w\overline{w}, \\
\mathbf{Z}_{1,\overline{z_3}} &= P_u w^2 + 2P_v w^2 w\overline{w} + 4P_w w^2 w\overline{w} + Q_u w^2 w\overline{w} + 2Sw^2 w\overline{w}.
\end{aligned}$$

We are now ready to work out the eigenvalues associated with $(dg)_{x_0}$

$\mathbf{V}_0$

One of the eigenvalues associated with the fixed point subspace is forced by the symmetry to be zero, the other is given by $2Re(\alpha)$ where

$$\alpha = \mathbf{Z}_{1,z_1} + \mathbf{Z}_{1,z_2} + \mathbf{Z}_{1,z_3} = P + 3P_u r^2 + 2Qr^2$$

to third order, so that $2Re(\alpha) = 2r^2 Re(P_u + Q)$, so that the sign of the eigenvalue is given by the sign of $Re(3P_u(0) + Q(0))$.

$\mathbf{V}_1$

The other isotypic component is four dimensional, and so $dg : \mathbf{V}_1 \to \mathbf{V}_1$ is of the form 5.2.15, where we have

$$a_1 = \mathbf{Z}_{1,z_1} - \mathbf{Z}_{1,z_3},\ a_2 = \mathbf{Z}_{1,\overline{z_1}} - \mathbf{Z}_{1,\overline{z_3}},\ a_3 = \mathbf{Z}_{1,z_2} - \mathbf{Z}_{1,z_3},\ a_4 = \mathbf{Z}_{1,\overline{z_2}} - \mathbf{Z}_{1,\overline{z_3}}$$

and

$$b_1 = \mathbf{Z}_{2,z_1} - \mathbf{Z}_{2,z_3},\ b_2 = \mathbf{Z}_{2,\overline{z_2}} - \mathbf{Z}_{2,\overline{z_3}},\ b_3 = \mathbf{Z}_{2,z_2} - \mathbf{Z}_{2,z_3},\ b_4 = \mathbf{Z}_{2,\overline{z_2}} - \mathbf{Z}_{2,\overline{z_3}}.$$

So, after carrying out these calculations we have that, up to third order

$$a_1 = Qw\overline{w},\ a_2 = Qw^2,\ a_3 = a_4 = 0$$

and

$$b_1 = b_2 = 0,\ b_3 = Qw\overline{w},\ b_4 = Qw^2.$$

This means that we are looking for the eigenvalues of the matrix

$$\begin{bmatrix} Qw\overline{w} & Qw^2 & 0 & 0 \\ \overline{Q}\overline{w}^2 & \overline{Q}w\overline{w} & 0 & 0 \\ 0 & 0 & Qw\overline{w} & Qw^2 \\ 0 & 0 & \overline{Q}\overline{w}^2 & \overline{Q}w\overline{w} \end{bmatrix}$$

which are just the eigenvalues of

$$\begin{bmatrix} Qw\overline{w} & Qw^2 \\ \overline{Q}\overline{w}^2 & \overline{Q}w\overline{w} \end{bmatrix}$$

twice, which gives us eigenvalues of $2Re(Qr^2)$ and 0.

Up to fifth order a_3, a_4, b_1 and b_2 are still all zero,

$$a_1 = P + 2Qw\overline{w} + 3R(w\overline{w})^2 \text{ and } a_2 = Qw^2 + 2Rw^2(w\overline{w}) - 2Sw^2(w\overline{w}).$$

If we substitute the branching equation

$$a_1 = Qw\overline{w} + 2R(w\overline{w})^2 - 2S(w\overline{w})^2$$

and so we still get two zero eigenvalues. therefore, up to fifth order, the signs of the eigenvalues corresponding to the isotypic component $\mathbf{V}_1$ are given by the sign of $Re(Q(0))$ and 0.

ii) Isotropy $\mathbf{W}_2$ has fixed point subspace $\mathbf{V}_0 = Fix(\mathbf{W}_2) = (w, w, 0)$ and two further, two dimensional, isotypic components given by $\mathbf{V}_1 = (w, -w, 0)$ and $\mathbf{V}_2 = (0, 0, w)$. The branching equation is now $P + Qr^2 + Rr^4 = 0$. Again, it turns out that it is sufficient to consider calculations to third order, but we begin by considering to fifth order.

$$\begin{aligned}
\mathbf{Z}_{1,z_1} &= P+P_u w\overline{w} + 2Qw\overline{w} + 2P_v(w\overline{w})^2 + 2P_w(w\overline{w})^2 + Q_u(w\overline{w})^2 + 3R(w\overline{w})^2 \\
\mathbf{Z}_{1,z_2} &= P_u w\overline{w} + 2P_v(w\overline{w})^2 + 2P_w(w\overline{w})^2 + Q_u(w\overline{w})^2, \\
\mathbf{Z}_{1,z_3} &= 0, \\
\mathbf{Z}_{1,\overline{z_1}} &= P_u w^2 + Qw^2 + 2P_v w^2 w\overline{w} + 2P_w w^2 w\overline{w} + Q_u w^2 w\overline{w} + 2Rw^2 w\overline{w}, \\
\mathbf{Z}_{1,\overline{z_2}} &= P_u w^2 + 2P_v w^2 w\overline{w} + 2P_w w^2 w\overline{w} + Q_u w^2 w\overline{w}, \\
\mathbf{Z}_{1,\overline{z_3}} &= 0.
\end{aligned}$$

$\mathbf{V}_0$

The only non-zero eigenvalue is given by $2Re(\alpha)$ where $\alpha = \mathbf{Z}_{1,z_1} + \mathbf{Z}_{1,z_2}$ so up to third order $\alpha = P+2P_u w\overline{w}+2Qw\overline{w} = (2P_u+Q)r^2$. Therefore the eigenvalue is given by $2r^2 Re(2P_u(0) + Q(0))$.

$\mathbf{V}_1$

We now consider $\mathbf{V}_1$ which being two dimensional means we need both the trace, $2Re(\alpha)$, and the determinant, $|\alpha|^2 - |\beta|^2$, where

$$\alpha = P + 2Qr^2 + 3Rr^4 = Qr^2 + 2Rr^4$$

and

$$\beta = Qw^2 + 2Rw^2 r^2.$$

This gives, up to third order a trace of $2r^2 Re(Q)$, but the determinant is zero, even up to fifth order.

$\mathbf{V}_2$

Next we consider the final isotypic component $\mathbf{V}_2$ given by $(0,0,w)$. We again require the trace and determinant where this time $\alpha = \mathbf{Z}_{3,z_3} = P$ and $\beta = \mathbf{Z}_{3,\overline{z_3}} = 0$. This gives, up to third order, a trace of $2Re(P) = -2r^2 Re(Q)$ and a determinant of $Q\overline{Q}r^4$ which must always (generically) be positive.

iii) Isotropy $\mathbf{W}_1$, with $Fix(\mathbf{W}_1) = (w,0,0)$ and isotypic component $\mathbf{V}_1 = (0,u,v)$, has branching equation $P + Qr^2 + Rr^4 = 0$. To fifth order we also have

$$
\begin{aligned}
\mathbf{Z}_{1,z_1} &= P + P_u w\overline{w} + 2Qw\overline{w} + 2P_v(w\overline{w})^2 + Q_u(w\overline{w})^2 + 3R(w\overline{w})^2\\
\mathbf{Z}_{1,z_2} &= 0,\\
\mathbf{Z}_{1,z_3} &= 0,\\
\mathbf{Z}_{1,\overline{z_1}} &= P_u w^2 + Qw^2 + 2P_v w^2 w\overline{w} + Q_u w^2 w\overline{w} + 2Rw^2 w\overline{w},\\
\mathbf{Z}_{1,\overline{z_2}} &= 0,\\
\mathbf{Z}_{1,\overline{z_3}} &= 0.
\end{aligned}
$$

$\mathbf{V}_0$

The non-zero eigenvalue is given by $2Re(\alpha)$ where $\alpha = \mathbf{Z}_{1,z_1}$. So the eigenvalue corresponding to $Fix(\mathbf{W}_1)$ is, up to third order,

$$2Re(P + P_u r^2 + 2Qr^2) = 2r^2 Re(P_u + Q).$$

$\mathbf{V}_1$

The other eigenvalues are those of the map $(dg)_{x_0} : \mathbf{V}_1 \to \mathbf{V}_1$, where $\mathbf{V}_1$ is four dimensional, and so the map is of the form 5.2.15. This case simplifies dramatically though, since many of the entries vanish. We have

$$a_1 = \mathbf{Z}_{2,z_2},\ a_2 = \mathbf{Z}_{2,\overline{z_2}},\ a_3 = \mathbf{Z}_{2,z_3},\ a_4 = \mathbf{Z}_{2,\overline{z_3}},$$

and

$$b_1 = \mathbf{Z}_{3,z_2},\ b_2 = \mathbf{Z}_{3,\overline{z_2}},\ b_3 = \mathbf{Z}_{3,z_3},\ b_4 = \mathbf{Z}_{3,\overline{z_3}}.$$

But up to *any* order we have

$$\mathbf{Z}_{2,z_2} = P,\ \mathbf{Z}_{2,z_3} = \mathbf{Z}_{2,\overline{z_2}} = \mathbf{Z}_{2,\overline{z_3}} = 0$$

and

$$\mathbf{Z}_{3,z_3} = P, \ \mathbf{Z}_{3,z_2} = \mathbf{Z}_{3,\overline{z_2}} = \mathbf{Z}_{3,\overline{z_3}} = 0.$$

All that we require then are the eigenvalues of

$$\begin{bmatrix} P & 0 & 0 & 0 \\ 0 & \overline{P} & 0 & 0 \\ 0 & 0 & P & \\ 0 & 0 & 0 & \overline{P} \end{bmatrix}$$

Which are just P (twice) and $\overline{P}$ (twice). Therefore for stability we require $2Re(P) > 0$, or, using the branching equation, $Re(Q) < 0$ (this follows since $Re(z) = Re(\overline{z})$).

iv) Isotropy $\widetilde{\mathbf{Z}_3}$, has a two dimensional fixed point subspace of the form $(w, \zeta w, \zeta^2 w)$ and a four dimensional isotypic component $\mathbf{V}_1$ of the form $(u, v, -(\zeta u + \zeta^2 v))$, where $\zeta = e^{2\pi i/3}$. Due to the eventual complexity of the calculations, we take all our preliminary calculations to only third order, giving a branching equation of $P + Qr^2 = 0$. We also obtain

$$\begin{aligned}
\mathbf{Z}_{1,z_1} &= P + P_u w\overline{w} + 2Qw\overline{w}, \\
\mathbf{Z}_{1,z_2} &= \zeta^2 P_u w\overline{w}, \\
\mathbf{Z}_{1,z_3} &= \zeta P_u w\overline{w}, \\
\mathbf{Z}_{1,\overline{z_1}} &= P_u w^2 + Qw^2, \\
\mathbf{Z}_{1,\overline{z_2}} &= \zeta P_u w^2, \\
\mathbf{Z}_{1,\overline{z_3}} &= \zeta^2 P_u w^2.
\end{aligned}$$

We also require

$$\begin{aligned}
\mathbf{Z}_{2,z_1} &= \zeta P_u w\overline{w}, \\
\mathbf{Z}_{2,z_2} &= P + P_u w\overline{w} + 2Qw\overline{w}, \\
\mathbf{Z}_{2,z_3} &= \zeta^2 P_u w\overline{w}, \\
\mathbf{Z}_{2,\overline{z_1}} &= \zeta P_u w^2, \\
\mathbf{Z}_{2,\overline{z_2}} &= \zeta^2 P_u w^2 + \zeta^2 Qw^2, \\
\mathbf{Z}_{2,\overline{z_3}} &= P_u w^2.
\end{aligned}$$

$\mathbf{V}_0$

Again, the only non-zero eigenvalue is given by $2Re(\alpha)$ where

$$\alpha = Z_{1,z_1} + \zeta \mathbf{Z}_{1,z_2} + \zeta^2 \mathbf{Z}_{1,z_3}$$

and so $\alpha = P + 2Qr^2 + 3P_u r^2$ so that the eigenvalue is given by $2r^2 Re(3P_u + Q)$.

$\mathbf{V}_1$

As already mentioned, $\mathbf{V}_1$ is four dimensional, and so $(dg)_{x_0} : \mathbf{V}_1 \to \mathbf{V}_1$ is again of the form 5.2.15, where now

$a_1 = \mathbf{Z}_{1,z_1} - \zeta\mathbf{Z}_{1,z_3}$, $a_2 = \mathbf{Z}_{1,\overline{z_1}} - \zeta\mathbf{Z}_{1,\overline{z_3}}$, $a_3 = \mathbf{Z}_{1,z_2} - \zeta^2\mathbf{Z}_{1,z_3}$, $a_4 = \mathbf{Z}_{1,\overline{z_2}} - \zeta^2\mathbf{Z}_{1,\overline{z_3}}$

and

$b_1 = \mathbf{Z}_{2,z_1} - \zeta\mathbf{Z}_{2,z_3}$, $b_2 = \mathbf{Z}_{2,\overline{z_1}} - \zeta\mathbf{Z}_{2,\overline{z_3}}$, $b_3 = \mathbf{Z}_{2,z_2} - \zeta^2\mathbf{Z}_{2,z_3}$, $b_4 = \mathbf{Z}_{2,\overline{z_2}} - \zeta^2\mathbf{Z}_{2,\overline{z_3}}$.

Therefore, up to third order, remembering that $\zeta^3 = 1$ and $1 + \zeta + \zeta^2 = 0$,

$$a_1 = (1-\zeta^2)P_u w\overline{w} + Qw\overline{w},\ a_2 = Qw^2,\ a_3 = (\zeta^2 - 1)P_u w\overline{w},\ a_4 = 0$$

and

$$b_1 = (\zeta - 1)P_u w\overline{w},\ b_2 = 0,\ b_3 = (1-\zeta)P_u w\overline{w} + Qw\overline{w},\ b_4 = \zeta^2 Qw^2.$$

We are therefore left with requiring the eigenvalues of

$$\begin{bmatrix} (1-\zeta^2)P_u w\overline{w} + Qw\overline{w} & Qw^2 & (\zeta^2-1)P_u w\overline{w} & 0 \\ \overline{Q}\overline{w}^2 & (1-\zeta)\overline{P_u} w\overline{w} + \overline{Q}w\overline{w} & 0 & (\zeta-1)\overline{P_u}w\overline{w} \\ (\zeta-1)P_u w\overline{w} & 0 & (1-\zeta)P_u w\overline{w} + Qw\overline{w} & \zeta^2 Qw^2 \\ 0 & (\zeta^2-1)\overline{P_u}w\overline{w} & \zeta\overline{Q}\overline{w}^2 & (1-\zeta^2)\overline{P_u}w\overline{w} + \overline{Q}w\overline{w} \end{bmatrix}.$$

This matrix has trace

$$\left[(2-\zeta-\zeta^2)(P_u + \overline{P_u}) + 2(Q + \overline{Q})\right] w\overline{w} = 2r^2 Re(3P_u + 2Q),$$

so we at least know that this has to be positive to ensure stability. By rearranging the characteristic equation we also find that one of the eigenvalues is zero (using Maple), and so the other three are found as roots to the cubic equation

$$\begin{aligned} &\lambda^3 - \left(3Pu + 3\overline{P_u} + 2Q + 2\overline{Q}\right)\lambda^2 \\ &+ \left(6Q\overline{P_u} + 3\overline{P_u}\overline{Q} + 6P_u\overline{Q} + 3P_uQ + 9P_u\overline{P_u} + 2Q\overline{Q} + Q^2 + \overline{Q}^2\right)\lambda \\ &- \left(9P_u\overline{P_u}Q + 9P_u\overline{P_u}\overline{Q} + 3P_uQ\overline{Q} + 3\overline{P_u}Q\overline{Q} + 3P_u\overline{Q}^2 + 3\overline{P_u}Q^2\right). \end{aligned} \tag{5.2.16}$$

By inspection, and then by division of polynomials, equation 5.2.16 can be factorised as

$$\left(\lambda - Q - \overline{Q}\right)\left(\lambda^2 - (3P_u + 3\overline{P_u} + Q + \overline{Q})\lambda + 3\overline{P_u}Q + 3P_u\overline{Q} + 9P_u\overline{P_u}\right)$$

giving roots, and so eigenvalues, of $Q + \overline{Q}$ and

$$\frac{1}{2}\left[3P_u + 3\overline{P_u} + Q + \overline{Q} \pm \sqrt{(3P_u + Q - 3\overline{P_u} - \overline{Q}) + 4Q\overline{Q}}\,\right].$$

The first of these is just $2Re(Q)$, the other two are

$$\frac{1}{2}\left[Re(3P_u + Q) \pm \sqrt{(2iIm(3P_u + Q))^2 + 4Q\overline{Q}}\,\right]$$

or

$$\tfrac{1}{2}Re(3P_u + Q) \pm \sqrt{Q\overline{Q} - Im(3P_u + Q)^2}.$$

In order for our $\widetilde{\mathbf{Z}}_3$ solution branch to be stable we require all the non-zero eigenvalues to have positive real part. We already know that we need

$$Re(3P_u + Q) > 0$$

(trace of our matrix), and so now we also need either $Im(3P_u + Q)^2 \geq Q\overline{Q}$ (makes the part inside the square root negative, and so only adds an imaginary part to both eigenvalues) or

$$\sqrt{Q\overline{Q} - Im(3P_u + Q)^2} < \tfrac{1}{2}Re(3P_u + Q)$$

where both sides of this inequality are positive, this is equivalent to requiring that

$$Q\overline{Q} < Im(3P_u + Q)^2 + \tfrac{1}{4}Re(3P_u + Q)^2$$

and so a condition for stability must be that

$$0 \leq Q\overline{Q} = |Q|^2 < Im(3P_u + Q)^2 + \tfrac{1}{4}Re(3P_u + Q)^2.$$

We have now proved the results of the Theorem.

∎

The results of Theorem 5.2.4 are summarised in table 5.2, and some representative bifurcation diagrams are shown in Figure 5.2.

We now consider the final case of Hopf bifurcation with $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry, which can be thought of as coupled cells with equivariant coupling.

Isotropy Σ	Fix(Σ)	Branching Equation	Signs of Eigenvalues
$\mathbf{S}_3$	(w,w,w)	$P+Qr^2+Rr^4+2Sr^4=0$	$Re(3P_u+Q)$ $Re(Q)(twice)$
$\mathbf{W}_2$	$(w,w,0)$	$P+Qr^2+Rr^4=0$	$Re(2P_u+Q)$ $trace=Re(Q)$, $det=0$, $trace=-Re(Q)$, $det=Q\overline{Q}$
$\mathbf{W}_1$	$(w,0,0)$	$P+Qr^2+Rr^4=0$	$Re(P_u+Q)$ $-Re(Q)$ (four times)
$\tilde{\mathbf{Z}}_3$	$(w,\zeta w,\zeta^2 w)$	$P+Qr^2+Rr^4-Sr^4=0$	$Re(3P_u+Q)$ $Re(Q)$ $Q\overline{Q}<$ $Im(3P_u+Q)^2+\frac{1}{4}Re(3P_u+Q)^2$

Table 5.2: Stability of branches of solutions in the presence of $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry arising from Hopf bifurcations.

5.3 Hopf Bifurcations With $\mathbf{Z}_2 \times \mathbf{S}_3$ Symmetry

As with the wreath product case we start with the irreducible action of $\mathbf{Z}_2 \times \mathbf{S}_3$, on $\mathbf{V}_R = \{x \in \mathbf{R}^3 : x_1 + x_2 + x_3 = 0\}$, and then extend it, in the natural way, to the action of $\mathbf{Z}_2 \times \mathbf{S}_3 \times \mathbf{S}^1$ on $\mathbf{V}_C = \{z \in \mathbf{C}^3 : z_1 + z_2 + z_3 = 0\}$. That is we let $\mathbf{S}_3$ act on $\mathbf{V}_C$ by permutation of indices, $\kappa \in \mathbf{Z}$ by $\kappa z = -z$ and $\theta \in \mathbf{S}^1$ by $\theta z = e^{i\theta} z$.

This action is Γ-simple, but the action of $\mathbf{Z}_2$ is precisely the same as the action of $\pi = e^{i\pi} \in \mathbf{S}^1$. This means that when a $\mathbf{Z}_2$ internal symmetry is present in three coupled cells, and the coupling produces a global $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry, then the isotropy subgroups that occur are exactly the same as those of $\mathbf{S}_3$, and on the Γ-simple subspace the two cases behave identically.

However, care must be taken when interpreting the results to an 'unreduced' situation, such as systems of coupled oscillators,as will be shown later.

5.4 Comparisons Between The Different Cases

As already noted, the case of $\mathbf{Z}_2 \times \mathbf{S}_3$, at least when reduced to its Γ-simple representation, is exactly the same as a system with $\mathbf{S}_3$ symmetry, the only additional isotropy being on the same group orbit as the existing $\tilde{\mathbf{Z}}_2$ isotropy subgroup. As will

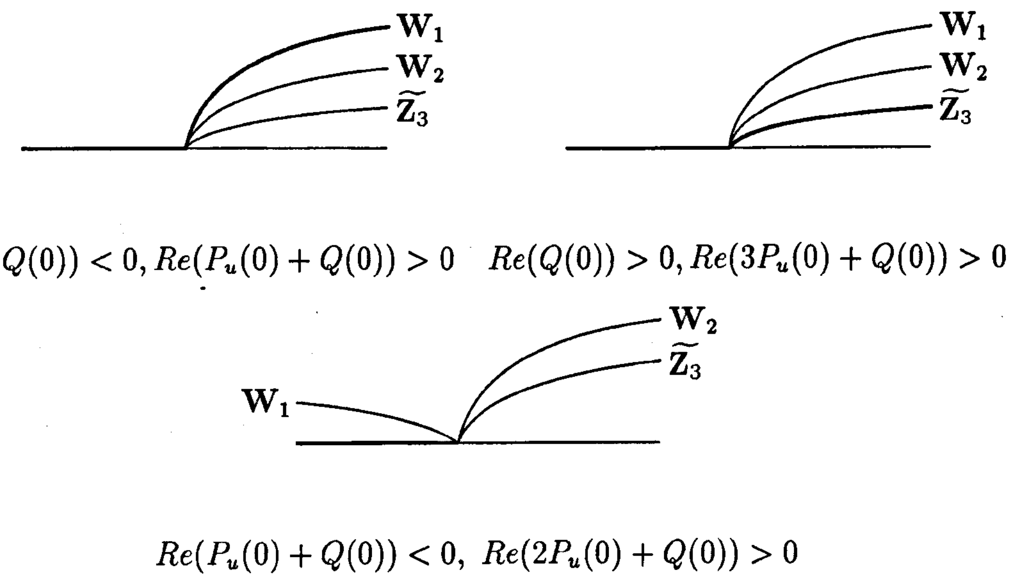


Figure 5.2: Representative bifurcation diagrams for $\mathbf{Z}_2 \wr \mathbf{S}_3$ Hopf bifurcations. Shown are the two cases where there is a stable branch, and one other representative. Thick lines denote stability, thin lines instability.

be shown later however, when we consider systems of coupled oscillators, the presence of the 'internal' $\mathbf{Z}_2$ symmetry *is* important when we come to interpret our results. One observation for instance, is that the element $[\kappa, \pi] \in \mathbf{Z}_2 \times \mathbf{S}_3$ acts as the identity, and so is contained in *every* isotropy subgroup. This has important consequences when considering the shape of the waveform of the individual oscillators.

Once we introduce the wreath product case however, the differences between $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $\mathbf{S}_3$ Hopf bifurcations becomes apparent. For instance, a solution branch with isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ is no longer guaranteed (generically), and, surprisingly, the conjugacy class of solutions, including the isotropy $\widetilde{\mathbf{Z}}_2$ which appears in both $\mathbf{S}_3$ and $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry cases, is now generically unstable, unlike previously.

We also obtain an entirely new branch of solutions, corresponding to isotropy $\mathbf{W}_1$, with which care must again be taken when interpreting the results of solutions of dynamical systems having this isotropy as their symmetry.

Chapter 6

Coupled Oscillators With Internal Symmetry

Now that we have considered the theoretical implications of the addition of a $\mathbf{Z}_2$ symmetry, we consider how this extra symmetry can affect the results of three coupled oscillators. We do this for both how the individual oscillators differ from the next, and also how the actual pattern of the individual oscillators is affected by the change. We begin by summarising the results of Golubitsky et al. [17] on three coupled oscillators with $\mathbf{S}_3$ symmetry, i.e. without any internal symmetries present.

6.1 Oscillators with $\mathbf{S}_3$ Symmetry

As in [17], we use a specific form of equation to illustrate a system of coupled oscillators, but the results still hold for a more general set of equations with $\mathbf{S}_3$ symmetry. For further simplicity we also consider the case where single oscillators live on the plane, so that our system of coupled oscillators lives on $\mathbf{R}^6$.

We consider a system of form

$$\frac{d}{dt}(x_i, y_i) = F(x_i, y_i, \lambda) + K(x_i, y_i, x_j, y_j, x_k, y_k, \lambda) \tag{6.1.1}$$

where $x_i, y_i \in \mathbf{R}$, $\{i,j,k\} = \{1,2,3\}$, $F(x_i, y_i, \lambda) : \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$ is the equation for the uncoupled oscillators, $K : \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}^2$ is the coupling term between the oscillators and λ is the bifurcation parameter.

$\mathbf{S}_3$ symmetry is obtained by the condition

$$K(u, v, w, \lambda) = K(u, w, v, \lambda)$$

where $u, v, w \in \mathbf{R}^2$. If we further assume that these oscillations have occurred through a Hopf bifurcation at $\lambda = 0$, then the expected patterns of oscillation at the bifurcation point are outlined in Table 6.1, and shown diagrammatically in Figure 6.1. For more detail see Golubitsky et al. [17], Theorem XVII 4.4. We use a letter A to denote a waveform, and the notation $A + \theta$ to denote the waveform A phase shifted by θ.

6.2 Three Coupled Oscillators With Internal Symmetries - The Equations

For this section we consider three identical oscillators, where each oscillator has a general internal Γ symmetry, and consider how the coupling leads to different global symmetries, and how the ensuing patterns of oscillation are affected by this coupling and its interaction with the internal symmetries.

A Note On Three Coupled Oscillators Without Internal Symmetries

In the case of three oscillators without internal symmetries, each oscillator is governed by an equation of the form

$$\frac{dx_i}{dt} = f(x_i) + g(x_i, x_j, x_k),$$

where $f(x_i)$ is the equation of motion for the uncoupled i^{th} oscillator and g is the coupling term. For three such oscillators, each having phase space $\mathbf{R}^n$, the whole system will lie in $\mathbf{R}^{3n}$. The form of each individual oscillator is then taken as being the projection of the full system onto the corresponding subspace $\mathbf{R}^n \subset \mathbf{R}^{3n}$.

We now allow the oscillators to possess internal Γ symmetries, i.e. each uncoupled oscillator is governed by the equation

$$\frac{dx_i}{dt} = f(x_i)$$

where $f(x)$ is Γ equivariant, and the oscillator lives in the plane, $\mathbf{R}^2$. We consider the global coupled system governed by the equations

$$\frac{dx_i}{dt} = f(x_i) + g(x_i, x_j, x_k) \tag{6.2.2}$$

where $\{i, j, k\} = \{1, 2, 3\}$, and each individual oscillator is taken to be the projection onto the appropriate $\mathbf{R}^2$ subspace. For the $\mathbf{S}_3$ symmetry we require only that

$$g(x, y, z) = g(x, z, y)$$

Isotropy	Oscillators			Comments
$\tilde{\mathbf{Z}}_2$	A	$A+\pi$	B	Two oscillators of identical waveform but π out of phase, the third oscillating with half the period of the other two.
$\tilde{\mathbf{Z}}_3$	A	$A+\frac{2\pi}{3}$	$A+\frac{4\pi}{3}$	Each oscillator $2\pi/3$ out of phase with the other two.
$\mathbf{S}_1 \times \mathbf{S}_2$	A	C	C	Two oscillators identical and in phase the third oscillating with different waveform but same period

Table 6.1: Patterns of oscillation observed in the presence of $\mathbf{S}_3$ symmetry.

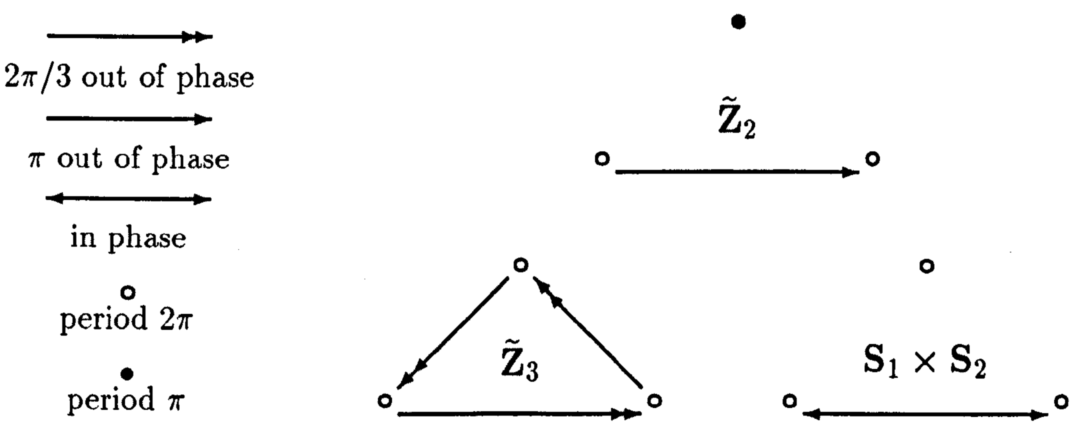


Figure 6.1: Patterns of oscillation observed in the presence of $\mathbf{S}_3$ symmetry.

and for the present we allow g to be in as general a form as possible, imposing more realistic conditions later.

In the section on steady-state bifurcations of coupled cells we considered how different coupling lead to different global symmetries. Here we do the reverse, beginning with the two possible symmetries $\mathbf{Z}_2 \times \mathbf{S}_3$ and $\mathbf{Z}_2 \wr \mathbf{S}_3$ and considering how we can achieve these symmetries with suitable coupling.

6.2.1 Wreath Product Coupling of Three Oscillators

Firstly we consider the case where the coupling leads to a global $\Gamma \wr \mathbf{S}_3$ symmetry. This is in fact the easiest case, since g *must* satisfy, for all $\gamma \in \Gamma$,

$$g(x_i, \gamma x_j, x_k) = g(x_i, x_j, x_k) \tag{6.2.3}$$

and

$$g(\gamma x_i, x_j, x_k) = \gamma g(x_i, x_j, x_k). \tag{6.2.4}$$

This means that the coupling leading to wreath product symmetry is precisely the same as the coupling being invariant under the internal symmetries.

6.2.2 Direct Product Coupling and Skew-Equivariance

We now consider the case when our system 6.2.2 has global $\Gamma \times \mathbf{S}_3$ symmetry. The only necessary criteria are that, for all $\gamma \in \Gamma$,

$$g(x_i, \gamma x_j, x_k) \neq g(x_i, x_j, x_k) \tag{6.2.5}$$

and

$$g(\gamma x_i, \gamma x_j, \gamma x_k) = \gamma g(x_i, x_j, x_k). \tag{6.2.6}$$

For example, linear coupling, $g(x_i, x_j, x_k) = x_i + x_j + x_k$, satisfies these conditions and so leads to direct-product coupling. For $\mathbf{Z}_2$ symmetry, as shown in the case of steady-state bifurcations with $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry, this can also be achieved by equivariant coupling, where

$$g(\gamma x_i, x_j, x_k) = \gamma g(x_i, x_j, x_k) \tag{6.2.7}$$

and

$$g(x_i, \gamma x_j, x_k) = \gamma g(x_i, x_j, x_k). \tag{6.2.8}$$

for all $\gamma \in \Gamma$. These conditions force that

$$g(\gamma x_i, \gamma x_j, \gamma x_k) = \gamma^3 g(x_i, x_j, x_k) = \gamma g(x_i, x_j, x_k)$$

and so we must have $\gamma^3 = \gamma$, and so $\gamma^2 = id$. That is, this can only work if the order of γ is two, and so $\Gamma = \mathbf{Z}_2$ or $\Gamma = \mathbf{D}_1$. We therefore consider a more general relationship for more complicated symmetries Γ.

Definition 6.2.1 *A map $g(x,y,z) : \mathbf{R}^3 \to \mathbf{R}^3$ is* Γ-Skew Equivariant *if for all $\gamma \in \Gamma$*

$$g(\gamma x, y, z) = \hat{\gamma} g(x,y,z),$$

$$g(x, \gamma y, z) = \tilde{\gamma} g(x,y,z)$$

and

$$g(x, y, \gamma z) = \overline{\gamma} g(x,y,z)$$

where $\hat{\gamma}$, $\tilde{\gamma}$, $\overline{\gamma} \in \Gamma$.

Remark 6.2.2 *This definition is easily extendible to maps of a general number of variables, though it becomes messy to define.*

Remark 6.2.3 *From this definition it is easy to see how both invariance and equivariance are special cases of skew-equivariance. Invariance is found by the condition $\hat{\gamma} = \tilde{\gamma} = \overline{\gamma} = id$ for all $\gamma \in \Gamma$ and equivariance by $\hat{\gamma} = \tilde{\gamma} = \overline{\gamma} = \gamma$.*

Remark 6.2.4 *We will normally drop the group dependence when it is clear which symmetry group we are dealing with, and so normally say that g is skew-equivariant.*

There are however some conditions that must be placed on skew-equivariant functions, regardless of the overall symmetries, to avoid inconsistencies. One of the most fundamental is that it should not matter in which order a γ acts on two or more variables. For example, if we have a function $g(x,y,z)$ then applying α to x and then δ to y (where $\alpha, \delta \in \Gamma$) or applying δ to y followed by α to x must both have the effect of mapping $g(x,y,z)$ to $g(\alpha x, \delta y, z)$. This forces that we must have that $\hat{\gamma}$, $\tilde{\gamma}$ and $\overline{\gamma}$ all commute with each other for each $\gamma \in \Gamma$.

Define the *order of an element* γ, denoted $o(\gamma)$, to be the lowest number l such that $\gamma^l = id$, and denote 'a divides b' by $a|b$. For consistency we must now also have

Proposition 6.2.5 *$o(\tilde{\gamma})|o(\gamma)$, $o(\hat{\gamma})|o(\gamma)$ and $o(\overline{\gamma})|o(\gamma)$.*

Proof: We show for $\hat{\gamma}$, the others follow similarly. Let $o(\gamma) = l$, so that $\gamma^l = id$. Then we have $g(\gamma x, y, z) = \hat{\gamma}(x,y,z)$ and so

$$g(\gamma^l x, y, z) = \hat{\gamma}^l g(x,y,z).$$

But $g(\gamma^l x, y, z)$ is just $g(x, y, z)$ and so we must have $\hat{\gamma}^l = id$ and so $o(\hat{\gamma})$ must divide $o(\gamma)$ and we are done. ∎

Next we examine how the skew-equivariance must be restricted to achieve $\Gamma \times \mathbf{S}_3$ symmetry. The first thing to note is that for the $\mathbf{S}_3$ symmetry all we need is

$$\tilde{\gamma} = \overline{\gamma}. \tag{6.2.9}$$

Now consider how one element $\gamma \in \Gamma$ acts on a system of equations 6.2.2. First let γ act on the first variable, this gives

$$\begin{aligned} \gamma\dot{x} &= \gamma f(x) + \hat{\gamma} g(x, y, z) \\ \dot{y} &= f(y) + \tilde{\gamma} g(y, x, z) \\ \dot{z} &= f(z) + \tilde{\gamma} g(z, x, y) \end{aligned} \tag{6.2.10}$$

then apply γ to the second variable

$$\begin{aligned} \gamma\dot{x} &= \gamma f(x) + \tilde{\gamma}\hat{\gamma} g(x, y, z) \\ \gamma\dot{y} &= \gamma f(y) + \hat{\gamma}\tilde{\gamma} g(y, x, z) \\ \dot{z} &= f(z) + \tilde{\gamma}^2 g(z, x, y) \end{aligned} \tag{6.2.11}$$

and then to the third variable

$$\begin{aligned} \gamma\dot{x} &= \gamma f(x) + \tilde{\gamma}^2\hat{\gamma} g(x, y, z) \\ \gamma\dot{y} &= \gamma f(y) + \tilde{\gamma}\hat{\gamma}\tilde{\gamma} g(y, x, z) \\ \gamma\dot{z} &= \gamma f(z) + \hat{\gamma}\tilde{\gamma}^2 g(z, x, y) \end{aligned} \tag{6.2.12}$$

If we consider the case where $\Gamma \subset \mathbf{O}(2)$, then the elements can be generated by the equivalent of rotations and reflections. Two rotations can then be generated by one rotation whose order is the lowest common multiple of the two rotations, and so it is enough to consider how only one element acts on the system, except when reflections are present which will be dealt with later.

We now consider direct product coupling, the first things to note being that we must have the criteria

A1 $\gamma = \hat{\gamma}\tilde{\gamma}^2$

A2 ($\hat{\gamma} = \gamma$ and $\tilde{\gamma} \neq id$) or ($\hat{\gamma} \neq \gamma$)

A3 ($\hat{\gamma}\tilde{\gamma} = \gamma$ and $\tilde{\gamma}^2 \neq id$) or ($\hat{\gamma}\tilde{\gamma} \neq \gamma$)

A1 comes directly from 6.2.12, A2 from 6.2.10 to prevent the whole system possessing $\Gamma \wr \mathbf{S}_3$ symmetry and A3 from 6.2.11 for the same reason, since otherwise if γ generates Γ, the symmetry would be $\Gamma \wr \mathbf{S}_3$.

However, in A2, if $\tilde{\gamma} = id$ then A1 implies that $\hat{\gamma} = \gamma$ when A2 implies that $\tilde{\gamma} \neq id$ so we must therefore have that $\tilde{\gamma} \neq id$. Also if $\hat{\gamma}\tilde{\gamma} = \gamma$ then $\tilde{\gamma} \neq id$ (A2), but $\hat{\gamma}\tilde{\gamma} = \gamma$ implies $\gamma = \hat{\gamma}\tilde{\gamma}^2$ (A1) $= \gamma\tilde{\gamma}$ so $\tilde{\gamma} = id$ which we cannot have, so we must have $\hat{\gamma}\tilde{\gamma} \neq \gamma$.

This gives us a revised set of conditions

B1 $\tilde{\gamma} \neq id$

B2 $\gamma = \hat{\gamma}\tilde{\gamma}^2$

B3 $\hat{\gamma}\tilde{\gamma} \neq \gamma$

which force a stronger condition onto the order of $\tilde{\gamma}$

Proposition 6.2.6 $o(\tilde{\gamma}) = o(\gamma)$

Proof: Suppose not, and let $o(\tilde{\gamma}) = m < k = o(\gamma)$ and apply γ^m to the first variable in our equations, then we get

$$\begin{aligned} \dot{y} &= f(y) + g(y, \gamma^m x, z) \\ &= f(y) + \tilde{\gamma}^m g(y, x, z) \\ &= f(y) + g(y, x, z) \end{aligned}$$

Since $\gamma \in \Gamma$, we have that $\gamma^m \neq id$ is also in Γ, call $\gamma^m = \delta \in \Gamma$. But then $\tilde{\delta} = id$ contradicting B1. Therefore we must have that $o(\tilde{\gamma}) = k = o(\gamma)$. ∎

We now make the following observations

Remark 6.2.7 *If $\Gamma \subset \mathbf{O}(2)$ consists entirely of rotations, then Γ can be generated by only one rotation, whose order is the lowest common multiple of the orders of all the rotations in Γ. We therefore only need specify three elements, γ, $\tilde{\gamma}$ and $\hat{\gamma}$ for the one generator γ and the rest will follow.*

Remark 6.2.8 *If Γ contains a reflection, then every member of Γ must have order 1 or 2. Suppose otherwise, and let κ be the reflection so that $o(\tilde{\kappa}) = 2$ so $\hat{\kappa}\tilde{\kappa}^2 = \hat{\kappa} = \kappa$ and if γ is the rotation of order k then $o(\tilde{\gamma}) = k$ but for $\tilde{\gamma}$ and $\hat{\kappa}$ to commute, as they must, we must have $o(\tilde{\gamma}) = 2$, causing a contradiction.*

Remark 6.2.9 *It immediately follows that if Γ contains an order 2 rotation π as well as an order k rotation, γ, then we must have $\tilde{\pi} = \pi$.*

Symmetry groups with more than one order-two element

Any group containing a reflection κ and either another reflection, τ, where the axis of the two reflections are perpendicular, or π, a rotation through π , must contain κ, τ *and* π (if τ not perpendicular to κ then $\kappa\tau$ is a rotation of order $k \geq 3$). Then $\tilde{\kappa}$, $\tilde{\tau}$ and $\tilde{\pi}$ must all be distinct, otherwise, if $\gamma \neq \delta$ and $\tilde{\gamma} = \tilde{\delta}$ then $o(\gamma\delta) = 2$ but $o(\tilde{\gamma}\tilde{\delta}) = o(id) = 1$ contradicting our conditions. Since $o(\tilde{\gamma}) = 2$ we must have $\hat{\gamma} = \gamma$, to satisfy B2, and so we must have $\hat{\kappa} = \kappa$, $\hat{\tau} = \tau$ and $\hat{\pi} = \pi$. This leaves us with only six possibilities for the choice of the form of skew-equivariance.

$$(\tilde{\kappa}, \tilde{\tau}, \tilde{\pi}) \in \{(\kappa,\tau,\pi),(\kappa,\pi,\tau),(\pi,\tau,\kappa),(\tau,\kappa,\pi),(\pi,\kappa,\tau),(\tau,\pi,\kappa)\} \qquad (6.2.13)$$

Remark 6.2.10 *If we have $(\tilde{\kappa}, \tilde{\tau}, \tilde{\pi}) = (\kappa, \tau, \pi)$ then we have purely equivariant coupling.*

We are left with five basic types of skew-equivariance for γ to achieve direct product coupling, where Γ is generated by one element γ.

Type I $\hat{\gamma} = id$ and $\tilde{\gamma}^2 = \gamma$;

Type II $\tilde{\gamma}^2 = id$ and $\hat{\gamma} = \gamma$;

Type III Neither I nor II but $\hat{\gamma} = \gamma^{-1}$ and $\begin{cases} \textbf{a)}\ \tilde{\gamma} = \gamma, \\ \textbf{b)}\ \tilde{\gamma} = \gamma^{-1}, \\ \textbf{c)}\ \tilde{\gamma} = \gamma^{(k+2)/2}; \end{cases}$

Type IV $\tilde{\gamma} = \gamma^{-1}$ but $\hat{\gamma} \neq \gamma^{-1}$ and not I or II;

Type V None of the above.

Theorem 6.2.11 *Necessary conditions on the generator γ, for the skew-equivariant coupling term g to produce direct product coupling $\Gamma \times \mathbf{S}_3$, with one of the Types above are*

i) *Type I is possible only if $o(\gamma)$ is odd;*

ii) *Type II is possible only if $o(\gamma) = 2$;*

iii) *Type IIIa is always possible;*

iv) *Type IIIb is possible only if $o(\gamma) = 2$ or 4;*

v) *Type IIIc is possible only if $o(\gamma) = k$ where k is even and $HCF\left(\frac{k+2}{2}, k\right) = 1$;*

vi) *Type IV is possible only if* $\widehat{\gamma} = \gamma^3$, *and if* $3 \not| o(\gamma) = k$ *then* $o(\widehat{\gamma}) = k$ *otherwise* $o(\widehat{\gamma}) = k/3$.

Proof: Let $o(\gamma) = k$.

i) If k is even then we must have $o(\widetilde{\gamma}) = k$ and $\widetilde{\gamma}^{k/2}\widetilde{\gamma}^{k/2} = \gamma^{k/2} = id$ so that $o(\gamma) = k/2$ which contradicts our assumption, and so k must be odd.

ii) Again we must have $o(\widetilde{\gamma}) = o(\gamma)$ so since $o(\widetilde{\gamma}) = 2$ the result follows.

iii) Satisfies B1, B2 and B3.

iv) If $\widehat{\gamma} = \gamma^{-1}$ and $\widetilde{\gamma} = \gamma^{-1}$ then $\widehat{\gamma}\widetilde{\gamma}^2 = \gamma^{-1}\gamma^{-1}\gamma^{-1} = \gamma$ so $\gamma^4 = id$ so $o(\gamma) = 1, 2$ or 4. We have assumed that $\gamma \neq id$ hence the result.

v) $\widehat{\gamma} = \gamma^{-1}$ and $\widetilde{\gamma} = \gamma^{(k+2)/2}$. We must necessarily have k even since $(k+2)/2$ must be an integer. If $HCF((k+2)/2, 2) \neq 1$ then there exists an m such that $\widetilde{\gamma}^m = id$ for $m < k$, i.e. $o(\widetilde{\gamma}) < o(\gamma)$.

vi) If $\widetilde{\gamma} = \gamma^{-1}$ and $\widehat{\gamma} \neq \gamma^{-1}$ then $\widehat{\gamma}\widetilde{\gamma}^2 = \widehat{\gamma}\gamma^{-1}\gamma^{-1} = \gamma$ so $\widehat{\gamma} = \gamma^3$. If $o(\gamma) = k = 3c$ then $\gamma^{3c} = id = \widehat{\gamma}^c$ and $o(\widehat{\gamma}) = k/3$, otherwise $o(\widehat{\gamma}) = k$ (clear).

∎

Type V Elements of Γ

Let $o(\gamma) = k$, then we must have that $k \geq 3$, otherwise we have $k = 1$ and so $\gamma = \widetilde{\gamma} = \widehat{\gamma} = id$ or $k = 2$ and $\widetilde{\gamma}^2 = id$, so is of Type II. In fact we now have two possibilities for $\widehat{\gamma}$:

i) $o(\widehat{\gamma}) = k$,

ii) $o(\widehat{\gamma}) = b < k$ where $b|k$.

Remark 6.2.12 *If k is prime then we can only have the first case, i).*

We shall now consider each of these cases in turn.

i) Let $o(\gamma) = o(\widetilde{\gamma}) = o(\widehat{\gamma}) = k$. Then we must have $\widetilde{\gamma} = \gamma^l$ for some $2 \leq l \leq k-2$ and $HCF(l,k) = 1$ (if $l = 1$ then $\widetilde{\gamma} = \gamma$ and $\widehat{\gamma} = \gamma{-}1$, Type IIIc, and if $l = k-1$ then $\widetilde{\gamma} = \gamma^{-1}$, Type IIIb or IV, and if $HCF(l,k) \neq 1$ then $o(\widetilde{\gamma}) \neq k$) and $\widehat{\gamma} = \gamma^m$ for some $2 \leq m \leq k-2$ and $HCF(m,k) = 1$ (again we have that if $k = 1$ then $\widehat{\gamma} = \gamma$ and so $\widetilde{\gamma} = \gamma^{-1}$, Type II, and $HCF(m, k) = 1$ needed for $o(\widetilde{\gamma}) = k$). Therefore to obtain $\widehat{\gamma}\widetilde{\gamma}^2 = \gamma$ we require the condition

$$m + 2l = nk + 1 \tag{6.2.14}$$

for some $0 < n \leq 2$. Since $m < k$ and $l < k$ we must also have the weaker condition $m + 2l < 3k$.

ii) Now let $o(\widehat{\gamma}) = b$ and $o(\gamma) = o(\widetilde{\gamma}) = k$ and let $k = bc$, where we must have $2 \leq b < k - 1$.

Claim 6.2.13 *We must also have that c is odd.*

Proof: If $\delta = \gamma^b$ then $\widehat{\delta} = \widehat{\gamma}^b = id$ giving us Type I, so $o(\delta)$ must be odd, and $o(\delta) = c$. ∎

Therefore we have that $\widetilde{\gamma} = \gamma^l$ for $2 \leq l \leq k - 2$ and $HCF(l, k) = 1$ as before and $\widehat{\gamma} = (\gamma^c)^m$ with $1 \leq m < b$ and $HCF(m, b) = 1$ (note that $o(\gamma^c) = b$). Therefore we require $\gamma^{cm}\gamma^{2l} = \gamma$ and so

$$cm + 2l = nk + 1 \tag{6.2.15}$$

for $n \leq k$. In particular, since

$$m + 2l/c = nb + 1/c$$

($(2l - 1)/c = m + nb \in \mathbf{Z}$) we must have that $c|(2l - 1)$.

We are now in a position to classify all possible combinations of skew-equivariance for coupling terms of systems of form 6.2.2 so as to produce direct product coupling with respect to the internal symmetry Γ of each uncoupled cell. We consider Γ to be a subset of $\mathbf{O}(2)$, for any order of generators for Γ. There are two possible cases, Γ is generated by one element γ which has order k, so giving us $\mathbf{Z}_k$, or Γ can be generated by two elements, one of order k and the other of order 2 (a reflection), giving us $\mathbf{D}_k$. The latter case has already been considered, either we have $\Gamma = \{id, \kappa\}$ where $o(\kappa) = 2$ and $\widehat{\kappa} = \kappa$ and $\widetilde{\kappa} = \kappa$, or $\Gamma = \{id, \kappa, \tau, \pi\}$ where τ is another reflection perpendicular to κ and π is a rotation through π. This leads to six possible configurations for the skew-equivariance to achieve direct product coupling given in equation 6.2.13.

We now consider those symmetry groups Γ isomorphic to $\mathbf{Z}_k$ up to and including $k = 9$. This should cover most practical applications, but larger values of k can be investigated by the same systematic approach adopted here.

Theorem 6.2.14 *When we have a system of form 6.2.2, where the uncoupled function of motion f is Γ-equivariant, the coupling term g is skew-equivariant, and Γ is generated by one element γ of order k, then the number of configurations of the skew-equivariance needed to achieve direct product coupling, up to $k = 9$, is as given below*

$o(\gamma)$	**Ways to construct skew equivariance**
1	*1*
2	*1*
3	*2*
4	*2*
5	*4*
6	*2*
7	*6*
8	*4*
9	*6*

Proof: The proof is simply a systematic check of the following list for a specific k.

- Can we have Type I; we need $o(\gamma)$ odd and $\tilde{\gamma} = (\gamma^{(k-1)/2})^{-1}$.
- Can we have Type II; we need $o(\gamma) = 2$.
- Can we have Type III;
  - $\tilde{\gamma} = \gamma$ always works,
  - $\tilde{\gamma} = \gamma^{-1}$ needs $o(\gamma) = 2$ or 4,
  - $\tilde{\gamma} = \gamma^{(k+2)/2}$ needs k even and $HCF((k+2)/2, k) = 1$.
- Can we have $\tilde{\gamma} = \gamma^{-1}$ and $\hat{\gamma} \neq \gamma^{-1}$ so that $\hat{\gamma} = \gamma^3$; need $o(\gamma) \neq 4$.
- Can we have $o(\gamma) = o(\hat{\gamma}) = o(\tilde{\gamma})$ without any of the above.
- Can we have $o(\gamma) = o(\tilde{\gamma}) = k$ and $o(\hat{\gamma}) = b < k$ and none of the above.

We now consider each case for k in turn, we specify a configuration for the skew equivariance by the triplet $(\gamma, \hat{\gamma}, \tilde{\gamma})$, and we denote the elements as rotations where a rotation of 2π is equivalent to the identity.

$o(\gamma)=1$: $\gamma = id$ so $\widehat{\gamma} = \widetilde{\gamma} = id$ and so the only possibility is $(\gamma, \widehat{\gamma}, \widetilde{\gamma}) = \underline{(id, id, id)}$.

$o(\gamma)=2$: We consider the case where γ is not a reflection but a rotation through π. Hence $\gamma = \pi$, $\widetilde{\gamma} = \pi$ and so we must also have $\widehat{\gamma} = \pi$ giving us $\underline{(\pi, \pi, \pi)}$ (we cannot have $\widehat{\gamma} = id$ since $o(\gamma)$ is not odd).

$o(\gamma)=3$: So that $\gamma = 2\pi/3$. We must also have $o(\widetilde{\gamma}) = 3$ and $o(\widehat{\gamma}) = 1$ or 3 $(o(\widehat{\gamma})|o(\gamma))$. If $o(\widehat{\gamma}) = 1$ then $\widehat{\gamma} = id$, this is allowed since k is odd. Then $\widetilde{\gamma} = (\gamma^{(k+1)/2})^{-1} = \gamma^{-1}$ and so we have $\underline{(2\pi/3, id, 4\pi/3)}$. We also have $\widetilde{\gamma} = \gamma$ and $\widehat{\gamma} = \gamma^{-1}$, giving us $\underline{(2\pi/3, 4\pi/3, 2\pi/3)}$. The only other possibility requires $m + 2l = 4$ or 7 where $2 \leq l \leq 1$ which is impossible.

$o(\gamma)=4$: So that $\gamma = \pi/2$. Then $o(\widehat{\gamma}) = 1$ or 4 (not 2 since $4 = 2 \times 2$, where 2 is even - see Claim 6.2.13). If $o(\widehat{\gamma}) = 1$ then $\widehat{\gamma} = id$, but $o(\gamma)$ is even so are no solutions. One solution is $\widetilde{\gamma} = \gamma$ and $\widehat{\gamma} = \gamma^{-1}$, giving $\underline{(\pi/2, 3\pi/2, \pi/2)}$ and since $o(\gamma) = 4$ we can also have $\widehat{\gamma} = \widetilde{\gamma} = \gamma^{-1}$, giving $\underline{(\pi/2, 3\pi/2, 3\pi/2)}$. In this case $(k+2)/2 = 3$ so that $\gamma^{(k+2)/2} = \gamma^{-1}$, so is no new solution there. The only other possibility is that $o(\gamma) = o(\widehat{\gamma}) = o(\widetilde{\gamma}) = 4$, in which case we need $2 \leq l \leq 2$ and $HCF(l, 4) = 1$ which is impossible, so no other solutions.

$o(\gamma)=5$: We now have $\gamma = 2\pi/5$, $o(\widetilde{\gamma}) = 5$ and $o(\widehat{\gamma}) = 1$ or 5. If $o(\widehat{\gamma}) = 1$, $\widehat{\gamma} = id$ then we must have $\widetilde{\gamma} = (\gamma^2)^{-1} = 6\pi/5$ giving $\underline{(2\pi/5, id, 6\pi/5)}$. Now assume that $o(\widehat{\gamma}) = 5$. $\widehat{\gamma} = \gamma^{-1} = 8\pi/5$ and $\widetilde{\gamma} = \gamma$ is one solution, $\underline{(2\pi/5, 8\pi/5, 2\pi/5)}$. We can also have $\widetilde{\gamma} = \gamma^{-1}$ and $\widehat{\gamma} = \gamma^3$ since $o(\gamma) \neq 4$, giving $\underline{(2\pi/5, 6\pi/5, 8\pi/5)}$. The only other possibilities must satisfy $m + 2l = 6$ or 11 for $2 \leq l, m \leq 3$, that is $\{l, m\} \subset \{2, 3\}$. The only solution is $m = l = 2$ giving $\widetilde{\gamma} = \widehat{\gamma} = \gamma^2$ and so $\underline{(2\pi/5, 4\pi/5, 4\pi/5)}$.

$o(\gamma)=6$: Now $\gamma = 2\pi/6 = \pi/3$. So $o(\widetilde{\gamma}) = 6$ and $o(\widehat{\gamma}) = 1, 2$ or 6 (if $o(\widehat{\gamma}) = 3$ then $k = 6 = 3 * c$ where c is even). If $o(\widehat{\gamma}) = 1$ we have that $o(\gamma)$ is even, so this cannot work. Now let $\widehat{\gamma} = \gamma^{-1} = 5\pi/3$, then $\widetilde{\gamma} = \gamma$ will work, giving $\underline{(\pi/3, 5\pi/3, \pi/3)}$ but $\widetilde{\gamma} = \gamma^{(k+2)/2}$ does not work since $HCF((k+2)/2, k) = HCF(4, 6) \neq 1$. Since $o(\gamma) \neq 4$ however, $\widetilde{\gamma} = \gamma^{-1}$ and $\widehat{\gamma} = \gamma^3$ does work, so we have $\underline{(\pi/3, \pi, 5\pi/3)}$. Now consider the case $o(\widehat{\gamma}) = o(\widetilde{\gamma}) = 6$. We then require $m + 2l = 7$ or 13 where $2 \leq l, m \leq 4$ but $HCF(c, b) \neq 1$ when $c = 2, 3$ or 4. The only other possibility is $o(\widehat{\gamma}) = 2$, when we must have $cm + 2l = 7$ or 13, where $c = 3$ and $l \neq 1, 2, 3, 4, 5$ or 6, giving no extra solutions.

$o(\gamma)=7$: Now $\gamma = 2\pi/7$, so $o(\widetilde{\gamma}) = 7$ and $o(\widehat{\gamma}) = 1$ or 7. If $o(\widehat{\gamma}) = 1$ then $\widetilde{\gamma} = (\gamma^3)^{-1} = \gamma^4 = 8\pi/7$ so we have one solution $\underline{(2\pi/7, id, 8\pi/7)}$. Now assume $o(\widehat{\gamma}) = 7$. Let $\widehat{\gamma} = \gamma^{-1} = 12\pi/7$ then $\widetilde{\gamma} = \gamma$ works, so we have

$\underline{(2\pi/7, 12\pi/7, 2\pi/7)}$. $\tilde{\gamma} = \gamma^{-1}$ and $\hat{\gamma} = \gamma^3$ also works, so $\underline{(2\pi/7, 6\pi/7, 12\pi/7)}$. The only other possibilities are in the case of $o(\hat{\gamma}) = o(\tilde{\gamma}) = o(\gamma)$ when we require $\tilde{\gamma} = \gamma^l$ and $\hat{\gamma} = \gamma^m$ and we must have $m + 2l = 8$ or 15 where $m, l \in \{2, 3, 4, 5\}$. This gives us three solutions $(l, m) = (2, 4)$, $(3, 2)$ or $(5, 5)$ corresponding to configurations $\underline{(2\pi/7, 8\pi/7, 4\pi/7)}$, $\underline{(2\pi/7, 4\pi/7, 6\pi/7)}$ and $\underline{(2\pi/7, 10\pi/7, 10\pi/7)}$.

$o(\gamma) = 8$: We now have $\gamma = \pi/4$, $o(\tilde{\gamma}) = 8$ and $o(\hat{\gamma}) = 1$ or 8 ($8 = 2\times 4 = even \times even$). We cannot have $o(\hat{\gamma}) = 1$ since $o(\gamma)$ is even, so we must have $o(\hat{\gamma}) = 8$. Let $\hat{\gamma} = \gamma^{-1} = 7\pi/4$ then $\tilde{\gamma} = \gamma$ works, giving us $\underline{(\pi/4, 7\pi/4, \pi/4)}$, and since $HCF(5, 8) = 1$ we can also have $\tilde{\gamma} = \gamma^5$ and so $\underline{(\pi/4, 7\pi/4, 5\pi/4)}$. We can also have $\tilde{\gamma} = \gamma^{-1}$ and $\hat{\gamma} = \gamma^3$ giving $\underline{(\pi/4, 3\pi/4, 7\pi/4)}$. That then leaves us needing solutions to $m + 2l = 9$ or 17 where $m, l \in \{3, 5\}$, which gives just one more solution corresponding to $m = l = 3$, namely $\underline{(\pi/4, 3\pi/4, 3\pi/4)}$.

$o(\gamma) = 9$: Now we have $\gamma = 2\pi/9$, $o(\tilde{\gamma}) = 9$ and $o(\hat{\gamma}) = 1, 3$ or 9. If $o(\hat{\gamma}) = 1$ then $\tilde{\gamma} = (\gamma^4)^{-1} = \gamma^5 = 10\pi/9$ works giving us $\underline{(2\pi/9, id, 10\pi/9)}$. Now assume that $o(\hat{\gamma}) = 9$, and firstly $\hat{\gamma} = \gamma^{-1} = 16\pi/9$. Then $\tilde{\gamma} = \gamma$ always works, giving $\underline{(2\pi/9, 16\pi/9, 2\pi/9)}$. Now let $\tilde{\gamma} = \gamma^{-1} = 16\pi/9$ and $\hat{\gamma} = \gamma^3 = 2\pi/3$ then we have the solution $\underline{(2\pi/9, 2\pi/3, 16\pi/9)}$. The other $o(\hat{\gamma}) = 9$ solutions come from the only solutions to $m + 2l = 10$ or 19 where $m, l \in \{2, 4, 5, 7\}$. The only possibilities are $(l, m) = (4, 2)$ or $(7, 5)$ corresponding to configurations $\underline{(2\pi/9, 4\pi/9, 8\pi/9)}$ and $\underline{(2\pi/9, 10\pi/9, 14\pi/9)}$. Finally consider the case of $o(\hat{\gamma}) = 3$, so that we require $cm + 2l = 10$ or 19 where $c = 3$ and $l \in \{2, 4, 5, 7\}$ and $m = 1$ or 2. But we also need $c|(2l - 1)$, i.e. $3|(2l - 1)$ so $l = 2$ or 5. This gives only one solution, $l = m = 2$, corresponding to $\underline{(2\pi/9, 4\pi/3, 4\pi/9)}$.

Count the number of solutions to each case, and we are done. ∎

6.3 Coupled Oscillators With Internal $\mathbf{Z}_2$ Symmetry

We now consider systems of three coupled oscillators, where each oscillator has its own internal $\mathbf{Z}_2$ symmetry, a well known example being the Van der Pol equation, which we shall use later. For now we consider a model-independent situation, and comment on the possible patterns that we would expect to see. Therefore the only restrictions we place on the system is that they are governed by the set of equations 6.2.2, that

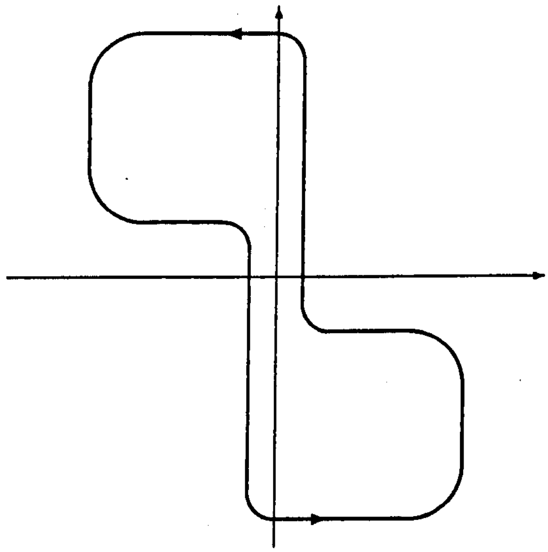


Figure 6.2: A possible form for a single oscillator with internal $\mathbf{Z}_2$ rotational symmetry.

f, the equation for uncoupled motion, is $\mathbf{Z}_2$-equivariant, and the coupling is designed to produce either $\mathbf{Z}_2 \wr \mathbf{S}_3$ or $\mathbf{Z}_2 \times \mathbf{S}_3$ global symmetry.

The natural way to go about this initially seemed to be, for each isotropy subgroup, to consider identical oscillators differing only by a phase-shift as a single oscillator coupled to the non-identical one(s). And then try to apply the work carried out on single planar oscillators earlier. For example, isotropy $\mathbf{Z}_2$ could be thought of as a two oscillator system, where one of the oscillators is really two identical oscillators π out of phase. It turns out however that thinking of the problem in this way produces many solutions to our problem that the group theory does not allow, since every solution must have an isotropy that is a subgroup of the global symmetry group.

Instead we use a property of the individual oscillators that must hold, mentioned earlier, that on the irreducible representations of both our cases the elements $[\kappa, \pi] \in \mathbf{Z}_2 \times \mathbf{S}_3 \times \mathbf{S}^1$ and $[(\kappa, \kappa, \kappa), \pi] \in \mathbf{Z}_2 \wr \mathbf{S}_3 \times \mathbf{S}^1$ act as the identity, and so must be present in *every* isotropy subgroup. This means that *every* oscillator *must* have a $[\kappa, \pi]$ symmetry, that is, applying the $\mathbf{Z}_2$ action and then a phase-shift of π leaves the oscillator unchanged. Such as the oscillator in Figure 6.2.

This is perhaps a surprising observation, which would only seem to hold for $\mathbf{Z}_2$ internal symmetry. For example the $\mathbf{S}_3$ solution present in all the cases, must now have this additional symmetry thrust upon it in the guise of three identical, in-phase oscillators, each with an internal $[\kappa, \pi]$ symmetry.

We now consider when each oscillator has an internal $\mathbf{Z}_2$ symmetry generated by a rotation through π, denoted $\pi \in \mathbf{O}(2)$. So if $x(t) \in \mathbf{R}^2$ then $\pi x(t)$ rotates the vector $x(t)$ a half-turn about the origin. The group theory then forces each oscillator to have $[\pi, \pi]$ symmetry, that is $x(t+\pi) = \pi x(t)$. In the following we denote the possible outcomes in two notations. The first is in the same way we did for the $\mathbf{S}_3$ case, that is, we denote a specific waveform by A, and a phase shift of θ by $A+\theta$, and so $\pi \in \mathbf{Z}_2$ acted on the waveform looks like πA, so that each oscillator must satisfy $A = \pi A + \pi$. A '0' denotes a quiescent oscillator.

The other notation we shall use is to denote the solution $X(t)$ as

$$X(t) = (x_1(t), x_2(t), x_3(t)) \in \mathbf{R}^6$$

so that $\pi x_i = -x_i = x_i(t+\pi)$ (where each $x_i(t)$ is the projection from $\mathbf{R}^6$ to the appropriate $\mathbf{R}^2$ corresponding to oscillator i).

Our aim is to compare the two different symmetries in the presence of $\mathbf{Z}_2$ symmetry, i.e. $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $\mathbf{Z}_2 \times \mathbf{S}_3$, and then to compare to the case of no internal symmetry, $\mathbf{S}_3$. We must therefore consider each type of coupling separately.

a) $\mathbf{Z}_2 \times \mathbf{S}_3$ Symmetry - Direct Product Coupling

Isotropy $\tilde{\mathbf{Z}}_3$ Again we have the same situation as in the $\mathbf{S}_3$ global symmetry case except for the $[\pi, \pi]$ symmetry of each oscillator. This gives us

$$[A,\ A+2\pi/3,\ A+4\pi/3] \text{ where } A+\pi = \pi A$$

or

$$(x(t), x(t+2\pi/3), x(t+4\pi/3)) \text{ where } x(t+\pi) = -x(t).$$

Isotropy $\tilde{\mathbf{Z}}_2$ Without the internal symmetry this leads to the solution

$$(x(t), x(t+\pi), y(t))$$

where $y(t)$ is π-periodic, that is , $y(t+\pi) = y(t)$, but with the internal $\mathbf{Z}_2$ symmetries present the third π-periodic oscillator is forced to cease oscillating, this is because we now have that the isotropy forces $y(t+\pi) = y(t)$ but the $[\pi, \pi]$ symmetry forces $y(t+\pi) = -y(t)$ and so we must have $y(t) = -y(t) = 0$, giving $y(t) \equiv 0$, so we have

$$[A,\ A+\pi,\ 0] \text{ where } A+\pi = \pi A$$

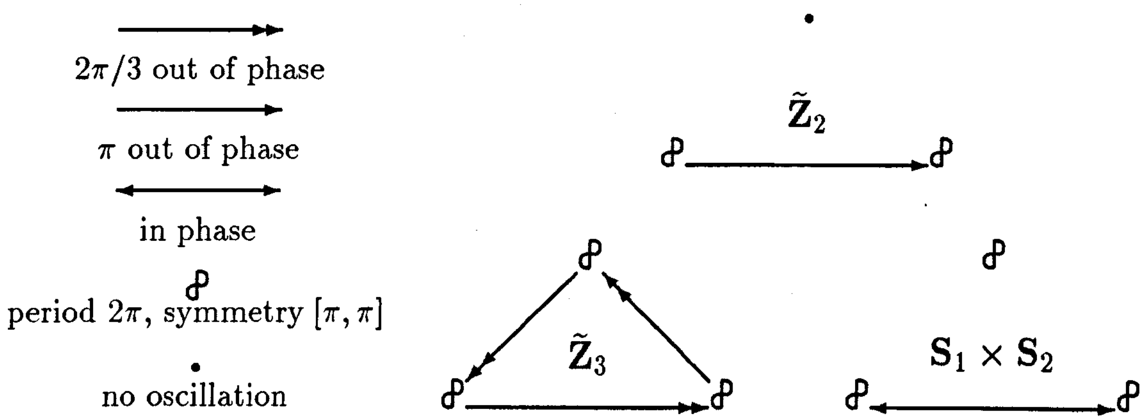


Figure 6.3: Three coupled oscillators with $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry.

or

$$(x(t), x(t+\pi), 0) \text{ where } x(t+\pi) = -x(t).$$

Isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ This is the same as the $\mathbf{S}_3$ global symmetry case except that we will now have an internal $[\pi, \pi]$ symmetry giving

$$[A,\ A,\ C] \text{ where } A + \pi = \pi A \text{ and } C + \pi = \pi C$$

or

$$(x(t), x(t), z(t)) \text{ where } x(t+\pi) = -x(t) \text{ and } z(t+\pi) = -z(t).$$

We show these possibilities diagrammatically in Figure 6.3.

a) $\mathbf{Z}_2 \wr \mathbf{S}_3$ Symmetry - Wreath Product Coupling

The main difference between this and the direct product case is that once we have one solution, the corresponding conjugacy class is large relative to the direct product case, and we can obtain another solution by applying the $\mathbf{Z}_2$ action to any one of the oscillators, which can lead to some remarkable results.

Isotropy $\mathbf{S}_3$ The obvious solution corresponding to isotropy $\mathbf{S}_3$ is given by

$$[A,\ A,\ A] \text{ where } A + \pi = \pi A$$

or

$$(x(t), x(t), x(t)) \text{ where } x(t+\pi) = -x(t).$$

But we may now also have the conjugate solution which is *not* seen in the case of global $\mathbf{S}_3$ symmetry.

$$[A,\ A+\pi,\ A] \text{ where } A+\pi = \pi A$$

or

$$(x(t), x(t+\pi), x(t)) \text{ where } x(t+\pi) = -x(t).$$

Isotropy $\widetilde{\mathbf{Z}_3}$ Again we see the pattern

$$[A,\ A+2\pi/3,\ A+4\pi/3] \text{ where } A+\pi = \pi A$$

or

$$(x(t), x(t+2\pi/3), x(t+4\pi/3)) \text{ where } x(t+\pi) = -x(t).$$

but now we also get a quite surprising conjugate solution by applying a π phase shift to any of the oscillators, for example the third oscillator, and then permuting. Since $4\pi/3 + \pi = \pi/3$ modulo 2π we find

$$[A,\ A+\pi/3,\ A+2\pi/3] \text{ where } A+\pi = \pi A$$

or

$$(x(t), x(t+\pi/3), x(t+2\pi/3)) \text{ where } x(t+\pi) = -x(t).$$

Isotropy $\tilde{\mathbf{Z}}_2$ and $\mathbf{W}_2$ As in the direct product case isotropy $\tilde{\mathbf{Z}}_2$ corresponds to the pattern

$$[A,\ A+\pi,\ 0] \text{ where } A+\pi = \pi A$$

or

$$(x(t), x(t+\pi), 0) \text{ where } x(t+\pi) = -x(t),$$

but we could also see the conjugate solution

$$[A,\ A,\ 0] \text{ where } A+\pi = \pi A$$

or

$$(x(t), x(t), 0) \text{ where } x(t+\pi) = -x(t).$$

Note however that the analysis of the generic solution to this isotropy on the irreducible representation, at the point of primary bifurcation, showed this solution to be generically unstable.

Isotropy $\mathbf{W}_1$ As for isotropy $\mathbf{W}_2$ the $[\pi, \pi]$ symmetry of the individual oscillator patterns forces two of the oscillators to cease oscillating, and so we are left with

$$[A,\ 0,\ 0] \text{ where } A + \pi = \pi A$$

or

$$(x(t), 0, 0) \text{ where } x(t+\pi) = -x(t).$$

The final two isotropies correspond to sub-maximal isotropies, but they may still occur in 'real' systems of oscillators (as opposed to generic solutions on the Γ-simple space).

Isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ As obtained previously we have

$$[A,\ A,\ C] \text{ where } A + \pi = \pi A \text{ and } C + \pi = \pi C$$

or

$$(x(t), x(t), z(t)) \text{ where } x(t+\pi) = -x(t) \text{ and } z(t+\pi) = -z(t).$$

except that now we also have the conjugate pattern

$$[A,\ A+\pi,\ C] \text{ where } A + \pi = \pi A \text{ and } C + \pi = \pi C$$

or

$$(x(t), x(t+\pi), z(t)) \text{ where } x(t+\pi) = -x(t) \text{ and } z(t+\pi) = -z(t).$$

Isotropy $\{(\kappa, \kappa, 0), \pi\}$ Since the third oscillator is again forced by the symmetry to be π-periodic it must cease to oscillate, and we're left with

$$[A,\ C,\ 0] \text{ where } A + \pi = \pi A \text{ and } C + \pi = \pi C$$

or

$$(x(t), z(t), 0) \text{ where } x(t+\pi) = -x(t) \text{ and } z(t+\pi) = -z(t).$$

We show these possibilities diagrammatically in Figure 6.4.

Summary of Three Coupled Oscillators, Each With Internal Z_2 Symmetry

We summarize the results of the two types of coupling in table 6.2. We let A and C denote 2π-periodic oscillators and B denote a π-periodic oscillator in the case of purely $\mathbf{S}_3$ global symmetry. In addition, when $\mathbf{Z}_2$ symmetry is present, we have that $A + \pi = \pi A$, and similarly for C. A '0' denotes a quiescent oscillator.

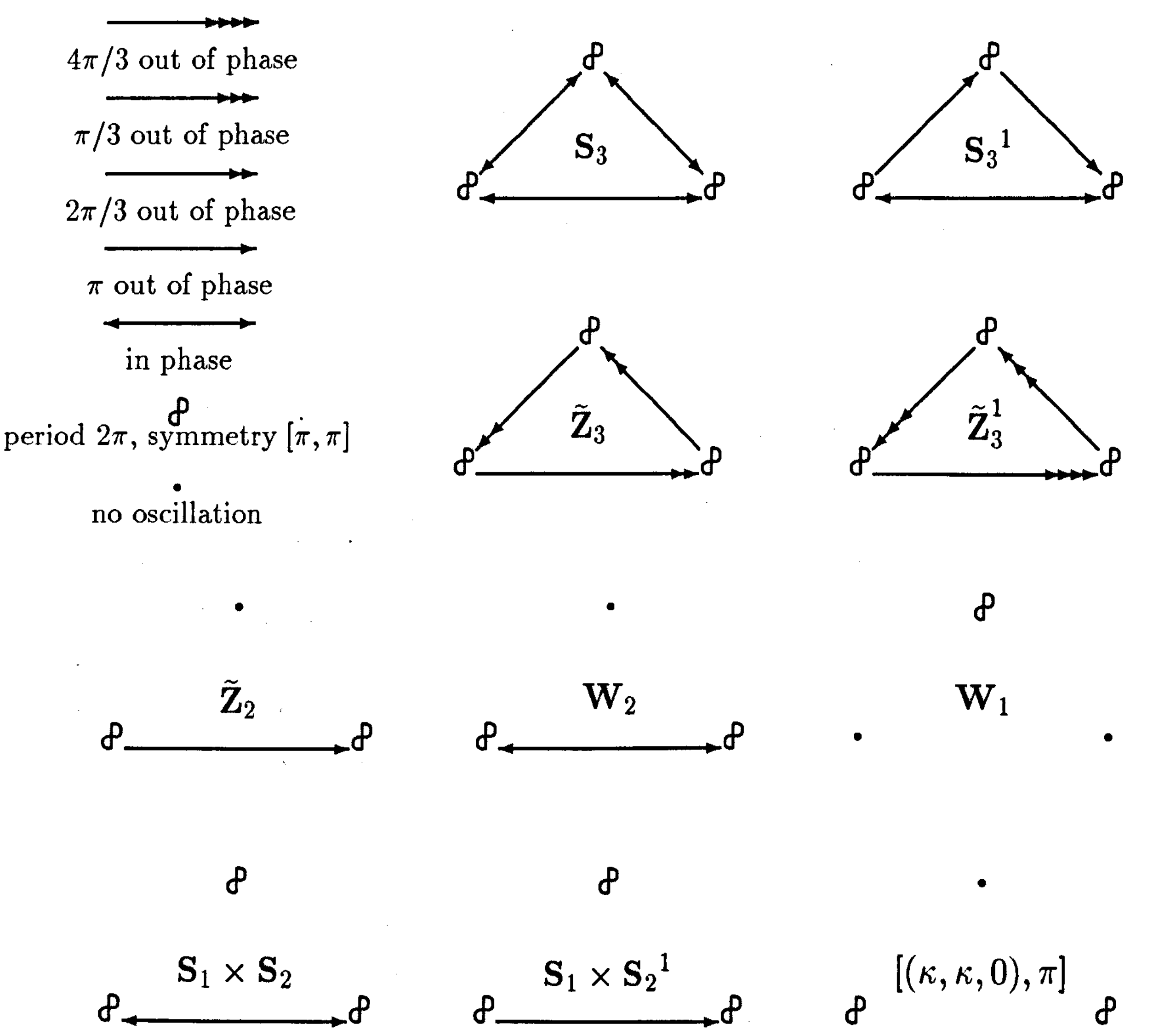


Figure 6.4: Three coupled oscillators with $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry.

Symmetry	Isotropy	Pattern Observed		
$\mathbf{S}_3$ No internal	$\tilde{\mathbf{Z}}_3$	A	$A+2\pi/3$	$A+4\pi/3$
	$\tilde{\mathbf{Z}}_2$	A	$A+\pi$	B
	$\mathbf{S}_1\times\mathbf{S}_2$	A	A	C
$\mathbf{Z}_2\times\mathbf{S}_3$	$\tilde{\mathbf{Z}}_3$	A	$A+2\pi/3$	$A+4\pi/3$
	$\tilde{\mathbf{Z}}_2$	A	$A+\pi$	0
	$\mathbf{S}_1\times\mathbf{S}_2$	A	A	C
$\mathbf{Z}_2\wr\mathbf{S}_3$	$\mathbf{S}_3$	A	A	A
		A	$A+\pi$	A
	$\tilde{\mathbf{Z}}_3$	A	$A+2\pi/3$	$A+4\pi/3$
		A	$A+\pi/3$	$A+2\pi/3$
	$\mathbf{W}_2$	A	$A+\pi$	0
		A	A	0
	$\mathbf{W}_1$	A	0	0
	$\mathbf{S}_1\times\mathbf{S}_2$	A	A	C
		A	$A+\pi$	C
	$(\kappa,\kappa,0),\pi$	A	C	0

Table 6.2: Coupled oscillators with internal $\mathbf{Z}_2$ (rotational) Symmetry

Direct Product Coupling

When each oscillator has an internal $\mathbf{Z}_2$ symmetry, acting as a rotation by π, and the coupling is set up so as to produce a global $\mathbf{Z}_2\times\mathbf{S}_3$ symmetry there are only two real differences between this set up and the original $\mathbf{S}_3$ symmetric scenario, both difference caused by the necessary $[\pi,\pi]$ symmetry of each individual oscillator:

i) Each oscillator is forced to have an internal $[\pi,\pi]$ symmetry, regardless of the isotropy of the solution, and

ii) Those isotropies causing frequency doubling in the $\mathbf{S}_3$ case now forces a quiescent oscillator.

Wreath Product Coupling

If however we let the coupling produce a wreath product between the global and internal symmetries, i.e. $\mathbf{Z}_2\wr\mathbf{S}_3$, then in addition to the new solution obtained in the direct product case (as outlined in ii) above) we see an additional six new possibilities, four of which arise from new conjugate solutions to previously observed patterns, and the other two are entirely new ones.

i) **New Solutions** correspond to isotropies $\mathbf{W}_1$ and $[(\pi, \pi, 0), \pi]$, the latter is a submaximal isotropy, the former however we would expect to occur generically, so is a 'likely' outcome.

ii) **Conjugate Solutions** however produce the most interesting results, such as two in-phase oscillators, and the third π out of phase ($\mathbf{S}_3$) or two in phase oscillators and a quiescent one ($\mathbf{W}_2$, although this is generically unstable on the irreducible subspace). The most surprising result however arises from one of the conjugate solutions to $\tilde{\mathbf{Z}}_3$, a solution which is seen in the $\mathbf{S}_3$ case. The conjugate solution of two phase-differences of $\pi/3$ and the third of $4\pi/3$ produces a pattern which at first sight does not appear to be a recognisable symmetry subgroup of our problem. It should be easy to see how coupling n oscillators together, each with an internal $\mathbf{Z}_2$ symmetry so as to produce wreath product coupling would produce even more unexpected patterns of this type.

Chapter 7

Numerical Simulations

We now present three coupled oscillator models to illustrate the properties of coupled oscillators with $\mathbf{Z}_2 \times \mathbf{S}_3$ and $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetries which we have predicted. It turns out that some of the results that are found may not necessarily occur *per se* from the mechanisms we have investigated, but if we add some more parameters it may be possible that they do. In all the cases it is also not clear whether the solutions observed would be stable *at the point* of bifurcation, or whether they become stable through a secondary bifurcation (see for example discussion on bifurcations of the Rivlin Cube in [17], Case Study 5). The results are still shown here however to show that the *patterns* predicted can occur in 'real' systems.

Two of the models are used to show that all the patterns that we have discussed can actually appear in model situations, and the third is used since it is a particularly well known, and frequently applied, oscillator model. In each case we describe the equations that were used and discuss how they are/might be related to the results which we have found earlier.

All the sample output was obtained using Dstool, 'A Dynamical System Toolkit with an Interactive Graphical Interface' (Guckenheimer, Myers, Wicklin, Worfolk, Center For Applied Mathematics, Cornell University). Initially the integrator routine used for calculations was a fourth order Runge-Kutta algorithm; the results were later checked using a Bulirsh-Stoer algorithm (see for example [27] for discussion on these algorithms).

7.1 Volume 2 Oscillator - Direct Product Coupling

We begin by repeating the calculations of the oscillator described in Golubitsky et al. [17] which we shall call the 'Volume 2' oscillator. The authors of this book perform numerical simulations on a system of three coupled oscillators

$$\frac{d}{dt}(x_p, y_p) = F(x_p, y_p, \lambda) + K(\lambda)(2x_p - x_{p-1} - x_{p+1}, 2y_p - y_{p-1} - y_{p+1}) \tag{7.1.1}$$

where each cell is described by the two state variables (x_p, y_p) and p is taken modulo 3. In particular they use

$$\begin{aligned} F(x, y, \lambda) &= \begin{bmatrix} 4 & 1 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + p(x^2+y^2)\begin{bmatrix} x \\ y \end{bmatrix} \\ &\quad + q(x^2+y^2)\begin{bmatrix} -y \\ x \end{bmatrix} - 2K(\lambda)\begin{bmatrix} x \\ y \end{bmatrix} + r\begin{bmatrix} (x^2+y^2)^2 \\ 0 \end{bmatrix} \end{aligned} \tag{7.1.2}$$

where

$$K(\lambda) = -\lambda\begin{bmatrix} -4 & 2 \\ -2 & -4 \end{bmatrix}.$$

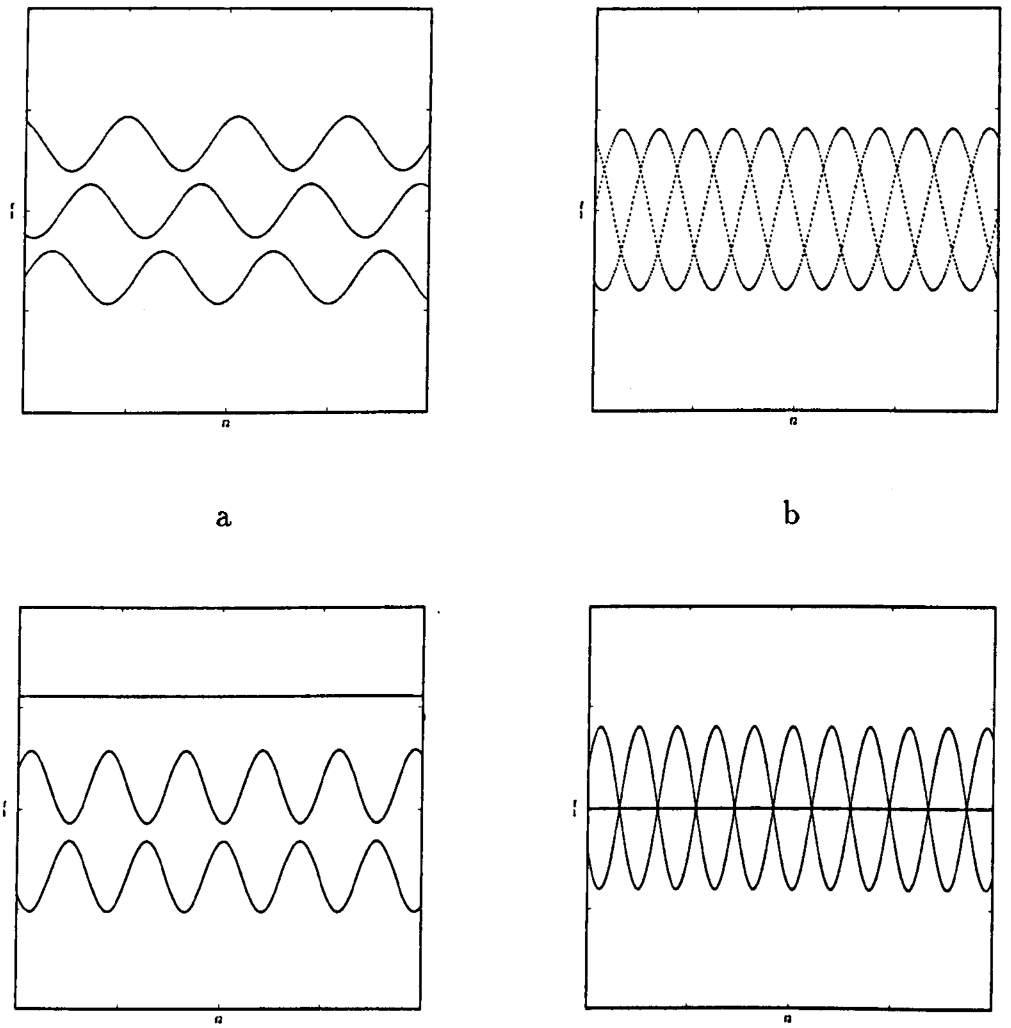
When $r = 0$ equation 7.1.2 is $\mathbf{Z}_2$ equivariant under the action $(x, y) \mapsto (-x, -y)$ and the full system 7.1.1 exhibits $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry (i.e. direct product coupling). The two patterns corresponding to the spatio-temporal isotropies predicted in Proposition 5.1.1 (since the $\mathbf{Z}_2 \times \mathbf{S}_3$ isotropies are in one-to-one correspondence with the $\mathbf{S}_3$ ones) are shown in Figure 7.1 where in the case of isotropy $\tilde{\mathbf{Z}}_2$ one oscillator is forced to become quiescent due to the internal $[\pi, \pi]$ symmetry of the oscillators (Chapter 6). In particular note that (pointed out in [17]) the solution corresponding to isotropy $\tilde{\mathbf{Z}}_2$ is certainly *not* stable at the point of bifurcation.

Allowing r to become non-zero breaks the $\mathbf{Z}_2$ symmetry of the oscillators and produces a $\tilde{\mathbf{Z}}_2 \subset \mathbf{S}_3 \times \mathbf{S}^1$ isotropy solution where one of the oscillators has double the frequency of the other two. This, along with the pattern corresponding to $\mathbf{S}_3$ is shown in Figure 7.2.

7.2 Van Der Pol Oscillators with Wreath Product Coupling

We now examine a case where we have wreath product coupling as opposed to direct product coupling or coupling that produces only $\mathbf{S}_3$ global symmetry.

In particular we use three coupled Van der Pol oscillators which, originally used to model a specific electronic circuit, must be the most frequently utilised for the purposes of modeling oscillatory phenomenon. It is for this reason that we show here how the oscillator patterns observed can be dramatically affected by the presence



a b

c d

Figure 7.1: Volume 2 oscillators with isotropies a) and b) $\tilde{\mathbf{Z}}_3$, where $(p, q, \lambda, r) = (-5, 30, 1.05, 0)$ and c) and d) $\tilde{\mathbf{Z}}_2$, where $(p, q, \lambda, r) = (-5, -50, 1.2, 0)$.

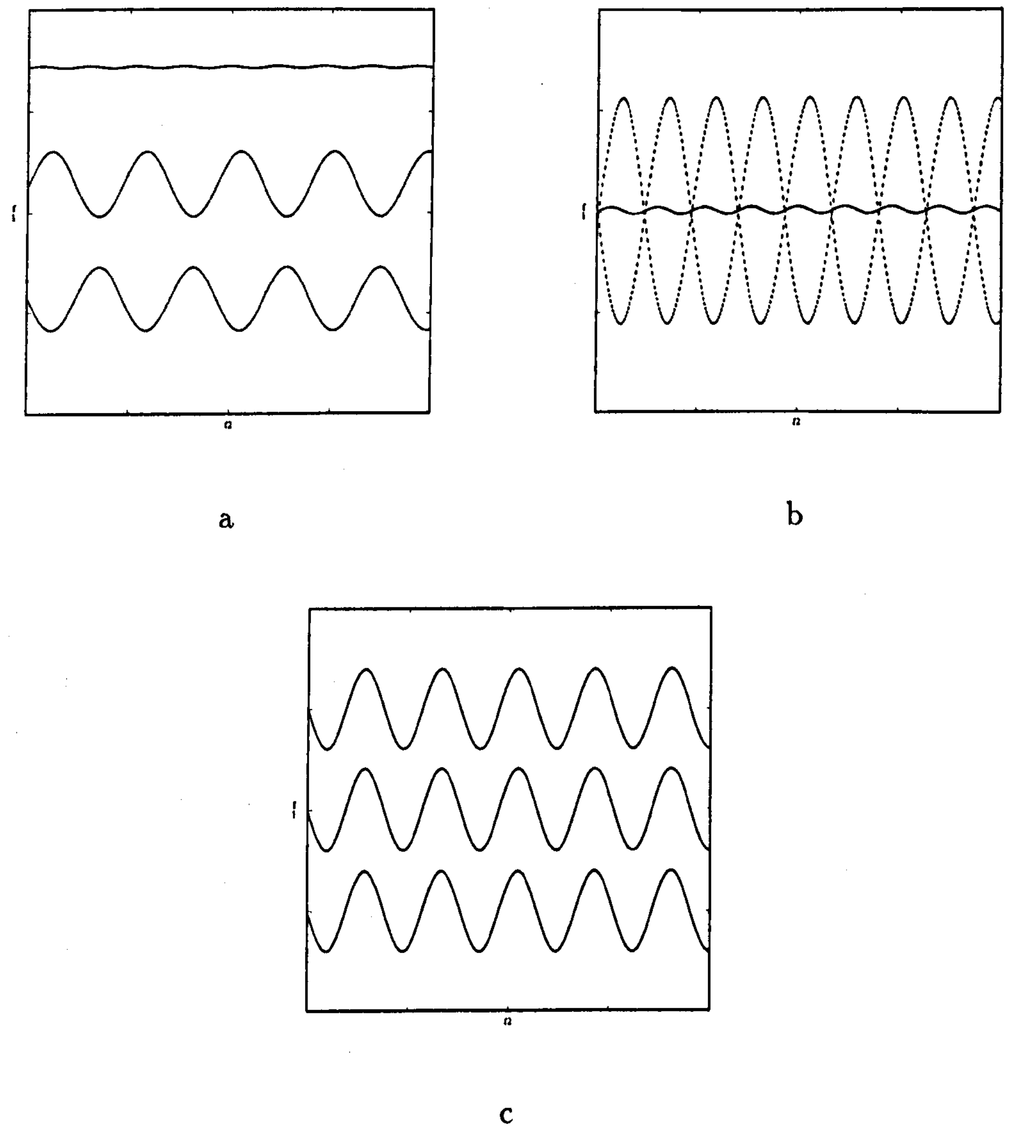


Figure 7.2: Volume 2 oscillators with isotropies a) and b) $\tilde{\mathbf{Z}}_2$, where $(p, q, \lambda, r) = (-5, -50, 1.1, 10)$ breaking the internal $\mathbf{Z}_2$ symmetries and c) $\mathbf{S}_3$ where $(p, q, \lambda, r) = (-5, -50, -0.6, 0)$.

of the internal $\mathbf{Z}_2$ symmetry, a fact that may not have been appreciated before this work.

The form of the equations we use here are, for a single oscillator, given by

$$\begin{aligned} \dot{x} &= y - \alpha x(x^2/3 - \delta) + \beta, \\ \dot{y} &= -x. \end{aligned} \tag{7.2.3}$$

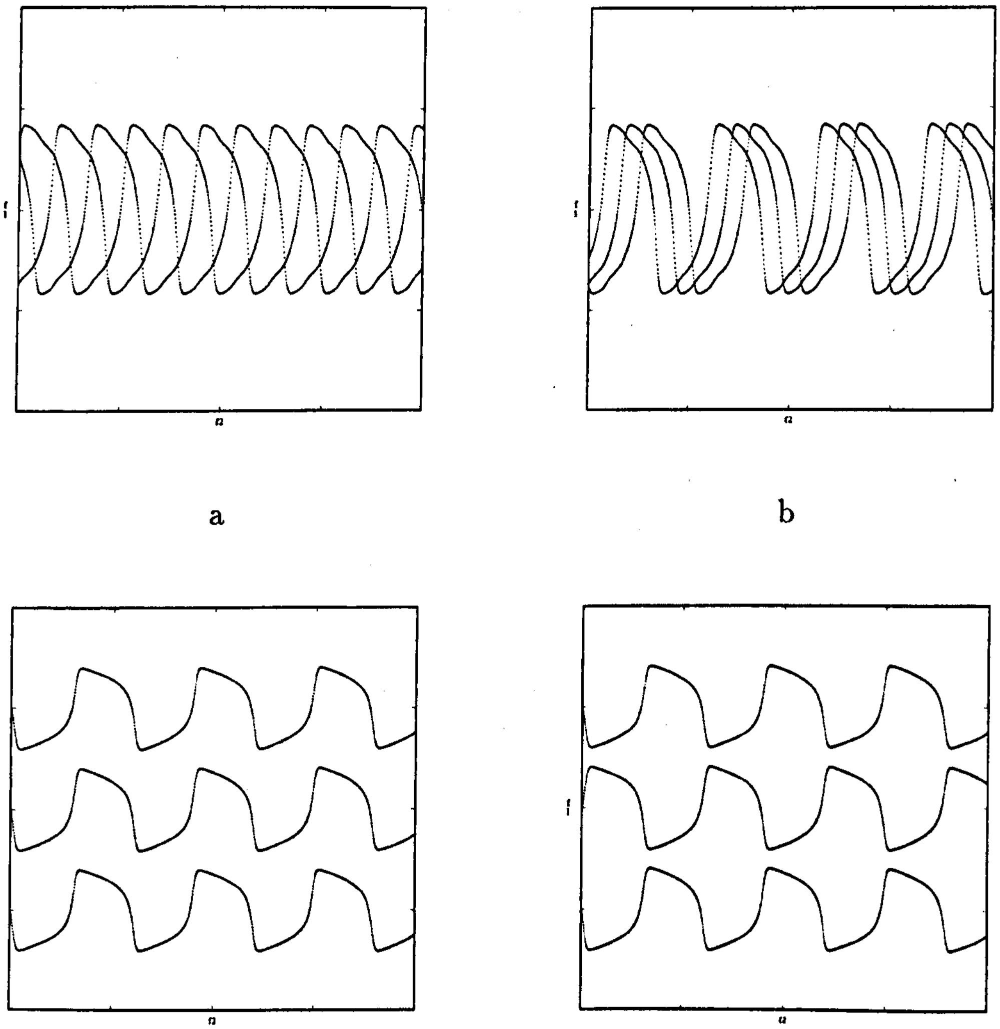
Here β breaks the $\mathbf{Z}_2$ symmetry, again given by $(x, y) \mapsto (-x, -y)$, and so for our simulations is set to zero. For the 'classical' equation we must also set $\delta = 1$. Wreath product coupling is achieved by coupling the equations thus

$$\begin{aligned} \dot{x_i} &= y_i - \alpha x_i(x_i^2/3 - \delta) + \beta + kx_i(2x_i^2 - x_{i-1}^2 - x_{i+1}^2), \\ \dot{y_i} &= -x_i. \end{aligned} \tag{7.2.4}$$

where i is taken modulo 3 and k represents the coupling strength.

It is not clear whether or not the resulting patterns observed when numerically integrated occur at a Hopf bifurcation, and remain stable from that bifurcation (see [19] for a discussion of the bifurcations that can occur for the Van der Pol equation), it may be possible however to augment the equations with an additional parameter, for which a suitable Hopf bifurcation occurs, and for some value is identical to the equations integrated. This will not be attempted here, rather we use the results of the numerical runs to show that the patterns with isotropies given by Proposition 5.2.1 *can* exist. Note also that all the solutions shown can be stable at the point of bifurcation by Theorem 5.2.4

Some of the resulting numerical patterns are shown in Figure 7.3. The most interesting solutions are those corresponding to $\tilde{\mathbf{Z}}_3$ and its conjugate, the latter of which shows the more unexpected (though predicted) result of two phase differences of $\pi/3$ and a third of $4\pi/3$. We also see the solution corresponding to isotropy $\mathbf{S}_3$ and its conjugate of two oscillators in phase, the third π out of phase.



a b

c d

Figure 7.3: Van der Pol Oscillators with isotropies a), b) $\tilde{\mathbf{Z}}_3$ and conjugate, where $(\alpha,\beta,\gamma,\delta) = (3,0,0.1,1)$ and c) and d) $\mathbf{S}_3$ and conjugate, where $(\alpha,\beta,\gamma,\delta) = (3,0,-0.1,1)$.

7.3 Parabolic Oscillator

Our third and final oscillator that we discuss has been chosen for the wide range of isotropies it is willing to exhibit. The details for this oscillator have been taken from Andronov et al. [2], but we briefly outline the derivation here. The model is based upon 'a material point of mass m allowed to move freely along a parabola determined by the equation $x^2 = 2pz$ and rotating with constant angular velocity Ω about the x axis'. This can be thought of as a marble being allowed to move freely about the inside of a 'parabolic tea-cup'. See Figure 7.4 for a diagrammatic representation.

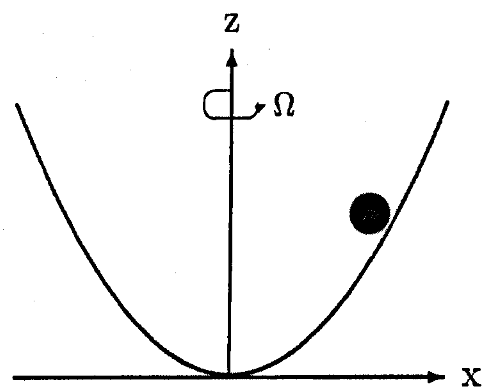


Figure 7.4: Parabolic Oscillator, see text for details.

The equations of motion can then be derived as

$$\begin{aligned} \frac{dx}{dt} &= y, \\ \frac{dy}{dt} &= -\frac{\left(\lambda + \frac{y^2}{p^2}\right) x}{\left(1 + \frac{x^2}{p^2}\right)} \end{aligned} \tag{7.3.5}$$

where $\lambda = g/p - \Omega^2$. This has a bifurcation at $\lambda = 0$, and for $\lambda > 0$ and so $\Omega^2 < g/p$ we achieve oscillations about the equilibrium point $(0, 0)$.

To increase the possibility of a greater number of stable solutions (by a 'trial and error' approach) we also introduce a new parameter, c, into the equation and then add coupling terms to give us the following coupled system

$$
\begin{aligned}
\frac{dx_i}{dt} &= y_i, \qquad\qquad (7.3.6)\\
\frac{dy}{dt} &= -\frac{\left(\lambda + \frac{y_i^2}{p^2}\right)x}{\left(1+\frac{x^2}{p^2}\right)} + \epsilon w y_i(2y_i^2 - y_{i-1}^2 - y_{i+1}^2) + dwy_i^3(2y_i^2 - y_{i-1}^2 - y_{i+1}^2)\\
&\quad +\epsilon(w-1)(2y_i - y_{i-1} - y_{i+1}) + d(w-1)y_i^2(2y_i - y_{i-1} - y_{i+1}).
\end{aligned}
$$

Where as usual i is taken modulo 3.

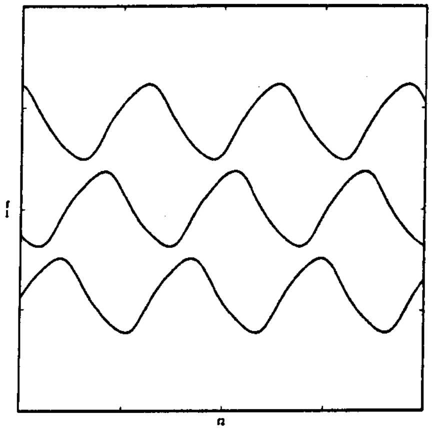
For $w = 0$ we have direct product coupling, and for $w = 1$ we have wreath product coupling. We also have four parameters λ, c, d and ϵ with which to seek stable solution branches of this oscillator system.

We note here however that as it stands the original (uncoupled) equations 7.3.5 do not achieve oscillatory behaviour by a Hopf bifurcation. Rather as λ is varied, in particular as it passes through zero in the positive direction, the system undergoes a bifurcation from one unstable state of equilibrium $x = y = 0$ of saddle type ($\lambda < 0$) to an infinite number of states of equilibrium corresponding to the straight line $y = 0$ ($\lambda = 0$) to an equilibrium of centre type ($\lambda > 0$, equilibrium $x = y = 0$) and so we do not even have an attracting limit cycle. We do however have oscillations about $x = y = 0$ with amplitude given by initial conditions for a fixed value of $\lambda > 0$..

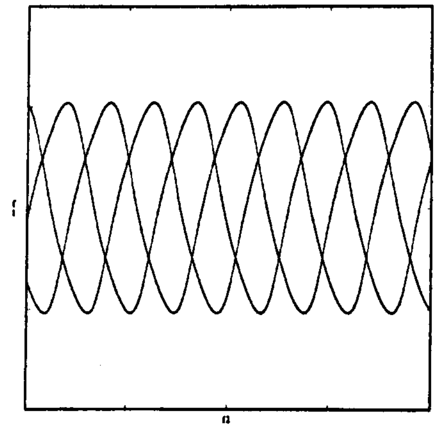
It is unclear how the extra parameter, or indeed the coupling terms affect this behaviour, but from the patterns observed through numerical analysis it would seem possible that the equations *could* be augmented so as to produce the appropriate Hopf bifurcations. In particular the behaviour *does* seem to suggest the existence of an attracting limit cycle (from the emergence of similarly sized oscillations).

The direct product case only seems to result in two stable patterns for a wide range of parameters and choice of initial conditions, namely patterns corresponding to isotropy $\widetilde{\mathbf{Z}_3}$ (isotropy guaranteed in the previous work by Proposition 5.1.1), as well as a solution with $\mathbf{S}_3$ symmetry which is *not* guaranteed by the work we have carried out earlier, but which we show here since it did ocur. Some representative output is shown in Figure 7.5.

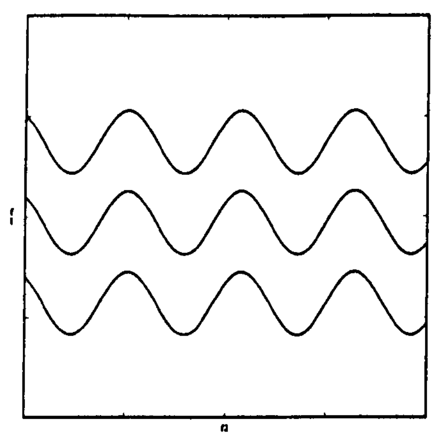
The wreath product case however leads to a very large variety of patterns, all of which were predicted in earlier work (Proposition 5.2.1 and Chapter 6). We do however see solutions which we have shown are generically unstable at the point of bifurcation in the general setting (Theorem 5.2.4), suggesting perhaps that they are the result of a secondary bifurcation if indeed the oscillations can be produced by a Hopf bifurcation in the necessary manner. Note also the sub-maximal isotropy $\mathbf{S}_1 \times \mathbf{S}_2$ which by Field and Richardson [14] will not generically exist at a Hopf bifurcation.



a



b



c

Figure 7.5: Parabolic Oscillators with direct product coupling showing a),b) isotropy $\widetilde{\mathbf{Z}}_3$, where $(\lambda, p, c, d, \epsilon) = (1, 1, 1.05, -1, 0.5)$ and c) symmetry $\mathbf{S}_3$ where $(\lambda, p, c, d, \epsilon) = (1, 1, 5, 0, -0.5)$.

The patterns observed are shown in Figures 7.6, 7.7 and 7.8.
In Figure 7.6 we see the solutions corresponding to isotropies $\widetilde{\mathbf{Z}_3}$ and it's conjugate, and again we see the 'unexpected' result of two phase differences of $\pi/3$ and one of $4\pi/3$. Figure 7.7 shows the $\mathbf{S}_3$ solution and its conjugate and also a solution corresponding to isotropy $\mathbf{S}_1 \times \mathbf{S}_2$, or rather it's conjugate where the two identical waveforms are π out of phase.

Finally Figure 7.8 shows the solutions corresponding to the isotropies $\mathbf{W}_2$, it's conjugate, and $\mathbf{W}_1$, where, respectively, one, one and two oscillators are forced to lie dormant. Note in particular that by Theorem 5.2.4 the solution branches corresponding to isotropy $\mathbf{W}_2$ would be generically unstable at the point of bifurcation in the general setting.

7.4 Comments

This Chapter has shown that it is possible to achieve the patterns predicted by the theory, and that some oscillators (namely our Parabolic Oscillator) are very susceptible to the differences in the type of coupling between oscillators.

In the case of the Van der Pol oscillators it shows that when using them for modeling purposes, unless an internal $\mathbf{Z}_2$ symmetry is specifically required, the presence of the internal symmetry could have an unwanted or unexpected outcome on the results.

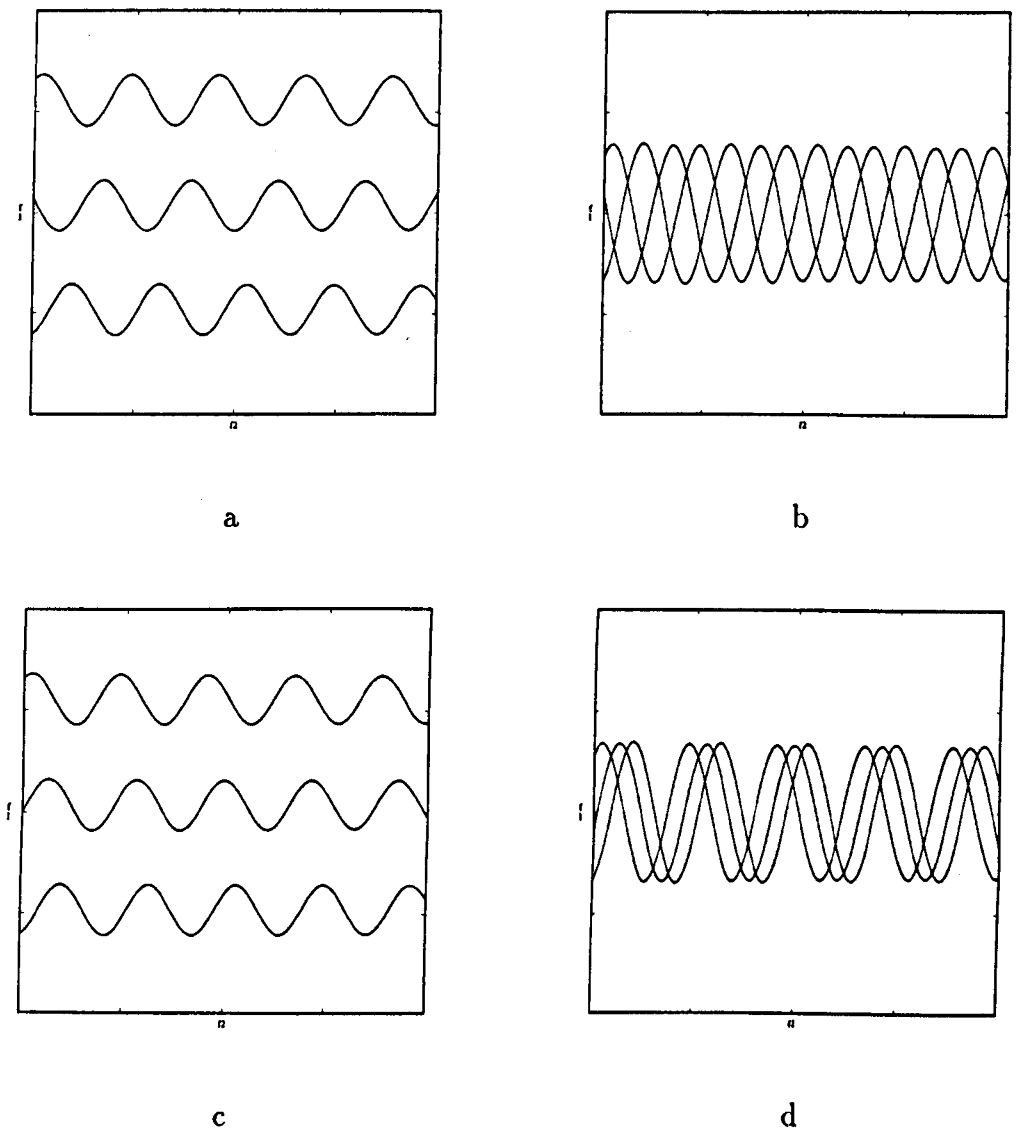


Figure 7.6: Parabolic Oscillators - wreath product coupling showing isotropies a) and b) $\widetilde{\mathbf{Z}_3}$ and c) and d) it's conjugate. All patterns obtained using values $(\lambda, p, c, d, \epsilon) = (1, 1, 1.05, -0.2, -0.2)$.

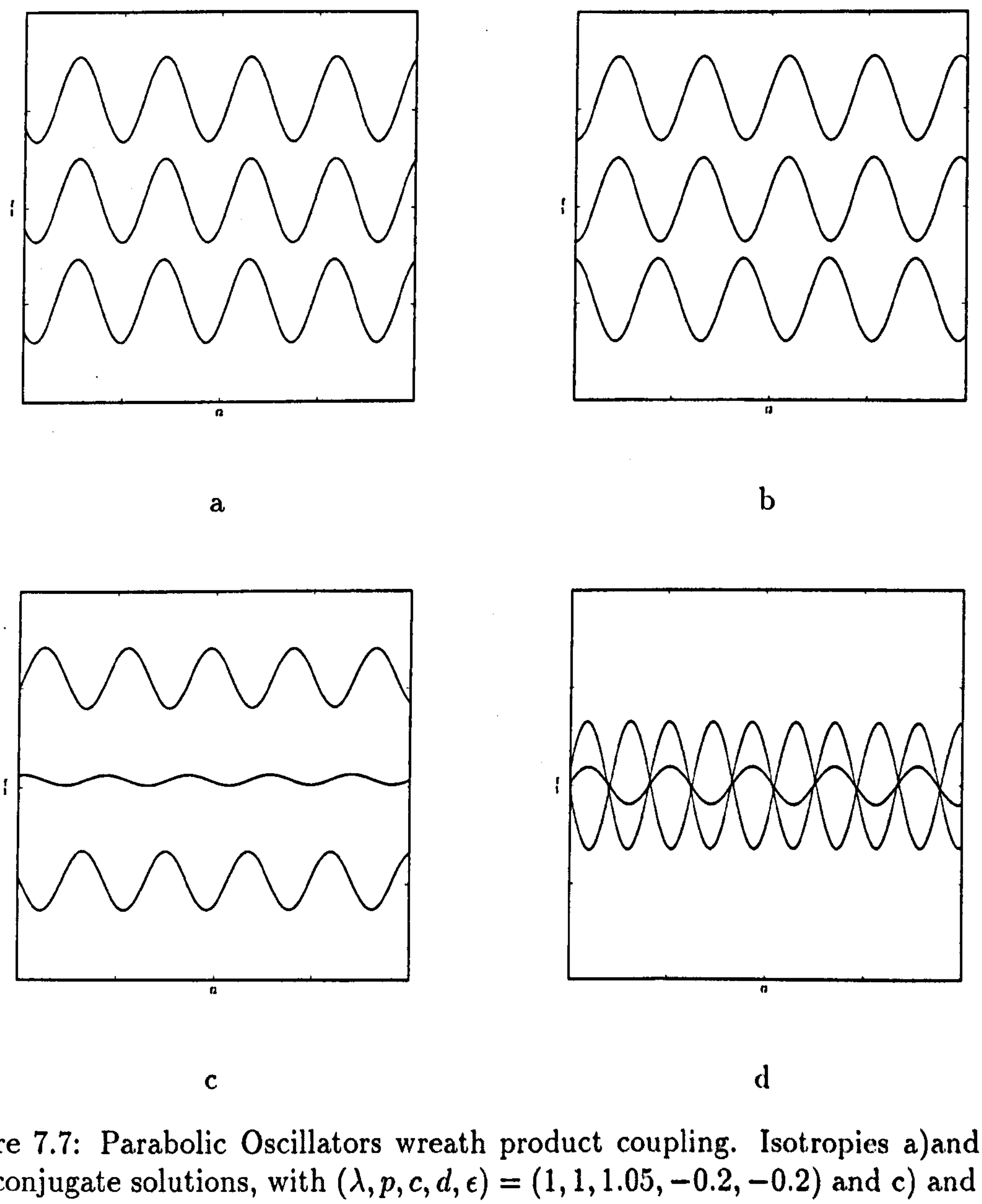
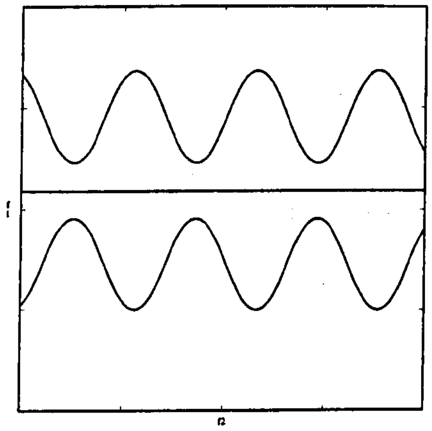
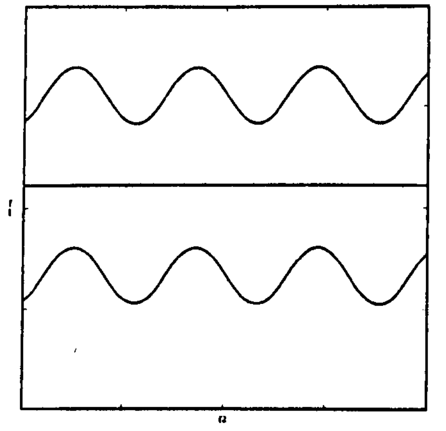


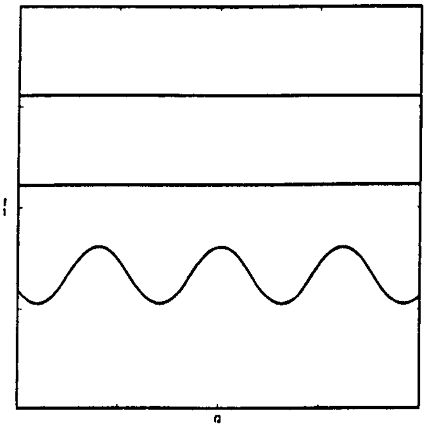
Figure 7.7: Parabolic Oscillators wreath product coupling. Isotropies a)and b) $\mathbf{S}_3$ and conjugate solutions, with $(\lambda, p, c, d, \epsilon) = (1, 1, 1.05, -0.2, -0.2)$ and c) and d) the sub-maximal $\mathbf{S}_1 \times \mathbf{S}_2$ (conjugate) solution with $(\lambda, p, c, d, \epsilon) = (1, 1, 1, -0.2, -0.1)$.



a



b



c

Figure 7.8: Parabolic Oscillators - wreath product coupling showing isotropies a) $\mathbf{W}_2$ b) it's conjugate and c) $\mathbf{W}_1$. All patterns achieved using values $(\lambda, p, c, d, \epsilon) = (1, 1, 5, -2, 0.5)$.

Chapter 8

Insect Gaits and Coupled Oscillators

In this Chapter we continue the work of Wood [30] and [31] where networks of coupled non-linear oscillators were investigated with respect to their suitability for modelling insect locomotion. The variables that describe the specific leg movements of an animal to achieve locomotion are collectively called the *gaits* of the animal.

This area of research has been enjoying a considerable amount of interest recently, with the main goal being both small scale robots, for example Kleiner [21] mentions a hexapod robot used to explore craters of active volcanos, and for legged vehicles for human passengers, see for example Waldron et al. [29]. Over rough terrain the advantages of legged locomotion over wheeled are clear, and a better understanding and modelling of how this is achieved in nature can only enhance the results that can be artificially achieved.

In particular we show how the addition of internal symmetries to current models can increase their realistic properties, an addition that appears to be present, although not realised, in Berkemeier [5], where the model presented appears to have $\mathbf{Z}_2 \wr \mathbf{D}_4$ symmetry, or similar. This symmetry is suggested by several patterns involving only differences of half period phase shifts being found for the same parameters.

By thinking of each single leg as a pendulum, it is easy to see how an internal $\mathbf{Z}_2$ symmetry can be justified in the modelling situation. We begin by outlining details on the actual stepping motions, or gaits, of insects that have been observed. These details have also been outlined in Wood [30], [31], where most of the information was based on Manton [23].

8.1 Insect Locomotion

The most important criterion is that the insect must be stable at all times during locomotion. There has been some locomotion observed that does not adhere to this, but then the insect's speed is fast enough to carry it through the unstable phases [23].

Manton suggested that to ensure stability at all phases during an insect's locomotion it must

M1 have at least three legs in contact with the ground disposed about the centre of gravity;

M2 paired legs must move in opposite phase (except when swimming or jumping);

M3 leg one must remain on the ground until the footfall of leg two, and similarly for legs two and three, where legs on each side are numbered from back to front.

In addition we have

M4 the most mechanically advantageous hexapod gaits giving least strain on the organisms are those in which the footfalls occur at equal intervals of time;

D1 the duration of the forward swing of the leg, off the ground, remains constant as walking speed changes.

We note here however that unlike most quadrupeds, such as horses, that change between specific gaits as they increase speed (such as canter to gallop), insects do not appear to make any sudden change of gaits. To move faster, they make smooth transitions between different gait patterns. Criterion **M4** however does suggest that there are some configurations that are more favoured than others.

Insect gaits can be best described by two variables, the *pattern of the gait*, the relative durations of the forward and back swings of each leg, and the *phase difference* between successive legs. For the purposes of the work presented here we concentrate only on the latter of these, the phase difference, since this is the most easily observable variable of the gait. We assume that each leg has a period of 2π to compare with our earlier work when considering models.

We label the legs of an insect as in Figure 8.1, numbering each side from back to front. This is backward to the normal convention, but we number in this way since insects tend to move legs in this manner.

Empirical observations on smooth surfaces (see for instance [23]) suggest that the phase differences along each side rarely fall outside the range $2\pi/5$ and π, where

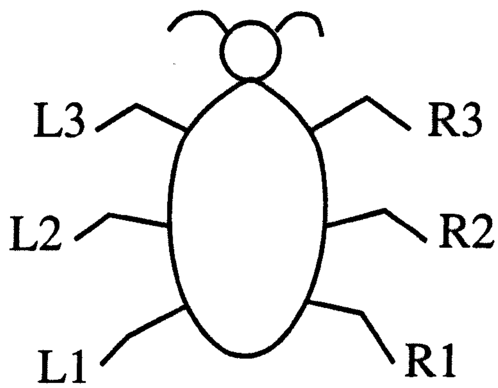


Figure 8.1: Labelling of an insect's legs.

phase differences of greater then π create unstable phases. Phase difference between the legs of a pair do generally satisfy **M2** being π out of phase with each other.

When off smooth surfaces (see for example [26]), or during activities such as swimming, the phase difference between legs of a pair has been seen to be less than π, for instance phase differences of $2\pi/3$ are very common (see [9]) as are legs moving in phase during swimming or jumping. For the later models however we begin by restricting ourselves to the case where legs in a pair move π out of phase with each other unless some external force is acting on the system.

The lower phase differences are seen at slower speeds where they appear as a wave from back to front along each side, R1R2R3 or L1L2L3. As the insect increases speed the phase differences increase towards a limit of π, when we have the so called *Tripod Gait* at the fastest (stable) speeds, where the legs move as

(L1R2L3)(R1L2R3)

where legs in each bracket are in phase, and the brackets are π out of phase.

To simplify the analysis further we consider only those gaits which satisfy **M4**, that is that footfalls must occur at equal intervals, which forces intervals of 2π, π, $2\pi/3$, $\pi/2$, $2\pi/5$ or $\pi/3$ between footfalls. This reduces the problem to considering the more 'favoured' gaits among the continuous change in phase difference case. We also assume each side of legs act identically to the other up to constant phase shift and that movement is achieved by a 'rear to front' wave. The reverse can then be found by 'reversing time'. We now consider each possible footfall interval in turn, and *list all* the possible gait patterns.

- A footfall interval of 2π means that all the legs are *moving in phase, and we* have the 'pronk'

  G1 (L1L2L3R1R2R3)

  where all legs inside the bracket move in phase. This gait does of course disobey both **M2** and **M3**.

- An interval of π gives us two possibilities,

  G2 (L1L2L3)(R1R2R3)

  G3 (L1R2L3)(R1L2R3)

  Where each bracket is π out of phase with the other, legs within the same bracket are in phase. Again **G2** does not obey **M3**, but **G3** gives us the Tripod Gait.

- Footfall intervals of $2\pi/3$ again gives us two possibilities

  G4 (L1R1)(L2R2)(L3R3)

  G5 (L1R2)(L2R3)(L3R1)

  where each bracket is now $2\pi/3$ out of phase with the next. **G4** then gives us each pair of legs in phase with each other, and **G5** gives us legs in a pair having a phase difference of $2\pi/3$.

- An interval of $\pi/2$ between footfalls gives the two possibilities

  G6 (L1)(L2R1)(L3R2)(R3)

  G7 (L1R3)(L2)(L3R1)(R2)

  where each bracket is $\pi/2$ out of phase with the next. Now we have a phase difference of $\pi/2$ between legs of a pair with gait **G6** and a π phase difference in gait **G7**.

- For a footfall interval of $2\pi/5$ we have only the one gait

  G8 (L1)(L2)(L3R1)(R2)(R3)

where each bracket is $2\pi/5$ out of phase with the next, and legs in a pair are $4\pi/5$ out of phase with each other.

- Finally we have the case of a footfall interval of $\pi/3$, which gives the two gaits

  G9 (L1)(L2)(L3)(R1)(R2)(R3)

  G10 (L1)(R1)(L2)(R2)(L3)(R3)

  with each bracket being $\pi/3$ out of phase with the next. Gait **G9** gives us a phase difference of π between legs of a pair, and gait **G10** a phase difference of $\pi/3$ between legs of a pair.

Of these Collins and Stewart [9] cite **G3**, **G5**, **G7** and **G9** as being commonly observed, **G9**, **G7** and **G3** in cockroaches moving at slow, medium and fast speeds respectively and **G5** in stick insect locomotion.

We note here that of these gaits, **G1** to **G10**, the only ones satisfying **M2** are **G2**, **G3**, **G7** and **G9** (π phase differences between legs of a pair), **G1** and **G4** (legs of a pair in phase).

Since this list is of the most mechanically advantageous, we assume that any other patterns observed in nature are due to sensory feedback, and that given ideal situations the gait patterns will fall back into one of these 10 gaits.

8.2 The Central Pattern Generator and Coupled Oscillator Models

The widely accepted mechanism for driving limb coordination during locomotion in animals is the *Central Pattern Generator* (CPG), Cohen et al. [7] giving indirect experimental evidence of such a system in a Lamprey.

The CPG is generally thought of as being a network of coupled neuronal oscillators, in particular Cohen et al. [7] suggest a model based on two coupled chains of oscillators, each chain controlling one side of the animal. A hypothetical relaxation oscillator network put forward by Gewecke (see Wood [30] for a description) also suggests that we could assume that each of these neuronal oscillators possesses a primitive sort of $\mathbf{Z}_2$ symmetry, by swapping individual cells of the oscillator.

We now propose to model the CPG of insects by symmetrical networks of coupled non-linear oscillators. This is a method used with considerable success in studies of quadruped gaits by Collins and Stewart [8], and also applied by these authors to hexapod gaits [9], as well as other similar work by Wood [30] and [31], again with a fair degree of success. Here though we use the work presented earlier, specifically adding internal $\mathbf{Z}_2$ symmetries to the individual oscillators, to try and provide and even better model.

We do however base our model on one of those considered before. Since Cohen et al. suggest that a CPG is based on two coupled rings of oscillators, we consider a model of two coupled rings of three oscillators. We present this model schematically in figure 8.2.

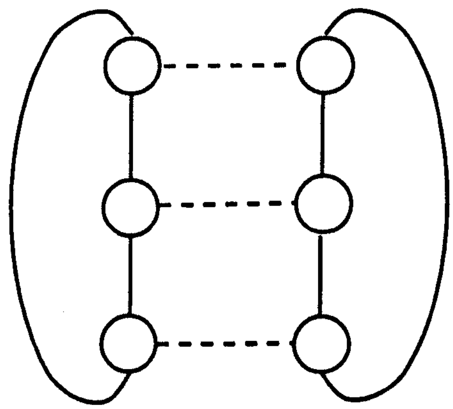


Figure 8.2: A suggested model for an insect's CPG.

In addition we assume that each oscillator has an internal $\mathbf{Z}_2$ symmetry, an obvious extension to the problem if you think of each leg, which each oscillator is controlling, as being, in its simplest form, a pendulum.

By noting that if each ring of three oscillators is itself thought of as an oscillator, giving a system of two coupled oscillators, we can then easily achieve an in-phase or a π out of phase solution, giving a system that obeys criterion **M2** (see for example Golubitsky et al. [17]). Therefore we consider each ring separately for now, knowing that we can then couple the two rings so as to produce a π phase difference between legs of a pair. We show later that given the correct coupling we should also be able to achieve legs of a pair being $\pi/3$ or $2\pi/3$ out of phase with each other.

This reduces the problem to considering a ring of three coupled oscillators where each oscillator has an internal $\mathbf{Z}_2$ symmetry, and so we can apply the earlier results.

Assume that the $\mathbf{Z}_2$ action acts on a planar oscillator as a rotation.

Without the internal symmetries present we are left with the case of $\mathbf{S}_3$ symmetry acting by permutation of oscillators, which will give possible phase differences between successive oscillators of 0, π and $2\pi/3$ which gives in turn possible gait patterns of **G1**, **G2**, **G3** and **G4**. If in addition we allow the coupling between the two *rings* to be a wreath product coupling with respect to the $\mathbf{S}_3$ symmetry of each ring then we can apply a permutation $\rho \in \mathbf{S}_3$ to one ring only and so achieve gait **G5** as a conjugate solution to **G4**.

If we now add an internal $\mathbf{Z}_2$ symmetry to each oscillator, and we let the coupling between the oscillators of a ring produce a global $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry then as well as the gaits possible with purely $\mathbf{S}_3$ symmetry we can also obtain a phase difference of $\pi/3$ between two oscillators of a ring as the conjugate solution to a phase difference of $2\pi/3$ between all of the oscillators. With the other ring of oscillators π out of phase this gives rise to gaits **G9**, and if, as before, we let the coupling between rings be wreath product coupling with respect to the $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry of each ring then we can also achieve gait **G10**.

This leaves the only gaits that we cannot realise by a model with these symmetries as **G6**, **G7** and **G8**. If, however, we let each individual oscillator have internal $\mathbf{S}_4$ symmetry, then from the earlier work it should be clear that we could realise both **G6** and **G7** as conjugate solutions to **G1** the 'all in phase' solution. We note though that of these only **G7** has been observed as a common gait.

Remark 8.2.1 (Quadruped Gaits) *The addition of internal $\mathbf{Z}_2$ symmetries with wreath product coupling within each ring also has the interesting property that if we take the isotropy subgroup $\mathbf{W}_2$ then we have a quiescent oscillator, and the remaining two oscillators act as two coupled oscillators either in phase or π out of phase. In this way we can recover the results of Collins and Stewart [8] on quadruped gaits, the gaits of which have been observed in insects with amputated legs (see for example Pearson et al. [26]).*

Remark 8.2.2 *This type of model also allows for solutions where the legs move in a wave from back to front on one side and a wave from front to back on the other (if the two rings are wreath product coupled). In this way a robot using an electronic CPG based on this model could turn on the spot in a similar fashion to caterpillar track vehicles.*

8.3 Methods of Gait Transition

In Collins and Stewart [8] the authors suggest that gait transitions could occur by a succession of bifurcations from one stable state to another. In the case of hexapod gaits however there is no evidence that insects allow their gaits to suddenly jump from one to the other, indeed the evidence points towards a smooth transition through the possible gaits.

Therefore we suggest here a possible alternative method for gait transitions which is particularly suitable for the model we have outlined previously. The mechanism could work something like this. We know that, generically, for systems of coupled oscillators with $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry, we can have stable solutions with isotropy $\mathbf{S}_3$ and $\tilde{\mathbf{Z}}_3$ for the same parameters, *and so therefore* we can have various stable limit cycles existing in phase space for the same parameters. It *may now be possible to achieve* gait transitions by moving our periodic cycle from the basin of attraction corresponding to one isotropy to the basin of attraction corresponding to another, while continuously changing the phase differences between successive legs, rather than by 'jumps' in phase difference which happens when we use gait change by bifurcations.

This method then produces a more realistic transition to the case of transitions by bifurcation. This mechanism then also allows for the fact that insects appear to favour certain gaits, as proposed by **M4**.

8.4 Comments

In this Chapter we have attempted to show how the addition of an internal $\mathbf{Z}_2$ symmetry into the oscillators of previously considered models for insect locomotion can considerably enhance the resulting gaits possible.

With a larger internal symmetry group it should be clear that an even greater number of gaits should be possible, but the case of only $\mathbf{Z}_2$ would appear to be sufficient, for the model considered, to cover the majority of gaits observed in nature.

It is also clear that the comments here leave many questions still to be answered, such as how the gaits during transitions would look, and indeed how such a transition could be efficiently modelled.

Chapter 9

An Application Of Skew-Equivariance

In this Chapter we give an application of skew equivariance to the system considered by Dangelmayer et al. [11] to show the possible use of this idea outside of that discussed in Chapter 6.

9.1 Equations with S_3 global symmetry

We apply the notion of Γ-skew-equivariance to the following set of equations:

$$\begin{aligned} \dot{x_1} &= f(x_1) + g(x_1, x_2, x_3) \\ \dot{x_2} &= f(x_2) + g(x_2, x_1, x_3) \\ \dot{x_3} &= f(x_3) + g(x_3, x_1, x_2) \end{aligned} \tag{9.1.1}$$

where each x_i is in $\mathbf{R}^k$, each $f(x_i)$ is Γ-equivariant and each $g(x_i, x_j, x_k)$ is both Γ skew-equivariant and invariant under permutation of it's last two variables, i.e.

$$g(x, y, z) = g(x, z, y).$$

Therefore the entire system lives on $\mathbf{R}^{3k}$ and is at least S_3-equivariant where the action is given by permutation of indices.These equations, for example, could represent three identical oscillators, each with internal Γ symmetry identically coupled so as to produce a global S_3 symmetry.

To make the working more manageable we consider the coupling term as $g(x, y, z)$ and as before assume the skew-equivariance manifests itself as

$$g(\gamma x, y, z) = \hat{\gamma} g(x, y, z) \tag{9.1.2}$$

and

$$g(x, \gamma y, z) = g(x, y, \gamma z) = \tilde{\gamma} g(x, y, z) \tag{9.1.3}$$

for each $\gamma \in \mathbf{\Gamma}$ where both $\hat{\gamma}$ and $\tilde{\gamma}$ are also in $\mathbf{\Gamma}$.

One of the first things we notice is that we must have

$$\widehat{id} = \widetilde{id} = id, \tag{9.1.4}$$

otherwise the equations will be inconsistent.

It is easy to obtain systems of the form 9.1.1 with $\mathbf{\Gamma} \wr \mathbf{S}_3$ symmetry by simply setting $\hat{\gamma} = \gamma$ and $\tilde{\gamma} = id$ for all $\gamma \in \mathbf{\Gamma}$. We have also shown that if $\mathbf{\Gamma}$ is cyclic, i.e. isomorphic to $\mathbf{Z}_k$, then it is also straight-forward to obtain $\mathbf{\Gamma} \times \mathbf{S}_3$ symmetry.

We also have found various relations between the elements of $\mathbf{\Gamma}$ and their corresponding skew-equivariant partners which are necessary to produce $\mathbf{\Gamma} \times \mathbf{S}_3$ symmetry in the whole system. Here we consider the more general case of what global symmetries are possible in the presence of $\mathbf{\Gamma}$-skew equivariant coupling. The forms that are possible do still however produce some necessary relations to avoid inconsistencies. To understand these restrictions we must consider the equations 9.1.1, 9.1.2 and 9.1.3 after we have applied the element $\alpha \in \mathbf{\Gamma}$ to x_1 and the element $\beta \in \mathbf{\Gamma}$ to x_2. Consistency in all variables will then follow by symmetry.

$$\begin{aligned} \alpha \dot{x_1} &= \alpha f(x_1) + \hat{\alpha}\tilde{\beta} g(x_1, x_2, x_3) \\ \beta \dot{x_2} &= \beta f(x_2) + \hat{\beta}\tilde{\alpha} g(x_2, x_1, x_3) \\ \dot{x_3} &= f(x_3) + \tilde{\alpha}\tilde{\beta} g(x_3, x_1, x_2) \end{aligned} \tag{9.1.5}$$

As before define the order of an element $\gamma \in \mathbf{\Gamma}$, $o(\gamma)$, to be the smallest number l such that $\gamma^l = id$. Then an immediate consequence of 9.1.2 is that

$$o(\hat{\gamma}) | o(\gamma) \tag{9.1.6}$$

and a direct consequence of 9.1.3 is that

$$o(\tilde{\gamma}) | o(\gamma). \tag{9.1.7}$$

Otherwise $\gamma^l = id$ but $\hat{\gamma}^l \neq id$ and $\tilde{\gamma}^l \neq id$, contradicting 9.1.4. Now turning our attention to 9.1.5 notice that it should not matter in which order we apply the elements α and β to different variables. This leads us to conclude that

$$\hat{\alpha}\tilde{\beta} = \tilde{\beta}\hat{\alpha} \tag{9.1.8}$$

i.e. $\hat{\alpha}$ and $\tilde{\beta}$ must commute. Similarly

$$\tilde{\alpha}\tilde{\beta} = \tilde{\beta}\tilde{\alpha}. \tag{9.1.9}$$

We are now ready to apply these ideas to a specific scenario, based on the papers [10] and [11].

9.2 Systems With Internal $\mathbf{D}_3$ Symmetry

We begin by considering a system of form 9.1.1, where the internal symmetry $\mathbf{\Gamma}$ is $\mathbf{D}_3$. For the purposes of the application later however we will think of this symmetry as $\mathbf{S}_3$, since it becomes the natural way of thinking of $\mathbf{D}_3$ in that scenario.

For now though we are not concerned with how the elements of $\mathbf{D}_3$ act on each variable, but more how the elements of $\mathbf{D}_3$ affect the coupling terms $g(x_i, x_j, x_k)$, and so we keep it as general as possible.

So, let $\mathbf{D}_3$ acting on $\mathbf{R}^k$ be generated by the elements ρ and κ where

$$\rho^3 = \kappa^2 = id$$

and

$$\kappa\rho = \rho^2\kappa$$

i.e. when thinking of $\mathbf{D}_3$ acting as a linear map on $\mathbf{R}^2$, ρ is a rotation through $2\pi/3$ and κ is a reflection, and when thinking of $\mathbf{D}_3$ acting as $\mathbf{S}_3$ on $\{0,1,2\}$ then ρ acts as (012) and κ as a transposition, for example (12).

We must now consider which elements of $\mathbf{D}_3$ the elements $\widehat{\rho}, \widetilde{\rho}, \widehat{\kappa}$ and $\widetilde{\kappa}$ must correspond to.

The first thing to notice is that because of 9.1.6 and 9.1.7 we must have

$$\widehat{\rho}, \widetilde{\rho} \in \{id, \rho, \rho^2\}$$

and

$$\widehat{\kappa}, \widetilde{\kappa} \in \{id, \kappa\}.$$

Also notice that if $\widetilde{\rho} \neq id$ and $\widehat{\kappa} = id$ then $o(\widetilde{\rho\kappa}) = o(\widetilde{\rho}\widetilde{\kappa}) = 3$ but $o(\rho\kappa) = 2$ contradicting 9.1.7. Similarly if $\widehat{\rho} \neq id$ and $\widehat{\kappa} = id$ then $o(\widehat{\rho\kappa}) = 3$ but $o(\rho\kappa) = 2$ contradicting 9.1.6. If $\widetilde{\kappa} = \kappa$ then we must have $\widetilde{\rho} = id$ so that $\widetilde{\kappa}$ and $\widetilde{\rho}$ commute, to satisfy 9.1.9, and we must also have $\widehat{\rho} = id$ to satisfy 9.1.8. Finally if $\widehat{\kappa} = \kappa$ then $\widehat{\rho} = id$, again to satisfy 9.1.8.

We have now proved

Lemma 9.2.1

$$\begin{aligned}(\widehat{\rho}, \widetilde{\rho}, \widehat{\kappa}, \widetilde{\kappa}) \in \{&(id,id,id,id),(id,id,id,\kappa),(id,id,\kappa,id),(id,id,\kappa,\kappa),\\ &(\rho,id,\kappa,id),(\rho^2,id,\kappa,id)\}\end{aligned}$$

∎

Note that with skew-symmetry we cannot obtain full $\mathbf{D}_3 \times \mathbf{D}_3$ symmetry. When we now consider the effect of these skew-symmetries on our equations 9.1.1 we find the following

Theorem 9.2.2 *The combinations of elements found in Lemma 9.2.1 correspond to the following global symmetries when applied to a system 9.1.1*

Ref.	$\widehat{\rho}$	$\widetilde{\rho}$	$\widehat{\kappa}$	$\widetilde{\kappa}$	Symmetry
1	id	id	id	id	$\mathbf{S}_3$
2	id	id	id	κ	$(\kappa,\kappa,0) \wr \mathbf{S}_3$
3	id	id	κ	id	$\mathbf{Z}_2 \wr \mathbf{S}_3$
4	id	id	κ	κ	$\mathbf{Z}_2 \times \mathbf{S}_3$
5	ρ	id	κ	id	$\mathbf{D}_3 \wr \mathbf{S}_3$
6	ρ^2	id	κ	id	$\mathbf{Z}_2 \wr \mathbf{S}_3$

Proof: Combinations **1** − **4** all have the pair $(\widehat{\rho}, \widetilde{\rho}) = (id, id)$ so that the (rotation) ρ does not enter into the global symmetry; any application of ρ will not alter the equations 9.1.1. Thus we are left with only (the reflection) κ playing a role and we recover the results of earlier, Chapter 6.

Combination **5** means that applying either, or both, ρ or κ leaves the equation 9.1.1 unchanged, and so we are left with $\mathbf{D}_3 \wr \mathbf{S}_3$ symmetry.

Combination **6** yields the same result as **3** since allowing $\widehat{\rho} = \rho^2$ does not introduce any more symmetry, since no number of applications of ρ can leave the equations 9.1.1 unchanged. ∎

Equations with $\mathbf{S}_3$-symmetry have been extensively studied, and the results are well known (see for example [17]), and so will not produce any particularly new results. Systems with $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry however we have only been applied to the simplest situation of three coupled oscillators where each oscillator has an internal $\mathbf{Z}_2$-symmetry. Here we consider another application.

9.3 A Hierarchical Network Of Nine Oscillators - Theoretical Results

We will apply the results that we have found for systems with skew-equivariant coupling terms to the system of oscillators considered in [10] and [11]. This system consists of three clusters of oscillators, where each cluster itself contains three oscillators. Let the state of the j^{th} oscillator of the i^{th} cluster lie in $\mathbf{R}^k$, and be denoted x_{ij}, then the entire system lives on $\mathbf{R}^{9k}$.

The oscillators are then coupled so that the three oscillators in *each cluster* are arranged so that they have $\mathbf{D}_3$ symmetry, and the *clusters* are arranged so that they too have $\mathbf{D}_3$ symmetry. We represent this schematically in figure 9.1.

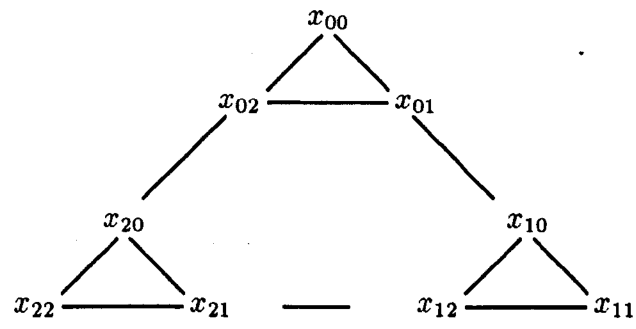


Figure 9.1: Schematic Representation of 3 Clusters of 3 Oscillators

In Dangelmayer et al. [10] the coupling chosen leads to $\mathbf{D}_3 \times \mathbf{D}_3$ symmetry. Here we arrange the coupling within each cluster to be skew-equivariant, and so produce $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetries, and we then compare the results to the case of $\mathbf{D}_3 \times \mathbf{D}_3$. To do this we consider each cluster to be modelled by the equation

$$\dot{x}_i = f(x_i) \tag{9.3.10}$$

where $i \in \{0, 1, 2\}$ and $x_i \in \mathbf{R}^{3k}$, and then the whole system to be modelled by

$$\begin{aligned} \dot{x_0} &= f(x_0) + g(x_0, x_1, x_2) \\ \dot{x_1} &= f(x_1) + g(x_1, x_2, x_0) \\ \dot{x_2} &= f(x_2) + g(x_2, x_1, x_0) \end{aligned} \tag{9.3.11}$$

where $x_i \in \mathbf{R}^{3k}$ and $g(x, y, z)$ is $\mathbf{D}_3$ skew-equivariant, and the whole system is $\mathbf{S}_3$-equivariant, acting by permutation of indices.

Let $\mathbf{S}_3$ be generated by the elements $\rho = (012)$ and $\kappa = (12)$, and the equivalent elements in the internal $\mathbf{D}_3$ symmetries be equal to a rotation through $2\pi/3$ (ρ) and a reflection (κ). When we interpret the results with the actual system, we then take the internal $\mathbf{D}_3$ symmetry as acting like the global $\mathbf{S}_3$ symmetry. We present a summary of the results of Dangelmayer et al. [10] at the end of this Chapter.

9.4 Hopf Bifurcations with $\mathbf{Z}_2 \times \mathbf{S}_3$ and $\mathbf{Z}_2 \wr \mathbf{S}_3$ Symmetries

Here we briefly recap the theoretical results we have found earlier.

Group orbit rep.	Isotropy Subgroup (Σ)	$dimFix(\Sigma)$
$(0,0,0)$	$\mathbf{Z}_2 \times \mathbf{S}_3$	0
$(z, \eta z, \eta^2 z)$	$\widetilde{\mathbf{Z}_3}$	2
$(z, -z, 0)$	$\widetilde{\mathbf{Z}_2}$	2
$(2z, -z, -z)$	$\mathbf{S}_1 \times \mathbf{S}_2$	2

Table 9.1: List of isotropy subgroups of $\mathbf{Z}_2 \times \mathbf{S}_3$ having 2-dimensional fixed point subspaces - where $\eta = e^{2\pi/3}$*(up to conjugacy).*

9.4.1 $\mathbf{Z}_2 \times \mathbf{S}_3$-Symmetry

Consider the representation of $\mathbf{Z}_2 \times \mathbf{S}_3 \times \mathbf{S}^1$ acting on

$$\mathbf{C}_0^3 = \{(z_1, z_2, z_3) \in \mathbf{C}^3 : z_1 + z_2 + z_3 = 0\}$$

given by ρ acting as the permutation (012) on the indices, κ acting as

$$\kappa(z_1, z_2, z_3) = (-z_1, -z_2, -z_3)$$

and $\theta \in \mathbf{S}^1$ acting as multiplication by $e^{i\theta}$. This representation is $\mathbf{\Gamma}$-simple, and so by the Equivariant Branching Lemma (2.2.2) there exist solutions with isotropy Σ which satisfy $dimFix(\Sigma) = 2$. However, since the action of $\mathbf{Z}_2$ is exactly the same as the action of $\pi \in \mathbf{S}^1$, in this $\mathbf{\Gamma}$-simple representation, the isotropy subgroups of $\mathbf{Z}_2 \times \mathbf{S}_3$ are precisely those of $\mathbf{S}_3$.

Those isotropy subgroups with two-dimensional fixed point subspaces, and their corresponding group orbit representative, are listed, up to conjugacy, in Table 9.1.

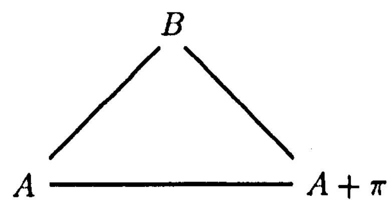
9.4.2 $\mathbf{Z}_2 \wr \mathbf{S}_3$-Symmetry

The $\mathbf{\Gamma}$-Simple representation of $\mathbf{Z}_2 \wr \mathbf{S}_3$ that we use is the action of $\mathbf{Z}_2 \wr \mathbf{S}_3$ on $\mathbf{C}^3$. Where $\mathbf{Z}_2$ acts on each component of $\mathbf{C}^3$ as multiplication by -1, and $\mathbf{S}_3$ acts by permutation of indices.

This time the list of isotropy subgroups is different to that of the purely $\mathbf{S}_3$-symmetric case, and those with two dimensional fixed point subspaces are listed, up to conjugacy, in table 9.2.

Group orbit rep.	Isotropy Subgroup (Σ)	$dimFix(\Sigma)$
$(0,0,0)$	$\mathbf{Z}_2 \wr \mathbf{S}_3$	0
(z,z,z)	$\mathbf{S}_3$	2
$(z,z,0)$	$\mathbf{W}_2$	2
$(z,0,0)$	$\mathbf{W}_1$	2
$(z,\eta z,\eta^2 z)$	$\widetilde{\mathbf{Z}_3}$	2

Table 9.2: List of isotropy subgroups of $\mathbf{Z}_2 \wr \mathbf{S}_3$ having 2-dimensional fixed point subspaces - where $\eta = e^{2\pi/3}$ *(up to conjugacy).*



where $B = B + \pi$.

Figure 9.2: Form imposed on each cluster by the $\widetilde{\mathbf{Z}_2}$ symmetry.

9.5 Networks Of Oscillators - Predicted Patterns

We begin by recapping notation to make the results easier to visualise later. We denote a waveform, representing a periodic solution to 9.3.10, by a capital letter, e.g. A. Unless otherwise stated we assume that each waveform has period 2π. We also denote phase shifts by a θ added to the letter. For example $A+\pi$ is the same as the waveform A phase-shifted by π.

We again note that both $[(\kappa,\pi)] \in \mathbf{Z}_2 \times \mathbf{S}_3$ and $[(\kappa,\kappa,\kappa),\pi] \in \mathbf{Z}_2 \wr \mathbf{S}_3$ act as the identity in their respective irreducible representations and so they are contained in *all* the isotropy subgroups, and so the symmetry is inherited by all the patterns.

In the case of three clusters of three oscillators, these symmetries correspond to each cluster having a $\widetilde{\mathbf{Z}_2}$ symmetry. That is, two of the oscillators of each cluster have identical waveform, but are π out of phase, while the third oscillator has half the period. Schematically this means that each cluster has the form shown in Figure 9.2. Remember that by convention we are taking $\mathbf{D}_3 = \mathbf{S}_3$ to be generated by the elements

$\kappa = (12)$ (reflection) and $\rho = (012)$ (rotation), and our diagrams are arranged so that '0' is at the top and elements are numbered clockwise modulo 3.

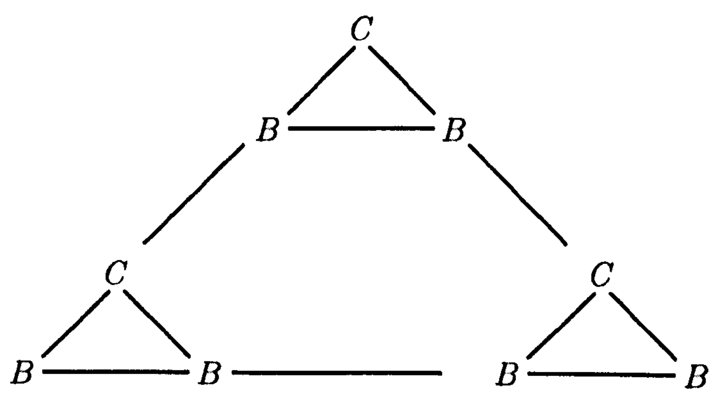
We now consider each global symmetry separately, and interpret the restrictions that each isotropy subgroup places on the possible patterns. The patterns are found by using the fact that an application of any element of the isotropy subgroup must leave the patterns unchanged, and also that each cluster must be left unchanged by an application of $[\kappa, \pi]$.

9.5.1 Predicted Patterns - $\mathbf{Z}_2 \times \mathbf{S}_3$

Although having the same isotropy subgroups as $\mathbf{S}_3$, the additional structure produces more complicated results, mainly through the internal $\widetilde{\mathbf{Z}_2}$ of each cluster.

- Isotropy $\mathbf{Z}_2 \times \mathbf{S}_3$

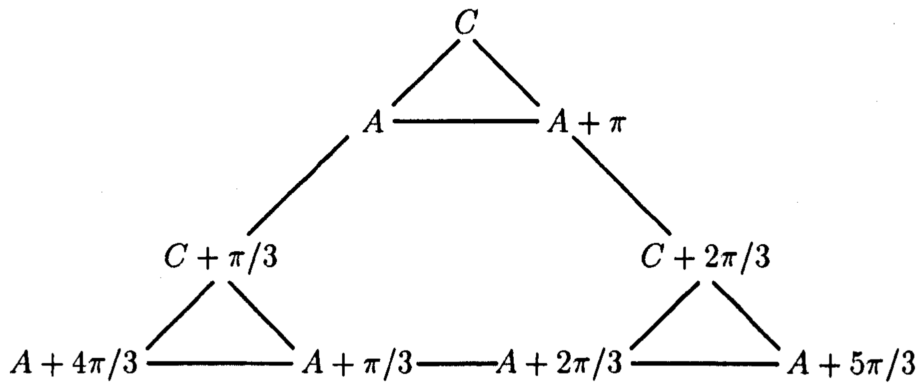
  Applying the $\mathbf{Z}_2$ to each cluster has to leave the oscillator pattern invariant, forcing all the oscillators in each cluster to oscillate with a half-period, since in the notation of figure 9.2 we must have $A = A + \pi$.



  where $B = B + \pi$ and $C = C + \pi$.

- Isotropy $\widetilde{\mathbf{Z}_3}$

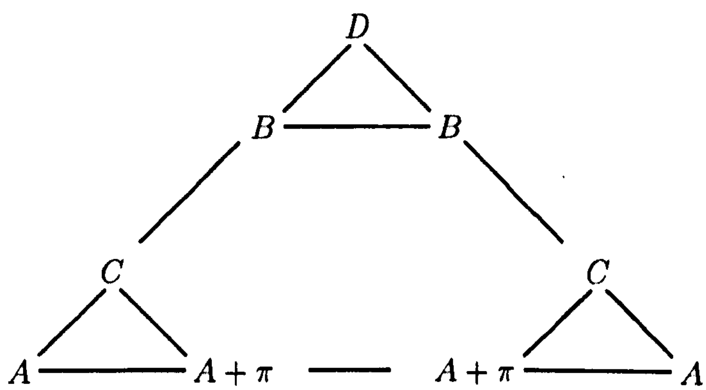
  In this case the symmetry simply forces each cluster to be $2\pi/3$ out of phase with the next, which along with the $\widetilde{\mathbf{Z}_2}$ within each cluster gives us



where $C = C + \pi$.

- Isotropy $\widetilde{\mathbf{Z}_2}$

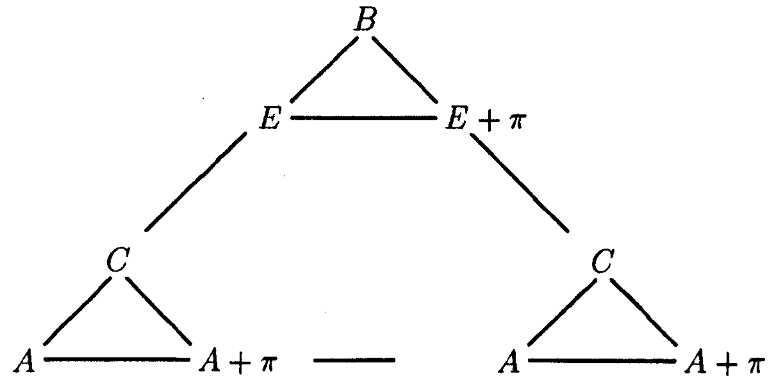
  Now the presence of a 'global' $\widetilde{\mathbf{Z}_2}$ forces all the oscillators of the 'top' cluster to oscillate with period π, and the other two clusters to be π out of phase with each other.



where $B = B + \pi$, $C = C + \pi$ and $D = D + \pi$.

- Isotropy $\mathbf{S}_1 \times \mathbf{S}_2$

  And, finally, this isotropy simply forces two of the clusters to be identical, the third being entirely different, except for the internal $\widetilde{\mathbf{Z}_2}$ symmetry.



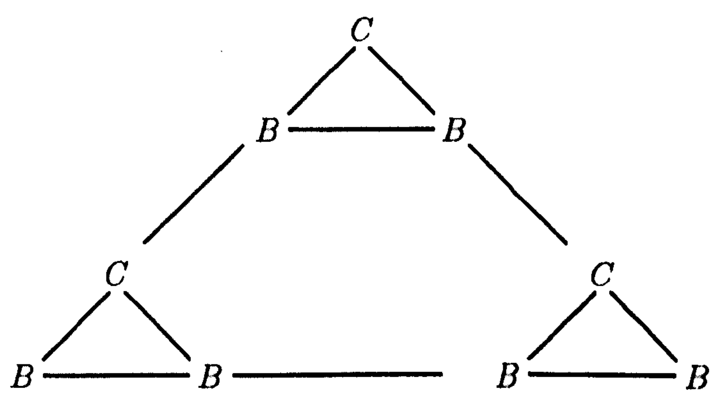
where $B = B + \pi$ and $C = C + \pi$.

9.5.2 Predicted Patterns - $\mathbf{Z}_2 \wr \mathbf{S}_3$

As has already been noted in, the wreath product case produces additional solutions through both a new isotropy ($\mathbf{W}_1$) and new conjugacy classes. In addition there may be solutions corresponding to sub-maximal isotropies $\mathbf{S}_1 \times \mathbf{S}_2$ and $[(\kappa, \kappa), \pi]$ which for clarity will not be considered here. For all the following patterns, conjugates can be found by applying a π-phase shift to *any* of the clusters.

- Isotropy $\mathbf{Z}_2 \wr \mathbf{S}_3$

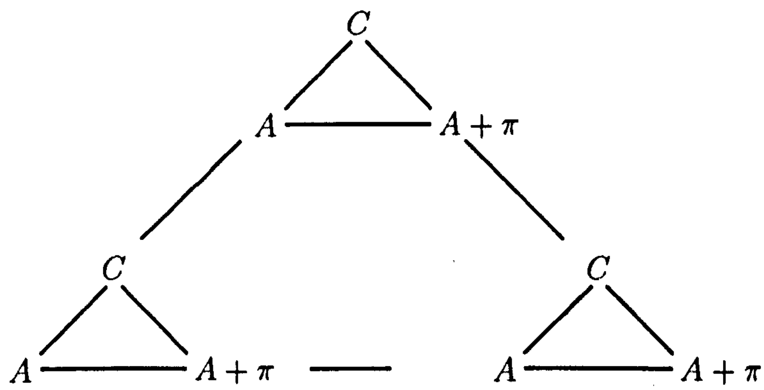
  This isotropy forces all the oscillators to have period π, and all the conjugates are obviously identical.



where $B = B + \pi$ and $C = C + \pi$.

- Isotropy $\mathbf{S}_3$

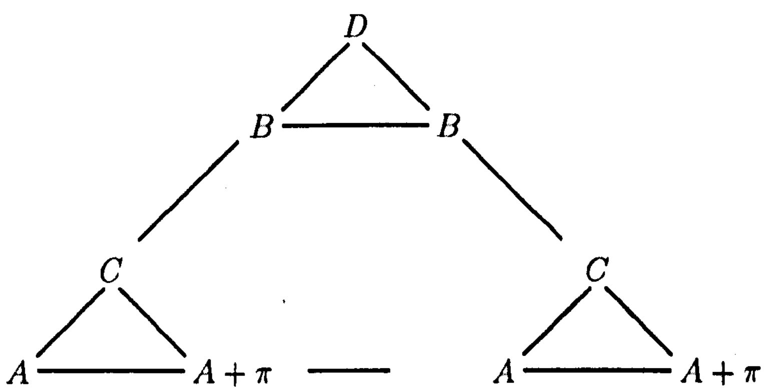
  Each cluster must be identical, and conjugates are found by swapping any $(A, A + \pi)$ pair in any cluster (i.e. a π phase shift to any cluster).



where $C = C + \pi$.

- Isotropy $\mathbf{W}_2$ $(\widetilde{\mathbf{Z}_2})$

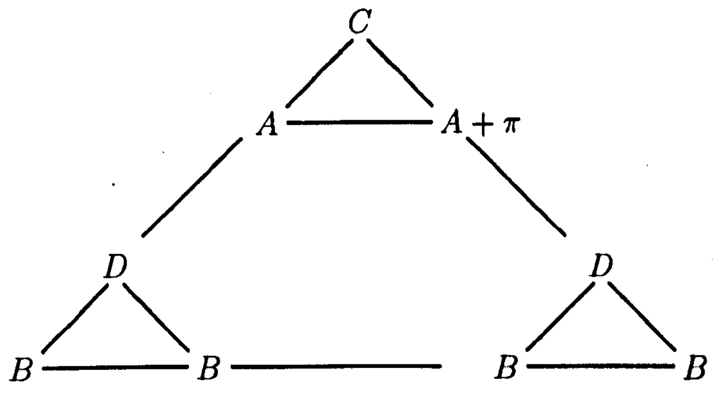
  The isotropy now forces all the oscillators of one cluster to oscillate with period π; conjugate solutions are found by applying a π phase shift to either of the other two clusters. Note how this differs from the standard case of three coupled oscillators when one of the oscillators becomes quiescent.



where $B = B + \pi$, $C = C + \pi$ and $D = D + \pi$.

- Isotropy $\mathbf{W}_1$

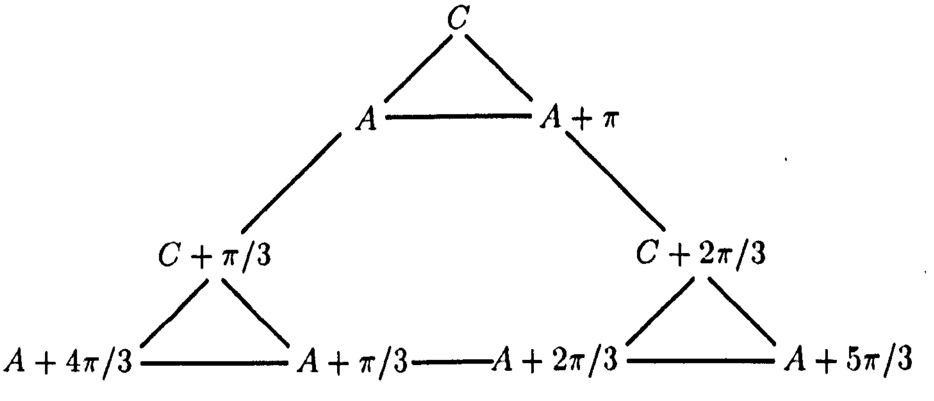
  Now the isotropy subgroup forces two of the clusters to contain only π periodic oscillators, and the third cluster has it's form forced by the internal symmetry inherent in all the clusters.



where $B = B + \pi$, $C = C + \pi$ and $D = D + \pi$.

- Isotropy $\widetilde{\mathbf{Z}_3}$

  And finally, isotropy $\widetilde{\mathbf{Z}_3}$ causes each cluster to be $2\pi/3$ out of phase with the others. Note that because one of the oscillators in each cluster has period π then you get a phase shift of $\pi/3$ between some oscillators.



where $C = C + \pi$.

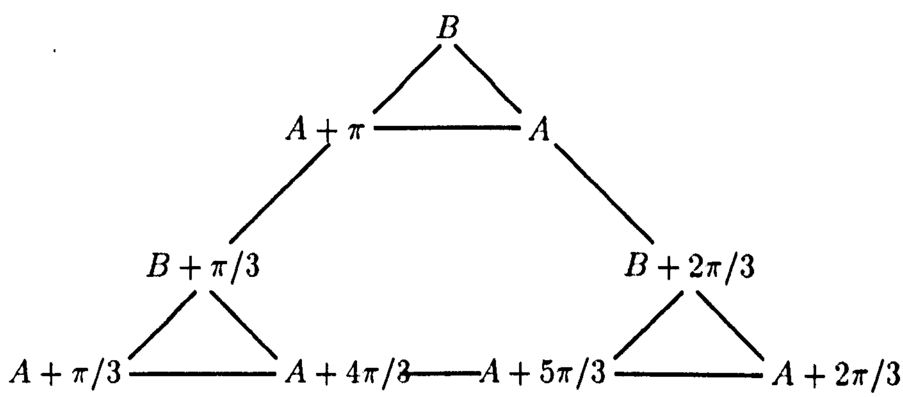
A Note On Stabilities

In the earlier work it has been shown that in fact the solutions corresponding to isotropy $\mathbf{W}_2$ are generically unstable, and so we would not expect such a solution to exist stably at the point of bifurcation.

9.5.3 Comments And Comparisons with $\mathbf{D}_3 \times \mathbf{D}_3$

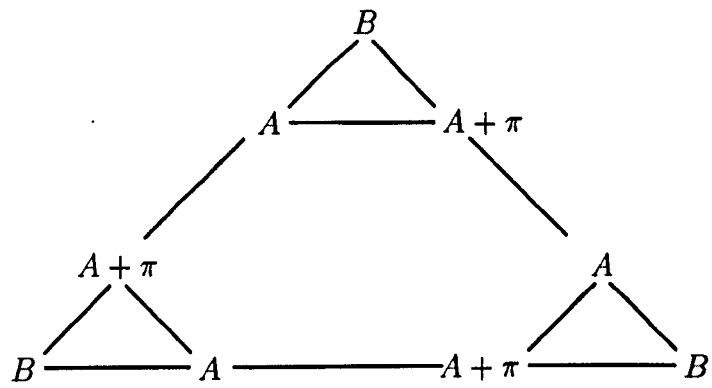
Since both $\mathbf{Z}_2 \times \mathbf{S}_3$ and $\mathbf{Z}_2 \wr \mathbf{S}_3$ are subgroups of $\mathbf{D}_3 \times \mathbf{D}_3$ it would seem plausible that all the possible patterns that can be observed in the latter case (see end of Chapter) would be reproduced by the new cases. This does not appear to be the case however. The first thing to notice is that since the element $[\kappa, \pi]$ must act on each cluster, there are only four of the patterns obtained with $\mathbf{D}_3 \times \mathbf{D}_3$ symmetry that could possibly be seen with $\mathbf{Z}_2 \times \mathbf{S}_3$ and $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetries. These are:

- Isotropy $\widetilde{\mathbf{Z}_2}^m \times \widetilde{\mathbf{Z}_3}^M$



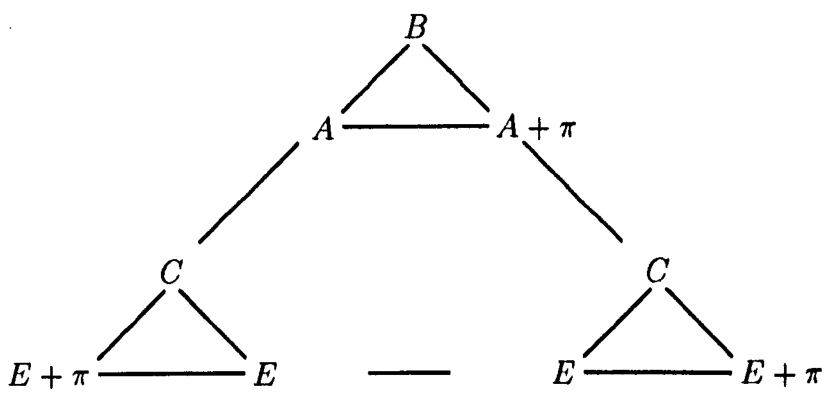
where $B = B + \pi$.

- Isotropy $\widetilde{\mathbf{D}_3}^{mM}$

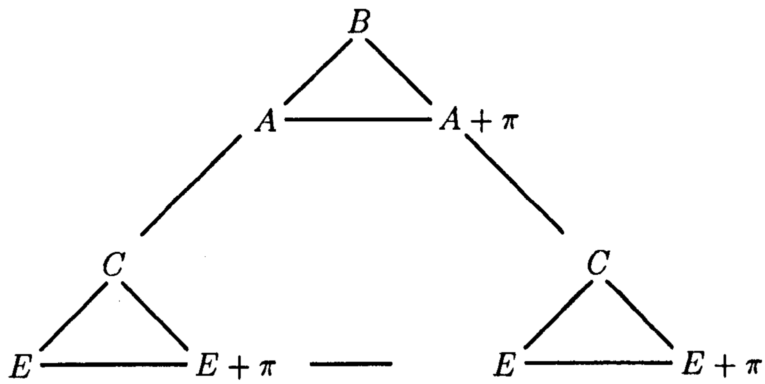


where $B = B + \pi$.

- Isotropy $\widetilde{\mathbf{Z}_2}^m \times \widetilde{\mathbf{Z}_2}^M$



- Isotropy $\widetilde{\mathbf{Z}_2}^m \times \mathbf{Z}_2{}^M$

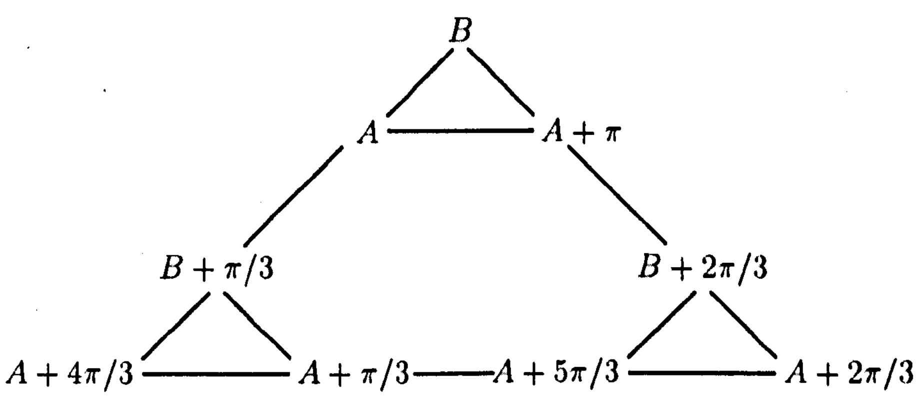


where $B = B + \pi$ and $C = C + \pi$.

We shall now consider each of these patterns in turn to compare with the $\mathbf{Z}_2 \times \mathbf{S}_3$ and $\mathbf{Z}_2 \wr \mathbf{S}_3$ cases.

Isotropy $\widetilde{\mathbf{Z}_2}^m \times \widetilde{\mathbf{Z}_3}^M$

This gives the same pattern as seen in isotropy $\widetilde{\mathbf{Z}_3}$ in both the $\mathbf{Z}_2 \times \mathbf{S}_3$ and $\mathbf{Z}_2 \wr \mathbf{S}_3$ cases. However, the $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry provides an extra conjugate solution not seen in the $\mathbf{D}_3 \times \mathbf{D}_3$ case, namely



Isotropy $\widetilde{\mathbf{D}_3}^{mM}$

The pattern seen with isotropy $\widetilde{\mathbf{D}_3}^{mM} \subset \mathbf{D}_3 \times \mathbf{D}_3$ is seen in neither the $\mathbf{Z}_2 \wr \mathbf{S}_3$ nor $\mathbf{Z}_2 \times \mathbf{S}_3$ cases due to the $\mathbf{Z}_3$ part of the isotropy 'twisting' the clusters with respect to each other. However, a *similar* pattern is seen in the $\mathbf{Z}_2 \wr \mathbf{S}_3$ case by relabelling oscillators within clusters, in effect 'removing the twist'. This solution corresponds to $\mathbf{S}_3 \subset \mathbf{Z}_2 \wr \mathbf{S}_3$.

Isotropy $\widetilde{\mathbf{Z}_2}^m \times \widetilde{\mathbf{Z}_2}^M$

As in the previous case the pattern seen here is no longer possible with either $\mathbf{Z}_2 \wr \mathbf{S}_3$ or $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry. The only similar pattern is when we consider the sub-maximal $\mathbf{S}_1 \times \mathbf{S}_2 \subset \mathbf{Z}_2 \times \mathbf{S}_3$ which gives the same pattern except for a π phase shift applied to one of the clusters.

Isotropy $\widetilde{\mathbf{Z}_2}^m \times \mathbf{Z}_2{}^M$

This isotropy gives precisely the same pattern as the sub-maximal isotropy

$$\mathbf{S}_1 \times \mathbf{S}_2 \subset \mathbf{Z}_2 \times \mathbf{S}_3.$$

Other Patterns

There are also entirely new patterns possible corresponding to $\widetilde{\mathbf{Z}_2}$ (and $\mathbf{W}_2 \subset \mathbf{Z}_2 \wr \mathbf{S}_3$) in which all of the oscillators in a cluster oscillate with minimal period π. Similarly $\mathbf{W}_1 \subset \mathbf{Z}_2 \wr \mathbf{S}_3$ produces a new pattern where all the oscillators in one of the clusters oscillate with minimal period π.

9.6 Applying The Results To $\mathbf{D}_3 \times \mathbf{D}_3$ Symmetric Hopfield Neurons

We now apply the notion of skew-equivariance to the equations considered by Dangelmayer et al. in [11]. In this paper they consider the case where the system has $\mathbf{D}_3 \times \mathbf{D}_3$ symmetry, we now consider how to change the coupling between clusters, and within clusters, to obtain the symmetries $\mathbf{Z}_2 \wr \mathbf{S}_3$ and $\mathbf{Z}_2 \times \mathbf{S}_3$, which we know are possible through the work carried out in the previous section.

9.6.1 The Equations

As in [11] we consider coupled neurons modelled by the equations

$$\begin{aligned} \dot{u} &= -u + h(\Lambda u) - kh(\Lambda v) \\ \dot{v} &= -v + h(\Lambda v) + kh(\Lambda v) \end{aligned} \qquad (9.6.12)$$

where u and v are considered as input voltages of an excitatory and an inhibitory neuron respectively, and h is given, in [11], as $h(u) = tanh(u)$.

Initially we will consider when these neural oscillators are coupled with linear coupling. For full $\mathbf{D}_3 \times \mathbf{D}_3$ coupling this means we have equations of the form

$$\begin{aligned}
\dot{u}_{00} &= -u_{00} + h(\Lambda u_{00}) - kh(\Lambda v_{00}) \\
&\quad + k_1 \sum_{j=1}^{2} h(\Lambda u_{0j}) + k_2 \sum_{i=1}^{2} h(\Lambda u_{i0}) + k_3 \sum_{i,j=1}^{2} h(\Lambda u_{ij}) \\
\dot{v}_{00} &= -v_{00} + h(\Lambda v_{00}) + kh(\Lambda v_{00}) \qquad (9.6.13)
\end{aligned}$$

where k_1 represents the couplings within the cluster, k_2 describes the coupling of corresponding oscillators in different clusters and k_3 couples oscillators with different micro- and macro- indices. Remember that the variable u_{ij} has macro index i and micro index j.

To try and achieve the symmetries detailed earlier by using skew-equivariant coupling, we rewrite these equations with the most general possible linear coupling, and then see what constraints must be placed on the various coefficients for the desired results. With this in mind we write

$$\begin{aligned}
\dot{u}_{00} &= -u_{00} + h(\Lambda u_{00}) - k_0 h(\Lambda v_{00}) + a_0 h(\Lambda u_{00}) + a_1 h(\Lambda u_{01}) + a_2 h(\Lambda u_{02}) \qquad (9.6.14) \\
&\quad + a_3 h(\Lambda u_{10}) + a_4 h(\Lambda u_{20}) + a_5 h(\Lambda u_{11}) + a_6 h(\Lambda u_{12}) + a_7 h(\Lambda u_{21}) + a_8 h(\Lambda u_{22}) \\
\dot{v}_{00} &= -v_{00} + h(\Lambda v_{00}) + k_0 h(\Lambda v_{00})
\end{aligned}$$

However, as a minimum, these equations must be $\mathbf{S}_3$ equivariant, and so this forces

$$a_3 = a_4, \ a_5 = a_7 \text{ and } a_6 = a_8.$$

Therefore our general, linear coupled equations are,

$$\begin{aligned}
\dot{u}_{00} &= -u_{00} + h(\Lambda u_{00}) - k_0 h(\Lambda v_{00}) + b_0 h(\Lambda u_{00}) + b_1 h(\Lambda u_{01}) + b_2 h(\Lambda u_{02}) \qquad (9.6.15) \\
&\quad + b_3(h(\Lambda u_{10}) + h(\Lambda u_{20})) + b_4(h(\Lambda u_{11}) + h(\Lambda u_{21})) + b_5(h(\Lambda u_{12}) + h(\Lambda u_{22})), \\
\dot{v}_{00} &= -v_{00} + h(\Lambda v_{00}) + k_0 h(\Lambda v_{00}).
\end{aligned}$$

i.e. the variables are paired so that a 'macro' permutation of (12) leaves the equations unchanged.

Now thinking of the equations as being of the form 9.3.11, then our coupling term for the zeroth cluster will look like

$$\begin{aligned}
g_{00} &= b_0u_{00} + b_1u_{01} + b_2u_{02} + b_3(u_{10} + u_{20}) + b_4(u_{11} + u_{21}) + b_5(u_{12} + u_{22});\\
g_{01} &= b_6u_{01} + b_7u_{02} + b_8u_{00} + b_9(u_{11} + u_{21}) + b_{10}(u_{12} + u_{22}) + b_{11}(u_{10} + u_{20});\\
g_{02} &= b_{12}u_{02} + b_{13}u_{00} + b_{14}u_{01} + b_{15}(u_{12} + u_{22}) + b_{16}(u_{10} + u_{20}) + b_{17}(u_{11} + u_{21}).
\end{aligned}$$

where

$$g_0 = \begin{bmatrix} g_{00} \\ g_{01} \\ g_{02} \end{bmatrix}$$

Now, remembering that our $\mathbf{S}_3$ action is generated by the elements (012) and (12) we attempt to find restrictions on the coefficients b_i that will give us the necessary skew-equivariance, and so symmetry.

9.6.2 $\mathbf{Z}_2 \wr \mathbf{S}_3$ Symmetry

From theorem 9.2.2 we know that the skew equivariance must be of one of two possible forms to achieve $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry, namely

$$(\hat{\rho}, \tilde{\rho}, \hat{\kappa}, \tilde{\kappa}) = (id, id, \kappa, id)$$

or

$$(\hat{\rho}, \tilde{\rho}, \hat{\kappa}, \tilde{\kappa}) = (\rho^2, id, \kappa, id).$$

We shall now consider each of these cases in turn.

Case $(\hat{\rho}, \tilde{\rho}, \hat{\kappa}, \tilde{\kappa}) = (id, id, \kappa, id)$

We consider how each element of the skew-equivariance must restrict our choices for the b_i's, and then look at how these restrictions combine in the full equations.

Element $\hat{\rho} = id$

Having $\hat{\rho} = id$ forces:

$$\begin{aligned}
b_0 &= b_1 = b_2\\
b_6 &= b_7 = b_8\\
b_{12} &= b_{13} = b_{14}
\end{aligned} \tag{9.6.16}$$

Element $\tilde{\rho} = id$

Gives us

$$\begin{aligned} b_3 &= b_4 = b_5 \\ b_9 &= b_{10} = b_{11} \\ b_{15} &= b_{16} = b_{17} \end{aligned} \tag{9.6.17}$$

Element $\hat{\kappa} = \kappa$

Forces us to have

$$\begin{aligned} b_1 &= b_2 \\ b_6 &= b_{12} \\ b_7 &= b_{14} \end{aligned} \tag{9.6.18}$$

But putting 9.6.18 into 9.6.16 means that

$$b_6 = b_7 = b_8 = b_{12} = b_{13} = b_{14}$$

and so $\hat{\kappa}$ actually acts as the identity, causing a contradiction. Therefore we cannot achieve the desired result, using linear coupling, with these choices.

Case $(\hat{\rho}, \tilde{\rho}, \hat{\kappa}, \tilde{\kappa}) = (\rho^2, id, \kappa, id)$

We now try the same thing for the other possible choices of elements.

Element $\hat{\rho} = \rho^2$

This means that $\hat{\rho} = (210)$ and so we must have

$$\begin{aligned} b_0 &= b_6 = b_{12} \\ b_1 &= b_7 = b_{13} \\ b_2 &= b_8 = b_{14} \end{aligned} \tag{9.6.19}$$

Element $\tilde{\rho} = id$

Gives us

$$\begin{aligned} b_3 &= b_4 = b_5 \\ b_9 &= b_{10} = b_{11} \\ b_{15} &= b_{16} = b_{17} \end{aligned} \tag{9.6.20}$$

Element $\widehat{\kappa} = \kappa$

Means that we must have

$$\begin{aligned} b_1 &= b_2 \\ b_6 &= b_{12} \\ b_7 &= b_{14} \end{aligned} \tag{9.6.21}$$

Element $\widetilde{\kappa} = id$

And finally we need

$$\begin{aligned} b_4 &= b_5 \\ b_{10} &= b_{11} \\ b_{16} &= b_{17} \end{aligned} \tag{9.6.22}$$

But this is already catered for in 9.6.20.

This gives us a coupling term that satisfies the required conditions, and so through the specified skew-equivariance we obtain $\mathbf{Z}_2 \wr \mathbf{S}_3$ symmetry. To see this define the following coefficients

$$\begin{aligned} c_0 &= b_0 = b_6 = b_{12} \\ c_1 &= b_1 = b_2 = b_7 = b_8 = b_{13} = b_{14} \\ c_2 &= b_3 = b_4 = b_5 \\ c_3 &= b_9 = b_{10} = b_{11} \\ c_4 &= b_{15} = b_{16} = b_{17} \end{aligned}$$

Then we have the coupling term for the zeroth cluster given by

$$\begin{aligned} g_{00} &= c_0u_{00} + c_1(u_{01} + u_{02}) + c_2(u_{10} + u_{20} + u_{11} + u_{21} + u_{12} + u_{22}) \\ g_{01} &= c_0u_{01} + c_1(u_{02} + u_{00}) + c_3(u_{11} + u_{21} + u_{12} + u_{22} + u_{10} + u_{20}) \\ g_{02} &= c_0u_{02} + c_1(u_{00} + u_{01}) + c_4(u_{12} + u_{22} + u_{10} + u_{20} + u_{11} + u_{21}) \end{aligned} \tag{9.6.23}$$

and the other coupling terms by symmetry.

9.6.3 $\mathbf{Z}_2 \times \mathbf{S}_3$ Symmetry

In this case we must have $(\widehat{\rho}, \widetilde{\rho}, \widehat{\kappa}, \widetilde{\kappa}) = (id, id, \kappa, \kappa)$. However, having both $\widehat{\rho}$ and $\widetilde{\rho}$ as the identity means that $\widehat{\kappa}$ and $\widetilde{\kappa}$ must also act as the identity, and so it is not possible to achieve $\mathbf{Z}_2 \times \mathbf{S}_3$ symmetry by skew-equivariance with linear coupling.

It would be possible to use linear coupling and skew-equivariance if we chose a different $\mathbf{Z}_2$ action within each cluster though. For example, if the action was to multiply all the variables by -1, then a skew-equivariance could easily be chosen to satisfy our requirements.

We can still use skew equivariance to produce the necessary symmetries however if we look beyond linear coupling. One such solution would be to use coupling of the form

$$\begin{aligned} g_{00} &= AB \\ g_{01} &= ACE + BDE + BCF + ADF \\ g_{02} &= BCE + ADE + ACF + BDF \end{aligned} \tag{9.6.24}$$

where

$$\begin{aligned} A &= u_{00}h(\Lambda u_{01}) + u_{01}h(\Lambda u_{02}) + u_{02}h(\Lambda u_{00}), \\ B &= u_{00}h(\Lambda u_{02}) + u_{02}h(\Lambda u_{01}) + u_{01}h(\Lambda u_{00}), \\ C &= u_{10}h(\Lambda u_{11}) + u_{11}h(\Lambda u_{12}) + u_{12}h(\Lambda u_{10}), \\ D &= u_{10}h(\Lambda u_{12}) + u_{12}h(\Lambda u_{11}) + u_{11}h(\Lambda u_{10}), \\ E &= u_{20}h(\Lambda u_{21}) + u_{21}h(\Lambda u_{22}) + u_{22}h(\Lambda u_{20}), \\ F &= u_{20}h(\Lambda u_{22}) + u_{22}h(\Lambda u_{21}) + u_{21}h(\Lambda u_{20}). \end{aligned}$$

It is easy to see that applying ρ to any cluster leaves all of these terms invariant, and that applying κ to any cluster will swap A with B, C with D and E with F, and so produces the required result in 9.6.24.

9.7 Comments

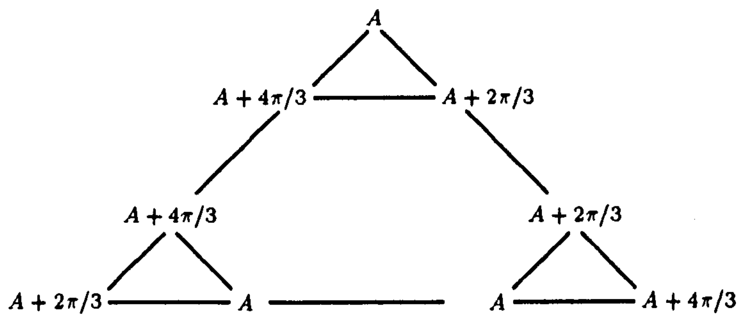
We have shown in this section that the notion of skew-equivariance can be a very useful tool. In particular, here we have 'picked out' the internal $\mathbf{Z}_2$ symmetry of the clusters and then used the earlier results to discover the new patterns of oscillation that could arise, from basically the same system we started with.

It is apparent that skew-equivariance could show some very useful general properties which have yet to be considered.

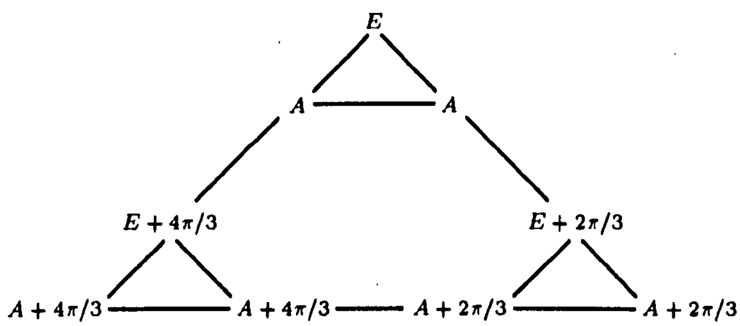
9.8 Patterns Seen in the Presence of $\mathbf{D}_3 \times \mathbf{D}_3$ Symmetry

In this final section we show all the patterns, up to conjugacy, that were predicted by Dangelmayer et al. in [10] to be possible. This section is designed to be for reference purposes only, and so we just present the patterns in diagrammatic form.

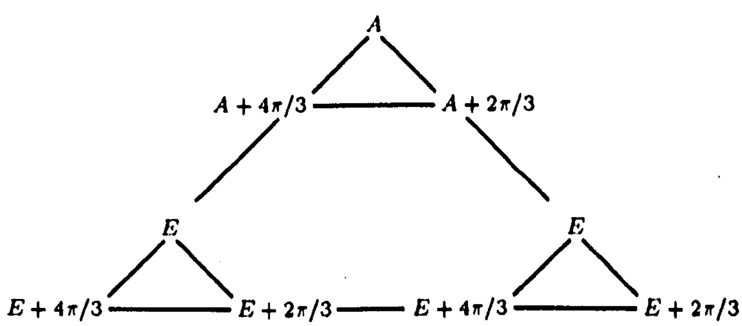
- Isotropy $\widetilde{\mathbf{Z}_3}^m \times \widetilde{\mathbf{Z}_3}^M$



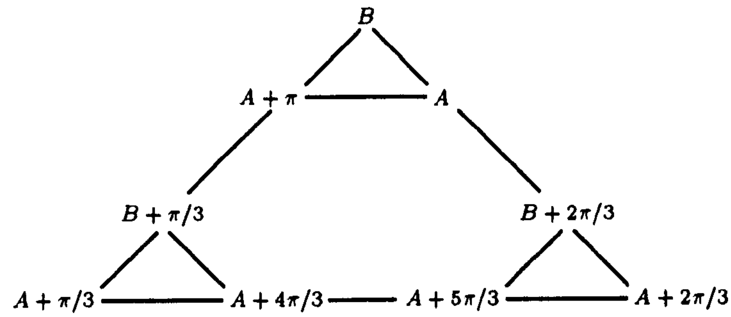
- Isotropy $\mathbf{Z}_2^m \times \widetilde{\mathbf{Z}_3}^M$



- Isotropy $\widetilde{\mathbf{Z}_3}^m \times \mathbf{Z}_2^M$

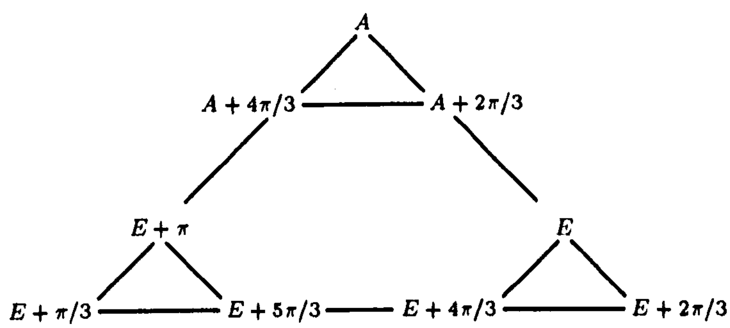


- Isotropy $\widetilde{\mathbf{Z}_2}^m \times \widetilde{\mathbf{Z}_3}^M$

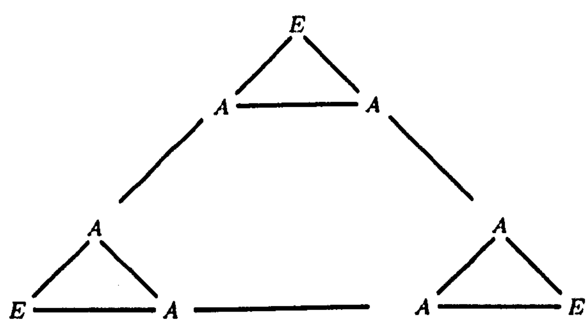


where $B = B + \pi$.

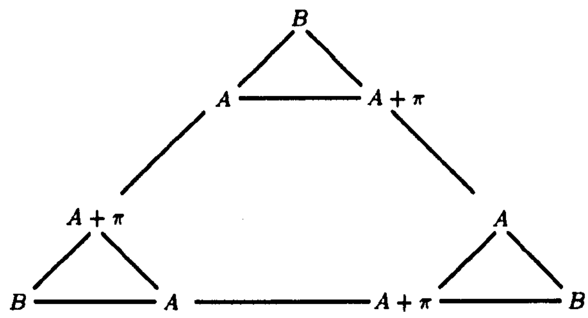
- Isotropy $\widetilde{\mathbf{Z}_3}^m \times \widetilde{\mathbf{Z}_2}^M$



- Isotropy $\mathbf{D}_3^{mM}$

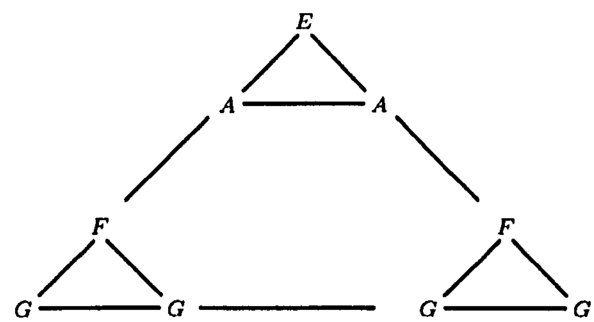


- Isotropy $\widetilde{\mathbf{D}_3}^{mM}$

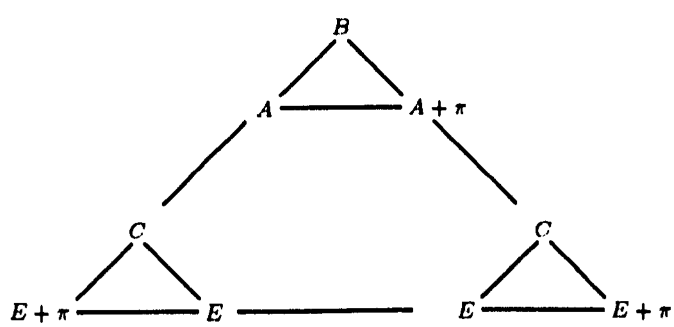


where $B = B + \pi$.

- Isotropy $\mathbf{Z}_2{}^m \times \mathbf{Z}_2{}^M$

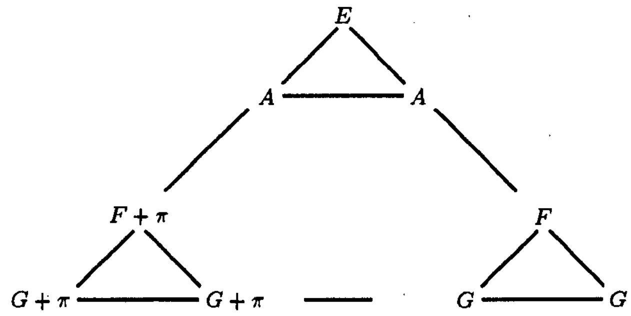


- Isotropy $\widetilde{\mathbf{Z}_2}^m \times \widetilde{\mathbf{Z}_2}^M$

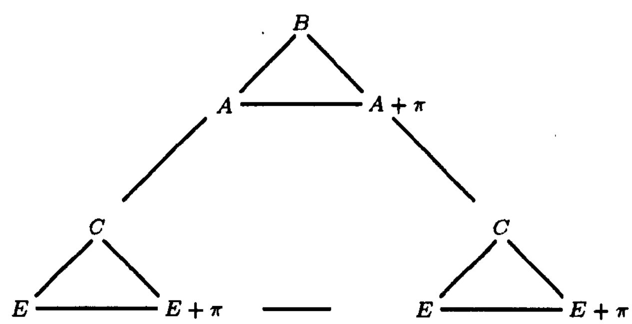


where $B = B + \pi$ and $C = C + \pi$.

- Isotropy $\mathbf{Z}_2{}^m \times \widetilde{\mathbf{Z}_2}^M$



- Isotropy $\widetilde{\mathbf{Z}_2}^m \times \mathbf{Z}_2{}^M$



where $B = B + \pi$ and $C = C + \pi$.

Chapter 10

Concluding Remarks

In this the final Chapter we summarize the main results that we have found and their implications, and suggest future research that may follow from the work presented here.

Summary of Results

We have shown that introducing an internal $\mathbf{Z}_2$ symmetry into systems of coupled cells can substantially alter the results of both steady-state and Hopf bifurcations in these systems. In particular we discover several entirely new branches of solutions when we couple the cells with respect to the internal symmetries so as to produce wreath product coupling. We also find differences to the no-internal-symmetries case when we couple the cells so as to produce direct product coupling, though the differences are not so obvious.

This complements the work of Dionne et al. [12] and [13] who considered the general theory of coupled cells with wreath and direct product coupling and found results along the same lines as those found here in a more general setting.

We have also shown that these symmetries can cause some quite unexpected results when we apply the theory to systems of coupled oscillators, as well as showing that we *can* achieve these results through numerical experiments.

Finally the two applications we have considered show that the work presented here has some useful consequences, which leaves plenty of room for further work to follow on from it.

Further Work

In the course of writing this thesis, many ideas for further applications and directions for further research have come to light, and we list here some topics for further investigation which the author hopes to tackle in the near future.

- The work presented here has only investigated the extra solutions that can be found in systems with 'all-to-all' coupling, and the addition of an internal $\mathbf{Z}_2$ symmetry. Given the motivation the work here could be repeated for both larger internal symmetries and, for example, coupling in a ring. There are also other symmetries that are realisable with internal $\mathbf{Z}_2$ and global $\mathbf{S}_n$ symmetries which we have not considered here.

- The work here on Hopf bifurcations could be extended to n coupled cells instead of just three, as well as investigating different internal and global symmetries.

- Much of the work carried out here on insect gaits (Chapter 8) is very speculative and model independent as far as a specific mathematical model is concerned. The author wishes to continue the work presented here to create a realistic model of an insect's Central Pattern Generator.

- The notion of skew-equivariance appears to have much deeper properties than those discussed here, and future investigation could lead to a theory that has many applications.

Bibliography

[1] J.Frank Adams, *Lectures on Lie Groups*, In particular, Chapter 3, *Elementary Representation Theory*, Univ. Chicago Press, 1969.

[2] A.A. Andronov, A.A. Vitt, S.E. Khaikin. *Theory of Oscillators*, Pergamon Press Ltd., 1966.

[3] D. G. Aronson, M. Golubitsky, M. Krupa. *Coupled Arrays of Josephson Junctions and Bifurcations of Maps with S_N Symmetry*, Nonlinearity **4**, pp861-902, 1991.

[4] P. Ashwin, J.W. Swift. *The Dynamics of n Weakly Coupled Identical Oscillators*, J. Nonlinear Science Vol 2, pp 69-108, 1992.

[5] M.D. Berkemeier. *Coupled Oscillators in a Platform with Four Springy Legs and Local Feedback*, Boston University Technical Report No. AM-95-009, 1995.

[6] W. Burnside, *Theory of Groups of Finite Order, 2nd edition*, Cambridge Univ. Press 1911, reprinted Dover, New Yprk 1955.

[7] A.H. Cohen, R. Rand, P.J. Holmes, *Systems of Coupled Oscillators as Models of Central Pattern Generators*, in *Neural Control of Rhythmic Movements in Vertebrates*, ed Cohen, Rossignal and Grillner, John Wiley and Sons inc., New York, 333-367, 1988.

[8] J.J. Collins, I. Stewart, *Coupled Non-linear Oscillators and the Symmetries of Animal Gaits*, J. Nonlinear Science **3** pp349-392, 1993.

[9] J.J. Collins, I. Stewart, *Hexapodal Gaits and Coupled Non-linear Oscillator Models*, Biol. Cybern. **68** 287-298, 1993.

[10] G. Dangelmayer, W. Güttinger, M. Wegelin. *Hopf Bifurcation with $\mathbf{D}_3 \times \mathbf{D}_3$-Symmetry*, Zangew Math Phys (ZAMP) **44**, pp 595–638, 1993.

[11] G. Dangelmayer, W. Güttinger, Jörg Oppenländer, Jörg Tomes, M. Wegelin. *Synchronised Patterns in Hierarchical Networks of Neuronal Oscillators with* $\mathbf{D}_3 \times \mathbf{D}_3$*-Symmetry*, Preprint, Institut für Informationverarbeitung, 1995.

[12] B. Dionne, M. Golubitsky, I. Stewart. *Coupled Cells with Internal Symmetry. Part I: Wreath Products*, Warwick Preprint 33/1994.

[13] B. Dionne, M. Golubitsky, I. Stewart. *Coupled Cells with Internal Symmetry. Part II: Direct Products*, Warwick Preprint 34/1994.

[14] M.J. Field, R.W. Richardson, *Symmetry Breaking and the Maximal Isotropy Subgroup Conjecture for Reflection Groups*, Arch. Rational Mech. and Anal. **105**, pp61-94, 1989.

[15] M. Golubitsky, J-M. Mao, M. Nicol. *Symmetries of Periodic Solutions for Planar Potential Systems*, accepted for publication in Proceedings of the AMS.

[16] M. Golubitsky, D. G. Schaeffer. *Singularities and Groups in Bifurcation Theory* vol.I, Appl. Math. Sci. **69**, Springer-Verlag, New York 1985.

[17] M. Golubitsky, I. N. Stewart and D. G. Schaeffer. *Singularities and Groups in Bifurcation Theory* vol.II, Appl. Math. Sci. **69**, Springer-Verlag, New York 1988.

[18] J.A. Green, *Sets And Groups*, Routledge and Kegen Paul, (1965) 1988.

[19] J. Guckenheimer, P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* Appl. Math. Sci. **42**, Springer Verlag, New York 1983.

[20] M. Hall, *The Theory Of Groups*, Chelsea Publishing Co., NewYork, 1976.

[21] K. Kleiner, *Look to the Insect*, New Scientist, 12th November, 1994.

[22] J. Knobloch, A. Vanderbauwhede. *Hopf Bifurcation at k-Fold Resonances in Equivariant Reversible Systems*, in *Dynamics, Bifurcation and Symmetry - New Trends and New Tools*, ed. P. Chossat, Kluwer Academic Publishers, 1994.

[23] S.M. Manton, *The Evolution of Anthropodan Locomoting Mechanisms Part 10: Locomoting Habits, Morphology and Evolution of the Hexapod Classes*, J. Linn. Soc. London, Zoology Series, 1973.

[24] I. Melbourne, *A Singularity Theory Analysis of Bifurcation Problems with Octahedral Symmetry*, Dyn. Stab. Sys. **1**, 293-321, 1986.

[25] I. Melbourne, *Bifurcation Problems with Octahedral Symmetry*, PhD. Thesis, University of Warwick, 1987.

[26] K.G. Pearson, R. Franklin, *Characteristics of Leg Movements and Patterns of Coordination in Locusts Walking on Rough Terrain*, Int. J. Robotics Res. **3** 101-112, 1984.

[27] W.H. Press *at al.*, *Numerical Recipes in Pascal, The Art of Scientific Computing*, Cambridge University Press, 1989.

[28] R.P. Stanley, *Invariants of Finite Groups and thier Applications to Combinatorics*, Bulletin of the AMS, Vol. 1, **3**, 1979.

[29] K.S. Waldron, V.J. Vohnout, A. Pery, R.B. McGhee, *Configuration Design of the Adaptive Suspension Vehicle*, Int. J. Robotics Res. **3** 37-48, 1984.

[30] D. Wood *Modelling Hexapod Gaits by Coupled Non-linear Oscillators*, Final Year Project towards the degree of BSc., University of Warwick, 1991.

[31] D. Wood *Towards Modelling an Insects Central Pattern Generator by Networks of Coupled Non-Linear Oscillators*, Dissertation towards the degree of MSc., University of Warwick, 1992.