Time-change and control of stochastic volatility

by

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Contents

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Declarations

I confirm that the thesis has not been submitted for a degree at another university.

Chapter 2 and parts of Chapter 4 are joint work with Sigurd Assing and Saul D. Jacka, *Monotonicity of the value function for a two-dimensional optimal stopping problem.* To appear in *Ann. Appl. Probab.*

Chapter 3 and parts of Chapter 4 are joint work with Saul D. Jacka. A version is available at arXiv:1309.1404, and a modified version is in preparation.

Abstract

The central theme of this thesis is the behavior of the value function of general optimal stopping problems under a stochastic volatility model when varying the volatility dynamics. We first use a combination of time-change and coupling techniques to show regularity properties of the value function. We consider a large class of terminal payoffs: when the first component of the model is a stochastic differential equation without drift we allow for general measurable functions, and when it has a drift we impose a mild condition which includes possibly unbounded and discontinuous functions. We also consider a running cost which can be any non-negative and bounded Borel function. Moreover, we derive the solution of a zero-sum game of stopping and control, which arises when considering some parameter uncertainty in the volatility dynamics. In both finite and infinite horizon, we exhibit the existence of a saddle point using stochastic control and martingale arguments as well as the probabilistic representation of solutions to free-boundary problems.

Overall, our approach in mainly theoretical, however we impose only verifiable conditions. We then discuss some examples arising in American option pricing where our results are applicable. In particular, we are able to compare American option prices under different volatility models in a variety of settings and we establish that the optimal exercise boundary for the associated option is a monotone function of the volatility.

Chapter 1

Introduction

The central theme of this thesis is the behavior of the value function of optimal stopping problems in a general stochastic volatility model when varying the volatility dynamics. Our techniques are mainly based on time-change, coupling, and control of stochastic processes.

To illustrate the mathematical objects we are interested in, consider the following situation. Suppose that we observe a random process $X = (X_t)_{t\geq 0}$ which evolves in continuous-time and that we wish to *(optimally) stop* the observation when the value $g(X_t)$ is maximal over a predetermined period of time, for some payoff function $g(x)$. Since X is random, each observation corresponds to a possible scenario, then we maximize in average over all random times (stopping rules). The *value function* is precisely the maximal expected payoff over all stopping rules.

In this thesis, the dynamics of X are initially determined by the stochastic differential equation

$$X_t = X_0 + \int_0^t a(X_s)\, Y_s \, dB_s, \tag{1.1}$$

where $a : \mathbb{R} \mapsto \mathbb{R}$ is a Borel function satisfying certain conditions, $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, and $Y = (Y_t)_{t\geq 0}$ is another stochastic process to be referred to as the *stochastic volatility* of X (the dynamics of Y are specified below in the Overview). We shall study this object with a so-called *discount (or killing) rate*, meaning that X vanishes at an independent, exponentially distributed time.

Given an initial condition $(X_0, Y_0) = (x, y)$, the value function $v(x, y)$ we are initially interested in is given by

$$v(x, y) = \sup_{0 \le \tau \le T} E_{x,y} [e^{-\alpha\tau} g(X_\tau)], \tag{1.2}$$

where τ denotes a (finite) stopping time of (X, Y), T is the time horizon (which may be infinite), $E_{x,y}$ denotes the expectation conditional on $(X_0, Y_0) = (x, y)$, $\alpha > 0$ is the discount rate, and the payoff (or gain) function $g : \mathbb{R} \to \mathbb{R}$ is measurable.

To ensure the well-posedness of this problem, we assume throughout this thesis that

$$E_{x,y} \left[\sup_{0 \le t \le T} e^{-\alpha t} |g(X_t)| \right] < \infty \tag{1.3}$$

for each initial condition (x, y), which is a common assumption in the context of optimal stopping problems.

For completeness, we discuss in Appendix A sufficient conditions on g and on the dynamics of (X, Y) in order for (1.3) to be satisfied.

Later on we will see that the functional properties of $v(x, y)$ in (1.2) are preserved in the more general case where there is cost of observation:

$$v(x, y) = \sup_{0 \le \tau \le T} E_{x,y} [e^{-\alpha\tau} g(X_\tau) - \int_0^\tau e^{-\alpha s} c(X_s) ds], \tag{1.4}$$

where c is a non-negative and bounded Lebesgue integrable function.

The main focus of the thesis is on the behavior of $v(x, y)$ when varying the dynamics of Y.

Overview and literature review.

Chapter 2. Time-change of stochastic volatility: in this chapter we mainly study monotone properties of $v(x, y)$ as a function of the initial volatility value y, when Y is either a Markov chain or a diffusion process. In the second case we also study continuity of $v(x, \cdot)$.

In Section 2.1 we account for the main results on time-changes that will be used throughout the thesis, and the proofs are provided at the end of the chapter.

In Section 2.2 we look at the case when Y is an irreducible continuous-time Markov chain (MC), with finite state space and is independent of the Brownian motion B driving equation (1.1).

In Section 2.3 we assume that Y solves a stochastic differential equation of the form

$$Y_t = Y_0 + \int_0^t \eta(Y_s)\, dB_s^Y + \int_0^t \theta(Y_s)\, ds \tag{1.5}$$

where $B^Y = (B_t^Y)_{t\geq 0}$ is a standard Brownian motion such that $\langle B, B^Y\rangle_t = \delta t$, for some real number $\delta \in [-1,1]$, and $\eta, \theta : \mathbb{R} \to \mathbb{R}$ are continuous functions.

In Section 2.4 we deal with an extension of the results obtained to the case with cost of observations in (1.4).

Specifically, the method of proof of the monotonicity of $v(x, \cdot)$ in (1.2) is as follows. First, by a suitable time-change device, we transform the "coupled" system (X, Y) (where the X component depends upon Y) into a "decoupled" one in the sense that each component is autonomous. Second, we reformulate the original value function in terms of (X, Y) into one in terms of the decoupled system by using the fact that the strong Markov property is preserved after time-change. The key feature of the new formulation is that the whole dependence on the volatility is placed on the discount factor only, which is a continuous and strictly increasing additive functional. Finally, the construction is made in such a way that there is no "overtaking" by the paths of the time-changed volatility processes.

The time-change technique proves to be very powerful and many authors take advantage of a time-change device in a variety of settings, for instance:

- Hobson [24] and Henderson [21] time-change volatility processes to compare European option prices.
- Cissé et al. [8] time-change a one-dimensional regular diffusion X to "modify" a reward function with a continuous additive functional as the discount factor into one with linear discounting (see Lemma 3.1 in [8]). We apply a similar technique in Lemma 4.4 below.
- Kyprianou et al. [34] time-change an α-stable process X to "erase" the negative components of X (see Proposition 3.2 in [34]).

- In [2], we time-change the stochastic volatility system (X, Y) to "transfer" the whole dependence of X on the volatility to the discount factor only (see Lemma 2.10 below).

Chapter 3. Control of stochastic volatility: in this chapter we derive the solution of a zero-sum game of stopping and control. The solution is presented in Section 3.3.2, under verifiable conditions. Examples where these conditions are satisfied are provided in the next chapter.

In Section 3.1 we establish the notation and state the problem. We allow for some *parameter uncertainty* in the dynamics of Y. This uncertainty is incorporated through the Q-matrix (MC case) or the drift of the volatility (diffusion case).

In Section 3.2 we derive smoothness for the value function of generic optimal stopping problems. The reason is because the proofs of our verification theorems in the subsequent section rely on analytical methods for which sufficient regularity of a candidate value function is required. The strong Markov property as well as the probabilistic representation of solutions to Dirichlet-type problems are the main tools. We remark that, despite the prominence of regime-switching models in recent years, there are no general results concerning the regularity properties of the value function $v(x, y)$ in (1.4) within this context, so these have been addressed here and in Section 4.2 as well.

In Section 3.3 we state and show the main results of this chapter, Theorems 3.16 and 3.17. These assert that the value of the game of stopping and control identifies with the value function of certain optimal stopping problem associated to an extremal scenario. Such a candidate value function is assumed to be continuous in some sense (depending on the setting) and monotone in y. In particular, we exhibit a saddle point when the space of control values is compact.

Some works involving the solution of a zero-sum game of stopping and control are the following:

- Karatzas and Sudderth [32] assume that the gain function is continuous, and that the state process is a linear diffusion in $[0, 1]$ with drift and diffusion coefficients affected by some control parameters.

- Weerasinghe [45] studies a game with running payoff, where the state process is a one-dimensional diffusion and the control corresponds to its diffusion coefficient (which may be degenerate).

- Karatzas and Zamfirescu [33] examine a differential game from a martingale approach and allowing terminal and running payoff. The state process solves a controlled functional/differential equation (coefficients may depend on the whole path) and the control only affects the drift.

Chapter 4. Applications to option pricing: our approach in the previous Chapters is theoretical, and we try to keep as much generality as possible, yet to impose verifiable conditions. This Chapter deals with the examples, the verification of such conditions, and deduce that the optimal stopping boundary of the stopping problems we work with is monotone in y.

In the context of mathematical finance, X stands for the stock price process. The model for X when its stochastic volatility Y is a (function of a) MC is referred to as *regime-switching* (or *Markov modulated*). The literature on optimal stopping with regime-switching models is mostly concentrated on specific examples. See for instance, Buffington and Elliot [7], Guo and Zhang [19], Jobert and Rogers [30], and Yao et al [47]. In the other formulation for Y, Heston [22], and Hull and White [25] (amongst many others) assume that Y solves an autonomous stochastic differential equation (SDE). See also a survey by Frey [14].

The monotonicity property of option prices has been studied by several authors in the diffusion case:

- Romano and Touzi [41, Theorem 3.1] deal with European options and the stochastic volatility model of Hull & White [25]. They work under the assumption that the volatility function is bounded by two constants and that the payoff g satisfies a logarithmic growth condition.

- Ekström [10, Theorem 4.2] compares prices of American options in the case that $Y \equiv 1$ and the gain function satisfies the condition $g(a\,x) \leq a\,g(x)$ for $a \geq 1$ and $x \geq 0$.

- Hobson [24, Theorem 6.4] applies time-change and coupling for comparing prices of European options in a general stochastic volatility model, and under the assumption that g is convex.

We adapt and combine the techniques of Ekström and Hobson to the case of American-type options in both, finite and infinite horizon. We place minimal constraints on the running and terminal payoff functions: when X is driftless (as in (1.1)) g is only assumed to be measurable (see Chapter 2). In the case that X has a linear drift (as in (1.6)), we impose a mild condition on g which includes non-increasing, possibly unbounded and discontinuous functions (see Section 4.1).

Whether it be a Markov chain or a diffusion process modeling Y, it is difficult to make precise the parameters driving its dynamics because of the uncertain nature of the volatility of stock prices. In the first case, the transition rates model the occurrence of sudden economic movements (*switches*) but, in practice, these rates are not fully observable (see Hartman and Heaton [20] and references therein). In the other case, the drift of volatility typically characterizes the choice of the pricing measure, but there is no definite criterion telling us which measure should be used (see Hobson [23], [24]).

There is some work on *model uncertainty* that takes account of uncertainty in the volatility model. For instance, Avellaneda et al. [3] and Frey [15] assume that the volatility is a predictable process which is only known to be bounded between two constant values. We allow for some *parameter uncertainty* in Chapter 3, incorporated through the Q-matrix or the drift of the volatility, and study a zero-sum game which can be interpreted as the stopper trying to maximize his payoff while *nature* plays against him and tries to minimize this payoff. The resulting value of the game, when it exits, is the worst-case scenario for the stopper in the presence of parameter uncertainty.

In Section 4.1 we adapt the results of Chapter 2 to the case where X has linear drift, because we deal with the dynamics for the stock price process:

$$X_t = X_0 + \int_0^t X_s \, Y_s \, dB_s + \int_0^t r \, X_t \, dt, \tag{1.6}$$

where $r > 0$ stands for the instantaneous interest rate.

In Section 4.2 we deal with the first example which is the regime-switching model used by Guo and Zhang [19] and Jobert and Rogers [30], where the drift of X is stochastic and depends on the volatility, and $g(x) = \max\{0, K - x\}$, for some positive constant K. An important consequence of our results is that the optimal thresholds characterizing the optimal stopping rule of this problem are monotone. This was suggested in the numerical examples provided in [19] and [30], without proof.

Next, under a more general model for X and only assuming that g is non-negative and continuous, we also show that the function $v(\cdot, y)$ is continuous for each y, in the regime-switching setting.

In Section 4.3 we examine two examples in the diffusion setting, based on Bessel processes. We consider the Hull & White [25] and Heston [22] models. Apart from verifying that these models satisfy all of our conditions in the previous chapters, we also establish that the optimal stopping boundary for American put options is monotone in the volatility. The last part of the section is dedicated to a brief review of Bessel processes and some path-comparison properties that are used.

Chapter 2

Time-change of stochastic volatility

2.1 Preliminaries

Consider a two-dimensional strong Markov process $(X, Y) = (X_t, Y_t)_{t\geq 0}$ given on a family of probability spaces $(\Omega, \mathcal{F}, P_{x,y}, (x, y) \in \mathbb{R} \times \mathcal{S})$, and adapted to a filtration $(\mathcal{F}_t)_{t\geq 0}$ of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions.

This chapter deals with the value function

$$v(x, y) = \sup_{0\leq\tau\leq T} E_{x,y} [e^{-\alpha\tau} g(X_\tau)], \quad (x, y) \in \mathbb{R} \times \mathcal{S}, \tag{2.1}$$

where $\alpha > 0$ is the discount rate, $T \in [0, \infty]$ is the time horizon, the gain function $g : \mathbb{R} \to \mathbb{R}$ is measurable, and the supremum is over finite stopping times of (X, Y). The precise dynamics of the pair (X, Y) are specified in Sections 2.2.1 and 2.3.1 below.

The main focus is on the monotonocity of the function $v(x, \cdot)$.

The time-change method is extensively used in this chapter, so we review the main results that are going to be used in the sequel. The proofs are provided in Section 2.5.

Definition 2.1 *A time-change A is a family $\{A_t; t \geq 0\}$ of stopping times such that $t \mapsto A_t$ is increasing and right-continuous.*

Proposition 2.2 *Consider a right-continuous, non-decreasing, adapted process* Γ *and set* $A_t = \inf\{s \geq 0 : \Gamma_s > t\}$. *Then* A *is a time-change. Moreover,* $\Gamma_s = \inf\{t \geq 0 : A_t > s\}$ *and* Γ_s *is an* $(\mathcal{F}_{A_t})_{t\geq 0}$*-stopping time for each* $s \geq 0$.

Corollary 2.3 *In the context of Proposition 2.2,*

(i) If Γ *is strictly increasing then* A_t *is continuous.*

(ii) If Γ *is continuous and* $\lim_{t\to\infty} \Gamma_t = \infty$ *a.s. then* A_t *is strictly increasing, finite and* $\lim_{t\to\infty} A_t = \infty$ *a.s.*

(iii) If Γ *is strictly increasing, continuous,* $\lim_{t\to\infty} \Gamma_t = \infty$ *a.s.* <u>*and*</u> Γ_t *is finite for all* $t \geq 0$ *a.s. then*

$$\Gamma_{A_s} = A_{\Gamma_s} = s, \quad \textit{for all } 0 \leq s < \infty \textit{ a.s.}$$

and

$$s < \Gamma_t \quad \textit{if and only if } A_s < t \textit{ for all } 0 \leq s, t < \infty \textit{ a.s.}$$

The following lemma is a consequence of the symmetric roles of A and Γ. Denote by $\mathcal{M}$ and $\mathcal{T}$ the families of finite stopping times relative to the filtrations $(\mathcal{F}_t)_{t\geq 0}$ and $(\mathcal{F}_{A_t})_{t\geq 0}$, respectively.

Lemma 2.4 *Let* Γ *be strictly increasing, continuous,* $\lim_{t\to\infty} \Gamma_t = \infty$ <u>*and*</u> Γ_t *is finite for all* $t \geq 0$ *a.s. If* $\rho \in \mathcal{M}$ *then* $\Gamma_\rho \in \mathcal{T}$ *and if* $\tau \in \mathcal{T}$ *then* $A_\tau \in \mathcal{M}$.

This lemma is used in the proof of Lemma 2.10. A similar statement can be found in [42, VIII.65.8].

The last part of this section looks at a very particular time-change. The result is used in an argument in Section 2.2.3. Let us fix the notation:

Suppose that W is a Brownian motion, adapted to $(\mathcal{F}_t)_{t\geq 0}$ and let Z be an $(\mathcal{F}_t)_{t\geq 0}$-adapted process satisfying that

$$\int_0^t f(Z_s)^2 ds < \infty, \qquad \text{a.s. for all } t \geq 0,$$

where f is a Borel measurable function. Then the stochastic integral M_t, with

$$M_t = \int_0^t f(Z_s) dW_s,$$

is well-defined and is an $(\mathcal{F}_t)_{t\geq 0}$-continuous local martingale. Further assume that $\lim_{t\to\infty}\langle M\rangle_t = \infty$ and define $A = (A_t)_{t\geq 0}$ by $A_t := \inf\{s \geq 0 : \langle M\rangle_s > t\}$, which is a time-change (see [39, Theorem V.1.6]).

Proposition 2.5 *Suppose that* $f(\cdot)^2 > 0$. *If* W *is independent of* Z *then* $B_\cdot = M_{A_\cdot}$ *is independent of* $Z_{A_\cdot}$.

2.2 The regime-switching case

2.2.1 Introduction

Let $\mathcal{S} = \{y_i : i = 1, 2, \ldots, m\}$ be a subset of $(0, \infty)$. We assume that the strong Markov process (X, Y) satisfies the following. For every $(x, y) \in \mathbb{R} \times \mathcal{S}$, there is a Brownian motion B on $(\Omega, \mathcal{F}, P_{x,y})$, adapted to $(\mathcal{F}_t)_{t\geq 0}$, independent of Y, and such that

$$X_t = x + \int_0^t a(X_s) Y_s \, dB_s, \quad t \geq 0, \; P_{x,y} - a.s. \tag{2.2}$$

where the process Y is a continuous-time, irreducible Markov chain on the finite state space $\mathcal{S}$ with Q-matrix $(q[y_i, y_j])$.

We assume that the stochastic differential equation $dX' = a(X')Y'dB'$, with B' and Y' related as in (2.2), admits a weakly unique solution.

We will show that, under Condition **C1** below (see page 13), for fixed $x \in \mathbb{R}$ and $y, y' \in \mathcal{S}$:

$$\text{if} \quad y \leq y' \quad \text{then} \quad v(x, y) \leq v(x, y'). \tag{2.3}$$

The proof of this result is mainly based on a combination of time-change and coupling arguments.

The statement in (2.3) supports the intuition that the larger the volatility of a diffusion the sooner this diffusion reaches the points where the gain function

g is large. Since the positive discount factor α *kills* the gain as time elapses, one expects that $v(x, y')$ is larger than $v(x, y)$ for $y' > y$.

The following theorem is the main result of Section 2.2, which asserts (2.3) when the gain function g is non-negative.

Throughout this thesis we say that a continuous-time Markov chain Y is *skip-free* if its Q-matrix is tridiagonal (e.g. when Y is a birth-death Markov chain).

Theorem 2.6 *Let Condition* **C1** *on page 13 be satisfied. Assume that the gain function g is non-negative, and that Y is skip-free. Then, for each $x \in \mathbb{R}$, $v(x, \cdot)$ is non-decreasing on $\mathcal{S}$.*

The proof is presented in Section 2.2.4, where we also discuss the case when g is possibly negative.

2.2.2 Heuristics: the time-changed dynamics

Fix $(x, y) \in \mathbb{R} \times \mathcal{S}$ and write

$$X_t \;=\; x + \int_0^t a(X_s)\, dM_s, \quad t \geq 0,\; P_{x,y} - a.s.,$$

where the stochastic integral $M_s = \int_0^s Y_u\, dB_u$ is well-defined because the paths of Y are piecewise constant and so $\int_0^s Y_u^2 du < \infty$, $P_{x,y}$ - a.s., for all $s \geq 0$ (see [39, IV.2.7]).

The quadratic variation $\langle M \rangle_\cdot = \int_0^\cdot Y_s^2\, ds$ is a continuous, $(\mathcal{F}_t)_{t\geq 0}$-adapted process and satisfies $P_{x,y}\left(\lim_{t\uparrow\infty} \langle M \rangle_t = \infty \right) = 1$ since $\min\{y_1, y_2, \ldots, y_m\} > 0$. Hence, the inverse of $\langle M \rangle_t$ given by

$$\langle M \rangle_t^{-1} = \inf\{s \geq 0 : \langle M \rangle_s > t\},$$

exists for each $t \geq 0$, and $\langle M \rangle$ defines a time-change.

Consider the following processes:

$$G_t := X \circ \langle M \rangle_t^{-1}, \qquad Z_t := Y \circ \langle M \rangle_t^{-1}, \qquad t \geq 0.$$

Using Proposition V.1.4 in [39] we can write, for each $t \geq 0$,

$$X \circ \langle M \rangle_t^{-1} = x + \int_0^{\langle M \rangle_t^{-1}} a(X_s)\, dM_s = x + \int_0^t a(X \circ \langle M \rangle_s^{-1})\, d(M \circ \langle M \rangle_s^{-1}).$$

This yields

$$G_t = x + \int_0^t a(G_s)\, dW_s \qquad t \geq 0,\ P_{x,y} - a.s., \tag{2.4}$$

where $W = M \circ \langle M \rangle^{-1}$ is an $(\mathcal{F}_{\langle M \rangle_t^{-1}})_{t \geq 0}$-Brownian motion by the Dambis-Dubins-Schwarz Theorem (see [39, V.1.6]).

Let us now consider Z. What is the generator of this time-changed process?

We claim that

$$Z \text{ is a Markov chain with } Q\text{-matrix } (y_i^{-2} q[y_i, y_j]). \tag{2.5}$$

Indeed, if we let L denote the infinitesimal generator of Y, then for each bounded and measurable function f and each $y \in \mathcal{S}$ ($x \in \mathbb{R}$ fixed),

$$\begin{aligned} L^Z f(y) &= \lim_{t \downarrow 0} \frac{1}{t} E_{x,y}[\, f(Y \circ \langle M \rangle_t^{-1}) - f(y)] \\ &= \lim_{t \downarrow 0} E_{x,y} \left[\frac{f(Y \circ \langle M \rangle_t^{-1}) - f(y)}{\langle M \rangle_t^{-1}} \; \frac{\langle M \rangle_t^{-1}}{t} \right] \\ &= \lim_{t \downarrow 0} E_{x,y} \left[\frac{1}{\langle M \rangle_t^{-1}} \left(\int_0^{\langle M \rangle_t^{-1}} L\, f(Y_s) ds \right) \frac{\langle M \rangle_t^{-1}}{t} \right] \end{aligned}$$

where the last equality is a consequence of the martingale problem for the continuous-time Markov chain Y.

Given that $d\langle M \rangle_t = Y_t^2\, dt$,

$$\langle M \rangle_t^{-1} = \int_0^{\langle M \rangle_t^{-1}} \frac{1}{Y_s^2}\, d\,\langle M \rangle_s = \int_0^t \frac{1}{(Y \circ \langle M \rangle_s^{-1})^2}\, ds = \int_0^t \frac{ds}{Z_s^2}, \qquad t \geq 0.$$

Then, a simple application of L'Hôpital's rule gives, $P_{x,y}$-a.s.,

$$\frac{1}{\langle M \rangle_t^{-1}} \int_0^{\langle M \rangle_t^{-1}} L\, f(Y_s) ds \quad \overset{t \to 0}{\longrightarrow} \quad L\, f(y), \qquad \text{and} \qquad \frac{\langle M \rangle_t^{-1}}{t} \quad \overset{t \to 0}{\longrightarrow} \quad \frac{1}{y^2}$$

Finally, by the bounded convergence theorem and the above limits, it is plain that $L^Z f(y) = y^{-2} L f(y)$, confirming the statement in (2.5).

2.2.3 Reformulation of the value function

An important part of our approach is to reformulate $v(x, y)$ in (2.1) so that we work on only one probability space, and this is where the coupling method comes into play.

The almost sure equality in (2.4) shows that the constructed processes $G = X \circ \langle M \rangle^{-1}$ and $W = M \circ \langle M \rangle^{-1}$ form a weak solution to the equation $dG = a(G)\, dW$. However, this solution may not be unique. The following condition on the coefficient a is imposed.

C1: We assume that a is a measurable function such that the stochastic differential equation $dG = a(G)\, dW$ driven by a Brownian motion W, has a weakly unique strong Markov solution with state space $\mathbb{R}$.

There are well-known sufficient conditions for **C1** to hold, for instance, it suffices that $a^2(x) > 0$, for all $x \in \mathbb{R}$, and $a^{-2}(\cdot)$ is locally integrable (see Theorem 5.15 in [31], p.341). This includes any non-zero continuous function. An example of a discontinuous function satisfying **C1** is $a(\cdot) = \text{sign}(\cdot)$ where $\text{sign}(x) = 1$ for $x \geq 0$ and $\text{sign}(x) = -1$ for $x < 0$ (see p.73 in [36]).

More generally, Engelbert and Schmidt [12] give necessary and sufficient conditions for **C1** to be verified: consider the sets

$$I(a) = \left\{ x \in \mathbb{R} : \int_{-\epsilon}^{\epsilon} \frac{dy}{a^2(x+y)} = \infty, \ \forall \epsilon > 0 \right\}, \quad N(a) = \{x \in \mathbb{R} : a(x) = 0\}.$$

The claim in **C1** holds if and only if $I(a) = N(a)$ (we refer to Theorem 5.7 in [31] for a proof).

Lemma 2.7 *For given $(x, y, y') \in \mathbb{R} \times \mathcal{S} \times \mathcal{S}$, there is a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ equipped with a filtration $\tilde{\mathcal{F}}_t$, $t \geq 0$, which is big enough to carry four basic processes G, W, Z, Z' such that:*

(i) (G, W) is a (weak) solution of $dG = a(G)dW$ starting from x.

(ii) Z *(resp.* Z'*) is a MC with* Q*-matrix* $(y_i^{-2}q[y_i,y_j])$ *and starting from* y *(resp.* y'*).*

(iii) (G,W) *is independent of* (Z,Z')*.*

(iv) (G,Z,Z') *is a strong Markov process with respect to* $(\tilde{\mathcal{F}}_t)_{t\geq 0}$*.*

Proof. The law of the strong Markov process G in **C1** is entirely determined by its semigroup of transition kernels of G. Multiply these transition kernels and the transition kernels of a continuous-time Markov chain on $\mathcal{S}\times\mathcal{S}$ both marginals of which are determined by the Q-matrix $(y_i^{-2}q[y_i,y_j])$. This results in a semigroup of transition kernels of a strong Markov process (G,Z,Z') with G being independent of (Z,Z').

With the aid of Kolmogorov's Existence Theorem, we choose a complete probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ such that (G,Z,Z') starts from fixed (x,y,y') in $\mathbb{R}\times\mathcal{S}\times\mathcal{S}$ (see [1, Th. 3.1.7] for instance). Let $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ denote the augmentation of the natural filtration of the triplet (G,Z,Z').

By the well-posedness of the martingale problem associated with the strong Markov process G, with $L\,f(x)=\frac{1}{2}a^2(x)\frac{\partial^2 f}{\partial x^2}$ denoting its infinitesimal generator acting on functions $f\in C^2(\mathbb{R})$, we have that

$$f(G_t)-f(G_0)-\int_0^t L\,f(G_s)ds \qquad \text{is an } (\tilde{\mathcal{F}}_t)_{t\geq 0}\text{- local martingale.}$$

In particular, setting $f(x)=x$, we obtain that G_t-x is a continuous $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-local martingale with quadratic variation $\int_0^\cdot a(G_s)^2ds$. Thus, by a well-known result going back to Doob (see [26, Th. II 7.1'] for example), there is a Brownian motion W on $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ (or on a canonical enlargement of it[1]) such that

$$G_t-x=\int_0^t a(G_s)dW_s, \quad t\geq 0,\ \tilde{P}-a.s. \tag{2.6}$$

The construction of W (as given in the proof of Th. II 7.1' in [26]) shows that the pair (G,W) is also independent of (Z,Z'). □

[1]Our convention is to use $(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{P})$ for the enlarged space, too.

Remark 2.8 The Brownian motion W in (2.6) might only be a Brownian motion with respect to a filtration $(\tilde{\mathcal{G}}_t)_{t\geq 0}$ larger than $(\tilde{\mathcal{F}}_t)_{t\geq 0}$. Then the stochastic integral in (2.6) can only be understood with respect to the larger filtration. However, in the sequel, it is only relevant to consider the filtration $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ because we are solely interested in the strong Markov property of (G, Z, Z'). The latter is used in the proof of Theorem 2.6 to deduce that, for a specific coupling of the chain (Z, Z'), the marginals (G, Z) and (G, Z') are also strong Markov with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$.

In the remainder of this section we *revert* to the original system in (2.2) but now on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. The procedure resembles (in fact, inverts) the one of the previous section.

Let G, W, Z, Z' be given on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ as described in Lemma 2.7 and define $\Gamma = (\Gamma_t)_{t\geq 0}$ by

$$\Gamma_t = \int_0^t Z_s^{-2}\, ds, \qquad t \geq 0. \tag{2.7}$$

This process is continuous and strictly increasing since Z only takes non-zero values. Moreover, it has the property that

$$\Gamma_t < \infty,\ t \geq 0,\ a.s., \qquad \text{and} \qquad \lim_{t\uparrow\infty} \Gamma_t = \infty \ \ a.s., \tag{2.8}$$

Thus A, the inverse of Γ, given by

$$A_t = \inf\{s \geq 0 :\ \Gamma_s > t\}, \qquad t \geq 0, \tag{2.9}$$

is also a continuous and strictly increasing process satisfying

$$A_t < \infty,\ t \geq 0,\ a.s., \qquad \text{and} \qquad \lim_{t\uparrow\infty} A_t = \infty \ \ a.s. \tag{2.10}$$

As a consequence, the two technical properties

P1: $\Gamma_{A_t} = A_{\Gamma_t} = t$ for all $t \geq 0$ a.s.

P2: $s < \Gamma_t$ if and only if $A_s < t$ for all $0 \leq s, t < \infty$ a.s.

must hold (see Corollary 2.3 above).

We can rewrite (2.6) to get

$$G_t = x + \int_0^t a(G_s) Z_s d\tilde{M}_s, \quad t \geq 0, \, a.s.$$

where the stochastic integral $\tilde{M}_s = \int_0^s \frac{dW_u}{Z_u}$ exists by (2.8), for each $s \geq 0$.

The inverse of $\langle \tilde{M} \rangle$, A, defines a time-change. Consider the processes

$$\tilde{X}_t := G \circ A_t, \quad \tilde{Y}_t := Z \circ A_t, \quad \tilde{B}_t := \tilde{M} \circ A_t, \qquad t \geq 0.$$

Arguing as before to obtain (2.4)-(2.5),

$$\tilde{X}_t = x + \int_0^t a(\tilde{X}_s) \tilde{Y}_s \, d\tilde{B}_s, \qquad t \geq 0, \, a.s., \tag{2.11}$$

where $\tilde{B}$ is a Brownian motion[2] by the Dambis-Dubins-Schwarz Theorem [39, V.1.6], and also

$$\tilde{Y} \text{ is a Markov chain with } Q\text{-matrix } (q[y_i, y_j]). \tag{2.12}$$

Moreover, $\tilde{B}$ and $\tilde{Y}$ are independent since W and Z are independent (see Proposition 2.5 with $f(z) = 1/z$).

The constructed processes $((\tilde{X}, \tilde{Y}), \tilde{B})$ give a weak solution to (2.2) starting from (x, y).

Remark 2.9 We can repeat the same constructions introduced for $(\tilde{X}, \tilde{Y})$ using Z, but now with Z'. We identify these objects with an apostrophe. The resulting pair $((\tilde{X}', \tilde{Y}'), \tilde{B})$, gives a weak solution to (2.2) starting from (x, y'). Notice that the process G is the same in the definition of both $\tilde{X}$ and $\tilde{X}'$. This fact is key in the rest of the arguments below.

In the remainder of this section we give an alternative expression for the value function in (2.1). We do this for $(\tilde{X}, \tilde{Y})$ only, because similar arguments will follow for $(\tilde{X}', \tilde{Y}')$.

The following lemma considers $\{\Gamma_t; t \geq 0\}$ and $\{A_t; t \geq 0\}$ as families of

[2] $\tilde{B}$ is actually an $(\tilde{\mathcal{G}}_{A_t})_{t \geq 0}$ Brownian motion, see Remark 2.8.

stopping times and is a consequence of their symmetric roles. Consider the families $\mathcal{M}$ and $\mathcal{T}$:

$$\mathcal{M} = \{\,\text{finite stopping times with respect to } (\tilde{\mathcal{F}}_t)_{t\geq 0}\}$$

and

$$\mathcal{T} = \{\,\text{finite stopping times with respect to } (\tilde{\mathcal{F}}_{A_t})_{t\geq 0}\}.$$

Lemma 2.10 *For any $T \in [0,\infty]$,*

$$\sup_{\tau\in\mathcal{T}_T} \tilde{E}\,[e^{-\alpha\tau} g(\tilde{X}_\tau)] = \sup_{\rho\in\mathcal{M}_T} \tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho)], \tag{2.13}$$

where $\mathcal{M}_T = \{\rho \in \mathcal{M} : 0 \leq \rho \leq A_T \text{ a.s.}\}$ and $\mathcal{T}_T = \{\tau \in \mathcal{T} : 0 \leq \tau \leq T \text{ a.s.}\}$.

Proof. Fix $\tau \in \mathcal{T}_T$ and observe that $\tilde{E}\,[e^{-\alpha\tau} g(\tilde{X}_\tau)] = \tilde{E}\,[e^{-\alpha\Gamma_{A_\tau}} g(G_{A_\tau})]$ by Property **P1**. Also A_τ is in $\mathcal{M}$ by Lemma 2.4 and by the increasing property of A_t as a function of t, we have that $0 \leq A_\tau \leq A_T$ a.s. Hence $A_\tau \in \mathcal{M}_T$ and

$$\tilde{E}\,[e^{-\alpha\tau} g(\tilde{X}_\tau)] \leq \sup_{\rho\in\mathcal{M}_T} \tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho)], \quad \forall\,\tau \in \mathcal{T}_T.$$

Similarly, given $\rho \in \mathcal{M}_T$, the equality $\tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho)] = \tilde{E}\,[e^{-\alpha\Gamma_\rho} g(\tilde{X}_{\Gamma_\rho})]$ and the fact that $\Gamma_\rho \in \mathcal{T}_T$ lead to

$$\tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho)] \leq \sup_{\tau\in\mathcal{T}_T} \tilde{E}\,[e^{-\alpha\tau} g(\tilde{X}_\tau)], \quad \forall\,\rho \in \mathcal{M}_T.$$

The proof is complete. □

Since the equation $X_t = x + \int_0^t a(X_s)Y_s\,dB_s$ admits a weakly unique solution, we have that

$$\underbrace{(\tilde{X},\tilde{Y}) = (G\circ A, Z\circ A)}_{\text{under } \tilde{P}} \overset{law}{=} \underbrace{(X,Y).}_{\text{under } P_{x,y}}$$

As a consequence, we obtain that

$$v(x,y) = \sup_{0\leq\tilde{\tau}\leq T} \tilde{E}\,[e^{-\alpha\tilde{\tau}} g(\tilde{X}_{\tilde{\tau}})], \tag{2.14}$$

where the stopping times $\tilde{\tau}$ are with respect to the filtration generated by $(\tilde{X}, \tilde{Y})$.

Proposition 2.11 *Assume that (G, Z) is a strong Markov process with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$. Then, for any $T \in [0, \infty]$,*

$$v(x, y) = \sup_{\rho \in \mathcal{M}_T} \tilde{E}\,[e^{-\alpha \Gamma_\rho} g(G_\rho)]. \tag{2.15}$$

Proof. Since (G, Z) is a strong Markov process with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$, the time-changed process $(\tilde{X}, \tilde{Y})$ must possess the strong Markov property with respect to $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$ (see Theorem 65.9 in [42]).

The stopping times used in (2.14) are with respect to the filtration generated by $(\tilde{X}, \tilde{Y})$ which might be smaller than $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$. However, given that $(\tilde{X}, \tilde{Y})$ is also strong Markov with respect to $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$, the corresponding suprema are the same (see Proposition B.2). In other words,

$$v(x, y) = \sup_{0 \leq \tau \leq T} \tilde{E}\,[e^{-\alpha \tau} g(\tilde{X}_\tau)]$$

where the finite stopping times τ are with respect to the filtration $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$.

The conclusion follows directly from Lemma 2.10. □

Of course, all the results above remain valid for $v(x, y')$, $\tilde{X}'$, $\tilde{Y}'$, $\mathcal{T}'_T$, $\mathcal{M}'_T$, A' and Γ' if these objects are constructed by using Z' instead of Z. Then the conclusion of Proposition 2.11 holds for $v(x, y')$ provided (G, Z') is a strong Markov process with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$.

2.2.4 Monotonicity in y

Proof of Theorem 2.6. Fix $x \in \mathbb{R}$ and $y, y' \in \mathcal{S}$ such that $y \leq y'$. We split the proof into two parts.

(i) While in Lemma 2.7 the coupling of the two chains Z and Z' was not specified any further, we now choose a particular coupling associated with a Q-matrix $\mathfrak{Q}$ which allows us to compare Z and Z' directly.

Denoting the Q-matrix corresponding to the independence coupling by $\mathfrak{Q}^{\perp}$, we set

$$\mathfrak{Q}\left[\begin{array}{c|c} y_i & y_j \\ y_k & y_l \end{array}\right] = \begin{cases} \mathfrak{Q}^{\perp}\left[\begin{array}{c|c} y_i & y_j \\ y_k & y_l \end{array}\right] & : \ i \neq k \\ y_i^{-2}\, q[y_i, y_j] & : \ i = k,\ j = l \\ 0 & : \ i = k,\ j \neq l \end{cases}$$

for $y_i, y_j, y_k, y_l \in \mathcal{S}$. That is, Z and Z' move independently until they hit each other for the first time and then they move together. It follows from the skip-free-assumption that Z cannot overtake Z' before they hit each other for the first time. Hence

$$Z_0 = y \leq y' = Z_0' \quad \text{implies} \quad Z_t \leq Z_t',\, t \geq 0,\, a.s., \tag{2.16}$$

which results in the inequality

$$\Gamma_t \;=\; \int_0^t Z_s^{-2} ds \;\geq\; \int_0^t (Z_s')^{-2} ds \;=\; \Gamma_t', \quad t \geq 0,\, a.s. \tag{2.17}$$

As a consequence, we also have that the inverse (increasing) processes $A = \Gamma^{-1}$ and $A' = (\Gamma')^{-1}$ must satisfy the relation $A_t \leq A_t'$, $t \geq 0$, a.s.

(ii) Notice that the above comparison and the fact that g is non-negative allow us to conclude that

$$\tilde{E}\,[e^{-\alpha \Gamma_\rho} g(G_\rho)] \;\leq\; \tilde{E}\,[e^{-\alpha \Gamma'_\rho} g(G_\rho)] \quad \text{for every } \rho \in \mathcal{M}_T. \tag{2.18}$$

Since $A_T \leq A_T'$ implies that $\mathcal{M}_T \subseteq \mathcal{M}_T'$, we obtain that

$$\sup_{\rho \in \mathcal{M}_T} \tilde{E}\,[e^{-\alpha \Gamma_\rho} g(G_\rho)] \;\leq\; \sup_{\rho' \in \mathcal{M}_T'} \tilde{E}\,[e^{-\alpha \Gamma'_{\rho'}} g(G_{\rho'})]$$

or equivalently, that $v(x, y) \leq v(x, y')$ thanks to Proposition 2.11, provided (G, Z) and (G, Z') are strong Markov processes with respect to $(\tilde{\mathcal{F}}_t)_{t \geq 0}$, the augmentation of the filtration generated by (G, Z, Z'). Let's check this property for (G, Z) only, as the same argument follows for (G, Z').

Given a bounded and measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ and a stopping time

τ with respect to $\tilde{\mathcal{F}}_t$, first note that

$$\tilde{E}[\, f(G_{\tau+t}, Z_{\tau+t}) \mid \tilde{\mathcal{F}}_\tau \,] = \tilde{E}[\, f(G_{\tau+t}, Z_{\tau+t}) \mid G_\tau, Z_\tau, Z'_\tau \,],$$

by the strong Markov property of the triplet (G, Z, Z') (simply set $\bar{f}(x, z, z') = f(x, z)$). Now, Z'_τ is either independent or equal to Z_τ, depending on whether τ is smaller or greater than the coupling time of the chains, respectively. Formally, if $C = \inf\{t \geq 0 : \; Z_t \geq Z'_t\}$ is the coupling time, then

$$\tilde{E}[\, f(G_{\tau+t}, Z_{\tau+t}) \mid G_\tau, Z_\tau, Z'_\tau, \{\tau < C\} \,] = \tilde{E}[\, f(G_{\tau+t}, Z_{\tau+t}) \mid G_\tau, Z_\tau, \{\tau < C\} \,]$$

by the independence of Z'_τ and the pair G_τ, Z_τ on $\{\tau < C\}$ (see for instance Section 9.7 in [46]); and

$$\tilde{E}[\, f(G_{\tau+t}, Z_{\tau+t}) \mid G_\tau, Z_\tau, Z'_\tau, \{\tau \geq C\} \,] = \tilde{E}[\, f(G_{\tau+t}, Z_{\tau+t}) \mid G_\tau, Z_\tau, \{\tau \geq C\} \,]$$

because $Z_\tau = Z'_\tau$ on $\{\tau \geq C\}$. We conclude that

$$\tilde{E}[\, f(G_{\tau+t}, Z_{\tau+t}) \mid G_\tau, Z_\tau, Z'_\tau \,] = \tilde{E}[\, f(G_{\tau+t}, Z_{\tau+t}) \mid G_\tau, Z_\tau \,]$$

as required. The proof is now complete. □.

The inequalities in (2.16), (2.17) and (2.18) are fundamental. With these inequalities in mind, we can obtain some variants of Theorem 2.6 to include the case where g is possibly negative in the infinite horizon case.

Corollary 2.12 *Let Condition* **C1** *on page 13 be satisfied and the time horizon $T = \infty$. Assume that g is non-positive, and that Y is skip-free. Then, for each $x \in \mathbb{R}$, the function $v(x, \cdot)$ is non-increasing on $\mathcal{S}$.*

Proof. Fix $x \in \mathbb{R}$ and $y \leq y'$. If g is non-positive then, instead of (2.18), we obtain

$$\tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho)] \;\geq\; \tilde{E}\,[e^{-\alpha\Gamma'_\rho} g(G_\rho)] \quad \text{for every } \rho \in \mathcal{M}_T \tag{2.19}$$

with $\mathcal{M}_T$ equals (up to versions) to $\mathcal{M}'_T$ since $T = \infty$. Hence the inequality $v(x, y) \geq v(x, y')$ can be deduced directly from Proposition 2.11. □

Notice that, in the case when $T < \infty$, there can be stopping times in $\mathcal{M}'_T$

which are not in $\mathcal{M}_T$ and so we could not deduce the monotonicity of v from the inequality in (2.19).

Let us now suppose that g takes both positive and negative values. In this situation, the conclusion of Theorem 2.6 remains valid provided the optimum $v(x, y)$, for (x, y) fixed, can be achieved by stopping at non-negative values of g only. Define $\mathcal{K}^{g+}_{x,y}$ to be the collection of all finite stopping times τ with respect to the filtration generated by (X, Y) with $(X_0, Y_0) = (x, y)$ and such that $g(X_\tau) \geq 0$.

Corollary 2.13 *Let Condition* **C1** *on page 13 be satisfied and the time horizon $T = \infty$. Assume that the gain function g is such that $\{x : g(x) \geq 0\} \neq \emptyset$, and that Y is skip-free. Further assume that, for (x, y) fixed,*

$$v(x, y) \;=\; \sup_{\tau \in \mathcal{K}^{g+}} E_{x,y}\,[e^{-\alpha\tau} g(X_\tau)], \tag{2.20}$$

where $\mathcal{K}^{g+} \equiv \mathcal{K}^{g+}_{x,y}$. Then, $v(x, y) \leq v(x, y')$ for all $y' \in \mathcal{S}$ such that $y \leq y'$.

Proof. Let $y' \in \mathcal{S}$ be such that $y \leq y'$.

Following the proof and notation of Theorem 2.6, part (i) remains valid (as it does not involve the payoff function g) so that $\Gamma_t \geq \Gamma'_t$ for all $t \geq 0$ a.s. Part (ii) of that proof is replaced by the next considerations. Define $\mathcal{M}^+ := \{\rho \in \mathcal{M} : g(G_\rho) \geq 0 \ a.s.\}$ and $\mathcal{T}^+ := \{\tau \in \mathcal{T} : g(\tilde{X}_\tau) \geq 0 \ a.s.\}$. To complete the proof it is sufficient to see that

$$v(x, y) \;=\; \sup_{\rho \in \mathcal{M}^+} \tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho)] \tag{2.21}$$

and that

$$\sup_{\rho \in \mathcal{M}^+} \tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho)] \;\leq\; \sup_{\rho \in \mathcal{M}} \tilde{E}\,[e^{-\alpha\Gamma'_\rho} g(G_\rho)] = v(x, y'), \tag{2.22}$$

where the equality on the right-hand side is due to Proposition 2.11.

To see (2.21) (cf. (2.15) above), first notice that the equality

$$\sup_{\tau \in \mathcal{T}^+} \tilde{E}\,[e^{-\alpha\tau} g(\tilde{X}_\tau)] = \sup_{\rho \in \mathcal{M}^+} \tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho)]$$

can be shown in exactly the same way that (2.13). Furthermore, under the condition $v(x,y) = \sup_{\tau\in\mathcal{K}^{g+}} E_{x,y}\,[e^{-\alpha\tau}g(X_\tau)]$, we must have that

$$v(x,y) \;=\; \sup_{\tau\in\mathcal{T}^+} \tilde{E}\,[e^{-\alpha\tau}g(\tilde{X}_\tau)]$$

because the law of $(\tilde{X},\tilde{Y})$ is equal to the law of (X,Y) under $P_{x,y}$, and $(\tilde{X},\tilde{Y})$ is strong Markov with respect to the filtration $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$.

To see the inequality in (2.22), just observe that (cf. (2.18))

$$\tilde{E}\,[e^{-\alpha\Gamma_\rho}g(G_\rho)] \;\leq\; \tilde{E}\,[e^{-\alpha\Gamma'_\rho}g(G_\rho)] \quad \text{for every } \rho\in\mathcal{M}^+$$

and that $\mathcal{M}^+ \subseteq \mathcal{M}$, which in turn implies (2.22). □

Remark 2.14 It is intuitively clear that all of our results will not change if the dynamics of X are instead

$$X_t \;=\; x + \int_0^t a(X_s)\,\sigma(Y_s)\,dB_s, \quad t\geq 0,\; P_{x,y} - a.s. \tag{2.23}$$

where $\sigma(\cdot) > 0$ and provided σ preserves order, in the sense that, for any $y_1 \leq y_2$, $\sigma(y_1) \leq \sigma(y_2)$. Of course, if for any $y_1 \leq y_2$, $\sigma(y_1) \geq \sigma(y_2)$, then the statements of the results change in order: *increasing* becomes *decreasing* and vice-versa.

Remark 2.15 The condition (2.20) is a technical one which allows us to get (2.22). A sufficient condition for (2.20) to hold is that

$$P_{x,y}\left(\,\inf\{t\geq 0 : g(X_t)\geq 0\} < \infty\,\right) \;=\; 1 \tag{2.24}$$

and it follows from the strong Markov property of (X,Y) (we are assuming that the time horizon is infinite). Indeed, if the process X always hits the set $\{x : g(x)\geq 0\}$ with probability one then it is quite natural that maximal gain is obtained whilst avoiding stopping at negative values of g. It is clear that $v(x,y) \geq \sup_{\tau\in\mathcal{K}^{g+}} E_{x,y}\,[e^{-\alpha\tau}g(X_\tau)]$. To show the reverse inequality, it suffices to check that for each finite stopping time τ we can find $\tau'\in\mathcal{K}^{g+}$ such that

$$E_{x,y}\,[e^{-\alpha\tau}g(X_\tau)] \;\leq\; E_{x,y}\,[e^{-\alpha\tau'}g(X_{\tau'})]. \tag{2.25}$$

Fix a finite stopping time τ and consider the event $B = \{\omega : g(X_{\tau(\omega)}(\omega)) \geq 0\ \}$. Also, define the stopping times

$$\tau_B(\omega) := \begin{cases} \tau(\omega) : \ \omega \in B, \\ +\infty : \ \omega \notin B, \end{cases} \qquad \text{and} \quad \tau_H := \inf\{t \geq \tau : \ g(X_t) \geq 0\},$$

and set $\tau' = \tau_B \wedge \tau_H$. By the strong Markov property and (2.24), $\tau_H < \infty$ and $g(X_{\tau'}) \geq 0$ $P_{x,y}$-a.s. so that $\tau' \in \mathcal{K}^{g+}$. Moreover, by definition of τ', it follows that $e^{-\alpha\tau}g(X_\tau) \leq e^{-\alpha\tau'}g(X_{\tau'})$ $P_{x,y}$-a.s., which in turn implies the desired inequality in (2.25) after taking expectation. In the case where $T < \infty$, it is very difficult to guarantee that X will hit a subset of the state space in finite time.

Consider the following example. Suppose that $a(x) = x$ in (2.2) so that

$$X_t = x \exp\left\{\int_0^t Y_s dB_s - \frac{1}{2}\int_0^t Y_s^2 ds\right\}, \quad t \geq 0,$$

and the payoff function g satisfies that the set $\{x : \ g(x) \geq 0\}$ is of the form $\{x : \ x \leq x^*\}$ for some $x^* > 0$ (for instance, $g(x) = (K - x)^+$ with $K > 0$ and $g(x) = e^{-2(x-2)} - 1$ for which $x^* = K$ and $x^* = 2$, respectively). In this situation, the condition (2.24) is satisfied. If $x \leq x^*$ this is clear, so we only have to verify the case where $x > x^*$. Let us fix (x, y) such that $x > x^*$ and y is an arbitrary point in $\mathcal{S}$. We know that the local martingale $M_t = \int_0^t Y_s dB_s$ is a time-changed Brownian motion. More precisely, there is a Brownian motion W such that $M_t = W_{A_t}$ where $A_t = \langle M \rangle_t$ (see [39, V.1.6]). Then the laws of X_t and $x \exp\left\{W_{A_t} - \frac{1}{2}A_t\right\}$ coincide. Given that $A_t \to \infty$ as $t \to \infty$, it follows that $X_t \to 0$ as $t \to \infty$ almost surely. Therefore, if the initial value $x > x^*$, X_t hits the set $\{x : \ x \leq x^*\}$ in finite time almost surely and (2.24) holds.

Now, the condition (2.24) is sufficient but not necessary as we next see. Suppose that X is as in the previous paragraph taking values in the positive half line, and g is given by $g(x) = -1$ if $x < 1$ and $g(x) = 1$ if $x \geq 1$. Let $0 < x < 1$ and observe that

$$x \exp\left\{W_{A_t} - \frac{1}{2}A_t\right\} \geq 1 \quad \Leftrightarrow \quad W_{A_t} - \frac{1}{2}A_t \geq \log(1/x).$$

Since $A_t \to \infty$ as $t \to \infty$ almost surely, we have that (see [6, p.251] for instance)

$$P_{x,y}\left(\sup_{t\geq 0}\left(W_{A_t} - \frac{A_t}{2}\right) \geq \log(1/x)\right) = P_{x,y}\left(\sup_{t\geq 0}\left(W_t - \frac{t}{2}\right) \geq \log(1/x)\right) = x < 1.$$

Setting $\tau_1 := \inf\{t \geq 0 : X_t \geq 1\}$ and using that X_t and $x\exp\left\{W_{A_t} - \frac{1}{2}A_t\right\}$ have the same law, we have that for $0 < x < 1$,

$$P_{x,y}(\,\tau_1 < \infty\,) = P_{x,y}\left(\sup_{t\geq 0}\left(W_{A_t} - \frac{A_t}{2}\right) \geq \log(1/x)\right) = x < 1,$$

and so condition (2.24) fails. However, for such a pair (x, y) with $0 < x < 1$ and y arbitrary, (2.20) holds. Indeed, given that for any (x', y')

$$v(x', y') \geq E_{x',y'}e^{-\alpha\tau_1}g(X_{\tau_1}) = g(1)\,E_{x',y'}I(\tau_1 < \infty) = P_{x',y'}(\tau_1 < \infty) > 0,$$

it is suboptimal to stop the observation of X_t when $X_t \in (0, 1)$. In other words, the maximal gain can only be obtained when we avoid to stop X at the set of negative values of g.

2.3 The diffusion case

2.3.1 Introduction

Fix $\delta \in [-1, 1]$. We assume that the strong Markov process (X, Y) satisfies the following. For every $(x, y) \in \mathbb{R} \times \mathcal{S}$ ($\mathcal{S} \subseteq (0, \infty)$), there is a pair (B, B^Y) of Brownian motions on $(\Omega, \mathcal{F}, P_{x,y})$, adapted to $(\mathcal{F}_t)_{t\geq 0}$, with covariation $\langle B, B^Y\rangle_t = \delta t$, $t \geq 0$, and such that

$$X_t \;=\; x + \int_0^t a(X_s)Y_s\,dB_s, \quad Y_t \;=\; y + \int_0^t \eta(Y_s)dB_s^Y + \int_0^t \theta(Y_s)ds \qquad (2.26)$$

for all $t \geq 0$, $P_{x,y} - a.s.$, where a, η, θ are continuous functions on $\mathcal{S}$.

We assume that the system in (2.26), with (X, Y) unknown, admits a weakly unique non-exploding solution.

It is appropriate to state a preliminary result, which is the analogue of Theorem 2.6. Recall that $v(x, y)$ is given in (2.1) above and the payoff function g is a measurable function satisfying (1.3).

Theorem 2.16 *Let Conditions* **C1'-C2'** *on page 26 be satisfied. Assume that the gain function* g *is non-negative. Then, for each* $x \in \mathbb{R}$, $v(x, \cdot)$ *is non-decreasing on* $\mathcal{S}$.

The proof of this and a more general version of it, Theorem 2.22, are presented in Section 2.3.4.

Since the volatility process Y has continuous paths, it is natural to ask to what extent continuity of $v(x, \cdot)$ holds. In Section 2.3.5 we further exploit the time-change and coupling techniques to address this question. The next theorem summarizes the results in that section, specifically part (i) (resp. (ii)) corresponds to Proposition 2.29 (resp. Proposition 2.30).

Theorem 2.17 *Let Conditions* **C1'-C2'** *on page 26 be satisfied. Then, for each* $x \in \mathbb{R}$, *the following assertions hold:*

(i) If $T = \infty$ *and (2.49) is satisfied, then* $v(x, \cdot)$ *is continuous.*

(ii) If $T < \infty$ *and* g *is continuous, then* $v(x, \cdot)$ *is continuous.*

2.3.2 Heuristics: the time-changed dynamics

Fix $(x, y) \in \mathbb{R} \times \mathcal{S}$.

Similarly as in Section 2.2.2, we consider the stochastic integral $M_s = \int_0^s Y_u \, dB_u$ which is well-defined because the paths of Y are continuous and so $\int_0^s Y_u^2 du < \infty$, $P_{x,y}$ - a.s., for all $s \geq 0$.

Assume for now that the quadratic variation $\langle M \rangle_\cdot$ satisfies the property that $P_{x,y}\left(\lim_{t\uparrow\infty} \langle M \rangle_t = \infty\right) = 1$. Then, the inverse of $\langle M \rangle_t$ exists for each $t \geq 0$.

Consider the time-change of the pair (X, Y) by the inverse of $\langle M \rangle$:

$$G_t := X \circ \langle M \rangle_t^{-1}, \qquad \xi_t := Y \circ \langle M \rangle_t^{-1}, \qquad t \geq 0.$$

Using Proposition V.1.4 in [39], we can write

$$G_t = x + \int_0^t a(G_s)\, dW_s, \qquad \xi_t = y + \int_0^t \eta(\xi_s)\xi_s^{-1} dW_s^\xi + \int_0^t \theta(\xi_s)\xi_s^{-2} ds,$$

for each $t \geq 0$, where

$$W_t = M \circ \langle M \rangle_t^{-1}, \quad \text{and} \quad W_t^\xi = \int_0^{\langle M \rangle_t^{-1}} Y_s\, dB_s^Y, \qquad t \geq 0,$$

are $(\mathcal{F}_{\langle M \rangle_t^{-1}})_{t \geq 0}$-Brownian motions by the Dambis-Dubins-Schwarz Theorem (see [39, V.1.6]).

The covariation $\langle W, W^\xi \rangle_t$ can be calculated as follows. First, by a property of stochastic integrals (see [39, IV.2.7]) and using that $\langle B, B^Y \rangle_t = \delta t$, we have that

$$\langle \int_0^\cdot Y_s\, dB_s, \int_0^\cdot Y_s\, dB_s^Y \rangle_t = \delta \int_0^t Y_s^2 d(\langle B, B^Y \rangle)_s = \delta \langle M \rangle_t.$$

Hence,

$$\langle W, W^\xi \rangle_t = \langle \int_0^\cdot Y_s\, dB_s, \int_0^\cdot Y_s\, dB_s^Y \rangle \circ \langle M \rangle_t^{-1} = \delta t.$$

2.3.3 Reformulation of the value function

In the previous section we showed that, for every $(x, y) \in \mathbb{R} \times \mathcal{S}$, there exists a weak solution to the system of stochastic differential equations

$$\begin{aligned} dG_t &= a(G_t) dW, \\ d\xi_t &= \eta(\xi_t)\xi_t^{-1} dW_t^\xi + \theta(\xi_t)\xi_t^{-2} dt, \end{aligned} \tag{2.27}$$

taking values in $\mathbb{R} \times \mathcal{S}$ and driven by a pair of Brownian motions with covariation $\langle W, W^\xi \rangle_t = \delta t$, $t \geq 0$. However, this solution may not be unique. The following condition is imposed.

C1': We assume that the continuous functions a, η, θ are such that the system in (2.27) has, for all initial conditions $(G_0, \xi_0) \in \mathbb{R} \times \mathcal{S}$, a unique non-exploding strong solution taking values in $\mathbb{R} \times \mathcal{S}$.

The continuity of the functions a, η, θ ensures the existence of a weak solution (possibly exploding). Sufficient conditions for **C1'** to hold (see e.g.

Theorem 3.1 in [26]) typically require that the coefficients of (2.27) satisfy a Lipschitz condition (for pathwise uniqueness of the solution) and a linear growth condition (for non-explosion of the solution). These conditions may be weakened in some particular cases. For example, when $a(x) = x$ (which will be assumed in Chapter 4), we only need to make sure that there is a unique non-exploding strong solution for the autonomous equation for ξ. Since ξ is one-dimensional, there are sufficient conditions for the preceding to hold which weaken the Lipschitz condition (see for instance [26, V.3.2]).

Remark 2.18 We want to show (2.3) in this context using a similar method to that applied in Section 2.3.4. The main difference is that here, instead of constructing $\tilde{X}$ by time-changing a solution of the single equation $dG = a(G)dW$, we now construct $\tilde{X}$ by time-changing a solution of the system in (2.27).

Now, we choose a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ big enough to carry a pair of Brownian motions (W, W^ξ) with covariation $\langle W, W^\xi \rangle_t = \delta t$. Denote by $(\tilde{\mathcal{F}}_t)_{t\geq 0}$, the augmentation of the filtration generated by (W, W^ξ).

Suppose that (G, ξ) is the unique solution of the system (2.27) given on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ by (W, W^ξ) and starting from $(G_0, \xi_0) = (x, y)$ in $\mathbb{R} \times \mathcal{S}$. Define $\Gamma = (\Gamma_t)_{t\geq 0}$ and $A = (A_t)_{t\geq 0}$ (compare to (2.7) and (2.9), resp.) by

$$\Gamma_t = \int_0^t \xi_s^{-2} ds, \quad t \geq 0, \quad \text{and} \quad A_t = \inf\{s \geq 0 : \Gamma_s > t\}.$$

These processes are continuous and strictly increasing since ξ does not hit zero (recall that the state space of ξ is assumed to be $\mathcal{S} \subseteq (0, \infty)$).

Assume the following:

C2': For each initial conditions $(x, y) \in \mathbb{R} \times \mathcal{S}$, the associated process Γ satisfies that $\tilde{P}\left(\lim_{t\uparrow\infty} \Gamma_t = \infty\right) = 1$.

Under **C2'**, the properties in (2.8) and (2.10) still hold in this context. As a consequence, the two technical Properties **P1** and **P2** on page 15 must also be valid.

Since ξ is $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-adapted, we see that

$$G_t = x + \int_0^t a(G_s)\xi_s \, d\tilde{M}_s, \tag{2.28}$$

$$\xi_t = y + \int_0^t \eta(\xi_s) d\tilde{M}_s^\xi + \int_0^t \theta(\xi_s) d\Gamma_s, \tag{2.29}$$

where the continuous local martingales $\tilde{M}$ and $\tilde{M}^\xi$ given by the stochastic integrals

$$\tilde{M}_s = \int_0^s \xi_u^{-1} dW_u \quad \text{and} \quad \tilde{M}_s^\xi = \int_0^s \xi_u^{-1} dW_u^\xi$$

exist for each $s \geq 0$ by (2.8).

Now consider the $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$ - adapted processes

$$\tilde{X} := G \circ A, \quad \tilde{Y} := \xi \circ A, \quad \tilde{B}_t := \tilde{M} \circ A, \quad \tilde{B}^Y = \tilde{M}^\xi \circ A, \qquad t \geq 0.$$

We have that $\tilde{B}$ and $\tilde{B}^Y$ are $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$ - Brownian motions by the Dambis-Dubins-Schwarz Theorem [39, V.1.6] and that

$$\langle \tilde{B}, \tilde{B}^Y \rangle_t = \langle \tilde{M}, \tilde{M}^\xi \rangle_{A_t} = \int_0^{A_t} \xi_u^{-2} d(\langle W, W^\xi \rangle)_u = \delta \Gamma_{A_t} = \delta t, \quad t \geq 0, \, a.s.,$$

by Property **P1** on page 15.

It follows from (2.28)-(2.29) that $(\tilde{X}, \tilde{Y})$ constitutes a non-exploding weak solution of the system (2.26) with $\tilde{Y}_t \in \mathcal{S}$, $t \geq 0$. Hence

$$\underbrace{(\tilde{X}, \tilde{Y}) = (G \circ A, \xi \circ A)}_{\text{under } \tilde{P}} \overset{law}{=} \underbrace{(X, Y)}_{\text{under } P_{x,y}}.$$

As a consequence, we obtain that

$$v(x, y) = \sup_{0 \leq \tilde{\tau} \leq T} \tilde{E}\,[e^{-\alpha \tilde{\tau}} g(\tilde{X}_{\tilde{\tau}})], \tag{2.30}$$

where the stopping times $\tilde{\tau}$ are with respect to the filtration generated by $(\tilde{X}, \tilde{Y})$.

Remark 2.19 Due to **C2'** and the fact that the system (2.26) has weakly unique solutions, we have that $P_{x,y}\left(\lim_{t\uparrow\infty}\int_0^t Y_s^2 ds = \infty\right) = 1$, which was assumed in Section 2.3.2. Indeed, given that $\tilde{Y} = \xi \circ A$ solves weakly the second equation of (2.26), it follows that the law of Y under $P_{x,y}$ is the same as the law of $\tilde{Y}$ under $\tilde{P}$. Thus, for each $t \geq 0$,

$$\int_0^t Y_s^2 ds \overset{\text{law}}{=} \int_0^t (\tilde{Y}_s)^2 ds = A_t.$$

In other words, $\int_0^\cdot Y_s^2 ds$ and $A_\cdot$ are modifications of each other. Moreover, both have continuous paths and so they must be indistinguishable. Therefore, $\lim_{t\to\infty} A_t = \infty$ a.s. implies $P_{x,y}\left(\lim_{t\uparrow\infty}\int_0^t Y_s^2 ds = \infty\right) = 1$.

Analogously to the families $\mathcal{M}$ and $\mathcal{T}$ of Section 2.2.3, we define here

$$\mathcal{M} = \{\text{ finite stopping times with respect to } (\tilde{\mathcal{F}}_t)_{t\geq 0}\ a.s.\},$$

$$\mathcal{T} = \{\text{ finite stopping times with respect to } (\tilde{\mathcal{F}}_{A_t})_{t\geq 0}\ a.s.\},$$

and $\mathcal{M}_T = \{\rho \in \mathcal{M} : 0 \leq \rho \leq A_T\}$, $\mathcal{T}_T = \{\tau \in \mathcal{T} : 0 \leq \tau \leq T\}$, for each $T \in [0, \infty]$.

Remark 2.20 With these definitions, Lemma 2.10 remains valid in this setting, but notice that the filtrations $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$ are of a different nature here.

Proposition 2.21 *For any $T \in [0, \infty]$:*

$$v(x, y) = \sup_{\rho\in\mathcal{M}_T} \tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho)]. \tag{2.31}$$

Proof. Following the proof of Proposition 2.11, we only have to argue that $(\tilde{X}, \tilde{Y})$ is strong Markov with respect to $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$. Then we obtain that

$$v(x, y) = \sup_{\tau\in\mathcal{T}_T} \tilde{E}\,[e^{-\alpha\tau} g(\tilde{X}_\tau)],$$

and Lemma 2.10 does the rest.

Given that (G, ξ) is a strong solution to (2.27), it is a strong Markov process with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$, the natural filtration of the pair of driving Brownian motions (W, W^ξ) (see [36, Theorem 7.1.2]). Then, the time-changed process $(\tilde{X}, \tilde{Y})$ is strong Markov with respect to $(\tilde{\mathcal{F}}_{A_t})_{t\geq 0}$ by [42, Theorem 65.9]. □

2.3.4 Monotonicity in y and drift of volatility

In this section we show Theorem 2.16 by comparing the values of the function $v(x, \cdot)$ using the <u>same model</u> (X, Y) but with different initial conditions for Y.

Later on we will also compare the values of two functions $v^{(i)}(x, y)$, $i = 1, 2$, associated to two <u>different models</u> $(X^{(i)}, Y^{(i)})$, $i = 1, 2$, which differ not only in the initial condition but also in the drift coefficient of the second component.

Proof of Theorem 2.16. Fix $x \in \mathbb{R}$ and $y, y' \in \mathcal{S}$ such that $y \leq y'$. We split the proof into two parts.

(i) Let (G, ξ) and (G, ξ') be the solutions to (2.27) starting from (x, y) and (x, y'), respectively, which are both given by (W, W^ξ) on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Remark that G is indeed the same for both pairs since (2.27) is a system of decoupled equations.

Let C denote the coupling time of ξ and ξ', that is, $C = \inf\{t \geq 0 : \xi_t \geq \xi'_t\}$ and set

$$\bar{\xi}_t = \xi_{t\wedge C} + (\xi'_t - \xi'_{t\wedge C}), \qquad t \geq 0,$$

so that $\bar{\xi}_t \leq \xi'_t$ for all $t \geq 0$ everywhere.

The pair $(G, \bar{\xi})$ solves the system (2.27) starting from (x, y) since

$$\begin{aligned}
\bar{\xi}_t = y &+ \int_0^{t\wedge C} \eta(\xi_u)\, \xi_u^{-1} dW_u^\xi + \int_0^{t\wedge C} \theta(\xi_u)\, \xi_u^{-2} du \\
&+ \int_{t\wedge C}^t \eta(\xi'_u)\, (\xi'_u)^{-1} dW_u^\xi + \int_{t\wedge C}^t \theta(\xi'_u)(\xi'_u)^{-2} du \\
= y &+ \int_0^t \eta(\bar{\xi}_u)\, (\bar{\xi}_u)^{-1} dW_u^\xi + \int_0^t \theta(\bar{\xi}_u)(\bar{\xi}_u)^{-2} du.
\end{aligned}$$

By strong uniqueness, $\xi_t = \bar{\xi}_t$ for all $t \geq 0$ a.s. Hence

$$\xi_t \leq \xi'_t,\ t \geq 0,\ \text{a.s.},$$

which results in the inequality

$$\Gamma_t = \int_0^t \xi_s^{-2} ds \geq \int_0^t (\xi_s')^{-2} ds = \Gamma_t', \quad t \geq 0, \ a.s. \tag{2.32}$$

(ii) The second part is exactly as that of Theorem 2.6. Here we use Proposition 2.21 instead of Proposition 2.11. □.

Let us now suppose that, for each $i = 1, 2$, $(X^{(i)}, Y^{(i)})$ is a strong Markov process given on a family of probability spaces $(\Omega^{(i)}, \mathcal{F}^{(i)}, P^{(i)}_{x,y}, (x, y) \in \mathbb{R} \times \mathcal{S})$. The pair $(X^{(i)}, Y^{(i)})$ satisfies the system (2.26) with functions a, η and $\theta^{(i)}$. Finally, also assume the system

$$dX_t = a(X_t) Y_t \, dB_t, \qquad dY_t = \eta(Y_t) \, dB_t^Y + \theta^{(i)}(Y_t) \, dt,$$

with (X, Y) unknown, admits a weakly unique non-exploding solution.

The value functions $v^{(i)}$ are given by

$$v^{(i)}(x, y) = \sup_{0 \leq \tau \leq T} E^{(i)}_{x,y} [\, e^{-\alpha \tau} g(X^{(i)}_\tau) \,], \qquad (x, y) \in \mathbb{R} \times \mathcal{S},$$

where the stopping times τ are with respect to the natural filtration of $(X^{(i)}, Y^{(i)})$.

Now consider the following system, for each $i = 1, 2$:

$$\begin{aligned} dG_t &= a(G_t) dW_t, \\ d\xi_t &= \eta(\xi_t) \xi_t^{-1} dW_t^\xi + \theta^{(i)}(\xi_t) \xi_t^{-2} dt, \end{aligned} \tag{2.33}$$

where (W, W^ξ) are Brownian motions with covariation $\langle W, W^\xi \rangle_t = \delta t$ on some complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

We proceed as in Section 2.3.3 with the natural notation, and impose the following assumptions:

D1: For each $i = 1, 2$, let $a, \eta, \theta^{(i)}$ be measurable functions such that the system in (2.33) has, for all initial conditions $(G_0, \xi_0^{(i)}) \in \mathbb{R} \times \mathcal{S}$, a unique non-exploding strong solution taking values in $\mathbb{R} \times \mathcal{S}$.

D2: For each $i = 1, 2$ and for all initial conditions $(x, y) \in \mathbb{R} \times \mathcal{S}$, the associated process $\Gamma_t^{(i)} = \int_0^t (\xi_s^{(i)})^{-2} ds$ satisfies that $\tilde{P} \left(\lim_{t \uparrow \infty} \Gamma_t^{(i)} = \infty \right) = 1$.

D3: Fix $(x, y^{(1)}), (x, y^{(2)}) \in \mathbb{R} \times \mathcal{S}$ such that $y^{(1)} \leq y^{(2)}$. Let $(G, \xi^{(i)})$ be the solutions to (2.33) starting at $(x, y^{(i)})$, $i = 1, 2$. The property

$$\tilde{P}(\, \xi_t^{(1)} \leq \xi_t^{(2)}, \quad \forall\, t \geq 0\,) = 1 \tag{2.34}$$

is satisfied.

There are well-known sufficient conditions on the coefficients η and $\theta^{(i)}$ which ensure the validity of (2.34). This is part of the so-called *comparison theorems*, like Theorem IX.3.7 in [39] or Theorem V.43.1 in [40]. However, these Theorems usually ask for at least one of the the drifts $b^{(i)}(y) = \theta^{(i)}(y)/y^2$ to satisfy a Lipshitz condition, that is, $|b^{(i)}(x) - b^{(i)}(y)| \leq K|x - y|$ for some constant $K > 0$. We do not impose such a condition here because, as we will see in Section 4.3, there are examples of SDE's with no Lipschitz drift and still satisfying all of the above assumptions.

The next theorem should be compared with Theorem 6.4 of Hobson [24]. Hobson also applies time-change and coupling for comparing prices of European options in a general stochastic volatility model, and under the assumption that g is convex.

Theorem 2.22 *Let Conditions* **D1**-**D3** *be satisfied. Also assume that the gain function g is non-negative. If $\theta^{(1)}(y) \leq \theta^{(2)}(y)$, then for each $x \in \mathbb{R}$,*

$$v^{(1)}(x, y^{(1)}) \leq v^{(2)}(x, y^{(2)}), \qquad \textit{for all } y^{(1)} \leq y^{(2)}. \tag{2.35}$$

Proof. Fix $x \in \mathbb{R}$. Let $(G, \xi^{(i)})$ be the solution to (2.33) starting from $(x, y^{(i)})$, $i = 1, 2$, which exist by Condition **D1**.

Condition **D2** yields the reformulation of $v^{(i)}(x, y^{(i)})$. Specifically, for each $i = 1, 2$, we have that

$$\underbrace{(G \circ A^{(i)}, \xi^{(i)} \circ A^{(i)})}_{\text{under } \tilde{P}} \stackrel{law}{=} \underbrace{(X^{(i)}, Y^{(i)})}_{\text{under } P^{(i)}_{x,y^{(i)}}}$$

where $A^{(i)}$ is the right-inverse of $\Gamma^{(i)}$.

Consequently,

$$v^{(i)}(x,y) = \sup_{\rho \in \mathcal{M}_T^{(i)}} \tilde{E}\,[e^{-\alpha \Gamma_\rho^{(i)}} g(G_\rho)],$$

where $\mathcal{M}_T^{(i)}$ is the family of all the finite stopping times ρ with respect to $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ such that $0 \leq \rho \leq A_T^{(i)}$.

Finally, Condition **D3** implies that $\Gamma_t^{(1)} \geq \Gamma_t^{(2)}$ for all $t \geq 0$ a.s. So,

$$\tilde{E}\,[e^{-\alpha \Gamma_\rho^{(1)}} g(G_\rho)] \leq \tilde{E}\,[e^{-\alpha \Gamma_\rho^{(2)}} g(G_\rho)], \qquad \forall\, \rho \in \mathcal{M}_T^{(1)}.$$

Since $A_t^{(1)} \leq A_t^{(2)}$ for all $t \geq 0$ a.s. necessarily $\mathcal{M}_T^{(1)} \subseteq \mathcal{M}_T^{(2)}$. This gives

$$\sup_{\rho \in \mathcal{M}_T^{(1)}} \tilde{E}\,[e^{-\alpha \Gamma_\rho^{(1)}} g(G_\rho)] \leq \sup_{\rho \in \mathcal{M}_T^{(2)}} \tilde{E}\,[e^{-\alpha \Gamma_\rho^{(2)}} g(G_\rho)],$$

or equivalently, the desired result in (2.35). □

We state the next results without proofs, as these are very similar to those of Section 2.2.4, with the obvious notation.

Corollary 2.23 *Let all the other assumptions of Theorem 2.22 be satisfied, but g is non-positive and the time horizon $T = \infty$. Then, for each $x \in \mathbb{R}$,*

$$v^{(1)}(x, y^{(1)}) \geq v^{(2)}(x, y^{(2)}), \qquad \text{for all } y^{(1)} \leq y^{(2)}.$$

Fix $(x,y) \in \mathbb{R} \times \mathcal{S}$ and let $\mathcal{K}^{g+} \equiv \mathcal{K}^{g+}_{x,y}$ be the collection of all finite stopping times τ with respect to the filtration generated by (X,Y) with $(X_0, Y_0) = (x,y)$ and such that $g(X_\tau) \geq 0$.

Corollary 2.24 *Let Conditions* **D1**-**D3** *be satisfied and the time horizon $T = \infty$. Assume that the gain function g is such that $\{x : g(x) \geq 0\} \neq \emptyset$. Further assume that $v^{(1)}(x,y) = \sup_{\tau \in \mathcal{K}^{g+}} E^{(1)}_{x,y}\,[e^{-\alpha\tau} g(X_\tau^{(1)})]$. If $\theta^{(1)}(y) \leq \theta^{(2)}(y)$, then*

$$v^{(1)}(x,y) \leq v^{(2)}(x,y') \quad \text{for all } y' \in \mathcal{S} \text{ such that } y \leq y'. \tag{2.36}$$

When $\theta^{(1)} = \theta^{(2)} \equiv \theta$ the setting in the above theorem reduces to that of Theorem 2.22 but with g possibly negative. In such a case, Conditions **D1-D2** are equivalent to **C1'-C2'**, while **D3** is a fact rather than a assumption.

2.3.5 Continuity in y

Throughout this section we assume that Conditions **C1'-C2'** on page 26 are satisfied. We also agree on the following notation:

Let $\{y_n\}_{n=0}^{\infty} \subseteq \mathcal{S}$ be a sequence in $\mathcal{S}$ such that $y_n \to y_0$ as $n \to \infty$. Denote by (G, ξ^n) the solution to (2.27), starting from $(G_0, \xi_0^n) = (x, y_n)$, given by a pair (W, W^ξ) of Brownian motions with covariation $\langle W, W^\xi \rangle_t = \delta t$, $t \geq 0$, on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

Using (G, ξ^n), define Γ^n, A^n, $n = 0, 1, 2, \ldots$, like Γ, A on page 27. By Proposition 2.21,

$$v(x, y_0) = \sup_{\rho \in \mathcal{M}_T} \tilde{E}\,[e^{-\alpha \Gamma^0_\rho} g(G_\rho)], \quad v(x, y_n) = \sup_{\rho \in \mathcal{M}_T^n} \tilde{E}\,[e^{-\alpha \Gamma^n_\rho} g(G_\rho)],$$

where $\mathcal{M}_T^n = \{\rho \in \mathcal{M} : \, 0 \leq \rho \leq A_T^n \; a.s.\}$.

We shall frequently use the assumed integrability of $\sup_{t \geq 0} e^{-\alpha\, t} |g(\tilde{X}_t)|$ for each initial point (x, y) (recall the condition in (1.3)). Notice that we can write

$$\sup_{t \geq 0} e^{-\alpha \Gamma_t} |g(G_t)| = \sup_{t \geq 0} e^{-\alpha\, t} |g(\tilde{X}_t)|$$

since $G_t = \tilde{X} \circ \Gamma_t$ and the range of Γ_t over $t \geq 0$ is $(0, \infty)$.

Discussion

Suppose that $y_n \downarrow y_0$ as $n \to \infty$. By the coupling argument in the proof of Theorem 2.16, without loss of generality, one may choose $\{\xi^n\}_{n=0}^{\infty}$ such that

$$\xi_t^1 \geq \xi_t^2 \geq \cdots \geq \xi_t^n \geq \cdots \geq \xi_t^0 \; > \; 0, \qquad t \geq 0, \quad a.s.$$

Hence the pathwise limit $\lim_n \xi_t^n$, $t \geq 0$, exists and satisfies that

$$\lim_n \xi_t^n \geq \xi_t^0 > 0, \qquad t \geq 0, \quad a.s. \tag{2.37}$$

Since the ξ-component corresponding to the unique strong solution to (2.27) has continuous coefficients (see Condition **C1'**), we have that ξ^n converges to ξ^0 weakly if $y_n \to y_0$ (see [44, Corollary 11.1.5]).

Thus, for each $t \geq 0$,

$$\lim_n \xi_t^n \overset{law}{=} \xi_t^0 \qquad \text{and} \qquad \lim_n \xi_t^n \geq \xi_t^0, \qquad a.s.$$

This assertion implies that $\lim_n \xi_t^n = \xi_t^0$ a.s for all $t \geq 0$ (see Lemma 2.36). Finally, the processes $(\xi_t^0)_{t\geq 0}$ and $(\lim_n \xi_t^n)_{t\geq 0}$ must be indistinguishable as a consequence of the almost sure continuity of their paths, (see Lemma 2.37). In other words, we have verified that

$$\lim_n \xi_t^n = \xi_t^0, \quad t \geq 0, \ a.s. \tag{2.38}$$

It is intuitively clear that the continuity of $v(x, \cdot)$ should follow after some limiting arguments using the key equation (2.38).

The next lemma is the crucial tool in the remainder of this section.

Lemma 2.25 *If the sequence $(y_n)_{n=1}^{\infty}$ is monotone, that is, either $y_n \downarrow y_0$ or $y_n \uparrow y_0$ as $n \to \infty$, then*

$$\Gamma_t^n \to \Gamma_t^0 \quad \textit{and} \quad A_t^n \to A_t^0 \quad \textit{as} \quad n \to \infty \qquad t \geq 0, \qquad a.s. \tag{2.39}$$

Proof. Suppose that $y_n \downarrow y_0$ as $n \to \infty$. By (2.38),

$$\Gamma_t^0 = \int_0^t (\xi_u^0)^{-2} du = \int_0^t (\lim_n \xi_u^n)^{-2} du = \lim_n \Gamma_t^n, \quad t \geq 0, \quad a.s. \tag{2.40}$$

by monotone convergence. More precisely, $\Gamma_t^n \uparrow \Gamma_t^0$ for all $t \geq 0$, *a.s.* Since A^n and A^0 are the right-inverses of the continuous increasing processes Γ^n and Γ^0, respectively, we have that $A_t^n \downarrow A_t^0$, $t \geq 0$, a.s. This concludes the proof in the case where the y_n decreases.

In the case where $y_n \uparrow y_0$ as $n \to \infty$, we see that $0 < \xi_t^1 \leq \cdots \leq \xi_t^n \leq \cdots \leq \xi_t^0$ and (2.38) also holds (the argument to obtain this equation is the same if the ordering in (2.37) is reversed). Hence, we obtain (2.40) by Lebesgue's dominated convergence theorem. This ensures that $\Gamma_t^n \downarrow \Gamma_t^0$, $t \geq 0$ a.s., and so $A_t^n \uparrow A_t^0$, $t \geq 0$, a.s. □

The infinite horizon case

We now discuss the continuity of the value function $v(x,\cdot)$ on $\mathcal{S}$, in the infinite horizon case. An advantage over the finite horizon case is that the families of stopping times $\mathcal{M}_T^n$ coincide with $\mathcal{M}$ for all n when $T=\infty$.

In the case where g is non-negative, left-continuity is an easy consequence of Lemma 2.25 and the monotonicity of $v(x,\cdot)$.

Proposition 2.26 *Assume that $T=\infty$ and that g is non-negative. Then, for each $x\in\mathbb{R}$, the function $v(x,\cdot)$ is left-continuous.*

Proof. Let $y_n \uparrow y_0$. Since g is non-negative, Theorem 2.16 implies that

$$\limsup_{n\to\infty} v(x,y_n) \le v(x,y_0),$$

so it remains to show that

$$v(x,y_0) \;\le\; \liminf_{n\to\infty} v(x,y_n). \tag{2.41}$$

Pick an arbitrary $\rho\in\mathcal{M}$. Since $y_n\uparrow y_0$ we must have $\Gamma_\rho^n \downarrow \Gamma_\rho^0$ as $n\to\infty$. It follows from Fatou's Lemma and Lemma 2.25 that

$$\tilde{E}e^{-\alpha\Gamma_\rho^0}g(G_\rho) \le \liminf_{n\to\infty} \tilde{E}\, e^{-\alpha\Gamma_\rho^n}g(G_\rho). \tag{2.42}$$

Now, since $v(x,y_n)=\sup_{\rho'\in\mathcal{M}}\tilde{E}\,e^{-\alpha\Gamma_{\rho'}^n}g(G_{\rho'})$ for each $n=0,1,2,\ldots$, we obtain that

$$\tilde{E}e^{-\alpha\Gamma_\rho^0}g(G_\rho) \le \liminf_{n\to\infty} v(x,y_n). \tag{2.43}$$

and taking the supremum over $\rho\in\mathcal{M}$ on the left-hand side of (2.43) completes the proof. □

Corollary 2.27 *With the assumptions of Proposition 2.26, the function $v(x,\cdot)$ is lower semi-continuous.*

Proof. The inequality in (2.41) was shown when $y_n\uparrow y_0$, whereas the case $y_n\downarrow y_0$ follows by Theorem 2.16. Hence $v(x,\cdot)$ is lower semi-continuous. □

To establish the right-continuity of $v(x,\cdot)$, we are naturally tempted to use the *reverse* Fatou's Lemma and apply a similar argument as in Proposition 2.26 to show that $\limsup_{n\to\infty} v(x,y_n) \le v(x,y_0)$ when $y_n \downarrow y_0$. But, although it is true that

$$\limsup_{n\to\infty} \tilde{E}\, e^{-\alpha\Gamma^n_\rho} g(G_\rho) \le \tilde{E} e^{-\alpha\Gamma^0_\rho} g(G_\rho) \le v(x,y_0), \quad \text{for all } \rho \in \mathcal{M},$$

it <u>does not</u> follow that

$$\limsup_{n\to\infty} \sup_{\rho\in\mathcal{M}} \tilde{E}\, e^{-\alpha\Gamma^n_\rho} g(G_\rho) \le \sup_{\rho\in\mathcal{M}} \limsup_{n\to\infty} \tilde{E}\, e^{-\alpha\Gamma^n_\rho} g(G_\rho)$$

and so a different argument has to be used.

We are able to show the right-continuity of $v(x,\cdot)$ upon assuming an extra integrability condition, originated from the estimate in (2.44). It is important to mention that, from now on, we shall only make use of Lemma 2.25 (recall that $g \ge 0$ is imposed in Theorem 2.16) and so we consider the case where g is possibly negative.

Lemma 2.28 *Assume that $T = \infty$. Fix $y, y' \in \mathcal{S}$ such that $y \le y'$. Then, for all $N \in \mathbb{N}$*

$$\begin{aligned} 0 \le v(x,y') - v(x,y) \le \tilde{E}\left[\left(1 - e^{-\alpha(\Gamma_{A'_N} - N)}\right) \sup_{t\le N} e^{-\alpha t} |g(\tilde{X}'_t)|\right] \\ + \tilde{E}\left[\sup_{t\ge N} e^{-\alpha t}|g(\tilde{X}'_t)|\right]. \end{aligned} \tag{2.44}$$

where $\tilde{X}' = G \circ A'$ as usual.

Proof. Fix $N \in \mathbb{N}$ and an arbitrary $\epsilon > 0$. Choose an ϵ-optimal stopping time $\rho'_\epsilon \in \mathcal{M}$ for $v(x,y') = \sup_{\rho\in\mathcal{M}} \tilde{E}\,[e^{-\alpha\Gamma'_\rho} g(G_\rho)]$ so that

$$0 \;\le\; v(x,y') - v(x,y) \;\le\; \epsilon + \tilde{E}\,[e^{-\alpha\Gamma'_{\rho'_\epsilon}} g(G_{\rho'_\epsilon}) - e^{-\alpha\Gamma_{\rho'_\epsilon}} g(G_{\rho'_\epsilon})]. \tag{2.45}$$

After factorizing $g(G_{\rho'_\epsilon})$ and using that $\Gamma_{\rho'_\epsilon} - \Gamma'_{\rho'_\epsilon} \ge 0$, it is easy to see that the right-hand side of (2.45) is dominated by

$$\epsilon + \tilde{E}\left[\left(1 - e^{-\alpha(\Gamma_{\rho'_\epsilon} - \Gamma'_{\rho'_\epsilon})}\right) e^{-\alpha\Gamma'_{\rho'_\epsilon}} |g(G_{\rho'_\epsilon})| I(\rho'_\epsilon \leq A'_N)\right] \\ + \tilde{E}\left[\, e^{-\alpha\Gamma'_{\rho'_\epsilon}} |g(G_{\rho'_\epsilon})| I(\rho'_\epsilon > A'_N)\right].$$

Moreover, on the event $\{\rho'_\epsilon \leq A'_N\}$, we have that

$$e^{-\alpha\Gamma'_{\rho'_\epsilon}} |g(G_{\rho'_\epsilon})| \;\leq\; \sup_{t \leq A'_N} e^{-\alpha\Gamma'_t} |g(G_t)| \;=\; \sup_{t \leq N} e^{-\alpha t} |g(\tilde{X}'_t)| \tag{2.46}$$

since $\{t \leq A'_N\} = \{\Gamma'_t \leq N\}$ and $\tilde{X}'_t = G_{A'_t}$. Also,

$$\begin{aligned} 0 \leq \Gamma_{\rho'_\epsilon} - \Gamma'_{\rho'_\epsilon} \;&=\; \int_0^{\rho'_\epsilon} (\xi_u^{-2} - (\xi'_u)^{-2}) du \;\leq\; \int_0^{A'_N} (\xi_u^{-2} - (\xi'_u)^{-2}) du \\ \;=\; \Gamma_{A'_N} - \Gamma'_{A'_N} \;&=\; \Gamma_{A'_N} - N. \end{aligned} \tag{2.47}$$

Similarly, on the event $\{\rho'_\epsilon > A'_N\}$

$$e^{-\alpha\Gamma'_{\rho'_\epsilon}} |g(G_{\rho'_\epsilon})| \;\leq\; \sup_{t \geq A'_N} e^{-\alpha\Gamma'_t} |g(G_t)| \;=\; \sup_{t \geq N} e^{-\alpha t} |g(\tilde{X}'_t)|. \tag{2.48}$$

Putting all together we obtain the estimate in (2.44) up to ϵ, but ϵ can be made arbitrarily small so the proof is complete. □

Proposition 2.29 *Assume that $T = \infty$. Then, for each $x \in \mathbb{R}$, the function $v(x, \cdot)$ is continuous provided the following condition holds: for each $y_0 \in \mathcal{S}$ there exists $\bar{y} > y_0$ such that*

$$\sup_{y_0 \leq y' < \bar{y}} \tilde{E}\left[\sup_{t \geq N} e^{-\alpha t} |g(\tilde{X}'_t)|\right] \to 0 \qquad \text{as} \quad N \uparrow \infty \tag{2.49}$$

where $\tilde{X}' = G \circ A'$ as usual.

Proof. Fix y_0. We split the proof of continuity of $v(x, \cdot)$ at y_0 into left- and right-continuity.

(i) Let $y_n \uparrow y_0$. In Lemma 2.28, replace y' and y by y_0 and y_n, respectively. By Lemma 2.25, $\Gamma^n_t \downarrow \Gamma^0_t$ for all $t \geq 0$ a.s. and so

$$\lim_{n \to \infty} (\Gamma^n_{A^0_N} - N) = 0 \quad a.s.$$

By (2.49), given an arbitrarily small $\epsilon > 0$ we can choose N large enough that

$$\tilde{E}\left[\sup_{t\geq N} e^{-\alpha t}|g(\tilde{X}_t^0)|\right] \leq \epsilon,$$

and so Lemma 2.28 and the Dominated Convergence Theorem imply that

$$\lim_{n\to\infty}\left(v(x,y_0) - v(x,y_n) \right) \leq \epsilon + \tilde{E}\left[\left(1 - e^{-\alpha(\lim_n \Gamma^n_{A^0_N} - N)}\right)\sup_{t\leq N} e^{-\alpha t}|g(\tilde{X}_t^0)|\right] = \epsilon,$$

which yields the left-continuity of $v(x,\cdot)$ at y_0.

(ii) Let $y_n \downarrow y_0$ and assume without lost of generality that the sequence $\{y_n\}$ is bounded above by $\bar{y}$ such that (2.49) holds.

We know that $\Gamma_t^{\bar{y}} \leq \Gamma_t^n \leq \Gamma_t^0$ and $A_t^{\bar{y}} \geq A_t^n \geq A_t^0$ for all $t \geq 0$ a.s. and so

$$\sup_{t\leq N} e^{-\alpha t}|g(\tilde{X}_t^n)| = \sup_{t\leq A_N^n} e^{-\alpha\Gamma_t^n}|g(G_t)| \leq \sup_{t\leq A_N^{\bar{y}}} e^{-\alpha\Gamma_t^{\bar{y}}}|g(G_t)| = \sup_{t\leq N} e^{-\alpha t}|g(\tilde{X}_t^{\bar{y}})|$$

where the right-hand side is integrable.

Now, given an arbitrarily small $\epsilon > 0$, choose N large enough that

$$\sup_{y_0\,\leq\, y_n < \bar{y}} \tilde{E}\left[\sup_{t\geq N} e^{-\alpha t}|g(\tilde{X}_t^n)|\right] \leq \epsilon,$$

and notice that N does not depend on $n = 1, 2, \ldots$. Hence, we obtain from Lemma 2.28 and the Dominated Convergence Theorem that

$$\lim_{n\to\infty}\left(v(x,y_n) - v(x,y_0) \right) \leq \epsilon + \tilde{E}\left[\left(1 - e^{-\alpha(\lim_n \Gamma^0_{A^n_N} - N)}\right)\sup_{t\leq N} e^{-\alpha t}|g(\tilde{X}_t^{\bar{y}})|\right] = \epsilon,$$

which imply the right-continuity of $v(x,\cdot)$ at y_0.

Since y_0 is arbitrary, we conclude that $v(x,\cdot)$ is continuous. □

The finite horizon case

The main part in the proof of the next proposition is in the spirit of Lemma 2.28, but now using T instead of N.

Proposition 2.30 *Assume that $T < \infty$ and that the gain function g is continuous. Then the function $v(x, \cdot)$ is continuous.*

Proof. **(i)** Fix $y, y' \in \mathcal{S}$ such that $y \le y'$. Fix an arbitrary $\epsilon > 0$ and choose an ϵ-optimal stopping time $\rho'_\epsilon \in \mathcal{M}'_T$ so that

$$0 \;\le\; v(x, y') - v(x, y) \;\le\; \epsilon + \tilde{E}\,[e^{-\alpha\Gamma'_{\rho'_\epsilon}} g(G_{\rho'_\epsilon}) - e^{-\alpha\Gamma_{\rho'_\epsilon \wedge A_T}} g(G_{\rho'_\epsilon \wedge A_T})]. \quad (2.50)$$

Compare with (2.45) and notice that $\rho'_\epsilon \wedge A_T$ is used here instead of ρ'_ϵ since one cannot conclude that $v(x, y) \;\ge\; \tilde{E} e^{-\alpha\Gamma_\rho} g(G_\rho)$ for stopping times ρ which may exceed A_T with positive probability.

By considering the partition $\{\rho'_\epsilon \le A_T\}$ and $\{A_T < \rho'_\epsilon \le A'_T\}$, the right-hand side of (2.50) can be dominated by

$$\begin{aligned}
\epsilon + \tilde{E}\,\Big[& \Big(1 - e^{-\alpha(\Gamma_{\rho'_\epsilon} - \Gamma'_{\rho'_\epsilon})}\Big)\, e^{-\alpha\Gamma'_{\rho'_\epsilon}} |g(G_{\rho'_\epsilon})|\, I(\rho'_\epsilon \le A_T)\Big] \\
&+\; \tilde{E}\,\Big(1 - e^{-\alpha(T - \Gamma'_{\rho'_\epsilon})}\Big)\, e^{-\alpha\Gamma'_{\rho'_\epsilon}} |g(G_{\rho'_\epsilon})|\, I(A_T < \rho'_\epsilon \le A'_T) \\
&+\; \tilde{E}\, e^{-\alpha T} |g(G_{\rho'_\epsilon}) - g(G_{A_T})|\, I(A_T < \rho'_\epsilon \le A'_T).
\end{aligned}$$

by "adding" $\pm e^{-\alpha T} g(G_{\rho'_\epsilon})$ in the case where $A_T < \rho'_\epsilon \le A'_T$.

Next, we argue similarly as we did in (2.46)-(2.47). We know that $\rho'_\epsilon \le A'_T$ a.s. and so both

$$e^{-\alpha T} |g(G_{\rho'_\epsilon})| \;\le\; e^{-\alpha\Gamma'_{\rho'_\epsilon}} |g(G_{\rho'_\epsilon})| \;\le\; \sup_{t \le A'_T} e^{-\alpha\Gamma'_t} |g(G_t)|, \quad a.s.$$

and

$$0 \le \Gamma_{\rho'_\epsilon} - \Gamma'_{\rho'_\epsilon} \le \Gamma_{A'_T} - T, \quad a.s.$$

hold. Meanwhile, since $\{A_T < \rho'_\epsilon \le A'_T\} = \{\Gamma'_{A_T} < \Gamma'_{\rho'_\epsilon} \le T\}$, we have that

$$0 \le T - \Gamma'_{\rho_\epsilon} \le T - \Gamma'_{A_T}, \quad \text{on } \{A_T < \rho'_\epsilon \le A'_T\}.$$

Putting all together, we arrive to the following estimate for $v(x, y') - v(x, y)$ with $y \le y'$:

$$
\begin{aligned}
0 &\le v(x,y') - v(x,y) \\
&\le \epsilon + \tilde{E}\left[\left(2 - e^{-\alpha(\Gamma_{A'_T} - T)} - e^{-\alpha(T - \Gamma'_{A_T})}\right) \sup_{t \le A'_T} e^{-\alpha \Gamma'_t} |g(G_t)|\right] \qquad (2.51) \\
&\qquad + \tilde{E}\left[e^{-\alpha T} |g(G_{\rho'_\epsilon}) - g(G_{A_T})| \, I(A_T < \rho'_\epsilon \le A'_T)\right].
\end{aligned}
$$

Notice that the integrands on the right-hand side of (2.51) are bounded above by

$$\sup_{t \le A'_T} e^{-\alpha \Gamma'_t} |g(G_t)|$$

because also $e^{-\alpha T}|g(G_{A_T})| = e^{-\alpha \Gamma_{A_T}}|g(G_{A_T})| \le \sup_{t \le A'_T} e^{-\alpha \Gamma'_t}|g(G_t)|$.

(ii) Let $y_n \uparrow y_0$. Replace y and y' by y_n and y_0, respectively in (2.51).

By Lemma 2.25, $\Gamma^n_t \downarrow \Gamma^0_t$ for all $t \ge 0$ a.s. and so

$$\lim_{n\to\infty} (\Gamma^n_{A^0_T} - T) = 0, \quad \text{and} \quad \lim_{n\to\infty} (T - \Gamma^0_{A^n_T}) = 0 \quad a.s.$$

and also, given that g is continuous,

$$\lim_{n\to\infty} |g(G_{\rho^0_\epsilon}) - g(G_{A^n_T})| \, I(A^n_T < \rho^0_\epsilon \le A^0_T) = 0 \quad a.s. \qquad (2.52)$$

as A^n_T can be made arbitrarily close to A^0_T.

Since $\sup_{t \le A^0_T} e^{-\alpha \Gamma^0_t}|g(G_t)|$ is integrable, it follows by dominated convergence that

$$0 \le v(x,y_0) - v(x,y_n) \to \epsilon \qquad \text{as } y_n \uparrow y_0.$$

(iii) Let $y_n \downarrow y_0$. Now replace y and y' by y_0 and y_n, respectively in (2.51). By Lemma 2.25, $\Gamma^n_t \uparrow \Gamma^0_t$ for all $t \ge 0$ a.s.

A symmetric argument to that of the part (ii) yields

$$0 \le v(x,y_n) - v(x,y_0) \to \epsilon \qquad \text{as } y_n \downarrow y_0.$$

Simply note that, without loss of generality,

$$\sup_{t \le A^n_T} e^{-\alpha \Gamma^n_t}|g(G_t)| \le \sup_{t \le A^1_T} e^{-\alpha \Gamma^1_t}|g(G_t)|, \quad \forall\, n = 1, 2, \ldots,$$

where the right-hand side is integrable, and so dominated convergence can be used again.

Since $\epsilon > 0$ can be made arbitrarily small so the proof is complete. □

Remark 2.31 The continuity of g is only used in the argument in (2.52). Notice that, although the limit of A^n_T is A^0_T as $n \to \infty$, the limit of the indicator functions $I(A^n_T < \rho^0_\epsilon \leq A^0_T)$ does not necessarily vanish.

2.4 Adding a running payoff

We are going to extend the monotonicity results on $v(x, \cdot)$ by adding a so-called running payoff or cost of observations. We do so for both cases, regime-switching and diffusion, simultaneously because the ideas involved are the same.

Suppose that the value function $v(x, y)$ in (2.1) is, instead, of the form

$$v(x,y) = \sup_{0 \leq \tau \leq T} E_{x,y}\,[e^{-\alpha\tau} g(X_\tau) - C_\tau], \quad (x, y) \in \mathbb{R} \times \mathcal{S}, \tag{2.53}$$

where

$$C_t = \int_0^t e^{-\alpha\, s} c(X_s)\, ds$$

and $c : \mathbb{R} \to [0, \infty)$ is a bounded Lebesgue integrable function.

Theorem 2.32 *The statement and result of Theorems 2.6 and 2.16 remain valid when $v(x, y)$ is as in (2.53).*

To start, recall the construction of the new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$, as well as the time-changed process $(\tilde{X}, \tilde{Y})$ in Sections 2.2.3 and 2.3.3. These constructions do not depend on g nor c, but on the dynamics of the pair (X, Y) only.

Lemma 2.10 now reads as follows. Using the corresponding definitions for $\mathcal{M}_T$ and $\mathcal{T}_T$:

Lemma 2.33 *For any $T \in [0, \infty]$,*

$$\sup_{\tau \in \mathcal{T}_T} \tilde{E}\,[e^{-\alpha\tau} g(\tilde{X}_\tau) - \int_0^\tau e^{-\alpha\,t} c(\tilde{X}_t)\,dt] = \sup_{\rho \in \mathcal{M}_T} \tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho) - \tilde{C}_\rho], \quad (2.54)$$

where

$$\tilde{C}_\rho = \int_0^\rho e^{-\alpha\,\Gamma_t} c(G_t)\, Z_t^{-2} dt$$

in the regime-switching case, or

$$\tilde{C}_\rho = \int_0^\rho e^{-\alpha\,\Gamma_t} c(G_t)\, \xi_t^{-2} dt$$

in the diffusion case.

Proof. In the regime-switching case, $(\tilde{X}_t, \tilde{Y}_t) = (G_{A_t}, Z_{A_t})$ and recall that $d\Gamma_t = Z_t^{-2} dt$ (see equation (2.7) above).

If $\tau \in \mathcal{T}_T$ then $\rho = A_\tau \in \mathcal{M}_T$, and

$$\begin{aligned}
\int_0^\tau e^{-\alpha\,t} c(\tilde{X}_t)\,dt &= \int_0^{\Gamma_\rho} e^{-\alpha\,t} c(G_{A_t}) dt = \int_0^\rho e^{-\alpha\,\Gamma_t} c(G_{A_{\Gamma_t}}) d\Gamma_t \\
&= \int_0^\rho e^{-\alpha\,\Gamma_t} c(G_t) Z_t^{-2}\, dt.
\end{aligned}$$

Complementing this fact with the first part of the proof of Lemma 2.10, we obtain that for any $\tau \in \mathcal{T}_T$,

$$\tilde{E}\,[e^{-\alpha\tau} g(\tilde{X}_\tau) - \int_0^\tau e^{-\alpha\,t} c(\tilde{X}_t)\,dt] \le \sup_{\rho \in \mathcal{M}_T} \tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho) - \tilde{C}_\rho].$$

Analogously, if $\rho \in \mathcal{M}_T$ then we can verify that

$$\tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho) - \tilde{C}_\rho] \le \sup_{\tau \in \mathcal{T}_T} \tilde{E}\,[e^{-\alpha\tau} g(\tilde{X}_\tau) - \int_0^\tau e^{-\alpha\,t} c(\tilde{X}_t)\,dt]$$

using a symmetric argument and the fact that $\tau = \Gamma_\rho \in \mathcal{T}_T$. The proof is then complete.

In the diffusion case we only need to change the notation (use ξ instead of Z), but the arguments are not affected. □

Following the proof of Theorem 2.6, with the natural change of notation, it is easy to convince ourselves that one only has to verify that (compare with (2.18))

$$\tilde{E}\,[e^{-\alpha\Gamma_\rho} g(G_\rho) - \tilde{C}_\rho] \;\le\; \tilde{E}\,[e^{-\alpha\Gamma'_\rho} g(G_\rho) - \tilde{C}'_\rho] \quad \text{for every } \rho \in \mathcal{M}_T. \qquad (2.55)$$

But (2.55) follows thanks to (2.16) and (2.17), the fact that both g and c are non-negative, and

$$e^{-\alpha\Gamma_t} \le e^{-\alpha\Gamma'_t} \qquad \text{and} \qquad \tilde{C}_t \ge \tilde{C}'_t, \qquad \forall t \ge 0,\ a.s.$$

The rest of the proof of Theorem 2.6 remains unchanged.

2.5 Proofs of auxiliary results

Lemma 2.34 *Let τ and τ_n, $n = 1, 2, \ldots$, be stopping times of a filtration $(\mathcal{F}_t)_{t\ge0}$ satisfying the usual conditions.*

(i) *If $\tau_n \downarrow \tau$ then $\mathcal{F}_\tau = \bigcap_n \mathcal{F}_{\tau_n}$.*

(ii) *If $\tau = c$ a.s where c is a non-negative constant, then $\mathcal{F}_\tau \subseteq \mathcal{F}_c$.*

(iii) *The event $\{\tau_1 \le \tau_2\} \in \mathcal{F}_{\tau_2}$.*

(iv) *If X is a right-continuous adapted process then X_τ is $\mathcal{F}_\tau$-measurable on the event $\{\tau < \infty\}$.*

Proof. (i) The fact that $\tau \le \tau_n$, $n = 1, 2, \ldots$, implies that $\mathcal{F}_\tau \subset \bigcap_n \mathcal{F}_{\tau_n}$. On the other hand, if $B \in \bigcap_n \mathcal{F}_{\tau_n}$ then $B \cap \{\tau_n < t\} \in \mathcal{F}_t$, $n = 1, 2, \ldots$, and so

$$B \cap \{\tau < t\} = B \cap \bigcup_n \{\tau_n < t\} = \bigcup_n B \cap \{\tau_n < t\} \in \mathcal{F}_t, \quad t \ge 0.$$

(ii) If $B \in \mathcal{F}_\tau$ then $B \cap \{\tau \le t\} \in \mathcal{F}_t$, $t \ge 0$. In particular, taking $t = c$ and noticing that $\{\tau > c\}$ is a null set we obtain

$$B = B \cap \{\tau \le c\} \bigcup B \cap \{\tau > c\} \in \mathcal{F}_c.$$

(iii) To see that $\{\tau_1 \le \tau_2\} \in \mathcal{F}_{\tau_2}$, note that for any $r \ge 0$

$$\{\tau_1 < \tau_2\} \cap \{\tau_2 < r\} = \bigcup_{q\in\mathbb{Q},\, q\le r} \{\tau_1 \le q\} \cap \{q < \tau_2\} \cap \{\tau_2 < r\} \in \mathcal{F}_r$$

and so $\{\tau_1 < \tau_2\} \in \mathcal{F}_{\tau_2}$. Hence, $\{\tau_1 < \tau_2 + \epsilon\} \in \mathcal{F}_{\tau_2+\epsilon}$ for each $\epsilon > 0$ and by part (i),

$$\{\tau_1 \le \tau_2\} = \bigcap_{\epsilon>0}\{\tau_1 < \tau_2 + \epsilon\} \in \bigcap_{\epsilon>0} \mathcal{F}_{\tau_2+\epsilon} = \mathcal{F}_{\tau_2}.$$

(iv) Let τ_n be the approximating sequence of τ given by $\tau_n = \frac{\lfloor 2^n\tau\rfloor+1}{2^n}$ where $\lfloor\cdot\rfloor$ is the *floor* function which returns the integral part of the argument. Then each τ_n takes values on $\{k/2^n : k = 1, 2, \dots,\}$ and $\tau_n \downarrow \tau$. By the right-continuity of X we have that $X_\tau = \lim_n X_{\tau_n}$.

We aim to show that for each Borel set B it holds true that $\{X_\tau \in B\} \in \mathcal{F}_\tau$. By part (i), it is enough to prove that $\{X_\tau \in B\} \in \mathcal{F}_{\tau_n}$ for each $n = 1, 2, \dots$ Note that

$$\{X_{\tau_m} \in B\} \cap \{\tau_m < t\} = \bigcup_{k/2^m<t} \{X_{k/2^m} \in B\} \cap \{\tau_m = k/2^m\} \in \mathcal{F}_{k/2^m} \subset \mathcal{F}_t$$

for each $t \ge 0$. Thus $\{X_{\tau_m} \in B\} \in \mathcal{F}_{\tau_n}$, $m > n$. Because $X_\tau = \lim_m X_{\tau_m}$ the latter implies that $\{X_\tau \in B\} \in \mathcal{F}_{\tau_n}$, $n = 1, 2, \dots$, as required. □

Proof of **Proposition 2.2.** It is clear that A_t increases with t.

Fix $t \ge 0$. We first show that A is a time-change. A_t is a stopping time because it is the first entry time of the right-continuous process $\Gamma_\cdot$ into the open set (t, ∞). Indeed, $A_t = \inf\{s \ge 0 : \Gamma_s \in (t, \infty)\}$ and if $\Gamma_s \in (t, \infty)$ then $\Gamma_u \in (t, \infty)$ for every $u \in [s, s + \epsilon)$ for some $\epsilon > 0$ allowing us to write

$$\{A_t < r\} = \bigcup_{s<r,\, s\in\mathbb{Q}} \{\Gamma_s \in (t, \infty)\} \in \mathcal{F}_r,$$

where we have also used that Γ is adapted. The filtration $(\mathcal{F}_t)_{t\ge0}$ is right-continuous and so A_t is a stopping time. To see that $t \mapsto A_t$ is right-continuous, simply write $\{\Gamma_s > t\} = \bigcup_{\epsilon>0}\{\Gamma_s > t + \epsilon\}$.

Next, we verify that $\Gamma_s = \inf\{t \ge 0 : A_t > s\}$. On the one hand, if $A_t > s$ then $s \notin \{u \ge 0 : \Gamma_u > t\}$ and so $\Gamma_s \le t$. The latter implies that

$\Gamma_s \leq \inf\{t \geq 0 : A_t > s\}$. On the other hand, it is clear from the definition of A that $A_{\Gamma_s} \geq s$ and so $A_{\Gamma_{s+\epsilon}} > s$. The previous fact together with the increasing property of A imply that $\Gamma_{s+\epsilon} \geq \inf\{t \geq 0 : A_t > s\}$ and the right-continuity of Γ gives $\Gamma_s \geq \inf\{t \geq 0 : A_t > s\}$.

Finally, Γ_s is an $(\mathcal{F}_{A_t})_{t\geq 0}$-stopping time. This follows from the right-continuity of the filtration $(\mathcal{F}_{A_t})_{t\geq 0}$. Indeed, we want $\{\Gamma_s \leq r\} \in \mathcal{F}_{A_r}$ for each $r \geq 0$. Now, the event $\{\Gamma_s = r\} = \{A_{r+\epsilon} > s$, for $\epsilon > 0$ arbitrarily small$\}$ is necessarily in $\mathcal{F}_{A_r}$ because the filtration $(\mathcal{F}_{A_t})_{t\geq 0}$ is right-continuous. □

Proof of **Corollary 2.3.** (i) The process A can only jump if Γ has intervals of constancy.

(ii) By definition of A_t, it is strictly increasing because of the continuity of Γ. It is finite and has the limit $\lim_{t\to\infty} A_\infty = \infty$ since $\lim_{t\to\infty} \Gamma_t = \infty$.

(iii) Note that, by the continuity of Γ,

$$A_{\Gamma_s} = \inf\{t \geq 0 : \Gamma_t > \Gamma_s\} = \begin{cases} s & \text{if } \Gamma_{s-} < \infty \\ \infty & \text{if } \Gamma_{s-} = \infty. \end{cases}$$

Thus, if Γ is also finite then $A_{\Gamma_s} = s$, $0 \leq s < \infty$. Arguing symmetrically and using parts (i)-(ii) we obtain that $\Gamma_{A_s} = s$, $0 \leq s < \infty$.

Finally, by the definition of A_s it is plain that $(A_s < t \Rightarrow s \leq \Gamma_t)$, but $\{s = \Gamma_t\}$ is null because on that event one has that $A_{\Gamma_t} = A_s = t$. Conversely, by Proposition 2.2 we have that $(s < \Gamma_t \Rightarrow A_s \leq t)$, but again $\{A_s = t\}$ is null because on that event one has that $\Gamma_t = \Gamma_{A_s} = s$. □

Proof of **Lemma 2.4.** Fix $\rho \in \mathcal{M}$. We want to show that

$$\{\Gamma_\rho \leq r\} \in \mathcal{F}_{A_r}, \qquad \forall\, r \geq 0. \tag{2.56}$$

By part (iii) of Corollary 2.3, we know that the event

$$\Omega_0 = \{\omega :\ s < \Gamma_t(\omega) \text{ if and only if } A_s(\omega) < t \text{ for all } 0 \leq s, t < \infty\}$$

is so that $P(\Omega_0) = 1$ and that, for each $r \geq 0$, A_r is a finite $(\mathcal{F}_t)_{t\geq 0}$-stopping time. It is also clear that

$$\{\Gamma_\rho \le r\} \cap \Omega_0 \;=\; \{\rho \le A_r\} \cap \Omega_0.$$

By part (iii) of Lemma 2.34, $\{\rho \le A_r\} \in \mathcal{F}_{A_r}$ since both ρ and A_r are $(\mathcal{F}_t)_{t\ge0}$-stopping times. Therefore the claim in (2.56) holds true, that is, $\Gamma_\rho \in \mathcal{T}$.

By the symmetry of Γ and A, the second assertion ($\tau \in \mathcal{T} \;\Rightarrow\; A_\tau \in \mathcal{M}$) is proved. □

Proof of **Proposition 2.5.** Define the so-called *big filtration* $\mathbb{F}^{big} = (\mathcal{F}_t^{big})_{t\ge0}$ by

$$\mathcal{F}_t^{big} := \mathcal{F}_t^W \vee \sigma(\{Z_s : s \ge 0\}), \qquad t \ge 0.$$

Since W is independent of Z, W is also an $\mathbb{F}^{big}$-Brownian motion and M is an $\mathbb{F}^{big}$-continuous local martingale. It follows by Theorem V.1.6 in [39] that $B_\cdot = M_{A_\cdot}$ is an $(\mathcal{F}^{big}_{A_t})_{t\ge0}$-Brownian motion. In particular, the property of independent increments of B yields that

$$B_t \text{ is independent of } \mathcal{F}^{big}_{A_0} = \mathcal{F}^{big}_0, \quad t \ge 0. \tag{2.57}$$

Next, notice that $\langle M\rangle_s$ is strictly increasing in s because $f(\cdot)^2 > 0$. Hence $\langle M\rangle_{A_s} = s$ for all $s \ge 0$ (recall part (iii) of Corollary 2.3), and so

$$A_t = \int_0^{A_t} \frac{d\langle M\rangle_s}{f(Z_s)^2} = \int_0^t \frac{1}{f(Z_{A_s})^2} ds.$$

That is, $A_\cdot$ (and so $Z_{A_\cdot}$) is a functional of Z.

Finally, by the definition of $\mathbb{F}^{big}$ and the last assertion,

$$\mathcal{F}_0^{big} \supseteq \sigma(\{Z_s : s \ge 0\}) \supseteq \sigma(\{Z_{A_s} : s \ge 0\}).$$

Therefore, using (2.57), we conclude that $B_\cdot$ is independent of $Z_{A_\cdot}$. □

The following elementary lemmas were used to show Lemma 2.25.

Definition 2.35 Two processes $U = (U_t)_{t\geq 0}$ and $V = (V_t)_{t\geq 0}$ defined on the same probability space are *modifications* of each other if

$$\text{for each } t \geq 0, \qquad U_t = V_t, \qquad \text{a.s.}$$

They are *indistinguishable* if

$$U_t = V_t, \qquad \text{for each } t \geq 0, \quad \text{a.s.}$$

Lemma 2.36 *Let U and V be two random variables on the same probability space satisfying $U \geq V$ a.s. with U having the same law as V. Then $U = V$ a.s.*

Proof. Let $\mathbb{Q}$ denote the set of rational numbers and P the probability measure. Then

$$\{V < U\} = \bigcup_{q\in\mathbb{Q}} \{V \leq q < U\} = \bigcup_{q\in\mathbb{Q}} \left(\{V \leq q\}\backslash\{U \leq q\}\right).$$

If $P(V < U) > 0$ then there exists $q \in \mathbb{Q}$ such that $P(V \leq q) > P(U \leq q)$, but this contradicts the assumption that U and V have the same law. □

Lemma 2.37 *If $U = (U_t)_{t\geq 0}$ and $V = (V_t)_{t\geq 0}$ are modifications of each other and have a.s. continuous paths then they are indistinguishable.*

Proof. Let $\mathbb{Q}^+$ denote all the non-negative rational numbers. Consider the event $A = \{U_q = V_q,\ q \in \mathbb{Q}^+\}$ and its complement (in $\mathbb{Q}^+$)

$$A^c = \bigcup_{q\in\mathbb{Q}^+} \{U_q \neq V_q\}.$$

Since U and V are modifications of each other, it follows that $\{U_q \neq V_q\}$ is null for each $q \in \mathbb{Q}^+$ and then it is clear that A happens a.s.

Finally, every $t \geq 0$ can be approximated by a sequence of rational numbers, say $\{q_n(t)\}_{n=1}^{\infty} \subset \mathbb{Q}^+$. Thus, by the a.s. continuity of the paths, we have that:

$$U_t = \lim_n U_{q_n(t)} = \lim_n V_{q_n(t)} = V_t, \qquad \forall t \geq 0, \quad \text{a.s.}$$

as required. □

Chapter 3

Control of stochastic volatility

3.1 Setting and problem statement

In this chapter we derive the solution of a zero-sum game of stopping and control. The solution is presented in Section 3.3.2, under verifiable conditions. Examples where these conditions are satisfied are provided in the next chapter.

Theorems 3.16 and 3.17 state that the value of the game identifies with the value function of certain optimal stopping problem associated to an extremal scenario. In particular, we exhibit a saddle point under the assumption that the space of control values is compact. The proof is based on analytical methods for which smoothness of such a candidate value function is required. To show the latter, the strong Markov property as well as the probabilistic representation of solutions to Dirichlet-type problems are the main tools used.

The setting is similar to that of Sections 2.2.1 and 2.3.1 with the main difference that here, we allow for some *parameter uncertainty* in the dynamics of Y. This uncertainty is incorporated through the Q-matrix (MC case) or the drift of the volatility (diffusion case), and is represented in either case by the parameter process $\pi = (\pi_t)_{t\geq 0}$. The standing assumption is that π is only known to lie within two level-dependent values at each time.

Let (X, Y) be defined on a probability space $(\Omega, \mathcal{F}, P)$, and adapted to a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ of sub-σ algebras of $\mathcal{F}$. The process X evolves with the dynamics

$$dX_t = a(X_t)\, Y_t\, dB_t + \mu(X_t)\, dt, \tag{3.1}$$

where B is an $\mathbb{F}$-adapted Brownian motion, and a, μ are continuous functions and $a^2(\cdot) > 0$. Now, $Y = Y^\pi$ is a continuous-time controlled Markov chain or a controlled diffusion process. More precisely, the sample paths of Y are affected by a *control* process $\pi = (\pi_t)_{t \geq 0}$ adapted to $\mathbb{F}$. Such a process belongs to a set $\mathcal{A}$ of *admissible controls* which we now define.

Definition 3.1 (Admissible controls)

(i) Let $\mathcal{S}$ be a finite subset of $(0, \infty)$ (which we assume to be $\{1, 2, \ldots, m\}$ without loss of generality). At time t, given $y, y' \in \mathcal{S}$, $\pi_t[y, y']$ represents the infinitesimal rate at which Y_t jumps from y to y'. We write π_t to denote the matrix $(\pi_t[y, y'])$.

Let $\mathfrak{U}_{MC}$ be the class consisting of all $\mathbb{F}$-adapted processes $\pi = (\pi_t)_{t \geq 0}$ with values in the space of Q-matrices of jump rates on the state space $\mathcal{S}$.

The set $\mathcal{A}$ of admissible controls is defined as

$$\mathcal{A} = \{\pi = (\, (\pi_t[y, y'])\,)_{t \geq 0} \in \mathfrak{U}_{MC} : \; \pi_t[y, y'] \in A_{y,y'}, \; y, y' \in \mathcal{S}, \, t \geq 0\},$$

where $A_{y,y'} = \{0\}$ if $|y - y'| > 1$ and $A_{y,y'}$ is a compact subset of $[0, \infty)$ if $|y - y'| = 1$. In other words if $\pi \in \mathcal{A}$ then, for each $t \geq 0$, $(\pi_t[y, y'])$ is a tridiagonal $m \times m$-matrix satisfying:

$$\pi_t[y, y'] = 0 \quad \text{if } |y - y'| > 1, \qquad \pi_t[y, y'] \geq 0 \quad \text{if } |y - y'| = 1, \qquad \sum_{y' \in \mathcal{S}} \pi_t[y, y'] = 0.$$

(ii) Let $\mathfrak{U}_D$ be the class consisting of all $\mathbb{F}$-adapted processes $\pi = (\pi_t)_{t \geq 0}$ with values in $\mathbb{R}$. Given $\pi \in \mathfrak{U}_D$, suppose that Y^π satisfies the equation

$$dY_t = \eta(Y_t)\, dB^Y_t + \pi_t\, dt,$$

where B^Y is an $\mathbb{F}$-adapted Brownian motion, $\mathbb{F}$-adapted, with $\langle B, B^Y \rangle_t = \delta\, t$ for some $\delta \in [-1, 1]$, and η is a continuous function with $\eta^2(\cdot) > 0$. Assume that Y^π has state space $\mathcal{S} \subseteq (0, \infty)$.

The set $\mathcal{A}$ of admissible controls is defined as

$$\mathcal{A} = \{\pi = (\pi_t)_{t\geq 0} \in \mathfrak{U}_{IP} : \ \pi_t \in A_{Y_t}, \ t \geq 0\},$$

where A_y is a compact subset of $\mathbb{R}$, for each $y \in \mathcal{S}$.

In either case, we denote the family of admissible controls by $\mathcal{A}$ and the state space $\mathbb{R} \times \mathcal{S}$ of (X, Y) by $\mathcal{E}$. The precise description of the sets will be apparent from the context.

Remark 3.2 If $\pi \in \mathfrak{U}_{MC}$ is constant, that is $\pi_t[y, y'] = q[y, y']$ for all $y, y' \in \mathcal{S}$ and all $t \geq 0$, then we are in the setting of Section 2.2.1. If $\pi \in \mathfrak{U}_D$ is such that $\pi_t = \theta(Y_t)$ for all $t \geq 0$, then it corresponds to the setting of Section 2.3.1.

For each $T \in [0, \infty]$, denote by $\mathcal{M}_T$ the family of all stopping times τ with respect to the filtration $\mathbb{F}$, which are no greater than T. If $T = \infty$ we simply write $\mathcal{M}$.

For each π and τ, define the objective function

$$J_{x,y}(\tau, \pi) = E_{x,y}\left[e^{-\alpha\tau} g(X^\pi_\tau) - \int_0^\tau e^{-\alpha\, s} c(X^\pi_s)\, ds\right], \quad (x, y) \in \mathbb{R} \times \mathcal{S}, \quad (3.2)$$

where $\alpha > 0$, g is a non-negative and continuous function, and c is a non-negative and bounded function.

Throughout this chapter, we assume that $\lim_{t\to\infty} e^{-\alpha t} g(X^\pi_t) = 0$ a.s. for each π, so that $e^{-\alpha\tau} g(X^\pi_\tau) = 0$ on the event $\{\tau = \infty\}$.

Problem. Find a pair $(\hat{\tau}, \hat{\pi})$ such that

$$\sup_\tau \inf_\pi J_{x,y}(\tau, \pi) = J_{x,y}(\hat{\tau}, \hat{\pi}) = \inf_\pi \sup_\tau J_{x,y}(\tau, \pi). \quad (3.3)$$

where τ is chosen from $\mathcal{M}_T$ and π from a set $\mathcal{A}$ of admissible controls to be specified below.

This Problem is associated to a zero-sum game of stopping and control with objective function $J_{x,y}(\tau, \pi)$, in which the maximizer chooses a stopping rule $\tau \in \mathcal{M}_T$ whereas the minimizer chooses a control $\pi \in \mathcal{A}$. It is common to refer to $(\hat{\tau}, \hat{\pi})$ as a *saddle point* and $J_{x,y}(\hat{\tau}, \hat{\pi})$ as the *value of the game*.

3.2 Optimal stopping and regularity

Suppose that (X, Y) is the strong Markov process taking values in $\mathcal{E} = \mathbb{R} \times \mathcal{S}$, given as in either Section 2.2.1 (regime-switching) or 2.3.1 (diffusion) setting, and X has drift as in (3.1).

Consider the value function:

$$v(x, y) = \sup_{\tau \in \mathcal{M}_T} E_{x,y} \left[e^{-\alpha\tau} g(X_\tau) - \int_0^\tau e^{-\alpha\, s} c(X_s)\, ds \right]. \tag{3.4}$$

When $T < \infty$, we will need to emphasize the dependence of the value function on the *time to expiration*: for each $t \in [0, T]$,

$$v(x, y, t) = \sup_{\tau \in \mathcal{M}_t} E_{x,y}[\, e^{-\alpha\tau} g(X_\tau) - \int_0^\tau e^{-\alpha\, s} c(X_s)\, ds\,]. \tag{3.5}$$

Notice that $v(x, y, 0) = g(x)$ and $v(x, y, T) = v(x, y)$.

Typically, there are two related approaches to study the value functions in (3.4)-(3.5): the martingale and the analytical one. The first one refers to a probabilistic interpretation of v, whereas the latter refers to regularity properties of v as a real-valued function. These approaches are linked through the probabilistic representation of the solution of an appropriate free-boundary problem, which coincides with v.

3.2.1 The infinite horizon case

Assume that the time horizon is infinite, i.e. $T = \infty$.

We know that (see Theorem B.3 and Remark B.4) the optimal stopping time in the problem (3.4) is given by

$$\tau^* = \inf\{t \geq 0 : \ (X_t, Y_t) \notin \mathcal{C}\} \leq \infty, \tag{3.6}$$

the first exit time of (X_t, Y_t) from the so-called *continuation region*

$$\mathcal{C} = \{(x, y) \in \mathcal{E} : \ v(x, y) > g(x)\}.$$

That is,

$$v(x,y) = E_{x,y}\left[e^{-\alpha\tau^*}g(X_{\tau^*}) - \int_0^{\tau^*} e^{-\alpha\, s}c(X_s)\,ds\right].$$

We proceed to derive some analytical properties of v. The approach relies on some regularity of the parameters of the problem, including *a priori* information on the value function itself, as we see next.

We start with the diffusion setting.

Proposition 3.3 (Diffusion case) *Suppose that the infinitesimal generator $\mathbb{L}$ of (X,Y), acting on functions $h:\mathcal{E}\to\mathbb{R}$ with $h\in C^{2,2}(\mathcal{E})$, is*

$$\frac{1}{2}a(x)^2y^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}\eta^2(y)\frac{\partial^2}{\partial y^2} + \mu(x)\frac{\partial}{\partial x} + \pi(y)\frac{\partial}{\partial y}. \tag{3.7}$$

Assume that the functions a,μ,η,π and c are locally Hölder continuous. If $v(x,y)$ is (jointly) continuous then v is the probabilistic solution of the Dirichlet-type problem

$$\begin{aligned}(\mathbb{L}-\alpha)h(x,y) &= c(x), &&\text{in } \mathcal{C}\\ h(x,y) &= g(x), &&\text{on } \partial\mathcal{C}.\end{aligned}$$

In particular, v is $C^{2,2}$ in $\mathcal{C}$.

Proof. The assertion $v(x,y)=g(x)$ on $\partial\mathcal{C}$ is clear.

Since both v and g are continuous, $\mathcal{C}$ is open. Fix $(x_0,y_0)\in\mathcal{C}$ and let U be an open ball centered at (x_0,y_0) which is strictly contained in $\mathcal{C}$. Now consider the revised Dirichlet-type problem

$$\begin{aligned}(\mathbb{L}-\alpha)h(x,y) &= c(x), &&\text{in } U\\ h(x,y) &= v(x,y), &&\text{on } \partial U.\end{aligned} \tag{3.8}$$

Thanks to the continuity assumption on a and η and that $a(\cdot)^2,\eta(\cdot)^2>0$, we have that $a(\cdot)^2,\eta(\cdot)^2$ are bounded away from zero in any bounded open interval. This fact together with $\mathcal{S}\subseteq(0,\infty)$, imply that the operator $\mathbb{L}$ (and so $\mathbb{L}-\alpha$) is uniformly elliptic in any bounded subset of $\mathcal{E}=\mathbb{R}\times\mathcal{S}$ (see [16] for definition of ellipticity).

Given that a,μ,η,π and c are locally Hölder continuous, that v is continuous, and the fact that $\mathbb{L}-\alpha$ is uniformly elliptic in U, there exists a unique solution h to (3.8) which belongs to $C^{2,2}(U) \cap C^0(\bar{U})$ (see [17, Theorem 6.13] or [16, Theorem 6.2.4]).

Applying Itô's formula to $e^{-\alpha t}h(X_t, Y_t)$, it follows that the probabilistic representation of h in U is given by

$$h(x,y) = E_{x,y}\left[e^{-\alpha\tau_U} v(X_{\tau_U}, Y_{\tau_U}) - \int_0^{\tau_U} e^{-\alpha s} c(X_s) ds\right], \ (x,y) \in U,$$

where τ_U is the first exit time of (X,Y) from U (see also [16, Theorem 6.5.1]).

Moreover, by the strong Markov property of the diffusion (X,Y), and using the fact that $\tau_U \leq \tau^*$, it follows that $h(x,y) = v(x,y)$ everywhere in U (see Lemma B.5). In particular, the partial derivatives v_{xx}, v_{yy}, v_x, v_y exist and are continuous at (x_0, y_0).

Since $(x_0, y_0) \in \mathcal{C}$ was arbitrary the claim of the proposition follows. □

We now proceed to verify the smoothness of $v(\cdot, y)$ in the regime-switching case. An important difference from the previous setting is that the associated Dirichlet-type problem is not in its classic form because the infinitesimal generator of (X,Y) *is not* a linear partial differential operator. We use a local argument to *freeze* the Markov chain Y_t at the first jump time so that X_t, up to this time, is an autonomous process and the associated generator *is* a linear partial differential operator. Although this technique is natural, we are not aware of its application in the literature.

For each $y \in \mathcal{S}$, let $\mathcal{C}_y$ be the y-section of the continuation set, that is,

$$\mathcal{C}_y := \{x \in \mathbb{R} : (x,y) \in \mathcal{C}\}.$$

Proposition 3.4 (Regime-switching case) *Suppose that the infinitesimal generator $\mathbb{L}$ of (X,Y), acting on functions $h : \mathcal{E} \to \mathbb{R}$ with $h(\cdot, y) \in C^2(\mathbb{R})$, is*

$$\frac{1}{2}a(x)^2 y^2 \frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x} + [h(x,y+1) - h(x,y)]\lambda_y + [h(x,y-1) - h(x,y)]\mu_y, \tag{3.9}$$

where $\lambda_y, \mu_y \geq 0$ are the upwards and downwards jump rates, respectively. Assume that the functions a, μ and c are locally Hölder continuous. If $v(\cdot, y)$ is locally Hölder continuous, then v is the probabilistic solution of the Dirichlet-

type problem

$$\begin{aligned}(\mathbb{L}-\alpha)h(x,y) &= c(x), \qquad \textit{in } \mathcal{C}\\ h(x,y) &= g(x), \qquad \textit{on } \partial\mathcal{C}.\end{aligned}$$

In particular, for each $y \in \mathcal{S}$, $v(\cdot, y)$ *is* C^2 *in* $\mathcal{C}_y$.

Proof. The assertion $v(x,y) = g(x)$ on $\partial\mathcal{C}$ is clear.

Since $v(\cdot, y)$ and g are continuous, $\mathcal{C}_y$ is open.

Fix $y \in \mathcal{S}$ and choose an arbitrary $x_0 \in \mathcal{C}_y$. Let $I = (l, u)$ be an open interval centered at x_0 such that $I \subset \mathcal{C}_y$.

Define $f(x) := \lambda_y v(x, y+1) + \mu_y v(x, y-1)$, which is a locally Hölder continuous function, and the linear ordinary differential operator $\tilde{L}$ by

$$\tilde{L} := \frac{1}{2}a(x)^2 y^2 \frac{\partial^2}{\partial x^2} + \mu(x)\frac{\partial}{\partial x} - \kappa(y), \tag{3.10}$$

where $\kappa(y) = \lambda_y + \mu_y$ (the rate of leaving y). Since a is continuous and $a^2(\cdot) > 0$, the operator $\tilde{L}$ is uniformly elliptic in I.

Consider the Dirichlet problem (in the variable x):

$$\begin{aligned}(\tilde{L}-\alpha)H(x) &= c(x) - f(x), \qquad \text{in } I\\ H(l) &= v(l,y),\\ H(u) &= v(u,y)\end{aligned} \tag{3.11}$$

Given that a, μ and $c - f$ are locally Hölder continuous in I, and that $\tilde{L} - \alpha$ is uniformly elliptic in I, there exists a unique solution H to (3.11) which belongs to $C^2(I) \cap C^0(\bar{I})$ (see [17, Theorem 6.13]).

Define h on $\bar{I} \times \mathcal{S}$ as follows: for each $x \in \bar{I}$, set $h(x,y) := H(x)$ and $h(x,y') := v(x,y')$ for $y' \neq y$. Now, we aim to give a probabilistic representation of $h(x,y)$, $x \in I$.

Let T_1 be the first exit time of X from I and T_2 be the first jump time of the Markov chain Y from y, and set $\tau = T_1 \wedge T_2$. We can apply Dynkin's formula to obtain, for all $x \in I$,

$$h(x,y) = E_{x,y}e^{-\alpha\tau}h(X_\tau, Y_\tau) - E_{x,y}\left[\int_0^\tau e^{-\alpha s}(\mathbb{L}-\alpha)h(X_s,Y_s)ds\right].$$

On the one hand notice that, for all $s < \tau$,

$$(\mathbb{L} - \alpha)h(X_s, Y_s) = (\tilde{L} - \alpha)h(X_s, Y_s) + f(X_s) = c(X_s).$$

On the other hand, $h(X_\tau, Y_\tau) = v(X_\tau, Y_\tau)$ because of the boundary condition on H and the definition of h. Putting the latter facts together,

$$h(x, y) = E_{x,y}\left[e^{-\alpha\tau}v(X_\tau, Y_\tau) - \int_0^\tau e^{-\alpha s}c(X_s)ds\right], \; x \in I.$$

Moreover, using the strong Markov property and the fact that $\tau \leq \tau^*$, it follows that $h(\cdot, y) = v(\cdot, y)$ in I (see Lemma B.5). But $h(\cdot, y) = H(\cdot)$ in I, which implies that $(\mathbb{L} - \alpha)v(x, y) = c(x)$, for all $x \in I$, as required. In particular, $v(\cdot, y) \in C^2(I)$.

Since $y \in \mathcal{S}$ and $x_0 \in \mathcal{C}_y$ were chosen arbitrarily, the claim of the proposition follows. □

Remark 3.5 The arguments in the proof of Proposition 3.4 are not restricted to the case where the Q-matrix of Y is tridiagonal. We only assumed this to tailor the result to our setting.

Remark 3.6 We know from Theorem B.3 in Appendix B that, defining

$$V_t := e^{-\alpha t}v(X_t, Y_t) - \int_0^t e^{-\alpha s}c(X_s)\, ds,$$

the stopped process $V_{\cdot\wedge\tau^*} = (V_{t\wedge\tau^*})_{t\geq 0}$ is a martingale under $P_{x,y}$, for each fixed $(x, y) \in \mathcal{E}$.

If moreover $V_{\cdot\wedge\tau^*}$ is known to be uniformly integrable (e.g. when g is bounded), then the assertion $h(x, y) = v(x, y)$ in the proofs of smoothness of v are readily seen. For instance, in the proof of Proposition 3.3, we obtain that

$$h(x, y) = E_{x,y}\, V_{\tau_U} = E_{x,y}\, V_{\tau^*} = v(x, y)$$

since $\tau_U \leq \tau^*$ (see Theorem II.3.2 in [39]). This holds even if $\tau^* = \infty$ with positive probability (see Appendix B for more details).

3.2.2 The finite horizon case

Let us now assume that the time horizon is finite, i.e. $T < \infty$.

Fix $t \in [0, T]$. We know that (see Theorem B.3 and Remark B.6) the optimal stopping time in the problem (3.5) is given by

$$\tau_t^* = \inf\{0 \leq s \leq t : (X_s, Y_s, t - s) \notin \mathcal{C}\} \leq t, \tag{3.12}$$

the first exit time of $(X_s, Y_s, t - s)$ from

$$\mathcal{C} = \{(x, y, u) \in \mathcal{E} \times [0, T] : v(x, y, u) > g(x)\}.$$

That is,

$$v(x, y, t) = E_{x,y}\left[e^{-\alpha\,\tau_t^*} g(X_{\tau_t^*}) - \int_0^{\tau_t^*} e^{-\alpha\, s} c(X_s)\, ds\right].$$

In the next statements we consider the parabolic operator $\mathbb{L} - \alpha - \frac{\partial}{\partial t}$, where $\mathbb{L}$ will be either the operator (3.7) in Proposition 3.7 or (3.9) (with $h(x, \cdot)$ replaced by $h(x, \cdot, t)$) in Proposition 3.8. Although the idea of the proof of smoothness of v is very similar to that of the infinite horizon case, we provide the proofs for completeness. Since $T < \infty$, we show instead that v identifies with the solution of a *revised* initial-boundary value-type problem by restricting the domain to a bounded cylinder contained in $\mathcal{C}$ with v as the boundary condition. Again, the strong Markov property of (X, Y) plays an important role for this identification to hold.

Proposition 3.7 *Let the assumptions of Proposition 3.3 be satisfied. If $v(x, y, t)$ is (jointly) continuous then v is the probabilistic solution of the initial-boundary value-type problem*

$$(\mathbb{L} - \alpha - \partial/\partial\, t) h(x, y, t) = c(x), \qquad \text{in } \mathcal{C},$$
$$h(x, y, t) = g(x), \qquad \text{on } \partial\mathcal{C},$$

where $\mathbb{L}$ is in (3.7). In particular, v is $C^{2,2,1}$ in $\mathcal{C}$.

Proof. The assertion $v(x, y, t) = g(x)$ on $\partial\mathcal{C}$ is clear.

Fix $(x_0, y_0, t_0) \in \mathcal{C}$. Since both v and g are continuous $\mathcal{C}$ is open, and so we can choose an open rectangle R around (x_0, y_0) and an open interval $I = (t_1, t_2)$ around t_0 such that the cylinder $Q = R \times I$ is open and contained in $\mathcal{C}$.

Let B^r denote the interior of $\bar{Q} \cap \{t = t_2\}$ (this is the area obtained by intersecting the closure of the cylinder with the *right* end-point of I, and without the edge) and consider the revised initial-boundary value-type problem

$$\begin{aligned} (\mathbb{L} - \alpha - \partial/\partial t)h(x, y, t) &= c(x), \qquad & \text{in } Q \cup B^r \\ h(x, y, t) &= v(x, y, t), \qquad & \text{on } \partial Q \backslash B^r. \end{aligned} \tag{3.13}$$

The operator $\mathbb{L} - \alpha - \partial/\partial t$ is uniformly parabolic in Q. This property is inherited from the uniform ellipticity of $\mathbb{L}$ in R and the fact that the functions a and η are time-homogeneous.

Given that, by assumption, the functions a, μ, η, π and c are locally Hölder (hence uniformly Hölder in $\bar{Q}$) and v is continuous, there exists a unique solution h to (3.13) which is continuous in $\bar{Q}$ and has continuous derivatives $h_x, h_y, h_{xx}, h_{yy}, h_t$ (see [16, Theorem 6.3.6]).

Take an arbitrary $(x, y, t) \in Q$ and let τ_Q be the first exit time of the process $(X_s, Y_s, t - s)$ from Q (notice that it cannot exit across B^r). Applying Itô's formula to $e^{-\alpha s} h(X_s, Y_s, t - s)$ we obtain that the probabilistic representation of h in Q is given by

$$h(x, y, t) = E_{x,y}\left[e^{-\alpha\tau_Q} v(X_{\tau_Q}, Y_{\tau_Q}, t - \tau_Q) - \int_0^{\tau_Q} e^{-\alpha u} c(X_u) du\right]$$

Finally, using that $\tau_Q \leq \tau_t^*$, one sees that $h(x, y, t) = v(x, y, t)$ everywhere in Q (see Lemma B.7). In particular, $v \in C^{2,2,1}(Q)$.

Since $(x_0, y_0, t_0) \in \mathcal{C}$ was arbitrary the proof is complete. □

In what follows, for each $y \in \mathcal{S}$, $\mathcal{C}_y$ denotes the set

$$\mathcal{C}_y := \{(x, t) \in \mathbb{R} \times [0, T] : (x, y, t) \in \mathcal{C}\}.$$

Proposition 3.8 *Let the assumptions of Proposition 3.4 be satisfied. If $v(\cdot, y, \cdot)$ is locally Hölder continuous then v is the probabilistic solution of the initial-boundary value-type problem*

$$\begin{aligned}(\mathbb{L} - \alpha - \partial/\partial t)h(x, y, t) &= c(x), \qquad &&\text{in } \mathcal{C}\\ h(x, y, t) &= g(x), \qquad &&\text{on } \partial\mathcal{C}.\end{aligned}$$

where $\mathbb{L}$ is in (3.9). In particular, for each $y \in \mathcal{S}$, $v(\cdot, y, \cdot)$ is $C^{2,1}$ in $\mathcal{C}_y$.

Proof. Fix $y \in \mathcal{S}$ and $(x_0, t_0) \in \mathcal{C}_y$.

Since $v(\cdot, y, \cdot)$ and g are continuous, $\mathcal{C}_y$ is open. Let Q be an open rectangle around (x_0, t_0) such that $Q \subseteq \mathcal{C}_y$, and B^r be the upper open edge of Q.

Define $f(x, t) := \lambda_y\, v(x, y + 1, t) + \mu_y\, v(x, y - 1, t)$, which is a continuous function, and recall the operator $\tilde{L}$ in (3.10).

Consider the classical initial-boundary value problem

$$\begin{aligned}(\tilde{L} - \alpha - \partial/\partial t)H(x, t) &= c(x) - f(x, t), \qquad &&\text{in } Q \cup B^r\\ H(x, t) &= v(x, y, t), &&\text{on } \partial Q \backslash B^r.\end{aligned} \tag{3.14}$$

The operator $\tilde{L} - \alpha - \partial/\partial t$ is uniformly parabolic in Q. Arguing as before, by Theorem 6.3.6 in [16], we conclude that there exists a solution H to (3.14) which is continuous in $\bar{Q}$ and such that $H(x, t) \in C^{2,1}(Q)$.

Recall that $y \in \mathcal{S}$ is fixed from the beginning. Now, for each $(t, x) \in Q$ and $y' \in \mathcal{S}$, set

$$h(x, y', t) := \begin{cases} H(x, t) & \text{if } y' = y\\ v(x, y', t) & \text{if } y' \neq y. \end{cases}$$

Using this function and a local argument (we freeze Y at its first jump), we will show that

$$v(x, y, t) \equiv H(x, t) \qquad \text{in } Q.$$

Take $(t, x) \in Q$. Consider the stopping times $T_1 = \inf\{s \geq 0 : (X_s, t - s) \notin Q\}$ and $T_2 = \inf\{s \geq 0 : Y_s \neq y\}$, and set $\tau = T_1 \wedge T_2$. Notice that $T_1 \leq t$ a.s.

Since $Y_s \equiv y$ up to the time τ, an application of Dynkin's formula yields

$$h(x,y,t) = E_{x,y}\left[e^{-\alpha\tau}h(X_\tau, Y_\tau, t-\tau)\right] - E_{x,y}\left[\int_0^\tau e^{-\alpha\tau}(\mathbb{L} - \alpha - \partial/\partial t)h(X_s, Y_s, t-s)ds\right].$$

Now, $h(X_\tau, Y_\tau, t-\tau) = v(X_\tau, Y_\tau, t-\tau)$ a.s. Indeed, if $\tau = T_1$ we use the boundary condition in (3.14), otherwise the equality still holds by the definition of h.

Moreover, after a simple algebraic manipulation it can be seen that, for each $(t,x) \in Q$,

$$(\mathbb{L} - \alpha - \partial/\partial t)h(x,y,t) = (\tilde{L} - \alpha - \partial/\partial t)h(x,y,t) + f(x,t) = c(x).$$

We then arrive to the expression

$$h(x,y,t) = E_{x,y}\left[e^{-\alpha\tau}v(X_\tau, Y_\tau, t-\tau) - \int_0^\tau e^{-\alpha\tau}c(X_s)ds\right].$$

Upon recalling Lemma B.5 it is seen that $h(\cdot, y, \cdot) = v(\cdot, y, \cdot)$ in Q. But $h(\cdot, y, \cdot) = H(x,t)$ in Q, which in turn yields that $(\mathbb{L}-\alpha-\partial/\partial t)v(x,y,t) = c(x)$ and in particular $v(\cdot, y, \cdot) \in C^{2,1}(Q)$.

Since $y \in \mathcal{S}$ and $(x_0, y_0) \in \mathcal{C}_y$ were arbitrary, the proof is complete. □

3.3 Characterization of the value of the game

Proposition 3.9 *Suppose that there exists $\hat{\tau} \in \mathcal{M}_T$ and $\hat{\pi} \in \mathcal{A}$, and a function $w(x,y) : \mathcal{E} \to \mathbb{R}$ such that*

$$w(x,y) \le J_{x,y}(\hat{\tau}, \pi), \qquad \text{and} \qquad w(x,y) \ge J_{x,y}(\tau, \hat{\pi}), \tag{3.15}$$

for all $\pi \in \mathcal{A}$ and $\tau \in \mathcal{M}_T$. Then w is the value of the game, that is,

$$\sup_\tau \inf_\pi J_{x,y}(\tau, \pi) = w(x,y) = \inf_\pi \sup_\tau J_{x,y}(\tau, \pi). \tag{3.16}$$

If $w(x,y) \equiv J_{x,y}(\hat{\tau}, \hat{\pi})$ is such a function, then $(\hat{\tau}, \hat{\pi})$ is a saddle point.

Proof. Fix an initial condition $(x, y) \in \mathcal{E}$.

The last assertion simply refers to the defining property of a saddle point.

The first inequality in (3.15) implies two things: first, that $w(x, y) \leq \sup_\tau J_{x,y}(\tau, \pi)$ for all π, which yields

$$w(x, y) \leq \inf_\pi \sup_\tau J_{x,y}(\tau, \pi);$$

second, that

$$w(x, y) \leq \inf_\pi J_{x,y}(\hat{\tau}, \pi) \leq \sup_\tau \inf_\pi J_{x,y}(\tau, \pi).$$

Similarly, the second inequality in (3.15) also implies two things: first, that $w(x, y) \geq \inf_\pi J_{x,y}(\tau, \pi)$ for all τ, and so

$$w(x, y) \geq \sup_\tau \inf_\pi J_{x,y}(\tau, \pi);$$

and finally, that

$$w(x, y) \geq \sup_\tau J_{x,y}(\tau, \hat{\pi}) \geq \inf_\pi \sup_\tau J_{x,y}(\tau, \pi).$$

Putting all together we can see that if w satisfies (3.15) then it is the value of the game. □

The saddle point, if it exists, may not be unique as the following example shows.

Example 3.10 Suppose that $g(x) \equiv 0$ and that $c(x) > 0$ for all $x \in \mathbb{R}$. Then $J_{x,y}(\tau, \pi) \leq 0$ for all τ and π.

Set $\hat{\tau} = 0$, and note that $J_{x,y}(\hat{\tau}, \pi) = 0$ for all π. Hence, for each π, the pair $(\hat{\tau}, \pi)$ is a saddle point and $w(x, y) \equiv 0$ is the value of the game.

3.3.1 Verification theorems

For each constant $\pi \in \mathcal{A}_y$, denote by $\mathbb{L}^\pi$ the operator

$$\mathbb{L}^\pi = \frac{1}{2}a(x)^2\,y^2\frac{\partial^2}{\partial x^2} + \frac{1}{2}\eta^2(y)\frac{\partial^2}{\partial y^2} + \mu(x)\frac{\partial}{\partial x} + \pi\frac{\partial}{\partial y}.$$

Theorem 3.11 (Diffusion case, $T = \infty$) *Assume that the horizon T is infinite. Suppose that $w : \mathcal{E} \to \mathbb{R}$ is a continuous function such that the following conditions hold:*

(i) Its restriction on the open set $\mathcal{C} = \{(x,y) \in \mathcal{E} : w(x,y) > g(x)\}$ is $C^{2,2}$, and satisfies

$$\begin{aligned} \inf_{\pi\in\mathcal{A}_y} (\mathbb{L}^\pi - \alpha)w(x,y) &= c(x), && \text{in } \mathcal{C}, \\ w(x,y) &= g(x), && \text{on } \mathcal{E}\backslash\mathcal{C}, \end{aligned} \tag{3.17}$$

where the infimum is taken over all constant and admissible values.

(ii) For each $\pi \in \mathcal{A}$, let $\hat{\tau} \equiv \hat{\tau}^\pi := \inf\{t \geq 0 : (X_t^\pi, Y_t^\pi) \notin \mathcal{C}\}$. The family $\{e^{-\alpha\tau}w(X_\tau^\pi, Y_\tau^\pi) :$ finite $\tau \leq \hat{\tau}\}$ is uniformly integrable.

(iii) For each $\pi \in \mathcal{A}$, $\lim_{t\to\infty} E_{x,y}e^{-\alpha t}w(X_t^\pi, Y_t^\pi) = 0$.

Then

$$w(x,y) \leq J_{x,y}(\hat{\tau}, \pi), \qquad \text{for all } \pi \in \mathcal{A}. \tag{3.18}$$

Proof. Fix an initial condition $(x,y) \in \mathcal{E}$.

Pick an arbitrary $\pi \in \mathcal{A}$ and define the process $N(\pi) = (N_t(\pi))_{t\geq 0}$ by

$$N_t(\pi) := e^{-\alpha t}w(X_t^\pi, Y_t^\pi), \qquad t \geq 0.$$

For each $R > 0$, let U_R denote the open ball centered at (x,y) of radius R, and let τ_R denote

$$\tau_R = \min\{\hat{\tau},\, \inf\{t \geq 0 : (X_t^\pi, Y_t^\pi) \notin U_R\}\} \leq \infty.$$

Notice that $\tau_R \to \hat{\tau}$ a.s. as $R \to \infty$.

Since w is a smooth function in $C^{2,2}(\mathcal{C})$, we can apply Itô's formula for semimartingales (see Theorem II.33 in [38]) to obtain

$$N_{t\wedge\tau_R}(\pi) - w(x,y) = \int_0^{t\wedge\tau_R} e^{-\alpha u}(L_u^\pi - \alpha)w(X_u^\pi, Y_u^\pi)du + M_{t\wedge\tau_R},$$

where

$$M_t = \int_0^t e^{-\alpha s} w_x(X_s^\pi, Y_s^\pi)\, a(X_s^\pi) Y_s^\pi dB_s + \int_0^t e^{-\alpha s} w_y(X_s^\pi, Y_s^\pi)\eta(Y_s^\pi) dB_s^Y,$$

and for each $\omega \in \Omega$,

$$L_u^\pi(\omega) = \frac{1}{2}a(x)^2\, y^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}\eta^2(y)\frac{\partial^2}{\partial y^2} + \mu(x)\frac{\partial}{\partial x} + \pi_u(\omega)\frac{\partial}{\partial y}.$$

Given that w_x, w_y, a and η are continuous, they are bounded in U_R. This implies that $M_{t\wedge\tau_R}$ has bounded quadratic variation for each $t \geq 0$, and hence the process $M_{\cdot\wedge\tau_R}$ is a true martingale.

Moreover, by (3.17) we have that

$$(L_u^\pi - \alpha)w(X_u, Y_u) \geq \inf_{\pi_u \in A_{Y_u}} (\mathbb{L}^\pi - \alpha)w(X_u, Y_u) = c(X_u), \qquad \text{in } \mathcal{C}, \quad (3.19)$$

which yields

$$\int_0^{t\wedge\tau_R} e^{-\alpha u}(L_u^\pi - \alpha)w(X_u^\pi, Y_u^\pi)du \geq \int_0^{t\wedge\tau_R} e^{-\alpha u}\, c(X_u^\pi)du.$$

Hence for each $R > 0$,

$$N_{t\wedge\tau_R}(\pi) - w(x,y) \geq \int_0^{t\wedge\tau_R} e^{-\alpha u}\, c(X_u^\pi)du + M_{t\wedge\tau_R},$$

and after taking expectation we obtain

$$w(x,y) \leq E_{x,y}\left[N_{t\wedge\tau_R}(\pi) - \int_0^{t\wedge\tau_R} e^{-\alpha u}\, c(X_u^\pi)du\right]. \qquad (3.20)$$

Using that $\tau_R \to \hat{\tau}$ a.s. and the continuity of w we obtain the limit

$$\lim_{R\to\infty} N_{t\wedge\tau_R}(\pi) = e^{-\alpha\hat{\tau}\wedge t} w(X^\pi_{t\wedge\hat{\tau}}, Y^\pi_{t\wedge\hat{\tau}}), \quad \text{a.s.}$$

and further

$$\lim_{t\to\infty} e^{-\alpha\hat{\tau}\wedge t} w(X^\pi_{t\wedge\hat{\tau}}, Y^\pi_{t\wedge\hat{\tau}}) = e^{-\alpha\hat{\tau}} g(X^\pi_{\hat{\tau}}), \quad \text{on } \{\hat{\tau} < \infty\} \text{ a.s.}$$

by the definition of $\hat{\tau}$.

Since the family $\{e^{-\alpha\tau} w(X^\pi_\tau, Y^\pi_\tau) : \text{finite}\, \tau \le \hat{\tau}\}$ is uniformly integrable by assumption, it follows by dominated convergence that, for each $t \ge 0$,

$$\lim_{R\to\infty} E_{x,y}\left[N_{t\wedge\tau_R}(\pi)\right] = E_{x,y}\left[e^{-\alpha\hat{\tau}\wedge t} w(X^\pi_{t\wedge\hat{\tau}}, Y^\pi_{t\wedge\hat{\tau}})\right].$$

On the one hand, by dominated convergence, we can see that

$$\lim_{t\to\infty} E_{x,y}[e^{-\alpha\hat{\tau}\wedge t} w(X^\pi_{t\wedge\hat{\tau}}, Y^\pi_{t\wedge\hat{\tau}}) I(\hat{\tau} < \infty)] = E_{x,y}[e^{-\alpha\hat{\tau}} g(X^\pi_{\hat{\tau}})]$$

since $e^{-\alpha\hat{\tau}} g(X^\pi_{\hat{\tau}}) = 0$ on $\{\hat{\tau} = \infty\}$ a.s.

On the other hand, the assumption in (iii) yields

$$\lim_{t\to\infty} E_{x,y}[e^{-\alpha t} w(X^\pi_t, Y^\pi_t) I(\hat{\tau} = \infty)] = 0.$$

Therefore, after taking the limit as $R \to \infty$ and $t \to \infty$ in (3.20), we obtain

$$w(x,y) \le E_{x,y}\left[e^{-\alpha\hat{\tau}} g(X^\pi_{\hat{\tau}}) - \int_0^{\hat{\tau}} e^{-\alpha u}\, c(X^\pi_u) du\right] = J_{x,y}(\hat{\tau}, \pi),$$

where we also used monotone convergence for the Lebesgue integral part. This verifies (3.18), since π was chosen arbitrarily, and the proof is complete. □

In what follows, each constant, admissible Q-matrix $(\pi[y,y'])$, denote by $\mathbb{L}^\pi$ the operator

$$\mathbb{L}^\pi w(x,y) = \frac{1}{2} a(x)^2\, y^2 w_{xx}(x,y) + \mu(x) w_x(x,y) + [w(x,y+1) - w(x,y)]\pi[y,y+1] + [w(x,y-1) - w(x,y)]\pi[y,y-1].$$

Theorem 3.12 (Regime-switching case, $T = \infty$) *Assume that the horizon T is infinite. Suppose that $w : \mathcal{E} \to \mathbb{R}$ is a function such that for each $x \in \mathbb{R}$, $w(x, \cdot)$ is bounded; and for each $y \in \mathcal{S}$, $w(\cdot, y)$ is continuous. Suppose that:*

(i) The restriction of $w(\cdot, y) : \mathbb{R} \to \mathbb{R}$ on the open set $\mathcal{C}_y = \{x \in \mathbb{R} : w(x, y) > g(x)\}$ is C^2, and that the function w satisfies

$$\begin{aligned} \inf_{\pi} (\mathbb{L}^{\pi} - \alpha) w(x, y) &= c(x), \qquad \textit{in } \mathcal{C}, \\ w(x, y) &= g(x), \qquad \textit{on } \mathcal{E}\backslash\mathcal{C}, \end{aligned} \tag{3.21}$$

where $\mathcal{C} = \{(x, i) \in \mathcal{E} : w(x, i) > g(x)\}$ and the infimum is taken over all constant and admissible Q-matrices.

(ii) For each $\pi \in \mathcal{A}$, let $\hat{\tau} \equiv \hat{\tau}^{\pi} := \inf\{t \geq 0 : (X_t^{\pi}, Y_t^{\pi}) \notin \mathcal{C}\}$. The family $\{e^{-\alpha\tau} w(X_{\tau}^{\pi}, Y_{\tau}^{\pi}) :$ finite $\tau \leq \hat{\tau}\}$ is uniformly integrable.

(iii) For each $\pi \in \mathcal{A}$, $\lim_{t\to\infty} E_{x,y} e^{-\alpha t} w(X_t^{\pi}, Y_t^{\pi}) = 0$.

Then

$$w(x, y) \leq J_{x,y}(\hat{\tau}, \pi), \qquad \textit{for all } \pi \in \mathcal{A}. \tag{3.22}$$

Proof. The only consideration to bear in mind in this setting is the form of the operator $L_u^{\pi}(\omega) w(x, y)$, which is now given by

$$\begin{aligned} &\frac{1}{2} a(x)^2 y^2 w_{xx}(x, y) + \mu(x) w_x(x, y) + [w(x, y+1) - w(x, y)] \pi_u[y, y+1](\omega) \\ &\quad + [w(x, y-1) - w(x, y)] \pi_u[y, y-1](\omega). \end{aligned}$$

The rest of the arguments are identical to those in Theorem 3.11. □

Remark 3.13 Condition (i) of Theorems 3.11 and 3.12 may also be stated using variational inequalities. That is, we may require instead that w satisfies the following:

$$w(x, y) \geq g(x) \quad \text{everywhere,} \quad \inf_{\pi} (\mathbb{L}^{\pi} w - \alpha\, w - c) \geq 0 \quad \text{in } \mathcal{C} \tag{3.23}$$

where $\mathcal{C} = \{(x, y) \in \mathcal{E} : w(x, y) > g(x)\}$, and

$$\min\left\{ \inf_{\pi} (\mathbb{L}^{\pi} w - \alpha\, w - c),\ w - g \right\} = 0. \tag{3.24}$$

The disadvantage of this formulation is that it needs w and g to satisfy enough regularity conditions to belong to the domain of $\mathbb{L}^\pi$, unlike the formulation in (3.17) and (3.21) which only assumes that w is C^2 in $\mathcal{C}$ and g is non-negative and continuous.

When $T < \infty$ the arguments in the preceding proofs are virtually the same, except for natural changes in the notation. Then we only give the corresponding statements.

Theorem 3.14 (Diffusion case, $T < \infty$) *Assume that the horizon T is finite. Suppose that $w : \mathcal{E} \times [0,T] \to \mathbb{R}$ is a continuous function such the following conditions hold:*

(i) Its restriction on the open set $\mathcal{C} = \{(x,y,t) \in \mathcal{E} \times [0,T] : w(x,y,t) > g(x)\}$ is $C^{2,2,1}$, and satisfies

$$\begin{aligned} \inf_\pi (\mathbb{L}^\pi - \alpha - \partial/\partial t) w(x,y,t) = c(x), \qquad & \textit{in } \mathcal{C}, \\ w(x,y,0) = g(x), \qquad & \textit{on } \mathcal{E} \times \{0\}, \\ w(x,y,t) = g(x), \qquad & \textit{on } \mathcal{E} \times (0,T] \backslash \mathcal{C}, \end{aligned} \tag{3.25}$$

where the infimum is taken over all constant and admissible $\pi \in \mathcal{A}_y$.

(ii) For each $\pi \in \mathcal{A}$, let $\hat{\tau} \equiv \hat{\tau}^\pi := \inf\{t \geq 0 : (X_t^\pi, Y_t^\pi, T-t) \notin \mathcal{C}\} \leq T$. The family $\{e^{-\alpha\tau} w(X_\tau^\pi, Y_\tau^\pi, T-\tau) : \tau \leq \hat{\tau}\}$ is uniformly integrable.

Then $w(x,y,T) \leq J_{x,y}(\hat{\tau}, \pi)$ for all $\pi \in \mathcal{A}$.

Theorem 3.15 (Regime-switching case, $T < \infty$) *Assume that the horizon T is finite. Suppose that $w : \mathcal{E} \times [0,T] \to \mathbb{R}$ is a function such that for each $(x,t) \in \mathbb{R} \times [0,T]$, $w(x,\cdot,t)$ is bounded; and for each $y \in \mathcal{S}$, $w(\cdot,y,\cdot)$ is continuous. Suppose that:*

(i) The restriction of $w(\cdot,y,\cdot) : \mathbb{R} \times [0,T] \to \mathbb{R}$ on the open set $\mathcal{C}_y = \{x \in \mathbb{R} : w(x,y,t) > g(x)\}$ is $C^{2,1}$.

Also, w satisfies (3.25) where the infimum is taken over all constant and admissible Q-matrices $\pi = (\pi[y,y'])$.

(ii) For each $\pi \in \mathcal{A}$, let $\hat{\tau} \equiv \hat{\tau}^\pi := \inf\{t \geq 0 : (X_t^\pi, Y_t^\pi, T-t) \notin \mathcal{C}\} \leq T$. The family $\{e^{-\alpha\tau} w(X_\tau^\pi, Y_\tau^\pi, T-\tau) : \tau \leq \hat{\tau}\}$ is uniformly integrable.

Then $w(x, y, T) \leq J_{x,y}(\hat{\tau}, \pi)$, for all $\pi \in \mathcal{A}$.

3.3.2 Existence of a saddle point

In this section, we characterize the value of the game problem, stated on page 51, as the value function of an optimal stopping problem associated to an *extremal scenario*.

In the context of Proposition 3.3, suppose that $\pi(y)$ is replaced by the admissible control,

$$\pi^{\min}(y) = \inf A_y, \qquad y \in \mathcal{S} \subseteq (0, \infty),$$

which is well-defined since A_y is a compact subset of $\mathbb{R}$. In particular $\pi^{\min} = (\pi^{\min}(Y_t))_{t \geq 0}$ is an admissible and Markovian control.

Let $(X_t, Y_t) \equiv (X_t^{\min}, Y_t^{\min})$ be the associated diffusion process and $v^{\min}$ the corresponding value function as in (3.4). From now on we avoid to write the superscript on $(X_t^{\min}, Y_t^{\min})$ for ease of presentation.

When $T = \infty$, the following convergence in probability will be assumed:

$$\text{For each } \pi \in \mathcal{A}, \qquad e^{-\alpha t} v^{\min}(X_t^\pi, Y_t^\pi) \xrightarrow{p} 0 \tag{3.26}$$

This condition holds, for instance, when g is bounded.

Theorem 3.16 (Diffusion case) *Let $T \in [0, \infty]$ be the time horizon. Suppose that the functions $a, \mu, \eta, \pi^{\min}$ and c are locally Hölder continuous, $v^{\min}$ is continuous, and $v^{\min}$ is non-decreasing as a function of y. If $T = \infty$ also assume that (3.26) holds. Then, for each $(x, y) \in \mathcal{E}$,*

$$\sup_\tau \inf_\pi J_{x,y}(\tau, \pi) = v^{\min}(x, y) = \inf_\pi \sup_\tau J_{x,y}(\tau, \pi).$$

That is, $v^{\min}$ is the value of the game and $(\tau^*, \pi^{\min})$ is a saddle point, where τ^* is the optimal stopping time for $v^{\min}$.

Proof. Set $\hat{\pi} = \pi^{\min}$, $\hat{\tau} = \tau^*$ and $w = v^{\min}$. We only have to verify (3.15).

To start, by the optimality of $\hat{\tau}$,

$$w(x,y) \geq J_{x,y}(\tau, \hat{\pi}), \qquad \text{for all } \tau \in \mathcal{M}_T$$

with equality for $\tau = \hat{\tau}$.

(I) Suppose that $T = \infty$. We will verify Conditions (i)-(iii) of Theorem 3.11.

Since $w(x, \cdot)$ is non-decreasing, we have that for each $y \in \mathcal{S}$

$$\hat{\pi}(y) = \arg\min_{\pi \in \mathcal{A}_y} \frac{\partial w}{\partial y}(x,y)\, \pi,$$

for all $x \in \mathbb{R}$. Using this and Proposition 3.3 it follows that Condition (i) is satisfied.

To check Conditions (ii)-(iii), first notice that the function w can be estimated by

$$\begin{aligned} w(x,y) &= E_{x,y}\left[e^{-\alpha\hat{\tau}} g(X_{\hat{\tau}}) - \int_0^{\hat{\tau}} e^{-\alpha s} c(X_s), ds \right] \\ &\leq E_{x,y}\left[\sup_{s\geq 0} e^{-\alpha s}\, g(X_s) \right] + E_{x,y}\left[\int_0^{\infty} e^{-\alpha s} c(X_s), ds \right] \\ &\leq E_{x,y}\left[\sup_{s\geq 0} e^{-\alpha s}\, g(X_s) \right] + D \; < \; \infty \end{aligned}$$

where we have used the triangle inequality and $D > 0$ is some constant due to the fact that c is bounded. The expectation is finite because of the integrability condition (1.3) on g.

Now, for each $t \geq 0$ and $\pi \in \mathcal{A}$, (X_t^π, Y_t^π) is $\mathcal{F}_t$-measurable. The strong Markov property of the diffusion (X, Y) (see for instance [36, Theorem 7.2.4]) when started from (X_t^π, Y_t^π) implies that,

$$\begin{aligned} e^{-\alpha t} w(X_t^\pi, Y_t^\pi) &\leq e^{-\alpha t}\, E_{X_t^\pi, Y_t^\pi}\left[\sup_{s\geq 0} e^{-\alpha s}\, g(X_s) \right] + D \\ &= E_{x,y}\left[\sup_{s\geq 0} e^{-\alpha(s+t)}\, g(X_{s+t}) \;\mid \mathcal{F}_t \right] + D \\ &\leq E_{x,y}\left[\sup_{s\geq 0} e^{-\alpha s}\, g(X_s) \;\mid \mathcal{F}_t \right] + D. \end{aligned}$$

Since (X^π_τ, Y^π_τ) is $\mathcal{F}_\tau$-measurable, we can even replace t by a finite stopping time $\tau \in \mathcal{M}$ with $\tau \le \hat{\tau}$, in the above inequalities.

Setting $S^* = \sup_{s\ge 0} e^{-\alpha s}\, g(X_s) + D$, the collection of conditional expectations $\{E_{x,y}[\, S^* \mid \mathcal{F}_\tau] :\ \text{finite } \tau \in \mathcal{M}\}$ is uniformly integrable. Indeed, setting $D_\tau = E_{x,y}[\, S^* \mid \mathcal{F}_\tau]$, D_τ is $\mathcal{F}_\tau$-measurable and

$$\begin{aligned}\lim_{K\to\infty} \sup_\tau E_{x,y}(\, D_\tau\, I(D_\tau \ge K)\,) &= \lim_{K\to\infty} \sup_\tau E_{x,y}[S^*\, I(\, E_{x,y}[S^* \mid \mathcal{F}_\tau] \ge K)\,] \\ &= \lim_{K\to\infty} E_{x,y}[S^*\, I(\, E_{x,y}[S^*] \ge K)\,] \\ &= E_{x,y}[\, \lim_{K\to\infty} S^*\, I(\, E_{x,y}[S^*] \ge K)\,] = 0,\end{aligned}$$

where we have used the tower property of conditional expectations, dominated convergence and the fact that $E_{x,y}[S^*] < \infty$.

As a consequence, the family $\{e^{-\alpha\tau} w(X^\pi_\tau, Y^\pi_\tau) :\ \text{finite } \tau \le \hat{\tau}\}$ also is uniformly integrable, that is Condition (ii) holds.

Finally, using Condition (3.26) together with the fact that $\{e^{-\alpha t} w(X^\pi_t, Y^\pi_t) :\ t \ge 0\}$ is uniformly integrable, we obtain (see Section 13.7 in [46]) that

$$e^{-\alpha t} w(X^\pi_t, Y^\pi_t) \xrightarrow{L^1} 0.$$

This proves Condition (iii).

Then

$$w(x,y) \le J_{x,y}(\hat{\tau}, \pi), \qquad \text{for all } \pi \in \mathcal{A},$$

and this concludes the proof when $T = \infty$.

(II) Suppose that $T < \infty$. Likewise, it suffices to verify Conditions (i)-(ii) of Theorem 3.14. This can be done in the same way as in part (I) with no substantial differences. □

We now turn to the regime-switching case.

In the context of Proposition 3.4, suppose that the upward and downward rates λ_y and μ_y are replaced by

$$\lambda^{\min}_y = \inf\, A_{y,y+1}, \qquad \text{and} \qquad \mu^{\min}_y = \sup\, A_{y,y-1}, \quad y \in \mathcal{S},$$

respectively. Denote by $\pi^{\min}$ the tridiagonal Q-matrix with entries $\lambda^{\min}_y, \mu^{\min}_y$,

$y \in \mathcal{S}$. In particular, $\pi^{\min}$ is admissible.

As before, let $(X_t, Y_t) \equiv (X_t^{\min}, Y_t^{\min})$ be the associated strong Markov process and $v^{\min}$ the corresponding value function.

Theorem 3.17 (Regime-switching case) *Let $T \in [0, \infty]$ be the time horizon. Suppose that the functions a, μ, c and $v^{\min}(\cdot, y)$ ($v^{\min}(\cdot, y, \cdot)$ when $T < \infty$) are locally Hölder continuous, and $v^{\min}$ is non-decreasing as a function of y. If $T = \infty$ also assume (3.26). Then, for each $(x, y) \in \mathcal{E}$,*

$$\sup_\tau \inf_\pi J_{x,y}(\tau, \pi) = v^{\min}(x, y) = \inf_\pi \sup_\tau J_{x,y}(\tau, \pi).$$

That is, $v^{\min}$ is the value of the game and $(\tau^*, \pi^{\min})$ is a saddle point, where τ^* is the optimal stopping time for $v^{\min}$.

Proof. We only have to show that $w = v^{\min}$ satisfies Conditions (i)-(iii) of Theorem 3.12 when $T = \infty$ and Conditions (i)-(ii) of Theorem 3.15 when $T < \infty$.

Here we set $\hat{\pi} = \pi^{\min}$ and use that the Q-matrix $\hat{\pi} = (\hat{\pi}[y, y'])$ is such that, for each fixed $y \in \mathcal{S}$,

$$\hat{\pi}[y, y+1] = \arg\min_{\lambda \in A_{y,y+1}} \left[w(x, y+1) - w(x, y) \right] \lambda,$$

$$\hat{\pi}[y, y-1] = \arg\min_{\mu \in A_{y,y-1}} \left[w(x, y-1) - w(x, y) \right] \mu,$$

and when $T < \infty$ simply replace $w(x, \cdot)$ by $w(x, \cdot, t)$.

Proposition 3.4 (resp. 3.8) and the expression for $\hat{\pi}$ above verify Condition (i) when $T = \infty$ (resp. $T < \infty$).

The rest of the proof does not change in comparison with the proof of the previous Theorem. □

Remark 3.18 (a). As stated in Theorems 3.16 and 3.17, the continuity of the value function $v^{\min}$ is required to identify it with the value of the game. This requirement goes back to Propositions 3.3 and 3.4 to guarantee that $v^{\min}$ belongs to the domain of the associated operator $\mathbb{L}^{\min}$, at least in the continuation region $\mathcal{C}$.

There are well-known results regarding the continuity of v when (X, Y) is a diffusion process. For instance, Krylov shows in Theorem 6.4.14 [35] that the region where v is continuous coincides with the set where the infinitesimal generator $\mathbb{L}$ of (X, Y) (see expression in Proposition 3.3 above) is elliptic, provided the drift and diffusion coefficients $\sigma(x, y)$ and $b(x, y)$ are Lipschitz and satisfy a linear growth condition. Here,

$$\|\sigma(x, y)\|^2 = a^2(x)y^2 + \eta^2(y), \quad |b(x, y)|^2 = \mu(x)^2 + \pi(y)^2.$$

We do not impose, however, these conditions because they rule out some classical examples. For instance, in the Heston model (S, V) appearing in Section 4.3.2, the volatility coefficient of each component has the factor $\sqrt{x}$ which is not a Lipschitz function, but is Hölder continuous.

(b). The non-decreasing property of the function $v^{\min}(x, \cdot)$ was studied in Chapter 2 in the case when X is driftless. Sufficient conditions are given for instance in Theorems 2.6 and 2.16 (see also Section 2.4), under the assumption that g is non-negative. We extend these results in the next chapter to the case when X has drift.

Chapter 4

Application to option pricing

4.1 American options setting

Suppose that the dynamics of X are given by

$$dX_t = X_t \, Y_t \, dB_t \tag{4.1}$$

and so that (X, Y) is like in the setting of Section 2.2 or 2.3, with the particular case $a(x) = x$.

Note that the stochastic integral $\int_0^t Y_s \, dB_s$, $t \geq 0$, is well-defined since Y is adapted and either a piecewise-constant or a continuous process. Then, X is an exponential local martingale of the form

$$X_t = X_0 \, \exp\left\{ \int_0^t Y_s \, dB_s - \frac{1}{2} \int_0^t Y_s^2 \, ds \right\}, \quad t \geq 0, \text{ a.s.} \tag{4.2}$$

The American options we are interested in, pay $g(e^{\alpha\tau} X_\tau)$ when exercised at a stopping time τ before the maturity $T \in [0, \infty]$. Assuming that the probability measure $P_{x,y}$ is used for pricing when $X_0 = x$ and $Y_0 = y$, $(x, y) \in \mathbb{R}_+ \times \mathcal{S}$, the price of such an option is

$$v(x, y) = \sup_{0 \leq \tau \leq T} E_{x,y} \, [e^{-\alpha\tau} g(e^{\alpha\tau} X_\tau)], \tag{4.3}$$

where the supremum is taken over all finite stopping times with respect to the filtration generated by (X, Y), and $\alpha > 0$ stands for the instantaneous interest rate.

The process $(e^{\alpha t} X_t)_{t \geq 0}$ describes a simple model for the price of an asset with stochastic volatility Y.

Since g is not applied to X_τ but to $e^{\alpha\tau} X_\tau$, some of the conditions for our results in Chapter 2 have to be slightly adjusted.

First, the condition

$$E_{x,y}\left[\sup_{0 \leq t \leq T} e^{-\alpha t} |g(e^{\alpha t} X_t)| I(t < \infty)\right] < \infty$$

for all $(x, y) \in \mathbb{R}_+ \times \mathcal{S}$ is now assumed throughout.

The analogue to what was obtained in Propositions 2.11 and 2.21, but now for the value function given in (4.3), is

$$v(x, y) = \sup_{\rho \in \mathcal{M}_T} \tilde{E}\,[e^{-\alpha \Gamma_\rho} g(e^{\alpha \Gamma_\rho} G_\rho)], \tag{4.4}$$

without imposing further conditions.

Following the proofs of Theorems 2.6 and 2.16, but now considering $v(x, y)$ in (4.4) instead, one can see that one only needs to verify that

$$\tilde{E}\,[e^{-\alpha \Gamma_\rho} g(e^{\alpha \Gamma_\rho} G_\rho)] \leq \tilde{E}\,[e^{-\alpha \Gamma'_\rho} g(e^{\alpha \Gamma'_\rho} G_\rho)] \quad \text{for every } \rho \in \mathcal{M}_T, \tag{4.5}$$

where $\Gamma_t \geq \Gamma'_t$, $t \geq 0$ a.s. This is the analogue of the crucial comparison in (2.18).

Notice that (4.5) is true, for instance, when g is non-increasing. More generally, if we assume that

$$\begin{aligned} &g \text{ is a non-negative and measurable function such that } g \neq 0, \\ &g(ax) \leq a g(x), \qquad \text{for all } a \geq 1,\ x \geq 0, \end{aligned} \tag{4.6}$$

then the inequality in (4.5) holds. Indeed, for each fixed $\rho \in \mathcal{M}_T$, simply choose $a = e^{\alpha(\Gamma_\rho - \Gamma'_\rho)} \geq 1$ and note that

$$g(e^{\alpha\Gamma_\rho} G_\rho) = g(a\, e^{\alpha\Gamma'_\rho} G_\rho) \leq a g(e^{\alpha\Gamma'_\rho} G_\rho), \quad a.s.$$

We obtain the following analogues of Theorems 2.6 and 2.16, respectively, for the value function in (4.3).

Theorem 4.1 (MC case) *Assume that g satisfies (4.6). Also assume that Y is skip-free. Then, for each $x \in \mathbb{R}_+$, $v(x, \cdot)$ is non-decreasing on $\mathcal{S}$.*

Proof. By Theorem 2.6 and the previous discussion, we only need to verify Condition **C1** on page 13 but this is plain thanks to the special case $a(x) = x$. □

Theorem 4.2 (Diffusion case) *Assume that g satisfies (4.6). Also assume Conditions* **C1'-C2'** *on page 26 with $a(x) = x$. Then, for each $x \in \mathbb{R}_+$, $v(x, \cdot)$ is non-decreasing on $\mathcal{S}$.*

Proof. The proof follows by Theorem 2.16 and the above discussion. □

Remark 4.3 Since $a(x) = x$, Conditions **C1'-C2'** on page 26 become a condition only on the autonomous process ξ in (2.27).

The class of functions in (4.6) was introduced by Ekström [10] to compare prices of American options when the volatility is stochastic but only depends on the stock price process and time.

The following classes of functions satisfy the condition in (4.6):

- Non-increasing: it is plain that (4.6) holds.
- Concave: since $g(0) \geq 0$, we have that $g(\lambda y) \geq \lambda g(y)$ for any $\lambda \in [0, 1]$ and $y \geq 0$. Now, for any $a \geq 1$ and $x \geq 0$, simply set $\lambda = 1/a$ and $y = ax$ to obtain the desired inequality.

4.2 A regime-switching model

Consider the regime-switching model of Guo and Zhang [19] and Jobert and Rogers [30] for the stock price process:

$$dS_t = S_t(Z_t \, dB_t + r(Z_t)dt),$$

and Z is an irreducible, continuous-time Markov chain with finite state space $\mathcal{S} \subset (0, \infty)$ and independent of B. It is assumed that $r(z) > 0$ for all $z \in \mathcal{S} = \{y_1, y_2, \ldots, y_m\}$. Here it is understood that $y_1 < y_2 < \cdots < y_m$.

The value of a perpetual American-type option with random discount is given by (cf. equation (13) in [30])

$$v(x, y) = \sup_{\tau \in \mathbb{F}} E_{x,y} \left[\exp \left(- \int_0^t r(Z_s) ds \right) g(S_\tau) \right], \tag{4.7}$$

where the payoff g is a measurable function, and $(x, y) \in \mathbb{R}_+ \times \mathcal{S}$. The supremum is over all stopping times τ with respect to $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ (we write $\tau \in \mathbb{F}$), the augmentation of the natural filtration of (S, Z), and $E_{x,y}$ denotes the conditional expectation $E[\cdot \mid S_0 = x, Z_0 = y]$.

4.2.1 From additive to linear discounting

In this section we show that the value function in (4.7) is non-decreasing in y, which has an impact in determining the shape of the optimal stopping boundary (see next section).

We shall further exploit the time-change technique to deduce that this value function can be reduced to one with a linear discount factor, so that we can apply Theorem 4.1.

Some of the ideas here are in the spirit of Cissé et al. [8].

Let us rewrite S_t as follows:

$$S_t = x + \int_0^t S_u \, r'(Z_u) dM_u + \int_0^t S_u \, r(Z_u) du$$

where

$$r'(z) = \frac{z}{\sqrt{r(z)}} \tag{4.8}$$

and

$$M_t = \int_0^t \sqrt{r(Z_u)}dB_u.$$

Define the random, additive functional $R = (R_t)_{t\geq 0}$ and its right-inverse $A = (A_t)_{t\geq 0}$ by

$$R_t := \int_0^t r(Z_s)ds, \quad A_t := \inf\{s \geq 0 : R_s > t\}.$$

The process R possesses all the properties as Γ in (2.8), and so does A. In particular, $\langle M \rangle_t = R_t$ and A defines a time-change with $A_t = \int_0^t \frac{1}{r(Y_s)}ds$.

Define $X_t := S \circ A_t$ and $Y_t := Z \circ A_t$ and $W_t = M \circ A_t$, $t \geq 0$. Then

$$dX_t = X_t(r'(Y_t)\, dW_t + dt), \quad X_0 = x, \tag{4.9}$$

where W is an $\mathbb{F}_A$-Brownian motion ($\mathbb{F}_A = (\mathcal{F}_{A_t})_{t\geq 0}$). Also, arguing very similarly to the way we did in Section 2.2.2, Y is a Markov chain with generator

$$L^Y f(y) = \frac{1}{r(y)}\, L\, f(y), \tag{4.10}$$

where L denotes the generator of Z, and f is any bounded and measurable function.

The proof of the following result is exactly as that of Lemma 2.10, but with different notation. We include the proof for completeness and ease of reference.

Lemma 4.4 *The following holds:*

$$v(x,y) \equiv \sup_{\tau\in\mathbb{F}} E_{x,y}\left[e^{-R_\tau} g(S_\tau)\right] = \sup_{\tau\in\mathbb{F}_A} E_{x,y}\left[e^{-\tau} g(X_\tau)\right] \tag{4.11}$$

Proof. First, for every $\tau \in \mathbb{F}$, $\tau' = R_\tau \in \mathbb{F}_A$, by Lemma 2.4. Since

$$e^{-R_\tau} g(S_\tau) = e^{-R_\tau} g(S_{A_{R_\tau}}) = e^{-\tau'} g(X_{\tau'})$$

we obtain

$$\sup_{\tau \in \mathbb{F}} E_{x,y} \left[e^{-R_\tau} g(S_\tau) \right] \leq \sup_{\tau \in \mathbb{F}_A} E_{x,y} \left[e^{-\tau} g(X_\tau) \right].$$

Next, for every $\tau \in \mathbb{F}_A$, $\tau' = A_\tau \in \mathbb{F}$, again by Lemma 2.4. Since

$$e^{-\tau} g(X_\tau) = e^{-R_{A_\tau}} g(S_{A_\tau}) = e^{-R_{\tau'}} g(S_{\tau'}),$$

the other inequality must hold. □

Starting with the strong Markov process (S, Z) with respect to $\mathbb{F}$, the time-changed process (X, Y) is also strong Markov but with respect to $\mathbb{F}_A$. Also, the stopping times on the right-hand side of (4.11) can take values on the entire half-line $(0, \infty)$ since $\lim_t R_t = \infty$. In other words, we are in the setting of Section 4.1 with: a modified volatility, namely $r'(Y_t)$ instead of Y_t; a unitary instantaneous interest rate $r \equiv 1$; and with infinite horizon $T = \infty$.

In [30], there was no assumption on the order of the values of $r(\cdot)$ whatsoever as the main focus of the authors is on pricing. But, for the purpose of monotonicity of the function $v(x, \cdot)$, the order of the values of $r(y)$ when comparing different y's becomes important.

Proposition 4.5 *Assume that g satisfies (4.6). Also assume that Z is skip-free and that $r(\cdot)$ is decreasing in its argument. Then, for each $x \in \mathbb{R}_+$, $v(x, \cdot)$ is non-decreasing on $\mathcal{S}$, where v is given in (4.7).*

Proof. The process Y, as a time-change of the original Markov chain Z, inherits the skip-free property of Z.

Also, notice that $r'(\cdot)$ in (4.8) satisfies that for any $y_1 \leq y_2$, $r'(y_1) \leq r'(y_2)$.

The result now follows from Remark 2.14, Theorem 4.1 and Lemma 4.4. □

4.2.2 Threshold ordering

We are going to apply Proposition 4.5 in the case that the gain function is $g(x) = \max\{K - x, 0\}$ where $K > 0$ is the *strike price*.

The authors of [19] and [30] verified that the value function is uniquely attained at a stopping time of the form

$$\tau^\star = \inf\{t \geq 0 : S_t < b[Z_t]\} \tag{4.12}$$

where the vector $b[y_i]$, $i = 1, 2, \dots, m$, is indexed by the states of the Markov chain Z. The values $b[y_i]$ are the so-called *thresholds*.

As part of the program to calculate the value of $v(x, y)$, there is an standing assumption: that $b[y_1] \geq \cdots \geq b[y_m]$ (see PROBLEM 1 in [30]). It is then stated in a footnote on the same page 2066 that "When it comes in practice to identifying the thresholds, no assumption is made on the ordering, and all possible orderings are considered."

This approach has exponential complexity as the number of states increases. However, our result in Proposition 4.5, reduces this complexity to choosing a unique ordering, namely

$$b[y_1] > \cdots > b[y_m]. \tag{4.13}$$

Indeed, since $\tau^\star$ is the unique optimal stopping time for the problem in (4.7), by general theory, it must coincide with the first time that the process (X, Y) enters the stopping region $\{(x, y) : v(x, y) = g(x)\}$. Thus, as it is not optimal to stop when g is zero in this example:

$$v(x, y_i) = g(x) \text{ for } x \leq b[y_i] \quad \textit{while} \quad v(x, y_i) > g(x) \text{ for } x > b[y_i]$$

for each $i = 1, \dots, m$. Now suppose that (4.13) does not hold. Then, there exists i such that $b[y_i] < x \leq b[y_{i+1}]$ for some x. So $v(x, y_{i+1}) = g(x) < v(x, y_i)$, but this contradicts the result of Proposition 4.5.

4.2.3 Continuity of the value function in x

We learned, in Section 4.2.1, that we can study functional properties of $v(x, y)$ in (4.7) under an equivalent model (X, Y) obtained by a suitable time-change. The new pair (X, Y) has the advantage that X has linear and deterministic drift (unlike S).

In this section, we assume that the model for X is given by

$$dX_t = X_t(Y_t\,dB_t + \mu\,dt),$$

with $X_0 \in \mathbb{R}_+$, where $\mu \in \mathbb{R}$ and Y is an irreducible, continuous-time Markov chain with finite state space $\mathcal{S}'$ and independent of B. Notice that X is explicitly given by

$$X_t = X_0 \exp\left\{\mu\,t + \int_0^t Y_s\,dB_s - \frac{1}{2}\int_0^t Y_s^2\,ds\right\}, \quad t \geq 0, \text{ a.s.} \tag{4.14}$$

We are interested in the (Lipschitz) continuity of

$$w(x,y) = \sup_{\tau\in\mathbb{F}} E_{x,y}\left[e^{-\alpha\tau} g(X_\tau)\right] \tag{4.15}$$

as a function of x. Here, $\alpha > 0$ and the supremum is over all stopping times τ with respect to the augmentation of the natural filtration of (X,Y), $\mathbb{F}$. As usual we assume that, for each initial value (x,y),

$$E_{x,y}\left[\sup_{t\geq 0} e^{-\alpha t} g(X_t)\right] < \infty. \tag{4.16}$$

Remark 4.6 The function w identifies with v in (4.7) via Lemma 4.4. Simply set $\alpha = \mu = 1$ in (4.14)-(4.15) and the state space of Y equals $\mathcal{S}' = \{r'(y) : y \in \mathcal{S}\}$.

We start with a simple result, which is a straightforward consequence of the form of X in (4.14).

Theorem 4.7 *Assume that g is a convex function. Then, for each $y \in \mathcal{S}'$, the function $w(\cdot, y)$ is convex in $\mathbb{R}_+$.*

Proof. Consider $\mathcal{E}_t^y := \exp(\mu t + \int_0^t Y_s dB_s - \frac{1}{2}\int_0^t Y_s^2 ds)$, where the superscript indicates that $Y_0 = y$. Then we can write $w(x,y) = \sup_\tau E[e^{-\alpha\tau} g(x\,\mathcal{E}_\tau^y)]$.

Let λ be in $[0,1]$. Then for each pair $x, x' \in \mathbb{R}_+$,

$$\begin{aligned}\lambda w(x,y) + (1-\lambda) w(x',y) &\geq \sup_\tau \left\{ \lambda E\left[e^{-\alpha\tau} g(x\,\mathcal{E}^y_\tau)\right] + (1-\lambda) E\left[e^{-\alpha\tau} g(x'\,\mathcal{E}^y_\tau)\right] \right\} \\ &\geq \sup_\tau \left\{ E\, e^{-\alpha\tau} \left(\lambda g(x\,\mathcal{E}^y_\tau) + (1-\lambda) g(x'\,\mathcal{E}^y_\tau) \right) \right\} \\ &\geq \sup_\tau \left\{ E\, e^{-\alpha\tau} g([\lambda x + (1-\lambda) x']\, \mathcal{E}^y_\tau) \right\} \\ &= w(\lambda x + (1-\lambda)x', y).\end{aligned}$$

That is, for each y, the function $w(\cdot, y)$ is convex in the positive half-line.□

Remark 4.8 An immediate by-product of the preceding proposition is that if g is convex then $w(\cdot, y)$ is locally Lipschitz continuous in $\mathbb{R}_+$.

Let us introduce some notation. From the general theory of optimal stopping for Markov processes (see Appendix B), the continuation region for the problem (4.15) is given by $\mathcal{C} = \{(x,y) \in \mathbb{R}_+ \times \mathcal{S}' :\, w(x,y) > g(x)\}$. For each $y \in \mathcal{S}'$, consider the y-section

$$C_y = \{x \in \mathbb{R}_+ :\, w(x,y) > g(x)\}.$$

In what follows we verify that $w(\cdot, y)$ is locally Lipschitz continuous in C_y and continuous across the boundary ∂C_y provided g is continuous. As a consequence, we have the following results.

Theorem 4.9 *Assume that g is non-negative and continuous. Then, for each $y \in \mathcal{S}'$, the function $w(\cdot, y)$ is continuous in $\mathbb{R}_+$.*

Proof. The result follows from Lemmas 4.11 and 4.12 below. □

Theorem 4.10 *Assume that g is non-negative and locally Lipschitz continuous. Then, for each $y \in \mathcal{S}'$, the function $w(\cdot, y)$ is locally Lipschitz continuous in $\mathbb{R}_+$.*

Proof. The result follows from Lemmas 4.12 and 4.13 below. □

Lemma 4.11 *Assume that g is non-negative and continuous. Then, for each $y \in \mathcal{S}'$, the function $w(\cdot, y)$ is continuous in $\mathcal{C}_y$.*

Proof. Fix $y \in \mathcal{S}'$ and denote by $q[y]$ the rate of leaving y, that is, $q[y] = \sum_{y' \neq y} q[y, y']$. Let τ^* be the optimal stopping time for the problem with initial condition (x, y). For any $\delta \in \mathbb{R}$ satisfying $x + \delta \in \mathbb{R}_+$, the form of X in (4.14) implies that

$$\begin{aligned} w(x+\delta, y) &\geq E\left[e^{-\alpha\tau^*} g(X_{\tau^*}) \mid X_0 = x+\delta, Y_0 = y \right] \\ &= E\left[e^{-\alpha\tau^*} g\left(\frac{x+\delta}{x} X_{\tau^*} \right) \mid X_0 = x, Y_0 = y \right]. \end{aligned}$$

Since g is non-negative and continuous, Fatou's Lemma yields the inequality $\liminf_{\delta \to 0} w(x+\delta, y) \geq w(x, y)$, i.e., $w(\cdot, y)$ is lower semi-continuous in $\mathbb{R}_+$. This in turn implies that the set C_y is open.

Let I be a bounded open interval contained in C_y. It is enough to show that $x \mapsto w(x, y)$ is continuous in I.

Denote by T_1 and T_2 the first exit of X from I, and the first jump time of the Markov chain Y from y, respectively. Set $\tau = T_1 \wedge T_2$. Then we have that $\tau \leq \tau^*$ and since (X, Y) is a strong Markov process it follows that

$$w(x, y) = E_{x,y}\, e^{-\alpha\tau} w(X_\tau, Y_\tau), \qquad \forall\, x \in I. \tag{4.17}$$

We shall show that the functions

$$\begin{aligned} F_1(x) &:= E_{x,y}[\, e^{-\alpha T_1} w(X_{T_1}, Y_{T_1})\, I\{T_1 < T_2\}\,], \quad \text{and} \\ F_2(x) &:= E_{x,y}[\, e^{-\alpha T_2} w(X_{T_2}, Y_{T_2})\, I\{T_2 < T_1\}\,] \end{aligned}$$

are continuous in I, so that the result follows because $w(x, y) = F_1(x) + F_2(x)$ (since $P_{x,y}(T_1 = T_2) = 0$).

We next use the following fact to show the continuity of F_1 and F_2. Let $\tilde{X}$ be the solution to the equation $d\tilde{X}_t = \tilde{X}_t(y\, dB_t + \mu\, dt)$ started at $\tilde{X}_0 = x$, and killed at an independent, exponentially distributed random time $e_\gamma \sim \mathrm{Exp}(\gamma)$. That is, $\tilde{X}$ is a geometric Brownian motion started at $\tilde{X}_0 = X_0 = x$ and $X_t = \partial$ for all $t \geq e_\gamma$, where ∂ is a cemetery point. We know that $\tilde{X}$ is a strong Feller process (see [44]) and therefore, for every bounded and measurable function ϕ

on $\bar{I} \cup \partial$, the functions

$$x \mapsto E\,[\phi(\tilde{X}_{T_1}) \mid \tilde{X}_0 = x] \quad \text{and} \quad x \mapsto E\,[\phi(\tilde{X}_t)\, I\{t < T_1\} \mid \tilde{X}_0 = x]$$

are continuous in I (see [5, Theorem 2.1 and Lemma 2.2]). By convention $\phi(\partial) = 0$.

Set $\phi(x) = w(x, y)$ and $e_\gamma = e_\alpha \wedge T_2$, where $e_\alpha \sim \mathrm{Exp}(\alpha)$ is independent of T_2 (that is, $\gamma = \alpha + q[y]$).

Since $\tilde{X}$ is independent of $Y_0 = y$, and on the event $\{T_1 < T_2\}$ one has that $\tilde{X}_{T_1} = X_{T_1}$, $Y_{T_1} = y$ a.s.,

$$\begin{aligned}
E\,[\phi(\tilde{X}_{T_1}) \mid \tilde{X}_0 = x] &= E\,[\phi(\tilde{X}_{T_1})\, I\{T_1 < e_\gamma\} \mid \tilde{X}_0 = x, Y_0 = y\,] \\
&= E\,[w(\tilde{X}_{T_1}, y)\, I\{T_1 < T_2\}\, I\{T_1 < e_\alpha\} \mid \tilde{X}_0 = x, Y_0 = y\,] \\
&= E\,[w(X_{T_1}, Y_{T_1})\, I\{T_1 < T_2\}\, I\{T_1 < e_\alpha\} \mid X_0 = x, Y_0 = y\,] \\
&= E_{x,y}\left[\int_0^\infty \left\{\, \alpha e^{-\alpha s} w(X_{T_1}, Y_{T_1})\, I\{T_1 < T_2\}\, I\{T_1 < s\}\right\} ds\right] \\
&= E_{x,y}\left[e^{-\alpha T_1} w(X_{T_1}, Y_{T_1})\, I\{T_1 < T_2\} \int_{T_1}^\infty \left\{\, \alpha e^{-\alpha(s-T_1)}\, ds\right\}\right] \\
&= E_{x,y}\,[\,e^{-\alpha T_1} w(X_{T_1}, Y_{T_1})\, I\{T_1 < T_2\}] = F_1(x).
\end{aligned}$$

Hence $F_1(\cdot)$ is continuous in I.

Let us now turn to $F_2(x)$. Since $T_2 \sim \mathrm{Exp}(q[y])$,

$$F_2(x) = \int_0^\infty q[y]\, e^{-q[y]\,t} E_{x,y}[\,e^{-\alpha\, t} w(X_t, Y_t)\, I\{t < T_1\} \mid T_2 = t]\, dt.$$

Now, take $y' \neq y$ and set $\phi(x) = w(x, y')$ and $e_\gamma = e_\alpha$, where $e_\alpha \sim \mathrm{Exp}(\alpha)$ is independent of T_2.

Since $\tilde{X}$ is independent of $Y_0 = y$ and T_2,

$$\begin{aligned}
E\,[\phi(\tilde{X}_t)\, I(t < T_1) \mid \tilde{X}_0 = x] &= E\,[\phi(\tilde{X}_t)\, I\{t < T_1\} \mid \tilde{X}_0 = x, Y_0 = y, T_2 = t] \\
&= E\,[w(\tilde{X}_t, y')\, I\{t < T_1\}\, I\{t < e_\alpha\} \mid \tilde{X}_0 = x, Y_0 = y, T_2 = t] \\
&= E\,[w(X_t, y')\, I\{t < T_1\}\, I\{t < e_\alpha\} \mid X_0 = x, Y_0 = y, T_2 = t] \\
&= E_{x,y}\,[e^{-\alpha t} w(X_t, y')\, I\{t < T_1\} \mid T_2 = t]
\end{aligned}$$

for each $t > 0$. The LHS of this chain of equalities (and so the RHS) is continuous in I with respect to x. Hence the conditional expectation $E_{x,y}[\, e^{-\alpha\, t} w(X_t, Y_t)\, I\{t < T_1\} \mid T_2 = t]$ in the expression for $F_2(x)$, given by

$$\sum_{y' \neq y} \frac{q[y,y']}{q[y]}\, E_{x,y}[\, e^{-\alpha\, t} w(X_t, y')\, I\{t < T_1\} \mid T_2 = t],$$

is also continuous in I with respect to x.

We conclude that $F_2(\cdot)$ is continuous in I by dominated convergence (recall that w is bounded), and so the proof is complete. □

Lemma 4.12 (Continuous-fit) *Assume that g is non-negative and continuous. Then, for each $y \in \mathcal{S}'$, $w(\cdot, y)$ is continuous on the boundary ∂C_y.*

Proof. Let us fix $y \in \mathcal{S}'$.

We know that C_y is open (see proof of Lemma 4.11) and so it is the union of countably many open intervals. We shall assume, for ease of presentation, that C_y is a half line of the form (b, ∞) for some $b \equiv b[y] \in \mathbb{R}$ (like in Section 4.2.2) so that $\partial C_y = \{b\}$. The reasoning below applies likewise to the case where C_y is a union of open intervals without substantial changes.

Since $w(x, y) = g(x)$ on $(-\infty, b]$ and g is continuous, it is clear that $w(\cdot, y)$ is left-continuous at b. It remains to see right-continuity.

Let $\delta > 0$. On the one hand, since $(b + \delta, y) \in C_y$ we have

$$w(b + \delta, y) - w(b, y) \geq g(b + \delta) - g(b)$$

and so the continuity of g yields

$$\liminf_{\delta \downarrow 0} w(b + \delta, y) - w(b, y) \geq 0. \tag{4.18}$$

On the other hand, let τ^δ denote the optimal stopping time for the problem with initial condition $(b + \delta, y)$. Then

$$\begin{aligned} w(b + \delta, y) &= E\left[e^{-\alpha \tau^\delta} g(X_{\tau^\delta}) \mid X_0 = b + \delta, Y_0 = y \right] \\ &= E\left[e^{-\alpha \tau^\delta} g\left(\frac{b + \delta}{b} X_{\tau^\delta} \right) \mid X_0 = b, Y_0 = y \right]. \end{aligned}$$

As τ^δ is suboptimal for the problem $w(b, y)$, we obtain that

$$w(b+\delta, y) - w(b, y) \leq E\left[e^{-\alpha\tau^\delta}\left\{g\left(\frac{b+\delta}{b}X_{\tau^\delta}\right) - g(X_{\tau^\delta})\right\} \mid X_0 = b, Y_0 = y\right].$$

Using Fatou's lemma (recall the assumption (4.16)) as well as the continuity of the paths of X,

$$\limsup_{\delta \downarrow 0} w(b+\delta, y) - w(b, y) \leq 0. \tag{4.19}$$

Putting (4.18) and (4.19) together we conclude that $w(\cdot, y)$ is right-continuous at b and the proof is complete. □

Lemma 4.13 *Assume that g is non-negative and continuous. Then, for each $y \in \mathcal{S}'$, the function $w(\cdot, y)$ is locally Lipschitz continuous in C_y.*

Proof. Fix $y \in \mathcal{S}'$. Let I be a bounded open interval contained in C_y.

Take $x, x' \in I$ with $x \neq x'$ and assume that $X = (X_t)_{t \geq 0}$ in (4.14) starts at x. Also denote by $X' = (X'_t)_{t \geq 0}$ the process satisfying (4.14) with driving Brownian motion $-B = (-B_t)_{t \geq 0}$ and started from $X'_0 = x'$. Suppose without loss of generality that $x > x' > 0$ (the case $x' > x > 0$ is covered by symmetry). Consider the coupling time

$$\tau(x, x') := \inf\{t > 0 : X_t = X'_t\}.$$

Let T_1 be the first time that either X or X' exits from the interval I, and T_2 be the first jump time of the Markov chain Y from y. If $\tau = T_1 \wedge T_2$ then, for each $x \in I$,

$$w(x, y) = E\, e^{-\alpha\tau} w(X_\tau, Y_\tau) \qquad \text{and} \qquad w(x', y) = E\, e^{-\alpha\tau} w(X'_\tau, Y_\tau).$$

The function $w(\cdot, y)$ is bounded on I (as it is continuous in C_y), say by $K/2 > 0$. Thus the difference in payoffs is bounded by K times the probability that the processes X and X' have not coupled by first exit time from I and by first jump of the chain. That is,

$$|w(x, y) - w(x', y)| \leq K\, P(\, T_1 \wedge T_2 < \tau(x, x')\,). \tag{4.20}$$

We next argue that $P(\,T_2 < \tau(x,x')\,)$ is $O(x-x')$ as $x - x' \to 0$.

After simple calculations, we see that $X_t = X'_t$ if and only if $\log(x) + \int_0^t Y_s dB_s = \log(x') + \int_0^t Y_s d(-B_s)$. Setting $r = \log(x/x') > 0$ and

$$R_t := r + \int_0^t 2Y_s dB_s, \qquad t \geq 0,$$

it follows that $\tau(x,x') = \inf\{t > 0 : R_t \leq 0\}$.

Let us consider $A_t := \int_0^t 4Y_s^2 ds$, $t \geq 0$. By the DDS Theorem, we know that there is a standard Brownian motion W such that R_t and $r + W_{A_t}$ coincide. Hence

$$\tau(x,x') = \inf\{t > 0 : R_t \leq 0\} = \inf\{t > 0 : W_{A_t} \leq -r\}. \tag{4.21}$$

Notice that by the continuity of the paths of A_t,

$$\{\,T_2 < \tau(x,x')\,\} \subseteq \{\,W_{A_s} > -r,\ \forall s \leq T_2\,\} = \{\,W_s > -r,\ \forall s \leq A_{T_2}\,\}.$$

Moreover, as the chain Y takes only a finite number of positive values, we have that A_t is bounded from below by $k(t) = 4m^2 t$ where m is the smallest state value. Hence,

$$\{\,W_s > -r,\ \forall s \leq A_{T_2}\,\} \subseteq \{\inf_{s \leq A_{T_2}} W_s \geq -r\} \subseteq \{\inf_{s \leq k(T_2)} W_s \geq -r\}.$$

Hence

$$P(\,T_2 < \tau(x,x')\,) \leq P\left(\inf_{s \leq k(T_2)} W_s \geq -r\right).$$

Since $k(T_2)$ has exponential distribution and is independent of B, it follows that the right-hand side is $O(r)$ as $r \to 0$ (see e.g. [6]). Using that $r = \log(x) - \log(x')$, we conclude that the right-hand side is $O(x-x')$ as $x - x' \to 0$.

The probability $P(\,T_1 < \tau(x,x')\,)$ can also be seen to be $O(x-x')$ as $x - x' \to 0$. Intuitively, as the starting points x and x' get arbitrarily close, the probability that either X or X' exits I *before* they couple gets very close to zero. □

4.3 Two Bessel-type models

4.3.1 The Hull & White model

Consider the model for the stock price S and its instantaneous variance V of Hull & White (cf. (a)-(b) on page 284 in [25]):

$$\begin{aligned} dS_t &= S_t(\sqrt{V}_t\, dB_t + r\, dt) \\ dV_t &= 2\eta\, V_t\, dB_t^Y + \kappa\, V_t\, dt, \end{aligned} \tag{4.22}$$

where η, κ are positive constants, and B, B^Y are independent Brownian motions. Setting $X_t = e^{-rt} S_t$ and $Y = \sqrt{V}$ transforms the above system into

$$\begin{aligned} dX_t &= X_t Y_t\, dB_t \\ dY_t &= \eta Y_t\, dB_t^Y + \theta Y_t\, dt, \end{aligned} \tag{4.23}$$

where $\theta = (\kappa - \eta^2)/2$. Assuming a positive initial condition $y > 0$, the equation for Y has a pathwise unique positive solution for every $\eta, \theta \in \mathbb{R}$.

The system in (4.23) is a particular case of (2.26) with

$$a(x) = x, \qquad \eta(y) = \eta\, y, \qquad \theta(y) = \theta\, y.$$

We shall verify that Conditions **C1'** and **C2'** on page 26 are satisfied in this context.

Calculating the equation for ξ in (2.27) gives a constant diffusion coefficient η and, if Z_t denotes ξ_t/η for each $t \geq 0$ then

$$dZ_t = dW_t^\xi + \frac{\theta}{\eta^2} Z_t^{-1} dt.$$

We could easily rewrite the last equation as

$$dZ_t = dW_t^\xi + \frac{\phi - 1}{2} Z_t^{-1}\, dt \tag{4.24}$$

which formally is an equation for a Bessel process with dimension $\phi = 1 + 2\theta/\eta^2 = \kappa/\eta^2$. This equation, and so the equation for ξ, only has a unique non-exploding strong solution if $\phi \geq 2$ and this solution stays positive when started from a positive initial condition. Hence Condition **C1'** holds true.

On the other hand, Condition **C2'** requires that $\lim_{t\to\infty} \Gamma_t = \infty$ a.s., where

$$\Gamma_t = \int_0^t \frac{1}{\xi_u^2} du = \eta^2 \int_0^t \frac{1}{Z_u^2} du, \quad t \geq 0.$$

This is true thanks to Proposition 4.24 with $p = 2$, provided $\phi \geq 2$.

Moreover, since Bessel processes are Feller processes (see [39, p446]), it follows that Z, and so ξ, is a Feller solution.

Therefore, assuming $\phi \geq 2$ (i.e. $\kappa \geq 2\eta^2$), the conclusion of Theorem 4.2 applies to American-type options (with either finite or infinite horizon) whenever the corresponding pay-off function g satisfies the stated conditions.

4.3.2 The Heston model

Consider the model for the stock price S and its instantaneous variance V of Heston [22]:

$$\begin{aligned} dS_t &= S_t(\sqrt{V}_t \, dB_t + r\, dt) \\ dV_t &= 2\eta\sqrt{V}_t \, dB_t^Y + \lambda(\kappa - V_t)\, dt, \end{aligned} \tag{4.25}$$

where η, λ, κ are positive constants and B, B^Y are Brownian motions with covariation $\delta \in [-1, 1]$.

The equation for V describes the so-called Cox-Ingersoll-Ross process (also known as the *square root process*) and it is well-known (see [9, p.391]) that, with a positive initial condition, this equation has a pathwise unique positive solution provided $\lambda\kappa \geq 2\eta^2$.

Setting $X_t = e^{-rt}S_t$ and $Y = \sqrt{V}$ transforms the above system into

$$\begin{aligned} dX_t &= X\, Y_t \, dB_t \\ dY_t &= \eta \, dB_t^Y + \left(\frac{\theta_1}{Y_t} - \theta_2 \, Y_t\right) dt, \end{aligned} \tag{4.26}$$

where $\theta_1 = (\lambda\kappa - \eta^2)/2$ and $\theta_2 = \lambda/2$. It is clear that the pathwise uniqueness of the equation for V ensures the pathwise uniqueness of positive solutions of the equation for Y.

The system in (4.26) is a particular case of (2.26) with

$$a(x) = x, \qquad \eta(y) = \eta, \qquad \theta(y) = \frac{\theta_1}{y} - \theta_2\, y.$$

Again, we shall verify that Conditions **C1'** and **C2'** on page 26 are satisfied in this context.

Calculating the equation for ξ in (2.27) yields

$$d\xi_t = \frac{\eta}{\xi_t} dW_t^{\xi} + \left(\frac{\theta_1}{\xi_t^3} - \frac{\theta_2}{\xi_t} \right) dt \tag{4.27}$$

and hence $Z_t = f(\xi_t)$, with $f(x) = x^2/(2\eta)$, satisfies

$$dZ_t = dW_t^{\xi} + \left(\frac{\phi - 1}{2Z_t} - \delta \right) dt \tag{4.28}$$

with $\phi = \theta_1/\eta^2 + 3/2 = \frac{\kappa\lambda}{2\eta^2} + 1$ and $\delta = \theta_2/\eta = \frac{\lambda}{2\eta}$.

Notice that the equation for Z above is that of a Bessel process with drift, and so by changing to an equivalent probability measure, Z is a Bessel process with dimension ϕ. In other words, up to a change of measure, if $\phi \geq 2$ then $Z_t = f(\xi_t)$ is the unique, non-exploding strong solution to (4.28) and is strictly positive a.s.

Since f is a strictly increasing and smooth function and so invertible on $(0, \infty)$, we have that $\xi_t = f^{-1}(Z_t)$ is a positive strong solution to (4.27). We claim that this is the only solution and so Condition **C1'** holds provided $\phi \geq 2$. Indeed, if ξ' also solves (4.27), with the same driving Brownian motion and same initial condition, then we must have that $f(\xi'_t)$ is a solution to (4.28). But this equation has pathwise uniqueness, that is, $f(\xi_t) = f(\xi'_t)$ for all $t \geq 0$ a.s. and this implies that $\xi_t = \xi'_t$ for all $t \geq 0$ a.s. because f^{-1} is injective on $(0, \infty)$.

Finally, the process

$$\Gamma_t = \int_0^t \frac{1}{\xi_u^2} du = \frac{1}{2\eta} \int_0^t \frac{1}{Z_u} du, \quad t \geq 0,$$

satisfies Condition **C2'** if $\phi \geq 2$, by Proposition 4.24 with $p = 1$.

The Feller property of the Bessel process Z carries over to ξ.

So, as in the previous example, the conclusion of Theorem 4.2 applies to American-type options (with either, finite or infinite horizon) whenever $\phi \geq 2$ (or $\lambda\kappa \geq 2\eta^2$) and the corresponding pay-off function g satisfies the stated conditions.

4.3.3 Price scenarios

In stochastic volatility models, the drift of volatility typically characterizes the choice of the pricing measure (which is not unique), but there is no definite criterion telling us which measure should be used (see Hobson [23], [24]).

In this section we deduce that increasing the drift of volatility yields larger American option prices. This is shown, for instance, by Hobson [24] for European options with convex payoff; and by Ekström [10] for American options but in the case $V_t = \sigma(t, S_t)$, so that there is not extra source of randomness.

Consider the system in (4.22) (or (4.25)) and suppose that $(S^{(i)}, V^{(i)})$, $i = 1, 2$, are the corresponding solutions which differ only in the coefficient $\kappa = \kappa^i > 0$ and the initial conditions $(S_0^{(i)}, V_0^{(i)}) = (x, y^{(i)})$ such that $y^{(1)} \leq y^{(2)}$.

For each $i = 1, 2$, the associated American option price is (cf. (4.3)):

$$v^{(i)}(x, y^{(i)}) = \sup_{0 \leq \tau \leq T} E_{x,y}^{(i)} \left[e^{-r\tau} g(S_\tau^{(i)}) \right]$$

In the previous sections, we verified Conditions **D1-D2** on page 31. Now consider the next situations:

Hull and White model. In the context of Section 4.3.1, assume that $2\eta^2 \leq \kappa^1 \leq \kappa^2$.

If $Z_t^{(i)} = \xi_t^{(i)}/\eta$ denotes the associated Bessel process with dimension $\phi^{(i)} = \kappa^{(i)}/\eta^2$ as in (4.24), then $2 \leq \phi^{(1)} \leq \phi^{(2)}$.

Heston model. In the context of Section 4.3.2, assume that $2\eta^2/\lambda \leq \kappa^1 \leq \kappa^2$.

If $Z_t^{(i)} = (\xi_t^{(i)})^2/(2\eta)$ denotes the associated Bessel process with dimension $\phi^{(i)} = \frac{\kappa^{(i)}\lambda}{2\eta^2} + 1$ (up to a change of measure) as in (4.28), then $2 \leq \phi^{(1)} \leq \phi^{(2)}$.

In either of the above cases, Condition **D3** on page 31 is also satisfied since, by Proposition 4.23,

$$0 < Z_t^{(1)} \leq Z_t^{(2)}, \qquad \forall t \geq 0,\ a.s.$$

Theorem 4.14 *Assume that g satisfies (4.6). If the coefficients $\kappa^{(i)}$, $i = 1, 2$, are related as above, then for each $x \in \mathbb{R}_+$*

$$v^{(1)}(x, y^{(1)}) \leq v^{(2)}(x, y^{(2)}), \qquad \textit{for all } y^{(1)} \leq y^{(2)}.$$

Proof. The idea of the proof is outlined in the proof of Theorem 2.22, for which all Conditions **D1-D3** hold in this context. There are some natural changes of notation. Here,

$$v^{(i)}(x, y) = \sup_{\rho \in \mathcal{M}_T^{(i)}} \tilde{E}\,[e^{-r\Gamma_\rho^{(i)}} g(e^{r\Gamma_\rho^{(i)}} G_\rho)], \tag{4.29}$$

for each $i = 1, 2$ and using that g satisfies (4.6) we obtain

$$\tilde{E}\,[e^{-r\Gamma_\rho^{(1)}} g(e^{r\Gamma_\rho^{(1)}} G_\rho) \ \leq\ \tilde{E}\,[e^{-r\Gamma_\rho^{(2)}} g(e^{r\Gamma_\rho^{(2)}} G_\rho)] \quad \text{for every } \rho \in \mathcal{M}_T^{(1)}.$$

The last inequality completes the proof because $\mathcal{M}_T^{(1)} \subseteq \mathcal{M}_T^{(2)}$. □

Remark that there is no continuity assumption on g to obtain the above comparison.

4.3.4 Properties of American-type option prices

Suppose that the stochastic volatility model (S, V) is either the Hull & White model in (4.22) or the Heston model in (4.25). In this setting, V stays positive a.s. and

$$S_t = S_0 \exp\left(rt + \int_0^t \sqrt{V_s} dB_s - \frac{1}{2} \int_0^t V_s ds \right), \quad t \geq 0,\ a.s. \tag{4.30}$$

Consider the value of a perpetual American-type option associated to

(S, V), represented by

$$v(x,y) = \sup_{\tau} E_{x,y}[e^{-r\tau} g(S_\tau)], \qquad (x,y) \in \mathbb{R}_+^2, \tag{4.31}$$

where the payoff function g is non-negative and the supremum is over all stopping times with respect to the augmentation of the natural filtration of (S, V).

We are implicitly assuming that the support of g restricted to the positive half-line is non-empty, and we write $\text{supp}(g) = \{x \in \mathbb{R}_+ : g(x) > 0\} \neq \emptyset$. For technical reasons, we further assume that $e^{-\alpha\tau} g(S_\tau)$ vanishes on the event $\{\tau = \infty\}$ (e.g. if g is bounded).

Proposition 4.15 *If g satisfies (4.6) then the function $(x,y) \mapsto v(x,y)$ is strictly positive on $\mathbb{R}_+ \times \mathbb{R}_+$.*

Proof. Let $(x,y) \in \mathbb{R}_+^2$ be an arbitrary initial condition for (S,V).

If $x \in \text{supp}(g)$ then $v(x,y) \geq g(x) > 0$, and we are done.

Let us now suppose that $x \notin \text{supp}(g)$. Given that g satisfies (4.6), necessarily $g(\bar{x}) = 0$ for all $\bar{x} \geq x$. Indeed, if $\bar{x} \geq x$ then we can find $a \geq 1$ such that $\bar{x} = ax$ and so $0 \leq g(\bar{x}) \leq ag(x) = 0$.

But $\text{supp}(g) \neq \emptyset$ by assumption, hence there must exist $\underline{\mathrm{x}} < x$ such that $\underline{\mathrm{x}} \in \text{supp}(g)$. Consider the stopping time

$$\tau = \inf\{t \geq 0 : S_t \leq \underline{\mathrm{x}}\} \leq \infty.$$

Using that $e^{-r\tau} g(S_\tau) = 0$ on the event $\{\tau = \infty\}$, we have that

$$\begin{aligned} v(x,y) &\geq E_{x,y} e^{-r\tau} g(S_\tau) \\ &= g(\underline{\mathrm{x}}) E_{x,y} e^{-r\tau} I(\tau < \infty) + E_{x,y} e^{-r\tau} g(S_\tau) I(\tau = \infty) \\ &= g(\underline{\mathrm{x}})\, E_{x,y} e^{-r\tau} I(\tau < \infty) > 0. \end{aligned}$$

The last strict inequality is due to the fact that $g(\underline{\mathrm{x}}) > 0$ and that $\tau > 0$ with positive probability on the event $\{\tau < \infty\}$. The latter is true because S has continuous paths and $S_0 = x > \underline{\mathrm{x}}$. The proof is now complete. □

Proposition 4.16 *Assume that g is a continuous function satisfying (4.6). Then, for each $x \in \mathbb{R}_+$, the American-type option value $v(x,y)$ is a continuous and non-decreasing function of y.*

Proof. From the calculations in Sections 4.3.1 and 4.3.2 we know that, by Theorem 4.2, the function $v(x,\cdot)$ is non-decreasing on $\mathcal{S} = (0,\infty)$. We also verified that the time-changed diffusion ξ in those examples is a Feller process. Upon assuming that g is continuous, the proof of the continuity of $v(x,\cdot)$ is along the lines of the proofs of Propositions 2.26 and 2.29, with some natural changes. We require continuous g because we deal with the payoff $g(S_\tau) = g(e^{r\tau}X_\tau)$ instead of $g(X_\tau)$, and recall that the limit arguments in the proofs of Propositions 2.26 and 2.29 only involve the discount factor. □

The following result complements Proposition 4.16 and its proof is identical to that of Theorem 4.7 so we omit it here.

Proposition 4.17 *Assume that g is a convex function. Then, for each $y \in \mathbb{R}_+$, the function $v(\cdot,y)$ is convex in $\mathbb{R}_+$.*

Combining Propositions 4.16 and 4.17, we now show that v is jointly continuous everywhere.

Let us first prepare a lemma borrowed from [4]. We shall use the following notation: for $\rho > 0$, $I(x;\rho)$ denotes the open interval $(x-\rho, x+\rho)$.

Lemma 4.18 *Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function, $x_0 \in \mathbb{R}$ and $\rho > 0$. If $\eta = \sup_{x \in I(x_0;\rho)} f(x)$ and $\alpha \in (0,1)$ then*

$$|f(x) - f(x_0)| \le \alpha(\,\eta - f(x_0)\,), \qquad \forall\, x \in I(x_0; \alpha\,\rho).$$

Theorem 4.19 *Assume that g is a non-negative, non-increasing and convex function. Then the function $v(x,y)$ is jointly continuous on $\mathbb{R}_+ \times \mathbb{R}_+$.*

Proof. Fix $(x_0, y_0) \in \mathbb{R}_+^2$. By Proposition 4.16, $v(x,\cdot)$ is continuous at y_0. Then it suffices to prove that $x \mapsto v(x,y)$ is continuous at x_0 uniformly over a neighborhood of y_0.

Fix $\rho > 0$ such that $I(x_0;\rho) \subset \mathbb{R}_+$. Since $v(\cdot,y)$ is convex by Proposition 4.17, it is also continuous in $I(x_0;\rho)$ and so

$$\eta(y) := \sup_{x \in I(x_0;\rho)} v(x,y) < \infty.$$

Moreover, by Lemma 4.18, for any $\alpha \in (0,1)$ the convexity of $v(\cdot,y)$ yields

$$|v(x,y) - v(x_0,y)| \leq \alpha(\,\eta(y) - v(x_0,y)\,), \qquad \forall\, x \in I(x_0;\alpha\,\rho). \tag{4.32}$$

Now, let U be a neighborhood of y_0 such that $U \times I(x_0;\rho) \subset \mathbb{R}^2_+$. It follows that the function $v(\cdot,\cdot)$ is bounded in $U \times I(x_0;\rho)$ because $v(\cdot,y)$ is continuous and $v(x,\cdot)$ is non-decreasing. This implies that

$$\eta := \sup_{y \in U} \eta(y) < \infty$$

and we can replace $\eta(y)$ in (4.32) by η.

Since $\alpha > 0$ can be made arbitrarily small and does not depend on y, we conclude that $v(\cdot,y)$ is continuous at x_0 uniformly in $y \in U$. $\square$

Proof of Lemma 4.18. Fix $x \in I(x_0;\alpha\,\rho)$. By the convexity of f we have that

$$\begin{aligned} f(x) - f(x_0) &= f\left(\alpha\left(\frac{x-(1-\alpha)x_0}{\alpha}\right) + (1-\alpha)x_0\right) - f(x_0) \\ &\leq \alpha f\left(\frac{x-(1-\alpha)x_0}{\alpha}\right) + (1-\alpha)f(x_0) - f(x_0) \\ &= \alpha\left(f\left(x_0 + \frac{x-x_0}{\alpha}\right) - f(x_0)\right) \leq \alpha(\,\eta - f(x_0)\,), \end{aligned}$$

since $|x - x_0|/\alpha \leq \rho$.

Now set $\alpha' = \alpha/(1+\alpha)$, and again by convexity it follows that

$$
\begin{aligned}
f(x_0) - f(x) &= f\left(\alpha' \left(\frac{(1+\alpha)x_0 - x}{\alpha} \right) + (1-\alpha')x \right) - f(x) \\
&\leq \alpha' f\left(\frac{(1+\alpha)x_0 - x}{\alpha} \right) + (1-\alpha')f(x) - f(x) \\
&= \alpha' \left(f\left(x_0 + \frac{x_0 - x}{\alpha} \right) - f(x) \right) \leq \alpha'(\eta - f(x)) \\
&= \alpha'([\eta - f(x_0)] + [f(x_0) - f(x)]),
\end{aligned}
$$

where we used that $|x_0 - x|/\alpha \leq \rho$. From here,

$$
(1-\alpha')(f(x_0) - f(x)) \leq \alpha'(\eta - f(x_0)).
$$

Finally, after multiplying the last inequality by $1+\alpha$, we obtain

$$
f(x_0) - f(x) \leq \alpha(\eta - f(x_0))
$$

which completes the proof. □

4.3.5 Monotone optimal boundary

As in the previous section, assume that the stochastic volatility model (S, V) is either the Hull & White model in (4.22) or the Heston model in (4.25).

In this section we concentrate on the American put option, that is, when $g(x) = \max\{K - x, 0\}$ for some $K > 0$ in (4.31). All the results of the previous section hold in this setting.

From the theory of optimal stopping (see Appendix B), the optimal stopping rule for $v(x, y)$ is given by

$$
\tau^* = \inf\{t \geq 0 : v(S_t, V_t) \notin \mathcal{C}\} \leq \infty, \tag{4.33}
$$

where $\mathcal{C} = \{(x, y) \in \mathbb{R}_+^2 : v(x, y) > g(x)\}$.

Now, for each fixed $y \in \mathbb{R}_+$, consider the y-section $\mathcal{C}_y = \{x \in \mathbb{R}_+ : v(x, y) > g(x)\}$, and define

$$
b(y) := \inf\ C_y. \tag{4.34}
$$

The aim of this section is to show that the optimal stopping boundary of v is characterized by the mapping $y \mapsto b(y)$ and that b is non-increasing and left-continuous in y.

Before stating and proving the main theorem, we show a further property of $v(x, y)$ in the direction of the variable x, which will be used to deduce the continuity of $b(y)$ (compare with Proposition 1 in [30]).

We use that $\max\{a - b, 0\} \geq \max\{a, 0\} - \max\{b, 0\}$.

Proposition 4.20 *For each $y \in \mathbb{R}_+$, the mapping $x \mapsto v(x, y) - g(x)$ is non-decreasing in* $(0, K)$.

Proof. Fix $x \in (0, K)$ and let $\epsilon > 0$ be such that $x + \epsilon < K$. Suppose that τ^* is optimal for $v(x, y)$. Then

$$\begin{aligned} v(x+\epsilon, y) &\geq E_{x+\epsilon,y}\, e^{-r\tau^*} g(S_{\tau^*}) \;= E_{x,y}\, e^{-r\tau^*} g\left(\frac{x+\epsilon}{x}\, S_{\tau^*}\right) \\ &= E_{x,y}\, e^{-r\tau^*} \max\left\{K - \frac{x+\epsilon}{x} S_{\tau^*}, 0\right\} \\ &= E_{x,y}\, e^{-r\tau^*} \max\left\{K - S_{\tau^*} - \frac{\epsilon}{x} S_{\tau^*}, 0\right\} \\ &\geq E_{x,y}\, e^{-r\tau^*} \max\left\{K - S_{\tau^*}, 0\right\} - \frac{\epsilon}{x}\, E_{x,y}\, e^{-r\tau^*} S_{\tau^*} \\ &= v(x, y) - \frac{\epsilon}{x}\, E_{x,y}\, e^{-r\tau^*} S_{\tau^*}. \end{aligned}$$

Now, recalling that $\int_0^t Y_s^2 ds < \infty$ a.s. for each $t \geq 0$ in our examples, we must have that the process $(e^{-rt} S_t)_{t\geq 0}$ is a non-negative continuous local martingale. Hence $(e^{-rt} S_t)_{t\geq 0}$ is a supermartingale and, in particular,

$$E_{x,y}\, e^{-r\tau^*} S_{\tau^*} \leq x.$$

The previous arguments yield $v(x + \epsilon, y) \geq v(x, y) - \epsilon$.

Finally, since $x + \epsilon < K$, we must have that $g(x) - g(x + \epsilon) = \epsilon$. From here we conclude that

$$v(x + \epsilon, y) - g(x + \epsilon) \geq v(x, y) - g(x)$$

as required. □

Remark 4.21 Remark that the inequality $v(x+\epsilon, y) \geq v(x,y) - \epsilon$ could have been obtained even if $x \geq K$. We only used that $x < K$ in the final part of the above proof. Instead, we obtain that $x \mapsto v(x,y)$ is non-increasing in $[K, \infty)$.

By adapting the arguments in [27], we are now ready to show the main result of this section.

Theorem 4.22 *The optimal stopping time* τ^* *is given by*

$$\tau^* = \inf\{t \geq 0 : S_t \leq b(V_t)\}, \tag{4.35}$$

where the boundary $b(y)$ *satisfies that* $0 < b(y) < K$*, and is a non-increasing and left-continuous function.*

Proof. We split the proof into four parts. We shall use that, by definition,

$$b(y) = \inf\ C_y = \inf\{x \in \mathbb{R}_+ : v(x,y) > g(x)\}.$$

Notice that if $x \geq K$ then necessarily $x \in \mathcal{C}_y$ because $g(x) = 0$ and $v(x,y) > 0$ by Proposition 4.15. Thus $\mathcal{C}_y \neq \emptyset$ and $b(y) \leq K$.

(I). The function $y \mapsto b(y)$ is non-increasing.

Since $v(x,\cdot)$ is non-decreasing by Proposition 4.16, we have that for each $\epsilon > 0$,

$$y \leq y + \epsilon \quad \Rightarrow \quad C_y \subseteq C_{y+\epsilon} \quad \Rightarrow \quad b(y) \geq b(y+\epsilon),$$

proving the claim.

(II). We establish (4.35). To this end it is enough to prove that, for each $y \in \mathbb{R}_+$,

$$\mathcal{C}_y = \{x \in \mathbb{R}_+ : x > b(y)\}.$$

We know that if $x \geq K$ then $x \in \mathcal{C}_y$.

On the other hand, suppose that $x \in \mathcal{C}_y$ is such that $x < K$. Such an x exists because $\mathcal{C}_y$ is open by the continuity of $v(\cdot, y)$ (see Proposition 4.17) and g, while $[K, \infty) \subseteq C_y$.

Using that $x \mapsto v(x,y) - g(x)$ is non-decreasing in $(0, K)$ by Proposition

4.20, we have that for each $\epsilon > 0$ such that $x + \epsilon < K$,

$$x < x + \epsilon \quad \Rightarrow \quad v(x+\epsilon, y) - g(x+\epsilon) > 0 \quad \Rightarrow \quad (x+\epsilon) \in C_y.$$

Since $\mathcal{C}_y$ is open, $b(y) \notin \mathcal{C}_y$. This completes the proof.

(III). The boundary $b(y)$ satisfies that $0 < b(y) < K$. Since $v(x,y) > 0$ everywhere and $b(y) \notin \mathcal{C}_y$,

$$v(b(y), y) = g(b(y)) > 0.$$

But $g(b(y)) = \max\{K - b(y), 0\} > 0$ if and only if $0 < b(y) < K$.

(IV). The function $y \mapsto b(y)$ is left-continuous.

We know that $(b(y), y) \in \mathcal{C}^c$ for all $y \in \mathbb{R}_+$.

Let $\{y_n, y\}$ be a sequence in $\mathbb{R}_+$ such that $y_n \uparrow y$.

By part (I), $b(y) \leq \liminf_n b(y_n)$. On the other hand, using that $\mathcal{C}^c$ is closed and that for each n, $(b(y_n), y_n) \in \mathcal{C}^c$, we obtain $(\limsup_n b(y_n)\,, y) \in \mathcal{C}^c$. This in turn yields $\limsup_n b(y_n) \leq b(y)$.

Putting all together, the claim is evident. □

4.3.6 Appendix: Bessel processes

We consider some well-known facts about Bessel processes that are used above. We refer to Chapter XI in [39], Section V.48 in [40], or the survey [18] for a deeper insight into this class of processes.

Consider the stochastic differential equation

$$X_t = x + 2\int_0^t \sqrt{|X_s|}\, dB_s + \phi\, t, \tag{4.36}$$

where $x \geq 0$ and $\phi \geq 0$. By Theorems IV.2.3 and IV.2.4 in [26] we know that for each $x \geq 0$, there exists a non-exploding weak solution to (4.36). This can be seen because the coefficients $\sigma(x) = 2\sqrt{|x|}$ and $b(x) \equiv \phi$ are continuous and satisfy the linear growth condition $|\sigma(x)|^2 + |b(x)|^2 \leq K(1+|x|^2)$ Also, by part (ii) of Theorem IX.3.5 in [39], we know that pathwise uniqueness holds for this equation, because

$$|\sqrt{x}-\sqrt{y}|^2 \le |x-y|, \qquad x,y\ge 0, \quad \text{and} \quad \int_{0+}^{\infty} \frac{da}{a} = +\infty,$$

which verify the conditions of that theorem.

In other words, the equation in (4.36) has a unique non-exploding strong solution $X = (X_t)_{t\ge 0}$ and it is referred to as the *square Bessel process with dimension* ϕ.

Since $X_t \equiv 0$ is the solution when $x = 0$ and $\phi = 0$, by a simple application of the comparison Theorem IX.3.7 in [39], we have that for $\phi \ge 0$,

$$X_t \ge 0, \quad \text{for all } t > 0 \text{ a.s.}$$

If $\phi = 2$ then X_t can be represented as the Euclidean norm $|W_t|^2$ of a two-dimensional Brownian motion W. Given that W does not hit $\{0\}$ in finite time with positive probability, the same is true for X. Hence, for $\phi \ge 2$,

$$X_t > 0, \quad \text{for all } t > 0 \text{ a.s.}, \tag{4.37}$$

again by Theorem IX.3.7 in [39].

When $\phi \ge 2$, we can apply Itô's formula to $Z_t = f(X_t)$, with $f(x) = \sqrt{x}$, by the fact in (4.37) so that we obtain the stochastic differential equation

$$dZ_t = dB_t + \frac{\phi-1}{2Z_t}\,dt, \qquad \phi \ge 2. \tag{4.38}$$

This equation has at least one solution for each initial condition $z = x^2 > 0$, namely $Z_t = \sqrt{X_t}$. It turns out that this is the only solution, because if Z' also solves (4.38) (with the same driving Brownian motion) and $Z'_0 = z$, then

$$Z_t - Z'_t = \int_0^t \frac{\phi-1}{2}\left(\frac{1}{Z_s} - \frac{1}{Z'_s}\right) ds.$$

Setting $\Phi(t) = |Z_t - Z'_t|$ and $\beta(t) = \frac{\phi-1}{2\,Z_t\,Z'_t} > 0$ we have that, for each $t \ge 0$,

$$0 \le \Phi(t) \le \int_0^t \beta(s)\,\Phi(s)ds.$$

By Gronwall's inequality we conclude that $\Phi(t) \equiv 0$, or equivalently, that $Z_t \equiv Z'_t$ for all $t \geq 0$.

The unique non-exploding strong solution $Z = (Z_t)_{t\geq 0}$ to (4.38) is referred to as the *Bessel process with dimension* ϕ.

We used the following elementary comparison result, which is a consequence of the above discussion:

Proposition 4.23 *Let ϕ^i, z^i, $i = 1, 2$ be real numbers such that $2 \leq \phi^1 \leq \phi^2$ and $0 < z^1 \leq z^2$. Let Z^i, $i = 1, 2$ be two Bessel processes with dimension ϕ^i and with respect to the same driving Brownian motion, such that $Z_0^i = z^i$. Then*

$$0 < Z_t^1 \leq Z_t^2, \qquad \forall\, t \geq 0 \quad a.s.$$

We also made use of Proposition A.1(ii)-(iii) in [24] in the examples, so we re-state it here for ease of reference.

Proposition 4.24 (Hobson [24]) *Let $Z = (Z_t)_{t\geq 0}$ be a Bessel process with dimension ϕ and such that $Z_0 > 0$. Define $\Gamma_t \equiv \Gamma_t^{(p)} = \int_0^t Z_s^{-p} ds$.*
(a) Suppose that $\phi > 2$. Then $\lim_{t\to\infty} \Gamma_t^{(p)} = \infty$ if and only if $p \leq 2$.
(b) Suppose that $\phi = 2$. Then $\lim_{t\to\infty} \Gamma_t^{(p)} = \infty$ for all p.

Appendix A

Estimates of moments and integrability

The main goal of this Appendix is to provide sufficient conditions on g and on the dynamics of (X, Y) in order to obtain

$$E_{x,y}\left[\sup_{0\leq t\leq T} e^{-\alpha t}|g(X_t)|\right] < \infty \tag{A.1}$$

where $T \in [0, \infty]$.

We first derive some estimates of the moments of the solution to an stochastic differential equation (SDE) with regime-switching coefficients. The proof of Proposition A.1 below is inspired by ideas in Kyrlov [35], and it is somewhat an extension of Corollary 2.5.12 in that book.

Estimates of moments of SDE's with regime-switching

The result of this section is of independent interest, and this is the reason why we assume the following general set-up.

Let $(W_t, \mathcal{F}_t)$ be a d_1-dimensional Brownian motion. Suppose that $r = (r_t)_{t\geq 0}$ is a continuous-time Markov chain, adapted to $(\mathcal{F}_t)_{t\geq 0}$, with finite state space $\mathcal{S} \subset \mathbb{R}$. The process r determines the regime-switching dynamics.

For $d \in \mathbb{N}$ and $x_0 \in \mathbb{R}^d$, $x = (x_t)_{t\geq 0}$ is a progressively measurable process in $\mathbb{R}^d$, with respect to $(\mathcal{F}_t)_{t\geq 0}$, satisfying that

$$x_t = x_0 + \int_0^t \sigma_s(x_s, r_s)dW_s + \int_0^t b_s(x_s, r_s)ds, \qquad a.s. \tag{A.2}$$

where $\sigma_t(x,r)$ is a random matrix of dimension $d \times d_1$; and $b_t(x,r)$ is a random vector of dimension d.

Proposition A.1 *Fix $T > 0$. Let there exist a constant $K > 0$ such that*

$$\|\sigma_t(x,r)\| + |b_t(x,r)| \leq K(1+|x|), \qquad \text{for all} \quad t \geq 0, x \in \mathbb{R}^d, r \in \mathcal{S}. \tag{A.3}$$

Then for all $t \in [0,T]$, $q \geq 0$, there exists a constant $N = N(x_0, K, t, q)$ such that

$$E \sup_{s\leq t} |x_s|^q \leq N < \infty. \tag{A.4}$$

Remark A.2 The particular case where $\mathcal{S}$ is a singleton corresponds to Corollary 2.5.12 in [35].

Proof. Fix an arbitrary $t \in [0,T]$ and $q \geq 0$.

We split the proof into three parts.

(I). Assume that $x_t(\omega)$ is bounded in ω and t.

Notice that

$$|x_t|^2 \leq 4\left[|x_0|^2 + \left|\int_0^t \sigma_s(x_s, r_s)dW_s\right|^2 + \left|\int_0^t b_s(x_s, r_s)ds\right|^2\right].$$

The linear growth condition in (A.3) implies the following. First, the stochastic integral $M_\cdot = \int_0^\cdot \sigma_s(x_s, r_s)dW_s$ satisfies that

$$E\langle M\rangle_t = E\int_0^t \|\sigma_s(x_s, r_s)\|^2 ds \leq 2K^2 E\int_0^t (1+|x_s|^2)ds < \infty$$

for all $t \geq 0$, since x_t is assumed to be bounded. Then M is a martingale. Second, using Hölder's inequality,

$$\left|\int_0^t b_s(x_s, r_s)ds\right|^2 \leq t\int_0^t |b_s(x_s, r_s)|^2 ds \leq 2K^2 t\int_0^t (1+|x_s|^2)ds.$$

Putting the last assertions together we obtain, after taking supremum over $[0,t]$ and expectation,

$$\begin{aligned}
E\sup_{0\le s\le t}|x_s|^2 &\le 4\left[|x_0|^2 + E\sup_{0\le s\le t}|M_s|^2 + 2K^2 t\, E\int_0^t (1+|x_s|^2)ds\right]\\
&\le 4\left[|x_0|^2 + 4E|M_t|^2 + 2K^2 t\, E\int_0^t (1+|x_s|^2)ds\right]\\
&\le 4\left[|x_0|^2 + (2K^2)4E\int_0^t (1+|x_s|^2)ds + 2K^2 t\, E\int_0^t (1+|x_s|^2)ds\right]\\
&\le 4|x_0|^2 + 8K^2(4+t)\int_0^t (1+E\sup_{0\le u\le s}|x_u|^2)ds
\end{aligned}$$

where we have used Doob's inequality, the fact that $M_t^2 - \langle M\rangle_t$ is a martingale (see for instance [39, II.1.7 and IV.1.3]), the linear growth condition in (A.3), Fubini's Theorem and the boundedness of x_t.

Now set $\varphi(t) = \sup_{0\le s\le t}|x_s|^2$, $a = 1+4|x_0|^2$, and $b = 8K^2(4+t)$, so that

$$1 + E\varphi(t) \le a + b\int_0^t \{\,1 + E\varphi(s)\,\}ds.$$

Then, by Grownwall's Lemma, we have that $1+E\varphi(t) \le a\,e^{bt}$, that is

$$E\sup_{0\le s\le t}|x_s|^2 \le \bar{N}(x_0, K, t)$$

where $\bar{N}(x_0,K,t) = (1+4|x_0|^2)e^{8K^2t(4+t)}$.

(II). Since x_t is continuous and bounded in t, it follows that $\sup_{s\le t}|x_s|^p = (\sup_{s\le t}|x_s|)^p$ for any $p\ge 0$. Using this equality with $p=q$ and then with $p=2$, we obtain that

$$E\sup_{0\le s\le t}|x_s|^q \le \left(E\sup_{0\le s\le t}|x_s|^2\right)^{q/2} \le N(x_0,K,t,q)$$

where we also used Hölder's inequality in the form $E\eta^q \le [E\eta^2]^{q/2}$. Here, $N \equiv N(x_0,K,t,q) = \bar{N}(x_0,K,t)^{q/2}$.

(III). We now assume the general case for $x_t(\omega)$.

For each $R>0$, consider the stopping time $\tau_R = \inf\{t\ge 0 : |x_t| \ge R\}$.

Then the stopped process $x_{t\wedge\tau_R}(\omega)$ is bounded in ω, t and moreover,

$$\begin{aligned} x_{t\wedge\tau_R} &= x_0 + \int_0^{t\wedge\tau_R} \sigma_s(x_s, r_s)dW_s + \int_0^{t\wedge\tau_R} b_s(x_s, r_s)ds \\ &= x_0 + \int_0^t I\{s < \tau_R\}\sigma_s(x_{s\wedge\tau_R}, r_{s\wedge\tau_R})dW_s \\ &\qquad + \int_0^t I\{s < \tau_R\}b_s(x_{s\wedge\tau_R}, r_{s\wedge\tau_R})ds. \end{aligned}$$

Notice that $x_{t\wedge\tau_R}$ solves (A.2) only that with the coefficients $\sigma_s(x,r), b_s(x,r)$ replaced by $I\{s < \tau_R\}\sigma_s(x,r), I\{s < \tau_R\}b_s(x,r)$, respectively. However, for each fixed ω,

$$\|I\{s < \tau_R\}\sigma_s(x,r)\| \le \|\sigma_t(x,r)\|, \quad \text{and} \quad |I\{s < \tau_R\}b_t(x,r)| \le |b_t(x,r)|.$$

Then the linear growth condition in (A.3) is satisfied for the coefficients of $x_{t\wedge\tau_R}$.

From parts **(I)-(II)**, we know that

$$E \sup_{0\le s\le t} |x_{s\wedge\tau_R}|^q \le N, \qquad \text{for each } R > 0.$$

Given that $\lim_{R\to\infty} \tau_R = \infty$ a.s, it follows that $\lim_{R\to\infty} |x_{s\wedge\tau_R}|^q = |x_s|^q$ a.s. by continuity of the paths of x_t. As this is true for each $s \le t$, we must have $|x_s|^q \le \lim_{R\to\infty} \sup_{u\le t} |x_{u\wedge\tau_R}|^q$ for each $s \le t$. Hence

$$\sup_{0\le s\le t} |x_s|^q \le \lim_{R\to\infty} \sup_{0\le s\le t} |x_{s\wedge\tau_R}|^q, \quad a.s.$$

Finally, Fatou's Lemma implies

$$E \sup_{0\le s\le t} |x_s|^q \le \liminf_{R\to\infty} E \sup_{0\le s\le t} |x_{s\wedge\tau_R}|^q \le N,$$

and the proof of Proposition A.1 is complete. □

Sufficient conditions for integrability

In this section we provide sufficient conditions in order for (A.1) to hold. To this end, we shall adapt Proposition A.1 to the setting in this thesis.

Fix the initial condition $(x, y) \in \mathbb{R} \times \mathcal{S}$ and a time horizon $T \in [0, \infty]$.

In the regime-switching case, we typically work with

$$X_t = x + \int_0^t a(X_s) Y_s \, dB_s + \int_0^t \mu(X_s) \, ds, \quad t \geq 0, \ P_{x,y} - a.s.$$

where the process Y is a continuous-time, irreducible Markov chain on the finite state space $\mathcal{S} = \{y_i : i = 1, 2, \ldots, m\} \subset (0, \infty)$ with Q-matrix $(q[y_i, y_j])$.

In the notation of Proposition A.1,

$$d = 1, \quad x_t = X_t, \quad r_t = Y_t, \quad \|\sigma(x, r)\|^2 = a^2(x) r^2, \quad |b(x, y)| = |\mu(x)|. \tag{A.5}$$

In the diffusion case, we typically work with the system

$$X_t = x + \int_0^t a(X_s) Y_s \, dB_s + \int_0^t \mu(X_s) \, ds, \quad Y_t = y + \int_0^t \eta(Y_s) dB_s^Y + \int_0^t \theta(Y_s) ds$$

for all $t \geq 0$, $P_{x,y}$- a.s. where $\langle B, B^Y \rangle_t = \delta \, t$ for some $\delta \in [-1, 1]$.

In the notation of Proposition A.1,

$$\begin{aligned} &d = 1, \quad x_t = (X_t, Y_t), \quad r_t = \text{constant} \\ &\|\sigma(x, y)\|^2 = a^2(x) y^2 + \eta^2(y), \quad |b(x, y)|^2 = \mu(x)^2 + \theta(y)^2. \end{aligned} \tag{A.6}$$

We assume the following:

(S1) The measurable gain function g has polynomial growth, that is,

$$|g(x)| \leq C(1 + |x|^q)$$

for some constants $C, q \geq 0$.

(S2) The measurable functions a, μ, η and θ satisfy a liner growth condition.

(S3) If $T = \infty$, further assume that

$$\lim_{t\to\infty} e^{-\alpha t}|g(X_t)| = 0 \qquad \text{a.s.}$$

With Assumption (S1), we have that for each $T' < \infty$,

$$E_{x,y}\left[\sup_{0\le t\le T'} e^{-\alpha t}|g(X_t)|\right] \le C + E_{x,y}\left[\sup_{0\le t\le T'} |X_t|^q\right]. \tag{A.7}$$

Denote by C^* the right-hand side of (A.7). Then we also have that

$$E_{x,y}\left[\sup_{t\ge 0} e^{-\alpha t}|g(X_t)|\right] \le C^* + E_{x,y}\left[\sup_{t\ge T'} e^{-\alpha t}|g(X_t)|\right]. \tag{A.8}$$

Under Assumption (S2), it is easy to see that the linear growth condition in (A.3) holds true in either of the cases (A.5) or (A.6). Hence C^* is finite, thanks to Proposition A.1.

If the horizon T is infinite, then Assumption (S3) implies that we can choose a sufficiently large T' such that

$$\sup_{t\ge T'} e^{-\alpha t}|g(X_t)| \approx 0 \qquad \text{a.s.}$$

This implies that the right-hand side of (A.8) is finite.

Appendix B

Key results from optimal stopping

Fix a time horizon $T \in [0, \infty]$. Assume that (X, Y) is a strong Markov process defined on $(\Omega, \mathcal{F}, P_{x,y}, (x, y) \in \mathcal{E})$, with (augmented) natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.

We shall use the common notation $\tau \in \mathbb{F}_T$ (resp. $\tau \in \mathbb{F}_{[t,T]}$) to mean a stopping time $\tau : \Omega \to [0, \infty]$ with respect to $\mathbb{F}$ subject to $0 \leq \tau \leq T$ (resp. $t \leq \tau \leq T$) a.s.

Consider the value function

$$v(x, y) = \sup_{\tau \in \mathbb{F}_T} E_{x,y} \left[e^{-\alpha \tau} g(X_\tau) - C_\tau \right], \quad (x, y) \in \mathbb{R} \times \mathcal{S}. \tag{B.1}$$

where

$$C_t = \int_0^t e^{-\alpha s} c(X_s) \, ds,$$

With this notation, v is like in (2.1) when $c \equiv 0$, or like in (2.53) when $c \geq 0$ and bounded.

Recall that we are assuming the integrability condition in (1.3).

We shall state some classical results from the theory of optimal stopping, mainly based on Theorem I.2.2 in [37]. But before that, let us introduce some notation:

Define the process $U = (U_t)_{t \geq 0}$ by

$$U_t = e^{-\alpha\, t} g(X_t) - C_t$$

and $V = (V_t)_{0 \leq t \leq T}$ to be the so-called *Snell envelope* of U with respect to the filtration $\mathbb{F}$,

$$V_t = \operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[t,T]}} E_{x,y} \left[U_\tau \mid \mathcal{F}_t \right].$$

We will also use $U(t, \tau)$, which is defined for each $\tau \in \mathbb{F}_{[t,T]}$ as

$$U(t, \tau) := \, e^{-\alpha\, \tau} g(X_\tau) - \int_t^\tau e^{-\alpha\, s} c(X_s)\, ds.$$

Notice that V_t may also be written as

$$V_t = \operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[t,T]}} E_{x,y} \left[U(t, \tau) \mid \mathcal{F}_t \right] - \int_0^t e^{-\alpha\, s} c(X_s)\, ds,$$

and that $V_t \geq U_t$ $P_{x,y}$-a.s. for each $t \geq 0$ and $V_0 = v(x, y)$.

Lemma B.1 *The process $V = (V_t)_{0 \leq t \leq T}$ is the smallest $\mathbb{F}$-supermartingale which dominates U.*

The proof of this Lemma is very similar to that of part 1^o of Theorem 2.2 in [37]. The only difference is that here, for each fixed $t \geq 0$, one shows the existence of a sequence $\{\tau_k : \, k \geq 1\}$ in $\mathbb{F}_{[t,T]}$ such that,

$$\operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[t,T]}} E_{x,y} \left[U(t, \tau) \mid \mathcal{F}_t \right] = \lim_{k \to \infty} E_{x,y} \left[U(t, \tau_k) \mid \mathcal{F}_t \right], \tag{B.2}$$

where the limit is monotonously increasing. The rest of the arguments are the same after the natural changes in notation.

The following result is used in the proof of Propositions 2.11 and 2.21 and states that the stopping times in (B.1) may be taken with respect to a larger filtration $\mathbb{G}$, provided (X, Y) is a $\mathbb{G}$-strong Markov process.

Proposition B.2 *Suppose that (X, Y) is a strong Markov process with respect a filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, and $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \geq 0$, with $\mathcal{G}_0 = \mathcal{F}_0$. Then*

$$v(x, y) = \sup_{\tau\in\mathbb{G}_T} E_{x,y}\,[e^{-\alpha\tau} g(X_\tau) - C_\tau].$$

Proof. Let $\tilde{V} = (\tilde{V}_t)_{t\geq 0}$ be defined as

$$\tilde{V}_t = \operatorname*{ess\,sup}_{\tau\in\mathbb{G}_{[t,T]}} E_{x,y}\,[U(t,\tau) \mid \mathcal{G}_t] - \int_0^t e^{-\alpha\, s} c(X_s)\, ds,$$

so that $\tilde{V}$ is the smallest $\mathbb{G}$-supermartingale which dominates U.

Since $\mathcal{F}_t \subseteq \mathcal{G}_t$, it is clear that $V_0 \leq \tilde{V}_0$.

We will show that V is *a* $\mathbb{G}$-supermartingale, which implies that $\tilde{V}_t \leq V_t$ a.s. for each $t \geq 0$. In particular, this yields that $V_0 = \tilde{V}_0$, proving the result.

For each $t \geq 0$, the terms on the right-hand side of (B.2) satisfy that

$$E_{x,y}\,[U(t,\tau_k) \mid \mathcal{F}_t] = E_{x,y}\,[U(t,\tau_k) \mid \sigma(X_t, Y_t)] = E_{x,y}\,[U(t,\tau_k) \mid \mathcal{G}_t], \qquad k \geq 1,$$

by the strong Markov property of (X, Y) with respect to both, $\mathbb{F}$ and $\mathbb{G}$.

If $u \leq t$ then

$$\begin{aligned}
E_{x,y}[\,V_t \mid \mathcal{G}_u] =& E_{x,y}\left[\lim_{k\to\infty} E_{x,y}\,[U(t,\tau_k) \mid \mathcal{F}_t] - \int_0^t e^{-\alpha\, s} c(X_s)\, ds \ \mid \mathcal{G}_u\right] \\
=& E_{x,y}\left[\lim_{k\to\infty} E_{x,y}\,[U(t,\tau_k) \mid \mathcal{G}_t] - \int_0^t e^{-\alpha\, s} c(X_s)\, ds \ \mid \mathcal{G}_u\right] \\
=& E_{x,y}\left[\lim_{k\to\infty} E_{x,y}\left[U(t,\tau_k) - \int_0^t e^{-\alpha\, s} c(X_s)\, ds \ \mid \mathcal{G}_t\right] \ \mid \mathcal{G}_u\right] \\
=& \lim_{k\to\infty} E_{x,y}\left[\, E_{x,y}\left[U(t,\tau_k) - \int_0^t e^{-\alpha\, s} c(X_s)\, ds \ \mid \mathcal{G}_t\right] \ \mid \mathcal{G}_u\right] \\
=& \lim_{k\to\infty} E_{x,y}\left[U(t,\tau_k) - \int_0^t e^{-\alpha\, s} c(X_s)\, ds \ \mid \mathcal{G}_u\right]
\end{aligned}$$

where we have used monotone convergence.

Now, by the linearity of the Lebesgue integral,

$$\begin{aligned}
&E_{x,y}\left[U(t,\tau_k) - \int_0^t e^{-\alpha\, s} c(X_s)\, ds \;\mid \mathcal{G}_u\right] \\
&\quad = E_{x,y}\left[U(u,\tau_k) + \int_u^t e^{-\alpha\, s} c(X_s)\, ds - \int_0^t e^{-\alpha\, s} c(X_s)\, ds \;\mid \mathcal{G}_u\right] \\
&\quad = E_{x,y}\left[U(u,\tau_k) - \int_0^u e^{-\alpha\, s} c(X_s)\, ds \;\mid \mathcal{G}_u\right] \\
&\quad = E_{x,y}\left[U(u,\tau_k) \;\mid \mathcal{G}_u\right] - \int_0^u e^{-\alpha\, s} c(X_s)\, ds \\
&\quad = E_{x,y}\left[U(u,\tau_k) \;\mid \mathcal{F}_u\right] - \int_0^u e^{-\alpha\, s} c(X_s)\, ds.
\end{aligned}$$

Putting all together,

$$\begin{aligned}
E_{x,y}[\, V_t \mid \mathcal{G}_u] &= \lim_{k\to\infty} E_{x,y}\left[U(u,\tau_k) \;\mid \mathcal{F}_u\right] - \int_0^u e^{-\alpha\, s} c(X_s)\, ds \\
&\leq \operatorname*{ess\,sup}_{\tau\in\mathbb{F}_{[u,T]}} E_{x,y}\left[U(u,\tau) \mid \mathcal{F}_u\right] - \int_0^u e^{-\alpha\, s} c(X_s)\, ds = V_u.
\end{aligned}$$

Hence V is a $\mathbb{G}$-supermartingale, and the proof is complete. □

It is interesting that in the previous statements there is no need to impose any assumption on the function g other than measurable. However, for the next result which is used in Section 3.2, we add the assumption that g is upper semi-continuous so that the *gain* process $U = (U_t)_{0\leq t\leq T}$ has upper semi-continuous paths.

Let τ^* be the first time that V coincides with U, that is,

$$\tau^* = \inf\{0 \leq t \leq T : V_t = U_t\}.$$

If $T < \infty$ then $V_T = U_T$ and so $\tau^* \leq T < \infty$ a.s. Otherwise, $\tau^* \leq \infty$.

Theorem B.3 *Suppose that $\tau^* < \infty$ $P_{x,y}$-a.s. and that g is an upper semi-continuous function. Then τ^* attains the supremum in (B.1), the process $V = (V_t)_{0\leq t\leq T}$ is the smallest $\mathbb{F}$-supermartingale which dominates U, and the stopped process $V_{\cdot\wedge\tau^*} = (V_{t\wedge\tau^*})_{0\leq t\leq T}$ is an $\mathbb{F}$-martingale.*

We refer to Theorem 2.2 in [37] for a proof: simply replace G_t and S_t by

U_t and V_t, respectively.

Remark B.4 When $T = \infty$, the strong Markov property of (X, Y) implies that

$$\begin{aligned}
\operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[t,\infty]}} E_{x,y}\left[U(t,\tau) \mid \mathcal{F}_t\right] &= \operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[t,\infty]}} E_{x,y}\left[e^{-\alpha\tau} g(X_\tau) - \int_t^\tau e^{-\alpha s} c(X_s) ds \mid \mathcal{F}_t\right] \\
&= e^{-\alpha t} \operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[t,\infty]}} E_{x,y}\left[e^{-\alpha(\tau - t)} g(X_\tau) - \int_t^\tau e^{-\alpha(s-t)} c(X_s) ds \mid \mathcal{F}_t\right] \\
&= e^{-\alpha t} v(X_t, Y_t).
\end{aligned}$$

So, in fact,

$$V_t = e^{-\alpha t} v(X_t, Y_t) - \int_0^t e^{-\alpha\, s} c(X_s)\, ds. \tag{B.3}$$

The latter implies that $V_t = U_t$ if and only if $v(X_t, Y_t) = g(X_t)$. As a consequence, τ^* takes the form

$$\tau^* = \inf\{\, t \geq 0 : v(X_t, Y_t) = g(X_t)\,\} \leq \infty.$$

In Section 3.2, we assume that $\lim_{t\to\infty} e^{-\alpha t} g(X_t) = 0$ a.s. In this case, it is readily seen that both, $U_\infty := \lim_{t\to\infty} U_t$ and $U(t,\infty) := \lim_{s\to\infty} U(t,s)$, exist because c is bounded, and so the result in Theorem B.3 is not affected when $\tau^* = \infty$ with positive probability.

The following Lemma is a by-product of the strong Markov property and it is used in the proofs of Propositions 3.3 and 3.4.

Lemma B.5 *Suppose that $T = \infty$. Fix $(x, y) \in \mathcal{E}$ and assume that g is upper semi-continuous and satisfies that $\lim_{t\to\infty} e^{-\alpha t} g(X_t) = 0$ a.s. Then for every finite stopping time τ such that $\tau \leq \tau^*$,*

$$E_{x,y} V_\tau = E_{x,y} V_{\tau^*} = v(x, y),$$

where V is of the form in (B.3).

Proof. By Theorem B.3 and Remark B.4, it follows that

$$v(x, y) = E_{x,y}\left(e^{-\alpha\tau^*} g(X_{\tau^*}) - \int_0^{\tau^*} e^{-\alpha s} c(X_s) ds\right) = E_{x,y} V_{\tau^*}.$$

Since $\lim_{t\to\infty} e^{-\alpha t} g(X_t) = 0$ a.s. we have that

$$E_{x,y}\, V_\tau = E_{x,y}\left[e^{-\alpha\tau}\, E_{X_\tau,Y_\tau}\left(e^{-\alpha\tau^*} g(X_{\tau^*}) I(\tau^* < \infty) - \int_0^{\tau^*} e^{-\alpha s} c(X_s) ds\right)\right] \\ - E_{x,y}\left[\int_0^{\tau} e^{-\alpha s} c(X_s) ds\right]$$

Let us now concentrate on the first expectation of the right-hand side of this equality. We have that

$$\begin{aligned} & E_{X_\tau,Y_\tau}\left(e^{-\alpha\tau^*} g(X_{\tau^*}) I(\tau^* < \infty) - \int_0^{\tau^*} e^{-\alpha s} c(X_s) ds\right) \\ &= E_{x,y}\left[\left(e^{-\alpha\tau^*} g(X_{\tau^*}) I(\tau^* < \infty) - \int_0^{\tau^*} e^{-\alpha s} c(X_s) ds\right) \circ \theta_\tau \mid \mathcal{F}_\tau\right] \\ &= E_{x,y}\left[e^{-\alpha(\tau^*-\tau)} g(X_{\tau^*}) I(\tau^* < \infty) - \int_0^{\tau^*} e^{-\alpha s} c(X_{s+\tau}) ds \mid \mathcal{F}_\tau\right] \\ &= E_{x,y}\left[e^{-\alpha(\tau^*-\tau)} g(X_{\tau^*}) I(\tau^* < \infty) - e^{\alpha\tau} \int_\tau^{\tau^*} e^{-\alpha s} c(X_s) ds \mid \mathcal{F}_\tau\right], \end{aligned}$$

where θ_τ shifts the paths by τ.

Given that $\int_0^\tau e^{-\alpha s} c(X_s) ds$ is $\mathcal{F}_\tau$-measurable, we obtain that

$$\begin{aligned} E_{x,y}\, V_\tau &= E_{x,y}\left[E_{x,y}\left[e^{-\alpha\tau^*} g(X_{\tau^*}) I(\tau^* < \infty) - \int_0^{\tau^*} e^{-\alpha s} c(X_s) ds \mid \mathcal{F}_\tau\right]\right] \\ &= E_{x,y}\left[e^{-\alpha\tau^*} g(X_{\tau^*}) I(\tau^* < \infty) - \int_0^{\tau^*} e^{-\alpha s} c(X_s) ds\right] \\ &= E_{x,y} V_{\tau^*}, \end{aligned}$$

which completes the proof. □

We remark that the optional sampling Theorem cannot be used in the case that τ^* is unbounded, unless the martingale $V_{\cdot\wedge\tau^*}$ is uniformly integrable.

Since we are assuming the integrability condition in (1.3) and that c is bounded, we can show that $V_{\cdot\wedge\tau^*}$ is indeed uniformly integrable. Then the following is an alternative proof to the above Lemma under (1.3), although the strong Markov property is again the main tool.

Alternative proof of Lemma B.5. It is enough to show that $V_{\cdot\wedge\tau^*}$ is uniformly integrable. Then we can use Theorem II.3.2 in [39].

The fact that c is non-negative and bounded together with the strong Markov property of the (X, Y) imply that, on the event $\{t < \tau^*\}$, $V_{t\wedge\tau^*}$ is bounded above by

$$\begin{aligned}
\left|e^{-\alpha t}v(X_t, Y_t) - \int_0^t e^{-\alpha s}c(X_s)ds\right| &\leq e^{-\alpha t}|v(X_t, Y_t)| + \int_0^t e^{-\alpha s}c(X_s)ds \\
&\leq e^{-\alpha t}E_{X_t,Y_t}\left(\left|e^{-\alpha\tau^*}g(X_{\tau^*}) - \int_0^{\tau^*} e^{-\alpha s}c(X_s)ds\right|\right) + \int_0^t e^{-\alpha s}c(X_s)ds \\
&\leq e^{-\alpha t}\, E_{X_t,Y_t}\left[\sup_{s\geq 0}\left\{e^{-\alpha s}\,|g(X_s)| + \int_0^s e^{-\alpha u}c(X_u)du\right\}\right] + \int_0^t e^{-\alpha s}c(X_s)ds \\
&= E_{x,y}\left[\sup_{s\geq 0} e^{-\alpha(s+t)}\,|g(X_{s+t})| + \int_t^\infty e^{-\alpha u}c(X_u)du \mid \mathcal{F}_t\right] + \int_0^t e^{-\alpha s}c(X_s)ds \\
&\leq E_{x,y}\left[\sup_{s\geq 0} e^{-\alpha s}\,|g(X_s)| + \int_0^\infty e^{-\alpha u}c(X_u)du \mid \mathcal{F}_t\right] \\
&\leq E_{x,y}\left[\sup_{s\geq 0} e^{-\alpha s}\,|g(X_s)| \mid \mathcal{F}_t\right] + D < \infty.
\end{aligned}$$

where we have used the triangle inequality, the integrability condition (1.3) on g, and $D > 0$ is some constant due to the assumption that c is bounded.

On the event $\{t \geq \tau^*\}$,

$$V_{t\wedge\tau^*} = V_{\tau^*} = e^{-\alpha\tau^*}g(X_{\tau^*}) - \int_0^{\tau^*} e^{-\alpha s}c(X_s)ds.$$

Therefore, setting $S^* = \sup_{s\geq 0} e^{-\alpha s}\,|g(X_s)| + D$, the collection of conditional expectations $\{E_{x,y}[\,S^* \mid \mathcal{F}_t] : \; t \geq 0\}$ is uniformly integrable. Moreover,

$$|V_{t\wedge\tau^*}| \leq E_{x,y}[\,S^* \mid \mathcal{F}_t],$$

from which we conclude that $\{V_{t\wedge\tau^*} \,:\, t \geq 0\}$ is uniformly integrable as well, as required. □

To finish this Appendix and to complete the exposition, let us consider the finite horizon case, $T < \infty$. In this case, we shall need to emphasize the dependance of the value function v on the *time to expiration*:

$$v(x,y,t) = \sup_{\tau \in \mathbb{F}_t} E_{x,y}[\, e^{-\alpha\tau} g(X_\tau) - C_\tau \,], \qquad 0 \le t \le T. \tag{B.4}$$

Notice that $v(x,y,0) = g(x)$ and $v(x,y,T) = v(x,y)$.

Remark B.6 When $T < \infty$, the strong Markov property of (X,Y) implies that

$$\begin{aligned}
\operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[t,T]}} E_{x,y}\left[U(t,\tau) \mid \mathcal{F}_t\right] &= \operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[t,T]}} E_{x,y}\left[e^{-\alpha\tau} g(X_\tau) - \int_t^\tau e^{-\alpha s} c(X_s) ds \mid \mathcal{F}_t\right] \\
&= e^{-\alpha t} \operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[t,T]}} E_{x,y}\left[e^{-\alpha(\tau - t)} g(X_\tau) - \int_t^\tau e^{-\alpha(s-t)} c(X_s) ds \mid \mathcal{F}_t\right] \\
&= e^{-\alpha t} \operatorname*{ess\,sup}_{\tau \in \mathbb{F}_{[0,T-t]}} E_{X_t,Y_t}\left[e^{-\alpha\tau} g(X_\tau) - \int_0^\tau e^{-\alpha s} c(X_s) ds\right] \\
&= e^{-\alpha t} v(X_t, Y_t, T-t).
\end{aligned}$$

So, in fact,

$$V_t = e^{-\alpha t} v(X_t, Y_t, T-t) - \int_0^t e^{-\alpha\, s} c(X_s)\, ds. \tag{B.5}$$

The latter implies that $V_t = U_t$ if and only if $v(X_t, Y_t, T-t) = g(X_t)$. As a consequence, τ^* takes the form

$$\tau^* = \inf\{\, 0 \le t \le T : v(X_t, Y_t, T-t) = g(X_t) \,\}.$$

Moreover, the previous reasoning and the form of $v(x,y,t)$ also yields that for each $t \in [0,T]$,

$$\tau_t^* = \inf\{\, 0 \le s \le t : v(X_s, Y_s, t-s) = g(X_s) \,\}$$

is optimal for $v(x,y,t)$.

Unlike the infinite horizon case in which the optimal time to stop may be infinite, the following Lemma is a straightforward consequence of Theorem B.3 and the optional sampling Theorem.

Lemma B.7 *Suppose that $T < \infty$. Fix $(x,y,t) \in \mathcal{E} \times [0,T]$ and assume that* *g is upper semi-continuous. Then for every stopping time τ such that $\tau \le \tau_t^*$,*

$$E_{x,y}\, V_\tau^t = E_{x,y}\, V_{\tau_t^*}^t = v(x,y,t),$$

where $V_\cdot^t$ is given by

$$V_s^t = e^{-\alpha s} v(X_s, Y_s, t-s) - \int_0^s e^{-\alpha u} c(X_u)\, du, \qquad 0 \le s \le t.$$

Proof. Theorem B.3 establishes that the process $(V_{s\wedge\tau_t^*}^t)_{s\le t}$ is a martingale. The result holds by the optional sampling Theorem, because $\tau_t^* \le t$ a.s. □

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