Equivalence

By

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A thesis submitted in partial fulfilment of the requirements

for the degree of

Doctor of Philosophy in Mathematics Education

University of Warwick

Institute of Education

December 2005

PAGE NUMBERING AS ORIGINAL

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Acknowledgments

Thanks to:

My supervisor, Prof. David Tall, for supporting me to intellectually grow,

My supervisor, Adrian Simpson, who with his canny questions helped me to sharpen my answers,

Prof. Ference Marton who virtually supervised me during my phenomenographical journey,

Dr. Bob Burn who with his letters encouraged me to accomplish this thesis,

Prof. Ian Stewart who helped me to overcome one of my long-lasting problems,

My logic teacher, Prof. Ali Enayat, for being always willing to answer my questions.

In addition, I am grateful to all the above people for giving their permission for me to use our personal communications in this thesis.

Thanks especially to the participants in this study who opened their mind to me, and for giving me the joy of learning.

Thanks also to my friend, Matthew Inglis, who literarily and critically read this thesis.

This thesis is **dedicated** to my beloved wife, Maryam, for loving me; my lovely daughter, Niloufar, for correcting my punctuation! And last but not least my smiley son, Artin.

Declaration

I, the author, declare that the work herein is my own and has not been submitted for a degree at another university. Part of the ideas implicitly or explicitly used in this thesis has been published in the following papers:

Asghari, H.A.: 2004a, 'Students' experiences of equivalence relations', in O. McNamara (ed.), *Proceedings of British Society for Research into Learning Mathematics*, 24, 1, 7-12.

Asghari, H.A.: 2004b, 'Organizing with a focus on defining, a phenomenographic approach', in M.J. Hoines & A.B. Fuglestad (eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 63-70.

Asghari, A. H.: 2005a, 'Examples: generating versus checking', in D. Hewitt and A. Noyes (eds.), *Proceedings of the sixth British Congress of Mathematics Education* held at the University of Warwick, pp. 25-32. Available from www.bsrlm.org.uk.

Asghari, A. H.: 2005b, 'A mad dictator partitions his country', in R. Barwell and A. Noyes (eds.), *Research in Mathematics Education*, 7, pp. 33-45.

Asghari, H.A. and Tall, D.O.: 2005, 'Students' experience of equivalence relations: a phenomenographic approach', in H.L. Chick and J.L. Vincent (eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, University of Melbourne, Melbourne, vol. 2, pp. 81-88.

Abstract

This thesis seeks to answer one single question: "what is an equivalence relation?" A more correct, though longer, version of this question is "what are the qualitatively different ways in which people experience an equivalence relation?" The second question is not simply a version of the first one. It has a completely different nature and consequently demands a completely different answer. *The* answer to the first question can be found in any textbook on the foundations of mathematics; while the second question can be answered only by conducting research where people are given a chance to reveal their conceptions of equivalence relations. These two questions embody two integrated phases of this thesis linked together with a transitory phase.

The first phase starts with a *definite* answer to the first question, i.e. the standard definition of equivalence relations. This definition is used to design a certain situation consisting of certain tasks embodying the corresponding notion. The initial intention of the situation is to get students to define certain predetermined concepts related to the notion of interest, and the effectiveness of the situation is characterized by the extent of students' success to do so. The tasks are tried out on a smallish sample of students. To put it bluntly, the situation fails to achieve its aim. In the process of interviewing the students it becomes clear that the standard definition is just *an* advanced means of organizing by which the given situation (and many others) can be organized. More importantly, there is a growing realization that the initial intention of the study ignores the richness of the students' ways of organizing the situation in favour of maintaining a narrow criterion for success. Relinquishing the latter in favour of the former is the turning point from the first phase to the second.

The second phase is a transitory phase in which more weight has been put on *what* students use to organize the given situation. Although the focus of this phase is not on the notion of equivalence relation, the students' works reveal some *unexpected* aspects of this notion. This suggests the possibility of using the original tasks for pursuing an unexpected purpose in the main (i.e. third) phase of this thesis.

The main phase of the thesis adopts a phenomenographic approach to reveal students' conceptions of equivalence relations. These conceptions are inferred from the ways that the students tackle the tasks, regardless of the extent to which they fit into the standard account. It is shown that these conceptions correspond to certain 'historical' counterparts, where some prominent mathematicians of the past have tackled certain situations that from the vantage point of today's mathematics embody the idea of equivalence relation. These correspondences put forward a critical distinction between "equivalence" as an experience and "equivalence" as a concept. This distinction calls into question the most popular view of the subject: that the mathematical notion of equivalence relation is the result of spelling out our experience of equivalence. Moreover, the findings of this study suggest that the standard definition of an equivalence relation is ill-chosen from a pedagogical point of view, but well-crafted from a mathematical point of view.

Introduction

This study is a journey of *learning*, both for me and the interviewees participating in the study. The interviewees are involved in a situation where they are asked to tackle certain tasks (see Appendix A). For them, learning takes place by a change in the way that they see the situation. This change is reflected in the way that they tackle the tasks involved. For me, learning takes place by a change in the way that I see the concepts *embedded* in the situation, i.e. *equivalence relations* and *partitions*. This change emerges from an investigation into *what* the participants learn.

With a few minor modifications, the above paragraph could be the last one in this thesis. It is a reflection on my journey at the end, rather than a road map adopted at the start. In effect, this thesis is an attempt to shed light on this opening paragraph.

One study, three phases

As far as my own experience is concerned, this study has three different, though integrated, phases.

In the first phase, the focus is on the process of *defining* a concept. This phase is reported under "preliminary study" (Chapter 1), where a concept is chosen as the intended concept to be defined. Then, certain tasks are designed around this concept. The aim of the tasks is to bring familiarity with the *new* concept while prompting students to define this newly emerged concept. A varied selection of concepts is chosen in this stage. The concept of *equivalence relation* is one of them; at the outset it is no more, no less important than the others. The tasks around this concept are given in Appendix A. Mad Dictator Task is a generic name for all these tasks (hereafter, "situation" and "Mad Dictator Task" are often used interchangeably until otherwise stated.) The tasks are the result of a library search about the subject. However, eventually the tasks are drawn on my own understanding of the subject. The initial aim of the Mad Dictator Task is to lead 'lay' students (unfamiliar with the mathematical definition of equivalence relations) to define an equivalence relation as traditionally defined, i.e. a relation possessing the three properties of being *reflexive*, *symmetrical* and *transitive*. The success of the tasks is determined by the extent to which the students notice and define these predetermined properties. In the process of interviewing students it is realized that the ideas that the students exploit to tackle the tasks are far better, though less *organized*, than expected. In other words, compared to the students' ideas, my initial ideas for handling the tasks appear to be quite artificial; however, while the former is spontaneous the latter is systematic. This leads to the second phase of the study in which the definition that I had in mind is regarded just as an advanced means of organizing the given situation.

The second phase of the study is a transitory phase in my thinking that is embodied in Chapter 2, "pilot study". *Organizing* is the main theme of this phase in which the aim is to investigate the ways that students *organize* the given situation. To achieve this aim the study adheres to *phenomenographic* methods. This phase is set in a frame based on some of Freudenthal's ideas about the organizing activities. The pilot study is quite open to the students' ideas, whether they are related to the notion of equivalence relation or not. Yet, the most important finding of this phase is a *new* definition, and accordingly a new representation for equivalence relations that seems to be overlooked in the literature. This opens the possibility of using the original tasks for pursuing an unexpected purpose in the main (i.e. third) phase of this thesis.

The third phase of the study evolves from the previous two phases. Equivalence relations from the first stage come back to the stage, and the idea of organizing from the second phase merges into some of the phenomenographical ideas. Here, the original tasks (the first three tasks given in Appendix A), together with two new tasks (the last two tasks given in Appendix A), serve a different purpose. That is, they are used to investigate *students' conceptions of equivalence relations and partitions.* I do stick to this aim in the remainder of the study. However, the very meaning of this aim, and the extent to which this bizarre situation (Mad Dictator Task) is suitable to achieve this aim, is revealed in the fullness of time. For the time being, suffice it to say that from the middle of Chapter 3 onwards this thesis progresses in two directions. One is investigating the *variation* in the ways that students tackle the tasks involved. The other is preparing the ground for interpreting this variation in terms of students' conceptions of equivalence relations and partitions. Chapter 3 makes a theoretical attempt to bring these two together. Afterwards, they seemingly diverge. The literature reviewed in Chapter 4 is about equivalence relations and partitions, or better to say, about the ways that these notions are understood by three prominent educators and/or mathematicians. In Chapters 5 and 6, where the results of the study are reported, these notions, or more accurately, my understanding of these notions recedes into the background, giving way to *what* students *learn* from their involvement in this rather peculiar situation. The next two chapters attempt to unite the students' experience of the situation and their experience of the notions of interest. However, this is not straightforward, mainly because the participants in this study are unfamiliar with equivalence relations and partitions as mathematical notions. To resolve this difficulty, it is shown that what has been portrayed in Chapter 5 and 6 also appears in some other situations that are commonly *believed* to embody the notions of interest. In this regard, the results of this study are used to read the 'history' of the subject (Chapter 7), and to reinterpret the results of other research in which the participants are mathematics students whom have been introduced to the standard definitions (Chapter 8). These different sources lend meaning to each other and gain meaning from each other, and all together give rise to an *unexpected* result, that of a distinction between "equivalence" as an experience and "equivalence" as a concept (i.e. equivalence relation). This is extended in the last section of Chapter 8 where the purpose of this study is revisited. Chapter 9 considers the implications of the findings for the teaching of the subject and the possibility of using the Mad Dictator Task as a teaching tool. And finally, the last chapter (Chapter 10) is a short epilogue that includes a summary of the thesis and suggestions for further research.

Part 1: Preliminary and Pilot study

Introduction

In part 1, I report the first two phases of this study in the first two chapters: Chapter 1 and Chapter 2 are concerned with the preliminary and the pilot study respectively. This part is a small-scale thesis. It includes the background, a review of literature related to the initial objectives of the research, the methodology and the results.

The first chapter is about situations that are devised to get students to *define* an intended concept. This chapter describes in detail the process of devising one of these situations. This situation is around the concept of *equivalence relation*. The initial aim of the situation is to lead students to define *symmetry* and *transitivity* which, together with *reflexivity*, constitute the defining properties of an equivalence relation. The situation is tried out on a small sample of undergraduate students. The students "successfully" meet the requirements of the situations without noticing the intended properties, and more importantly, without defining *what* they use. This entails a change in my perception of the role of definitions: that a definition is just *an* advanced means of *organizing*. This realization opens a new chapter in my study embodied here in the second chapter of this thesis, the pilot study.

The pilot study is characterized by two features: first, the so-called turn to organizing and second, my introduction and subsequent adherence to *phenomenography*. The corresponding chapter (Chapter 2) includes my first phenomenographic practice, my first phenomenographic interviews and analysis. At the end of Chapter 2, I show how the pilot study informed the main study.

Chapter 1: Preliminary study

1.1 The seed of the preliminary study[1]

This research implicitly started twelve years ago, while I was preparing for my first day at school as a so-called teacher assistant. I had to work in a mathematics class that was a marginal class in the students' mathematics course. On the one hand, the main teacher would cover the whole subject and he would grade the students, and on the other hand, from the school point of view, my class was an opportunity to fill the empty hours in their timetable aimed at keeping students in the class. Therefore, as an inexperienced teacher it seemed I would have many difficulties, the smallest one, having control over the students' behaviours to satisfy the school rules. So I decided to devise or find problems that attract the students' attention, so as to let me focus my attention on controlling communication of ideas related to those problems rather than controlling students' behaviour within the school rules. At the end of that academic year, even though I couldn't satisfy the school in having a silent class characterized by the monotone voice of the teacher and glazed eyes of the students, my students and I felt that we did a lot of meaningful activities.

The next six years, as a teacher, I held to the same ideas even in those courses in which I had to cover a predetermined subject and I must convey a normative way of seeing that subject.

During those years, I occasionally had a chance to work on activities that had been designed to bring familiarity with new concepts. Furthermore I was engaged in the class endeavours in its journey from that familiarity to the fully-fledged definitions of those concepts. But being involved as a teacher and having

[1] As a small sign of the extent of the drastic changes occurred in this study it is interesting to say that the original title of this section was "the seed of the research"!

responsibility of bridging the starting activities and the fully-fledged definitions, I had little opportunity to investigate students' experience. This opportunity is what I found from the outset in the preliminary study of my doctoral thesis.

1.2 What was the preliminary study about?

Following my experience as a teacher, the preliminary study was devoted to devising situations, which on the one hand, could bring familiarity with new concepts, and on the other hand, could prompt students to define these newly emerged concepts. Acknowledging the literature, those situations were called *defining situations*.

Literature

Two approaches were recognizable in the literature that at the time was akin to my research. First, the research that had started with familiar concepts (mainly, geometrical ones) and then had led the students to appreciate *good* definitions of those concepts; and second research that had engaged students in *problematic situations* where the concepts to be defined would emerge in the course of solving certain problems. The first group appreciated nothing less than a definition having *standard requirements for mathematical definitions* (Borasi, 1994, p.175), the second group allowed a definition *consistent* with the formal one, i.e. a statement characterising the concept to be defined and eliminating the often concrete aspects of the situation (see, for example Mariotti and Fischbein, 1997). "Defining" had been used by both groups for what students had been engaging in. As an example of the first group, I cite de Villiers (1998):

> The construction of definitions (defining) is a mathematical activity of no less importance than other processes... (ibid, p. 249)

And as an example of the second group I just cite the title of Mariotti and Fischbein's paper (ibid, p.219), "defining in classroom activities".

Though this surface similarity is enough to justify the name that I chose (defining situation), defining situation had still much to learn and much to inherit from these approaches. Perhaps, the most important one was the spotting of communication and proof as prompting elements in both of them.

1.2.1 Communication

> Students need to develop an appreciation of the need for precise definitions and for communication power of conventional mathematical terms by first communicating in their own words.
>
> (NCTM, E-Standards, Communication section)

The exploitation of communication as a prompting element in the situation aimed at "defining" could be seen in different forms:

- Communication between participants in the situation (between researcher/teacher and students, and between students) that for example could be seen when students had been invited to work in groups as in Furinghetti and Paola's (2000) study, "definition as a teaching object: a preliminary study"; or in Mariotti and Fischbein's study, as a *collective discussion* between students and teacher to harmonise between a "spontaneous defining process and a mathematical defining process".
- Communication as a meta-cognition element, between an individual and a mental "supposed others" (Shimizu, 1997) that like an "internal enemy" (following Mason et al, 1982) ask questions and make critiques "in the process of making a mathematical definition".
- Communication inherent in the situation as could be seen in the following problem:

> How would you explain in words, without making a sketch, what these quadrilaterals are to someone not yet acquainted with them? (de Villiers, 1998; following students' search for a definition of "rhombus")

Moreover, within and beyond these educational (research) contexts, communication as a social aspect shapes the norms and standards of mathematics. Communication, though important, misses its point if we do not consider that "in mathematics a definition does not just serve to explain to people what is meant by a certain word" (Freudenthal, 1973, p.416). There is also something deeper and structurally more important about them, that, "in mathematics definitions are links in deductive chains" (Freudenthal, 1973, p.416). It seems that for the latter aspect those educational (research) situations, having an interest in defining, had been relating more or less to *proof*.

1.2.2 Proof

> Mathematics differs from all other sciences in requiring that its propositions be proved...But you cannot prove a proposition unless the concepts employed in formulating it are clear and unambiguous, and this means that the concepts used in a proof either must be basic concepts...or must be rigorously defined in terms of such basic concepts. Mathematics, therefore, since it is about proof is also about definition. (Mayberry, 2000, p.3)

Alongside of this meta-mathematical dependence of proof on definition, a brief review of the literature showed that, in one way or another, understanding of proofs is related to understanding of definitions. It seems also reasonable that the students' competence in dealing with proofs is somehow affected by their ability to handle the related definitions. For example, in a transition course entitled "An Introduction to Higher Mathematics", Moore (1994) found students' inability to state the definitions was a source of difficulties in writing out proofs. Bills and Tall (1998) working with students in a university lecture course in Analysis, suggested that students' treatment of definitions and examples could affect their "comprehension of systematic proof". Again, using Analysis, Alcock and Simpson (1998) reported that different ways of understanding the dual role of definitions (both that if the definitional property holds, then the objects belongs to the class under consideration, and that if the object is known to belong to that class, then the definitional property can be taken as a consequence) could have different effects on students' work with certain sorts of proofs involving this dual use.

However, as soon as "defining" comes into play, definitions that so far have served a taken-for-granted, but important, role (in carrying out the proofs) come into focus. Accordingly, proofs that so far were mere users of definitions provide the necessary motives for defining what deserves to have a definition.

> The theorems of mathematics motivate the definitions as much as the definitions motivate the theorems. A good definition is "justified" by the theorems that can proved with it, just as the proof of the theorem is "justified" by appealing to a previously given definition. (Rota, 1997, p.97)

In some way, this inherent relationship of definitions and proofs was well-appreciated in the situations having an interest in defining .For example, in de Villers' (1998) study, the students had been engaging in making a list of properties of a familiar figure, and then choosing certain properties as defining properties to provide a definition for the concept involved. In addition, de Villers aimed at leading students to an appreciation and construction of "economical" definitions that in addition to having a more economical form (due to being more "hierarchical" than "partitional"), were characterized by their potential power to shorter and easier proof. In de Villers' activities, "proofs" served two different purposes: First, in the deductive phase of the activities where the chosen definition—as carrier of "sufficient information for the accurate construction" of the figure—could be used to *logically deduce* all other properties from it; and second, in de Villers' usage of economy in proof as a mobilizing element to look for economical definitions.

Unlike de Villers who, by introducing different stages in his study, had kept defining separate from, though related to, the processes of proof, Borasi (1994) had appreciated a more interwoven relationship between those two processes. Following Lakatos (1976), Borasi (ibid, pp. 180-183) used a theorem and its proof to refine "the tentative definition" of the concept (a "polygon") her students had been proving the theorem about.

In brief, the literature underlined the importance of communication and proof in the process of defining, although they were in use in different forms and for different purposes. In the next section, I will explain the way that I was to exploit them.

1.3 Defining situation

The notion of defining situation was characterised by certain problems and tasks around the target concepts that were unfamiliar to students who would be involved in the situation. As it has already been mentioned, a defining situation aimed at, on the one hand, bringing familiarity with new concepts, and on the other hand, prompting students to define newly emerged concepts. Having been informed by the literature, I came to appreciate communication and proof as essential parts of a defining situation.

Regarding communication, in addition to communication between students (in the case of working in group) and between students and the researcher, a defining situation involved a certain communication inherent in the situation and embodied in certain problems and tasks. For example, the task could be a game that would be played by two students, or it could demand explaining something to others. An example of the latter can be seen in de Villers (see Section 1.2.1), or in the following:

> Explain it to your peer as parsimoniously as you can. Give her or him the least information... (See the details in Section 1.5.)

Regarding proof, I shall say that what I had in mind was a very mild role, something like the role of the argument in Mariotti and Fischbein's (1997) study: a compelling argument aimed at making explicit the *reasons* for certain choices.

Generally speaking, a defining situation was a *non-standard situation* where the mathematics that students could experience is a form of *non-standard mathematics* (Burn et al, 1998, p.82) in which "Math can be seen to be created, rather than discovered...in that the reasons why certain choices are made (rather than having them given *ex cathedra*) can be explored".

1.4 The first defining situations, the first interviews

A varied selection of concepts (e.g. the highest common factor, the limit and so on) was chosen as target for the first defining situations. But, on the one hand, the length of the designed situations mainly consisted of several problems or tasks around the target concept, and on the other hand, having access to only an opportunistic sample of students made it difficult to have a reliable data about each situation as a whole. As a result, the first interviews had been devoted to only a certain part of each situation, mainly one problem or task from each situation in each interview. Therefore the preliminary study had no chance to approach one of its main objectives; that was revising the situations for the next stages of the study. In sum, regarding the short access to students, the preliminary study turned to a situation that seemed to be more suitable for only one course of interviews.

1.5 One defining situation, many interviews

The first defining situation that had a chance to be examined on several occasions was a situation based on *equivalence relations*. Here is the outline of the first version of the situation. I shall stress that it is the first version of a task that will be the sole task in this study. This first version "remains a crazy example" (to quote one of my supervisors). Nonetheless, it had a vital effect on my understanding of the situation. It also shows to what extent my knowledge shaped the way that I was interpreting the situation.

I planned to give the following information to the students. See Textbox 1.

> Ten painters should paint ten doors such that:
>
> - Each painter has to paint his own door (painter No1 paints door No1, painter No2 paints door No2, painter No3 paints door No3 and so on).
> - Either no two painters paint the same door or all the doors they paint are the same.
>
> Before going to the next page to see an example of an acceptable painting, please write here anything that has just come to your mind.

Textbox 1

Then through a milieu of examples and non-examples starting with the example shown in Figure 1, the situation would lead students to another activity (see Textbox 2) that was supposedly the main part of the situation.

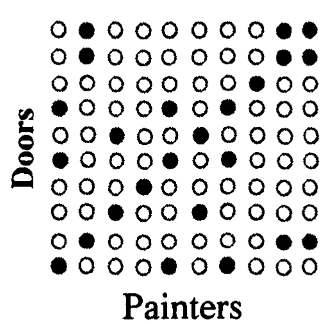

Figure 1: An Acceptable Painting

Create your own painting, and then explain it to your peer as parsimoniously as you can. Give her or him the least information, but still be sure that with given information she or he would recreate your painting.

Textbox 2

1.5.1 The pre-interview analysis of the situation

As mentioned above, the situation was based on the concept of equivalence relation, i.e. a relation having three properties: reflexivity, symmetry and transitivity (see Section 1.6.2 for a formal definition). But unlike an equivalence relation that is a particular relation between the elements of only one set, the situation seems to be about a relation between two sets, i.e. the set of painters and the set of doors. Nonetheless, these two sets have "the same number of elements", even though their elements belong to two different contexts. Therefore, there should be some tools to lead students to a context-free situation, viz. the relation between the elements of the set of natural numbers from one to ten. In order to achieve a context-free situation, the blank grids presented to students to make their own examples were label-free grids as the grid shown in Figure 2.

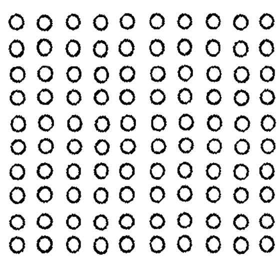

Figure 2: A blank grid

In addition to this slight change in the appearance of the grids, there was a deeper reason that could potentially lead students to a context-free situation: *symmetry of each example*. It was supposed that the symmetry of each example is more accessible as a property of the dotted grid rather than something about the painters and the doors. Therefore, giving the least amount of information—the most parsimonious one—could simultaneously guide students to a context-free situation and grasping the symmetry of each example.

Supposing all these assumptions, the study turned to the first interview on the situation related to equivalence relations.

1.5.2 The first interview

The first interview on equivalence relations (the situation regarding equivalence relations) took place with Shion, at that time a graduate student in mathematics. The interview aimed at seeing the situation through the eyes of a colleague to revise it for interviewing with students that had not been taught equivalence relations. But interestingly and surprisingly the situation turned to be a novel situation for her. That novelty had at least two consequences; first, providing a "successful" record of what was expected of the situation, and second, calling my attention to the context of the problems including the "real world" elements such as painters and doors.

The first "successful" record

If we define the success of the situation in terms of the achievement of its objectives, the first result was a success. In other words the outcome of Shion's engagement in the situation partly met my expectations: she spotted the "symmetric" property of each example; and while creating her own examples drew the symmetry from the given conditions.

Although, as we will see, the formulation of the situation had been changed after the Shion's interview, it is worth pausing to consider her interview to justify the subsequent modifications.

Shion spotted the symmetry property while checking Figure 1 to see whether it is an example or not.

`Shion: Why you can say this is symmetry.`

She immediately realized that Figure 1 is a symmetric figure. But, she was still in doubt whether Figure 1 is an example of an acceptable painting or not. Nonetheless, she replaced the given conditions by the symmetry, and then used the symmetry to check the examplehood of other figures. For example, she used it to reject the following figure (Figure 3).

```
○○○○○○○●○●
●○○○○○○○●○
○○○○○○○●○●
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○○●○○○○○○○
○●○○○●○○○○
●○○○○○○○●○
```

Figure 3: A Non-Symmetric Non-Example Figure

> Shion: Maybe I can omit this one [first condition], and if I think about this condition [second condition], they can cover the whole things; for my image, symmetry is like they can cover whole things, so if the graph is symmetric...referring to this graph, it is not symmetric, so maybe, this figure cannot cover the whole condition.

As I expected Shion came up with the idea of symmetry. Thus, I implicitly encouraged her to argue for her idea. Hereafter, when I refer to the interviews, I will use "interviewer", "I" and "Amir" interchangeably. Moreover, square brackets are used to introduce my comments.

> Interviewer: But symmetry is not in the conditions, so how you are sure about symmetry? You have only one figure that symmetry is correct in it, but if it isn't an example so, maybe symmetry is not correct.

> Shion: So if I want to say this one isn't symmetry, so this figure [Figure 3] can not express this one, then first I need to confirm the first figure [Figure 1] can express or can not express.

As yet the idea of symmetry had been based on only one example-in- trust (it is supposed to be an example because the interviewer has presented it). Therefore, the interviewer pinpointed the difficulty of this assumption.

Interviewer: Maybe symmetry is correct in this figure [**Figure 1**], not another figure, because it's a special example.

Accordingly, Shion gave her very first argument on symmetry.

Shion: Now I am thinking from this expression, I can think about some symmetric figure or not.

Shion: Now I am thinking, from the beginning, painters and doors are different, but there is no meaning, I don't need to have this difference. I can swap doors and painters; now I am thinking if I choose two painters the doors are the same or not, but then if doors are the same then painters must be the same. That's why, I can think about some symmetry property because I can swap or I can change the coordinates. It means symmetry.

Having given her first argument on symmetry, Shion referred to symmetry to accept the following non-example as an example (Figure 4).

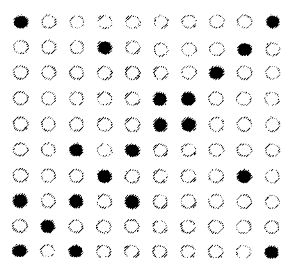


Figure 4: A Symmetric Non-Example Figure

Although she applies symmetry in every step that she takes, there is something about symmetry that still bothers her. Not satisfied with her own first argument, while making her own example she is still looking for a reason for that.

Shion: Totally black is also acceptable; it means if all of the circles are black, this condition is "or", it's also okay, because it's "or". Why everyone is symmetric?

And after a long discussion that is mainly about the "real world" context of the situation, she again returns to that bothersome question.

Shion: we have to satisfy both conditions; why everyone is symmetric.

And eventually, while struggling to make another example she comes to the following reason.

> Shion: [She turns (2, 7) into black, and all at a sudden] because the first sentences, (7, 2) must be black, because from the first sentences (7, 7) is black, then suddenly we need this. That's why the graph has symmetry property.

This very first interview had several important effects on the present study, among them, recognising the situation's defects and acknowledging the importance of making examples. Accordingly, the situation was modified.

1.6 The situation, revisited

As mentioned, and it can be seen from Shion's excerpts quoted above, the situation unexpectedly turned to be a novel situation for Shion. Furthermore, she was struggling with the situation more than was expected. As the following excerpt shows, it seems that her struggling was partly related to the real world context of the situation.

> Shion: The purpose of the painters is a goal, so if there are ten doors, these ten doors must be painted by someone; so the first sentence is the goal; so my question is, this is a goal, if we have ten doors we want to paint all, if each painter paints one then they can paint all of the doors; the first line we can satisfy this goal.

Shion's long-term struggling with the situation caused the first reflection on the situation.

1.6.1 Reflection on the situation, unity and plurality

It seemed that the obscurity of the situation was mainly reflected in the conflict between the real world's demands on the painters (and/or doors) and the situation's demands on them. In the former the job is done when each painter paints one of the doors, and in the latter, one door could be painted by several painters!

The situation was intended to relate the painters to each other by the doors that they paint, but the individuality of painters and doors prevailed over that intention. In particular, as numbered painters and doors they looked like single individuals characterized by their names (painter No 1, painter No2...; door No1, door No 2...). Therefore, while the painters painted their own doors as individual and unit (and the doors were painted as individual and unit), the situation demanded plurality, i.e. a painter as a person that paints something, a door as something that can be painted. In a more formal vein, the situation imposed being a member of the set of painters on each individual painter, aimed at repeatedly referring to him or her as the painter of certain doors (and similarly seeing each individual door in the set of doors could mean that it can be repeatedly painted by certain painters). The intended plurality let the painters relate to each other by the doors that they paint; now they were not only single individuals, but also someone that can fill one of the two sides of a relationship, or more importantly fill both sides of a relationship; they were simultaneously unit and plural.

This plurality can be implicitly seen in the eloquent and still informal introductory paragraph of the chapter on relations in Stewart and Tall (2000, p.62):

> The notion of a relation is one that is found throughout mathematics and applies in many situations outside the subject as well. Examples involving numbers include 'greater than', 'less than', 'divides', 'is not equal to', examples from the realms of set theory include 'is a subset of', 'belongs to'; examples from other areas include 'is the brother of', 'is the son of'. What all these have in common is that they refer to two things and the first is either related to the second in the manner described, or not.

Each one of the 'two things' in Stewart and Tall examples implicitly belongs to a set (regardless of the difficulties that the examples from the realms of set theory could make); therefore, even though, for example, 1 in $2 > 1$ is treated as an individual, being in the set of integer gives an infinite access to it and illuminates its plurality.

As a particular relation, an equivalence relation inherits all the above peculiarities in a more remarkable way. When we are looking for a concrete example of equivalence relation, we are apt to define a relation between *two* different things or people, say, *both* have the same colour, *both* live in the same street; we can check the possession of the given relationship between those two things or people by pointing to those two; we can do that in a more concrete level, or using Dienes' words (1976, p.9), in 'first order attributes' realms, say, they are *both* green, for the first relation, and they *both* live in Oxford Street, for the second. However, as Dienes pointed out, the former described by 'second order attributes' is more *abstract* and more difficult than the latter:

> To *have the same colour as something* else is a much more sophisticated judgement than to say that they are both green. (ibid, p.9)

I should add that passing to 'second order attributes' realms seems inextricable from grasping the reflexive property. To grasp the reflexive property, first we must go one step further in the situation, and look at the situation as '...having the same colour as...', '...living in the same street as...', and so on; that demands, on the one hand, a transfer from unity to plurality in the sense described for relations in general, and on the other hand, a transfer from plurality to unity, i.e. coming from *both* to *each.*

Bringing plurality and unity together is what I tried to achieve in the next version of the situation.

1.6.2 The Mad Dictator Task

Following the above considerations the context of the situation had been changed as follows. See Textfigure 1.

A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.

A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.

A visiting law must obey two conditions to satisfy the mad dictator:

1. When you are in a particular city, you are allowed to visit other people in that city.

2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

The dictator asks different officials to come up with valid visiting laws, which obey both these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below (Figure 5).

You may visit other people here

```
OOOOOOOOOO
OOOOOOOOOO
OOOOOOOOOO
OOOOOOOOOO
OOOOOOOOOO
OOOOOOOOOO
OOOOOOOOOO
OOOOOOOOOO
OOOOOOOOOO
OOOOOOOOOO
```

You are here

Figure 5: A grid to represent a visiting law

Textfigure 1: The Mad Dictator Task

And the second task changed to the following. See Textbox 3.

The mad dictator decides that the officials are using too much ink in drawing up these laws. He decrees that, on each grid, the officials must give the least amount of information possible so that the dictator (who is an intelligent person and who knows the two rules) could deduce the whole of the official's visiting law.

Looking at each of the examples you have created, what is the least amount of information you need to give to enable the dictator to deduce the whole of your visiting law.

Textbox 3: The least amount of information task

As far as the order of the tasks is concerned, a preference was given to the student-made examples demanded in the first task; accordingly, the situation aimed at leading students to the symmetry and transitivity through creating their own examples and giving the minimum amount of information demanded in the second task.

The underlying structure of the Mad Dictator Task

As repeatedly mentioned before, the Mad Dictator Task is based on the notion of equivalence relation. I shall give the standard definition of an equivalence relation for future reference. According to the standard account, an equivalence relation is a relation $\sim$ on a set S that has three properties: *reflexivity* ($a \sim a$ for all a in S), *symmetry* (if $a \sim b$ then $b \sim a$), *transitivity* (if $a \sim b$ and $b \sim c$ then $a \sim c$). Suppose $\sim$ is an equivalence relation on S and a is an arbitrary element of S, then the set of all elements of S that are related to a is called the *equivalence class* of a. It follows from reflexivity, symmetry and transitivity that for each pair of elements of S, say a and b, either the equivalence class of a is *equal* to the equivalence class of b or the intersection of the equivalence class of a and the equivalence class of b is the empty set. Given this and also considering that every element, say a, belongs to an equivalence class, namely the equivalence class of a, it can be seen that an equivalence relation on a set S *partitions* the set into equivalence classes, i.e. the set is divided into mutually exclusive classes. Conversely, suppose there is a set of mutually exclusive subsets of S, such that each element of S belongs to one of them, then the following relation is an equivalence relation:

$a \sim b$ if and only if a and b belong to the same subset.

In the context of the Mad Dictator Task, let $a \sim b$ if b is a visiting-city of a. Then $\sim$ would be an equivalence relation, providing that we have satisfied the second conditions of a visiting law:

> For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

Consider that by giving a "metonymical[2] definition" in which, a city is used to refer to people in the city, the new formulation had brought plurality and unity together. That is to say, "each city is its own visiting-city" metonymically stands for "in each city you can visit other people". Thus, the first condition of a visiting law guarantees that each city has at least one visiting-city, i.e. that city itself. It is the *reflexive* law that is visible as the main diagonal of the grid.

Having captured the reflexivity, the situation aimed at leading students to the two other properties of an equivalence relation, symmetry and transitivity.

1.7 Further sample

Having considered such details, the preliminary study went further with a small opportunistic sample of students comprising two first year undergraduate mathematics students, and two second year undergraduate physics students, all were students at the University of Warwick, one of the top five ranked universities in UK.

The two mathematics students have been taught equivalence relations and related subjects four weeks before the interview in a course entitled 'Foundations' held at the University. The two physics students had no previous 'formal' idea about equivalence relations and related subject.

[2] According to Lakoff and Johnson (1980, p. 36), metonymy "has primarily a referential function, that is, it allows us to use one entity to *stand for* another.

Regarding the nature of the second task involving communication between an official and the dictator, and aiming at making this communication an actual communication between two people (presumably one of them an official and the other the dictator), the students were invited to be interviewed in pairs, i.e. two mathematics students together, and two physics students together.

As mentioned before, the main change in the structure of the interviews was giving a preference to making examples, compared to the first interview in which making examples on the one hand, and checking certain figures to see whether they are example or not on the other hand, had no preference order. But still it was not the dynamic of making examples that was considered as important; as a result, it was taken for granted, while student-made examples, the product, were regarded as the core of the first task. The reason for this ignorance of the former in favour of the latter was the so-called importance of the second task for leading students to consider symmetry and transitivity, the two concepts seemingly embedded in the situation.

Practically, the interviews had a simple structure; the two tasks (generating an example of a visiting law, and giving the minimum amount of information) were posed in order. As soon as one of the students had made one or two examples agreeable to all (interviewer and interviewees), we started the second task. On the other hand, the interviewer, more like a teacher aiming at teaching certain predetermined concepts, sought those concepts in students' utterances. The interview with the physics students illuminates the nature of the initial interviews.

Andy is one of the two physics students who made his own examples, noticed symmetry of each figure, and gave a reason for that, and suggested the following information to convey his examples to another student:

> Andy: I am going to tell you some groups, each group visits all the other ones in the group and hence it is visited by all the other ones in the group, it visits them and it is visited by them.

Soon afterwards, I helped him to separate the given conditions from the symmetry (in the following, interviewer and interviewee agreed to take the first condition for granted):

> Andy: The second one implies the symmetry,
>
> Interviewer: The second one implies the symmetry or the symmetry property implies the second one,
>
> Andy: They imply each other,
>
> Interviewer: But you have an example here [pointing to a symmetric non-example figure],
>
> Andy: This is symmetrical, oh no, it isn't, and this one doesn't work.

And then, desperately looking for the transitivity the interviewer asked the following direct question:

> Interviewer: What condition must you add to have the second one?
>
> Andy: The second law implies symmetry; the symmetry doesn't imply the second law [while scrutinizing the present examples], *I am not sure what sort of answer you are looking for.*

And shortly after that:

> Andy: I can't think of any way to say the second law better than it is already said.

So far I have only given a glimpse of one of the interviews. More surprises were to come when I compared the results of the two interviews (interview with the physics students and interview with the mathematics students).

Results

Hargi and Shakil, the two mathematics students, came to talk about symmetry when they were looking for the minimum amount of information:

> Hargi: If you just give me a symmetrical half, you can give me half, then I can do it by symmetry.
>
> Interviewer: Why?
>
> Hagri: Because two can visit six, two can visit five, five must be allowed to visit two and six, two must be allowed to visit five and six, six must be allowed to visit two and six, [asking Shakil] you just give me mirror.
>
> Shakil: Why? It isn't a requirement.

Having come from a geometric expression of symmetry to a more algebraic one, while asking interviewer:

> Hagri: Is it that a requirement that if two goes to five then five goes to two?
>
> Interviewer: Is it that part of the conditions?
>
> Hagri: I'm not sure; [murmuring the conditions] no, it's not a condition.

Just returned to the task of giving the minimum amount of information:

> Hagri: I'm still worried about if two goes to five, doesn't necessarily mean that five goes to two.
>
> Shakil: No.
>
> Hagri: It doesn't mean necessarily, because five can go to five, two can go to five, so they have.
>
> Shakil: Yah, the city in common,
>
> Hagri: Visiting-city in common, so five doesn't necessarily go to two, because, um, five does go to two, five does go to two; because if five goes to two, they go to each other, two can go to five, and five can go to five, then, um, you can deduce that from conditions, we've got this identity, *a* goes to *b* then *b* goes to *a*.

Hagri then employed this newly confirmed property to reject Shakil's figure as an example:

> Hagri: I am saying picture has to be symmetric, now what he's got there, it is not symmetric, it 's not valid, it does not obey the rules, it's not an example.

And soon after, they accepted a symmetric non-example figure as an example; and they gave "half of the information" (the symmetric half) as the minimum amount of information, while considering other properties:

> Hagri: I ignore the diagonal; I'll give half of the information.
>
> Interviewer: Does it work always?
>
> Hagri: I guess if you had like a rotating group, if you like, two can go to five, five can go to seven, then,
>
> Shakil: Two has to go to five.
>
> Hagri: Seven has to go to five, and ...

While in the course of the consecutive events there was no sign of relating the symmetry idea and grouping idea to each other on the one hand, and separating them from each other on the other hand, all of a sudden they realized the equivalence relation in the situation:

> Shakil: Basically it's an equivalence relation; this is reflexive, symmetric and transitive.

But surprisingly, not even the successful deduction of these three properties from the two given conditions was of help to them to separate the different concepts encountered:

> Interviewer: What about symmetry, if one is symmetric, it has necessarily both properties or not.
>
> Shakil: If it has symmetry, yes it is.

Hagri: That means if you come up with a symmetric picture it must be an example, symmetry is equal to that two happen [the two given conditions].

In comparison, Andy, one of the two physics students, related the symmetry idea to the other ideas when only two cities were involved in:

Andy: Because it's gonna be each city that you visit can visit you, it's gonna be symmetrical about that line, that's what implied by this, so it's gonna be *symmetrical* down there, um, three and six are *grouped*.

Even so, when more than two cities were involved, still those two ideas appeared as isolated ones:

Andy: Each in a group visits all the other ones in the group, because one is the same group as five, so then it visits one, five and seven, five is in the same group as one and seven, and five so visit one, five and seven.

While in the last excerpt there is no sign of symmetry, in the following there is only a loose reference to symmetry:

Andy: If you go along the columns one by one, and you see, you know symmetry, you can see that two visit nine and ten you know that they must visit each other, so you check that ten visits nine...

In general, he referred to many different ideas without maintaining those ideas from one time to the other, and without necessarily relating those ideas to each other as the mathematics students did.

These two interviews radically affected my criterion for success and failure of the situation.

1.8 Discussion: success or failure

When I started the preliminary study, my criterion for success was whether students engaging in the situation could spot certain predetermined concepts or not; in particular, whether they could spot symmetry and transitivity or not.

Having that criterion in mind let me scrutinize the last two interviews in terms of success or failure.

Both groups of students successfully noticed symmetry, while only the mathematics students who had already been taught equivalence relations noticed transitivity. Therefore, as far as symmetry is concerned, the situation could be taken as a success, and regarding transitivity, as a failure, particularly, since the study would eventually target students who have not been taught the subject, i.e. students that are more like the physics students than the mathematics students.

Moreover, the students brought to my attention something I had previously ignored, something that was not then a yardstick to assess the success of the situation, but turned out to be as effective as symmetry and transitivity when tackling the requirements of the situation, i.e. the idea of 'grouping'. To be precise, despite the fact that I was aware of the idea of 'grouping' (or in a certain sense, 'partitioning') as a logically closely related concept to equivalence relations, the presence of symmetry and transitivity prevented me from realizing the extent to which the other concepts were involved.

Beyond those individual concepts, this first study revealed something that certainly played a crucial role in preparing me for relinquishing my criterion for success, i.e. the ways that those individual concepts could be related to each other. While for me those individual concepts had certain relations informed by the formal treatment of them, for students, they were mainly related to each other by their functionality within the situation.

Altogether, considering the data it seemed that my original view of success which was mainly based on tracing certain predetermined concepts in students' utterances was very narrow. Nonetheless, there was still one possibility to keep that view by modifying the tasks and/or adding certain new tasks in such a way that they would facilitate (1) the students' grasp of the missing concept of 'transitivity', and (2) the predetermined ways of connecting all the concepts involved in the situation. Furthermore, any measures of success should take account of two criteria, the extent to which those predetermined concepts would be brought up in each interview, and the extent to which their logical relations would be matched with the standard ones. To put it more simply: the more standard the outcome, the more successful the situation. On the other hand, by narrowing the situation in order to make success more likely, the more restricted, involved and artificial it would be. Nonetheless, there would still be no guarantee that it would achieve what it had been designed to achieve.

Given this, a certain fact that was then in the background of my reading of Mariotti and Fischbein (1997) had been come to the foreground, i.e. students' unforeseen difficulties. Quite akin to this study, they introduced a problem situation in which 'the concept to be defined functionally emerges from the solution of a problem'. Despite the indispensable and involved role of the teacher in their experiment to guide students to overcome the conflict between 'the spontaneous process of conceptualization and the theoretical approach to definitions', they repeatedly report the students' unforeseen difficulties to transcend the concrete situation to reach to the *intended* 'systematic organization of concepts'. If in their experiment, with such an involved role of the teacher, leading students to define certain intended concepts was so problematic, reducing the role of the teacher (as I aimed for) would result in a more complex situation. But, above all, the most important issue that I did not take into account in my first reading of Mariotti and Fischbein—the very same issue that was raised by the preliminary study—was 'systematic organization of concepts':

> In fact, theoretically, a definition relates the new object to all the others, in such a way that a chain (system) of definitions is built up; this system is an organic and coherent whole. The gap between a spontaneous defining process and a mathematical defining process concerns both the origin of the concepts and their organization within a theoretical system.
>
> (Mariotti and Fischbein, 1997, p.225)

The preliminary study opened a new chapter in my thinking in which I *learned* to consider defining in the realm of organizing. The initial stages of this chapter in my thinking deserve a chapter in this thesis: Pilot Study.

Chapter 2: Pilot study

2.1 Introduction

As the initial data revealed students' spontaneous ways of tackling the given situation (those that were not necessarily intended by me), they also shook my concerns about success and failure. Accordingly, the study gradually began a process of revision thoroughly embracing every aspect of it; the *intention* of the study turned to be an investigation of the ways that students *organize* the given situation, and following that, the methods of interviewing and analyzing were naturally revised.

This chapter looks at some aspects concerning the organizing activities. Particularly, Freudenthal's works on this issue are mentioned. After this, the methodology (i.e. *phenomenography*) to be used is described briefly. I also give the first data that were collected and analysed under my *new* perspective. Finally, based on the students' works on the Mad Dictator Task, I report a 'new' definition of equivalence relations, and consequently a new representation for them, which seems to be overlooked in the literature.

2.2 Didactical Phenomenology

A continuous change in my reading of the literature was the least result of the analysis of the data all through the study. I have already mentioned one of these *new* readings, i.e. my reading of Mariotti and Fischbein (see section 1.8); now I discuss another one, i.e. my reading of Freudenthal.

Freudenthal's works has certainly had a great effect on the literature as to defining. He himself favoured defining as a learning activity, though only as the finishing touch of an organizing activity:

> Most often definitions are not preconceived but the finishing touch of the organizing activity... In the course of these activities the student learns to define, and he experiences that defining is more than describing, that it is a

means of the deductive organization of the properties of an object.
(Freudenthal, 1973, p. 417)

Ten years on (1983), in Freudenthal's monumental work, defining does not retain its earlier importance in the organizing activities. In *Didactical Phenomenology* the goal is "the constitution of *mental objects*" rather than making concepts explicit.

> Our mathematical concepts, structures, ideas have been invented as tools to organise the phenomenon of the physical, social and mental world. Phenomenology of a mathematical concept, structure, or idea means describing it in its relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind, and, as far as this description is concerned with the learning process of the young generation, it is didactical phenomenology, a way to show the teacher the places where the learner might step into the learning process of mankind. (ibid, p. ix)...what a didactical phenomenology can do is... starting from those phenomena that *beg to be organized* and from that starting point teaching the learner to manipulate these means of organizing[3]. (ibid, p.32, emphasis added)

Looking at the initial data, the idea of organizing that was then in the background of my reading of Freudenthal came to the foreground. I learnt to see the situation that the students were engaged in as a situation that "begs to be organized". Given this, I shall add that I only have partly adopted Freudenthal's plan; I became interested in the ways that students organize the given situation, rather than teaching them any particular ways of organizing the given situation and/or "teaching them to manipulate any particular means of organizing". Now I needed a *methodology* that was fit for what I was about to investigate. At the time, *phenomenography* seemed an ideal choice.

2.3 Phenomenography

> Experiences are reflected in statements about the world, in acts carried out, in artifacts produced. Now, in the light of what we know about the world, such statements can appear more or less valid or consistent or useful, the acts more

[3] He then stresses that as far as teaching and learning are concerned, the means of organizing are primarily as mental objects and only secondarily as concepts, and that didactical phenomenology is mainly concerned with the material for the constitution of mental objects.

or less skilled, the artifacts more or less functional...we have to bracket[4] such judgments. We have to look at the statements, acts, and artifacts to find out what ways of experiencing particular aspects of the world they reflect, regardless of their validity, skilfulness, or functionality.

(Marton and Booth, 1997, p.120)

All I did towards the end of the preliminary study had a phenomenographic ring. For example, I had spontaneously started to bracket my judgment, to see the situation through students' eyes, and to consider the variation in the students' experience of the situation, as phenomenographers do. However, it was only under the pilot study that I explicitly, and as far as I could, truly, put this approach into practice. Foreshadowing a more thorough account of phenomenography in the methodology chapter (Chapter 3), here I give a snapshot of it in the context of two interviews.

2.4 Pilot Study

The pilot study occurred with two students having no formal previous experience of equivalence relations, partitions and related concepts usually used to define it. Tyler is an undergraduate computer science student and Jimmy is a sixth form student studying mathematics. I interviewed them individually. In the course of the interview, I invited them to think and talk aloud in order to audiotape their utterances. Also, they were encouraged to write their ideas. The interviews had a simple structure; the two tasks (generating an example of a visiting law, and giving the minimum amount of information) were posed in order, but the timing and questions were contingent on students' responses. The interviews aimed at reaching a mutual understanding between interviewer and interviewee (in the sense of Booth et al, 1999, p.69) of the situation and the ways that interviewee organized it. Therefore the interviewer did not judge the interviewees' utterances

[4] Somewhere else (p.119), as a footnote, they add: "To *bracket* is a term from phenomenology, meaning to suspend judgment."

as to his own understanding, and insisted on the students giving transparent reasons for their decisions, mainly, as Marton and Booth (1997,p.130) say, *"through offering interpretations of different things that interviewee has said earlier in the interview"*. The verbatim transcribed tapes and the students' written works were treated as data; and they analyzed according to the phenomenographic analysis method in which each *pertinent* extract was being inspected, as Marton and Booth (ibid, p.133) say, '*against the two contexts: now in the context of other extracts drawn from all interviews that touch upon the same and related themes; now in the context of the individual interview'*.

2.5 Results

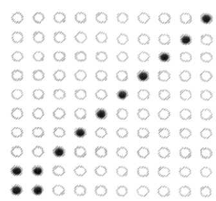
Regarding Jimmy's work and Tyler's work, three differences appeared to me more *critical* than the others:

- The difference in what they did to organize the situation;
- The difference in their outcomes;
- The difference between what they were aiming for.

2.5.1 The difference in what students did to organize the situation

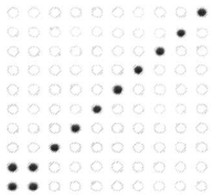
To satisfy the first condition of the given situation, Jimmy and Tyler blacked the diagonal and continued as follows (Textfigure 2 and Textfigure 3 respectively):

Jimmy: Now we have to satisfy the second condition, for each pair of cities, either their visiting-cities are identical, if you have the city one, if you can visit two, you have to, in city two either you can visit city one, like that, you have to because otherwise, they have something in common already, so you have to be able to visit.



Textfigure 2: Jimmy's initial approach to the task of generating an example

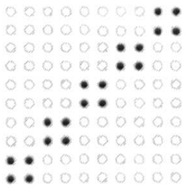
Tyler: If I am in city one, and we allow to visit city two, how the other things need to change, to keep the rules consistent and see either they are completely the same or completely different, so aha, so city two now have to be able to visit city one…



Textfigure 3: Tyler's initial approach to the task of generating an example

Jimmy "has a rule to apply"; he suspends his reasoning and replicates the result. In other words, he replicates a two by two block-square (Textfigure 4).

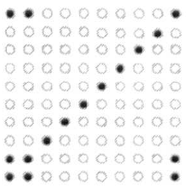
Jimmy: And likewise, if you go like that in pairs…It's like paired-up, so if you compare one and two, they have every thing in common, identical, if you compare one and three, one and four, one and five, or one and six, they have nothing in common…



Textfigure 4: Jimmy replicates a block square

On the other hand, Tyler considers two things, "mirroring in y equals x" and "box" (square), and then "to see what was happening" he decides to make city one visit city ten (Textfigure 5).

Tyler: And I realised first that, city ten has to visit city one…so that the second law …city ten has to visit city two…now I look at the city two, now I realised they are different from city one…so I copy number one on to number two also just to keep them the same…



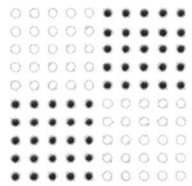
Textfigure 5: Tyler examines the necessity of the block square

As a result, Tyler abandons the "block square", keeps the "mirroring" and *proves* it as a "general pattern of these dots" (if (x, y) then (y, x)). In addition, the way that he proves "mirroring", gives him a new insight, i.e. considering the relationship between any two individual cities:

Tyler: If you allow a city to visit any other city, then it's gonna end up with having the same visiting-rules as that city that's allowed to visit and vice versa.

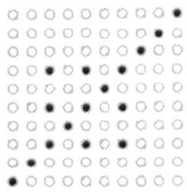
Jimmy still keeps the "block square" to generate his next examples (Textfigure 6), while Tyler uses "mirroring" and its proof (Textfigure 7).

Jimmy: I think there is something to do with square along this line of one and one, two and two, three and three, four and four, five and five; along this line ...if you draw a square...people from this city, this city and this city are able to visit each other, they will have identical connection, but other people will not be able to visit them...so people from this group and this group haven't anything in common, but inside, then they are identical.



Textfigure 6: Jimmy retains the block square

Tyler: city three can visit city five and seven...so I think of course it's gonna be reflected in y equals x...No, this is not I want to finish, because now I have cities that have dots in common and they aren't the same...what I'm missing...what I'm saying here is a sort of square...



Textfigure 7: Tyler uses his proof of the "mirroring" property

Then Tyler draws out, from the big block squares and "a sort of square" appeared in his last example (presented in Textfigure 7), the concept of the group of cities:

Tyler: I completely lost of this sort of way of representing the laws (on grid) because I think they start showing what cities are reachable...in sort of groups you can reach one of the other by travel down the road, you allow to pass the cities between to get from one to other...

Although Jimmy uses the "group" of cities to organize the given situation, his way is qualitatively different from Tyler's. While Jimmy's experience of a group, to a large extent, remains perceptually inseparable from the 'block square' as an incidental element, Tyler's experience of a group is seemingly free from such incidental elements.

2.5.2 The difference in their outcomes

While the result of Jimmy's work is many individual examples, Tyler transcends the situation by introducing new concepts. Particularly, he introduces a new concept with general applicability (the 'box concept'):

> Tyler: How do I say that columns must be the same mathematically? [He writes]
>
> If (x_1, y_1) and (x_1, y_2) and (x_2, y_1) then (x_2, y_2)
>
> Interviewer: Could you explain?
>
> Tyler: I think it's a mathematical way of saying ...if a column has two dots, and there is another column with a dot in the same row, then that column must also have the second dot in the same row...I take maybe a *box* of four dots...I use the coordinate because that makes it very general, and so if I made that my second law, for a mathematician might be easier to follow.

Has Tyler explicitly generated a new definition? It depends on what we decide to count as an act of defining. For the moment, it is much safer to say Tyler has explicitly generated a new concept (and, for us, unexpected) in order to *locally organize* this situation. Interestingly, using this new concept (hereafter, the *box concept*) *we* could offer a new definition for equivalence relations (see Section 2.8).

2.5.3 The difference between their goals

Although reflection on the interviewees' aims was an unexpected part of the interviews, phenomenographically it is a salient aspect of the experience of the situation and it refers to what students are trying to achieve. In addition, that indirect object (the students' aims) and the students' direct attention to the situation are different facets of their experience of the situation. For example, Jimmy and Tyler, both engaged in the same situation, but they were apparently trying to achieve two different objectives, Jimmy was trying to "apply those rules that he has learnt", while Tyler was trying to transcend his knowledge. Therefore, Tyler repeatedly asks himself "why"; the notion of the 'box concept' is the result of one of these questions.

2.6 Discussion

To avoid missing the trend in such a tangle of students' excerpts and sparse interpretation it is worth putting together the main changes from the preliminary study to the pilot study. As was mentioned the main change that consequently affected all the different aspects of the study is the change in the intention of the study towards an investigation of the ways that students organize the given situation. However, beneath this change there is an underlying vital shift, that, while the preliminary study was about *defining* per se, the pilot study placed that aspect within the framework of *organizing*. Given this and having adhered to phenomenographic approach, I have made the following methodological changes:

- Bracketing my judgment that was based on my understanding of equivalence relations and related concepts, the interview course has been transformed from a teacher-wise interview to an interview aiming at reaching a mutual understanding between interviewer and interviewee.
- In parallel with the change in interviewing, when analyzing I started to bracket my understanding of equivalence relations and related concepts as the yardstick for success and failure, and I started to search for what students experience when tackling the situation.

Furthermore, it is worth saying that as a phenomenographic study there is a dialectic relationship between interviewing and analyzing data, '*as a result of which the researcher's picture inevitably gains details, and finds new structure while new perspective reveal distant unsuspected figures*', as Marton and Booth (1997, p.132) put it. We have already seen this aspect where I discussed the rectification of the situation as such, in which I came from 'painters and doors' to 'cities and people'. An additional example that has continuously proven important is the distinction between '*generating* an example' and '*checking* the status of something for being an example'. While at the outset the interviewer demanded the former or the latter without any particular preference and regardless of whatever the interviewees had experienced, I gradually learned to appreciate the distinction between these two activities.

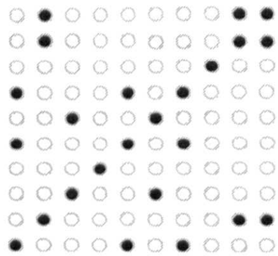


Figure 6: A prepared example

Let me turn to the course of Jimmy's interview to show how I put this appreciation into practice. Having encountered with Jimmy's examples (Textfigure 2, 4 and 6) in the course of the interview, I found them markedly different from one of my prepared examples (Figure 6); as a result, Jimmy was not asked to check the examplehood of such figures until he completed the two activities (giving examples of a visiting-law and giving the minimum amount of information).

In subsequent studies, when I collected more data, the distinction between generating and checking proved to be more subtle and involved. Although I shall postpone a detailed account of the distinction between generating and checking, I shall now mention that making it, or coming to learn about it, is an example of a more general aspect of phenomenographic research, i.e. the dual role of a phenomenographer as a researcher and as a learner. This thrilling aspect in a sense reflects the unwritten grounds for what I have done so far and what I am about to do. Thus let me add a few words about it.

2.7 Research as a learning experience

> We also see research as a learning experience: The researcher is finding something out, and to one extent or another, the research subjects are also learning. Remember then that in discussing the phenomenographic research effort we are considering a learner (the researcher) learning about a certain phenomenon (how others experience the phenomenon of interest) in a situation (the research situation) that is of her own molding. That molding or structuring, as in the other cases of learning, has an effect on the outcome of the leaning, both of the researcher (what she is able to bring out of the research effort) and of the people being studied (what they are able to reflect on in the research situation). (Marton and Booth, 1997, p.129)

Now, at the end of this rollercoaster part (including Chapters 1 and 2), perhaps the most decent question to ask is what "the phenomenon of interest" is in this research. What is that thing that I am about to learn how others experience it? Whatever the answer to these questions is, I am still faced with another challenging problem, that, what about the intention of the study, the very same intention that led the pilot study and in a sense embodied the turn from the preliminary to the pilot study; the one that says "the *intention* of the study turned to be an investigation of the ways that students *organize* the given situation". Would this vague intention, satisfactory enough for carrying out the pilot study, determine also the direction of the main study? What about the differences that I gave as the result of the pilot study? Are they *critical* enough to stand all the way through? After all, and above all, what about the notion of equivalence relation? These are questions or kind of questions that I will try to give their answer in the next chapter. Revealing the answer of one of these question (probably, the most important one), I shall say that in the next chapter the intention of the study will be defined anew! At the end of the present chapter (Section 2.9), I will give some initial reasons for doing so. Before that, let me expand on one of the remarks I made in Section 2.5.2. This provides the grounds for doing a reflection on this phase of study.

2.8 Box concept is a *local* concept

In Section 2.5.2, it was mentioned that Tyler has explicitly generated the *box* concept in order to *locally organize* the situation. The previous sentence has been inspired by two persons, Tyler who used the box concept for the first time, and Freudenthal who developed the ideas of the *global organization* and the *local organization*. In this section, I will discuss the box concept in the light of the latter ideas.

In the preface of "Mathematics as an Educational Task", Freudenthal (1973) says how as a mathematician he found it hard to rearrange his old ideas about teaching for his book. He explains that:

> The problem was not the dialectic instead of the deductive style, and the local organization of the subject matter was not a problem either. But the *global organization* was the sore point. I could not use the formal organization of a mathematics course or treatise where the author says, or writes things like "because of theorem... (cp. p. ...), applied under the condition of corollary... (p. ...), it appears that the definitions of ...on p. ...and on p. ...are equivalent." I could not use this method nor could I invent another form of organization. Thus the present book is, from the view point of a mathematician, badly organized. (ibid, p. IX)

The tension between global and local organization starts right from the beginning of the book and goes through to the end. Examples of this tension and/or the local and global organizations per se are abundant all over the book. However, if we look for a *definition* that could cover all those examples we could only find certain sparse attempts to give a definition instead; still even those rare attempts are somehow anchored to the examples or the context that they originate from. One of these attempts is the following, made when Freudenthal discusses the case of (teaching) geometry.

> [The student] learns the global organization, that is organizing not a system internally, but a category of systems by looking from outside- he learns to axiomatize. (ibid, p. 454)

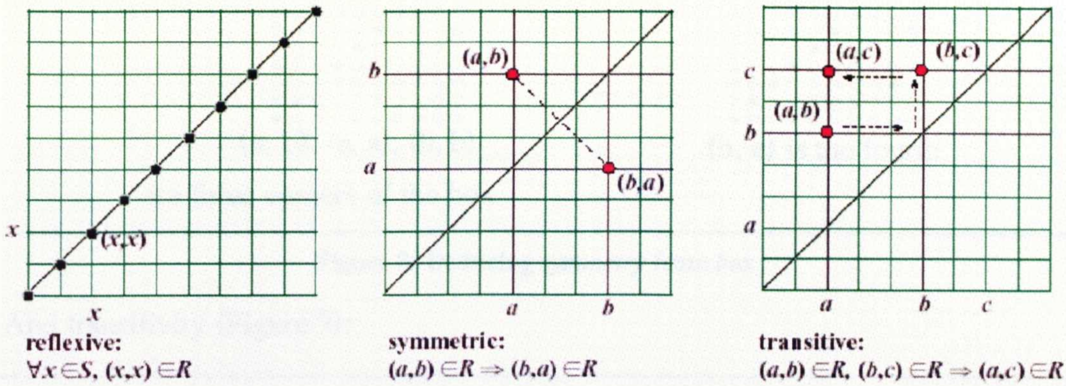
But globally organizing is not axiomatizing; if it was, his book itself was not subject to that. Axiomatization is only an example, an extreme case of a global organization.

Though Freudenthal does not say explicitly what he means by the local and global organizations, he is explicit about "mathematics as an activity" in which organizing in its different forms plays a vital role. He also explicitly warns us not to underestimate the importance of organizing locally; for "in general, what we do if we create and if we apply mathematics is an activity of local organization" (ibid, p. 461).

In the light of this distinction it is now possible to add the box concept to Freudenthal's examples. The box concept only gives us a local organization while the standard account of equivalence relations provides us with a global one in which two important types of relations, equivalence relations and order relations can be seen as particular types of transitive relations. However, it seems that the box concept has one or two interesting consequences, as I will now discuss.

Box Concept

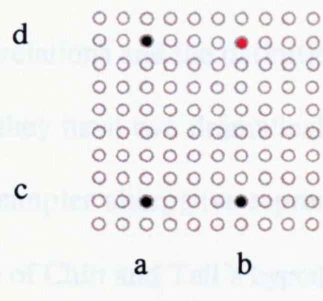
To see some of the *consequences* of the box concept I have to resort to an even more formal treatment of equivalence relations, i.e. their formulation as a set of *ordered pairs* having certain properties (The first formal treatment was given in Section 1.6.2). To give this formal and *partially pictorial* formulation, let me refer to a research paper that interestingly suggests a "visual representation" for *all* the defining properties of an equivalence relation. According to Chin and Tall (2001, p.245; this paper will be discussed in more detail in Section 8.3), these visual representations are as follows:



As it can be seen the first two (reflexivity and symmetry) have an easy visual representation that can also be found in the textbooks. But the visual representation of the transitive law is not given in the textbooks. Following the picture Chin and Tall say:

> The transitive law (a, b), (b, c) ∈ R implies (a, c) ∈ R is a little more sophisticated. (The transitive law moves horizontally from (a, b)—maintaining the second coordinate b—to the diagonal then vertically to the point (b, c), completing the rectangle to give the third point (a, c).) (ibid, p.245)

Compare that complexity with the simplicity of the visual representation of the box concept. (Figure 7)



If (a, c) and (a, d) and (b, c) then (b, d)

(Box concept)

Figure 7: Visual representation of the box

We only need a few pictures to see how having reflexivity and box concept, we can deduce symmetry (Figure 8):

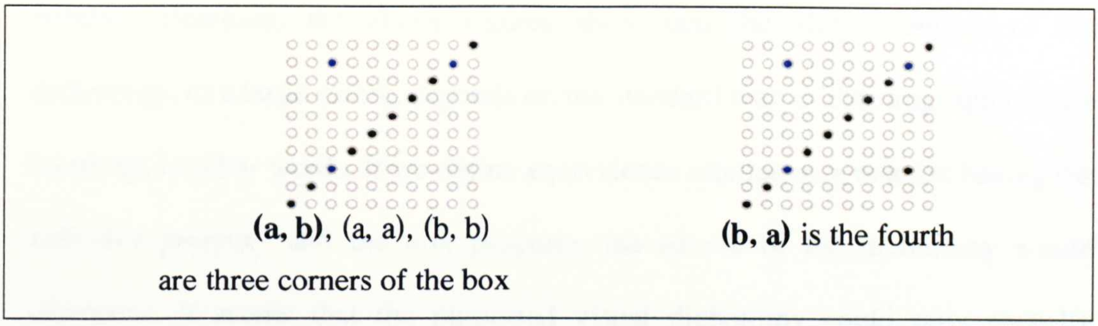


Figure 8: Deducing *symmetry* from *box*

And transitivity (Figure 9):

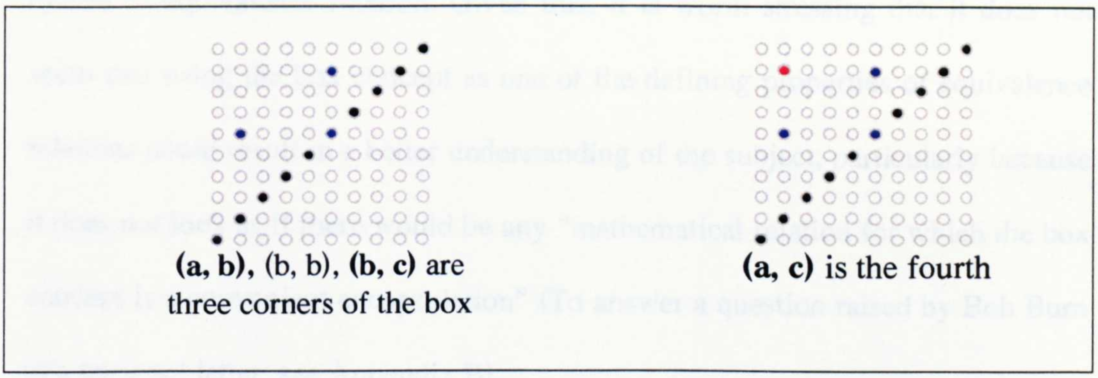


Figure 9: Deducing transitivity from box

It can also be seen that having the normative definition of equivalence relations, based on reflexivity, symmetry and transitivity, we can deduce the box concept. Hence, the standard definition of equivalence relations and the definition based on the box concept are *logically equivalent*, but they have two dramatically *different representations.* More importantly, having a simpler alternative representation of an equivalence relation calls into question one of Chin and Tall's hypotheses.

Regarding the visual representation of the standard account, Chin and Tall (ibid, p. 245) hypothesize that *"the complexity of the visual representation"* is a source of a *"complete dichotomy between the notion of relation* (interpreted as a subset of S × S) *represented by pictures and the notion of the equivalence relation* *which is not"*. Accordingly, they suspected the stated dichotomy inhibits students from grasping the notion of relation encompassing the notion of equivalence relation. However, the above figures show that the alleged source of the dichotomy, to a large extent, depends on the standard way of defining equivalence relations; in other words, if we define equivalence relation as a relation having the reflexive property and the box property, the source of the dichotomy would disappear. It seems that the purported visual dichotomy could only partially explain the *cognitive dichotomy* between the general notion of relation and the notion of equivalence relation. Given this, it is worth stressing that it does not seem that using the box concept as one of the defining properties of equivalence relations could result in a better understanding of the subject, particularly because it does not look as if there would be any "mathematical relation for which the box concept is a convenient encapsulation" (To answer a question raised by Bob Burn in a personal letter; see Appendix B).

2.9 Summary and Reflection

In the first part, I have reported the first two phases of my study. The aim of the first phase (the preliminary study) was to lead lay students to define certain predetermined concepts, i.e. symmetry and transitivity. The Mad Dictator Task was initially designed as a "problem situation" serving this aim. However, the students' experience of the situation brought some of my unsuspected assumptions to the fore. That is, leading students to define certain predetermined concepts is tantamount to taking it for granted that (1) there are certain fixed concepts that the students are trying to negotiate, and (2) there is a fixed way of relating these concepts to each other. The preliminary study showed, however, that the Mad Dictator Task, despite being designed around an intended concept, fails to guarantee either of these taken-for-granted aspects. This realization led to the second phase of the study (the pilot study) in which constructing a definition was regarded as the last part of an organizing activity.

Organizing was the main theme of the pilot study in which the aim was to investigate the ways that students *organize* the given situation. On reflection, making use of the idea of organizing was a kind of reaction against the thwarted expectations of the first phase of the study in which it was expected that students would see in the situation what I saw in it. As a result, the *different* ideas that students use to organize the situation were given more weight than the ways that they related these ideas to each other. Thus, some of the ideas that were treated as the means of organizing may remain quite unorganized. For example, it was claimed that Tyler generated the box concept in order to locally organize the situation, but it was not discussed how he himself related this concept to the other concepts that he *experienced* in the situation. Yet, his ideas revealed some *unexpected* features of the notion of equivalence relation. Tyler who has originated this new definition was not aware that he has done so. He was unfamiliar with the notion of equivalence relation and consequently with the way that his idea was related to it as customarily axiomatized. Yet, the way that he saw the situation informed us of some aspects of equivalence relations, regardless of whether he himself was aware of this connection or not. This opens the possibility of using the Mad Dictator Task for pursuing a new objective. This possibility will be examined in the next chapter where the idea of organizing is replaced by the more neutral idea of *experiencing*, and subsequently the intention of the study is defined anew.

Part Two: Methodology and Literature Review

Introduction

Since we cannot be on a rollercoaster until the end of this thesis (though I was all through the study!) in this part (Chapters 3, 4) I put together those issues that will be the points of reference in the rest of the thesis.

Chapter 3 concerns itself with methodological issues. In particular, I give a more detailed account of phenomenography. I also discuss the purpose of the study. This new intention *is* to investigate students' conceptions of equivalence relations (partitions). At first glance, this looks like a turn from the pilot study! However, I argue that it is a kind of delimiting of focus rather than a turn. Following the specification of the purpose, I described the means and the methods employed in the study. In particular, I add two other tasks to the tasks used in the pilot study. After this, I revisit "The least amount of information" task and question its role as the means of achieving the purpose of the study. I also introduce the participants.

Now that I have specified the purpose of the study I can look through the related literature. Consequently, Chapter 4 concerns itself with the literature as to equivalence relations and partitions. In particular, the ideas of Skemp, Dienes and Freudenthal about these concepts are described and compared with each other.

Chapter 3: Methodology

3.1 Introduction

In the previous chapter, when discussing the pilot study, I articulated certain reasons to adhere to phenomenography to mould my research endeavours. I also gave a brief account of this research tradition (see Sections 2.3 to 2.7). In this chapter, we learn much more about phenomenography. In particular, I show that it has given to this study much more than a methodology could give, i.e. it embraces the whole aspects of the work from the purpose of the study to the representation of the results (Chapters 5 and 6). I distinguish between the research methods and the theoretical base of phenomenography. The latter separates phenomenography from being simply a methodology (if it is a methodology at all), and it is the very same aspect that helps me to clarify and sharpen the vaguely stated intention, mainly, investigating the ways that the students organize the given situation (as stated in the pilot study).

3.2 Phenomenography, a *definition*

It seems appropriate to have the story of the origin of phenomenography in the words of Marton who was one of the leading pioneers of this research specialisation:

> Phenomenography is a research specialisation which was developed by a research group in the Department of Education at the University of Göteborg in Sweden in the early 1970s. The word 'phenomenography' was coined in 1979 and it was first appeared in Marton's (1981) work. Etymologically, it is derived from Greek words "phainemenon" and "graphein", which mean appearance and description, and phenomenography is thus about the description of things as they appear to us. (Marton and Fai, 1999, p.1)

And closer to a definition, Marton says:

> Phenomenography is a research method adapted for mapping the qualitatively different ways in which people experience, conceptualise, perceive, and understand various aspects of, and phenomena in, the world around them.
> (Marton, 1986, p.31)

Even on the surface of these two quotations it is noticeable that phenomenography has had many changes and has been interpreted differently from time to time. Notice that it has been regarded as a *research specialisation* in the more recent version, and as a research *method* in the older one. However, I shall say that it is mostly treated as a research specialisation, in particular, by Marton himself.

Generally speaking, since its birth, phenomenography has had a dual life, partly as a research specialisation, partly as a theory of learning[5]. As a research specialisation it is defined in terms of the object of research. As a theory of learning it is defined in terms of certain ontological and epistemological assumptions. Any research situation is regarded as a learning situation in which the participants (and in a sense the researcher(s)) are all learners, where learning is "becoming capable of discerning and separating aspects of a phenomenon the learner has not been able to discern and separate previously, and of being simultaneously and focally aware of aspects she has not been able to be simultaneously and focally aware of previously" (Marton and Booth, 1997, p. 145), and where the object of research is basically capturing "the differences in the experienced structure and meaning of the phenomenon in question" (ibid, p. 139).

Furthermore, the research has been used to shape and reshape the theory, and the theory in turn has been used to carry out the research. However, there is a '*dilemma*' in this fairy tale.

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[5] Today phenomenography has also been extended to teaching; even, its *methods* have been used outside the field of education. However, the main concern in this section is phenomenography as a theory of learning.

3.3 Phenomenography, a dilemma

Biggs (1999, p. 11) describes the original phenomenographic studies[6] as a breakthrough that rectified psychologists' concerns with developing "the One Grand Theory of Learning" emptied of *what* is learned. Since those seminal works the growing body of phenomenographic literature has been fluctuating between empirical studies on the one hand, and developing a *general theory of learning* on the other hand. But as a research tradition drawn out from—and mainly contributing to—the content-specific learning situations, phenomenography is seemingly faced with a dilemma, i.e. if Biggs' comment on the 'history' of phenomenography is correct, offering 'the one grand theory of learning' seems to contradict the very nature of phenomenography. Maybe, there is a way out of this dilemma where Marton and Booth (1997, p.115) suggest that 'general ideas and principles need to be developed anew in specific contexts and contents of learning and teaching'. Whether we accept the existence of this dilemma or not, there are certain theoretical elements governing all the empirical studies adhering to this research approach. Thus let me turn to one of these elements that in a sense characterize any phenomenographic study.

3.4 First-order perspective vs. second-order perspective

If there is one element in common between all the phenomenographic studies, it is the distinction between the *first-order perspective* and the *second order perspective*:

> A fundamental distinction is made between two perspectives. From the first-order perspective we aim at describing various aspects of the world and from the second-order perspective ...we aim at describing people's experience of various aspects of the world. (Marton, 1981, p.177)

[6] In particular, he refers to Marton and Säljö's (1976a, b) study that as a matter of fact had taken place before calling them a phenomenographic study.

As an example, let me briefly remind you of my perspective when I was conducting the preliminary study. Simply in the preliminary study I saw the situation through certain predetermined concepts. Moreover, I had certain predetermined ways of relating those concepts to each other. In other words, I saw the *figural* aspects of the situation through certain normative concepts and their normative conceptual relations. These predetermined norms were what I judged the respondents' works with; according to my understanding of the situation I considered them either as a success, a 'correct' answer, or as a failure, a 'wrong' answer. In sum, it seems that I had taken a first-order perspective in which a "respondent's statement would be judged in the light of other valid, consistent, and useful predetermined statements about the situation" (Marton and Booth, 1997, p.119). But, as a by-product of the preliminary study, I started to change my perspective. Correctness or incorrectness of the answers, degrees of success or failure, and basically what those judgements stem from all were bracketed[7]. I started to "consider the very same statement as reflecting the students' way of experiencing the problem, making sense of it" (ibid, p.119). I started to see the situation and the phenomena under study through the eyes of students. In other words, I adopted a second-order perspective. But, after all, what it is that I am to see through the students' eyes. What are the phenomena that I am interested in?

3.5 The Purpose of the Study

"The intention of the study is to investigate the ways that students *organize* the given situation." So far I have written the previous sentence many times in many different forms. At first glance it seems quite straightforward and familiar. However, in such a complex situation, the focus of the study could not be

[7] See Section 2.3 for the phenomenographic meaning of this term.

maintained only by having a vague sense of familiarity (see Section 2.9); such a focus is the first condition of any phenomenographic research, or any research at all. Thus, let me put it as bluntly as possible: *the purpose of this study is to investigate the ways in which students understand equivalence relations (and partitions) or equivalently, to investigate students' conceptions of equivalence relations (and partitions).* In the following sections I will argue that it is not a turn from one perspective to the other (as I had from the preliminary study to the pilot study); it is merely a kind of delimiting the focus within the same perspective. In this respect, there are certain questions that I should answer; the most important ones are (1) what about the concepts of organizing and organization? Have they simply vanished along the way? (2) Is it *valid* to use the devised situation to investigate students' understating of equivalence relations (and partitions)?

3.5.1 Organizing

To answer our first question I shall introduce some of the phenomenographic terminology. To do so, I have recourse to the evolution of phenomenography during its short history.

In one of the pioneering phenomenographic studies, when this approach had only a few terminologies as such, and as a matter of fact had not got any name at all, Marton and Säljö (1976a) examined what a group of Swedish university students had understood and remembered from reading a newspaper article dealing with a curriculum reform in the Swedish universities. Marton and Säljö identified four different conceptions of the intentional content of the passage. They called those four different types of answer *levels of outcome* constituting the *outcome space* for that particular text. The following four categories present what was understood at each of those levels. The categories form a hierarchy where the answers in each category grasp the point of the article better than the ones in the next categories.

> Level A: *Selective measures*. (Meaning that measures were to be taken only for those groups of students that did not fulfil the necessary requirements.)
>
> Level B: *Differential Measures*. (Measures to be taken which allow for differences between the various groups.)
>
> Level C: *Measures*. (Measures to be taken only.)
>
> Level D: *Differences*. (Differences between groups only.)
>
> (Marton & Säljö, ibid, p. 8)

It seems that the above outcome space is too content-related to be of any use. However, this content-related outcome space is exactly what Marton and Säljö used to make their point, that, "learning has to be described in terms of its content. From this point, differences in what is learned, rather than differences in how much is learned, are described (ibid, p. 4)."

Furthermore, they argued that if there are qualitative differences in what is understood and remembered from the text (the outcome of learning) it seems very likely that there are corresponding differences in the way different people set about reading (the process of learning). Having been motivated by this line of argument they also distinguished between two different *levels of processing, surface-level and deep-level processing.* In surface-level processing the subject focuses on the *sign* (i.e., the discourse itself or the recall of it), whereas in deep-level processing the student concentrates on *what is signified* (i.e., what the discourse is about).

But, how on earth all of these are related to the title of this section? To reveal the answer let me go further in this short 'history'.

After those seminal works, many other studies took place in different *learning situations*, though within the same perspective; some investigated the outcomes of learning, some studied the processes of learning, and some had an interest in both, however, all were content-specific. But, in many cases, such content-specific descriptions did not seem to be of any use, except that it could probably satisfy some theoretical curiosity. Moreover, and strangely enough, phenomenographers observed that the categories constituting the outcome space were often hierarchically ordered. These facts, together with a natural tendency towards generalizing, gave enough reasons for looking for some aspects to describe the differences in the outcomes and the processes of learning irrespective of the content! The *organizational* (later on, it was termed *structural*[8]) aspect and the *referential* aspect (the meaning) were the result of this so-called search. Both of these aspects have been used to describe different levels of the outcome and the process in a way to be applicable in every experience of learning.

For the time being, suffice it to say that SOLO[9] taxonomy, though it is outside the phenomenographic tradition, could be regarded as an attempt to describe the structural (organizational) differences in the different levels of the outcome space (Dahlgren, 1984); and the surface/deep dichotomy could be regarded as an attempt to describe the referential aspect of students' experiences (Marton and Säljö, 1984).

Furthermore, after they started the ball rolling, day after day, more theoretical elements were added to what basically was a simple but important ambition, that of describing "*the outcome space* of essential concepts and principles". I do not intend to give an account of what the phenomenography

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[8] According to Marton and Booth (1997, p. 87), the structural aspect of a way of experiencing something is twofold: discernment of the whole from the context on the one hand and discernment of the parts and their relationships within the whole on the other hand.

[9] "SOLO" stands for the Structure of the Observed Learning Outcome. See for example Biggs (1999, pp. 37-40)

edifice looks like nowadays. However, I do intend to remind ourselves of that simple original ambition. For this end, I shall refer to the closing paragraph of Marton & Säljö original paper.

> The most important conclusion we draw from our research is that learning should be described in terms of its content. A highly significant aspect of learning is, in our opinion, the variation in what is learned, i.e., the diversity of ways in which the same phenomenon, concept or principle is apprehended by different students. By gaining knowledge about how students comprehend, for instance, various scientific principles and ideas, we should obtain information which would undoubtedly prove fruitful for teaching. Consequently, we believe, it worthwhile to describe *the outcome space* of essential concepts and principles. The various *levels of outcome* will probably reveal the distinctive features and prerequisites of comprehension in these specific instances. We think, that, apart from what it may tell us about the general properties of cognition, it is of interest in its own right to describe what it takes, from a psychological point of view, to understand, for instance, the concept of scarcity in economics, or the law of diminishing returns.
>
> (Marton & Säljö, 1976a, p.10)

In effect, my ambition is simply to put this (closing) line of argument into practice, where the concepts of interest are as important as equivalence relations and partitions. In other words, my aim is to describe the outcome space of these two concepts. Foreshadowing the results of this study, I shall say that I have not used that *analytical* distinction between structural aspect and referential aspect, except in a few cases that I find them naturally useful to describe the results. In a sense, this study is very much the same as phenomenography was in the mid-1970s!

Regarding terminology, I continue using the terms "organizing", "organization" and so on in Freudenthal's sense and in the same way applied to the box concept (see Sections 2.2 and 2.8). However, unlike the pilot study, I do not use these terms to describe and categorize the students' works, mainly because to organize one needs to *reflect* on his or her "previously unreflected activity", making it *conscious* and the subject of *reflection* (Freudenthal, 1991, pp. 96-102).

But, as I will show, many of what is *experienced* in this study is not easily accessible to the learner's reflection (if it is accessible at all). This strongly applies to equivalence relations and partitions as two concepts that are *embedded* in the devised situation, the Mad Dictator Task. But, and this is a very important "but", are these concepts *really* embedded in the situation? Is it *valid* to use the Mad Dictator Task in order to explore the variation in the students' conceptions of equivalence relations and partitions? In the next section, I will try to answer these questions.

3.5.2 Validity

In this section we deal with the very two questions with which I finished the last section.

The first question was: Are the concepts of equivalence relation and partition *really* embedded in the situation? If by "really" it is meant that those concepts are hidden "out there" (by someone else) waiting to be discovered by the learner (i.e., discovery learning), the answer is 'no'; if by "really" it is meant that those concepts are "out there" manifesting themselves as the *constraints* that could be experienced by the student "only as the break-down of an action or thought"[10] (i.e. *radical constructivism*), the answer still is 'no'. Generally speaking, if by "really" it is meant that those concepts are really embedded in the situation as existing independently of the learners, the phenomenographical answer to our question is 'no'.

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[10] Only the part quoted is from von Glasersfeld (1990). The whole sentence is my opinion that is indeed a very rough description of radical constructivism. "Constructivism merely asserts that it [mind-independent, ontological reality] is not accessible to rational knowledge because it manifests itself only through the constrains that make some of our ways of acting and thinking unsuccessful; and, from the subjects' perspective, any such constraint is experienced (and therefore knowable) only as the break-down of an action or though" (ibid, p.37; square brackets added).

But if we change our perspective from *dualistic* (above examples) to *non-dualistic*, where the action (acted by the student) and that which is acted on are taken to be inseparably intertwined, we can assert that *certain qualities* in the unordered situation, the situation which in some way "asks to be ordered" (read it as "begs" to be organized!), actually help students to create the required order[11]. In turn, by studying the ways that students deal with the situation we can learn about those qualities as they experience them. Practically, we can say that experiencing certain concepts in our situation can help students to generate an example, to check the status of something for being an example or to give proofs. In turn, by analysing the ways that our students tackle those activities we can learn about those concepts. Thus, in a sense, I have *partially* given a positive answer to our question i.e., yes, there are certain concepts embedded in the situation, but not separated from one who experiences them through the situation. Having said so, it is also clear that I have not answered the very question that I started with! As a matter of fact, it seems that the question has been divided into two different, but interrelated questions, that, (1) whether certain concepts are embedded[12] in the situation or not (2) whether those concepts are equivalence relations (and partitions) or not.

In line with the phenomenographic general assumptions I have already given a positive answer to the former. The answer to the latter does not directly emerge from the former (though it certainly related to it). We should notice that this kind of question indeed is very situation-related. As a result, any possible answer is also, to a great extent, situation-related. This time, let me give the

[11] This is in line with the way in which Marton and Neuman (1990) argued for, though they did not use the terms dualistic and non-dualistic.
[12] In the sense of 'embedded' described above.

answer first, and then argue for that. As it is expected, the answer is 'yes'; but again, as it is expected, this positive answer should be warranted.

First and foremost, it is 'yes' because I have decided to explore the students' conceptions of equivalence relations and partitions[13]. In other words, it is *my* choice in a way; however, it is not completely a free choice. For example, taking it to the extreme, I could not make use of the devised situation to investigate the students' conceptions of the weather! Giving a much less extreme case and being closer to the meaning of 'choice' and its connotations, I did not devise the situation to investigate the students' *conceptions of dictatorships*. However, as a matter of fact, I had cases in which that conception (whatever it is) affected the ways that the student tackled the problem. For example, one participant took the diagonal as a *good example* (of a visiting-law) because it could truly reflect the meaning of a brutal dictatorship! Or, some students hesitated to accept the whole-grid figure, in which there is no visiting-limitation on the people, as an example; even though, they themselves had generated that figure! The reader who feels that "this is an argument against saying that an equivalence relation is embedded in the situation" (to quote one of my supervisors) may, if he wishes, now turn to Section 8.3.2. Here I will further this discussion in a *new* direction that to a large extent accords with the way that this investigation has been carrying out. Let me now turn to the methods employed in this study.

3.6 Methods

Under this headline, I put together those methodological issues peculiar to this study.

[13] This line of argument came out during a personal communication with Ference Marton.

3.6.1 Participants

As a phenomenographic study aiming at vividly mapping the variation in the students' conception of equivalence relations (and partitions), I tried to *maximize that variation as much as possible.* To this end, in the main study I interviewed thirteen students with varied background experience, seven in Iran[14] (all Iranian) and six in England (all English). One of the participants was female and all the others were male.

Participants were quite varied in terms of educational level: one first year politics students (Peter[15]), one first year law student (Dickon), four middle school Iranian students (Hess, Kord, Piro and Arash), three high school students (Ali, Poya and Pouria, all Iranian) and four first year mathematics students (Sarah, Amit, Chris, Ben). All the university students were students of the University of Warwick. The four middle school students were studying at a special state school for 'gifted and talented students'[16]. All the secondary school students were studying at state schools.

The sample was mainly opportunistic. Simply, participants were friends of mine (Ali, Pouria), friends of friends of mine (Peter, Dickon and Poya), those who replied to an invitation sent to the all first year mathematics students (Sarah and Amit), two of my supervisees (Ben and Chris) and finally the four middle school students (Arash, Hes, Kord and Piro) who were introduced to me by their

[14] School system in Iran is 5-3-4, the first five years is primary school, the middle three, middle school and the last four, high school which covers three years and a one-year pre-university programme. Children enter primary school when they are about seven years old; at the outset of middle school, they are about twelve and at the outset of high school they are about fifteen.

[15] All the names are pseudonym mainly chosen by participants.

[16] In Iran there are middle and secondary schools for such students, but not primary schools. Only a minority of students who are considered as high achievers (their marks at the last year of their primary school or the last year of their middle school must be above 18 out of the possible 20) are allowed to take the entrance exam. Among those high achievers who choose to take the exam only a tiny minority could go through.

supervisor. Although this is certainly an opportunistic sample, at the time it seems to be a satisfactory sample aimed at maximizing the variation in the students' *approaches*. Moreover, I did consider such a sample to be a natural part of the study in which such a varied range of students participated.

After all, it is worth stressing that despite being opportunistic within an educational level, I was quite selective when choosing the level itself. For example, I did try to have middle school students, high school students and university students. Furthermore, in a deeper stage of the study, when I realised that dealing with one of the activities (the Intersection Task[17] that involves *proof*) was more difficult than what I had envisaged, I did invite first year mathematics students who had been taught equivalence relations and related concepts. However, I received only two replies (Sarah and Amit); accordingly, such a few replies condemned me to be opportunistic within that educational level. Let me now turn to the actual course of the interviews.

3.6.2 Interviews

The study was conducted by holding individual task-based interviews in which five tasks were used. The tasks were devised in advance, the first three when doing the pilot study (see Section 2.4 and 2.6) and the last two were added for the main study (see below). Each interview took about one hour, except for two of them (Hess and Piro) that lasted for about one hour and a half. At the beginning of each interview, I let the interviewee know that his or her work will be used for research purposes. I also told the interviewees that they may choose a pseudonym, if they wish.

[17] See Section 3.6.2.

The way of interviewing remained more or less as it was in the pilot study (see Section 2.4). However, some changes have been made in the structure of the interviews. Some of these alterations were simply imposed by an unwanted limitation of access to the participants; and some of them were decided by ongoing changes in *my* own understanding of the situation as a whole, together with its component parts and the ways that they all together could serve the aim of the research. But, it is worth stressing that regardless of all the changes made, the focus of the research remained the same. In other words, all the way through I was investigating student's understanding of equivalence relations (partitions), though the situation that I was doing so was subject to some alterations. Since these changes could call into question the *validity* of this research, a thorough discussion of them is in order. To do so, I shall give a detail account of the interviews.

The Content

According to Bowden (2000, p. 8), "phenomenographic interviews usually begin with interviewees being asked to respond to a planned question or a given situation. Two common types of questions are (i) problem questions in the field under study, and (ii) questions of the 'what is X?' kind." Glancing through the section regarding the participants in this study, it can be seen that most of them were lay students in terms of their knowledge of *the* formal account of equivalence relations and related concepts. Given this, it can be seen that 'what is X' questions could not be asked (take X as "an equivalence relation" or "a partition"). Thus, I had no option but to employ the first type of questions, though with an extended sense of them.

As mentioned I started with three tasks, generating an example (of a visiting-law), checking the status of a figure for being an example (of a visiting-law) and "the least amount of information" task, and then for the main study, I added the following two tasks:

> The intersection task[18] : One of the officials, to create an example, uses other officials' examples: he takes two valid examples and put their common points in his own grid. Is the grid that he makes a valid example?
>
> The union task: Another official takes two valid examples and puts all of their points in his own grid. Is the grid that he makes a valid example?

I had some simple reasons for adding these two tasks. At the very least, they could give me more opportunities to investigate the students' experience of equivalence relations (partitions). But, the usefulness of these new tasks could only be *proved* in the actual course of the interviews. As I will show in many of the following sections, they both proved to be satisfactory. However, regarding one of the other tasks, "the least amount of information" task, it was not the case.

At the outset of this thesis (see Section 1.6.2), in *my* view "the least amount of information" task was highly relevant to the situation inasmuch as it could bring about some of the most relevant concepts. However, while for me that task was quite relevant to the other tasks, most students found it quite irrelevant to the other tasks, something that was there to be done only because it was there! As a result, towards the end of the data collection, that task was not used anymore! Thus, overall I worked with four tasks, two concerning examples—using the given definition to generate some examples and to check the status of some prepared figures for being an example—and the other two concerning proof. Now that I have discussed the content more or less used in each interview let me turn to the structure of the interviews.

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[18] The titles of the tasks are only used here. They were not used in the interviews.

The Structure

In general, the task regarding generating an example was used as the first task of each interview. At the very least, it allowed the interviewees to become *familiar* with the situation. But, the interviewees could also acquire this familiarity if at the outset I asked them to distinguish between prepared examples and non-examples of a visiting-law. However, the generating task seemed to be a better starter than the checking task, for the students' generated examples could bear something different from what I had in mind. A case in point was when the students' generated examples having certain special characteristics not common to *all* potential examples, or at least, not in common with the examples prepared by *me*. Thus, if I wanted to bracket my understanding of the situation it was better to start with generating task. Furthermore, as the data analysis progressed I realised that *there is a trace of checking in any act of generating*. In sum, the generating task was used as the first task of each interview, while the checking task was completely contingent on the actual course of the interviews. As a result, the checking task sometimes was used immediately after the generating task, sometimes after all the other tasks, and sometimes it was not used at all. The other two tasks, the intersection task and the union task, were used when the interviewee had at least two or three examples at hand.

3.6.3 Analysis

The audiotapes were transcribed verbatim. The grids were reconstructed based on the students' detailed descriptions of them. The verbatim transcribed tapes and the students' written works (including the generated and regenerated grids) were treated as data; and they were analyzed according to the phenomenographic analysis *method* in which:

> All of the material that has been collected forms a pool of meaning. It contains all that the researcher can hope to find, and the researcher's task is simply to find it. This is achieved by applying the principle of focusing on one aspect of the object and seeking its dimension of variation while holding other aspects frozen. The pool contains two sorts of material: that pertaining to individuals and that pertaining to the collective. It is the same stuff, of course, but it can be viewed from two different perspectives to provide different contexts for isolated statements and expressions relevant to aspects of the object of research. The researcher has to establish a perspective with boundaries within which she is maximally open to variation, boundaries derived from her most generous understanding of what might turn out to be relevant to depicting differences in the structure of the pool. The analysis starts from searching for extracts from the data that might be pertinent to the perspective, and inspecting them against two contexts: now in the context of other extracts drawn from all interviews that touch upon the same and related themes; now in the context of the individual interview.
>
> One particular aspect of the phenomenon can be selected and inspected across all of the subjects, and then another aspect, that to be followed, maybe, by the study of whole interviews to see where these two aspects lie in the pool relative to the other aspects and the background. In a study that involves a number of problems for solution, for instance, the analysis might start by considering just one of the problems as tackled and discussed by all the subjects, and then a selection of whole transcripts that include particularly interesting ways of handling the problem. This process repeated will lead to vaguely spied structure through and across the data that our researcher/learner can develop, sharpen, and return to again and again from first one perspective and then another until there is clarity. (Marton and Booth, 1997, p. 133)

It is longwinded, but I could not resist the temptation to quote it here because more or less it reflects all the aspects of a phenomenographic analysis. Consider that, it is like the outline of an idea, rather than a technique.

Having said all of this, I shall also add a few words on one extremely important phenomenographic assumption underlying this kind of analysis, that, students' conceptions and the ways they tackle a certain task are inseparably intertwined. This assumption had two vital consequences for the way that I went through the transcripts. First, all the way through I had to keep in mind (though it was not simple) that each statement serves a certain purpose for handling a certain task. To make this point as clear as possible I shall give a concrete example.

When I was analysing the students' work on the intersection task, I came to one particular piece of work (Arash's) that at first glance seemed so meaningless that I could not make any sense of it[19]. Words were so idiosyncratic and sentences were so seemingly unrelated to each other that any attempt to understand them seemed doomed to failure. Though idiosyncratic and seemingly unrelated, there was one important point that I could not ignore, that, those idiosyncratic words and those seemingly unrelated sentences had been expressed within a particular context (the intersection task) and with an intention (i.e., *proving* that the intersection of two examples is an example). Thus, I read that piece again and again (as I did so for all the other transcripts), to the extent that I knew it by heart (as I did so for many other excerpts). Again and again I studied it against two different contexts (1) Arash's own work where he had tackled the other tasks, (2) the other students' works where they had dealt with the very same task (the intersection task). The first context was used to reveal the meaning of his idiosyncratic words constituting his yet unrelated sentences, while the second context was used to gain a picture of his proof where those unrelated parts served the purpose of proving what he had set to prove. Eventually, after one year, my persistence paid off! Everything came to a new light; those seemingly unrelated parts were related to each others when I realized that how they had served his proof. It was a moment of *discovery* and the moment that I appreciated the beauty of his proof! It was a moment in which both of us learnt something new.

Let me turn to the second consequence of the interconnection between students' conceptions and the ways they tackle a certain task. Perhaps this second

[19] As a matter of fact, I was faced with the same situation in the course of the interview with him. Regardless of all our attempts, we failed to reach a *mutual understating*.

consequence practically is more important than the first. It suggests that a certain conception makes a difference in the way of dealing with a certain task. Hence, to uncover the students' conceptions I was looking for where they had been handling a particular task (say, generating an example) differently. The results of this study exemplify this approach. In this regard, the reader may also read Section 8.3.2.

As the result of analysis, the collected data were grouped into qualitatively different categories. Each category manifests my description of a particular way of experiencing the concept(s) of interest. When reading the descriptions (Chapter 5 and 6), it should be borne in mind that:

> We do not believe there is any uniform technique which would allow other researches to go from "the pool of meaning" to the emerging pattern of a hierarchy of similarities and differences. It is a discovery procedure which can be justified in terms of results, but not in terms of method.
>
> (Marton & Säljö, 1984, p.39)

CHAPTER 4: LITERATURE as to EQUIVALENCE RELATIONS

4.1 Introduction

In two of the previous sections (Sections 1.6.2 and 2.8) I discussed (defined), rather mathematically, the notions of equivalence relation and partition. I also made some sparse references to the literature concerning these concepts (Section 2.8). Nevertheless, since the next chapter concerns itself with the students' conceptions of these notions, having a more thorough discussion on them is in order.

Surprisingly, despite the fact that the notion of equivalence relation is one of the most fundamental ideas of mathematics, students' conceptions of it have attracted little attention as a research subject. As a result, what I cite here as the literature seems mainly like some speculations in which equivalence relations and partitions serve to exemplify certain theories and/or (psychological) principles. For example, Skemp (1971) used these ideas when attempting to "illustrate the development of mathematical schemas, starting with a conceptual analysis of some of the most basic ideas" (ibid, p. 140). Dienes (1971) used them to exemplify the application of his theory of mathematical-learning (1960, p. 44). I intend to focus on their treatment of these notions rather than their theoretical interests. However, I will inevitably have recourse to their theoretical perspective here and there, but only to the extent that it may facilitate understanding of their views on the equivalence relations (partitions). Before giving more details about these accounts, I shall stress that they have regarded an equivalence relation as a relation that is:

reflexive;

symmetric;

transitive.

However, as they have exploited these concepts for different purposes the orders with which they are introduced are different from one account to the other. Skemp's focus is on the transitivity. Thus, for him, an equivalence relation is (i) transitive, (ii) reflexive, (iii) symmetric (1971, p. 184). Dienes has used the same order as in most textbooks in mathematics. For him, an equivalence relation is (a) reflexive; (b) symmetric; (c) transitive (1976, p. 60). Moreover, as far as this review is concerned, this difference in focus has an important consequence. Consider that if I only look at the conclusion of their lengthy discussion, that is to say, if I only say that 'according to X an equivalence relation is Y', then I would have the same Y for different Xs; in other words, I would have the standard account of equivalence relations with its standard defining properties arranged in order of X's priority. Thus, to bring more fundamental differences to the fore, I shall scrutinize each treatment of equivalence relation in more detail. I shall go through the examples that they have provided for us and read what have been told between the lines of these examples rather than what have been *abstracted* from them.

After this detailed discussion of Skemp's and Dienes' account of the subject, an *alternative definition* of an equivalence relation is introduced. The latter definition is based on Freudenthal's (1966) textbook on logic.

It is worth stressing that my choice of the relevant literature shows that I am interested in the ways that people experiences equivalence relations and partitions. I shall also add that although I read these works before, during and after analysing the data, the following review was written after the next two chapters concerning the results of this study. As a result, my reading of the literature has been shaped by what I have learned from the students' experiences of the subject.

4.2 Skemp

> This [Equivalence] is one of the ideas which helps to form a bridge between the everyday functioning of intelligence and mathematics...
>
> (Skemp, 1971, p.173; square bracket added)

To reveal the meaning of this claim I shall refer back to Skemp's account of another *basic* idea that proves to be closely related to the idea of equivalence, i.e. the idea of a set. Earlier on, Skemp (ibid, p. 144) says:

> The importance of the idea of a set, and of the related ideas ... is that they form a bridge between the everyday function of intelligence, and mathematical thinking. From one side, the concept of a set is simply a recognition of something we do all the time, when we classify the things we encounter. 'What's this?' means 'To what class, or set, of objects does this belong?' But once made explicit, the idea of a set... will be found among the most helpful of all in clarifying the elementary, and many of the advanced, ideas in mathematics.

Let me separate out and adapt some parts of the above quotation and use it as a template for the idea of equivalence.

It forms a bridge between the everyday function of intelligence, and mathematical thinking. From one side, the concept of equivalence is simply a recognition of something we do quite often, *when we sort a given set of objects further into sub-sets which are alike in some way*. But once made explicit, it proves to be fundamental.

Taking this dual functionality into account and in accordance with his first principle of the learning of mathematics[20], Skemp provides us with (everyday) examples before defining the intended ideas mathematically.

[20] According to Skemp (1971, p.32), the two principles of the learning of mathematics are:

- Concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by arranging for him to encounter a suitable collection of examples.
- Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner.

> {Coins in my pocket} can be sorted into sub-sets of coins having the same value. {Pots of paint in a certain shop} can be sorted into sub-sets of the same colour. {Novels in the local library} can be sorted into sub-sets of novels by the same author. A method of sorting will be incomplete if there are any objects in the parent set which do not belong to one of the sub-sets; and ambiguous if any object can be assigned to more than one sub-set. So we say that every object in the present set must belong to one, and only one, sub-set. A set of sub-sets which satisfies this requirement is called a partition of the parent set. (ibid, p. 173)

Then he (ibid, p.174) considers two sorting methods with which the elements of the parent set can be sorted into sub-sets.

First, "we can start with some characteristic properties, and form our sub-sets according to this".

Second, "we can start with a particular matching procedure, and sort our set by putting all objects which match in this way into the same sub-set...this method is frequently used when encountering new objects". The particularity of this matching procedure is in its "exactness", i.e. having an exact measure for the sameness; a necessity that if it is achieved, the matching procedure is called an *equivalence relation*. The exactness of the matching procedure also accounts for the *transitive* property. And the importance of the latter is that "*any* two elements of the same sub-set in a partition are connected by the equivalence relation" (ibid, p.175).

Here, Skemp refers us to the appendix (ibid, p. 184), where in addition to the transitive property we can find two further properties of an equivalence relation in a more *formal* fashion:

> Let R stand for any relation, and let a, b, c be any of the set of objects for which this relation is defined. Then R is an equivalence relation iff, whichever objects we choose,
>
> a R b and b R c implies a R c
>
> a R a
>
> a R b implies b R a

In words, an equivalence relation is (i) transitive, (ii) reflexive, (iii) symmetric. Foreshadowing the discussion in Section 6.2, I shall say that there is a subtle conceptual shift between Skemp's explanatory account of the transitive property and the standard account of the transitivity. In the former the focus is on the *function* of the transitivity, while in the latter the focus is on the formal *definition* of it. As we will see later on, in the context of equivalence relations, this distinction is extremely important. However, for the time being, I shall continue with Skemp's explanatory ideas eloquently leading to a *fundamental theorem* that relate equivalence relations to partitions. This fundamental theorem shows that the two sorting methods mentioned above are two different ways of guaranteeing that "*any* two elements of the same sub-set in a partition are connected by the equivalence relation". And it is guaranteed "whether the sorting is done by method one or by method two. If by method two, it follows directly from the transitive property. If by method one, we can always find an equivalence relation between any two elements of the same sub-set (ibid, p. 175)." For example, for {Coins in my pocket}, partitioned into sub-sets of coins having the same value, our equivalence relation is 'having the same value as...'. In general, for any set at all, and for any partition at all, the equivalence relation is:

'...is in the same sub-set as...' .

Now, we only need one additional term to remind us of the close connection between an equivalence relation and a sub-set belonging to a partition; to indicate this connection, a sub-set belonging to a partition is called an *equivalence class*.

To summarize:

> Any equivalence relation, which can be applied to all elements of a given set, partitions the set into equivalence classes. And any partition of a set can be used to define an equivalence relation. (ibid, p. 175)

As it can be seen the heart of Skemp's account of the subject is a "particular matching procedure". But, he does not explicitly explore the subtle interconnection and interference between "a matching procedure" and the "similarities" which enable us to *abstract*. The only mention of this is given earlier as a warning in a parenthesis:

> *Abstracting* is an activity by which we become aware of similarities (in the everyday, not mathematical, sense) among our experiences. (ibid, p. 22)

Eight years on (1979), his descriptions add to confusion where he introduces *'functional equivalence'* as a kind of similarity which underlies our ability of successive abstraction: "that of repeating the process of concept formation at a higher level" (Skemp, 1979, p. 119).

> To buy a ticket on the underground, we need a 10p coin. Any 10p coin will do...It does not matter which particular 10p ticket we have...Almost every goal state to which we direct our actions, and the sub-goals from which we choose our paths, can be achieved within an equivalence class which usually offers more than one possibility: often many more. The examples used to introduce this concept of functional equivalence are equivalence classes of objects. (ibid, p. 129)

As an endnote, he adds:

> The term 'equivalence class' as here used is...consistent with its mathematical use, and the relationship between any two members of the same equivalence class is an equivalence relation in the mathematical sense. (ibid, p. 142)

Sometimes functional equivalence and *perceptual similarity* come together:

> They often meet at the perceptual level: when we want an apple (as against that apple), we identify members of this equivalence class by their physical resemblances to each other.

Sometimes they diverge:

> The man behind a window in the ticket office, and the coin-in-the-slot machine, are ...different perceptually. Their functional equivalence is recognized by the class-concept ticket-vendor. (ibid, p. 130)

It seems that the idea of equivalence that was supposed to exemplify Skemp's learning theory also underlies his theory! We are again faced with this circularity in Dienes' theory.

4.3 Dienes

For Dienes, the notion of relation is not simply one of the basic concepts of mathematics; it is what mathematics is all about.

> If any concise definition could be given of mathematics, it would probably be that it is the study of relations, by which we mean that we are relating something to something else. (Dienes, 1976, p. 1)

As far as equivalence relations and partitions are concerned, what he says is quite similar to what we have already learned from Skemp. However, he occasionally attends to some aspects that were not present in Skemp's explanatory account of these ideas; in particular, he has a much lengthy discussion on the *reflexive* and *symmetric relations*, two kinds of relations that together with the transitive relation constitute the formal account of equivalence relations. More interestingly, even when Skemp and Dienes seemingly talk about the *same* thing, they occasionally bring forward different aspects of that thing. Moreover, since Dienes applies a different learning theory, the nature of the examples that he employs are rather different from Skemp's examples. For all these reasons, it is of great interest and importance to compare these *different* accounts of the *same* ideas. Doing so, in a way, I phenomenographically analyze the literature on equivalence relations and partitions.

I shall start with saying a few words about Dienes' theory of mathematics learning. In effect, as far as the *foundation* of mathematical learning is concerned, Dienes and Skemp have much in common; both of them are basically concerned with the formation of mathematical concepts. Moreover, more or less both have the same account of what a mathematical concept (or a concept in general) *is*; a concept is the *defining property* (Skemp, 1971, p. 22) or the *common feature* (Dienes, 1960) of a *class*. However, they have slightly different views on what constitutes a suitable class for the formation of the intended concept. While for Skemp a suitable class (for the formation of the intended concept) is mainly a collection of examples accompanied by non-examples, for Dienes, a suitable class is mainly comprised of certain *games* where the ingredients of the intended concept are *played* with. As seen above, by collecting a suitable collection of examples Skemp kept his principle of the learning of mathematics intact when illustrating the ideas of equivalence relations and partitions. In a similar vein, Dienes examines the learning of the concepts of equivalence and partition from his own vantage point. His approach allows me to address his theory in the context of the concepts of my interest, as I did so regarding Skemp's theory. However, in doing so, I could not attend to all pro and cons of his theory, as I could not do so regarding Skemp's theory. Having this limitation in mind, I shall turn to Dienes' *programme* to teach the ideas of equivalence relations and partitions.

For him an equivalence relation is a very particular kind but a very important kind of relation in *the science of relations* (see the opening quotation at the beginning of the present section). Moreover, it is equally important to notice that "mathematics is the study of relationships in the abstract" (Dienes, 1971, p. 130). Thus, if mathematics is the study of relationships in the abstract, the aim of teaching mathematics is to get children to handle relations between abstractions. However, as soon as Dienes takes such a radical stand, equivalence relations and the very process of abstracting share a common feature, that is to say, both emerge from the idea of *likeness*. Having said so, I shall also add that Dienes himself does not explicitly point to this important common feature, though he explicitly separates the *likeness relations* (equivalence relations) that give rise to a partition from the others. However, his examples imply that abstracting (*isolating* a certain *property*) has more in common with the former than the latter. As a matter of fact, his programme starts with various games to get "children discover in which ways objects are similar to each other and in which ways they are different from each other" (Dienes, 1976, p. 3). Some of these games are aimed at isolating a certain property, and some are aimed at partitioning a certain set into disjoint sub-sets. As I have an interest in the idea of partition (and the related idea of equivalence relation), I shall continue by considering the latter games; though, from time to time, these two kind of games are inevitably tangled together.

The first set of games is basically to put Skemp's first method of sorting into practice. These games are aimed at getting children to split a given set by taking a likeness relation (in Skemp's words, a characteristic property) into account. For example:

> We could ask each child to write the name of the street in which he lives and then say a child is like another child if he lives in the same street as this other child, and is unlike another child if he lives in a different street. Clearly, unless you have two houses, one in one street and one in the other, and you are living in both of them, the class will be divided into as many sets as there are streets in which children in the class live. If there are seven streets on the list, then there may be five or six children in the same street; but there might be just one child living in another street. This does not matter, since those children whom I pick from the same set will be living in the same street, and if I pick a child from one set and a child from another, they will of course be living in different streets because if they were not they would have been put in the same set. [In this way] we have found a likeness relation which splits the objects (in this case, children) into disjoint sets or classes. We have made a *partition* of our universal set into disjoint sets or classes. (Dienes, 1976, p. 5)

Then, to induce children to gather the *mathematical essence* of these ideas it is desirable to use *the multiple embodiment principle* or *perceptual variable principle*, that is to say, it is desirable to "vary the perceptual representation, keeping the conceptual structure constant" (Dienes, 1971, p. 30). For example, if we wished to use geometrical attributes:

> We could draw a maze on the floor in which there was only one 'inside' and one 'outside'. We could then place certain children in certain parts of the maze. It could then be asked which children could be reached by which children without going over the separating lines. It is best to draw the maze in chalk on the floor and not at first on the blackboard. If it is drawn on the blackboard the children cannot stand on it. The 'universe' here would consist of some selected positions of the children, and the relation would be one of accessibility. A child would be *accessible* to another child if he could go to this other child without crossing a line. It would become clear after a while that some children are accessible to all other children in a certain subset of the selected positions. In another subset of the selected positions a different set of children would be accessible to each other. *No member of one subset would be accessible to any member of the other subset.* This means simply that some children have been place *inside* and other children have been place *outside*, and the set of children *inside* are inaccessible to the set of children *outside*. But the children belonging to the set of children inside are all accessible to each other, and the children belonging to the set of children outside are likewise all accessible to each other. (Dienes, 1971, pp. 137-138)

Consider that to turn sorting by the characteristic properties to a *sorting game* that can be played in a *concrete fashion*, Dienes not only needs to start with the intended criterion for partitioning, but also needs to partition the universal set before getting children to do so. This aspect of his games is embodied in the 'maze game' by deciding the characteristic properties *inside* and *outside* beforehand, and in the 'street game' by asking each child to write the name of the street in which he or she lives. Later on, Dienes suggests another game that in a sense is an attempt to break this unfortunate situation in which the intended equivalence classes are dictated to the children in a way. Before discussing this new game that put Skemp's second sorting method into practice I shall point to a subtle difference between what Dienes sees in his splitting games and what Skemp sees in his sorting examples.

Dienes has a *global* view of the elements of each sub-set (equivalence class), that is to say, for him *all* the elements of each sub-set are connected to each other. For example, in the 'maze example', he says that "*some* children are accessible to *all* other children in a certain subset of the selected positions", and "the children belonging to the set of children inside are *all* accessible to each other, and the children belonging to the set of children outside are likewise *all* accessible to each other" (emphasis added). Compared to Dienes' global view, Skemp provide us with a *local* view of each equivalence class in which the focus is on the connection between *any two* elements. For example, adapting Skemp's language for the 'maze example', we can say that *any two* children belonging to the set of children inside are connected (accessible) to each other. Having said so, I shall also add that there is no difference between Skemp's and Dienes' account of an equivalence class per se. Here are their *definitions* of an equivalence class:

> Skemp (1971, p. 175): A sub-set belonging to a partition is called an *equivalence class*.
>
> Dienes (1976, p. 7): The sets into which the universal set is split by a partition are called *equivalence classes*.

Let me come back to Dienes' games. As mentioned before, the first set of Dienes' games (that were based on certain predetermined characteristic properties) suffers from the fact that certain intended equivalence classes are imposed on the children. Although Dienes does not explicitly mention this aspect of his first set of games, the following games implies a *partial* cure for this unmentioned aspect. In effect, this game is in line with Skemp's second sorting method, i.e. sorting by an *exact matching procedure*.

> One way of getting children to realize how equivalence classes are constructed out of equivalence relations is to get them to construct these classes themselves, out of an equivalence relation. We do not even have to give the equivalence relation, they can find this relation by relating an object to another by a likeness. For example, we might pick two people out of the PEOPLE LOGIC SET[21]. Let us take a small green standing girl and a large green sitting girl. We

[21] People Logic Set—the set embodies 48 elements, each piece having one of the following attributes: male or female; adult or child; sitting, standing or walking; green, red, blue or yellow. (Dienes, 1976, ix)

could ask the children in what way these two people alike. They will probably say straight away that they are both girl and they will soon add that they are both green, but that one is sitting and one is standing, and one is small and one is large...let us suppose that a group of children has decided that *small green standing girl* and the *large green sitting girl* are alike in two ways, namely that they are both *green* and that they are both *girls*. Then we can ask if there are some other people who are like the two who have been chosen. In other words, are there any other green girls? They will soon collect all the rest of the green girls. Some will be small, some will be large, some will be sitting, some standing, and some walking. (Dienes, 1976, pp. 7-8)

Now comes "the most difficult part" of the game that completely distinguishes it from the first set of the games. Assuming that the children have already collected all the green girls, Dienes (ibid, p. 8) asks the following question:

'Children, try to make another set of people in which the people *are like each other in the same way as* the people are like each other in the first set we have just made.' (Emphasis added)

Unlike the first two games in which the dividing likeness relations were embedded in the structure of the games, here the instruction asks for *building up* a likeness relation that is of a *higher order* than those which the children have already played with. To put it simply, while the green girls have been placed in a set because they are all *green*, and they are all *girl*, the instruction asks for a likeness relation in terms of *colour* and *sex*. Having *greenness* and *girlness* in mind, the game has already come to an end, since the children have already placed all the green girls together. But, if they *jump* from the realm of *first order attributes* (e.g. greenness, girlness) to the realm of *second order attribute* (e.g. colour, sex), then it may occur to them that:

The way in which the people are alike in our first set is that they are all the same colour and they are all of the same sex. This *can be repeated*. The children could for example, put together all the blue boys, or all the red girls, and so on. (ibid, p. 8)

Although Dienes warns us to be patient with children while they are passing from first order to second order attributes, he does not suggest any games to help getting them discover this higher order likenesses. Moreover, it seems that Dienes entirely ignores how grasping the *reflexive* property is relying on passing to second order attributes. See section 1.6.1 in this regard. As a matter of fact, as soon as Dienes starts to study *the properties* of an equivalence relation his discussion leaves us with even more clearer untouched issues. This will be discussed in the next sub-section.

Reflexivity, Symmetry and Transitivity

Let us suppose that (1) the children have just successfully finished the three mentioned games, i.e. they have just split the given universal set into disjoint sets, (2) in each case they have found (or have been explicitly told) the likeness relation which has partitioned the set, (3) they have also learned that not all likeness relations give rise to a partition.

Now comes the most unsatisfactory part of applying Dienes' theory to this particular situation. How can we get children to realize the properties of the likeness relations that partition a set? What is the *play stage* for the properties of equivalence relations? Dienes tackled this question at least twice; once in the third edition of his book, "Building up Mathematics" (1963); once thirteen years later (1976), in another book devoted entirely to relation, "Relations and Function". However, in none of them does he have a satisfactory answer for these questions. In both of them, by using the equivalence relations of his sorting games he simply introduces the properties of equivalence relations one by one. For example, consider "the relation of living in the same street". He simply says that it is a *reflexive* relation because "*we* will find that every person lives in the same street as himself" (Dienes, 1976, p. 57; emphasis added). Moreover, it is also a *symmetric* relation; "if Bill lives in the same street as Dick, then Dick will live in the same street as Bill. This is true of any two people who live in the same street.

So 'living in the same street' is a symmetric relation" (ibid, p. 58). It is also *transitive* because "if a person lives in the same street as a second person, and the second person lives in the same street as the third person, it is obvious that the first person lives in the same street as the third" (ibid, p. 60).

In a more or less similar fashion, he *shows* each equivalence relation in his sorting games possess these properties. Doing so, he certainly violates his own principle by which the educators are invited to give the child a *"freedom of choice"*:

> It is impossible to allow a child to select his own mathematical curriculum or methodology. What he must have is freedom to act within a certain discovery situation. Of course, we are none of us really free since most of our actions are predetermined by circumstances and by various restrictions and commands; none the less, within these limits we have a power of choice. The job of the educator is to give the child a freedom of choice through which, whatever he chooses, mathematical learning will take place. (Dienes, 1971, pp. x-xi)

I am not criticizing Dienes' theory, since I have not provided enough ground to do that. I have only tried to see a part of his work in the light of another. In this regard, it is reasonable to ask what "freedom of choice" might be in the case of equivalence relations. To have a tentative answer, let me go through a possible scenario. *Assume* that Dienes' programme has encouraged and accustomed the children to consider the properties of the equivalence relation at hand. Now, the children are free to choose from some of the properties that they have supposedly noticed, taking it into account that their choices should amount to the fundamental function of an equivalence relation, i.e. it splits the universal set into disjoint sub-sets. Using the terminology of Dienes' six-stage theory of learning mathematics (see for example, Dienes, 1971 or 1973), we can say that the children are now playing in the sixth stage, i.e. the stage of *formalisation* where "some properties described are chosen as fundamental, the choice being to some extent arbitrary.

Then results are found according to which we can 'reach' the other properties from the fundamental ones" (Dienes, 1971, p. 36).

The above scenario is mine rather than Dienes. As a matter of fact, Dienes himself supposes that the so-called properties of an equivalence relation are embodied in the sorting games. That is to say, he supposes that each likeness relation in each one of the sorting games embodies the three properties of being reflexive, symmetric and transitive. Using his six-stage terminology, the children are in the third stage where they are encouraged to find the *common structure* embodied in the games. To help them see the *commonality of structure*, Dienes makes use of the *dictionary technique*. "This is a way of 'translating' one embodiment into another, while leaving the abstract properties embodied unchanged by the translation" (1971, pp. 32-33). Regarding the mentioned games, here is a "dictionary".

Reflexivity	Symmetry	Transitivity
Every person lives in the same street as himself.	If A lives in the same street as B, then B lives in the same street as A.	If A lives in the same street as B and B lives in the same street as C, then A lives in the same street as C.
Any child is accessible to himself.	If A is accessible to B, then B is accessible to A.	If A is accessible to B, and B is accessible to C, then A is accessible to C.
Each person has the same colour and sex as himself.	If A has the same colour and the same sex as B, then B has the same colour and sex as A.	If A has the same colour and the same sex as B and B has the same colour and the same sex as C, then A has the same colour and the same sex as C.

Given this, many questions remain unanswered. Why on earth should the children come up with the *standard* properties of an equivalence relation, i.e. reflexivity, symmetry and transitivity? What kind of games can be played to get children to realize these properties in the first place? For example, what aspect of the 'maze game' could lead a child to consider that he or she is accessible to himself or herself? And the same kind of questions can be asked in regard to the symmetry and transitivity. These questions have been asked within Dienes' framework, but Dienes does not provide us with any possible answer. Instead of providing any tasks to get the children to *form* the *intended* properties, he simply attributes the intended properties to the relevant likeness relation in each one of the sorting games. It seems a serious gap in Dienes' programme. However, foreshadowing the result of the present study, I shall say that he could not fill that gap even if he tried to do so. In general, neither his programme nor any other can do so if they are based on the sorting games. This claim will be examined more closely in the next chapters. For the time being, I end this section with another game that not only embodies all the ideas in the present section, but also unintentionally plants seeds of doubts about such programmes.

> A similar game could be played with younger children by taking sets of objects as members of the universe. Each set of objects could be put into a container and the sets in the containers would form the members of our universe of discourse. We could then say that two sets, whose members can be paired off, are related to each other by our relation. Every container with two objects is related to every other container with two objects. Any set with just one object is related to any other set with just one object. But our set of two objects will not be related to any set with three objects or four objects or any number other than two. This game creates in a concrete fashion the equivalence classes from which the natural numbers 1, 2, 3, 4, and so on, are constructed. In every equivalence class the sets are equivalent to each other, and the common property binding the members of these equivalence classes together is their number property. This is what we call a *natural number*. It will readily be seen that this relation is symmetric, since if container B has as many objects as

container A, then container A has as many objects as container B. it is reflexive, since container A has as many objects in it as container A. it is also transitive, as if A has as many as B, and B has as many as C, then A has as many as C. We are thus dealing with an equivalence relation, and we have constructed the corresponding equivalence classes which lead to the concept of natural numbers. (Dienes, 1971, pp. 139-140)

In the following chapters we will see that the phrase "it will readily be seen that" should be replaced by "it will deliberately be chosen that"! Dienes' examples unintentionally give some grounds for this replacement. For example, where he says "Every container with two objects is related to every other container with two objects" he plants the seeds of the property of *F-transitivity* that can be used as one of the defining properties of an equivalence relation. This latter property is introduced in the next section.

4.4 Freudenthal

In the process of changing the direction of this study perhaps nothing was as influential as Freudenthal's (1966, p.17) definition of an equivalence relation, i.e. a relation (say $\sim$) that possesses the following two properties:

(1) $A \sim A$

(2) If $B \sim A$ and $C \sim A$ then $C \sim B$.

Expressed in words, the two laws (1) and (2) will be:

(1') Every object is equivalent to itself (*reflexivity*).

(2') If two objects are equivalent to a third, then they are also mutually equivalent (*transitivity*).

And immediately he mentions that from (1) and (2) also follows:

(3) If $A \sim B$ then $B \sim A$.[22]

In words:

(3') If an object is equivalent to a second object, then the second object is also equivalent to the first (*symmetry*).

[22] As we see on replacing B by A, A by B and C by B in (2').

He again emphasizes that "actually, the first two properties are sufficient" to define an equivalence relation.

At first glance, it seems just an alternative definition. However, as far as the chronology of this study is concerned this definition came as a discovery. Remember that the initial aim of this study was leading the students to *the* defining properties of an equivalence relation. When doing the preliminary study I realized that as far as the transitivity is concerned the situation is far from being satisfactory. Then, I attempted to design some supplementary tasks in order to direct the student's attention to the missing concept of transitivity. But, as discussed in Section 1.8, I *gradually* gave up these unfortunate attempts and started bracketing my understanding of the situation (see Section 2.3). Meanwhile, in the process of interviewing and simultaneously analysing the data I observed time and again that certain elements were assumed to be *matched* without being *directly* compared. At the time, I used the term "triangularity" (Asghari, 2004b, p. 69) to capture this particular way of matching the elements involved. Generally speaking, it was a way of guaranteeing the equivalence between two elements based on their equivalence with a third element. "Triangularity" could mean the standard transitivity (i.e. If a is related to b and b is related to c then a is related to c), or something else that I had *discovered* in the students' way of tackling the situation (i.e. If a is related to b and a is related to c then b is related to c). Now in a state of great excitement I had found the latter, in Freudenthal's textbook on logic, as one of the defining properties of an equivalence relation. The immediate consequence of these more or less simultaneous discoveries was recognizing that not only in our situation, but also *in any other situation based on 'equivalence* *relations', there is no way to bring 'transitivity' up unless it is taught* (Asghari, 2004b, p. 70). This was the final nail in the coffin of my first plan.

As far as the terminology was concerned, later on (following a private communication from Bob Burn) we named this latter property 'F-transitivity'[23] (If B ~ A and C ~ A then C ~ B) to distinguish it from the standard transitivity, although Freudenthal himself called that 'transitivity'. The importance of F-transitivity will be revealed in the fullness of time. For the time being, suffice it to say that F-transitivity is equivalent to standard transitivity when dealing with equivalence relations, but it is not satisfied by an *order* relation.

Talking of transitivity and order, I shall also say a few words about the *local grasp of the linear order by the law of transitivity.*

Transitivity is the link between local and global order

Five years after his discussion on 'global and local organisation' (see Section 2.8), Freudenthal (1978) introduces another pair, global and local perspective. The former "is only an extreme case, at quite high a level" (ibid, p. 252) of the latter. However, both pairs are being discussed at the same level, i.e. at the level of examples. I shall mention one of these examples and add another example from the present study, hoping that they make my later usage of this distinction clear. Freudenthal's example concerns itself with the linear order:

> Everybody grasps and experiences globally the linear order on the number line...For millennia mankind and even mathematicians have been content with the global grasp of linear order, and for the great majority this has not even changed today. Mathematicians, of course, know that this globally given order can be locally grasped by the law of transitivity and a few others, and can be axiomatically described. From this cognition many didacticians draw the conclusion that the linear order would and should be constituted starting with transitivity...In a mathematical system the law of transitivity might be at the basis of linear order; developmentally transitivity is a consequence of linear

[23] 'F' in F-transitivity stands for Freudenthal.

> order, and the axiomatic view is one of those inversions I called anti-didactic. The mathematician is right to be proud that by the local grasp of the linear order he makes the extension possible to partial order, but didactically this is entirely irrelevant. (ibid, pp. 254-255)

In the light of the distinction between local and global perspective it is now possible to give an interesting interpretation of the box concept. Let us follow Tyler on his way of putting forward the box concept. He starts with a global grasp of the sameness of two columns.

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Tyler:  How do I say that columns must be the same
        mathematically?
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Then he answers this global question locally, because a local property seems to be more mathematical.

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If (x1, y1) and (x1, y2) and (x2, y1) then (x2, y2)
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However, he is not quite in agreement with Freudenthal regarding his preference for the local definition.

> Yet in modern mathematics global definitions are much preferred above local ones provided they are as exact. So if one would start today from scratch, each mathematician would choose the global definition and at most mention the second definition as a cheaper one, but it seems that even in mathematics a thousand-year-old tradition is not easy to break. (Freudenthal, 1978, p. 256)

In the sections to come I will add another example to the Freudenthal's collection. As a matter of fact, the example that I will give is somehow related to the idea of F-transitivity. But, I was not aware of this example until I analyzed the students' works. Thus let me reveal that in due course. It is interesting that Freudenthal himself missed this latter example in his search of examples of local and global perspective.

> Examples of the relation between local and global perspective in the learning processes or in steering learning processes might mean great progress. It is a big problem how to find them. (ibid, p. 257)

In Section 9.9.2, I will discuss one possible reason for his ignorance of such an example that is closely related to his own definition of equivalence relations.

4.5 Summary and Afterword

In this chapter mainly three complementary ways of experiencing equivalence relations has been discussed. In particular, an alternative, and nowadays non-standard, definition of equivalence relations was introduced. According to this definition, an equivalence relation is a relation that possesses the properties of reflexivity and F-transitivity. It was also mentioned that the latter property first was discovered among the students' ways of tackling the Mad Dictator Task. Thus it is expected that different aspects of the F-transitivity will be examined in the chapters to come, and in particular, in the context of the students' works in this particular situation.

As an afterword, I shall add that any discussion about equivalence relations may not be complete without taking a rather more fundamental concept into account. This latter concept is the concept 'set'. In today's mathematics, the notion of set precedes the notion of relation in general, and the notion of equivalence relation in particular. Though it is mathematically related, my brief discussion on sets is closely tied to the students' experiences of the Mad Dictator Task. Thus I have decided to postpone this discussion until I have given a detail account of the students' experiences in the present study (see Section 5.4).

Part three: Results

Introduction

In this part, the results of the study will be presented. It is the core of this study that everything else is somehow related to it. It is an end in itself, but at the same time it is a start for the rest of this thesis.

Overall this part concerns itself with the variation in the ways that the students tackled the tasks involved. Here I will take advantage of the literature in two mutual ways: first, to transcend this particular situation as such by using the concepts applicable across similar situations, second, to give a clear picture of what seems to be overlooked in the literature. Once again, it is worth saying that my use of the literature is not to measure students' ways of experiencing the given situation by the possible standard ways introduced in the literature, I only use the literature to situate, as properly as possible, the present discussion in a more familiar context.

This part includes two chapters, Chapter 5 or Results I, and Chapter 6 or Results II.

In a sense, and only in a sense, Chapter 5 has echoes of 'partitions'. Here I will enumerate and examine three overarching ways of handling the situation. The first one is characterized by the lack of the idea of 'grouping' (or in a certain sense, 'partitioning'). The second one is distinguished by the presence of one, and only one, single group of related elements. The third one resembles *our* notion of 'partition'. Towards the end of the chapter I scrutinize the idea of "group" per se and discuss to what extent it is different from a "set".

In Chapter 6, I turn my attention from 'a *group* of related elements' to the *relations* between the *elements* of each group. Hence what was in the background in Chapter 5 will be brought to the fore in Chapter 6. Here I will illuminate certain ways of matching the elements of each group. In particular, three important ways of connecting the elements to each other will be discussed. These ways are *transitivity*, *F-transitivity* and *symmetry*. The last section of this chapter is a discussion about the results as a whole, linking Chapter 6 to Chapter 5.

Chapter 5

Results (I): Grouping

5.1 Introduction

This chapter concerns itself with three overarching ways of handling the situation. I introduce three main conceptions and the ways that they are related to each other. These conceptions are: *matching conception*, *single-group conception* and *multiple-group conception*.

I illuminate the above conceptions and their relationship in the light of students' utterances and deeds.

Towards the end of this chapter, it will gradually be disclosed why in this particular situation the term 'group' has an advantage over the standard term 'set'.

5.2 Grouping

> A question that the reader might well have posed when reading phenomenographic studies...is : "At what level do the descriptions offered, the ways of experience, apply to the subjects of the respective studies? Do they apply to the individuals or to the group of individuals or to a wider population?" The answer lies in the fact that phenomenography focuses on variation. The objective of a study is to reveal the variation, captured in qualitatively distinct categories, of ways of experiencing the phenomenon in question, regardless of whether the differences are differences between individuals or within individuals. (Marton and Booth, 1997, p.124)

It is the answer that should well be kept in mind when reading my descriptions in the present chapter and the next chapter. They are not meant to categorize students per se. They only reflect my interpretation of the students' conceptions and the ways that they are different from each other. An individual could be the bearer of different conceptions (i.e., could be in different categories) at different times; Indeed, if it was other than this, *learning* (at least in its phenomenographic sense) could not happen.

As a result of analyzing the verbatim transcripts three qualitatively distinct ways of experiencing the tasks were identified; these three have been captured by the following three categories:

Matching Conception, in which a pair of elements (in our case, a pair of points, cities or columns), one at a time, matter to the students. At the best, the focus is on what is applicable to *any* two elements.

Single-group Conception, in which the focus is on only one single group[24] while all the other elements that do not fall into that group are treated as individuals. The elements in the focal group in one way or another are related to each other while all other elements are in the background as individual elements.

Multiple-group Conception, in which, "disjoint groups" are experienced; the groups have no elements in common and the elements of each group are related to each other in one way or another.

My main endeavour in this chapter is to illuminate the above categories and their relationship in the light of students' utterances and deeds.

5.2.1 Matching Conception

In this category, the focus is on the matching *procedure* between individual elements; what students experienced and described is in terms of the elements involved, without resort to a group and/or groups of elements. Before giving an example, it is worth saying that somehow or other the defining properties of an equivalence relation determine an exact matching; so do the defining properties of a visiting law. Consider that both of them are presented in a *group-free* manner. For example, concerning the latter, the second condition is "*for each pair of cities*, either their visiting-cities are identical or they mustn't have any visiting-cities in common"; or interpreting that in terms of columns, it says "either we have two

---

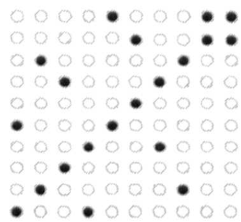
[24] The term "group" is used by mathematician in a very different way. In mathematics, a group is a set endowed with a "binary relation" having certain properties. However, as it clear from the context, my usage of this term is completely different from its mathematical usage. As a matter of fact, choosing this term has been inspired by the students who used the term when grouping (in its vernacular sense) certain elements with each other.

matching columns, or we have two completely distinctive columns" (Amit). Whatever the elements in focus are, these conditions are applicable (in action or in theory) to *any* two elements; no group is in sight.

The students exemplifying this category are quite different. For example, Ali (a first year high school student) mainly experienced the situation as a matching procedure, while Hess (a first year middle school student) manifested multiple ways of experiencing the situation. They both serve to illustrate this category because, as mentioned before, these categories map what has been experienced rather than one who has experienced.

I shall start with Ali when he was GENERATING an example. To generate an example he started putting some points on the grid haphazardly besides the points on the diagonal (Textfigure 8).

Ali: I choose the very first things [points] haphazardly, and then I am going to match the things that have not been matched up yet.



Textfigure 8: The first stage of Ali's matching procedure

Then he "started again". He paired up city 1 (column 1) with all the other cities (columns), one-by-one; if two focal cities (columns) had something in common, he matched them up, and if they had been already matched or they had nothing in common, he left them as they were. For example, in the above figure (see Textfigure 8), city 1 and city 2 have nothing in common and neither have city 1 and city 3. Thus, so far, there is no need for making any change in the figure. But, city 1 and city 4 have a visiting-city in common, namely, city 1. Hence, Ali added two new points to the original figure to match city 1 and city 4: the red points in Figure 10.

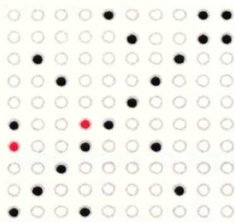


Figure 10: Ali matches city 1 and city 4

Then he did the same process on city 2 and paired it up and matched it up (if necessary) with all the other city after city 2, and so on. The result of this long process was Figure 11.

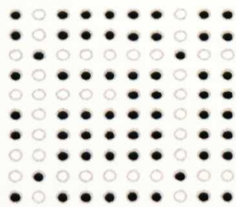


Figure 11: A stage of Ali's matching procedure

Not being sure that it is an example, he decided to "check from the start". More importantly, for the lack of any kind of grouping, he could not anticipate anything about the result; in particular, whether it would be a full grid (whole-grid example) or not.

```
Ali:      Now, we are checking from start; it is
          going to be full [having all points].
```

And he did so. Eventually the process ended with Figure 12 (below).

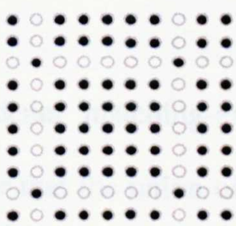


Figure 12: An example generated by Ali by using a matching procedure

Consider that the matching conception is reflected by the way that Ali GENERATED his own example, namely, a matching procedure.

The matching procedure exploited by Ali seems only a simple straightforward interpretation of the given conditions in terms of the columns. A less clear case is matching a pair of points. Hess exemplifies this latter kind of matching.

Hess is one of the students who tried to replace the given conditions with the *simpler* one(s). While generating a few examples including the diagonal, the whole-grid and Figure 13, he stated his quest for something more *general*, a certain "property that *all* of them have".

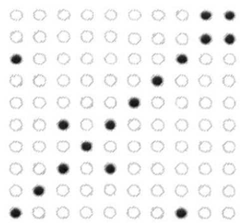


Figure 13: One of Hess' examples

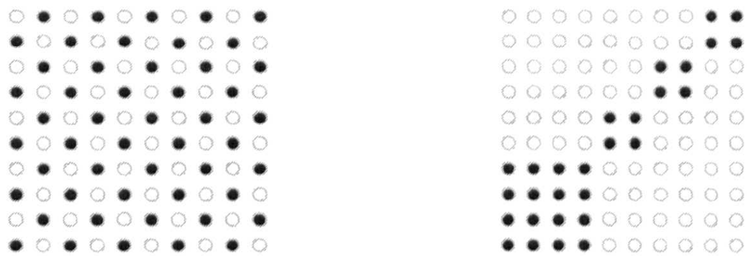
Hess: `In general, any two` *`symmetric`* `points that we choose we have one [example] about this [the diagonal].`

Unlike Ali, Hess envisages what an example might look like. However, like Ali, he only groups together two elements (in his case, two points), ignoring the possible connections that each one of these two elements might have with the other elements. As a result, some non-examples are counted as examples (see Section 6.4.1).

Let me also mention the students' matching-experiences of the union and the intersection tasks. First consider that the union of two examples may or may not be an example, but the intersection of two examples is an example. The union task proved to be readily in the realm of the matching conception. That is to say, all the students successfully tackled the union task, using a matching procedure. By contrast, the intersection task hardly was tackled solely based on a matching procedure, although it is potentially possible to give a purely matching argument for it.

Peter (first year politics students) exemplifies a way of handling the union task, using a matching procedure (Textfigure 9).

Peter: You are not going to (get a right example), one here, 9 and 10 are completely un-identical, in example four [the following figure on the left]; in the example nine [the figure on the right], they are identical. If you put those together those two cities are neither completely un-identical nor identical.



Textfigure 9: Peter tackles the union task

No students in this study developed a fully-fledge matching argument for the intersection task, although most of them attempted to do so at one time and another. Here is one of these attempts exemplifying the present category and also leading us to the next, i.e. single-group conception.

Arash (a middle school student) argues for his affirmative answer to the intersection task. That is, he wants to show that the intersection of any two examples is an example.

Arash: We can say that it is correct, since the columns appear in pairs. Now, in this pairs, I mean, in A and B [example A and example B] either this pairs are identical or not. If they are identical it means, in general, if in A and B two columns are identical then all the columns related to them are also identical. They again give us a correct example, because they are in pairs. That means there could be even three columns. In this way, the columns appear in pairs in the intersection. So, it's nearly been proved.

Interviewer: Nearly!

Arash: Yes, nearly! I have a little bit of difficulty in the last part.

So far, it seems that he has given an interpretation of the task rather than an argument. It is a kind of making sense of the situation that plays a crucial role in the argument to come. In particular, consider his shift from a pair of columns to three columns.

Arash: Consider one of these examples; remove completely *a kind of column* [a set of identical columns] from it. For example, we remove three columns; the example will come to no harm. That means it is still a correct example. Suppose there are three columns, here we had two columns like that; still we can remove one of the columns and still no harm will come to the example.

Although he shifted from a pair of matching columns to three columns, he did not maintain the connections between the cities constituting these columns. As a result, he envisaged the possibility of removing one of the columns without causing any damage to the examplehood of the figure. Meanwhile, he applied his plan to one of the examples. Soon he realized that it does not work in his favour.

Arash: No! No! It will be damaged. We must remove all of them with each other.

Before continuing with his argument I shall stress that each one of his words has several idiosyncratic meanings. As a result, I could barely follow his argument in the course of the interview. However, when analysing I had more chance to reveal the meaning of his idiosyncratic words and to gain a picture of his proof (see Section 3.6.3). Let me now come back to his argument. He does not simply ignore his failed plan. He amends it and exploited it again for the underlying cities constituting each one of the identical columns.

Arash: Aha! This is a set that has certain members. If we change it to a one-dimensional set [the underlying cities constituting each identical columns], now, certain members are removed. The other members—I mean those that are not removed—they are common to the two members [two underlying sets of related cities]; their kind is common [the columns including them are identical]. Now, if a member is removed completely there is no problem. Ok, we remove from this set certain members. In fact, we do this.

Overall he shifted his attention twice, first from a single matching pair of columns to three matching columns, and second, from the matching columns to the underlying cities constituting a single set of the members of these columns. In other words, he shifted from a matching conception to a single-group conception. The latter is the subject of the next section.

As the summary of the section, I shall say that the matching conception is characterized by the students' focus on the matching elements, one pair at a time. It could be somehow related to any other conception; after all, the second defining condition of a visiting-law is a matching condition. However, it alone is not an effective way of tackling the tasks involved.

5.2.2 Single-Group Conception

In the previous section, we saw that the simplicity of the matching procedure has a reverse effect on the way of dealing with the situation. That is to say, it makes it hard, though not impossible, to handle the different requirements of the situation. For example, adopting a matching procedure, Ali needed to check many different pairs of columns to see whether a figure at hand is an example or not. Arash's first impression of the CHECKING task also illustrates this point. Textfigure 10 shows how he first approached the CHECKING task, namely, by using a matching procedure. He is about to check Figure 14.

Arash: Ok, the *algorithm* for solving all of this examples is to make ten cities, then we say, for example, 1 has 1 and 9, 2 will be 2, 5 and 6; 4 will be…and so on, we say all of them like that, then we examine them *two by two*, it would be two to the power of ten cases, I don't know if *we can simplify it or not*.

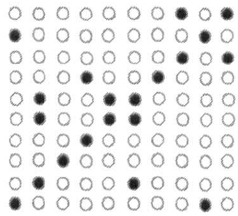


Figure 14: A non-example

Textfigure 10: Arash CHECKS Figure 14

Since Figure 14 is not an example, he soon found a non-matching pair (namely, city 2 and city 5) saving him from the cumbersome matching algorithm. However, the next figure to be checked is an example (Figure 15; Textfigure 11) demanding a full use of the matching algorithm (Textfigure 11).

Arash: I'm looking at the first column, then I'm checking to the end, then, I go down, the same two must… that means like this, ten to the power of ten, hundred cases, we must check hundred rooms, we check the first one, some of them are white…

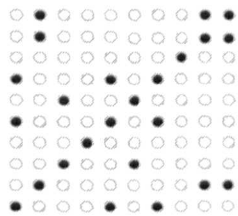


Figure 15: An example

Textfigure 11: Arash—CHECKING

Seeking a *better* way of dealing with the task, Arash finds a shortcut: he groups together certain elements.

> Arash- No, we consider those that are black [black points], we check them… then we check the whites [white points]…now we can do this in order, I say, for example, we check it [the first column], the blacks are here, for example, in these two columns [columns 5 and 7] the blacks are here too… now, we see that the blacks are complete… we check the whites as well, we put the blacks aside, and we are checking the whites… if they are completely identical, then they are going with each other, that means 1, 5 and 7.

As it can be seen he shifted from a matching procedure to a single-grouping procedure. The group of the related black points against a background of the unrelated white points is a metaphor for the category of the single-group conception, which *is delineated by the existence of one single 'group' while all the other elements that do not fall into that group are treated as individuals. The elements in the group in one way or another are related to each other.*

Each student in the present study could exemplify this category, except for Ali who all the way through experienced the situation as a matching procedure (see Section 5.2.1). I shall continue with some other examples. The first two examples are two students CHECKING the same figure (Figure 15; Textfigure 11). The next two examples illustrate how GENERATING can be also facilitated by a single-grouping procedure. Finally, I finish this section with a single-group argument for the intersection task.

Sarah (first year mathematics student) is checking Figure 15 (Textfigure 11).

> Sarah: I just gonna look through exhaustion. 1 visits 1, 5 and 7, because of the diagonal, therefore 5 must visit 1, 5 and 7; 7 must visit 1, 5 and 7 and *nothing else* can visit 7, nothing else can visit 5, nothing else can visit 1, which is just true; so they all are identical, 2 visits 2, 9 and 10; 9: 2, 9 and 10; 10: 2, 9 and 10, so they are all in common. 3 visits 3 and 6; 6 visits 3 and 6; 4 visits only 4, nothing else can visit 4, and all the others. I know this is an example just by exhaustion, *but I don't know whether there must be some kind of formula...*
>
> Interviewer: But you have for each pair of cities either...
>
> Sarah: If I take like, there I took 1 and 3; then 1 and 3 have nothing in common, say I take *one other town*, so anything that's not filled in on 1, when I go along, then I can see that *none of them* can visit 1 either, along the bottom, nothing, 1 can't visit *any those towns* either, *so it's distinct...*

The next example (Hess—CHECKING) in a sense has nothing to add to the previous two examples (Arash—CHECKING and Sarah—CHECKING). I have deliberately chosen this particular student (Hess) to stress once again that it is possible for the same student to experience different things at different times. Remember that Hess exemplified the category of the matching-conception. Now, he is exemplifying the category of the single-group conception. Later on, he will exemplify the category of the multiple-group conception!

Here is the way that he checked Figure 15 (Textfigure 11).

> Hess: I check [that] for 1, the columns 5 and 7 are for it; *they are as it is, and they are completely different from others.* 2 has commonality with 9 and 10, so I control 9 and 10, I see they are there. For 3, because it has 6, I control 6, and the same for 4, and the same for 5, and we don't need to check the rest.
>
> Interviewer: Why don't we need the rest?
>
> Hess: Because it is *symmetrical*, 6 is the same...Okay we check that [laughing], really what for... because if 6 was there previously, *if 6 was there in the five previous ones*, it's been checked before, if it is not there, so it has no commonality.
>
> Interviewer: But it could have commonality with the next points.
>
> Hess: Alright, we check it [laughing]; [see] each one has been checked before.

Consider that for *us* Figure 15 is composed of certain disjoint groups. However, it seems that what the above students were doing when CHECKING Figure 15 is basically a *sequential packing* (grouping) in which first certain elements are grouped together while all the other elements recede into the background, then some other elements hitherto being in the background are packed while all the other elements—including the elements of the previous group(s)—recede into the background, and so on until all the elements are *exhausted* (see Section 5.2.3).

Let me now continue with less controversial cases where a single-group is clearly dominant. The next two examples are concerned with GENERATING rather than CHECKING.

Kord (a middle school student) generated Figure 16 (see Textfigure 12) in his quest for a "*model*" that works.

```
Kord: Alright, now if we want to change city 1
and give it another city, certainly it finds a
commonality with that city; that means we must
give to that other city, city 1, so we put like
this [Figure 16]… this [very doubtfully], I think
it is (a model), let me think a little bit
more…I compare them one by one, I know the
first condition has been satisfied, we
satisfied it at the outset; now I'm looking at
the second condition by comparing the cities;
city 1 has no similarity with cities 3, 4, 5
and so on, city 2 also has no similarity, those
cities themselves, cities 3 to 10, have no
similarity with each other, as a pair, only
city 1 and 2 have similarity, okay it is for
each two cities, 1 and 2 are two cities, [thus]
it is correct.
```

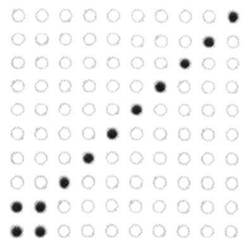


Figure 16: A two-by-two block square

Textfigure 12: Kord—GENERATING

Consider that city 1 *and* city 2 has not yet *formed* a group as such; in other words, as far as comparison between cities is concerned they are still treated as individual. However, the presence of a square led him to experience the points (making a square) as a *unit*, and subsequently relate the corresponding cities to each other without the need of checking each pair, one at a time. As a result, *GENERATING was facilitated* (Textfigure 13).

Kord: Now by drawing a diagonal line it satisfies the first condition; now the second condition … I'm coming from that side, from that corner upper right, if I modify my previous example, would it be a correct example. [Without hesitation] This [Figure 17] is also possible.

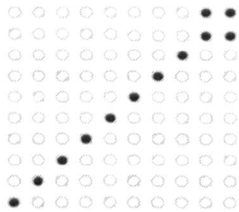


Figure 17

Textfigure 13: Kord—GENERATING

And he immediately continues (Textfigure 14):

Kord: Even we can do it like this [Figure 20], because it satisfies the conditions, it satisfies the first condition, and the second condition. See *all the three cities of 8, 9 and 10 are common [identical] with each other;* That means, this one can be an example [Figure 20], this one, if we take four of them, can be an example [Figure 19], this one can be an example [Figure 18]

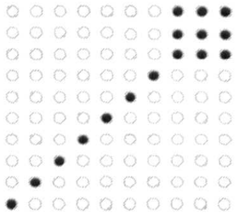


Figure 20

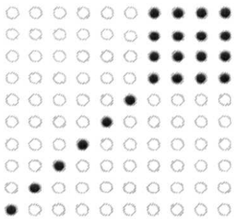


Figure 19

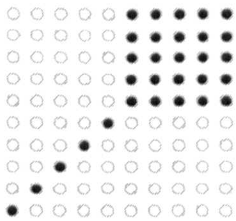
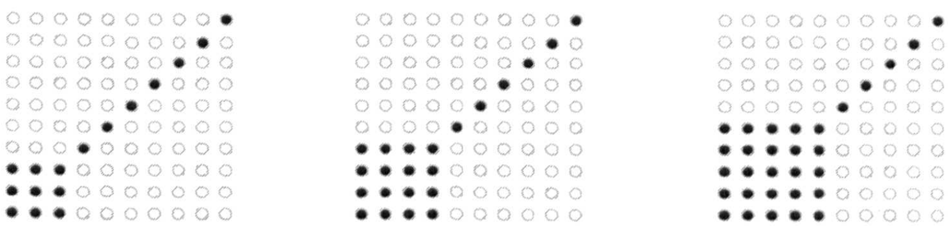


Figure 18

Textfigure 14: Kord—GENERATING

He shifts his attention to the other corner of the grid (Textfigure 15).

Kord: The same from that corner [lower left]… that means this squares are being filled, these squares…



Textfigure 15: Kord—GENERATING

Kord: Now by drawing a diagonal line it satisfies the first condition; now the second condition … I'm coming from that side, from that corner upper right, if I modify my previous example, would it be a correct example. [Without hesitation] This [Figure 17] is also possible.

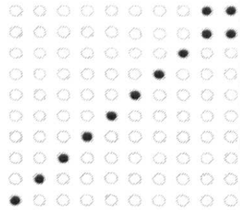


Figure 17

Textfigure 13: Kord—GENERATING

And he immediately continues (Textfigure 14):

Kord: Even we can do it like this [Figure 18], because it satisfies the conditions, it satisfies the first condition, and the second condition. See *all the three cities of 8, 9 and 10 are common [identical] with each other;* That means, this one can be an example [Figure 18], this one, if we take four of them, can be an example [Figure 19], this one can be an example [Figure 20].

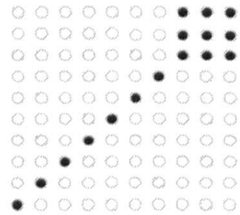


Figure 18

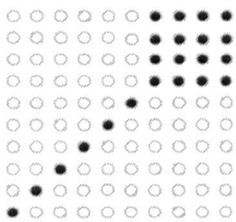


Figure 19

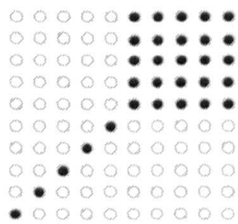
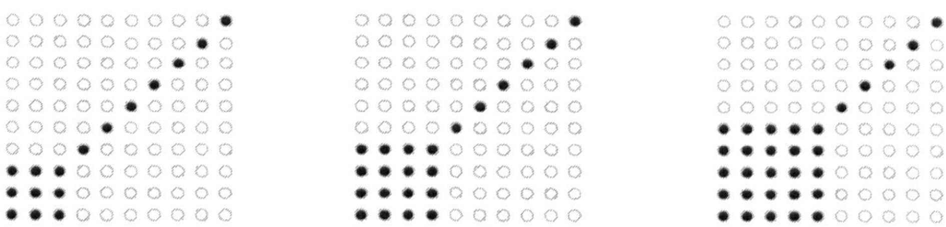


Figure 20

Textfigure 14: Kord—GENERATING

He shifts his attention to the other corner of the grid (Textfigure 15).

Kord: The same from that corner [lower left]… that means this squares are being filled, these squares…



Textfigure 15: Kord—GENERATING

Each of these has a square in one corner (lower left or upper right) but in no case did he put together a picture with squares in both corners (nor did he put a square somewhere else!). It was only the "union task" that compelled him to put two squares in the opposite corners of the grid (Textfigure 16). Even when confronted with the 'union task', he found it necessary to focus on one square after the other; while he checked whether the square that he has been focusing on has been properly packed, he unpacks the other square and treated its elements on a par with all other individual elements.

Kord: I think, this square and this square [make an example; Figure 21], because it has obeyed all the rules. See cities 1, 2 and 3, all cities that can visit [each other] are the same, are common [identical]; city 3 have no commonality with city 4, and *with other cities*. Cities 8, 9 and 10 like city 3, all visiting-cities, those that can visit each other are identical and they have no commonality *with other cities*, so this is correct.

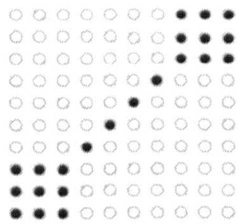


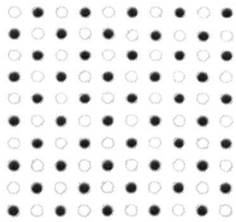
Figure 21: The union of two examples

Textfigure 16: Kord—Checking the union of two examples

Consider how, *for a moment*, city 3 stands for (represents) the cities it is related to. However, it is not the *group* of cities 8, 9 and 10 which is distinct from city 3; it is cities 8, 9 and 10 which on a par with the other individual cities (e.g. city 4) have no commonality with city 3; they have no commonality simply because there is no other point in the third column but those points in the first three rows.

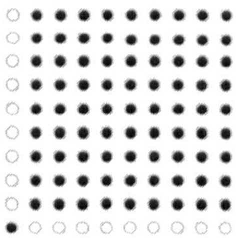
Let me give one more example in which the task at hand is GENERATING. This case in a sense is in the *border* of this category and the next category (multiple-group conception).

Chris (a first year mathematics student) has already generated the whole-grid and the diagonal as his first two examples. He is about to generate his third example (Textfigure 17).

> Chris: Only the city they are in, yeah. Then we can have, so if you took err, 1 and 2 again and said they were going to be different, you could, if you filled up all of 1, so then you can say 1 and 2 are different, but then 1 and 3 were similar; so *different and similar*, so that means that 1 would have 3 in it, and 3 would have 1 in it. And then if you did 4, you could say that, if you did 2 and 4, you could say they were the same, similar, so if 2 and 4 were similar, you'd have, 4 would have 2, and 2 would have 4. Like that. And then you could say that 1 and 4; therefore are different, because 1 and 2 are different, and 2 and 4 are the same. So you could have, an alternating pattern then, of well, so 1, 3, 5, 7, 9 are the same; and then, 2, 4, 6, 8, 10 are the same [Figure 22].
>
> 
>
> **Figure 22**

Textfigure 17: Chris—GENERATING

Then he immediately continued by generating another example (Textfigure 18).

> Chris: You could have 1 as, by itself and have all the rest the same, and *the total opposite*, 1... Ermm, if you said that 1 was just having 1 by itself, then, you could have that 2 to 10 contains 2 to 10... So then, 1 is completely different to all the rest, but then 2 and 10 are identical [Figure 23].
>
> 
>
> **Figure 23**

Textfigure 18: Chris—GENERATING

Now, he tries to *generalize* and verbalize what he has already observed (done). To do so, he also implicitly and partially underlies why the single-group conception is so pervasive!

> Chris*:* So what have I done? I've done you can visit every city (the whole-grid), you can visit only the city they are in (the diagonal), erm, that is the

> pattern of alternate, so 1, 3, 5, 7, 9 are the same, 2, 4, 6, 8, 10 are the same... So people from 1, 3, 5, 7, 9 can visit each other, people from 2, 4, 6, 8, 10 can visit each other, so you have to have *distinct sets. If you are starting to get some in common, you have to have distinct sets of what's in the set, and what's out of the set.* So if you pair two numbers up, like 1 and 3, then they are a set by themselves; and then the rest of them cannot be [related] to 1 and 3.

Consider, when the focus is on a single-group, or better to say, when the focus is on *relating* certain elements together (packing them with each other), all that first matters is what is (should be, decided to be) *in* the group and what is *out* of the group. *Outsiders* are simply outsiders having the same status defined in their relation to the focal group, that, they are not *in* it.

> Chris: *So you've got two sets, one's the common set, so that's where they're identical, if everyone's identical. And then you have one, which is the, erm, they've got none in common, and that's where they've all got to be...*

However, the outsiders as a whole, a whole that can be seen and *expressed* relatively easily, compel him to group the outsiders together. Thus, for example, if the focal group is comprised of the *numbers 1 to 5*, the outsiders, comprised of *the numbers 6 to 10,* could also form a group together.

> Chris: You can have, a mixture of one common set, and one not common set, where you say had 1 to 5 where the same and the 6, 7, 8, 9, 10 were completely different. So that means, 6, 7, 8, 9, 10 could only feature themselves in it...(or) you can have two sets which are common to each other, the odd numbers and the even numbers, or you could have... 1 to 5 and 6 to 10, they just happen to be the same... So you only get, basically you get two choices with your sets, and *how you do it.*

If one has the concept of *partition,* now it seems that Chris is only one moment away from exemplifying the next category (multiple-group conception). However, this moment never occurred in the course of his interview.

Let me finish this section with a single-group argument for the intersection task. There are many cases in which a single group implicitly or explicitly features in the students' affirmative argument for the intersection task. As an example I have chosen Amit who at the time was a first year mathematics student (for another example see Arash; Section 5.2.1). In order to make his final argument accessible I should give some *historical* details that in a sense are the backbone of his argument.

Amit's first argument for the *validity* of the intersection was based on the symmetry. He *proved* that the intersection of two valid examples is a symmetric figure, and assuming that the symmetry "is equivalent to the second law" he concluded that the intersection also is a valid example.

> Amit: I think the answer is true, that if we take any two valid examples and take the common points we also get a valid example.
>
> Interviewer: why?
>
> Amit: I think the reason is if we have valid examples, because of the second law, the second law ensures that we have some symmetry here… so if take this symmetry law as truth for every example, then if we take the combination of two examples, or their common points, then it'll also have symmetry as well.
>
> Interviewer: Why?
>
> Amit: Ok. Suppose we have, suppose that the points common to both of them do not have a symmetry, suppose we have a point which is on this side, but which doesn't appear on this side, that means in the two original maps, we must also have something which is on this side and not on this side, but that is not possible, because we started with the valid maps, we must have the symmetry, which means when we take the common points the symmetry also holds.

Consider that his argument is a matching argument in which the focus is on a pair of symmetric points regardless of the possible connections that each one of these two points might have with the others. My question about the validity of his argument led him to consider these connections besides generating a non-example symmetric figure (Textfigure 19).

> Interviewer: I understand that the common points must have symmetry, but I couldn't understand why it must be an example.

Amit: Ok! Suppose that we have symmetry, these two columns are identical or they are not identical, if they are identical then the second law holds, so we suppose that they are not identical, so let me take another example. Ok! We know we have symmetry, so if we've got two-six, we also have six-two [Figure 24], now suppose that these are not identical, as I said, that means there is something in this column which is not in this column , so suppose I introduce number two-eight, then this means because we know symmetry I have to introduce eight-two [Figure 25], *um, is this a valid example*, [whistling] this is not a valid map, because six and two both have this thing in common, but six-eight is not there, and similarly, eight-two is there, but eight-six is not there, so this is not a valid law.

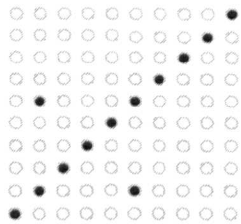


Figure 24

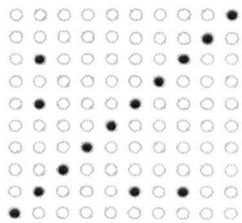


Figure 25

Textfigure 19: Amit—Generating a non-example symmetric figure

Having explicitly distinguished between the symmetry and the conditions of a visiting- law, he is about to amend his first argument. This *new* argument more or less has the same structure as the earlier one, i.e. "if we are going to breach the law here (in the intersection), then we must breach the law somewhere before (in one of the original examples"[25]. However, there is a vital difference between the two arguments. In the first one the "law" is a matching law, while in the second one the "law" is a grouping one. Here is his single-group argument.

Amit: Okay. If I suppose that this is the third example, which is the combination of the first two, I am going to suppose that we contain two-four and four-two, and we contain two-six and six-two [Figure 26], now notice that this is not a valid example, because we do not include six-four and we do not include four-six, which we have to include this to be a valid example.

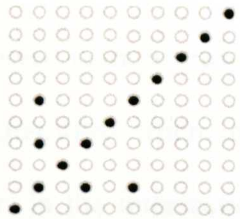


Figure 26

Now I am saying this comes from another two, the intersection of two other examples, now the fact that these are common points means that all these points were in the first example and they were in the second example, now the first example was a valid example, so if these points were in the first example then it follows that this point[(4,6)] was also in the first example and this point[(6,4)] was also in the first example, because the first example was valid, now the black points were all in the second example as well, but because the second example was valid it means that these red points[Figure 27; he actually uses red colour] both of them were also in the second example, otherwise the second won't be valid, that means these red points that I've newly marked in, these red points, both were in the first example and they were in the second example, this means that they are also in the third example because the third example is the points which are common to the first and second example, so instead of being red these should be in black because there are no difference from the rest of the points, they are common... And these now satisfy the second law; this means that the common points will always satisfy the second law.

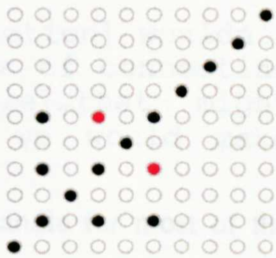


Figure 27

[25] This actually was expressed by Amit when tackling the union task. It is interesting to say that only two students thought that the union of two examples is an example. Both students were first year mathematics students who had been taught the subject!

In brief, the single-group conception is characterised by the students' focus on a certain group of elements while all the other elements are treated as individual. I showed that the students in this category exhibit a *better* way of dealing with the tasks at hand, GENERATING, CHECKING and PROVING. This way is effective particularly because only a *finite* number of elements are involved in the situation. This latter fact manifests itself in the GENERATING and CHECKING where the students may tackle the task "just by exhaustion". However, in the intersection task, where the students need to show that the intersection of *any* two examples is an example, a single-group conception could result in an argument limited to a particular collection of the single-group examples (see Amit's argument above), because the students could not see how the other connections (among the elements outside each focal single-group) could affect the situation. This latter aspect can be seen in Peter's explanations where he frustratingly explains why he initially thought that the intersection would not be an example.

> Peter: I was thinking it [the intersection] wouldn't [be an example] because there is ways of connecting the numbers [in the first example]…that might contradict another connection that we would make later on [in the second example].

Although the intersection task proved to be difficult to handle for most of the students, it was an opportunity to bring about the next conception, i.e. multiple-group conception.

5.2.3 Multiple-Group Conception

In this category, "disjoint groups" are experienced; the groups have no elements in common and the elements of each group are related to each other in one way or another. There are only two students who exemplify this category, Andy (a second year physics student; see Section 1.7) and Hess (a middle school student). Both these students *explicitly* express this idea and made use of it. Andy exploited it when conveying one of his examples to his co-interviewee.

> Andy: I am going to tell you some groups, each group visits all the other ones in the group and hence it is visited by all the other ones in the group, it visits them and it is visited by them.

Let me here follow Hess as he dealt with the problem of giving the least amount of information for Figure 28 (Textfigure 20), which then was abbreviated to Figure 29: ("abbreviated" is the way that Hess describes the figure with the least amount of information).

> Hess: for example, 1, 5 and 7 make a group [it is the first time that he uses the word "group"] with each other, so I only draw 5 and 7, It doesn't need [to do something] for 5 and 7, then I see 2, 9 and 10 make a group with each other, I do for 2 these, it doesn't need for 9 and 10; 3 and 6 make a group too, 4 nothing, *it make a group for itself*, for 5, 1, no 5 has been done [so far this reflect a sequential packing, but all of a sudden he shifts his attention to the *groups* involved],*how many groups are they*? It's been finished, 8, it's been finished; that's it [Figure 29].

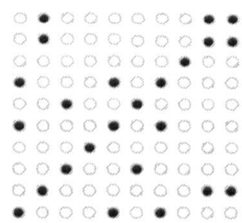


Figure 28

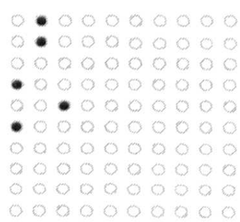


Figure 29

Textfigure 20: Hess—The least amount of information task

Although for a moment he experienced the *groups* involved (see Textfigure 20), he came back to the level of sequential grouping when explaining how the dictator could create the original example.

> Hess: First dictator draws these [the points on the diagonal], then he sees, 1,5 and 7 make a group, so he does the same thing for 5 and 7, 1, 5, 7; for 7 he put 1, 5 and 7 too; then, he sees 2,9 and 10 makes a group [he put necessary points for 9 and 10]; then he comes to 3 and 6 [he draws]; for 4 itself, for 5, 5 has been done before, they have been become *symmetric*.

And again he turned to the multiple-group conception when explaining why this abbreviated figure *uniquely* determines the original figure:

> Hess: there is only one case, *when we draw the diagonal, the groups are determined; and when the groups were determined there is only one case.*

Later on, in the course of the intersection task, where he skilfully drew on these ideas (to prove that the intersection is an example), he completed what he had started in the course of the above task. After examining different arguments for the intersection problem he decided to work on the abbreviated figures, since "their abbreviations are themselves" and by using them "our way would be simpler":

> Hess: if we prove that, if we have an abbreviation and reduce it a bit, still it is an abbreviation, it proves [the intersection task], when we take the intersection it is a part of it [one of the original examples], it is like that we omit a part of it.
>
> Interviewer: aha!
>
> Hess: suppose [we have] 9, 5 and 6; 10, 7, 3 and 2, suppose it is like this ... [Figure 30],

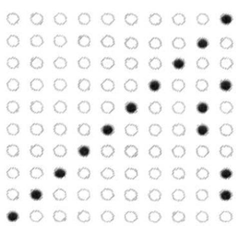


Figure 30

To lessen the interviewer's confusion about making use of abbreviations for finding the intersection of two examples he switches to the following two examples (Figures 31 and 32):

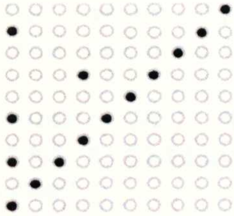


Figure 31

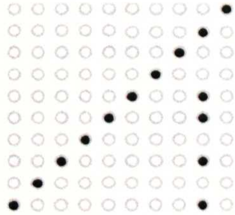


Figure 32

Hess: Now, I am saying what their intersection is, now; 1, 3 and 5 have commonality with each other, so their intersection becomes these two… [The red points on Figure 33; using the red colour is mine]

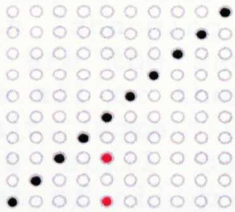


Figure 33

Hess: And from these two it is deducible that this one is [in], this one is [in], and this one is [in], so that's the figure [Figure 34].

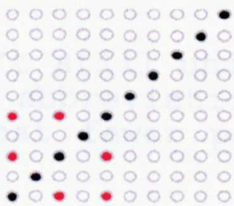


Figure 34

And shortly after this clarification (for the interviewer) he brought back on the same track.

Hess: Suppose this is an abbreviations, here, 9, 5, 6 make a group which has no relation to, for example, the group of 2, 3, 7 and 10; *they have been divided into some groups that have no intersection with each other*, certain different groups have been created… when we make it bigger [putting all the related points, like the shift from Figure 33 to 34] they'll have no relation with each other, so if we *reduce this one, nothing happens to that one, then we must only investigate that into this one itself, it causes disruption or not*… for example we say, 2, 3 and 10, with 7, for example, if we omit 3, we would have no problem, its reason is very clear.

Interviewer: No! No! When I was a student, every time I had a difficulty in my exam, I was writing it is obvious!

Hess: this is not difficult, because here, for example, *we omit a member of the group*; these groups have no relation with each other…this is an example. 5, 6 and 9 are with each other, 2, 3, 7 and 10 are with each other, when this is it, if we omit one of these ones, there is no relation to those, when I omit one of this ones, I omit one of its members, when I draw it again [changing it to unabbreviated figure], again they are the same members but 3, 3 have be omitted from all of them, you should accept that … if I omit some [points] from each abbreviation, it is still correct *…because the abbreviation is a set of all groups*, so if …we have an abbreviation, then we omit some of [its points], even randomly, *this remaining figure is again the abbreviation of another example*, so it doesn't matter if we have reduce it with a rule, It makes no difference, whatever we reduce, it is still an abbreviation.

Interviewer: I can't still understand one point, you say it is the abbreviation of that figure, how do you know that fewer points can't create that picture.

Hess: It is the minimum possible number; *I only determined the groups*.

Let me highlight a piece of his argument:

Hess: they have been divided into some groups that have no intersection with each other.

This reminds us of the notion of partition *explicitly expressed in words*. This fact markedly distinguishes the multiple-group conception from the single-group conception. This distinction is based on a very deep methodological issue. There are many different places that I have claimed and I will claim that this or that student exemplifies this or that conception. In all of these cases, my claims are based on two different, but interrelated sources: (1) the ways that they tackled certain tasks (GENERATING, CHECKING and PROVING) (2) what they verbalized, whether they openly expressed the concepts involved or not. But, as far as the multiple-group conception is concerned, I could only rely on the latter source in the most direct and explicit form, i.e. whether the students explicitly spoke of disjoint groups or not. For example, should I claim something only based on Hess' explanation of the way that the dictator could create the original example (see above, after Textfigure 20), I would claim that it reflects a single-group conception. Consider that what he (the dictator or Hess!) was doing is basically a sequential packing. In line with this argument, I shall add that we have many other cases that when *we* look at what the students had generated as an example *we* could see several disjoint groups (see for example, Amit in Section 6.4); however, in none of these cases I did claim that they exemplified the multiple-group conception, as I did not so for Ali (Section 5.2.1, Figure 12). Overall I am inclined to think that the multiple-group conception can be only expressed in words, while the single-group conception can be expressed either in words or in action. In the next section, where the meaning of a group is examined, I will show that the distinction between the single-group and multiple-group conception is also closely related to the '*unity*' of a group (Section 5.3.3).

5.3 Groups

Looking at the examples reported so far and even the name of the last two categories, single-group conception and multiple-group conception, clearly suggest that the idea of a group (of related elements or to be related element) plays a vital role in the ways that the students tackled the tasks involved. Although when describing the main categories I have equivocally used the term "group", there are subtle differences in the ways that each group is experienced. These differences reflect three aspects of a group (1) the number of the elements of the group, (2) the inside and the outside of the group, (3) the unity of the group.

In the following sub-sections I discuss these different ways that a group is experienced. As usual, I depict these conceptions in the context of the students' work; in particular I make use of the work that in a way are in the border of two different categories.

5.3.1 Two to Many

In this section I show the number of the elements of a group is a vital factor in the way that the group may or may not be experienced. In particular, it seems that the change from "two to many" appears to be a turning point between the first two categories, the matching conception and the single-group conception.

Let me highlight the main differences between the first two categories, the matching conception and the single-group conception:

- In the matching conception the focus is on the *pairs* of elements, one at a time, where all the other elements recede into the background.
- In the single-group conception, a single-group of *equivalent* elements is experienced, while all other elements recede into the background.

Looking at the way that I put these two categories, it appears that there is an overlap between the two. The overlap is where the single group is comprised of only two elements. After all, as far as the dictionary definition of "equivalent" or "equivalence" is concerned we have *two* equivalent elements or equivalence between *two* things. As a matter of fact, in many cases taking two elements *identical* is among the first things that the students experienced when tackling the tasks, and sometimes it is the only thing all the way through. Here are some examples.

Sarah (a first year mathematics students) is generating her third example (Figure 35 below). Her first two examples were the whole-grid and the diagonal respectively.

> Sarah: You must always have the diagonal filled in to satisfy first part. For each pair of cities, either they mustn't have any in common, so I could make 1 and 2 different; 1 and 3 the same. So 1, 3 the same; that would be another [example; Figure 35]… for each pair of cities, such as 2 and 4, they have nothing in common.

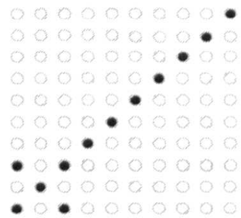


Figure 35

> 2 and 5 such as 2 and 4, all up to 10, they have nothing in common; and 1 and 3 both have exactly the same, so 1 can visit 1 and 3 and 3 can visit 1 and 3. So for that pair of cities … yes 1 and 3 as a pair… if you take this pair they are identical, but anything else is distinct, they have nothing in common.

To make another example, she matched two other cities.

> Sarah: Another one again, um… just pair of any other cities…so I can like choose two and four.

Although her first example was the whole-grid where "everything would be identical", that example remained marginal. It took a long time for her to realized that she had "narrowed her reasoning" by taking only a pair of cities into account. It was only in the course of CHECKING one of the prepared figures that she realized that "she always had worked only on a pair of cities".

> Sarah: I always work on pair of cities, yah, there is still pair of cities, because *in my head I've just doing two*, but you can have anything up to ten, *they must be in common…*

Another interesting case in point is Ali. Remember that he was mainly generating his examples starting with some haphazardly chosen points on the grid and pairing up the cities. Both Figure 12 (see Section 5.2.1) and Figure 36 were GENERATED by the same strategy, i.e. a matching procedure.

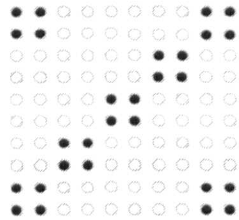


Figure 36

As he was starting with some haphazardly chosen points having the same status as each other and without any kind of grouping, he could not anticipate anything about the result; in particular, whether it would be a full grid (whole-grid example) or not (Section 5.2.1). After generating some examples, the interviewer somehow encouraged him to anticipate the result.

```
Interviewer: You were afraid that the whole grid is going to be full [referring to Figure 12], how could you be sure that the way that you do [he interrupted the interviewer],
```

`Ali: it wouldn't always be full,`

`Interviewer: yes,`

Ali: all right, now I can put in them, for example, *something that is the same*.

`Interviewer: what do you mean?`

`Ali: For example, like this [pointing to Figure 36], it would be square by square, only in this middle, and this middle.`

`Interviewer: Then how could you be sure that all of them wouldn't be full?`

`Ali: Shall I draw it?`

`Interviewer: Explain it!`

Ali: Now I must choose *the cities that match with each other from the start*, but it needs a little bit of thought!

Consider that each square in the middle (Figure 36) determines a pair of cities and is determined by a pair of cities; moreover, "something that is the same", "the cities that match with each other" all refer to a pair of cities. This aspect becomes clearer when he actually generates his example as follows.

Interviewer: What do you mean by "matching"?

Ali: Not as before where I firstly and haphazardly chose the cities, points, and then started drawing.

Interviewer: So what are you going to do?

Ali: first, the cities, they visit themselves, so it is going to be full [the diagonal], then, here [for] city 1, city 10 is going to be full, for city 10, its city 1 is going to be full, since *it is going to be identical with 1* [Figure 37]

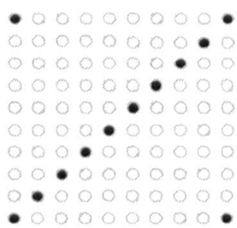


Figure 37

The rest speaks for itself.

Ali: Then, in the second city, I fill this city[3], then, um, in the third city, um, um, the third city I fill this one[2], the fourth city, um, I fill this one[5], the fifth city it[4] is going to be filled, the sixth city, this one is going to be filled, this one, its seventh city, the seventh city, its sixth city would be filled, the eighth city, its ninth city would be filled, the ninth city, its eighth city would be filled, the tenth city, the city, u...m [hesitating], *let's check it.*

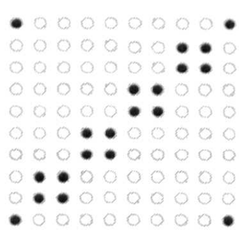


Figure 38

When he came to city 10 he hesitated for a while. As a result, the interviewer asked him what he was up to.

Interviewer: You wanted to do something but you change your mind, what did you want to do?

Ali: I wanted to say that let city 10 have city 9.

Interviewer: Then why did you change your mind?

Ali: Um, I thought maybe that would happen...that all of them are going to be filled in.

Consider that if he put that point, the point (10, 9), the only thing happening was having *more than two identical elements* where cities 1, 8, 9 and 10 all became identical. However, neither in this example nor in the next example generated by making use of the same idea, he did not experience a group comprised of more than two identical elements. Moreover, his hesitation for adding an extra element to an already packed group (adding 9 to the *group* comprised of 1 and 2) suggests that at least for a while there was confusion about *inside* and *outside* of that group (and the group comprised of 8 and 9), and also that group lost its *unity*[26].

Before closing this section it is worth referring to Kord again. Remember how the presence of a block square helped him to shift from the identicalness of two elements to the identicalness of more than two elements (Section 5.2.2). However, in his case the identicalness of more than two cities was experienced tightly bound to the consecutive cities determined by the corresponding block square. To make this aspect of his work clear we shall follow him as he dealt with the problem of giving the least amount of information for a figure having a three-by-three block square in the lower left corner (Figure 39).

Kord: if I want to show this square with this diagonal line [Figure 39], I put [a point] here [Figure 40].

[26]The next two sections are concerned with these two last aspects, in and out, and unity.

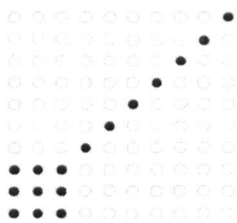


Figure 39

Figure 40

Kord: Exactly, that means only one point…I'm writing city 3 visits city 1; I'll give him only this information. First he'll draw the diagonal line and then he will see that city 3, city 3 is common with city 1, their visitors, so he will make the visitors of *city 1 identical with city 3*…that means he'll put here and here, it is here, then, *no he'll draw another thing, another example has been created*! [Figure 41]

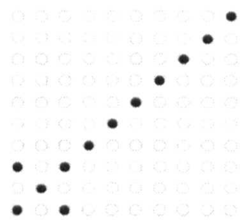


Figure 41

Kord: that means one example was added to the previous examples! Because it's obeyed all conditions; condition one, the diagonal line, condition two, city 3 and city 1, now all are in common, but for example city 3 has nothing in common with other cities.

Although he had also generated another example comprised of the pairs of nonadjacent identical cities [Figure 42, below], he never put that idea into practice with more than two cities; in other words, he never generated an example with three or more freely distributed cities.

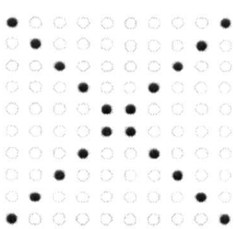


Figure 42

Referring back to the way that he first gave the purported least amount of information for figure 39, it is noticeable how the presence of a square, once helpful to relate more than two cities to each other, when limited to the square itself and not to the underlying cities, reshaped *in* and *out* of the original *group*. Moving to a new example [Figure 41], a new group of two identical elements, with its own inside and outside, was made. Then this new group gained its unity where one of its members (city 3) represented the group itself.

In sum, when a group is comprised of only two identical elements it is in the border of the first two categories, the matching conception and the single-group conception; it is properly in the second category when it has more than two identical elements. In both cases, two vital aspects determine whether a group is a group or not. These two aspects are the *extension* of the group (i.e. what is inside the group) and the unity of the group. Let me now turn to the first aspect.

5.3.2 In and Out

The cases that I (or the participants) have spoken of "in the group" and/or "out of the group" are already so abundant that it seems I do not need any further explanations or, in particular, any further examples. However, this aspect is of such great importance that I need to add a few words and to give some other examples. Doing so, I also try to relate this aspect to the unity of the group.

First, it is noticeable that a 'group' is asymmetric to what are 'in the group' and what are 'out of the group'. In other words, while what are in the group are related to each other in one way or the other, what are out the group are simply individuals having the same status defined in their relation to the group, that, they are not in it (see Section 5.2.2; in particular, Chris' case).

Second, so far I have equivocally spoken of the idea of group and its constituting *elements* without paying enough attention to what actually constitute the group, cities, points or columns. But, as far as tackling the tasks is concerned, this distinction is of great importance. As an example, let me once again refer to

Kord (see Section 5.3.1) where he chose only the corners of a 3 by 3 square to represent the corresponding square. Consider that in a normal sense (i.e. geometric sense), a square is perfectly determined by its corners whether it is a block square or not. But, this is not the case when the square is comprised of certain *equivalent* points determining some equivalent cities and not the others, and determined by those cities and not the others; in this case the points in the corner are not more important than other points, neither are they less important than them. As we saw, when tackling "the least amount of information" task, Kord lost the equivalence of the points of the 3 by 3 block square, and consequently, the *extension* of the group of the underlying cities was disturbed. Nonetheless, the block square had an opposite effect somewhere else, when it compelled Kord to consider the identicalness of more than two cities (see Section 5.2.2). In other words, an *easy-to-see* structure (when CHECKING) and an *easy-to-impose* structure (when GENERATING) helped him to shift from the group of equivalent points to the group of the underlying equivalent cities (though in a very limited sense). In reverse, we have cases in which an *easy-to-express property* of certain cities (e.g. even cities) or a group of cities (the first five cities) not only relate those cities to each other, but also determine the corresponding points. A case in point is Chris (section 5.2.2). Another case is Peter.

```
Peter: You could do it in odd numbers and
even numbers, so if all odd number cities
would be able to visit other odd number
cities, and all even number cities would
be able to visit all even number cities,
that way they are either identical or they
haven't got anything in common.[Figure 43]
```

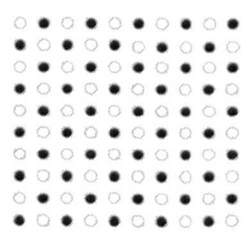


Figure 43

Putting these cases together it appears that the *extension* of a group of identical elements has been defined by its *intension*, whether loosely applied and in danger of losing its unity (Kord's case) or firmly expressed (Peter's case) and potentially convenient to experience its unity.

5.3.3 Unity

"Unity" is a term serving to capture those experiences maintaining a group as a group without losing the relations once established between its members. To clarify the idea I shall give an example. This example, in a sense, includes all that we have already had about a group. Piro (a middle school student) is about to GENERATE an example 'between the example with the minimum coloured points (the diagonal) and the example with the maximum coloured points (the whole-grid).

Piro: Now, for cities 1 and 2, I take these two identical. Now, for cities 1 and 3, I take them "no one"; that means the second part of the condition[27]. [Figure 44]

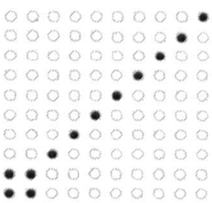


Figure 44

Piro: Now, for 1 and 3 I put "no one", um, for 1 and 4 I put "no one", for 1 and 5, for example, I put thing, um, common, that means, I colour 5, 5 in the first row, and 1 in the fifth row [Figure 45], then for 6, 7 and 8, I leave them empty, for 9 again, I make it common, and again I leave 10 [Figure 46].

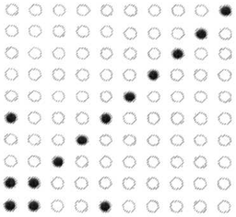


Figure 45

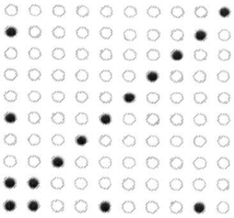


Figure 46

[27] That is the second part of the second condition.

So far, a pair of elements, one at a time, is in the fore, while all the other elements are in the background. There is an attempt to 'make the visiting-cities (of the focal pairs) identical'; however, when doing so, all the other cities recede into the background. Upon the interviewer's question about the ways that the visiting-cities are identical, he started to see the other visiting-cities of the two focal cities apart from those two cities themselves.

> Piro: No! No! I did them wrong…they are not the same, now, what I must do, I must choose a city for that, so I must colour this one, for making the visiting-cities identical, because, um, now, the visiting-cities of, for example, 9, are 1 and 9, for 1, is also 1 and 9, *but there is a difficulty, when for 1 I colour 1, 2, 5 and 9, I must colour them for 9 as well*…for becoming identical, I must colour them [Figure 47].

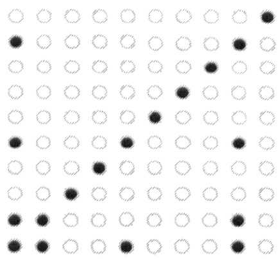


Figure 47

Although he now considers the other visiting-cities of the two focal cities, the focal pair is given more weight than the cities that they are related to. In other words, they do not form a group in which all members have the same status where each two are related to each other. As a result, 2 and 5 still retain their independent identity, although they have been already counted among visiting-cities of 1 and 9.

> Piro: Ok, now, I consider city 2, for city 2, I've already counted 2 and 1, for example, we leave 2 and 3, 2 and 4, 2 and 5, and 2 and 6, we take 2 and 7 common, such that for both of them, we take 2 and 7 [Figure 48], that means for 7, now again, for visiting-city of 2, 1, 2 and, *Aha, again a difficulty.*

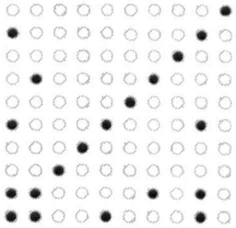


Figure 48

> Interviewer: What is it?

Piro: The problem is that *one is among visiting-cities of 2*…Ok, I've chosen it; so I must also put it for 7. [Figure 49]

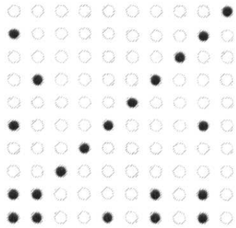


Figure 49

He is now experiencing the extension of the *original group* that he was working within. As a result, he is rubbing out whatever is related to 7 (of course, except for 7 itself), and from there, completing the figure is only a matter of putting the related points.

Piro: When I do this, the visiting-cities of 7 and 2 are identical, but they are not identical with 1, [he is rubbing them off]; that means, now I colour I colour 5 and 2 in the first place and again we will come back, now 2 has this and this [5 and 9; Figure 50]

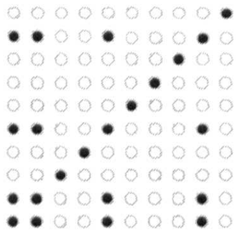


Figure 50

Now, there is a group there determined by its extension; however, the extension itself is not determined by its intension. The related points in Figure 50 have not got that much structure as a block square in the figures generated by Kord; the underlying cities also have not got an easy-to-express property as in Chris or Peter's case. Nonetheless, the underlying cities have something in common, that, they all have the same visiting-cities.

Piro: …the visiting-cities of 1 are 1, 2, 5 and 9, and for 9, it must be 1, 2, 5 and 9, so for 2 it must also be 1, 2, 5 and 9, and also for 5, it must be 1, 2, 5 and 9, and the same for 9, now, um, now this is an example…

Working only with a finite number of elements allows him to *coordinate the extension and the intension* of the group that he is experiencing at this moment.

But, are these elements, 1, 2, 5 and 9, also experienced as a *unit*? Let us follow him to find the answer.

Piro spontaneously continues by making some changes in Figure 50 to create another example.

Piro: But, in this way, I can make some changes in the rooms that I haven't touched yet; for examples, 3, 4, 6, 7, 8 and 10. For example, I can take 6 and 10 equal. That means I must put 6, the visiting-city 6, for 10. For city 6, I must take the visit 10 [Figure 51]

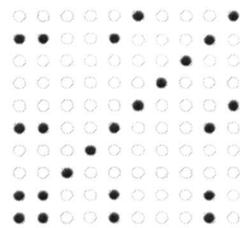


Figure 51

The elements that were untouched (3, 4, 6, 7, 8 and 10) are now coming to the fore quite separated from the rest (forming the original single-group). The new to-be-related-cities (3, 6 and 10) are being chosen from the previously untouched elements, and the rest (7 and 8) recedes into the background alongside the original ones (1, 2, 5 and 9).

Piro: Then, um, for example, we can take both of them, 6 and 10, equal to 3. So, for city 3, we must put visiting-cities 6 and 10. For city 6, we have already had 10, so we must put 3; also 3 was previously there [in the tenth column], so we must add 3. Now, this is an example [Figure 52].

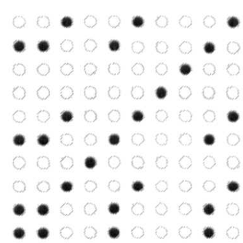


Figure 52

Now the already disturbed original group (1, 2, 5 and 9) together with the newly made one (3, 6 and 10), both completely *lose their unity* as a group when Piro checks the examplehood of the new figure (Figure 52, above).

```
Piro: [it is an example because] it is symmetric
about the diagonal, therefore both conditions are
satisfied. The first condition is satisfied by
drawing the diagonal; the second condition is
satisfied, for example for city 5 we put the
visiting-city 2, for city 2, I put the visiting city
5. That means…here, for example we're choosing the
point 5 and 2, because we've coloured the point 2
and 5. So, they are symmetric about this thing [the
diagonal], therefore the condition is satisfied.
```

Consider how the point 2 and 5, once a member of a group of related points, has been detached from the original group, and has become a member of *an undivided whole* where there is no difference between it and the point, say 3 and 6, which was originally a member of a different group.

Losing the unity of the groups involved (if they had one) raises doubts about whether Piro experienced the disjoint groups *present* in Figure 52. It is also the case in many other examples in which certain disjoint groups are present in the generated figure, but it does not reflect students' multiple-group conception. In a way, experiencing a group as a unit is the dividing line between single and multiple-group conception (see also Section 5.2.3). I shall also add that I did not fully realize the connotations of the term "unity" until I read 'the history of the subject'. In this regard, the reader may wish to read Section 7.4.

In sum, although I have equivocally used the term 'group' in many different cases in which some only reflect the extension of the group, a group is a group when both its extension and its unity are experienced. Both these aspects could remind us of the idea of a 'set'. This resemblance brings me to the last section of this chapter where I discuss the distinction between a set and a group (as exploited throughout the present chapter).

5.4 Group or Set

In Chapter 4, when reviewing the literature, we saw that equivalence relations and partitions are mathematically based on a more fundamental idea, the idea of a set. But, in that chapter I did not independently discuss this latter idea because only through a simultaneous analyses of the data and reading the literature I gradually became aware of the different aspects of a set, and more importantly the distinction between a set and a group. Had I discussed these ideas in that chapter, I would have referred to many sections of the next chapter. At that time, this seemed to me putting the cart before the horse. Thus, I decided to discuss it here at the end of Chapter 5 where I can refer back to some familiar ideas.

All the way through the present chapter I was reluctant to use 'sets', and only in one or two places I used the related notion of 'partition' to describe the students' experiences in this particular situation. Instead, I used 'groups' for the former, and the 'multiple-group' for the latter. However, it does not mean that a 'set' and a 'group' (or a 'partition" and a 'multiple-group') could be used interchangeably. In this section I underline the distinction between these notions. To do so, I make use of the literature and the data of this study.

A quick look at the literature shows some significant similarities between a set and a group. For example, Skemp's account of sets reminds us of our discussion of "in and out" as to a group:

> For mathematical purpose we must...agree to confine our attention to *well-defined* collections of objects; that is, collections for which we can say, of any chosen object, whether it is in the collection or not. Such a collection we shall call a *set*: and the objects which are in it we shall call its *members*.
>
> (Skemp, 1971, p. 142)

The very first two sentences of Hausdorff's famous text Grundzüge der Mengenlehre (1914, translated as Set Theory by Aumann et al, 1962) states another aspect of a set that reminds us of the unity of a group. It even bears a remarkable resemblance to the 'language' used throughout this study.

> A set is formed by the grouping together of single objects into a whole. A set is a plurality thought of as a unit. (ibid, p.11)

But then, what it is that distinguishes between a group and a set. Freudenthal provides us with an answer:

> Except in artificial examples and exercises, sets are usually endowed with, and are dependent on, *structures* and can be grasped through these structures only. As a substratum a set becomes *explicit* if the structure is recognised and consciously eliminated. (Freudenthal, 1983, p.53; emphasis added)

Thus, it seems that a group (as used here) is a set plus a *structure*. I shall illuminate this with some examples.

In Section 5.2.2 we saw how the presence of a *block square* helped Kord to relate more then two cities to each other. Then, in Section 5.3.1 it was shown how Kord's focus on this structure (a block square), rather than the underlying cities, hindered him in generating an example with three or more freely distributed cities.

As another example, let me once more refer to Hess. In Section 5.2.3 we saw that some of Hess' descriptions—when tackling the intersection task—were very close to the standard account of partitions. Moreover, we saw that he easily could shift his attention from points to cities or from a group of points to a group of related cities and vice versa. For example, having the following figure (Figure 53) he easily shifted to the underlying related cities, saying that "1, 5 and 7 make a group with each other" and so on.

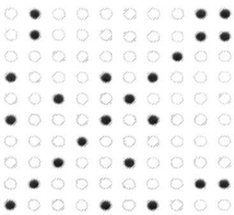


Figure 53

And conversely, to make a new example or a new abbreviated figure he may start from the underlying cities and then shift to the related points on the grid. However, as mentioned before, during the course of the interview I was puzzled by Hess' use of the abbreviated figures in his argument for the intersection task, in particular, by the way of finding the intersection of two original figures from their abbreviations. To clarify this latter aspect, I asked the following question accompanied by the following figure (Figure 54, below).

> Interviewer: Now, suppose you want to give the abbreviation, you give these points [pointing to the points on the first column of Figure 53], now suppose I want to give the abbreviation, and I give these points [figure 54], how can I understand from the figure that they [two figures that we are going to find their intersection] have any commonality.

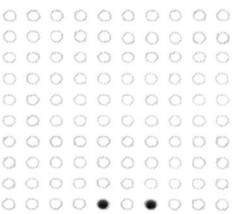


Figure 54

But, to my great surprise, first he did not regard Figure 54 as an abbreviated figure at all! He rejected that, paused for a moment, and then said:

> Hess: Yes, it is [an abbreviation]. All of them have 1, so all of them must be equal to 1, alright; all of them are equal to 1, so all of them are equal to each other.

I just realized that the way that he experienced an abbreviated figure was attached to the vertical structure of each column. And unintentionally my question led him to recognize this unnecessary structure and eliminate it.

The above examples may give the impression that as soon as the students have experienced the underlying group of cities (constituting a group of points or columns), they experienced a set. However, there is one subtle aspect calling into question this conclusion. Consider that a group, as used here, is comprised of certain elements (two elements or more than two elements) having a mutual relation with each other; even stronger, they are pair-wise identical, or they are *all* identical. Thus, the students experienced a set comprised of the identical elements. That is a group.

It is worth saying that *mathematically* there is no need to distinguish between a set and a group, because they have been masterly unified in the standard account of sets, equivalence relations and partitions (see also Section 9.2.1). To quote Quine (1940, p.121):

> This is the end; no abstract object other that classes [sets] are needed—no relations, function, numbers, etc., except insofar as these are constructed simply as classes.

However, it seems that as far as the learning is concerned, it is just a start. It remains to be answered when a student needs to eliminate the structure for making the underlying set explicit. What kind of situation can bring this conception about? It does not seem that a situation based on equivalence relations would be a satisfactory one, at least if it is used alone.

Chapter 6

Results II: Matching

6.1 Introduction

In Chapter 5 we saw that a group, when experienced, freed the students from matching any two possible pairs and it facilitated the ways that they dealt with the tasks at hand. For example, comparing Kord and Ali, it can be seen that how a square of certain points (though a very special square, i.e. a block square in one corner; Section 5.2.2, single-group conception) helped the former (Kord) not to compare all the possible pairs, while the latter (Ali) needed to check *all* the possible pairs, one at a time (Section 5.2.1, matching conception). This simple comparison between the two ways of generating an example conceals something important beneath its simplicity. This hidden thing is a very subtle *shift*. Consider that Ali failed to shift from a pair-wise matching to the underlying group of related (or to-be-related) elements while Kord easily shifted from a group (embodied as a square) to a complete match where each two elements in the focal group were matched together. These two are two different shifts between two different ends.

At one end, there are certain elements (two or more than two) matched in pairs, at the other end there is the group of these identical elements. In the previous chapter the experience of grouping (single or multiple) was brought to the fore, while the connections between the elements were in the background. In the present chapter these connections come to the fore. In particular, I discuss three important kinds of connections that had a vital effect on the ways that the students tackled the tasks involved. These connections are *transitivity*, *F-transitivity* and *symmetry*.

6.2 Transitivity

> The importance of the transitive property is that any two elements of the same sub-set in a partition are connected by the equivalence relation.
>
> (Skemp, 1971, p. 175)

This suggests that the transitive property is what that makes the vague phrase used in the second (and third) category clear; where I said that "the elements in the group in one way or another are related to each other." However, using the term 'transitivity' is misleading as it reminds us of the standard account of the subject where the transitive property is formalised as follows:

$$a \sim b \text{ and } b \sim c \text{ implies } a \sim c.$$

It is misleading since no student in this study experienced the transitivity as such, although many of them experienced a group of related elements in which any two elements were related to each other! Thus there must be some other way by which the students did connect (match) any two elements of a certain group to each other *without actually matching any two possible pairs*. In this section I intend to bring these ways to the fore.

First, it is not only through transitivity by which any two elements of a group could be related to each other without actually matching any two. Sometimes, each two elements are taken to be identical because there is a certain property common to each individual element. That is to say, the intended connections are guaranteed by the intention of the group. Here is an example (for more examples see Section 5.3.2).

Peter (a first year politics student) is generating his first example.

Peter: if you could visit any city that fulfils both conditions… say, people in city one are allowed to visit people in every city, and *the same for every one*, then it fulfil the criteria… if you are in one, you can visit people in one, and then if I get two, they can visit everyone, for each pair of cities [he reads the second condition] in this way *they all will be identical, for any pair of cities.* [Figure 55]

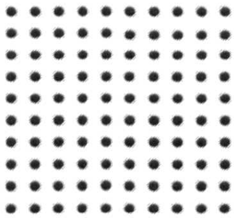


Figure 55

Second, even when certain pairs of elements are being actually matched, there is no reason to match them in *any particular order* as implied by the standard account of the transitivity. A case in point is the way that Peter (the very same student who exemplified the first point of this section) GENERATED one of his examples.

In his quest for generating an example having a certain predetermined number of points, he has just generated an example having 12 points (Figure 56, below).

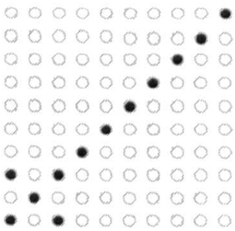


Figure 56

Now, he is about to generate an example with 14 points.

Peter: city 3 can visit city 1 and vice versa, and that's completely identical for those. If you've got city 4 has to visit city 1 which then I can get the *can of worms* and I've to say city 4 can visit city 3 and vice versa…[Figure 57]

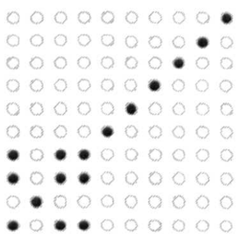


Figure 57

City 4 can visit city 3 because city 4 can visit city 1 and city 1 can visit city 3, or city 4 can visit city 3 because city 4 can visit city 1 and city 3 can visit city 1. It is the former that we can happily call an experience of the transitive property; it has the order demanded by the standard account of the transitivity. But, 'The can of worms' is a metaphor eloquently showing that the order does not really matter. In a similar vein with this example and this metaphor, there is another illuminating example and two other elegant metaphors yet to come.

Peter who generated the following figure (Figure 58) as an example with 20 points is now looking for an example with 22 points.

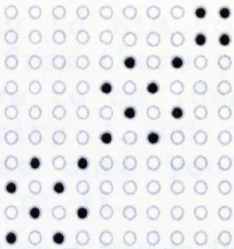


Figure 58

> Peter: I don't think you can have 22 [points] from them [i.e. starting from Figure 58] because you've reached a saturation point; as soon as you add another dot, it's not just linked to the two cities that you put dot to. In example 6 [pointing to Figure 58] 10 can also visit 8 [Figure 59], and then 8 can also visit 10… [Figure 60],

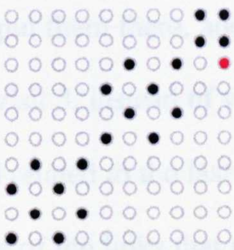


Figure 59: 10 can also visit 8

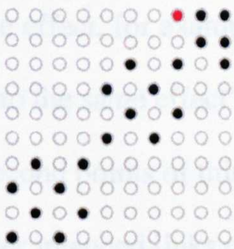


Figure 60: Then 8 can also visit 10

> …but still you're relying on 10 be identical to 9, so that needs to be filled in, and you're relying on 8 being identical to 6, 6 has to visit 10 as well, and *the whole thing like a pack of cards, dominos*, are being identical now, they have to be identical. [Figures 61-63]

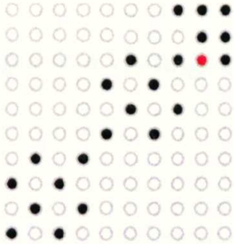


Figure 61: Still you're relying on 10 be identical to 9.

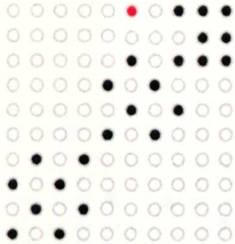


Figure 62: and you're relying on 8 being identical to 6

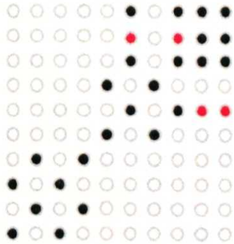


Figure 63: The whole thing like a pack of cards, dominos

What he means by the domino (effect) becomes clear when he dealt with the problem of giving the least amount of information for his first example (the whole grid).

> Peter: …if I just say to him you can visit city 1 from all ten cities, from there you would be able to work out…they all have to be identical because they all share visiting-city 1…and that would be *the domino effect*, because once you've filled in 1 it start rolling because of it…

It works because "you need to have a way to link (them) to each others". Expressing this link seems to be *easier* when all of them are linked to a focal member. However, when factual conclusions are concerned it is only a matter of choice what way they are linked together. For example, regarding giving the least amount of information for the whole-grid example, Peter added: "you could have ten dots and not put them there, and still have the same effect, for example, if you put 4 could also visit 3, instead of visiting 1 that leads to the same effect".

Third (and related to the last paragraph of the second points of this section), though the choice of the links that guarantee some other intended links could vary from situation to situation, choosing a *focal link* seems to be not only more easily expressible, but also more easily generalizable. The following illustrates this point.

Hess is about to explain why Figure 64 that he has just generated is an example of a visiting law.

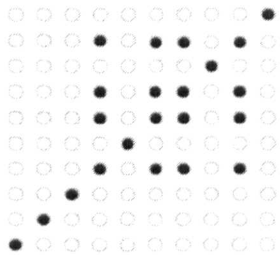


Figure 64

```
Hess: I am going to show that those that have
commonality with four are equal to it.
```

And he did so. And shortly after that, while generalizing his argument he added:

```
Hess: For each column we check that those that are
equal to it, those that must be equal to it, are
they equal to it or not.
```

It is a global grasp of F-transitivity (see Section 4.4). We can argue that such a conception in which the identicalness of certain elements is based on their identicalness with a focal element has arisen from this particular situation and/or a particular task that one was involved in. Later on, based on the history of the subject (see Chapter 7) I will show that although this conception has arisen from this particular situation, it is not peculiar to that. Moreover, this particular situation and/or a particular way of tackling the task in hand could also impose an order on the elements that are being matched together and thus reflect the transitive property. But, in this case I can only report a sequential use of transitivity, rather than a local grasp of transitivity in Freudenthal's sense (see 4.4). This brings us to the fourth point of this section.

Fourth, following Freudenthal (see 4.4) I distinguish between a *global* grasp of transitivity and a *local* one. In the former, the elements are sequentially matched together and based on that they all are taken to be identical or at least without the need for further matching, while in the latter, transitivity is verbalized and formalized as it is in the standard account where *three* generic elements capture the sequentiality. Let me give an example.

Amit (a first year mathematics student) is generating his very first example. We join him in the middle of his work when he has just made "2 identical to 1" and "3 different to the others". (Figure 65)

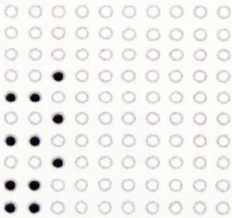


Figure 65

> Amit: For 4 we have to have 4, if you take *the pair 2 and 4*, because they both have this one in common *they must be identical*. [Figure 66]

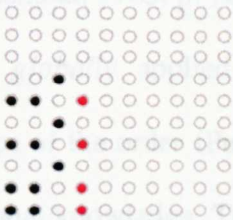


Figure 66

As it can be seen the way that he is generating his example put an order on the elements being matched together; 4 and 2 are identical, and 2 and 1 are identical, consequently 4 and 1 are identical or all of them (1, 2 and 4) are identical.

> Amit- 5, we have to have 5, therefore must be identical with 3. [Figure 67]

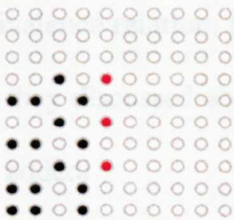


Figure 67

He is capable of doing a sequential matching when more than three elements are involved.

Amit: Ok, for 6 we put in *6, which is similar to 4.* [Figure 68]

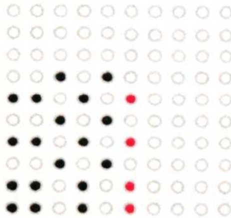


Figure 68

He is also certainly capable of repeating this idea on a different set of elements.

Amit: And in the same way, 7 is the same as 5, so have to be identical. [Figure 69]

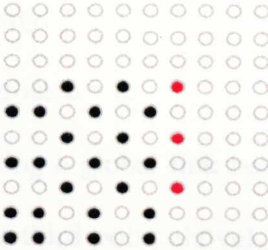


Figure 69

However, neither he nor any other students in the main study showed anything more than an "operational knowledge of transitivity" (Freudenthal, 1983, p. 12).

We have already had an example in which one *globally* applied the transitivity (Amit) and we had an example of the irrelevance of the order (Peter), now I need to show that what could be beyond this operational knowledge. To do so, I need to make use of an excerpt not from the main study, but from the preliminary study where Hargi and Shakil (see Section 1.7), two first year mathematics students, were struggling with this question that whether the conditions of a visiting-law can be replaced by the symmetry property or not. As mentioned, they suddenly realised that "basically it's an equivalence relation; this is reflexive, symmetric and *transitive*". And later on, when they discussed about the examplehood of a non-example symmetric picture, Hargi added:

Hargi: …that picture has symmetry, but it's not an example, because it's missing the third thing. It's reflexive over the diagonal, it's symmetric like this way, you need to satisfy the first two, I have

> a third one, which I'd say a_1 goes to a_2 and a_2 goes to a_3 then I want a_1 go to a_3. That's my third property, if I would satisfy this, my picture works.

It is a local grasp of the transitivity. Consider that here we have a shift and a choice, both imposed by their formal knowledge of the subject. The shift is the shift from the global grasp of the transitivity to a local one. The choice is the choice of a particular order. As mentioned earlier (Section 4.4) we can have the following local property, instead of the standard transitivity:

F-transitivity: $a \sim b$ and $a \sim c$ implies $b \sim c$.

It is a local property that is much in harmony with making a global link between certain elements and a focal one (see the second and the third point above). It is important to notice that as far as equivalence is concerned the above property and the transitivity amount to the same thing, i.e. the equivalence of two things can be drawn from their equivalence with a third. This *logically* equivalence stems from the most *natural* properties of a matching procedure, *symmetry*.

6.3 symmetry

Symmetry seems to be the most *natural* property of a matching procedure; simply *two* things are matched together. To see how natural it is, let me recall the example given in the matching conception category where Ali matched up *all* possible pairs to guarantee examplehood of his figure (see Section 5.2.1); however, not quite all possible pairs! Taking symmetry of the matching procedure for granted, he only needed to match forty-five pairs of cities not ninety pairs, as he did so by pairing city 1 with all the other cities after that (nine cities), and then city 2 with all the other cities after that (eight cities), but not any more with city 1, and then city 3 with all the other cities after that (seven cities), but not any more with city 1 and city 2, and so on.

Having the standard account of symmetry in mind, the expression "*symmetry* of the matching" could be misleading. Consider that in the normative account of the subject *symmetry is dealt with first by its breakage*! In other words, by saying "if a ~ b then b ~ a" two different weight are put on "a" and "b" and consequently on "a ~ b" and "b ~ a". For example, by saying that '1 can visit 9', '1' is a visitor and '9' is a visiting-city, and consequently it is quite another matter whether '9 can visit 1' or not.

> Chris:... *if 1 can visit 9*, then by the common cities, 9 contains itself already, so that means if 9 is in 9, so they must, to make them, it, valid, they must be the same. Because if what you've chosen, that 1 can visit 9, 9 already can visit itself, *so then 9 must be able to visit 1* to satisfy the rule.

The above excerpt has been taken from a longer one in which Chris was trying to explain the origin of *geometrical symmetry* (the symmetry about the diagonal). The immediate lines before what was quoted above are:

> Chris: Oh yeah, I know where it came from. It came from, if you, if you say, so when you compare cities 1 and 10, I'll make it easier, *1 and 9 because they are the same* [referring to Figure 70].

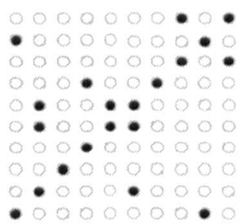


Figure 70

"1 and 9 because they are the same", this is what I refer to as the symmetry of matching in which two elements, say 1 and 9, simply are taken as a pair, *an identical pair*. Adapting one of his earlier utterances[28] we can see that when 1 and 9 are taken to be identical, they have an equal status: "If you take 1 and 9, as a pair, er ...if they are identical, then 1 has to be able to go to 9 *and* 9 has to be able to visit 1". And for another pair when generating another example:

[28] Originally, it was about the pair 1 and 2 and it was said when generating his first example (the whole-grid). The generic proof of the symmetry property (geometric symmetry) in which the pair 1 and 9 was used as a generic pair was given towards the end of the interview.

Chris: ...3 could be, 3 and 4, 4 would then have to be 4 and 3... 3 and 4 are identical.

This equal status is clearer when more than two elements are involved:

Chris: ... so you could do, 1, 3, 5, 7, 9 (for 1), and then for the 2 you could do 2, 4, 6, 8, 10 and then you'd do the same again, so 3 would be ... 1, 3, 5, 7, 9 and *4 would be the same as 2...*

Moreover, even the equality of status of the pair involved is reflected in the way that he experienced the geometrical symmetry. As a matter of fact he realised that each example is geometrically symmetric only as he dealt with the union task that was the last task in his interview. Here are some quotes from his work before dealing with the union task.

Chris: ...2 and 4 were similar, you'd have, 4 would have 2 *and* 2 would have 4...

Chris: ...If you add a point into that in the first grid, then you have to add two points in,

Chris: ...Say if you had the diagonal up there, and then so 5, 2, 2, 5...those two points ringed.

Each one of these quotes gives the impression that he was experiencing certain symmetry in the figure that he was working on at the moment. However, he was not consciously aware of geometric symmetry of each example until the union task. When tackling the union task probably because he took into account more than a pair of points at a time, he realized that each example is symmetric about the diagonal.

Chris: We say that that I've only got point 4 and 8. So if he said that 4 and 8, so eight's got 4 in it and four's got 8 in it, so that's a valid grid there. And if you do a second one, and if you said it had 2, 4 and 8, 2, 4 and 8, so you say that 4 and 8 are the same but then if you have any point in 2, which is in 8 or 4, it will have to be the same as that, but you have to have 8 and 4. You'd have to have 8 and 4. So it's symmetric, *why haven't I noticed that before? Oh I've been asleep.* Ok its symmetric about the diagonal, I've got that finally. Ok, so then if you take all the points off that, its still going to be a valid example, as that's a valid example, and although *you've got two extra points there and there,* that aren't in that grid, *because of the way the grid's work out, that if that line is in, if that is in there, then that also has to be in there.*

As far as Chris' experience of the situation is concerned, we can now turn back where we did start, i.e. when he consciously broke the symmetry to prove the symmetry! However, this latter symmetry, geometric symmetry, is such a recurring idea that deserves a section of its own.

6.4 Geometrical Symmetry

Considering that all the students in this study were familiar with geometrical symmetry from their early years of their formal education and/or their experiences outside school, it was not very surprising if they could notice the geometrical symmetry of each individual example, i.e. it is symmetric about the diagonal (regardless of their wording). However, as Chris' experience shows, there exist certain subtleties beneath this seemingly simple idea. The next two sub-sections concern themselves with these subtleties; in particular, I discuss different ways that the geometric symmetry was experienced and the ways that it was related to the other conceptions. To do so, I make use of certain semi-formal and formal formulations; none of them is used as a yardstick to judge students' work, *they are only a means of presenting the differences*. As a matter of fact, all of the differences that I talk about have been drawn out from students' work and I was not aware of most of them before having the data.

6.4.1 Logically Equivalent, *Structurally* Different

Let me start with the very often experienced form of symmetry as a property between *two* points or *two* elements:

For every x and y,

If (x, y) is on the grid, then (y, x) is on the grid.

Only few students used symbols to formulate symmetry. Even Tylor (a second year computer science student; see Section 2.5) who as a matter of fact was the only student that used symbols to formulate some of his ideas (including symmetry) did not write the quantifier part (for every x and y) and took it for granted that the points are on the grid (as I took the context of x and y for granted in the quantifier part of the above formulation of symmetry!). In short, this is Tyler's way of putting it:

If (x, y) then (y, x) $\longleftarrow$ SYMMETRIC ABOUT Y=X

Regardless of all these subtleties, this semi-formal formulation (that is very akin to the standard formulation[29]) has a very crucial characteristic in common with the way that our students experienced it, though it is not the way they put it. Generally speaking, in an undivided whole (the underlying *set* or the underlying grid), the main focus of both of them is on *two* elements, a *pair* at a time. In other words, x and y are two elements of the undivided underlying *set* regardless of any other possible relations that each one of them could have with the other elements, and in a similar vein, (x, y) and (y, x) are two points of the undivided underlying grid regardless of considering the likely group of points that they could possibly belong to. This conceptualization can be read between the lines of the following excerpt as a typical example.

Hess (a middle school student) is generating some of his earlier examples.

[29] In the standard formulation, x and y could be the same element or, in terms of points, (x, y) and (y, x) could be the same point. The possibility of equality of x and y, or equivalently (x, y) and (y, x), is a crucial aspect of standard treatment of symmetry that more often than not is overlooked by the students (see also Section 8.3.2).

> Hess: Now, if for every, for example, this point, 1, if I put everywhere, for example, I put 8, I must put for 8, 1...3, I put 5 and for 5 I put 3... 10, I put 9, also 9, I put 10...about this (diagonal), *each point we put this side, we put its symmetric on that side.*

Now let us read our semi-formal formulation as follows:

For every x,

For every y, if (x, y) is on the grid, then (y, x) is on the grid.

As it can be seen, *logically* these two semi-formal statements amount to the same thing. However, they do not focus on the same thing. While the focus of the first one is on *two* elements (a pair at a time), the second one gives weight to an element, and then, its likely relations with the other elements. To illuminate this *shift of focus*, let me give another excerpt from the very same interview from which I gave an example for the first conceptualisation.

> Hess: if 10 has 9, 9 certainly has 10, that means if 10 have something, that one has ten too... The same for 9, everything that 9 have, they have 9, too... and the same for 8 and so on.

It is worth considering how this shift of focus plants seeds of a different *structure* in which the underlying undivided whole is to be divided into two "groups", one group of elements with each one related to the focal element, and another "group" including the remaining elements that are not related to the focal element. In other words, in one way or another, it plants the seeds of a single-group conception. However, *within a group*, symmetry can have a rather different formulation than the one that we know (the standard formulation). Within a group it can be experienced as a matching symmetry.

6.4.2 Geometrical Symmetry as a Matching Symmetry

Let me recall Tyler's way of putting the symmetry about the diagonal:

If (x, y) then (y, x)

This was experienced by the students when tackling the different tasks involved. And it was expressed more or less as Peter expressed it:

> Peter: if you fill in any other point from there, then you have to fill in this point.

However, it was not the only way of experiencing this symmetry. An *alternative* way was the following:

(x, y) *and* (y, x)

Let me give some examples in the context of the tasks.

One way of GENERATING an example is to determine identical-to-be cities in advance. The most basic and most practiced form of this way of generating an example is taking two cities identical. Here is a familiar case:

> Chris: Ok, so for cities; if you take 1 and 2, as a pair, err, they both can be identical, or they mustn't have any in common. So if they are identical, then 1 has to be able to go to 2, *and* 2 has to be able to visit 1.

If we use a grid as the medium of presenting our example, as our students did so, this identicalness of the cities one and two means that the two points (1, 2) *and* (2, 1) are on the grid.

Here is another example of generating an example in which more than two cities are involved:

> Piro: ... as usual the first job that I do is drawing the diagonal. Now, for example, I take city 10 and, say city 6, equal. Now, *I colour them symmetrically.* [Figure 71]

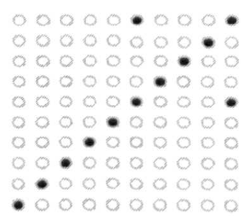


Figure 71

...That means when I take them identical...when *I choose city 6 as a visiting city of city 10, quickly on that side I choose 10 as the visiting city of 6...thus, we are going to save symmetry.* Now, I take 1 and 6; I take them identical... [Figure 72]

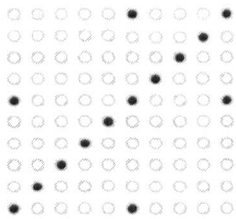


Figure 72

...Because 6 and 10 were identical themselves, the points *1-10 and 10-1 are also coloured...* [Figure 73]

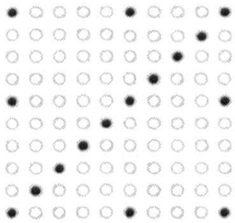


Figure 73

As it can be seen in both cases the identical-to-be cities were determined in advanced. In other words, at the outset it was determined which cites are to be grouped together. But in the background of a single group of certain to-be-related cities the following two accounts of symmetry *are experienced interchangeably*:

If (x, y) then (y, x)

(x, y) and (y, x)

The interchangeability of these two can be seen in Piro's way of generating the above example. By choosing the point (10, 6) and *then quickly* choosing the point (6, 10) on the other side, he "maintains" the first form of symmetry. And by colouring the points (1, 10) *and* (10, 1), he maintains the second form.

Consider that what quantifies these two accounts is determined by the pairs of the points that are to be on the grid (examples above) or are actually on the grid (Piro—CHECKING below). As a result, the replacement of one with the other does not make any difficulty[30], at least as far as an 'operational knowledge of symmetry' is concerned. In particular, this statement applies when the symmetric pairs are already out there, for example, when they CHECK the examplehood of a symmetric figure. Here is an example.

Piro is checking the examplehood of Figure 74.

> Piro: It is symmetric. Now the points 3 and 1 with 1 and 3, the points 5 and 3, 3 and 5, the point 1 and 10, the point 10 and 1, the point 7 and 6, the point 6 and 7…

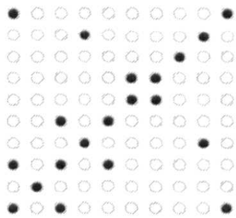


Figure 74

In general, either of the two accounts of symmetry has been used in this situation without causing any problem in the students' realization of the symmetry of a figure. In other words, no one regarded a non-symmetric figure as a symmetric one or a symmetric figure as a non-symmetric one. However, until the students do not differentiate the symmetry (with either of the two accounts) from the underlying group(s), it could affect the way that they tackle the tasks. For example, Piro first thought that Figure 74 is an example. Another interesting case in point is Sarah who for a long time was "just kind of stuck in the idea" of taking any symmetric figure for an example until she was asked to CHECK the above figure (Figure 74).

> Sarah: Yah that is an example too… um, as the example before, because it is symmetrical, one can visit one and three, three can visit one and three, and, sorry one can visit three and ten [silence] oh! Um, I am not sure whether my whole stand here, my whole symmetrical thing, one can visit three and ten, then that must mean that three must be able to visit one and ten also, in order for them to be identical, but three can not visit ten, so therefore my symmetrical idea isn't right at all, so I've been arguing wrong, I knew symmetry wouldn't be the way to go, I just couldn't think a *counterexample* on my head, *because I always works on pair of cities*, yah, there are still pair of city, because in my head

[30] If they are quantified by the universal quantifier on the elements of the underlying set, the "and" account implies that all elements are related to each other. In the context of a grid, it says that all the points are on the grid.

> I've just doing two, but you can have anything up to ten…I thought one visits three *and* three visits one, but I hadn't take into account of them visiting the other towns…

Then she tried to make a counterexample for her own. But since she was working within a single-group undifferentiated from the symmetry she failed to do so.

> Sarah: if I hadn't taken such a simple one, if I tried to do one more, say, if I said one visits three and five, then in my head I might think that all that must mean that as it visits three and five, then three must visit one, three and five, five must visit one, three and five, *that's symmetric*, but I don't think it works [as a counterexample]… this one [Figure 74] is symmetrical but clearly it doesn't work. So this wouldn't be my counterexample.

Sarah did not differentiate the background group(s) from the symmetry in the course of the interview. Every time she attempted to generate a non-example symmetric figure she experienced a single-group together with *all* the possible symmetrical connections between the members of the group. As a result, she could not make a counterexample by herself although she did try on a number of occasions; thus *my* counterexample [Figure 74] remained the only one all the way through. Consider that to generate a non-example symmetric figure she needed to distinguish between a single group of identical elements and the very same group comprised of matching pairs. In other words, she needed to see the same thing in two different ways. She needed to be explicitly aware of the unity of the group involved to be able to break the unity. In this way she could generate the non-example that she was looking for by choosing some of the pairs and removing the others. In this regard, the reader may wish to read again the way that Amit generated a non-example symmetric figure (Section 5.2.2, Textfigure 19).

In sum, the geometric symmetry was one of the ideas that the majority of the students experienced in this situation and even they used the term "symmetry" to describe what they experienced. However, as Sarah's work (above) suggests, to tackle the tasks the students need to discern this aspect of the situation and relate it to the other aspects. But, this applies to any other aspect of the situation. The next section furthers this discussion.

6.5 The integration of grouping and matching

> Maria[31]...has brought about a new way of experiencing the task...it is not a conceptual change—we are not suggesting that she was incapable of experiencing division in this way before, nor that she will continue to be able to do so in the future, but rather in this task she has become capable of experiencing the task in a new way. This is highly situated claim.
>
> (Booth et al, 1999, p. 76)

In Chapters 5 and 6, *I* focused on the grouping experience and the matching experience respectively. However, as far as the students' conceptions are concerned, these two are interwoven. As a result, in chapter 5 we had cases exemplifying the conceptions discussed in chapter 6 and vice versa. In this section, I integrate these two chapters.

Looking back, it can be seen that, according to a methodological choice, the focus of this study has been on the outcomes of learning (learned) rather than on the learners. As a result, the data floated between different categories regardless of whom the data were collected from. However, even the fragmented experiences described in Chapter 5 and 6 suggest something in line with what Marton and Booth call *the path of learning*: "that learning proceeds from a vague undifferentiated whole to a differentiated and integrated structure of ordered parts...the more that this principle applies in the individual case, the more successful is the learning that occurs" (Marton and Booth, 1997, p.138).

To exemplify this principle we need to give more weight to the individuals' voice, something that I avoided doing all the way through. However, I deliberately, and sometimes inevitably, chose a particular student (Hess) to exemplify different conceptions. Indeed, his work manifests a development from "a vague undifferentiated whole to a differentiated and integrated structure of

[31] Maria is an 11-year old girl participating in a phenomenographic study concerning students' understanding of division.

ordered parts". It also shows a development from the matching conception (Section 5.2.1) to the single-group conception (Section 6.4.1), then to a sequential grouping (Section 5.2.2 and 5.2.3), and eventually to the multiple-group conception (Section 5.2.3). A global grasp of F-transitivity saved him from practically matching *all* the possible pairs in a focal group (Section 6.2). He consciously eliminated the connections between the elements of a focal group and then reconnected some of them in order to give (1) a non-example symmetric figure (Section 6.4.1) and, (2) a proof of the intersection task (5.2.3).

Having such a case that in a sense is a representative of the "collective intellect" seems quite exceptional. In most cases, one is aware of a certain aspect when tackling a certain task and is aware of another aspect when tackling another task, and these aspects are not necessarily simultaneously present in his or her focal awareness. As an example, let us follow Amit (a first year mathematics students) across the different tasks. In Section 6.2 we left him in the middle of generating his first example where he came up with the following figure (Figure 75).

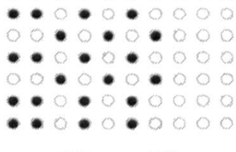


Figure 75

So far, he has proceeded with comparing each *new* city with the previous ones, making them identical (if necessary) or making it different (if possible). Applying the same way on the remaining cities (8, 9 and 10), he completed his first example.

Amit: 8 has to include 8, and if I include any one below 8, then it has to be identical to one of these, but I can make it different by putting in 9 as well, which is allowed [Figure 76],

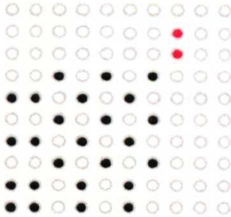


Figure 76

...so 9 has to include number 9, then has to include number 8 as well...[Figure 77]

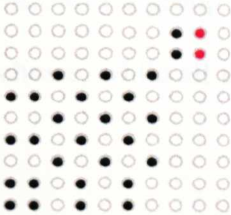


Figure 77

...and in 10 I can just include 10, and leave it [Figure 78].

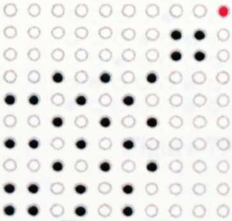


Figure 78

As mentioned in Section 6.2 he exemplified an "operational knowledge of transitivity". The way that he GENERATED his very first example also exemplifies a sequential grouping (see Section 5.2.3). His example per se is comprised of several disjoint groups of related elements. But, none of these is reflected in the way that he is CHECKING his figure (Figure 78).

> Amit: It is an example because, um, all, the main diagonal is filled, which means that, it corresponds to the first law that, when you are in a particular city you are allowed to visit other people in that city, the main diagonal is filled, for each pair of cities, so if you take any two cities, either their visiting cities are identical, so either we are two matching rows, matching columns , or we are two completely distinctive columns, and this satisfies the rules.

Although he did not talk about what he is experiencing when generating an example, there are certainly certain things that *change* from one example (generating) to the other. For example, as a result of *familiarity* with the situation, his competence as an example GENERATOR rose. On the surface, he is now able to *distinguish* between the two conditions of a visiting law.

> Amit: We know from start that we have to have every thing in the diagonal, so I can fill this to start…

On a deeper level, he is now able to determine the *extension* of a group-to-be in advance and fill the necessary points in accord with his choice, while all the other cities (points or columns) are staying in the background.

> Amit: …so suppose we create another example which contains 6 can visit 1, 2 and 4…[Figure 79]

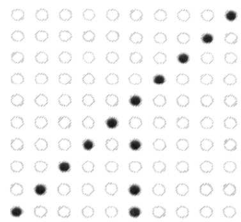


Figure 79

> … and to make it a valid example we know that 1, because 6 can visit 1, 1 can also visit 2 and 4, and because this contains 2 , 2 can also visit 1 and 4 and can also visit 6, and the same is number 4, um, that means this is now a valid example [Figure 80]

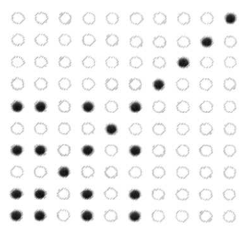


Figure 80

In a similar vein, he can 'add a few extra points' choosing from yet untouched points.

> … but we can change it if we like by adding a few extra points, if you want, suppose that 7 can visit 9, that means that 9 can visit 7, and that's now a valid example [Figure 81].

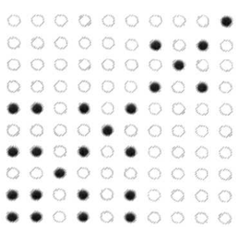


Figure 81

However, still none of these *changes* in the way that he is generating his examples are reflected in the way that he is checking them.

Amit: ...and they do form a valid law because all the main identity, the main diagonal is filled, which means the first law is satisfied, and also *in any case where a pair of cities can be visited by the same city they are also identical.*

It is worth saying that what he said as the reason for the examplehood of his figure is the only account of F-transitivity *locally* expressed in all the data. Probably his previous experience as a mathematics student being taught the relevant subject-matter was implicitly of help to him to come up with this locally-expressed F-transitivity. Whatever brought that conception about, neither it nor any others so far experienced were reflected in the intersection task where he attempted to generalise the argument to an example-free situation—an argument that works for *any* two examples rather than two concrete examples. Instead, it is a newly spoken-out concept that matters, viz. geometric symmetry.

Amit: I think the answer is true, that if we take any two valid examples and take the common points we also get a valid example...I think the reason is if we have valid examples, because of the second law, the second law ensures that we have some symmetry here... so if we take this symmetry law as truth for every example, then if we take the combination of two examples, or their common points, then it'll also have symmetry as well.

It seems that each single-group *implied* by the way that he was generating his examples lost its unity (if it had that) when proving his affirmative answer to the intersection task. As a result, the symmetry was experienced as a property of an undivided whole, although *we* can see in his examples (Figures 78 and 81) a divided whole. The following two figures schematically represent these two different ways of seeing an example, say Figure 78. Figure 82 focuses on the groups, while Figure 83 focuses on the symmetry.

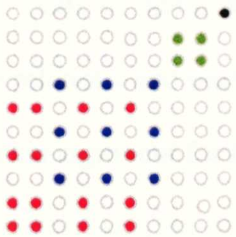


Figure 82

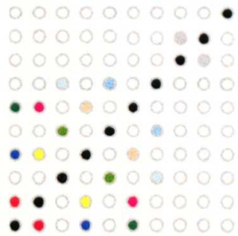


Figure 83

It is in doubt whether Amit himself saw Figure 78 as Figure 82. Even if we argue that he somehow manifests a multiple-group conception when generation his examples, we can not say that he had a *conceptual change* (in the sense of Booth et al, 1999, p. 76; see above) say from the sequential grouping to the multiple-group conception in which the symmetry is differentiated form the underlying groups. Remember that generating a non-example symmetric figure took him by surprise (Section 5.2.2). To generate this he learned to distinguish between a single group of identical elements and the very same group comprised of matching pairs. As a result, he generated a non-example symmetric figure by choosing some of these pairs and removing the others (Section 5.2.2). Then he used this *new* way of experiencing the task to revise his proof of the intersection task. To do so, even in the context of the single-group example, he needed to go one step further. That is to say, he needed to *eliminate* even the symmetric connection between any two elements of the underlying group, see each element as an individual, and eventually rematch some of them (the ones in the intersection) with each other. But he failed to do so. As a result, his proof seems to be a complex way of arguing that the intersection of two *identical* single-group examples is an example (Section 5.2.2). It is interesting that he could not successfully accomplish the union task for the same reason that he could not do so the intersection task. He was one of the two students who thought that the union of any two examples is an example (noteworthy both of them were mathematics students having been taught equivalence relations and partitions!). To illuminate the above explanations let me mention his argument for the union task.

Amit: If I do the union, I think it is also a valid example.

Interviewer: could you explain?

Amit: the reason is if I take all the points, which mean the union of the first two, *then for every point, which is in the first example, It had to also include all the points, which are necessary to maintain the fact that one is an example,* and similarly for two,

Then he decided to continue with two concrete examples. Accordingly, he easily generated the following two examples:

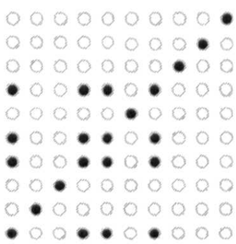


Figure 84

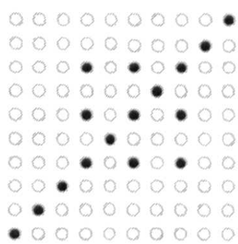


Figure 85

The union of Figure 84 and 85 is not an example. However, instead of practically finding the union of these two, Amit himself carried a *thought experiment* as follows. Doing so, he could not see a counterexample in front of his eyes.

Amit: this is a valid example, taking the union of this, the union will be a valid example because, suppose it is not a valid example, that means one of two laws is violated, we know that the first law is not violated because we have the main diagonal, so the only possibility is that the second law is violated, um, now the second law states that for each pair of cities either their visiting-cities are identical or they are not in common, so the only way that this can be violated, is it, two columns have an element [a point] in common but they are not identical; can this happen?

To answer this question we need to take a point, break its structure as a point, see its constituting elements as individuals, and check whether they are matched again as two members of a new group in the union. In this regard, the reader may wish to read the way that Peter tackled the union task (Section 5.2.1). Coming back to Amit, it can be seen in the following that he does not eliminate the structure of a point and he does not separate a point from its original group. Each point is attached to its original group and it is carried to the union together with its original group. As a result, Amit envisages that the union is a valid example:

> Amit: When I take the union, I think the union will be a valid example because, um, for every point that I include, I mean, I can include one-four, because one-four is in the first example, I know I will also, that are not going changing the rules because all the other points which are required to avoid breaching the rules are definitely present, because they are present in the first example, the fact that one-four was present in the first example means that four-one must also present in the first example, because this is a valid example, and if I take a point from union then the same applies to every points, so for example I know that four-seven appears in the first example, because this is a valid example I know that seven-four also appeared, because this is a valid example I know that, um, since one can visit four and seven can visit four, I know that one, seven must be able to visit one, I know that one must be able to visit seven, so I can build up this way and I will never breach the laws of being a valid example, because I am taking the elements from one, the elements from the other, *if I am going to breach the law here, then I must breach the law somewhere before.*

It is worth stressing that to *successfully* tackle these particular tasks he needed to be simultaneously aware of many different kinds of connections among the elements of the original examples. These connections are (1) the connections between all the constituting elements of each group, (2) the connection between each two symmetric points in the group, (3) the connection between each point and its constituting elements. Being aware of the first two kinds of connection he could successfully generate a non-example symmetric figure. But as a result of missing the last one, he suggested a very limited argument for the intersection task and basically gave a *wrong* answer to the union task.

Amit's experience, in a sense, reflects any other individual's experience of the Mad Dictator Task. That is, on the one hand, an experience of grouping, de-grouping and regrouping, and on the other hand, an experience of matching, de-matching and re-matching. At the outset, these form "a vague undifferentiated whole". Then, in the course of tackling different tasks they are differentiated and integrated. The more that this principle applies in the individual case, the more successful is the way of tackling the task at hand. However, it does not mean that the aspects that have been differentiated and integrated when handling a certain task are also being carried to another task. They are *usage-specific*. That is to say, specific to the activity at hand, that, GENERATING an example, CHECKING the status of something as an example of something else, or PROVING. This suggests a way to further this study. That is to investigate the possible interconnection between each one of these activities and the students' conceptions. To do so, we need to give more weight to the individuals' voice. However, again it is not the individuals per se that matter; the focus will be on what they reveal about the activities. Although it is a worthwhile way to continue this study (see Asghari, 2005a), I have chosen a different way, i.e. to investigate students' conceptions of equivalence relations and partitions!

If we assume that matching and grouping somehow resemble making equivalent and partitioning respectively, the picture given in this study, and in particular in this section, comes as a non-standard picture. Generally speaking, in a standard setting, a successful student is the one who is able to unite the notions of equivalence relation and partition, while in the setting of this study, a successful student is the one who is able to differentiate them. But, this comparison is based on this fundamental *assumption* that the students' experiences in this particular situation reveal something about their experiences of equivalence relations and partitions. Chapter 8 is to scrutinize this assumption once more. To do so, I show what has been portrayed as the students' conceptions in this particular situation can be also observed in some other situations that are commonly *believed* to embody the notions of equivalence relation and partition. In this regard, the 'history' of the subject provides a valuable source. This history will be dealt with in Chapter 7. These interconnections between different sources in a way support my assumption besides exemplifying a fundamental aspect of the phenomenographical results.

> Conceptions and ways of understanding are not seen as individual qualities. Conceptions of reality are considered rather as categories of description to be used in facilitating the grasp of concrete cases of human functioning. Since the same categories of description appear in different situations, the set of categories is thus stable and generalizable between the situations even if individuals move from one category to another on different occasions. The totality of such categories of description denotes a kind of collective intellect, an evolutionary tool in continual development. (Marton, 1981, p. 177)

Part 4: Towards a unifying picture

Introduction

This part mainly concerns itself with what *I* learned about equivalence relations and partitions. I base my interpretations on the results given in Chapters 5 and 6, on the literature reviewed in Chapter 4, on the papers of Chin and Tall (2000, 2001 and 2002) and on the 'history' of the notions of interest. From these four different, but related, sources the first one is more problematic than the others; any use of it will inevitably lead to this question that to what extent the students' experiences of this particular situation (the Mad Dictator task) reveal something about their conceptions of equivalence relations and partitions, if it reveals something in that direction at all. Talking of the sources, it seems that making use of the so-called historical instances of equivalence relations creates more or less the same problem as the one that I am faced with when using the students' experience of the Mad Dictator Task. For example, to what extent and in what sense does Euclid's account of the notion of equality or ratio say something about equivalence relations. Consider that the latter is a 20^{th}-century notion, more than two millennia after Euclid's Elements.

To resolve the above problems I show that these sources together give us a picture that seems to be more complete than any picture that can be gained from each single source alone. Thus, in a sense, the *choice* of the content of interest is supported indirectly by the extent of the usefulness of the results of the study. In this regard, I particularly show that each one of these sources can shed light on the others, and all together, they produce useful insights into teaching and learning of the subject. After all, what ultimately matters is the teaching and the learning of the subject matter rather than what I have learned or what the reader(s) of this thesis will learn from it.

This part includes three chapters, Chapters 7, 8 and 9.

Chapter 7 concerns itself with some historical instances of equivalence relations and equivalence classes. In this chapter, I will use the students' conceptions described in the previous two chapters to interpret some historical events as to the subject of interest. I will argue that there are many similarities between the students' conceptions and the ways that some great mathematicians of the past have tackled certain situations. More importantly, the history of the subject shows that the picture drawn so far lacks at least one important aspect related to the notion of equivalence relation. This fundamental aspect is making use of an equivalence relation to *create* a new entity, i.e. the so-called "definition by abstraction". I will use this new aspect to clarify and subsequently modify the meaning of the so-called "unity" of a group.

In Chapter 8, the interpretive scope of the findings of this study will broaden by reinterpreting a set of data gathered by Chin and Tall (2000, 2001 and 2002) and originally interpreted from the vantage point of the standard axiomatization of equivalence relations. The feasibility of making a comparison between this study and Chin and Tall's study suggest that these two *different* studies both refer to the same *content*. Exactly what this content is will be the theme of the remainder of Chapter 8. At the end of this chapter, I will directly address the extent to which this study has investigated what it was intended to investigate.

Chapter 9 explores some of the pedagogical implications of this research.

Finally, Chapter 10 concludes this thesis with a short epilogue.

CHAPTER 7: HISTORICAL INSTANCES

7.1 Introduction

> Some early examples of equivalence-class-style arguments do occur in contexts where some specific well-defined concrete set underlies the subject—Gauss' number theory in his Disquisitiones Arithmeticae is an outstanding example—but the general technique of appealing to equivalence classes appeared only around the time of Dedekind, when set theory began to be introduced as a basis for mathematics in general, and it took some time to become established. This gives a good example of a piece of mathematics that has been popularised only recently, but which has already been retrospectively written back into the 'history' of the subject, even back to Euclid, and all this has happened almost within my own lifetime, in a process that is every bit as efficient and thorough as the work of the thought police in George Orwell's *Nineteen Eighty-Four.* (Fowler, 1999, p. 371)

Let me clearly state the scope of this chapter. I shall start with what this chapter is not about, and what I will avoid doing.

First, this chapter does not concern itself with the chronological order of the historical 'accounts' as to the ideas of equivalence relation and partition; it takes a thematic rather than chronological approach to the ideas. I will not intend to answer the questions about the origins of this and that idea, or this and that term. Having said this, I shall briefly add that it is very strange that there is no general agreement on the origin of such recent ideas and recent terms, even if we confine ourselves to the origin of what today is called an equivalence relation (For an illuminating debate, see the online postings in Historia Mathematica under the title "concept of equivalence relation". Hereafter Historia Mathematica will be abbreviated as HM). For example, if you ask who originated the combination 'equivalence relation' or 'equivalence class', you can hardly find an answer. Regarding the idea per se, the situation is barely better than the term. David Fowler knowledgeably suggests we should carefully separate the notion of equivalence relation from the notion of equivalence class, (I shall add) at least as far as the history of these *two* notions is concerned:

> I think that the equivalence-relation-idea is very old and relatively unproblematic. But to then do the equivalence-class-step, we need an idea of a class/set/..., and there's nothing in general like that before Dedekind. However, where some clear kind of set is already there, in the nature of the topic, equivalence class-style arguments may be used. Big example: Gauss, Disq Arith, who formulates perfect such procedures in N. (Fowler. "Equivalence classes as object." Online posting. 22 Aug 1998. HM)

However, Fowler himself immediately mentions that probably others disagree with crediting Dedekind with having enunciated and used "perfect equivalence class arguments". Indeed, others disagree. Besides Dedekind, at least Peano, Frege and Russell are among the candidates. Each one of them that we choose would be rejected in favour of *the* other. For example, Dummett (1991), in his book on Frege's philosophy of mathematics, while obviously discrediting Dedekind in favour of Frege for his definition of number (the one that Fowler praises Dedekind for), writes:

> One of the mental operations most frequently credited with creative powers was that of abstracting from particular features of some object or system of objects, that is, ceasing to take any account of them. It was virtually an orthodoxy, subscribed to by many philosophers and mathematicians..., that the mind could, by this means, create an object or system of objects lacking the features abstracted from, but not possessing any others in their place. It was to this operation that Dedekind appealed in order to explain what the natural numbers are. ...Frege devoted a lengthy section of *Grundlagen*, §§29-44, to a detailed and conclusive critique of this misbegotten theory; it was a bitter disappointment to him that it had not the slightest effect.
>
> (Dummett, 1991, p.50)

On the other hand, Rodriguez-Consuegra (1991, pp. 155-156), in his book on the mathematical philosophy of Bertrand Russell, credits Russell who independently from Frege gave his definition of cardinal numbers:

> It is usually accepted that, a little after attending the Paris Congress of 1900, Russell obtained (independently from Frege) his famous definition of cardinal number as a class of classes for the first time...

Russell himself has contributed to this view on his "introduction to mathematical philosophy":

> The question "what is a number?" is one which has been often asked, but has only been correctly answered in our own time. The answer was given by Frege in 1884, in his *Grundlagen der Arithmetik.* Although his book is quite short, not difficult, and of the very highest importance, it attracted almost no attention, and the definition of number which it contains remained practically unknown until it was rediscovered by the present author in 1901.
>
> (Russell, 1919, p. 11)

Here and there, Russell also reminds us of the role of Peano in introducing '*the principle of abstraction*', underlying the process exploited to define numbers:

> Peano has defined a process which he calls definition by abstraction, of which, as he shows, frequent use is made in Mathematics. This process is as follows: when there is any relation which is transitive, symmetrical and (within its field) reflexive, then, if this relation holds between u and v, we define a new entity Ø (u), which is to be identical with Ø (v). (Russell, 1903, p. 219-220)

To define a number, Russell makes use of the principle of abstraction. He first defines the similarity between two classes, and then shows that the similarity is a relation possessing the three properties of being reflexive, symmetrical and transitive (see Section 7.2). Although the properties of similarity are clearly the defining properties of what today is known as an equivalence relation, the next sentence shows that this name was not in use at that time:

> Relations which possess these properties are an important kind, and it is worth while to note that similarity is one of this kind of relations. (ibid, p.16)

Let me now mention the **second** realm that I do not intend to enter in this chapter. I avoid engaging myself in the philosophical discussions interwoven with every aspect of the subject. Only naming Frege and Russell among the possible originators is enough to show how keeping away from such discussions is difficult, and the extent to which I will be exposed to missing some valuable information. Nonetheless, I will try to confine myself to the mathematical texts or the mathematical parts of the philosophical texts.

Third, I will try to avoid a common tendency towards the 'history' of the subject. That is, I will not "retrospectively write back" the standard account of the subject into the history. It is a normal practice to read the so-called historical examples in the light of *the* standard account of equivalence relations. For example, Euclid's first Common Notion, "Things which are equal to the same thing are also equal to one another", is followed by statements like these: This expresses that equality is *transitive*. It can also easily be seen that it is *reflexive* and *symmetric*. Then, each one of these properties is followed by its standard definition. For example, in David Joyce's[32] guide to the first Common Notion (http://aleph0.clarku.edu/~djoyce/java/elements/elements.html) we read:

> The first Common Notion could be applied to plane figures to say, for instance, that if a triangle equals a rectangle, and the rectangle equals a square, then the triangle also equals the square.

Compare this with the way that Euclid himself uses the first Common Notion in the proof of Proposition I of Book I:

> ...each of the straight lines CA, CB is equal to AB. And things which are equal to the same thing are also equal to one another: therefore CA is also equal to CB. (Heath[33], 1926, Volume I, p.241)

Neither Euclid's first Common Notion per se nor the way that he uses it directly conveys the same thing as Joyce's interpretation (Joyce also immediately shifts to the *standard* account of the subject).

Joyce is not alone in giving such interpretations. In this case, he has simply carried on the tradition of "retrospectively writing back into the 'history' of the subject". In effect, any other single piece that *recently* has been written on the subject could exemplify this unfortunate attitude. In one of my earlier writings with Tall (Asghari and Tall, 2005, p. 86) we wrote:

---

[32] David Joyce has created an award-wining web version of Euclid's Elements including the text of all 13 Books followed by Joyce's guides. Hereafter his website will be referred to by DJ.

[33] My references to Euclid are cited from Heath's edition of Euclid's Elements unless otherwise stated. Hereafter the quotes from Heath's edition will be referred only by the volume number and the page number.

> ... in some of the earliest formal notions relating to equivalence, the Greek notion of two lines l, m being 'parallel' is shown to satisfy the two properties 'a P b implies b P a' and 'a P c and b P c implies a P b'. But a is not parallel to itself. (How could it be? Two parallel lines have no points in common but a has all its points in common with itself).

Even though we were careful enough about the notion of transitivity, we missed the notion of symmetry. Consider that it is not shown by Euclid that 'a P b implies b P a'. As a matter of fact, regarding parallel lines, he treats them simply as a *pair* of parallel straight lines (see Section 7.3).

Let me now turn to what this chapter is intended for. Although some (if not all) of the things-to-be-done have already been mentioned in parallel with the things-not-to-be-done, it is worth collecting them together. In particular, I shall expand on my comments on reading the 'history' of the subject.

> **First**, this chapter takes a thematic rather than chronological approach to the ideas.
>
> **Second**, it will be confined to mathematical texts (and/or textbooks) or the mathematical parts of philosophical texts.
>
> **Third**, it will replace one construct with another for reading the history of the subject.

The third point certainly needs more explanation. I have already, though implicitly, applied this *approach* in my discussion on the historians' approach to the subject (see above, the third thing of the things-not-to-be-done). Consider that I have used something that is not among historians' tools when approaching the subject. That is, the students' conceptions of the subject. I have used the students' conceptions to draw a picture of some historical events, and I will do so in more details through this chapter.

As a matter of fact, the mathematics education community is no stranger to a certain kind of *relation* between students' conceptions and historical ones. In the literature, this relation is embodied in the alleged relation between *ontogenesis* and *phylogenesis* i.e. generally speaking, between "the *development* of students' mathematical thinking" and "the historical conceptual mathematical *development*" respectively (Furinghetti and Radford, 2002, p. 633; emphasis added). Furinghetti and Radford examine different interpretations of the relations between ontogenesis and phylogenesis, and in particular, different interpretations of the so-called "recapitulation law". That is, "in their intellectual development our students naturally traverse more or less the same stages as mankind once did" (ibid, p. 633). Let me only focus on the latter that is *seemingly* more related to the theme of this section.

Furinghetti and Radford consider the interpretations of "recapitulation law" given by three important mathematicians, all concerning the use of history of mathematics in teaching. I shall mention only one of them and add something similar from Freudenthal who more or less states the same idea as to learning more than eighty years later.

> Zoologists claim that the embryonal development of animals summarizes in a very short time all the history of its ancestors of geologic epochs. It seems that the same happens to the mind's development. The educators' task is to make children follow the path that was followed by their fathers, passing quickly through certain stages without eliminating any of them. In this way, the history of science has to be our guide. (Poincaré, 1899, p. 159; translated by Furinghetti and Radford, 2002, p. 638)

In his discussion on the Socratic Method, Freudenthal writes in a similar fashion:

> To acquire knowledge is re-discovering not what others knew before me but rather what I myself knew when my soul stayed in the realm of the ideas. We need not devour Socrates to the last morsel and we need not share his belief in pre-existence. What then remains is learning by re-discovery, where now the "re" does not mean the learner's pre-history but the history of mankind. It may seem as though the learner is repeating the development of his ancestors in

> rediscovering what they knew. Therefore I would prefer to call it re-invention, but this is an unimportant point of terminology. (Freudenthal, 1973, p. 102)

It is, on the face of it, a tempting claim in particular as to the subject of my interest. As my quick journey into the history will show there are many similarities between the students' conceptions described in the previous chapter and the ways that the great mathematicians of the past have tackled certain situations. However, to claim that these similarities are developmental parallelism—between the "development of students' mathematical thinking and historical conceptual mathematical developments"—certainly needs further research and a different methodology, although the existence of such methodology has been called into question.

> One of the problems with the recapitulationist approach is that...the idea of history is reduced to a linear sequence of events judged from the vantage point of the modern observer. In all likelihood, the extremely low number of studies that attempt to check the validity of recapitulation law is evidence of the impossibility of reproducing the conditions in which ideas developed in the past. (Furinghetti and Radford, 2002, p. 650)

Recalling the students' conceptions elaborated in Chapters 5 and 6, I shall stress that they show the variations in the students' conceptions rather than the students' conceptual development. Likewise, I am looking into the history of the subject to pinpoint the historical variations rather than the historical development. Thus, in a sense, "the relations between conceptual and historical developments" has been replaced with the relations between conceptual and historical variations. Let me call the latter *variational approach* to history and start my journey after such a long introduction. All the way, I will make use of the students' conceptions as a window through which I make sense of the historical situations that "from the vantage point of the modern observer" embody the idea of equivalence relation.

7.2 Historical Conceptions of Equivalence Relations

I have already mentioned Russell's treatment of numbers. It is worth completing it as it captures many of the conceptions discussed in Chapter 5 and 6. Russell starts with the definition of "similarity":

> Two classes are said to be "similar" when there is a one-to-one relation which correlates the terms of the one class each with one term of the other class, in the same manner in which the relation of marriage correlates husbands with wives. (Russell, 1919, pp. 15-16)

Then he informs us of the properties of similarity:

> It is easy to prove (1) that every class is similar to itself, (2) that if a class α is similar to a class β, then β is similar to α, (3) that if α is similar to β and β to γ, then α is similar to γ. A relation is said to be reflexive when it possesses the first of these properties, symmetrical when it possesses the second, and transitive when it possesses the third. It is obvious that a relation which is symmetrical and transitive must be reflexive through its domain. (ibid, p. 16)

Then he defines the number of a class:

> *The number of a class is the class of all those classes that are similar to it.* (ibid, p. 18; italic in original)

His definition of the number of a class reflects a single-group conception in which all those classes that are similar to a focal class are grouped together. Then, to define *a* number, he eliminates the centrality of that focal class:

> A number will be a set of classes such as that any two are similar to each other, and none *outside* the set are similar to any *inside* the set. (ibid, pp. 18-19; emphasis added)

It still reflects a single-group conception with a clear distinction between inside and outside. Although a number is a set of sets (or class of classes) it does not reflect a multiple-group conception by the description given in Section 5.2.3. Thus, a number is a single-group among an infinite number of other single groups, all together constituting a multiple-group. As a single-group there is a clear *comparison* between what is outside the group and what is inside. Each focal group (each number in focus) gains its unity from the unity of the underlying class (in the sense of Hausdorff; see Section 5.4) that has been established earlier on by using the metaphor "bundle[34]":

> We can suppose all couples in one bundle, all trios in another, and so on. In this way we obtain various bundles of collections, each bundle consisting of all the collections that have a certain number of terms. Each bundle is a class whose members are collections, i.e. classes; thus each is a class of classes. The bundle consisting of all couples, for example, is a class of classes: each couple is a class with two members, and the whole bundle of couples is a class with an infinite number of members, each of which is a class of two members.
>
> (Russell, 1919, p. 14)

Let me expound even more on Russell's definition of a number. We saw that:

A number will be a set of classes such as that any two are similar to each other, and none outside the set are similar to any inside the set.

Immediately Russell (ibid, p.19) introduces a different way of describing the above set in terms of the number of *one* of its members.

In other words, a number (in general) is any collection which is the number of one of its members (i.e. the class of all those classes that are similar to it).

Let us see how these two definitions are related to each other. Because of the *transitivity* and *symmetry* of similarity 'any two classes similar to the assigned class are similar to one another also' (thus any two are similar to each other). Moreover, by definition, none outside the set is similar to the assigned class inside the set. Hence it is not similar to any inside the set, because 'if two classes are inside the set, and the one of them is not similar to any class, the remaining one will not also be similar to the same' (thus none outside the set are similar to any inside the set).

[34] Russell himself discusses 'the unity of a class' in a more direct way somewhere else, for example, by distinguishing "the many from the whole which they form" (1903, p. 70) or by considering "*the* in the plural"(1919, p. 181): "In the present chapter [Classes] we shall be concerned with *the* in the plural... In other words, we shall be concerned with *classes*".

The phrases enclosed in the inverted commas in the above paragraph are the result of my deliberate attempt to link Russell's claims to Propositions 12 and 13 in Euclid's Book X (X12 and X13). They are as follows (Vol III; pp. 34-36)

> Proposition 12: Magnitudes commensurable with the same magnitude are commensurable with one another also.
>
> Proposition 13: If two magnitudes be commensurable, and the one of them be incommensurable with any magnitude, the remaining one will also be incommensurable with the same.

And Definition 1 in Book X reads:

> Those magnitudes are said to be **commensurable** which are measured by the same measure, and those **incommensurable** which cannot have any common measure. (Vol 3, p. 10)

These bear a remarkable resemblance to Russell's ingredients. However, *structurally* speaking, it does not seem that they have made a similar thing out of it. To see where they differ I shall devote a separate section to Euclid.

7.3 Euclid (and Hilbert)

We saw that there is a startling similarity between Euclid's notion of commensurability in Book X and what Russell calls "this important kind of relations" (equivalence relations). Here I discuss to what extent this similarity holds.

As far as Book X is concerned let me rely on Fowler's judgment:

> We have...a straightforward instance of transitivity in X12...and the relation is clearly reflexive and symmetric; so will we not find here an equal natural appeal to the equivalence class containing a given magnitude, corresponding to our important idea of the set of rational numbers with the size of that magnitude as unit? (Fowler, 1999, p.370)

Fowler's answer is negative, "a firm though complicated 'No'". He suggests two different reasons for his negative answer. Both reasons take me into Book X and involve spending quite a lot of time on some *unfamiliar* definitions. Thus, let me only give the core of his reasons.

> First, the underlying *equivalence class* in Book X, if we insist on imposing this point of view, is not the one that seems natural and familiar to us.
>
> Second, Euclid always considers *individuals*, never sets.
>
> (ibid, p. 370; emphasis added)

It seems that at best Euclid has conceptualized a *single group*, "a class R of straight lines...whose members are called rational" (Mueller, 1981, p. 267). After that, he describes some kinds of *irrationals* lines, all treated as individuals (Consider that Euclid's usage of the terms *rational* and *irrational* differ from *us*. For example, for Euclid, the diagonal of a square with rational side is rational. For a thorough description see, Fowler 161-188, or van der Waerden, 1954, pp, 168-172).

Before leaving Book X and turning to more familiar realms I shall once more refer to it and Fowler's interpretation of Proposition 12. We read:

> We have...a straightforward instance of transitivity in X12...and the relation is clearly reflexive and symmetric. (Fowler, 1999, p.370)

In my terminology we have a straightforward instance of F-transitivity in X12 rather than transitivity. As a matter of fact, it is only in Euclid's proof of Proposition 13 that we can find a straightforward instance of the transitivity. In the middle of his proof he says:

> For, if B is commensurable with C, while A is also commensurable with B, A is also commensurable with C. (Vol III, p. 36)

In every other place that from our point of view some kind of equivalence is concerned Euclid has a preference for the F-transitivity when he *formalises* a general statement. However, when he comes to practice, transitivity and F-transitivity are often used interchangeably. Thus, his very first Common Notion reads:

> Things which are equal to the same thing are also equal to one another.
>
> (Vol I, p. 155)

Even in practice, up to Proposition 29 any time that he makes use of the first Common Notion (e.g. in Propositions 1, 2, 3, 13, 14, 15) he sticks with F-transitivity. I have already given one example in which Euclid has applied the F-transitivity on *three* straight lines (see Section 7.1). Here is an example that he applies it on angles. In his proof of Proposition 13 of Book I we read:

> Therefore the angles DBA, ABC are equal to the three angles DBE, EBA, ABC.
> But the angles CBE, EBD were also proved equal to the same three angles;
> and things which are equal to the same thing are also equal to one another;
> therefore the angles CBE, EBD are also equal to the angles DBA, ABC.
> (Vol I, p. 275)

Thus, so far Euclid's use of the first Common Notion complies with its formulation. It is only in his *proof* of Proposition 30 that Euclid does not obey his formulation of the first Common Notion, although the formulation of Proposition 30 per se still follows the same structure as the first Common Notion:

> Straight lines parallel to the same straight line are also parallel to one another.
> (Vol I, p. 314)

The first line of the proof, where Euclid specifies three generic straight lines, complies with the form of the proposition:

> Let each of the straight lines AB, CD be parallel to EF; I say that AB is also parallel to CD.

Then, in the other part of the proof he makes use of the transitivity:

> The angle GHF is equal to the angle GKD.
> But the angle AGK was also proved equal to the angle GHF;
> Therefore the angle AGK is also equal to the angle GKD. (Vol I, p. 314)

It is interesting that before this proof any use of the first Common Notion is followed by stating it; here for the first time Euclid does not state it.

Finally, in a similar fashion, In Book V, where Euclid deals with ratios and proportions, Proposition 11 has the same style as all the other general statements claiming the equivalence of two things based on their equivalence to a third:

> Ratios which are the same with the same ratio are also the same with one another. (Vol II, p. 158)

Unlike Proposition 30 of Book I and Proposition 12 of Book X (see above), right at the start of the proof of this proposition he interprets it in terms of the so-called transitivity.

> For, as A is to B, so let C be to D,
> and, as C is to D, so let E be to F;
> I say that, as A is to B, so is E to F. (Vol II, p. 158)

Let me put all of this *evidence* together. When stating a general statement, Euclid makes use of an F-transitivity style. On the other hand, where operational use of that statement is concerned he freely switches from F-transitivity to transitivity and vice versa.

It could be argued that whenever Euclid deals with some kind of *equivalence* the F-transitivity and transitivity amount to the same thing because of the *symmetry* embedded in the situation. Having said that, I shall stress that "Euclid missed symmetry" (DJ). However, the symmetry that he missed is the standard account of symmetry as known today. It is a fact that Euclid never used an "if-then" account of symmetry. But, more interesting fact is that he used what I called "symmetry of the matching" (Section 6.3). That is, an "and" account of symmetry: simply two things are matched together. Generally speaking, he works with a pair of *two* things that are *equivalent* to each other. The symmetry in the *status* of the *two* things involved is maintained by the given definitions: "**Parallel** straight lines are...", "Those magnitudes are said to be **commensurable** which ..." and so on.

To highlight this distinction I shall compare Euclid's treatment with a *modern* axiomatization of geometry. That is Hilbert's (1971, pp. 10-11). It is only after establishing the *symmetry* of segment congruence that we read:

> Due to the symmetry of segment congruence one may use the expression "Two segments are congruent to each other."

And the symmetry of segment congruence, as we may expect, has been *defined* as follows:

If	$AB \equiv A'B'$
Then	$A'B' \equiv AB$

It seems what comes first when conceptualizing is last when formalizing.

There are still some startling points in this fruitful comparison. Interestingly, Hilbert also starts with the F-transitivity of segment congruence and then moves to the transitivity. He takes as primary the notion of "congruence" (or "equal") between segments. His first axiom of congruence "requires the **possibility of** ***constructing*** **segments**". Then in the second axiom he establishes *F-transitivity*.

> III, 2. If a segment A'B' and a segment A"B", are congruent to the same segment AB, then the segment A'B' is also congruent to the segment A"B", or briefly, if two segments are congruent to a third one they are congruent to each other.

The so-called properties of equivalence relations follow from the axioms. However, Hilbert only names two of them, namely, symmetry and transitivity.

> Since congruence or equality is introduced in geometry only through these axioms, it is by no means obvious that every segment is congruent to itself. However, this fact follows from the first two axioms on congruence if the segment AB is constructed on a ray so that it is congruent, say, to A'B' and Axiom III, 2 is applied to the congruences $AB \equiv A'B'$, $AB \equiv A'B'$.
>
> On the basis of this the *symmetry* and the *transitivity* of segment congruence can be established by an application of Axiom III, 2; i.e., the validity of the following theorems:
>
> | | |
> |---|---|
> | If | $AB \equiv A'B'$ |
> | then | $A'B' \equiv AB$ |

If	$AB \equiv A'B'$
and	$A'B' \equiv A''B''$
then	$AB \equiv A''B''$

Such an argument in which the transitivity or F-transitivity are applied to only *two objects* never occurs in Euclid's Elements. Whenever Euclid applies one of these properties *three different* elements are involved. In this regard it seems that even Hilbert has a selective memory. Consider the following two definitions of one of *today's* classical example of an equivalence relation, i.e. the relation of being parallel:

> Euclid's Definition 23, Book I: Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction. (Vol I, p. 154)
>
> Hilbert's Definition of parallels: Two lines are said to be parallel if they lie in the same plane and do not intersect. (Hilbert, p. 25)

Both definitions do not allow that a line be parallel to itself. Both give an equal (symmetric) status to both lines. Now compare the following two statements:

> Euclid's Proposition 30, Book I: Straight lines parallel to the same straight line are also parallel to one another. (Vol I, p. 314)
>
> Hilbert (a requirement equivalent to the Axiom of Parallels): If two lines a, b in a plane do not meet a third line c in the same plane then they also do not meet each other. (Hilbert, p. 25)

Now, if we apply the same argument that Hilbert once used for *two* congruent segments, we are faced with what has been ruled out, i.e. the reflexivity of the relation of being parallel: $a \parallel a$ (a is parallel to a) follows from $a \parallel b$, $a \parallel b$. Thus, it is evident that when parallels are concerned, Hilbert only can apply *F-transitivity* to *three* different elements. Moreover, neither him nor Euclid allow that **a line be parallel to itself**. In other words, the relation of being parallel is not *reflexive* in either of these treatments. However, as regards the reflexivity of the notions that are more *visibly* related to *equivalence*, Euclid distinctly differ from Hilbert and all the others whom I have cited (or will cite).

Generally speaking, Euclid never expresses a phrase like, 'a is equivalent to (read it "is equal to", "is parallel to", "is commensurable with") itself'. He employs a rather *alternative* approach. That is, in a sense, applying 'one and the same thing twice'. In this regard, let me examine those propositions in Book I where there is a *need* for the *reflexivity* of "equality", namely, the propositions that Heath has unequivocally decided that Euclid has used the following Common Notions:

> C. N. 2. If equals be added to equals, the wholes are equal.
> C. N. 3. If equals be subtracted from equals, the remainders are equal.
> (Vol I, p. 155)

In I2 (Book I, Proposition 2) we read: (In the following the square bracket is Heath's; A and B are two points on DL and DG respectively)

> DL is equal to DG.
> And in these DA is equal to DB;
> therefore the remainder AL is equal to the remainder BG. [C. N. 3]
> (Vol I, p. 244)

As the link in the square bracket suggests, this is a straightforward application of C. N. 3 in which (two) equals are added to (two) equals.

In I13 we have:

> ...the angle CBE is equal to the two angles CBA, ABE,
> let the angle EBD be added to each;
> therefore the angles CBE, EBD are equal to the three angles CBA, ABE, EBD. [C. N. 2] (Vol I, 275-276)

If we interpret these lines as an application of the second Common Notion, the middle line (let the angle EBD be added to each) implies the reflexivity of equality, i.e. the angle EBD is equal to itself; even in more absurd form, the angle EBD is equal to the angle EBD (though it seems absurd, it is the way that I have been taught Euclidian geometry!). Euclid himself does not use either of these two forms. As mentioned before, in a sense, he applies a "common" thing twice. Even he uses the word "common". However, in this case, for the sake of fluency, Heath has decided to remove the word:

> **Let the angle EBD be added to each**, literally "let the angle EBD be added (so as to be) common,"... "let the common angle be subtracted" as a translation...would be less unsatisfactory, it is true, but, as it is desirable to use corresponding words when translating the two expression, it seems hopeless to attempt to keep the word "common," and I have therefore said "to each" and "from each" simply. (Vol I, p. 276)

On the one hand, Euclid's insistence on using the latter approach rather than the reflexivity of "equality", and on the other hand, the very fact that he never appeals to a reflexivity-style-phrase , both suggest the difficulty of applying a relation that is basically experienced between *two* objects on only *one* object (see Section 1.6.1 and also Section 9.2.1).

Let me finish my short historical journey with another great and historically influential piece of writing. That is "Disquisitiones Arithmeticae".

7.4 Gauss

The first four pages of Gauss' "Disquisitiones Arithmeticae" (1801; translated by Clarke, 1966) in a way reflect all the main conceptions described in Chapter 5, the matching conception, the single-group conception and the multiple-group conception. Moreover, it provides a global grasp of F-transitivity (Section 6.2) besides another example of the "missing symmetry". In the following, I shall use my terminology alongside Gauss' work.

In the first definition he states a matching condition on the integers:

> If a number a divides the difference of the numbers b and c, b and c are said to be *congruent relative to* a; if not, b and c are *noncongruent*. The number a is called the *modulus*. If the numbers b and c are congruent, each of them is

> called a *residue* of the other. If they are noncongruent they are called *nonresidues*. (p.1)

As it can be seen at the very start he gives an equal status to *two* matching numbers: "each of them is called a residue of the other". Then he collects together all the numbers that are matched with (congruent to) a focal element (Section 5.2.2 and 6.2) to form a single-group.

> Given a, all the residues module m are contained in the formula a + km where k is any integer. (p.1)

Although this statement clearly states when a given number is inside the group and when it remains outside, it does not directly say anything about the relation between any two numbers inside the group. Thus, Gauss needs to establish another property. That is the F-transitivity of number concurrence, globally stated:

> If many numbers are congruent to the same number relative to the same modulus, they are congruent to one another (relative to the same modulus). (p.2)

The way that Gauss continues suggests that the picture drawn in Chapter 5 and 6 lacks (at least) one important aspect. Let me reveal it when discussing Gauss' work.

Gauss' first "theorem" and its immediate consequence suggest a multiple-group conception. For example, the latter reads:

> Each number therefore will have a residue in the series 0, 1, 2,...m – 1 and in the series 0, –1, –2, ... – (m – 1). We will call these the *least residues*... (p.2)

Thus each number is inside one (and only one) of the groups determined by its least residues (or one of them). And, we already know that any two numbers in each group are congruent to one another. Thus, we have a picture closely resembling the picture drawn in the previous chapters. But, both of these pictures lack an important aspect that somehow is related to the groups involved, in particular, to the so-called "unity" of a group.

As far as the students' experiences are concerned, "unity" was a term serving to capture those experiences maintaining a group as a group without losing the relations once established between its members. Now, having this sense of "unity" in mind, it is certain that Gauss exemplifies the "multiple-group conception" in which the groups involved maintain their unity all the way through.

But, it is of paramount importance to say that each group is experienced *together with* its members in all the above cases, whether it is Gauss' number theory or it is the experience of one of the students with the Mad Dictator Task! Maybe the best way to explain this aspect is to look at a modern textbook dealing with the same subject in the context of equivalence relations.

Stewart and Tall (2000) use the concept of equivalence relation to set up 'the integers mod n'. They start with the modulus 3 and define the *relation* $\equiv_3$ of congruence modulo 3 quite similar to Gauss. Then they check that the three standard properties of an equivalence relation are satisfied by the given relation.

> We know that the equivalence classes (known as *congruence classes* mod 3) partition Z. What are they? (ibid, p. 75)

To answer this question they show "every integer is either of the form 3k, 3k + 1, or 3k + 2 (accordingly as it leaves remainder 0, 1, or 2 on division by 3)". Eventually, they conclude that there are exactly three equivalence classes:

$$E_0 = \{\ldots -9, -6, -3, 0, 3, 6, 9 \ldots\}$$

$$E_1 = \{\ldots -8, -5, -2, 1, 4, 7, 10 \ldots\}$$

$$E_2 = \{\ldots -7, -4, -1, 2, 5, 8, 11 \ldots\}$$

As it can be seen the above lines can be translated into Gauss' *language* and vice versa. For example, using Gauss' language in this particular case we can say "each number has a residue in the series 0, 1, 2". However, there is a small, though very important, phrase in Stewart and Tall's account of congruent numbers that has no counterpart in Gauss'. That is 'the integers mod n'. The idea underlying this subtle phrase needed another hundred years or so (after Disquisitiones Arithmeticae) to come into use. To make the idea more transparent, I shall continue comparing the two.

Stewart and Tall change from E_0, E_1 and E_2 to 0_3, 1_3 and 2_3; and then start doing arithmetic with these *new entities* (the integers mod 3). Consider that 0_3, 1_3 and 2_3 are new entities having different properties from the integers numbers used to define them. These new entities have been obtained by the "principle of abstraction" (see Section 7.1).

Gauss himself uses *the least residues* of each number. The least residues are two integer numbers that are in a particular relation with another integer number, and all of them belong to a certain group containing all the other integer numbers congruent to the same residues relative to the given modulus. In other words, Gauss continues to use the *old* entities endowed with a new structure.

> We can also discover the underlying principles governing the rules that are ordinarily used to verify arithmetic operations. Specially, if from given numbers others are to be derived by addition, subtraction, multiplication, or by raising to powers, *we substitute least residues in place of the given numbers relative to an arbitrary modulus* (usually we use 9 or 11 because in our decimal system the residues are easily found). The resulting numbers, as we shall soon see, must be congruent to those deduced from the given numbers, otherwise there is a defect in the calculation. (ibid, p. 4)

Thus, in a way, Fowler's point about Euclid (see above) to the same extent applies to Gauss. That is, Gauss always considers (already defined) *individuals*. However, this comment should be moderated by saying that Gauss clearly *structures* the old realm; something that is opaque in Euclid's Elements to say the least.

Coming back to the data of the present study, I shall say that the Mad Dictator Task (as a generic name for all the tasks used in this study) fails to investigate an important aspect relating to the notion of equivalence relation, i.e. the so-called "definition by abstraction" by which a new entity is *created* (to use the word so dear to Dedekind). However, the absence of this fundamental aspect clears the way for clarification of the meaning of the 'unity' of a group. In this regard, I shall mention that Russell *informally* captures the unity of a focal single-group (a number) with the unity of the underlying set (set of sets), but *formally*, when he defines that number, he reverses this order; that is to say, he captures the unity of the underlying set with the unity of the corresponding single-group.

My 'variational' investigation of 'the history of the subject' has come to an end. It is certain that I have not mentioned some eminent mathematicians (and philosophers) who took part in building up our today's knowledge of the notion of equivalence relation. I have not even looked thoroughly into the works of those whom I have mentioned. However, I have tried to suggest a way of looking into the 'history' that could draw a distinction between different accounts of the *same* notion; a distinction that seems to be finer than the 'standard' one.

In brief, the *different* notions that come under the umbrella of the notion of equivalence relation have been skilfully treated in their own context in the course of the history without recourse to the latter notion as known today. More interestingly, and less obvious, there is some remarkable resemblances between the ways that the students (in the present study) tackled a bizarre task (the Mad Dictator Task) and the ways that some great mathematician of the past organized their field of study.

In the next chapter, reading Chin and Tall's papers I will show that even after being taught the formal account of the subject, we can observe some of these conceptions in the students' responses to questions of the "what is X?" kind.

Chapter 8: Discussion

8.1 Introduction

In Section 3.5.2, under the heading "validity", I discussed the sense in which the concepts of interest are embedded in the Mad Dictator Task. But, in a seemingly radical turn, in Chapters 5 and 6, I described "the variation in the ways that the students tackled the tasks involved". This boils down to this question: to what extent and in what sense this research has investigated what it was intended to investigate (Section 3.5), i.e. students' conceptions of equivalence relations and partitions. As often as not, this question and the like have created a sense of unease about this research among its audience including my supervisors, some of the anonymous reviewers of the papers based on this research, some of the participants in the research seminars where I reported my results, some of my colleagues (other PhD students) and probably the reader(s) of this thesis. Even, quite often I personally experienced a feeling of unease in the different stages of the study!

Generally speaking, there are two different kind of uneasiness about the result of this study. One accepts the *content*, but questions the *scope* of the study; the other radically denies the content (hence it becomes absurd to be discussing the scope of the study).

An example of the first group is the following in which one of the reviewers of one of my paper (Asghari, 2005b) expresses his concern about the implications of the research.

> *I am left asking the question of what are the implications of this research that has only looked at a few (3?) students. Emerging, or changing, views of how some students represent "one" class of equivalence relation, i.e. using elements of the real numbers, is an interesting piece of work, but will it impact teaching or practice, for example?*

This paper had reported the drastic change in my personal perspective from the preliminary study to the pilot study (see Section 1.8). As the reviewer rightly says, in that paper I only looked at a few students without any substantial discussion of the possible implications. These details aside, it is important to notice that the reviewer basically is in agreement with the content that the paper seeks to address. Even, he or she makes a further *generalisation* towards the class of equivalence relations on the real numbers.

On the contrary, the second group questions the Task and the extent to which it addresses what it is intended to address. An example of this group is the following in which the reviewer of a more substantial paper (Asghari and Tall, 2005) expresses his concerns about the different interpretations of the Task.

> *The visiting law is not comprehensible: Both parts of it can be understood in several different non-equivalent ways. For example, "they mustn't have any visiting cities in common" admits the following situation: City A has visiting cities B and C. City B has visiting cities A, D and E. Then clearly A and B do not have any visiting cities in common. The mathematically informed reader, who knows what an equivalence relation is, knows that A should be included in the set of visiting cities of A and B should be included in the set of visiting cities of B. She or he also knows that the formal definition of an equivalence relation starts with $A \sim A$, and concludes that this is presumably what the authors have meant with the first part. But it is not what the first part says and I would not expect anybody to understand the first part this way.*

The above critique is in a sense a mild version of a more radical stance taken by the others; they ask, in effect, why not refer to students' experiences of the Mad Dictator Task, instead of their experiences of equivalence relations (partitions)?

Consider that quoting the above reviews is by no means an attempt at criticising them. They have been used here to exemplify the types of question that the present chapter seeks to answer.

I intend to answer these questions under two headings "scope" and "content". Although for facilitating the discussion I have separated the scope of the study from its content, in effect, the former is a means to justify the latter.

My discussion of the scope of the study is twofold: interpretive and practical. The present chapter concerns itself with the former and the next chapter with the latter. In this chapter, I use the findings of this study as a guide to interpret the result of another study, as I did so when interpreting some historical situations. I argue that the remarkable similarities between the variations in the ways of tackling these *different* situations should point to the same phenomenon. In making these interconnections between different situations, I show that the *choice* of the content of interest was (is) appropriate rather than 'right' or 'wrong' in the objective sense of these words.

In the section entitled "content", based on some phenomenographical arguments I will address the problem of the content more directly.

At the end of this chapter, I will pull together the threads of my argument, examining the extent to which this study has addressed its purpose.

8.2 Interpretive Scope

In the previous chapter, I showed how the results of this study can be used to interpret some of the so-called historical instantiations of equivalence relations and equivalence classes. This comparison has exemplified one of the most important aspects of phenomenographical results or the so-called "categories of description", i.e. these categories refer to the "collective level". In the words of Marton and Booth:

> We may not have identified the most typical or the most advanced way in which a person can experience a phenomenon, and we may not have described a generalizable distribution of the different ways of experiencing it, but we

> may still very well have identified the variation in terms of which we can characterized the different ways the phenomenon appears to the particular person in different situations. When we talk about "categories of description" we usually do so in terms of qualitatively different ways a phenomenon may appear to people of one kind or another. Thus, categories of description refer to the collective level. (Marton and Booth, 1997, p. 128)

It is important to notice that this line of arguments is based on this fundamental assumption that a certain phenomenon is embedded in different situations. This assumption mainly relies on the judgment of informed individuals. It was based on this kind of judgments that I chose the historical situations discussed in the previous chapter. In the same way, I will scrutinize the Mad Dictator Task in Section 8.4.1. In the present section, I further a complementary reason to ascertain that these different situations are somehow similar. To do so, I turn around Marton and Booth's argument.

As mentioned above, Marton and Booth assume that a certain phenomenon is embedded in different situations. Based on this assumption, they say that the variation (in how the phenomenon is experienced) might turn out to be generalizable across these different situations; it seems that the "converse" is also true. That is to say, if the variation in the ways that people experience different situations is identical, this variation should point to the same phenomenon. Using this argument, we can now say that the similarities between the findings of this study and some historical situations point to the variation in how people experienced the same phenomenon, though the situations in which they experienced that phenomenon are remarkably different from each other. Thus, if the different ways that one of these situations has been experienced are commonly *believed* to reveal something about a certain phenomenon, people's experiences of the other situations also reveal something about the same phenomenon.

In the following Section, in line with the above argument, I will link the results of this study to a series of papers by Chin and Tall (2000, 2001 and 2002) in which they use the questions of the 'what is X?' kind (take X as "an equivalence relation" or "a partition"). Thus, the question of content is less problematic in their study. Their data reveal some remarkable examples of what I have described as the students' conceptions. This resemblance is remarkable, particularly because Chin and Tall worked with students who have already been exposed to the formal treatment of equivalence relations and partitions. This extends the interpretive scope of the findings of this study to the more standard situations. More importantly, it, together with the history of the subject, underlies what has been experienced in the situation of the present study.

8.3 The far end of the bridge

> [Equivalence] is one of the ideas which helps to form a bridge between the everyday functioning of intelligence and mathematics. (Skemp, 1971, p.173)

Chin & Tall (2000, 2001 and 2002) consider the mathematical concept development of novice university students introduced to formal definitions and formal proof, with empirical data collected on "equivalence relation" and "partition" (ibid, 2000, p.177). Unlike the present study, in their study the content (equivalence relations and partitions) is secondary to the process ("the shift to the formal mathematical register", ibid, 2000, p.178); in a sense, the content more or less is just a means to exemplify their theory in which "informal mathematics becomes formalised by introducing definitions, proving theorems and compressing formal concepts into cognitive units appropriate for powerful formal thinking" (ibid, 2001, p.241). As it can be seen there is no explicit mention of the content any more. Nonetheless, since they base their interpretations on the students' responses to certain problems about equivalence relations and partitions, I consider their data to be a reflection of their students' conceptions of these notions. Thus, my reading of Chin and Tall's data can be regarded as an alternative interpretation of a set of data gathered in a different setting. Let me now re-read some of their data.

8.3.1 Partitions

The first example that bears a striking *qualitative* resemblance to the single-group conception is some of their students' responses when asked "to write down two different **partitions** of the set with four elements, X = {a, b, c, d}". Chin and Tall report that "three[35] of the students giving unsatisfactory responses shared the same *misconception*: that the term 'partition' referred to *each individual subset*, not to the collection of all subsets (ibid, 2001, p .247; emphasis mine)." For example, the student whom Chin and Tall report his response gave the following two sets as two different partitions:

$$P_1 = \{a\} \quad , \quad P_2 = \{b, c, d\}$$

It seems that at least two aspects led Chin and Tall to regard the above response as a misconception. First, they compared it with the *correct* response based on the standard definition of a partition. After all, the students whom they worked with had been taught the standard definitions. Second, their use of what they term 'informal thinking' or 'informal thought' is mainly restricted to 'the way in which language is used':

> The introduction of formal proof in mathematics involves a significant shift...This shift alters the way in which language is used from an everyday informal register to a formal mathematical register (in the sense of Halliday, 1975), recently described by Alcock and Simpson (1999) as the "rigour prefix". This changes register from informal "loosely" speaking to formal "strictly speaking" in mathematics. The shift from informal to formal thinking is by no means easy. (Chin and Tall, 2000, p.177)

[35] It is three out of fifteen; the other twelve "gave satisfactory answers".

As a result, the above response has been classified as an incorrect response at the expense of what it can reflect about the student's evolving images of the notion of partition. Chin and Tall themselves somehow suggest the possibility of the latter interpretation where they compare the "informal and formal thinking" with "the use of images and the use of definitions" respectively.

> The research of Moore (1994) ... gives many fascinating insights into the usage of images and definitions in formal mathematics. In a sense his distinction between the use of images and the use of definitions has similarities with our focus on informal and formal thinking. However, his paper uses an interpretation of "concept image" which contrasts definition and image as distinct entities. For us the concept image includes the definition and its resulting related imagery. This allows us to formulate *an ongoing change of the total concept image* that steadily builds up the formal register.
>
> (Chin and Tall, 2000, p. 183; emphasis added)

However, this alternative interpretation appears not to have reached the above example. This suggests that the distinction between informal and formal is open to debate and various interpretations to say the least. Scrutinizing the possible interpretations of this distinction is outside the scope of the present study. For the time being, suffice it to say that this distinction can not be found in the phenomenographic literature in which what matters is simply students' conceptions (of something). Hence, for the sake of clarification, I will avoid giving any interpretation based on this distinction, though it is tempting to claim that the present study has furthered Chin and Tall's study by investigating the opposite end of the bridge, i.e. *informal* conceptions of equivalence relations and partitions. Let me now turn back to their data.

In a more direct fashion (compared to the above question) they asked each student to 'say what "partition" means to him or her'. One of the students gave the following response (ibid, 2000, p, 180):

$$\left\{ \begin{array}{l} A \cap B = \varnothing \\ \\ A \cup B = C \end{array} \right. \quad \Rightarrow \quad \text{A, B is a partition on C.}$$

Chin and Tall classify this response as "Example": "giving a single specific or general example" (ibid, 2000, p.180). I prefer to put it in the border of the single-group conception (exemplified by the first excerpt of this section and the one given in Section 5.2.2; Chris-GENERATING) and the multiple-group conception. It is worth stressing that these are just alternative interpretations based on two different approaches to the same data. More importantly, there are some important points that the results of this study appear to be a complement to those of Chin and Tall. Recall my claim that the multiple-group conception can only be inferred if the students explicitly verbalize it. Chin and Tall (2000 and 2001) report that even after being taught the subject the majority of their student found it difficult to verbalize the definition of partition. They also add that "looking closely at the responses reveals that the majority of students tried to use *their own language* to interpret the definition of 'partitions' so that their answers were highly varied" (ibid, 2001, p.247). Unfortunately, they do not report any examples of these responses. Thus, let me turn back to the excerpts reported in their papers. It is interesting that a few examples that Chin and Tall have considered to be worthy of mentioning resembles the findings of this study. I shall continue with these comparisons.

8.3.2 Equivalence Relations

In the last paper of the series (2002), Tall and Chin 'focus on the following question to investigate how the students understand the definition of "equivalence relations"':

> Let X = {a, b, c} and the relation $\sim$ be defined where $a \sim b$, $b \sim a$, $a \sim a$, $b \sim b$, but no other relations hold. Is this an **equivalence relation**? If not, say why?
>
> (ibid, p. 275)

The initial purpose of the authors was to investigate how the students use the universal quantifier which occurs in the reflexive law, i.e. $a \sim a$ *for all* $a \in S$. They classified an answer as "Correct" when it expressed that the reflexive property does not hold, or specifically noted that $c \sim c$ does not hold. As usual the most interesting answers fall into the other category, "Incorrect". By their definition of "Correct", the students who missed the universal quantifier in the reflexive property fall into the "Incorrect" group. Generally speaking, the latter group based their response on the other two standard properties of an equivalence relation, symmetry and transitivity.

The students mainly thought that "it needs three different elements to make transitivity hold" (ibid, p. 278). Moreover, when asked, 'If the two relations $a \sim b$, $b \sim a$ are removed from this question, what will happen?' they replied 'symmetry will not hold either' (ibid, p.279). In effect, it means it needs two elements to make symmetry hold. More interestingly, in some of the excerpts the symmetry property is not dealt with as it was formulated in the course, namely, in the form of an implication: if $a \sim b$ then $b \sim a$ for all $a, b \in S$. Not this, but the common theme which the symmetry property is dealt with is interesting, namely, in the 'and' form. Here is the response of one of the students as to the symmetry property:

$a \sim b$ & $b \sim a$ hence symmetric.

All of these remind us of something familiar. That is, they echo some of the students' works presented in the previous chapter; the students who had a different background and were involved in a complete different setting, compared to the students in Chin and Tall's studies. It also reminds us of Euclid and Gauss. To highlight these occurrences I shall start with the symmetry first.

In Section 6.4.2, I gave an example of using the 'and' account of the symmetry, rather than 'if-then' account. I argued that the replacement of the latter by the former applies in particular when the symmetric points are already on the grid. Arguably, we are faced with the same phenomenon in Chin and Tall's study where $a \sim b$ and $b \sim a$ are already out there. Recall that Euclid and Gauss also "missed" symmetry (Sections 7.3 and 7.4).

In the section 6.4.1, it was mentioned that the possibility of equality of x and y, is an aspect of the standard treatment of symmetry that more often than not is overlooked by the students. This clearly manifests itself in the students' responses in Chin and Tall's study. That is, 'it needs two elements to make symmetry hold'. It is worth saying that in the present study only one student took the equality of x and y into account, though in a very *indirect* way. Hess, the very same student who exemplified different conceptions introduced in the previous chapter, started with the diagonal and the whole-grid as his very first two examples of a visiting-law. Putting those two examples aside, he turned his attention to the symmetric points on the grid and conjectured that "in general, any two symmetric points that we choose we have one [example]". At this time, his examples are divided into three different categories, the diagonal, the whole-grid, and the category of the symmetric figures. Soon the last two categories merged into one:

> Hess: I am going to give more explanation. We have two cases, or all of them are different, that means only this diagonal; or when they are not different...they are these symmetric points. So they are all the possible cases.

Then he decided to count all the *examples*. His way of counting led him to merge the diagonal with the symmetric figures.

> Hess: let me count them, 8, no, 9 points. It will be 45; there is 45 points, each point could be there or not, so the number of cases will be 2 to the power of 45, and we have one case there [the diagonal], so 2 to the power of 45 plus 1; no, no, 2 to the power of 45; because each one of these points could be full or not, [moreover] this side is the same as that side [he is referring to the two sides of the diagonal]; *so there is no need for making a distinction*. So it is 2 to the power of 45.

It is the closet account of counting the diagonal among the symmetric figures in my data—a degenerated symmetric figure in which there is no points on the two sides of the diagonal. However, it does not mean that he also extended the extension of the term 'symmetry' to include the diagonal.

Let me give another illuminating example of the students' tendency to keep the diagonal separate from the symmetric figures. Sarah (at that time, first year mathematics student) is tackling the intersection task. At this time, her examples are divided into three different categories, the diagonal, the whole-grid and the examples *like* the following in which only two cities are related to each other.

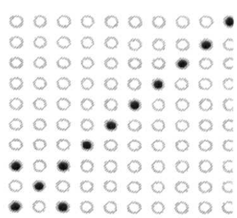


Figure 86

She argued that "*anything* he put with that one (the whole-grid), take the common points, [it] would be a valid example", and as for the diagonal, "if he take that with anything else, then again the diagonal line would be in common, so it would be again a valid example". And finally, "you *always* have either the diagonal line *or* one of the solutions" because:

> Sarah: If you had like, if you had just one dot filled in, on one side of the diagonal, it has to be *symmetrical* (it is the first time that she refers to 'symmetry'), she has to have that dot on the other side too, and if one of this dot was common to both, then both dots would be common to both, both examples…therefore the result has to be a valid example…as it would be symmetrical about the diagonal.

Consider that *mathematically* the different pieces of her argument can be united by saying that 'the intersection of two symmetric figures is a symmetric figure'. However, to say that, the diagonal, the whole-grid and her other examples have to be regarded as the special cases of a symmetric figure. But, she did not show anything in that direction, in particular, in regard to the diagonal.

Let me now turn to Chin and Tall's third observation. That is, the vast majority of their student (82% of the "Incorrect" category, ibid, 2002, p. 279) thought that "it needs three different elements to make transitivity hold". In the section 5.3.1, I wrote that when a group is comprised of only two identical elements it is in the border of the first two categories, the matching conception and the single-group conception; it is properly in the second category when it has *more than two identical elements*. Then in the section 6.2, I argued that the functionality of the transitivity or F-transitivity can only be realized when more than two elements are involved; It is only in these cases that the transitivity and F-transitivity help the students to realize the equivalence of two elements without directly and practically matching them. In this regard, again recall that whenever Euclid applies one of these properties *three different* elements are involved. Thus, even though the students' responses in Chin and Tall's study were not based on a 'correct' application of the standard formulation, they arguably were not in conflict with their "operational knowledge of transitivity" (see Section 6.2). Chin and Tall (2002, p.279) suggest a rather different (though complementary) idea, pointing to what misled the students:

> In our data, approximately half the students are unable to handle the definition in a simple example using only three elements. The reasons are diverse, but 82% of those giving an incorrect reason for the example not being an equivalence relation focus on the transitivity law where there is a sense that 'the transitivity law must involve three elements' and even that the transitive law is interpreted using an embodiment that is the same as the axiom in an order relation... In the case of an order relation, it is a natural thought process to imagine the elements ordered in a line, and, in the absence of an embodied image of the notion of equivalence relation, in using the transitive law, it is natural to link to the self-same image.

In sum, it seems that as far as handling the definitions is concerned, many of the students (in Chin and Tall's study) rightly fall into the 'Incorrect' category. However, their 'incorrect' ways of tackling that particular question appears to stem from certain deeper conceptions. In particular, it may come as a surprise that some of these *formally* 'incorrect' ways (if not all) were simply among the ways of *successfully* tackling the Mad Dictator Task, and more importantly, they have some historical counterparts. As mentioned before, these resemblances between the ways that different people of one kind or another have tackled these *different* situations should point to their experience of the same phenomenon or phenomena. This brings me back to the 'content' of this study.

8.4 Content

I started the present chapter aiming to justify the 'content' of the study. So far, I have addressed the problem of the content by linking the results of this study to some other situations that are commonly *believed* to embody the notions of equivalence relation and partition. In this section, I deal with the problem more directly. To do so, I divide it into two equally important sub-questions. The first sub-question is whether the Task has anything to do with the notion of equivalence relation (partition). The second sub-question, and certainly related to the first, is whether it could (can) be used to investigate students' conceptions of these notions. The affirmative answer to the former mainly relies upon the acceptance of the *experts*. And the answer to the latter is deeply rooted in some phenomenographic assumptions. Let me discuss these issues one by one.

8.4.1 *Experts'* accounts of the Task

This section addresses the first sub-question mentioned above, i.e. whether the Task has anything to do with the notion of equivalence relation (partition). Beyond doubt, the Mad Dictator Task was originally designed based on my understanding of equivalence relations and partitions. Then, some students in the preliminary study (Section 1.7) saw those notions in the Task, some reviewers criticized the Task for failing to address some aspects related to the notions, Dr Bob Burn sent me a letter reminding me that "he had himself played with such tables" for equivalence relations (see Appendix B), and finally, a lecturer used the Task as a means to teach the subject (see Section 9.4). These suggest that from the vantage point of an informed person the Task somehow embodies the ideas of equivalence relation and partition. However, as soon as we start to see the situation through the eyes of a lay person, we get a complete different picture. In effect, this study was all about this picture. And, this chapter was all about this question that to what extent we can relate this picture to the conceptions of equivalence relations and partitions. In the next section (8.4.3) I discuss this issue from a phenomenographical point of view. Here, at the end of this section I would like to add two other views to the above list.

The first one is from Dr Ali Enayat, my old logic teacher and currently a Professor of Mathematics at American University. Since his solution to the Task is rather *formal*, it is given in the Appendix C. The second one is from Prof. Ian

Stewart. It is given here. Neither of them needs any further explanation. I shall only mention that their solutions are the by-products of a discussion about the first condition of a visiting-law. After Prof. Stewart's solution I will immediately switch to the next section where I discuss the problem of the content from a phenomenographical standpoint.

A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.

A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.

A visiting law must obey two conditions to satisfy the mad dictator:

all

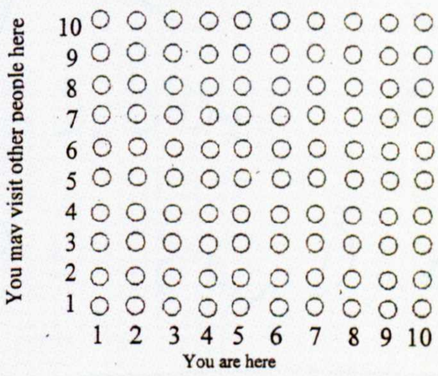
1. When you are in a particular city, you are allowed to visit other people in that city.

2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

The dictator asks different officials to come up with valid visiting laws, which obey both of these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below.

"PARTITION"

"visit"

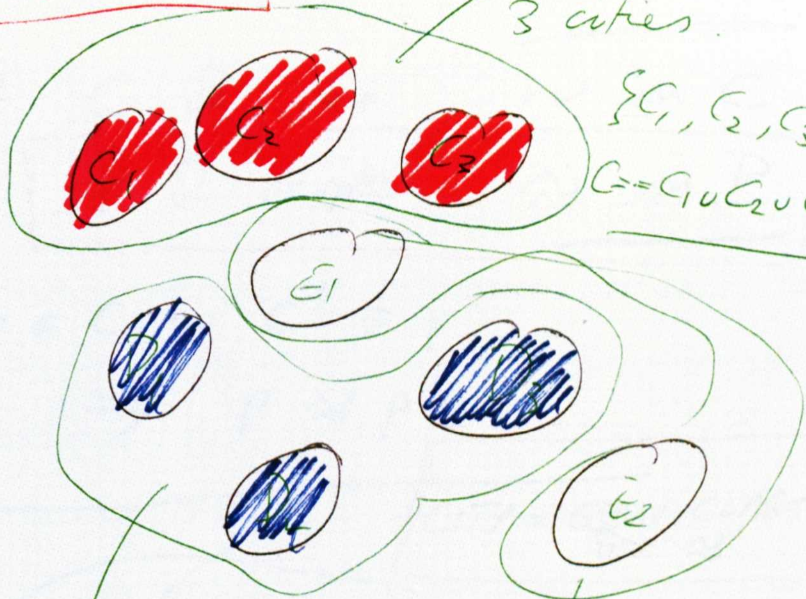


p $\underline{X(p)}$ people they can visit.

p can visit himself.

$\boxed{\text{"City" means not } \emptyset.}$

union of 3 cities



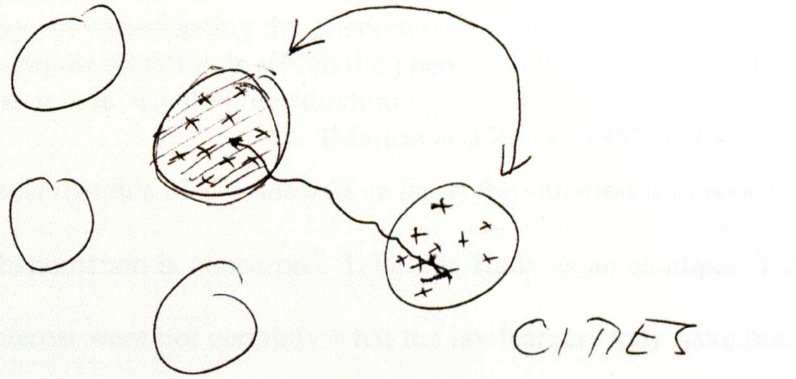
$\{C_1, C_2, C_3\}$

$C = C_1 \cup C_2 \cup C_3$

$D = D_1 \cup D_2 \cup D_3$

$E = E_1 \cup E_2$

$$P = \{C, D, E\}.$$



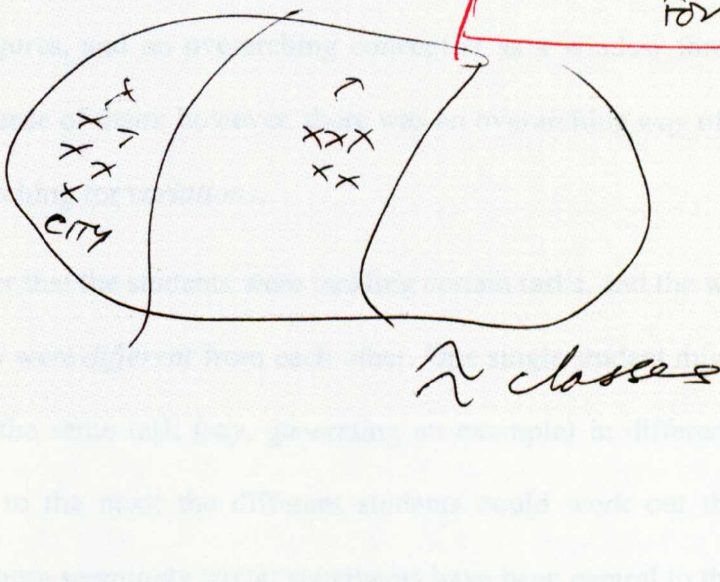
Set $C \ni$ cities $\qquad \sim$ on C

$\boxed{P \ni \text{people} \qquad \approx \text{ on } P}$

$p \in C \sim C' \ni p'$

$\Rightarrow p \approx p'$

$\boxed{\text{CITY} = \text{EQUIV. CLASS FOR} \approx}$



8.4.2 What is embedded[36] in the Mad Dictator Task?

> The researcher may thus opt to focus mainly on ways of experiencing the situation or on ways of experiencing the phenomenon. But the learner, as well, may focus mainly on the situation in which the phenomenon is embedded or on the phenomenon as it is revealed in the situation...
>
> (Marton and Booth, 1997, p.83)

At first glance, it seems that this idea holds true as far as the situation is concerned but will fail if the phenomenon is concerned. Take this study as an example. The phenomena of my interest were not certainly what the lay learners may have been focusing on. For them, "equivalence relations" and "partitions" were non-existent. The learners simply were tackling the tasks, presumably as best they could. Meanwhile, I was interviewing them, and I was to analyse their works, bracketing[37] my understating of equivalence relations and partitions, as best I could. Having put my understanding of the subject to one side, I was plainly faced with certain points on each grid where there were (are) incredible *blind choices* (there are 2^{100} different ways of putting the points on the grid). It was not looking promising to have more than three hundred pages of transcripts packed with words and figures, and no overarching concept(s) as a window through which I could make sense of them; however, there was an overarching *way* of looking into them, i.e. searching for *variations*.

Consider that the students were tackling certain tasks, and the ways that they were doing so were *different* from each other. One single student might have been dealing with the same task (say, generating an example) in different ways from one moment to the next; the different students could work out the same task differently. These seemingly trivial statements have been central to this study, and indeed to any phenomenographic study.

[36] See section 3.5.2.
[37] See Section 2.3.

When phenomenographers say "the learner may focus on the phenomenon", they mean he or she may do and/or say something that makes a difference to the way of handling the situation per se *and* can transcend the situation and link it with other situations and lend meaning to it. In this regard, the term "focus" is misleading. The learner may or may not be *explicitly* aware of the phenomenon. He or she also may or may not be familiar with any other situation in which the phenomenon (or a certain aspect of it) is embedded. Nevertheless, still the researcher's role is capturing the variation in the ways that the phenomenon (of his or her interest) is experienced in a certain situation (if he or she opts to focus on the phenomenon). Paradoxically, the researcher should also avoid imposing his or her understanding of the phenomenon on the learners' accounts. To avoid doing so, the researcher undertakes a 'double act of abstraction' (to use Cantor's memorable phrase). First, he or she describes *all* the aspects which make a difference in handling the situation at hand. Then, the researcher sieves the first collection of descriptions and discards those which seem to be specific to the situation at hand (see Appendix D). The result is customarily called *categories of description*, or synonymously *ways of experiencing* of the phenomenon of interest. The former gives weight to the researcher's role and the latter to the learners'. Whatever the name of the outcome, it is by no means an "exhaustive system".

> Inasmuch as a phenomenographic study always derive its descriptions from a smallish number of people...the system of categories presented can never be claimed to form an *exhaustive* system. But the goal is that they should be complete in the sense that nothing in the collective experience as manifested in the population under investigation is left unspoken.
>
> (Marton and Booth, 1997, p. 125; emphasis mine)

Coming back to the present study, it can now be seen that it has already given an example of a system that is not certainly exhaustive. Gauss' experience of equivalence relations showed that at least one vital aspect is lacking in the outcome of this particular situation (see Section 7.4).

Talking of Gauss, I shall also add that my reference to him, or to any other mathematician who somehow is connected with today's understanding of equivalence relations, shows the importance of *experts*' roles in deciding upon a certain situation. Consider that the problems that Euclid and Gauss dealt with were not *related* to equivalence relations more than was the Mad Dictator Task. It is only from today's vantage point that it can be said that equivalence relations are embedded in these situations. However, overrelying on the experts' account could be misleading. This account (or at least, my account) has been challenged by the students in this study many times. More important than all, their works have called into question the extent to which I have addressed the purpose of this study.

8.5 The purpose of the study revisited!

This present chapter set out to answer to what extent this study has addressed its purpose: investigating students' conceptions of equivalence relations (and partitions). So far, I have approached this problem from two different, though related, angles.

First, I showed that there are certain similarities between the ways that different people from different background have tackled certain situations including the Mad Dictator Task, some historical situations and the setting of Chin and Tall's study. I argued that these similarities should point to the same phenomenon or phenomena. Second, I showed that from the vantage point of an informed observer the first two situations embody the ideas of equivalence relation and partition. The third situation was less problematic since it was more directly related to our today's understanding of the subject.

What I called "double act of abstraction" is an attempt to interconnect the above approaches. Carrying out this double act, the ways that a certain situation is tackled are used to map the ways that a certain phenomenon is experienced. In doing so, there is a danger of "overabstracting". This study exemplifies this point!

I have argued that the notions of equivalence relation and partition are embedded in certain situations, then by carrying out a double act of abstraction I have jumped to people's conceptions of these notions as the phenomena transcending those situations. But, this jump has been challenged by the data gathered from different sources.

First, from the vantage point of today's mathematics, equivalence relations and partitions are differentiated from each other, and connected with each other by a *fundamental theorem*. In a "standard" course, it is expected that the learner grasp these two differentiated notions and pass at will from one to the other. Chin and Tall further this view by taking a cognitive step, saying that an "able" student is the one who uses "a *compressed concept*, encompassing both equivalence relation and partition" (Chin and Tall, 2000, p.183). On the other hand, the other sources suggest the possibility of starting with a "compressed concept" rather than finishing with it. Consider that, for a long time, the notions of equivalence and partition have not been differentiated from each other in the course of history. Moreover, in the setting of the present study, the in-action counterparts of these two notions, matching and grouping respectively, have not been experienced as two different things. Both of these contrasting views lead one to question whether we are faced with two different phenomena or an overarching phenomenon being experienced in different forms at different times.

Second, mathematically an equivalence relation is a relation endowed with certain properties. But, before being made explicit as a relation it is (historically was) experienced simply as 'equivalence'. The matching conception and all the discussion about 'missing' symmetry boil down to this conclusion. It also supported by our *everyday* experiences. For example, Johnson (1987, pp. 96-98) argues that "equivalence" manifest itself in most of our mundane experience of "balance" (e.g., standing upright, or walking, without falling over; carrying an equal load in each of our hand). Balance involves symmetry.

> In our daily lives we are constantly experiencing symmetries and asymmetries of forces relative to axes and points of various kinds...there is a single image-schema present in all such experiences: *a symmetrical arrangement of force vectors relative to an axis.* (ibid, p.97)

In the very next paragraph he makes an unwarranted jump from balance as an experience to balance as a concept saying:

> Corresponding to this recurring structure, there is a logic to our experiences of balance. [It] has a definite internal structure. This structure has three important properties: symmetry, transitivity, and reflexivity.
>
> *Symmetry*. A balances B if and only if B balances A.
>
> If we hold A in the left hand and B in the right hand, and they balance, then interchanging A and B will preserve the balance. This is neutral, since our understanding of balance involves a symmetry of forces.
>
> *Transitivity*. If A balances B, and B balances C, then A balances C.
>
> Suppose A is in the left hand and B is in the right hand, and they balance. Now suppose A by C and the balance is maintained, that is B balances C. Then I will know immediately that A and C will balance, even though I have not performed the act of weighing one against the other.
>
> 3. *Reflexivity*. A balances A.
>
> Obviously, this is not experienced directly. However, it follows from (a) the understanding of balance in terms of symmetry, and (b) the understanding of symmetry in terms of rotations around an axis to yield a perfect fit. If you could have exactly the same element or object on both sides, they would have the same weights and exert the same forces. From an experiential perspective, such a relation never holds, which is why it seems to us such a strange relation. We are, after all, never in a position to balance something with itself. But we know from our understanding of balance in general that this would be true if we could do it.

In the next chapter, I will show that it is the standard account of equivalence relations, with its *chosen* properties, that has given rise to Johnson's "inherent" properties of the balance, not the other way around, as he claims:

> In considering abstract mathematical properties (such as "equality of magnitudes"), we sometimes forget the mundane bases in experience which are both necessary for comprehending those abstractions and from which the abstractions have developed. The abstract concept of equality of magnitude can only be understood in terms of experience of balance. *It is no accident that the properties of the balance schema are just what mathematicians call the "equivalence relations."* (Emphasis added)

Although Johnson goes too far to claim that the properties of symmetry, transitivity, and reflexivity are *the* properties of "balance", his idea of balance, or roughly specking, *symmetrical relations*, once more highlights the vast variety of situations in which "equivalence" (or in Johnson's term, "balance") is experienced between a pair of two things *having* the same status (being in balance).

Third, mathematically a partition is a set of mutually exclusive classes. At the outset, the elements of each one of these classes are not equivalent with each other, far from that. The only aspect that relates the elements of each class to each other is that they are in the same class. Then, based on a fundamental theorem, a partition can be seen as a set of mutually exclusive equivalence classes in which any two elements of each equivalence class are equivalent with each other. But, as the distinction between a "group" and a "set" suggests (Section 5.4), in this study the students started where the above process ends. In other words, they started with a group of equivalent elements (two elements or more than two elements). Consider that this "reverse experience" is not peculiar to this study. Another case in point is Skemp's (1971, p. 173) first sorting method with which we start with the elements that are "alike" in some way. In this case, *each* element has a certain "characteristic property" (or certain characteristic properties) and the elements of the same class have the same characteristic property (or characteristic properties). What these cases have in common is that in both of them the equivalence of objects is experienced directly. In other words, an "equivalence class" experienced as a class of equivalent elements is an extension of "equivalence" of two elements in the sense discussed above. Using the language of this study, the single-group conception is an extension of the matching conception.

One could argue that the third point above is a complex way of saying that the mathematical notion of partition is a mathematical generalization of our experience of equivalence classes (as the groups of equivalent elements). This reminds us of Johnson saying that the notion of equivalence relation has been abstracted from our experience of equivalence (see the second point above). These two together bring us back to the first point discussed above that of whether we are faced with two different phenomena or an overarching phenomenon. The answer appears to be "two" mathematically, but "one" experientially. To put it another way, equivalence relations and partitions are two different mathematical notions defined to capture different aspects of our experience of equivalence. However, the latter goes too far in this direction, and as I will show in the next chapter, the former is too well-designed! Let me now turn to the fourth point of this section.

Fourth, it was mentioned that the notion of equivalence relations is in a sense a mathematization of our experience of equivalence. But, still the importance of this notion lies somewhere else; It mainly lies in mathematically *imposing* "equivalence" when it is absent, or it is pretended that it is absent! I shall repeat that from a mathematical point of view, an equivalence relation is first and foremost a relation. The latter is defined so that for any two elements (of the underlying set) it is known whether or not *the first is related to the second.* It is then shown that the given relation satisfies certain properties, and because of that we can have different aspects of our experience of equivalence. For example, the property of symmetry shows that the initial *order* was redundant and allows us to say that "two objects are equivalent to each other". In this regard, recall the distinction between Euclid and Hilbert. "Euclid missed symmetry" because the equivalence of the (two) objects was maintained from the outset. For example, in effect, he started by saying that "two segments are congruent to each other" while Hilbert finished by saying the same thing after establishing the symmetry of segment congruence.

Let me now pull together the threads of my arguments. It was argued that equivalence relations and partitions are two mathematical notions that have been abstracted from different aspects of our experience of equivalence. As mathematical concepts, they have been fitted into a network of other concepts including the concept of "set" and the concept of "relation" (mathematically, the notion of relation is formulated in set-theoretic terms). Thus, if we opt to investigate students' conceptions of equivalence relations and partitions we have to do so within a network of other related concepts. This is something that this study has certainly failed to do. However, the ways that students tackled the Mad Dictator Task seem to point to a more fundamental phenomenon, i.e. equivalence.

The above argument also applies to the historians approach to the "history of the subject", in particular to the history of equivalence relations. That is to say, if we insist on thinking of equivalence relations only in a set-theoretic context, then the history of the subject only starts around the end of nineteen century. But, as mentioned before, for many historians this history backs to Euclid. Thus, there should be something in the "relevant" historical situations that allows an informed observer to impose his or her view on them. It again seems that in all of these historical situations some kind of equivalence is involved. However, in reading the "history", more often than not "equivalence" and "equivalence relation" have not been differentiated from each other. As a result, the latter, with its standard and "recently" *chosen* properties, has been directly written back into the "history". But, the interchangeability of transitivity and F-transitivity, the "missing" symmetry and for example Euclid's reluctance to use reflexivity, show that the standard properties of equivalence relations are more than a simple indication of people's experience of equivalence. These properties have been *chosen* to *recreate* this experience. Intriguingly, each one of these chosen properties is at odds with our experience of equivalence. This issue will be addressed in the next chapter.

CHAPTER 9: IMPLICATIONS

9.1 Introduction

The previous chapter ended with an *unexpected* result: that the more I try to converge the ways of experiencing the Mad Dictator Task with the ways of experiencing equivalence relations, the more they diverge. To a certain extent this result also applies to the 'history' of the subject. That is to say, the ways that the so-called historical instances of equivalence relations were experienced in the course of history do not directly comply with today's understanding of the subject. To take account of these observations, a distinction was made between "equivalence" as an experience and "equivalence" as a concept (i.e. equivalence relation).

In this chapter, I conclude this thesis by discussing the pedagogical implications of the above distinction and the other findings of this study. Inevitably, my discussion will be closely related to the notion of equivalence relation as mathematically axiomatized. I also return to Skemp's and Dienes' account of the subject. And finally, I will discuss the possibility of using the Mad Dictator Task as a teaching tool.

9.2 Some first-order perspectives

> What can be seen from a first-order perspective could—and, we think, should—be informed by that which can be seen from the second-order perspective. (Marton and Booth, 1997, p. 121)

It is allegedly assumed that the standard properties of the notion of equivalence relation naturally occur in our experience of equivalence (see Section 8.5). In this section, I will examine this assumption. It will be shown that each one of those properties results from certain *mathematical* choices, rather than spelling out our experience of equivalence. To discuss some of these choices—and more importantly, the reasons for these choices—the sole important theorem of the subject is analyzed. In particular, the role of reflexivity in the standard form of this theorem is discussed. I also return to the transitive property.

9.2.1 A fundamental but otherwise useless theorem!

One particular aspect of the students' works was puzzling me time and again when analysing the transcripts. If, as the literature (mathematical, pedagogical, but not necessarily historical) suggests, the theorem linking partitions to equivalence relations is of such fundamental importance, why can the students *successfully* "partition" the elements involved without taking the *defining properties* of an equivalence relation into account? In effect, it seems that the whole of this thesis is an attempt to answer this question and debunk the customarily God-ordained defining properties of an equivalence relation. The properties of symmetry, transitivity and F-transitivity, have already been discussed in several sections and will be discussed again in the sections to come. But, the most troublesome property is not either of them; it is the property of *reflexivity*. This section examines this property and its role in the sole important theorem of the subject.

On the face of it, it appears that reflexivity should be among the least of the concerns in this particular situation, where at the outset it has been imposed on the situation by the first condition of a visiting-law (see Section 1.6.2). Furthermore, the majority of the students started with the diagonal as their first example of a visiting-law (see Appendix D). Thus, we (Asghari & Tall, 2005, p. 86) wrote:

> In many natural contexts, reflexivity is not made explicit. Family relationships allow A to be a brother of B, but A is not his own brother... In the case of the example of visiting cities represented on a grid, however, the reflexive law is visible as the main diagonal of the array.

Then we added in parenthesis:

> The matter is a little more subtle as the idea of 'matching' usually means matching *two* things.

Likewise, the reviewer of one of my earlier papers (Asghari, 2004b) wrote:

> Some questions for me are still open. The "box property" is suggested as a good candidate for substituting transitivity, yet the major problems with relations are usually with reflexivity, that is quite unnatural. In this case the artificiality of the situation seems to have given a natural flavour to it, but other cases are to be discussed before taking a firm position.

Indeed, "the matter is a little more subtle" and "other cases are to be discussed before taking a firm position".

As a matter of fact, as soon as a certain group of related elements was formed, the other elements, whether the points on the diagonal or the cities, were treated as the *individuals*. For example, the cities 1, 3 and 8 were grouped together and all the other cities receded into background as the individual cities, or, certain points on the grid formed a square and all the other points on the diagonal were treated as the individuals. This kind of treatment is abundant in Chapter 5 and 6. In contrast, there is a scarcity of a treatment in which a single city *forms its own group* and the only member of that group *is related to itself*. In fact, only one single case exemplifies this latter treatment; the one that has been reported in the section 5.2.3 (see Textfigure 20). Thus, the students could partition a collection without recourse to reflexivity. The key to the idea is to group *two* or *more than two elements* together while leaving the individuals alone as if they have no relation with the others, in particular, with themselves. More drastically, it seems that this idea works in general. That is to say, we can ignore the property of reflexivity and still collect together those *different* things which are related to each other. But, if we can do that, there should be something *wrong* with our fundamental theorem.

I could not find a better and more direct way to explain this surprising (at least to me) *defect* than to refer to my personal communication with Professor Ian Stewart. However, before reporting that, I shall quote Stewart and Tall's (2000, p. 75) "remark". Immediately after stating the fundamental theorem they write:

> REMARK. This theorem allows us to pass at will from an equivalence relation to a partition or back again, by a procedure which, when done twice, leads back to where we started.

Let me now use my personal e-mails to Prof. Stewart. I cite the e-mails in chorological order word for word.

> Amir: I remember you told me something like this: "mathematicians could go without reflexivity". At that time, I missed my chance to ask what you meant by that. I missed that probably because I had the same idea at the back of my mind. May I now ask you what you meant by that?
>
> Ian: Imagine a world in which the textbooks define an equivalence relation by omitting reflexivity. It wouldn't be hard to make everything work. For example, we would redefine the equivalence class [x] to be the set of everything equivalent to x, **together with x itself**. Then all the usual theorems would work. So we could get round the lack of reflexivity by building it into that definition.
>
> Amir: But it seems that there would be a subtle difficulty. Take "is a sibling of". Assuming that nobody is his or her own brother or sister, it is not a reflexive relation. Now, we find the equivalence classes as you redefined them. We will have many disjoints classes of brothers and sisters, and a lot of singletons, so far so good. Now, we turn back and try to remake the original relation starting from our partitioned world. Will we find the same relation as the one that we started?
>
> Ian: Don't think that matters. Equivalence relations are almost always used to set up equivalence classes - hence partitions. They seldom have any other use.

There are at least three points worthy of attention here.

First, there is no mathematical flaw in our fundamental theorem (what a relief!). Indeed, it skilfully connects the theory of (equivalence) relations to the theory of sets. Let me point to one connection that I like as my little discovery. That is a little reason for distinguishing between the object a and the set {a}.

> A distinction must certainly be made, at least conceptually, between the object a and the set {a} consisting of only this one element (even if the distinction is of no importance from a practical point of view)... (Hausdorff, 1914, p. 12)

Suppose we have an equivalence relation (by the standard definition). Because of the property of reflexivity, each member is *at least* related to itself. Suppose there is an element being only related to itself. Then try to construct its equivalence class. We should accept to have a set consisting of only that one element, or we should consider different cases when stating our theorem. It is well known that mathematicians almost always avoid the latter. Our fundamental theorem is full of this kind of connections. However, let me leave them here and turn to the second point.

Second, it seems that our theorem is fundamental only in one direction. That is to say, it is important because it verifies that equivalence classes are mutually disjoint. In turn, equivalence relations are important because they can be used to define mutually disjoint equivalence classes. The crux of the problem lies here. In the so-called everyday experiences there is hardly any need for making the underlying sets explicit (Section 5.4). To the same extent, there is hardly any need to take reflexivity into account. But, still we can split the universe of our discourse into disjoint groups (not sets). Take the family relationship "...is a brother of...." It can easily be seen that we can split people into disjoint groups, each group consisting of the brothers (it is plural). Let us see what happens.

You are not your own brother.

There are certain groups of brothers.

Now applying the converse of our fundamental theorem (as it is), all of a sudden:

You become your own brother!

It seems that it is only in the mathematics classrooms and mathematics textbooks that 'you are your own brother' or 'you have the same surname as yourself'! (The latter example has less difficulty when applying the converse of the theorem. However, we can do well without considering the 'underlying sets' and the property of 'reflexivity').

To think of mathematics without sets is far beyond the scope of this thesis. It is also too late to define an equivalence relation without taking reflexivity into account. However, just being aware of the possible variations makes us wiser and ready to encounter our students' possible difficulties. This brings me to the third point.

Third, the property of reflexivity has been *chosen* for certain reasons. One of these reasons has been discussed above. It is worth stressing that this property is not experienced directly. In this regard, recall Euclid's reluctance to use a reflexivity-style-phrase whenever some kind of equivalence is concerned. This fact becomes more interesting when we consider that Eulid's definitions are "indications of what is intuitively given" (Weyl, 1949, p. 19). In the case of our interest, it seems that what is intuitively given is that at least *two* objects are involved in our experience of equivalence, and even sometimes the equivalence of these two (or more than two) objects per se is experienced directly. Thus, the property of reflexivity is not simply the result of spelling out our experience. Even, it is somehow in conflict with this experience. To examine this point from another angle than the above let me once more refer to Chin and Tall's (2002) paper. Recall the question that they used in their study:

> Let X = {a, b, c} and the relation ~ be defined where a ~ b, b ~ a, a ~ a, b ~ b, but no other relations hold. Is this an **equivalence relation**? If not, say why?
> (ibid, p. 275)

Imagine what will happen if we add the relation c ~ c to the ones already defined, a ~ b, b ~ a, a ~ a, b ~ b. It seems that all of a sudden two hitherto different and inequivalent elements, a and b, become equivalent. It shows to what extent the problem could be at odds with the students' previous experience of equivalence. If they had been taught that, in effect, "the word 'equi-valent' suggests the meaning 'worth the same'" (Skemp, 1971, p. 173), what their reactions would be to the sudden change in the status of two *different* elements from 'not worth the same' to 'worth the same', in particular when this change occurs after adding a new relation that seemingly has nothing to do with the relation between those two elements.

Indeed, the reflexive property is not a straightforward *choice* as one of the standard properties of equivalence relations. However, it is in good company: "Transitivity".

9.2.2 Transitivity and F-transitivity

As mentioned before, whenever some kind of equivalence is concerned, the two properties of transitivity and F-transitivity can be used interchangeably. In this section, I briefly compare these two with each other.

"Transitivity" *is* one of the defining properties of an order relation. As quoted from Freudenthal (Section 4.4): "In a mathematical system the law of transitivity might be at the basis of linear order". But, as often as not, "transitivity" also is *chosen* as one of the defining properties of equivalence relations. Tall and Chin (2002, p. 280) point to one of the result of this pedagogically unfortunate choice. They report that some of the students in their study when dealing with the notion of equivalence relation used an embodiment of the transitive law that was compatible with an order relation rather than an equivalence relation.

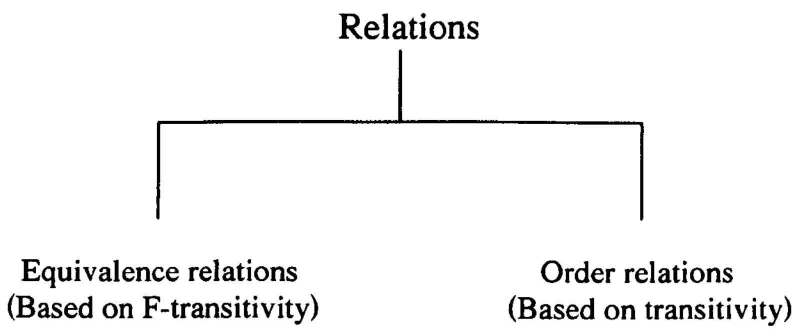
> In the case of an order relation, it is a natural thought process to imagine the elements ordered in a line, and, in the absence of an embodied image of the notion of equivalence relation, in using the transitive law, it is natural to link to the self-same image.

Unlike "transitivity", "F-transitivity" separates equivalence relations from order relations. As mentioned, "F-transitivity" first took me by surprise when analyzing the ways that the students had been tackling the Mad Dictator Task. At that time, it had no name and I was not aware of its frequent appearances in the history of the subject. The first (and the last in the context of equivalence relations) *modern* text that I found it was Freudenthal's (1966) textbook on logic. Accordingly, and following a private communication from Bob Burn, it was called "F-transitivity", although Freudenthal himself called it "transitivity". In the light of the history of the subject (Chapter 7) we can see that it could be called 'E-transitivity' in honour of Euclid, 'G-transitivity' for Gauss, or 'C-transitivity' for Cantor (1895, see "Contributions to the founding of the theory of transfinite numbers" translated by Jourdain, 1915). It was also shown that quite often the two properties of F-transitivity and transitivity has been used interchangeably whenever some kind of equivalence was concerned. It is interesting to say that even Freudenthal equivocates somewhat in the use of transitivity and F-transitivity. When defining equivalence relations, he uses the term "transitivity" for what we have called "F-transitivity". Then, a few pages on, when considering *order* he again uses the term transitivity for what is usually known as transitivity:

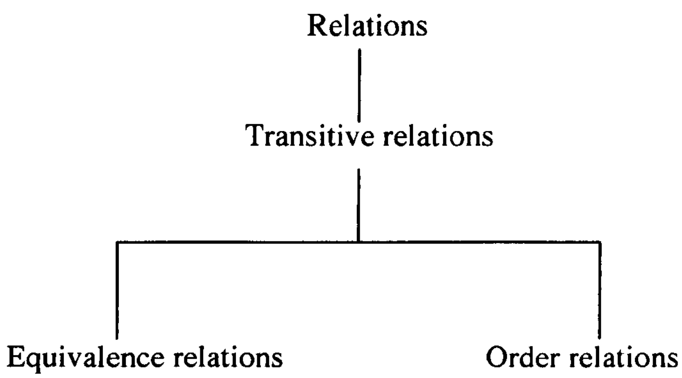
> ...and if, for every three *different* members a, b, c, of Z it follows from $a< b$ and $b< c$, that $a <c$ (*transitivity* of the $<$-relation). (ibid, p.19; emphasis mine)

There are at least two other intriguing points about the property of F-transitivity. First, if it is somehow guaranteed by the context that each object in the universe of discourse is *equivalent* at least to another object in the same universe, then we can define an equivalence relation by stating *only one* property, i.e. the property of F-transitivity. It is what Hilbert did to verify that the segment congruence satisfies the standard properties of an equivalence relation (Section 7.3).

Second, more importantly, the property of F-transitivity is in harmony with the standard way of defining an *equivalence class* consisting of everything *equivalent* to a *focal* element. Having established the F-transitivity of the given relation, it becomes obvious that any two members of the equivalence class are also equivalent to one another. In a way, the latter is nothing more than repeating the former. Considering that equivalence relations hardly have any other use except for constructing equivalence classes, it seems that giving a definition based on F-transitivity is *pedagogically* a sound idea. However, mathematically, this gives us a less organized structure (or a local organization). The following figure depicts this structure.



By comparison, the standard definition of equivalence relations (based on transitivity) provides us with a global organization. The following figure depicts this structure:



As it can be seen, the two important types of relations, equivalence relations and order relations *can be (logically) seen* as particular types of transitive relations. In this regard, the standard definition has a *mathematical* advantage over the definition based on F-transitivity. However, the transitive law is more in harmony with the right branch of the above figure than the left one.

It is also worth noticing that both of the figures above treat an equivalence relation as a particular relation. Considering that the mathematical definition of relations starts by distinguishing the order, it can be seen that even the standard account of relations is more in harmony with the right branch of the above figures than the left one (or at least with what an equivalence relation is intended for, i.e. recreating our experience of equivalence).

To simplify the discussion, when comparing the two properties of transitivity and F-transitivity I ignored all the other subtleties involved. However, even this simplified discussion shows the extent of the difficulties in making a balance between pedagogical and mathematical needs.

9.3 Pedagogical needs

> In fact, it will be found that in most textbooks in mathematics, equivalence relations are defined as any relation which possesses these three properties. We find it is much easier to introduce children to equivalence relations through the family idea. The idea of a likeness relation is easy to grasp. All we need to add

> is that sometimes such a relation will split the universe into disjoint families. The members of these families are related to each other by an equivalence relation, and members of different families are not. The reader should not go beyond this point unless he or she has understood clearly how we can pass from the fact that we have split our universal set into disjoint families by a likeness relation, to the conclusion that such a relation has the properties of being reflexive, symmetric, and transitive. (Dienes, 1976, p. 60-61)

It was argued that the importance of equivalence relations is mainly dependent on their role in constructing equivalence classes. It was also mentioned that the converse procedure, that allow *us* to pass from equivalence classes to equivalence relation, mathematically is of little importance, and even, when applying to the so-called everyday example it is somehow problematic. However, as the first two sentences of the above quotation suggests, when the teaching of the subject is concerned it seems that the order of importance of these two procedures is reversed.

Let us once more look at the textbook written by Stewart and Tall (2000). One of the co-authors of this textbook told me that in the process of writing it they were thinking of, and discussing the possible ways that students would think about the subject. Indeed, as far as equivalence relations and partitions are concerned, what Stewart and Tall thought of complies with Dienes' finding. They start with reminding the reader of the distinction between odd and even integers. Changing the *language*, they say that the set of integers is split into two disjoint subsets. Then they shift to a relation that splits the same set into those two pieces:

$m \sim n$ if and only if $m - n$ is a multiple of 2 (ibid, p. 72)

Looking for *some* "general properties" (of this relation) that make this splitting procedure possible, they compare it with another relation that fails to do so ('is a divisor of'). Then, they ask:

What is it that makes the original ~ work, whereas the others go wrong?

(ibid, p. 73)

The answer goes without saying. Considering that it is a textbook, I shall say that they have taken an appropriate approach. At the least, they were looking for "some general properties" before coming up with *the* general properties.

Dienes himself takes this approach to extremes. As mentioned in Section 4.3, Dienes' programme is aimed at getting children to *discover* the God-ordained properties of an equivalence relation that are *embodied* in certain games. I do not intend to go into details of his programme again. I shall only paraphrase what I wrote there:

Neither Dienes' programme nor any other situation based on certain sorting games (splitting a *set* into disjoint sub-sets) can get the students to come up with *the* standard properties of an equivalence relation.

Let me put together the reasons for this conclusion:

First, in many situations (whether mathematical or not) reflexivity is not made explicit. More strongly, there is hardly any direct need to take it into account.

Second, to conceptualize the standard account of symmetry (i.e. "if-then" account) students (and we) need to break the equivalent status given to each element of an equivalent pair (i.e. "and" account of symmetry). But, the idea of equivalence reinforces the latter rather than the former. Moreover, as far as a factual sorting is concerned, neither of the sorting methods discussed by Skemp (Section 4.2) and used by Dienes (Section 4.3) can spontaneously lead to the standard account. In the method two (i.e. sorting by a particular matching procedure), a criterion for matching is determined in advance. Accordingly, any two elements are either matched together in the manner described, or not. In the method one (i.e. sorting by certain characteristic properties), there is barely any need to consider a binary relation between any two elements of each assortment (sub-set, group of related elements). By default, symmetry that is intended to be a property of this relation does not arise at all.

Third, whenever some kind of equivalence is concerned, the two properties of transitivity and F-transitivity can be used interchangeably. Again, neither of the sorting methods discussed above can spontaneously lead to a distinction between these two properties.

Moreover, in the method one, the instructor inevitably needs to start with "equivalence classes" usually formed by taking certain "characteristic properties" into account. Thus, he or she needs to find a way to shift the students' attention to the equivalence relation between any two elements of the same sub-set. Again, it seems that there is no other way of doing so except saying that "see, any two elements of the same sub-set are equivalent to each other." Even if the instructor *successfully* draws students' attention to the equivalence relation between any two elements, he or she has put the cart before the horse, considering that the whole point of defining an equivalence relation is to construct equivalence classes. As a cure for this defect, the instructor may wish to "get [students] to construct these classes themselves, out of an equivalence relation" (Dienes, 1976, p.7; also see Section 4.3). That is putting the second sorting method into practice. Let us follow Dienes' Instruction. We ask the students to relate an object to another by a likeness. For example, we start with *two* green objects (from a given "set") that are different in other respects (e.g. size). Suppose a group of students has decided that the chosen objects are alike in the intended way, namely, they are both green.

Then we can ask if there are some other objects which are like the two which have been chosen. According to Dienes, they will soon collect all the rest of the green objects. Using the language of this thesis, they will come up with a "single-group". Now, there are two different ways to continue this "game". First, we can *repeat* the game by choosing two non-green objects having the same *colour*, say, blue. That is a "sequential grouping". Second, we can do as Dienes himself suggests. That is to say, we can get our students to realize that the way in which the objects are alike in our first set is that they are all the same colour. "This can be repeated" (see Section 4.3). Using Skemp's terminology, we help them to realize that the characteristic properties (green, blue...) themselves belong together—"they form a set which itself has an easily seen characteristic property" (Skemp, 1971, p. 174). In our example, each characteristic property is a colour. Thus, we have come back where we started, i.e. the sorting by method one!

Overall, these difficulties arise because it is assumed that an equivalence relation has been defined to mathematize our experience of equivalence. It is also assumed that *the* properties of an equivalence relation are embodied in certain situations. Accordingly, the role of the teacher is to get students to *capture* and mathematize the intended properties. But, from the outset, some kind of equivalence is directly experienced in the situations and the students have no reason to detach themselves from their experience. To make this point clearer, let me compare this approach with a somewhat different approach.

I have already mentioned Stewart and Tall's approach to introduce equivalence relations. We can now see that there is a subtle difference between their approach and the one mentioned above, although they are on a par with each other for taking the standard defining properties as given. Recall that, Stewart and Tall start with two disjoint sets (odd and even integers). Then they look for a *relation* that splits the set of integers into those disjoint sets. As the textbook authors, they inevitably need to give that relation. But, it is important to notice that they initially see an equivalence relation as a relation that is intended to *recreate* something anew.

Stewart and Tall's approach to a certain extent exemplifies a more general approach in which an equivalence relation is seen as a relation that is intended to mathematically recreate different aspects of our experience of equivalence, i.e. constructing equivalence classes, and based on that, discarding, and at the same time recreating, the so-called characteristic property by the principle of abstraction. Weyl succinctly expresses the main feature of this principle, distinguishing it from what he calls the "originary abstraction":

> In looking at a flower I can mentally isolate the abstract feature of color as such. This act of abstraction would here be primary while the statement that two flowers have the same color 'red' would be based on it; whereas in mathematical abstraction it is the equality which is primary, while the feature with regard to which there is equality comes second and is derived from the equality relation. (Weyl, 1949, p. 11)

It now can be seen that this distinction is lacking in the first sorting method (discussed above) and to some extent the second one (when limited to the so-called everyday situation; see above).

The above distinctions (between two different approaches to teaching the subject and between two different abstractions) also shed light on the Mad Dictator Task. Consider that the task does not start by dictating certain characteristic properties (the first sorting method). Neither does it start by *specifying* the manner with regard to which two cities (numbers) are equivalent (the second sorting method). However, it puts a crucial constraint on the situation by saying that, in effect, two elements are equivalent if and only if they have the same equivalence class. This gives the students an incredible freedom within the given constraints; they can create 115975 *different* visiting-laws! Mathematically speaking, the students start with something that in a standard setting they finish with, i.e. constructing equivalence classes. But, there is a fundamental difference between the ways that they construct these equivalence classes in these two different settings. In a standard setting, a relation is defined on a set. Thus, from the outset, there is a criterion with regard to which two elements are related to each other, or not. Accordingly, the students form the equivalence classes based on a uniform criterion throughout. This uniformity opens the ways for the most important reason for defining an equivalence relation, i.e. creating a new entity by the mathematical abstraction. This feature is completely lacking in the Mad Dictator Task, at least when being used without any guide. This brings me back to the original aim for devising the task, i.e. making use of it as a teaching tool.

9.4 The Mad Dictator Task as a Teaching Tool

I used the Mad Dictator Task only as a research device. A lecturer who was aware of my study suggested using the Task in his class (at Warwick University) consisting of fifteen prospective teachers. Before using the Task, we discussed together the wording of it, and in particular, about the first condition of the task (see Appendix D). He decided to help his students to grasp the intended meaning of the first condition. Moreover, he only used the first task, GENERATING an example of a visiting-law. Following his sessions he reported:

> I allocated about 25 minutes of an hour-long session to the task. The problem was given out to the students who were given a few minutes to read it and try

> to understand it. I asked the one of the group to explain what the first condition meant and as a group we decided that the diagonal had to be coloured in.
>
> The students then worked in groups to try to invent new visiting laws. They quickly discovered that just the diagonal, and the whole grid were valid laws. Although the vast majority of the students engaged and were interested in the task, one in particular didn't really understand what was happening and I spent ten minutes working alone with her to help her grasp what the task was about.
>
> After twenty or so minutes I invited one group to present their findings to the rest of the class. Two students produced a generic visiting law where each identical equivalence class was coloured (or shaded) the same. Independently of any prompting, the students had identified that if a~b then a and b had exactly the same equivalence class (although they didn't use this terminology of course). They also noted that if the law was changed to bring about two equivalence classes overlapping, the new equivalence class would be the union of the previous two.
>
> In the next seminar I formally introduced the notion of a relation, of transitivity, symmetry and reflexivity in a standard manner. I then asked the group whether the relation "a~b if a is a visiting city of b" was an equivalence relation. It was easy for them to see that it was, and as a group we checked it for transitivity, symmetry and reflexivity.
>
> The example that the students had constructed with different colours made it entirely natural to talk about equivalence classes, and I emphasised that the remarks that they made last time could be turned into theorems. Notably that a~b => equiv(a)=equiv(b), and a not~ b => equiv(a) intersect equiv(b) = empty set. We then proved both theorems together as a group.
>
> The mad dictator task proved more effective than I could have hoped for. Within 30 minutes of the students having been given the task, they had independently 'discovered' the notion of equivalence classes and had come up with the two main theorems I had on the next seminars lesson plan. Equivalence relations are a fundamental part of mathematics, and are notoriously difficult for students to master, so this task really is very helpful indeed. (Personal e-mail, cited word for word)

It would be clearly a mistake to *prescribe* the Mad Dictator Task based only on one single "successful" case. However, the richness of the students' works in this study, together with the report of this lecturer, suggests that it is worth pausing to consider it. Saying that, it is also important to question what a teacher (lecturer) intends to do with the Mad Dictator Task. As can be seen, for this lecturer, a visiting-law has served to exemplify *the* standard properties of an equivalence relation. Thus, he lost his chance to introduce *the* standard properties as a *choice* amongst other, leading the students "to understand why some organization, some concept, some definition is better than another (Freudenthal, 1973, p. 418)." Having said so, I shall also add that in the case of equivalence relations it is not easy to practice what we preach.

Each one of the defining properties of the mathematical notion of equivalence relation has been *chosen* for certain mathematical reasons. These reasons can only be revealed in the context of a global picture that took more than two thousands year to be drawn. It does not seem to be practical (or necessary) to bring all these reasons to the fore in a short time. However, being aware of these reasons enables us to focus our attention on—and draw our students' attention to—those critical features that might otherwise be taken for granted (by us and by our students).

I hope that the material contained in this thesis makes a contribution to better understanding of these critical aspects that took mankind such a long time to differentiate the notion of equivalence relation from its direct experience of equivalence and integrate it into a global picture.

Chapter 10: EPILOGUE

The journey back

It seems that learning (a concept) proceeds in the opposite direction of the course of this thesis. That is to say, it proceeds from experiencing to organizing, and then to defining. Theses three aspects are not mutually exclusive. A concept is experienced in a certain situation being organized by the concept or the concepts embedded in the situation. To organize—in Freudenthal's sense—one needs to reflect on one's "previously unreflected activity", making it conscious and the subject of reflection (see Section 3.5.1). But, what is experienced may not be easily accessible to the learner's reflection (if it is accessible at all). Yet, making explicit the means of organizing is both the means and the purpose of any act of defining. Thus, it appears that defining and organizing are contingent on experiencing. After all:

> What we can do with something, what we can possibly know about something is contingent on what this something is for us, what meaning it has for us, how we can experience it. On such grounds we claim that the capability for experiencing X in a certain way—or in certain ways—is more fundamental than the capability for knowing something about X or the capability for doing something with X. (Marton and Booth, 1997, p. 208)

In line with the above argument, as evidenced by the data, this thesis relinquished defining in favour of organizing and then, departed from both of them in favour of experiencing. However, all the way through, the X of interest remained the same, i.e. "equivalence relation".

The above consideration allowed me to investigate people's conceptions of equivalence relation even in those cases that the notion as such was non-existent, e.g. Euclid's experience of equivalence relation (consider that the notion of equivalence relation has been defined more than two millennia after Euclid), or "lay" students' conceptions of equivalence relation (i.e. those who were unfamiliar with the mathematical definition of equivalence relation). To do so, I analysed the ways these people tackled certain situations that—from the vantage point of an informed observer— embody the idea of equivalence relation. From this starting point, *I* deduced three alternative ways of defining equivalence relations and compared them in organizational terms (see Sections 2.8 and 9.2.2). However, as a drawback, one very important pedagogical question remained untouched, i.e. how we can get *students* to realize an equivalence relation as an organizational means. On the face of it, this question has a straightforward answer, i.e. involve them in certain situations embodying the idea of equivalence relation, and then, get them to realize the common feature of these situations. However, this thesis has called into question this teaching strategy, mainly because what is experienced in these situations is "equivalence" rather than "equivalence relation". This fundamental distinction gives our pedagogical question a distinctive feature.

We grow up, experiencing equivalence all the time, implicitly or explicitly. "Equivalence relation" is a mathematical notion defined to capture different aspects of our experience of equivalence. But, this mathematical notion is not simply the result of spelling out our experience of equivalence (see Section 8.5). Thus, to understand what we gain and what we lose with *different possible definitions* of equivalence relations, it is worthwhile to study different ways that different people in different situations experience "equivalence". But, if we only focus on different ways of experiencing equivalence regardless of the *background* of the experiencers—as this study did—we cannot pinpoint the needs of a particular leaner or a particular population learning the notion of equivalence relation. In the case of equivalence relations, this population is readily available: first year undergraduate mathematics students.

Chin and Tall (2001, p. 241) mention that in the Foundation Course, whoever taught it, there was a common concern that the section on relations (including equivalence relations) was considered the hardest by the students in their course evaluations. Although the present study may shed light on the source of these difficulties, it is certainly a way ahead to address first year undergraduate mathematics students more directly. That is, in a way, moving towards 'developmental phenomenography' in which the concepts under scrutiny are confined to a formal educational setting and the purpose of the study is to help the subjects of the research, or others with the *similar educational background* to learn (Bowden, 2000, p.3). This suggests a direct step forward. Yet, there are also some other steps that might be developed from this study.

The journey ahead

If we accept that organizing and defining is contingent on experiencing, the underlying method of this study might be used to study peoples' experience of a mathematical concept before it is defined (by the people involved, or for the people involved) , and even long before it is given a name (again, by the experiencers or for the experiencers). To do so, we may use the so-called historical situations, or design certain situations, where the concept (or concepts) under study might be experienced. However, this approach has some problems, many of which, emerge from the problematic distinction between experiencing a situation and experiencing the concept (or concepts) supposedly embedded in the situation.

One tends to satisfy the requirements of a situation at hand. One may attend to different aspects of the situation from one moment to the next, without being simultaneously aware of these different aspects. Moreover, one aspect, experienced in one situation, is not necessarily being carried to a *similar* situation (i.e. similar from the vantage point of an *informed* person). These aforementioned statements apply even more strongly when we consider different people in different, though similar, situations. Yet, the researcher's role is to capture and describe different ways of experiencing the concept (or the concepts) of his or her interest. To do so, the researcher inevitably ignores situational aspects of peoples' experience of the situation in favour of those aspects transcending it, linking it with other situations. The *outcome* of the researcher's endeavour is something free from situational elements and deprived of the individuals' voice, describing the *variation* in which the concept embedded in the situation under study may be experienced. The outcome is the researcher's construct. But, why on earth this construct should be credited with any value if it is not saying anything about the subjects of the study. This shall be judged by the ways it may be used and the insights it may give. The following gives some idea of this nature.

First, it can be asked to what extent *a* definition of the concept capture peoples' experience of it. This question can be approached by turning it on its head, i.e. describing the ways in which the informed persons—who know a definition—experience the concept. This can be done by asking questions of 'what is X?' kind, or again by using problematic situations. This gives us an idea of why a concept has been defined in a particular way.

Second, although the outcome is mainly the researcher's construct, it still might account for the *quality* of what has been done by the subjects in the situation under study. That is to say, it might be used to explain why some are better than others in handling the situation at hand. Here, experiencing the situation and experiencing the concept embedded in it meet each other again.

The researcher's construct is drawn upon the learners' endeavour to handle certain situations. Sometimes these situations per se are of quite importance. For example, consider the tasks used in the present study: exemplification and proving. Now, a natural question to be asked is if there is any relation between the ways that students handle these two activities and their experience of the concept (or the concepts) embedded in the situation. In the case of "lay" learners, this question should be considered very carefully, otherwise it bring us to the so-called "learning paradox", i.e. how they can generate an example of something, or can prove about it, if they do not know what it is. The answer to both questions is: they provide an example of, or prove something about a situation embodying the concept. This brings us back to the nature of these situations and the ways that they may be designed; in other words, how one can design a situation in which the concept of interest may be experienced.

One may have in mind some general conditions when designing one of these situations. The conditions like the following:

> The teacher [or the researcher] should stage situations for learning in which students meet new abstractions, principles, theories, and explanations through events that create a state of suspense. The events...serve to present a shadowy whole, a partial understanding that demands completion and challenges the learner to accomplish it. The whole needs to be made more distinct, and the parts need to be found and then fitted into place, like a jigsaw puzzle that sits on the table half-finished inviting the passerby to discover more of the picture. (Marton and Booth, 1997, p. 180; square bracket added)

Although such conditions give us some general ideas worthy of attention when designing a problematic situation, it shall be kept it mind that the extent of validity of a situation for what it has been designed can only determined by the results of the work of those who are actually involved in the situation. This is why there is no way to prescribe a procedure to design a situation bringing about certain conceptions for *all* involving in the situation.

As it can be seen the above speculations are mainly inspired by some phenomenographical assumptions and the actual course of this study. I hope that this study serves as an example of the feasibility and usefulness of the aforementioned ideas. This can be only revealed in the fullness of time.

Part 5: References and Appendices

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Appendices

Appendix A: The Mad Dictator Task

A country has ten cities. A mad dictator of the country has decided that he wants to introduce a strict law about visiting other people. He calls this 'the visiting law'.

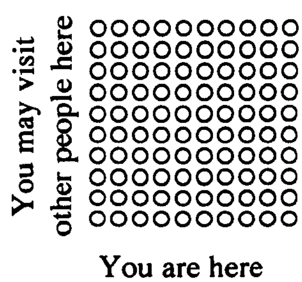
A visiting-city of the city, which you are in, is: A city where you are allowed to visit other people.

A visiting law must obey two conditions to satisfy the mad dictator:

1. When you are in a particular city, you are allowed to visit other people in that city.

2. For each pair of cities, either their visiting-cities are identical or they mustn't have any visiting-cities in common.

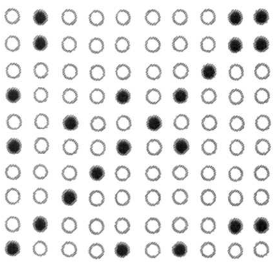
The dictator asks different officials to come up with valid visiting laws, which obey both these rules. In order to allow the dictator to compare the different laws, the officials are asked to represent their laws on a grid such as the one below.

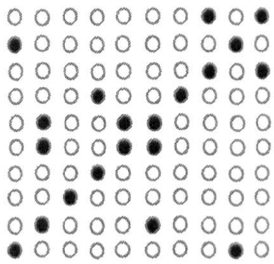


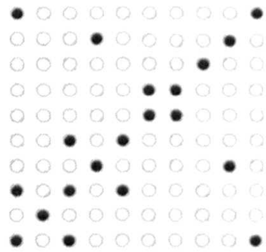
Suppose that you are an official. **Generate** an example of a visiting law.

Checking task:

Which of the following figures[38] is an example of a visiting law?







The least amount of information task:

The mad dictator decides that the officials are using too much ink in drawing up these laws. He decrees that, on each grid, the officials must give the least amount of information possible so that the dictator (who is an intelligent person and who knows the two rules) could deduce the whole of the official's visiting law.

Looking at each of the examples you have created, what is the least amount of information you need to give to enable the dictator to deduce the whole of your visiting law.

[38] These figures have frequently used in the interviews.

For the "main" study, the following two tasks were added to the above tasks.

The intersection task:

One of the officials, for creating an example, uses other officials' examples: he takes two valid examples and put their common points in his own grid. Is the grid that he makes a valid example?

The union task:

Another official takes two valid examples and puts all of their points in his own grid. Is the grid that he makes a valid example?

Appendix B: Dr Bob Burn's letter

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Sunnyside
Barrack Road
Exeter EX2 6AB
22 April 2004

Dear Mr Asghari

I was very interested in your BSRLM contribution *Students' Experiences of Equivalence Relations*, because I had myself played with such tables in my *Groups: a path to geometry*, chapter 8, page 89, question 5. Their great virtue, as I see it, is the ease with which the independence of the Reflexive, Symmetric and Transitive laws may be established.

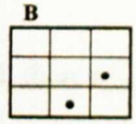
Denoting **R** = reflexive, **S** = symmetric, **T** = transitive and **B** = box, we have the following illustrations, using your convention for co-ordinates.

RST and **B** = equivalence relations

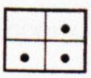
ST and B

T and B

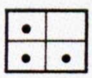
B



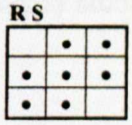
RT



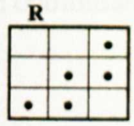
S



R S



R



So far as I can see, the box property is not equivalent to any combination of the other laws, since we can have both **RT** and **RS** with box failing. Set inclusion has **RT**, but not box.

The reflexive law is essential for your presentation. The transitive law is needed to describe many of the examples of relations which you cite.

Can you think of a mathematical relation for which the box property would be a convenient encapsulation?

There is a bogus proof that **S** and **T** imply **R**, which goes like this: **S** gives $(a, b) \Rightarrow (b, a)$; **T** gives (a, b) and $(b, a) \Rightarrow (a, a)$; so **S** and **T** give (a, a). The little square arrays exhibit what is wrong with this argument. Do you have anything similar with the box law?

Yours sincerely

R.P.Burn

Appendix C: Dr Ali Enayat's solution to the Mad Dictator Task

Dear Mr. Asghari,

It seems to me that the problem you are describing belongs squarely to graph theory:

The visiting laws can be viewed as follows:

Given a set C (of "cities"), there is a function f: C --> P(C), where P is the power set operation, such that:

(a) For each c in C, c is a member of f(C).

(b) For each pair c, and c' of members of C, either f(c) = f(c'), or f(c) intersection f(c') = the empty set.

This can be also described in terms of "multigraphs" [which then will begin to sound like multivalued logic], a topic thoroughly studied in combinatorics and graph theory.

It is clear from the visiting laws of the dictator, that the visiting laws dictate that the relation aRb, defined by "a is a member of f (b)" is an equivalence relation, and

therefore the dictator's decree will end up partitioning the set of cities into disjoint "visiting clumps".

Regards,

Ali Enayat, October 2, 2003

Appendix D: Double act of abstraction

> [The mathematically informed reader] knows that the formal definition of an equivalence relation starts with A ~ A, and concludes that this is presumably what the authors have meant with the first part. But it is not what the first part says and I would not expect anybody to understand the first part this way. (An anonymous reviewer; see Section 8.1)

This appendix provides an example of what I have called a "double act of abstraction" (see Section 8.4.2). In parallel, it also gives some examples of the ways that the first condition of a visiting-law was understood by the students.

The Mad Dictator Task is about peoples and their cities. But, a look at the students' excerpts used so far shows that there is no reference to the peoples. As a matter of fact, all the students discarded the "peoples" as soon as they could. Generally speaking, they started with the peoples and their cities, and then turned their attention to the cities and eventually to the numbers. Here are the very first lines of Chris' interview exemplifying this process.

CHRIS:	Ok so I've read it through once, and I'll read it through again.
CHRIS:	Ok, yep. So just produce anything, any solution.
AMIR:	Yeah.
CHRIS:	Ok. So,
AMIR:	You can use that paper; I have a lot of them.
CHRIS:	Its alright, I'll do it on this. 1 to 10,
CHRIS:	So, first one you're gonna have a diagonal across the middle.
AMIR:	Ok.
CHRIS:	Because they are all allowed to visit their own, so we have a diagonal like that.
AMIR:	So! Can you speak a little bit loudly?
CHRIS:	So *yeah got a diagonal, across the middle there.*

AMIR:	You don't have to shout!
CHRIS:	So that *satisfies condition number 1*, so everybody in city 1 can visit city 1, and everybody in city 7 can visit city 7. So the next part is that each pair of cities, either the ones they can visit are identical, or they mustn't have any visiting cities in common. Ok, so for cities, if you take 1 and 2, as a pair, errr, they both can be identical, or they mustn't have any in common. So if they are identical, then 1 has to be able to go to 2, and 2 has to be able to visit 1.

As it can be seen not only he shifted from the original formulation of the Task to the numbers very quickly, but also he soon realized that the first condition gives him the diagonal. However, there were also cases that it took the interviewee about ten minutes to come to these particular ways of handling the situation. For example, after about five minutes struggling to make sense of the conditions of the Task, in particular the first condition, Hess said:

Hess:	I was thinking that it shows the visiting person; a person can't visit himself.

Even in some cases, after the interviewee realized that the first condition gives him (or her) the diagonal, they initially found it a useless condition. Ben exemplifies this at the first moments of his interview:

Ben:	This is quite difficult to get your head round... I'm just reading it a few times,
Ben:	Am I on a time limit here or,
Amir:	No
Amir:	Please explain anything you want to do or you are thinking; suppose you are one of the officials...
Ben:	Right... so I'm an official, and I've got to, like, fill in this grid so it satisfies... For each pair of cities, there exists an

identical... so that means, yeah, that means people from 1 can visit other people in 1, and people in 2 can visit other people in 2... right...So... straight away, er, I wouldn't pair 1 with 1, or 2 with 2, or 3 with 3, or 4 with 4, etc, up to 10... up to 10 with 10, because *that would be a waste*, because you can already visit people from... for example if you live in 10, you can already visit people from 10, so there's no point creating them as a distinct pair.

Ignoring the peoples in favour of the cities has a vital effect on the way of tackling the task. It is also deeply related to the first condition of the task. More importantly, the students handled the first condition differently at least initially. But, these differences did not find any place in the picture drawn in this study. I decided that they are specific to the situation at hand.

It is also worth saying that despite the students' *spontaneous* shift from the peoples to the cities *I* did not ignore the possibility of *mathematically* formalizing the Task in terms of both the peoples and cities. It was the very reason that I contacted some mathematicians and logicians. But, we failed to do that. In the end, I accepted Prof. Stewart's advice, in effect, saying that when we *mathematically* model an everyday situation we ignore some aspects in favour of the others.

It is intriguing that the students spontaneously took Prof. Stewart's advice and treated the diagonal as an embodiment of the first condition. However, their works show that the interpretation of the diagonal as an embodiment of the reflexive law is more complex than my original interpretation of it (see Section 1.6.2 and Section 9.2.1). This shows that deciding whether an aspect should be abstracted or not is a complex process. Coming to such a decision can be facilitated by studying other situations in which the phenomenon of interest might be experienced.

> The researcher has a responsibility to contemplate the phenomenon, to discern its structure against the backgrounds of the situations in which it might be experienced, to distinguish its salient features, to look at it with others' eyes, and still be open to further developments. (Marton and Booth, 1997, p. 129)

This process reflects the evolving nature of the "categories of description".

Regarding the interconnection between the reflexive property and the first condition of the visiting-law, this means, it remains to be seen whether what has been downgraded here will reveal some critical aspects of the ways that the reflexivity might be experienced.