HARMONIC MAPS OF SPHERES AND EQUIVARIANT THEORY

by

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Summary:

In Chapter I we produce many new harmonic maps of spheres by the qualitative study of the pendulum equations for the join and the Hopf construction. In particular, we obtain

Corollary 1.7.1.

Let $\phi_1 : S^p \longrightarrow S^r$ be any harmonic homogeneous polynomial of degree greater or equal than two, and let ϕ_2 be the identity map id.: $S^q \longrightarrow S^q$ Then the (q+1)-suspension of ϕ_1 is harmonically representable by an equivariant map of the form $\phi_1 * \phi_2$ if and only if q=o ....5.

Corollary 1.11.1.

Let $[f] \in \pi_p^s$ be a stable class in the image of the stable J-homomorphism $J_p : \pi_p(\mathcal{O}) \longrightarrow \pi_p^s$, $p \geqslant 6$.
Then there exists $q > p$ such that $[f]$ can be represented by a harmonic map $\phi : S^{p+q+1} \longrightarrow S^{q+1}$.

In Chapter II we illustrate equivariant theory and study the rendering problems: in particular, we show that the restriction q=o ...5 in Corollary 1.7.1. can be removed provided that the domain is given a suitable riemannian metric; then, for istance, the groups $\pi_n(S^n) = \mathbb{Z}$ can be rendered harmonic for every n.

In Chapter III we describe applications of equivariant theory to the study of Dirichlet problems and warped products; and extensions of the theory to spaces with conical singularities.

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INTRODUCTION

This thesis deals with existence problems for harmonic maps.

Harmonic maps are the critical points of the energy functional: given a map $\phi:(M, g)\to(N, h)$ between two riemannian manifolds, its energy E is defined by

$$E(\phi)=\frac{1}{2}\int_M |d\phi|^2 dv_M$$

and the function $e(\phi)=\frac{1}{2}|d\phi|^2$ is called the energy density of ϕ.

The Euler-Lagrange equation associated with the energy functional E is described by a semilinear second order elliptic system of differential equations, in divergence form.

In local charts, such a system takes the form

(1) $$\Delta\phi^\gamma + g^{ij}\,{}^N\Gamma^\gamma_{\alpha\beta}\frac{\partial\phi^\alpha}{\partial x_i}\frac{\partial\phi^\beta}{\partial x_j}=0 \qquad \gamma = 1\ \ldots.\mathrm{dim}\ N.$$

where ${}^N\Gamma^\gamma_{\alpha\beta}$ are the Christoffel symbols of N.

Particular cases of harmonic maps are harmonic functions, geodesics, minimal surfaces, holomorpic maps between kähler manifolds: then the study of properties of harmonic maps is closely related to many fundamental problems in differential geometry and analysis on manifolds; in the last years, the interest in harmonic maps has widened to physicists, especially in connection with the theories of non-linear σ-models and liquid crystals (BCL).

The basic existence problem for harmonic maps can be formulated as follows:

Let $\phi_0 : M \longrightarrow N$ be a map of riemannian manifolds. Is there a harmonic map $\phi : M \longrightarrow N$ homotopic to ϕ_0 ?

Assuming both M and N to be compact, the most general existence result was proved by Eells-Sampson in 1964 (ES), in a fundamental paper that started the systematic study of harmonic maps: they proved, by using heat-flow tecniques, that the answer to the previous question is affirmative provided that the sectional curvature of N is non-positive. Moreover, the assumption of compactness of the range can be weakened to include a large class of complete target manifolds; we also mention that the same existence theorem was proved later by K. Uhlenbeck (U2) by using a more direct method of calculus of variations.

On the other hand, if the range has positive curvature, no general theory provides a complete answer to the existence question; in this case, according to the special class of manifolds under consideration, different methods have been successfully introduced to study existence and classification of harmonic maps: we mention a few

i) harmonic maps between surfaces (EW): in this case relationships with conformal and holomorphic maps have been exploited.

   In particular, let T^2, S^2 be the 2-dimensional torus and sphere : it has been shown that there is no harmonic map $\phi : T^2 \rightarrow S^2$ of degree 1. Then the answer to the existence question is not always affirmative.

ii) Harmonic maps from riemann surfaces to complex Grassmannians (BW);

Harmonic maps from the 2-dimensional sphere to the unitary group (Va), (U1);

Twistorial construction of harmonic maps (ESa).

In these cases, harmonic maps are described in terms of holomorphic data and almost complex structures.

iii) Harmonic maps with image contained in a not-too-large ball: existence and uniqueness results in this direction have been obtained in (H.K.W.) by using more direct methods in P.D.E.

iv) Equivariant theory. This will be the object of this thesis: roughly speaking, equivariant maps are maps with prescribed symmetries that force the elliptic system (1) to reduce to an <u>ordinary second order differential equation.</u>

Equivariant maps were first introduced by Smith in 1972 (S1, S2) and aimed to construct harmonic maps between spheres: in fact, there are two important classes of equivariant maps of spheres:

1) The join $\phi_1 * \phi_2 : S^{p+q+1} \longrightarrow S^{r+s+1}$ of two harmonic homogeneous polynomials $\phi_1 : S^p \longrightarrow S^r$, $\phi_2 : S^q \longrightarrow S^s$. (see sec. 1, Chapter I).

2) Maps $H : S^{p+q+1} \longrightarrow S^{r+1}$ produced, via Hopf's construction, by a map $F : S^p \times S^q \longrightarrow S^r$ which is harmonic with constant energy density in each variable separately (see sec. 8, Chapter I).

In both cases, the question of existence can be reduced to the qualitative study of the differential equation of a pendulum with variable gravity and damping; and positive answers can be achieved.

The study of these pendulum equations will be the object of Chapter I; now, in order to give a first description of our main results, we recall something from the previous works on the subject.

The problem of producing harmonic maps between spheres was raised in 1964 in the above mentioned work of Eells-Sampson: in particular, they asked whether the groups $\pi_n(S^n) = \mathbb{Z}$ can be harmonically represented, $n \geq 3$ (In the case $n = 1, 2$ the answer is affirmative and elementary).

In 1972 Smith studied the pendulum equations associated with the equivariant maps in 1), 2), and proved

A) The join $\phi_1 * \phi_2$ can be represented by a harmonic map provided that ϕ_1 and ϕ_2 satisfy certain damping conditions.

B) Non-existence for the Hopf construction in the case when $H : S^3 \to S^2$ has Hopf invariant $k \cdot 1$, $k \neq 1$.

As an application of the result in A), we have harmonic maps of each degree in $\pi_n(S^n)$, $n \leq 7$, $n = 9$; the dimension restriction is a consequence of the damping conditions.

In Chapter I we will improve A) and B) by showing

A') Existence of relevant solutions of the differential equation for the join <u>if and only if</u> certain <u>less restrictive</u> damping conditions are satisfied (Theorem 1.1.1.).

B') Sufficient conditions for the existence for the Hopf construction (theorem 1.8.1.).

As an application of A'), we will construct many new harmonic maps of spheres, including maps $\phi_1 * \phi_2 : S^m \longrightarrow S^m$, $m > 10$; but no complete results for $\pi_n(S^n)$, $n = 8$, $n \geqslant 10$. Roughly, positive answers for the equivariant harmonic join are to be confined to low dimensions.

By contrast, the result in B') will enable us to construct non-polynomial harmonic maps of spheres of large dimensions (Corollary 1.11.1.).

Smith's existence results were proved basically by comparing the pendulum equation with linear equations with constant coefficients; in our work, we introduce new comparison arguments and a method which shows when there are obstructions to the existence of special solutions.

It is worth noticing that, apart from quite a few harmonic homogeneous polynomials, the above equivariant maps are the only known examples of harmonic maps between spheres.

In Chapter II we establish a general setting for equivariant theory and take up a program of equivariant deformations of metrics: the main motivation for this is the above mentioned restriction to represent harmonically the groups $\pi_n(S^n)$.

We will show (sections 3 and 6) that, for every n, each class of the group $\pi_n(S^n)$ can be harmonically represented provided that the domain is deformed by using a suitable riemannian metric; moreover, the same holds for maps into ellipsoids (sec. 7).

An interesting feature of our approach to the rendering problem will be to relate its solution to the qualitative study of a <u>first</u> order <u>linear</u> differential equation.

Chapter II, and Chapter III as well, are much in the spirit of the P. Baird's work (B1) which can be taken, together with the Smith's works, as the best reference for backgrounds in equivariant theory. As far as less specialistic references to harmonic maps theory, we refer to (EL1),(EL2) and attached bybliography.

In Chapter III we describe some applications of equivariant theory to warped products and Dirichlet problems; we also illustrate extensions of equivariant theory to spaces with cone-like singularities and some further possible developments.

Each chapter is divided in sections and has a short introduction that summarizes the contents of the sections.

<u>Notations:</u>

Unless differently specified, when we refer to formula 2.7)(for instance), this formula is to be found in the second section of the chapter in which the reference is made.

Theorem (Proposition, Corollary, Lemma) 2.3.1. (for instance) is to be found in section 3 of Chapter II.

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CHAPTER I

In sections 1 - 7 of this chapter we study the join of two harmonic homogeneous polynomials of spheres; sections 8 - 11 deal with the Hopf construction.

Section 1 is devoted to introduce the problem of the join; it contains a description of the problem, the statement of the main result and information about how to read sections 2 - 7.

Section 8 plays for the Hopf construction an analogous role to section 1 for the join.

Section 1

In order to describe the problem of the join, we write every point $z \in S^{p+q+1} \subset \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ in the form $z = \sin s\, x + \cos s\, y$, with $x \in S^p$, $y \in S^q$ and $0 \leqslant s \leqslant \frac{\pi}{2}$.

The loci $s = o$ and $s = \frac{\pi}{2}$ are called focal varieties: we will perform our analysis on S^{p+q+1} minus the focal varieties and extend it to the whole S^{p+q+1} by using standard regularity arguments.

Let $\phi_1 : S^p \longrightarrow S^r$, $\phi_2 : S^q \longrightarrow S^s$ be two continuous maps of spheres: the homotopy class of their join $\phi_1 * \phi_2 : S^{p+q+1} \longrightarrow S^{r+s+1}$ can be represented by maps of the following type:

$$1.1) \qquad (\phi_1 * \phi_2)(z) = \sin\alpha(s)\,\phi_1(x) + \cos\alpha(s)\,\phi_2(y)$$

where α is a continuous function defined on $\left[0, \frac{\pi}{2}\right]$ such that

$$1.2) \qquad \alpha(0)=0,\ \alpha\left(\frac{\pi}{2}\right)=\frac{\pi}{2},\ \alpha(s)\in\left(0,\frac{\pi}{2}\right),\ s\in\left(0,\frac{\pi}{2}\right).$$

From now on we specialize to the case of harmonic homogeneous polynomials: let $\Phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a map whose components are harmonic polynomials which are homogeneous of common degree K and suppose that the restriction of Φ carries spheres to spheres, i.e. $\Phi/_{S^{m-1}} = \phi : S^{m-1} \longrightarrow S^{n-1}$.

Then ϕ is said to be a harmonic homogeneous polynomial of degree K.

The simplest examples of harmonic homogeneous polynomials are: the identity map id. : $S^p \longrightarrow S^p$, the standard K-fold rotation $i_K : S^1 \longrightarrow S^1$, the Hopf fibrations; other interesting examples arise as gradients of isoparametric functions and from orthogonal multiplications $F : S^p \times S^p \longrightarrow S^r$ via the Hopf construction.

Any harmonic homogeneous polynomial $\phi: S^{m-1} \longrightarrow S^{n-1}$ of degree K is a harmonic map with constant energy density $e(\phi) = \frac{1}{2} \cdot K(K+m-2)$.

Now, let $\phi_1 : S^p \longrightarrow S^r$ and $\phi_2 : S^q \longrightarrow S^s$ be two harmonic homogeneous polynomials: the constant energy density of ϕ_1 and ϕ_2, together with the special symmetries of the join map $\phi_1 * \phi_2$, imply that the condition of harmonicity for maps as in 1.1) reduces to a second order ordinary differential equation for the function α.

This reduction to an ordinary differential equation is the typical procedure of equivariant theory, as we will see in Chapter II, where we also develop the tools for the derivation of equation 1.3) below.

After the substitution $\tan s = e^{t}$, $t \in \mathbb{R}$, the relevant equation for α takes the form of a pendulum equation with variable gravity and damping

1.3)

$$\alpha''(t) + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right] \alpha'(t) + \left[\frac{\lambda_2 e^{t} - \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right] \sin \alpha(t) \cos \alpha(t) = 0 \qquad t \in \mathbb{R}$$

where
$$\lambda_1 = K_1(K_1 + p - 1), \quad K_1 = \deg(\phi_1)$$
$$\lambda_2 = K_2(K_2 + q - 1), \quad K_2 = \deg(\phi_2)$$

After the previous substitution, conditions 1.2) have become

1.4) $$\lim_{t \to -\infty} \alpha(t) = 0, \qquad \lim_{t \to +\infty} \alpha(t) = \frac{\pi}{2}, \qquad \alpha(t) \in \left(0, \frac{\pi}{2}\right), \quad t \in \mathbb{R}.$$

<u>Remark 1:</u>

More generally, the form of the gravity in 1.3) makes it reasonable to ask whether equation 1.3) possesses special solutions of the form

$$1.5)\quad \begin{cases} \lim_{t\to-\infty}\alpha(t) = 0, \quad \lim_{t\to+\infty}\alpha(t) = \dfrac{\pi}{2} + m\pi, \quad m\in\mathbb{N} \\ \alpha(t) \in \left(0, \dfrac{\pi}{2} + m\pi\right) \qquad t\in\mathbb{R}. \end{cases}$$

Such solutions would be relevant for our purpose because a function as in 1.5) can be used in 1.1) to define a continuous map of spheres.

Now we can state our main theorem, in which the non-existence statement is to be understood to hold for the class of maps 1.5) of Remark 1.

Theorem 1.1.1.

Let $\phi_1 : S^p \longrightarrow S^r$ and $\phi_2 : S^q \longrightarrow S^s$ be two harmonic homogeneous polynomials of degree K_1, K_2, and let $\lambda_1 = K_1(K_1+p-1)$, $\lambda_2 = K_2(K_2+q-1)$.

Then $\phi_1 * \phi_2 : S^{p+q+1} \longrightarrow S^{r+s+1}$ can be harmonically represented if and only if the following generalised damping conditions (G.D. C.) hold:

$$\text{G.D.C.} = \begin{cases} \text{a)} \quad (q-1)^2 < 4\lambda_2 \text{ or } \quad 2K_1 < (q-1) - \sqrt{(q-1)^2 - 4\lambda_2} \\ \\ \text{b)} \quad (p-1)^2 < 4\lambda_1 \text{ or } \quad 2K_2 < (p-1) - \sqrt{(p-1)^2 - 4\lambda_1} \end{cases}$$

Remark 2

We recall that, according to the notations above, Smith's theorem gives solutions provided that

$$\text{D.C.} = \begin{cases} \text{a)} \quad (q-1)^2 < 4\lambda_2 \\ \\ \text{b)} \quad (p-1)^2 < 4\lambda_1 \end{cases}$$

$$\text{or} \quad p = q, \quad \lambda_1 = \lambda_2 .$$

The proof of theorem 1.1.1. is the object of sections 2 - 6 and consists in the qualitative study of equation 1.3): for this purpose, we introduce a method that can also be applied to the study of the general equation of a pendulum with variable gravity and damping: our method may involve computer estimates as indicated in Remark 3 of section 6.

Our analysis enables us to determine the precise combination of the 4 parameters p, q, λ_1, λ_2 that separates existence from non-existence: this interesting feature has a geometrical application as indicated in Remark 6 of section 7.

The proof of Theorem 1.1.1. is rather long: in a first reading, the proof of Lemma 1.3.1., Lemma 1.4.1. and some computational arguments of section 5 can be omitted. By contrast, it is vital to get familiar with the meaning of the area H-N-D introduced in section 4 and with the conclusions of section 5.

The applications of theorem 1.1.1. are discussed in section 7.

Section 2

We investigate existence and non-existence of special solutions of 1.3) of the form 1.4): only in section 6 we will generalise the non-existence result to the class of functions 1.5) of Remark 1.

In order to accomplish our study in full, we consider p, q, K_1, K_2, and consequently λ_1, λ_2, as real parameters, with $1 \leq p, q$, $o < K_1, K_2$.

First we need to recall some facts from pages 48-53 (S2):

let t_o be the point in which the gravity $\left[\frac{\lambda_2 e^t - \lambda_1 e^{-t}}{e^t + e^{-t}}\right]$ vanishes.

We indicate with $\alpha(a_o, \dot{a}_o)$ the solution of 1.3) distinguished by initial data $\alpha(t_o) = a_o$ $\dot{\alpha}(t_o) = \dot{a}_o$.

If $a_o \in (o, \frac{\pi}{2})$, let $A^+(a_o)$ be the collection of $\dot{a}_o$ such that $\alpha(a_o, \dot{a}_o)$ increases monotonically to $\frac{\pi}{2}$ in finite time as t increases from t_o. Similarly, let $A^-(a_o)$ be the set of $\dot{a}_o$ such that $\alpha(a_o, \dot{a}_o)$ decreases monotonically to 0 in finite time as t decreases from t_o.

We define two functions $\dot{\alpha}_o^+$, $\dot{\alpha}_o^-$ on $(o, \frac{\pi}{2})$ as follows:

$$\dot{\alpha}^+(a_o) \overset{\text{def.}}{=} \inf. \left\{ A^+(a_o) \right\}$$

$$\dot{\alpha}^-(a_o) \overset{\text{def.}}{=} \inf. \left\{ A^-(a_o) \right\}$$

Proposition : $\dot{\alpha}_o^+ , \dot{\alpha}_o^-$ are well-defined and continuous with values in $(0,+\infty)$: moreover $\alpha(a_o, \dot{\alpha}_o^+(a_o))$ increases asymptotically to $\frac{\pi}{2}$ as t increases from t_o to $+\infty$.
Analogously, $\alpha(a_o, \dot{\alpha}_o^-(a_o))$ decreases asymptotically to 0 as t decreases from t_o to $-\infty$.

The existence of a special solution as in 1.4) is equivalent to the existence of a point $\hat{a} \in (0,\frac{\pi}{2})$ such that

$$\dot{\alpha}_o^+(\hat{a}) = \dot{\alpha}_o^-(\hat{a})$$

We will study when

i) there exists a point $a_1 \in (0,\frac{\pi}{2})$ such that $\dot{\alpha}_o^-(a_1) \leq \dot{\alpha}_o^+(a_1)$.

ii) there exists a point $a_2 \in (0,\frac{\pi}{2})$ such that $\dot{\alpha}_o^-(a_2) \geqslant \dot{\alpha}_o^+(a_2)$.

Because of the continuity of $\dot{\alpha}_o^+, \dot{\alpha}_o^-$, the validity of both i) and ii) is equivalent to the existence of $\hat{a}$ as above.

More precisely, we will prove what follows:

2.1) $\begin{cases} \text{if G.D.C. a) holds, then i) holds.} \\ \text{if G.D.C. a) does \underline{not} hold, then we have the} \\ \text{non-existence statement of the theorem.} \end{cases}$

2.2) $\begin{cases} \text{if G.D.C. b) holds, then ii) holds.} \\ \text{if G.D.C. b) does \underline{not} hold, then we have the} \\ \text{non-existence statement of the theorem.} \end{cases}$

Clearly the theorem follows from 2.1) and 2.2).
The proof of 2.2) is perfectly analogous to that of 2.1), thus we will just occupy ourselves with 2.1).

Section 3

Our non-existence results will not require any comparison argument;

By contrast, for establishing existence under G.D.C. it is vital the following comparison lemma:

Lemma 1.3.1.

Consider the differential equation

$$3.1)\quad \alpha''(t) + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right]\alpha'(t) + G(t)\sin\alpha(t)\cos\alpha(t) = 0 \qquad t \in \mathbb{R}$$

Suppose that

$$3.2)\quad G(t) < \left[\frac{\lambda_2 e^{t} - \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right] \qquad t \in \mathbb{R}$$

and that

3.3) Equation 3.1) has a solution F(t) as in 1.4).

Then i) of section 2 holds, i.e. there exists a point $a_1 \in (0, \pi/2)$ such that $\dot{\alpha}_0^-(a_1) \leq \dot{\alpha}_0^+(a_1)$.

Proof of Lemma 1.3.1.:

We will use the following comparison theorem for ordinary differential equations (see CL pag. 210):

Let L_i, $i=1,2$ denote the differential operator that acts on functions f by $L_i f = (p_i f')' + g_i f$, where p_i', g_i are continuous functions on $[t_0, T]$, such that:

$$3.4)\quad 0 < p_2(t) \leq p_1(t) \qquad t \in [t_0, T]$$

$$3.5)\quad g_2(t) > g_1(t) \qquad t \in (t_0, T)$$

Let f_i, $i=1,2$ be such that $L_i f_i \equiv 0$ on $[t_0, T]$ and let us call

$$w_i = \tan^{-1}\left(\frac{f_i}{f_i' p_i}\right)$$

Finally suppose that $w_2(t_0) \geq w_1(t_0)$

Then
$$w_2(t) > w_1(t) \qquad \forall\, t \in (t_0, T]$$

We want to use this comparison theorem for comparing equation 1.3) and equation 3.1): to do this we put

$$p_1(t) = p_2(t) = \exp\left(\int_{t_0}^{t} \left[\frac{(p-1)e^{-s}-(q-1)e^{s}}{e^{s}+e^{-s}}\right] ds\right)$$

$$g_1(t) = G(t)\,\frac{\sin(F(t))\cos(F(t))}{F(t)}\,p_1(t)$$

$$g_2(t) = \left[\frac{\lambda_2 e^{t} - \lambda_1 e^{-t}}{e^{t}+e^{-t}}\right]\frac{\sin(\bar{\alpha}(t))\cos(\bar{\alpha}(t))}{\bar{\alpha}(t)}\,p_2(t)$$

where $\bar{\alpha}(t)$ is the solution of 1.3) that we describe below.

Now we prove the following assertions:

3.6) $\quad \dot{\alpha}_0^{+}(F(t_0)) \geq F'(t_0)$

3.7) $\quad F'(t_0) \geq \dot{\alpha}_0^{-}(F(t_0))$

Lemma 1.3.1. follows immediately from 3.6) and 3.7) by taking $a_1 = F(t_0)$.

The physical meaning of 3.6) and 3.7) is the following: the equations 1.3) and 3.1) describe the motion of two pendulums with the same damping; but, because of the assumption 3.2), the gravity in 1.3) pushes always the pendulum toward 0 more than the gravity in 3.1).

Consequently, in order to reach the same final position, the pendulum 1.3) must be pushed harder than the pendulum 3.1) toward $\frac{\pi}{2}$

(this gives 3.6)), and softer toward 0 (3.7)). (see (R1))

Condition 3.7) is the corresponding of 3.6) in backward time, so we just give the detailed proof of 3.6).

Let us suppose that 3.6) does not hold: then there exists $\varepsilon > 0$ such that

3.8) $$\dot{\alpha}_0^+ \left(F(t_0) \right) + \varepsilon < F'(t_0)$$

Let $\bar{\alpha}(t) \overset{\text{def.}}{=} \alpha\left(F(t_0), \dot{\alpha}_0^+(F(t_0)) + \varepsilon \right)$; $\bar{\alpha}(t_0) = F(t_0)$ and $\bar{\alpha}'(t_0) < F'(t_0)$: then $\bar{\alpha}(t) < F(t)$ for $t > t_0$, t small; but, directly from the definition of the function $\dot{\alpha}_0^+$, we have that $\bar{\alpha}(t)$ must reach $\frac{\pi}{2}$ in finite time: $F(t)$ satisfies 1.4) by the hypothesis 3.3), then it makes sense to call T, $T > t_0$, the first point in which $\bar{\alpha}(T) = F(T)$.

We take $f_1(t) = F(t)$ and $f_2(t) = \bar{\alpha}(t)$. Now we check that all the hypotheses of the comparison theorem are satisfied on $[t_0, T]$: the only thing that it is not immediate is 3.5): but $\bar{\alpha}(t) < F(t)$ on (t_0, T) implies

$$\frac{\sin \bar{\alpha}(t) \cos \bar{\alpha}(t)}{\bar{\alpha}(t)} > \frac{\sin(F(t)) \cos(F(t))}{F(t)}$$

This fact, together with the assumption 3.2), gives 3.5).

So all the hypotheses of the comparison theoremare fulfilled, but the thesis of the theorem is violated: infact $\bar{\alpha}(t)$ reaches $F(t)$ in T from below: then $\bar{\alpha}'(T) \geq F'(T)$ and so $w_2(T) \leq w_1(T)$: this is the contradiction that we were after and ends the proof of Lemma 1.3.1

Section 4

We develop some material to construct differential equations as in Lemma 1.3.1.

Let $F : \mathbb{R} \to \left(0, \frac{\pi}{2}\right)$ be any differentiable function: we define $G_F(t)$ by

4.1) $$G_F(t) \overset{\text{def.}}{=} \frac{-F''(t) - \left[\dfrac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}} \right] F'(t)}{\sin(F(t)) \cos(F(t))}$$

Then

$$4.2)\quad F''(t)+\left[\frac{(p-1)e^{-t}-(q-1)e^{t}}{e^{t}+e^{-t}}\right]F'(t)+G_F(t)\sin(F(t))\cos(F(t))=0 \qquad t\in\mathbb{R}.$$

To apply Lemma 1.3.1, we need a special function $F:\mathbb{R}\to\left(0,\frac{\pi}{2}\right)$ such that

$$4.3)\qquad G_F(t)<\left[\frac{\lambda_2 e^{t}-\lambda_1 e^{-t}}{e^{t}+e^{-t}}\right] \qquad t\in\mathbb{R}$$

$$4.4)\qquad \lim_{t\to+\infty}F(t)=\frac{\pi}{2}\ ,\quad \lim_{t\to-\infty}F(t)=0$$

It follows from the definition of $G_F(t)$ that 4.3) can be reformulated as

$$4.3')\ F''(t)+\left[\frac{(p-1)e^{-t}-(q-1)e^{t}}{e^{t}+e^{-t}}\right]F'(t)+\left[\frac{\lambda_2 e^{t}-\lambda_1 e^{-t}}{e^{t}+e^{-t}}\right]\sin(F(t))\cos(F(t))>0 \qquad t\in\mathbb{R}$$

It is convenient to write $F(t)$ in the form

$$4.5)\qquad F(t)=\tan^{-1}\left(f(t)\right)\ ,\quad f:\mathbb{R}\to(0,+\infty)$$

In terms of $f(t)$ 4.3') and 4.4) become respectively

$$4.6)\ f''(t)-\left[\frac{2f(t)f'^2(t)}{1+f^2(t)}\right]+\left[\frac{(p-1)e^{-t}-(q-1)e^{t}}{e^{t}+e^{-t}}\right]f'(t)+\left[\frac{\lambda_2 e^{t}-\lambda_1 e^{-t}}{e^{t}+e^{-t}}\right]f(t)>0 \qquad t\in\mathbb{R}$$

and

$$4.7)\qquad \lim_{t\to+\infty}f(t)=+\infty \qquad \lim_{t\to-\infty}f(t)=0$$

The espression in 4.6) is obtained from 4.3') simply by direct substitution and use of the identity $\sin x\cos x=\left[\dfrac{\tan x}{1+\tan^2 x}\right]$

In order to simplify the notations, we will write:

$$D(t) \overset{\text{def.}}{=} \left[\frac{(p-1)e^{-t}-(q-1)e^{t}}{e^{t}+e^{-t}}\right]$$

$$\Gamma(t) \overset{\text{def.}}{=} \left[\frac{\lambda_2 e^{t}-\lambda_1 e^{-t}}{e^{t}+e^{-t}}\right]$$

$$\text{cond}(f)(t) \overset{\text{def.}}{=} f''(t)-\left[\frac{2f(t)f'^2(t)}{1+f^2(t)}\right]+D(t)\cdot f'(t)+\Gamma(t)\cdot f(t)$$

$$\text{cond}_L(f)(t) \overset{\text{def.}}{=} f''(t)+D(t)\cdot f'(t)+\Gamma(t)\cdot f(t)$$

In these notations, 4.6) is just $\text{cond}(f)(t) > 0$, $t \in \mathbb{R}$; moreover it is clear that $f(t)$ determines a solution of 1.3) (via 4.5)) if and only if

4.8) $\quad \text{cond}(f)(t) = 0 \quad , \quad t \in \mathbb{R}$

For our purposes,it is convenient to start with studying a necessary condition for 4.6) and 4.8) to hold,that is

4.9) $\quad \text{cond}_L(f)(t) \geq 0 \quad , \quad t \in \mathbb{R}$

We write

4.10) $\quad f(t) = \exp\left(\int_{t_0}^{t} H(s)\,ds + c\right) \quad c \in \mathbb{R}$.

where $H(t) = \left[\lg(f(t))\right]'$; the function $H(t)$ will play a fundamental role in our program.

In terms of $H(t)$, 4.9) is

4.11) $\quad H'(t) + H^2(t) + D(t)\cdot H(t) + \Gamma(t) \geq 0 \quad , \quad t \in \mathbb{R}$

In order to hope that 4.11) holds, it is clear that $H(t)$ can decrease only when

$$4.12) \quad H^2(t) + D(t)\cdot H(t) + \Gamma(t) > 0$$

The idea of the theorem is the following:
suppose that $\alpha(t)$ is a special solution of 1.3) of the form 1.4); take $f(t)$ as in 4.5), then 4.8) holds; consequently 4.9) holds and so 4.11) for the corresponding $H(t)$.

By the analysis of the asymptotic hehavior of a special solution, we will deduce that

$$\lim_{t\to-\infty} H(t) = K_1 \quad \text{and} \quad \lim_{t\to+\infty} H(t) = K_2$$

Now suppose that G.D.C. a) does not hold: roughly speaking, this implies that K_1 is quite bigger than K_2:
we will show that this would force $H(t)$ to decrease somewhere when 4.12) does not hold, contradicting the existence of a special solution.

On the contrary, if G.D.C. a) holds, we will be able to build up differential equations as in Lemma 1.3.1. and conclude the existence.

It is fundamental to study in detail when $H(t)$ can decrease without affecting 4.11):

Let $V_t(X)$ be the parabola

$$4.13) \quad V_t(x) \overset{\text{def.}}{=} X^2 + D(t)\cdot x + \Gamma(t)$$

Then $V_t(x) \leq 0$ only when X belongs to a certain interval, possibly empty, that we denote by $[X_{1_t}, X_{2_t}]$.

Therefore, when t ranges over $\mathbb{R}$, the intervals $[X_{1_t}, X_{2_t}]$ describe an area that we call H-N-D : if 4.11) holds, then $H(t)$ can not cross H-N-D decreasing, because $H(t) \in [x_{1_t}, x_{2_t}]$ forces $H'(t) \geq 0$.

The explicit description of H-N-D is long and therefore given separately in section 5:

but first we establish the necessary information about the asymptotic behavior of $H(t)$:

Lemma 1.4.1

Let $\alpha(t)$ be a special solution of 1.3) as in 1.4):
then the function $H(t)$ distinguished by $\alpha(t)$ via 4.5), 4.10), that is to say by

$$4.14)\quad \alpha(t) = \tan^{-1}\left[\exp\left(\int_{t_0}^{t} H(s)ds + c\right)\right]$$

satisfies

$$4.15)\quad \begin{cases} a) & \lim_{t \to -\infty} H(t) = K_1 \\ b) & \lim_{t \to +\infty} H(t) = K_2 \end{cases}$$

Proof of Lemma 1.4.1

The proof of this Lemma is just expressing in terms of $H(t)$ the well-known asymptotic behavior of a special solution: infact we know from pag. 61 (S2) that:

$$4.16)\quad \left[K_1 - \mathcal{O}(e^{2t})\right]\sin(\alpha(t))\cos(\alpha(t)) \leq \dot{\alpha}(t) \leq \left[K_1 + \mathcal{O}(e^{2t})\right]\sin(\alpha(t))$$

t sufficiently close to $-\infty$.

Also we have

$$4.17)\quad \begin{cases} \sin x \cos x = x + \mathcal{O}(x^3) \\ \sin x = x + \mathcal{O}(x^3) \\ \tan^{-1} x = x + \mathcal{O}(x^3) \end{cases} \qquad x \text{ small}$$

The direct substitution of 4.14) in 4.16), joined with 4.17) and the fact that $\lim\limits_{t \to -\infty} \alpha(t) = 0$, gives 4.15 a).

Analogously one proves 4.15 b).

Section 5:

In this fundamental section we collect the necessary information about the area H-N-D introduced in section 4: the proof of the statements of this section is mainly a computational argument, but it requires some effort.

The relevant results of this section are well visualized in figures 1.....8 below.

First we need to introduce the function

$$5.1)\quad \Delta(t) \overset{\text{def.}}{=} D^2(t) - 4\,\Gamma(t)$$

The explicit expression for $\Delta(t)$ is

$$5.2)\quad \Delta(t) = \frac{\left[(q-1)^2 - 4\lambda_2\right]e^{2t} + \left[4(\lambda_1 - \lambda_2) - 2(p-1)(q-1)\right] + \left[(p-1)^2 + 4\lambda_1\right]e^{-2t}}{\left[e^t + e^{-t}\right]^2}$$

If $\Delta(t) < 0$, then $\left[x_{1_t}, x_{2_t}\right]$ is empty.

If $\Delta(t) \geq 0$, then

$$5.3)\quad \begin{cases} x_{1_t} = \dfrac{-D(t) - \sqrt{\Delta(t)}}{2} \\[2ex] x_{2_t} = \dfrac{-D(t) + \sqrt{\Delta(t)}}{2} \end{cases}$$

Lemma 1.5.1

If $(q-1)^2-4\lambda_2<0$, then there exist $\bar{t}\in\mathbb{R}$, $\varepsilon>0$ such that

$$\Delta(t)<-\varepsilon \quad , \quad t\geqslant\bar{t} .$$

Proof of Lemma 1.5.1

This Lemma follows immediately from 5.2); under this hypothesis, H-N-D is empty for $t\geqslant\bar{t}$.

Now we come to the relevant part of our analysis and suppose that $(q-1)^2-4\lambda_2\geqslant 0$.

A computation shows that $\Delta(t)<0$ for some t if and only if

$$5.4)\quad \begin{cases} \text{a)} & 4(\lambda_1-\lambda_2)-2(p-1)(q-1)<0 \\ \text{b)} & (\lambda_1+\lambda_2)^2+(p+q-2)\cdot\left[(p-1)\lambda_2-(q-1)\lambda_1\right]>0 \end{cases}$$

Now, just recalling the fact that λ_1,λ_2 are related to K_1,K_2 by $\lambda_1=K_1(K_1+p-1)$, $\lambda_2=K_2(K_2+q-1)$, we notice that

$$5.5)\quad K_1=\frac{1}{2}\left[-(p-1)+\sqrt{(p-1)^2+4\lambda_1}\right]$$

and

$$5.6)\quad K_2<\frac{1}{2}\left[(q-1)-\sqrt{(q-1)^2-4\lambda_2}\right]$$

Lemma 1.5.2.

Let us suppose that $(q-1)^2-4\lambda_2\geqslant 0$.

Then we have

a) If $2K_1 > (q-1) - \sqrt{(q-1)^2 - 4\lambda_2}$, then $\Delta(t) > 0$, $t \in \mathbb{R}$.

b) If $2K_1 = (q-1) - \sqrt{(q-1)^2 - 4\lambda_2}$, and $\sqrt{(q-1)^2 - 4\lambda_2} > 0$,

then there exists $\tilde{t} \in \mathbb{R}$ such that

5.7) $$\begin{cases} \Delta(t) > 0 & t \neq \tilde{t} \\ \Delta(\tilde{t}) = 0 \end{cases}$$

c) If $2K_1 < (q-1) - \sqrt{(q-1)^2 - 4\lambda_2}$, and $\sqrt{(q-1)^2 - 4\lambda_2} > 0$,

then there exist $t_1, t_2 \in \mathbb{R}$ such that

5.8) $$\begin{cases} \Delta(t) > 0 & t \in (-\infty, t_1) \cup (t_2, +\infty) \\ \Delta(t_1) = \Delta(t_2) = 0 \\ \Delta(t) < 0 & t \in (t_1, t_2) \end{cases}$$

d) If $2K_1 = (q-1) - \sqrt{(q-1)^2 - 4\lambda_2}$, and $\sqrt{(q-1)^2 - 4\lambda_2} = 0$,

then $\Delta(t) > 0$, $t \in \mathbb{R}$ and $\lim\limits_{t \to +\infty} \Delta(t) = 0^+$.

e) If $2K_1 < (q-1) - \sqrt{(q-1)^2 - 4\lambda_2}$, and $\sqrt{(q-1)^2 - 4\lambda_2} = 0$,

then there exists $t_1 \in \mathbb{R}$ such that

5.9) $$\begin{cases} \Delta(t) > 0 & t \in (-\infty, t_1) \\ \Delta(t_1) = 0 \\ \Delta(t) < 0 & t \in (t_1, +\infty) \end{cases} \quad \text{and} \lim_{t \to +\infty} \Delta(t) = 0^- .$$

Moreover, in all cases, we have

5.10) $$\lim_{t \to -\infty} X_{2_t} = K_1 \quad , \quad \lim_{t \to -\infty} X_{1_t} = -K_1 - (p-1)$$

In the cases a), b), c), d), we have

$$5.11)\quad \begin{cases} \lim\limits_{t \to +\infty} x_{1_t} = \dfrac{1}{2}\left[(q-1) - \sqrt{(q-1)^2 - 4\lambda_2}\right] \\ \lim\limits_{t \to +\infty} x_{2_t} = \dfrac{1}{2}\left[(q-1) + \sqrt{(q-1)^2 - 4\lambda_2}\right] \end{cases}$$

<u>Proof of Lemma 1.5.2.</u>

Let us start with a): by substituting 5.5) in the hypothesis a), we have

$$5.12)\quad \sqrt{(p-1)^2 + 4\lambda_1} + \sqrt{(q-1)^2 - 4\lambda_2} > (p-1) + (q-1)$$

By elevating left and right-hand sides of 5.12) to the square, we obtain

$$5.13)\quad 2\sqrt{\left[(p-1)^2 + 4\lambda_1\right] \cdot \left[(q-1)^2 - 4\lambda_2\right]} > 4(\lambda_2 - \lambda_1) + 2(p-1)\cdot(q-1)$$

Now we observe what follows:

If the right-hand side of 5.13) is negative, then 5.4) a) does not hold.

If the right-hand side of 5.13) is greater or equal than zero, by elevating again to the square we have

$$(\lambda_1 + \lambda_2)^2 + (p+q-2)\cdot\left[(p-1)\lambda_2 - (q-1)\lambda_1\right] < 0$$

Then 5.4) b) does not hold.

Thus we can conclude that, under the hypothesis of Lemma 1.5.2. a), we have $\Delta(t) > 0 \;,\; t \in \mathbb{R}$.

Similarly, one checks that, under the hypothesis b), 5.7) holds; and 5.8) holds in the case c).

As for parts d), e), just looking at the explicit expression 5.2) of $\Delta(t)$ tells that it is enough to check that, under the hypothesis d), we have

$$5.14)\quad 4(\lambda_1-\lambda_2)-2(p-1)(q-1)=0$$

and that, under e), we have

$$5.15)\quad 4(\lambda_1-\lambda_2)-2(p-1)(q-1)<0$$

Infact, by using the assumption d) and 5.5), we have

$$\sqrt{(p-1)^2+4\lambda_1}=(q-1)+(p-1)\quad,\ (q-1)^2=4\lambda_2$$

And the hypothesis e) and 5.5) give

$$\sqrt{(p-1)^2+4\lambda_1}<(q-1)+(p-1)\quad,\ (q-1)^2=4\lambda_2$$

Then 5.14) and 5.15) follow easily.

Moreover, just writing the explicit expression of X_{1_t}, X_{2_t} as in 5.3) and remembering again 5.5), one easily obtains 5.10) and 5.11) and the proof of Lemma 1.5.2 is ended.

To simplify the notations, we will write

$$\tilde{\Delta}(t)=\Delta(t)\cdot[e^t+e^{-t}]^2$$

We need to study the derivative of the functions X_{1_t} and X_{2_t} : under the hypothesis of Lemma 1.5.2. b),c),e) the following statements and calculations are to be understood where they make sense, i.e. where $\Delta(t)>0$. From 5.3) we have

$$5.16)\qquad X_{1_t} = \frac{1}{2}\left[\frac{(q-1)e^{t}-(p-1)e^{-t}-\sqrt{\tilde{\Delta}(t)}}{e^{t}+e^{-t}}\right]$$

$$5.17)\qquad X_{2_t} = \frac{1}{2}\left[\frac{(q-1)e^{t}-(p-1)e^{-t}+\sqrt{\tilde{\Delta}(t)}}{e^{t}+e^{-t}}\right]$$

The computation of the derivative of 5.16), 5.17) gives

$$5.18)\qquad X'_{1_t} = \frac{\left[(p+q-2)\cdot\sqrt{\tilde{\Delta}(t)}\right]-A(t)}{\sqrt{\tilde{\Delta}(t)}\left[e^{t}+e^{-t}\right]^{2}}$$

$$5.19)\qquad X'_{2_t} = \frac{\left[(p+q-2)\cdot\sqrt{\tilde{\Delta}(t)}\right]+A(t)}{\sqrt{\tilde{\Delta}(t)}\left[e^{t}+e^{-t}\right]^{2}}$$

where

$$5.20)\qquad \begin{cases} A(t) = \left[(q-1)^{2}-2(\lambda_1+\lambda_2)+(p-1)(q-1)\right]e^{t}+ \\ \qquad +\left[-2(\lambda_1+\lambda_2)-(p-1)^{2}-(p-1)(q-1)\right]e^{-t}. \end{cases}$$

Now we study

$$5.21)\quad \left[(p+q-2)\sqrt{\tilde{\Delta}(t)}\right]^2 - A^2(t)$$

A long but straight forward calculation shows that 5.21) can be re-written as

$$5.22)\quad -4\cdot R_{p,q,\lambda_1,\lambda_2}\cdot\left[e^{2t}+2+e^{-2t}\right]$$

where

$$5.23)\quad R_{p,q,\lambda_1,\lambda_2} = (\lambda_1+\lambda_2)^2+(p+q-2)\cdot\left[(p-1)\lambda_2-(q-1)\lambda_1\right]$$

Now suppose that $(q-1)^2-4\lambda_2>0$: the following facts hold:

5.24) If $2K_1>(q-1)+\sqrt{(q-1)^2-4\lambda_2}$, then $R_{p,q,\lambda_1,\lambda_2}>0$

5.25) If $2K_1=(q-1)+\sqrt{(q-1)^2-4\lambda_2}$, then $R_{p,q,\lambda_1,\lambda_2}=0$

5.26) $\begin{cases}\text{If } (q-1)-\sqrt{(q-1)^2-4\lambda_2}<2K_1<(q-1)+\sqrt{(q-1)^2-4\lambda_2}, \\ \text{then } R_{p,q,\lambda_1,\lambda_2}<0\end{cases}$

5.27) If $2K_1=(q-1)-\sqrt{(q-1)^2-4\lambda_2}$, then $R_{p,q,\lambda_1,\lambda_2}=0$

5.28) If $2K_1<(q-1)-\sqrt{(q-1)^2-4\lambda_2}$, then $R_{p,q,\lambda_1,\lambda_2}>0$

The proof of 5.24) .....5.28) is omitted because again it is just a computation based on 5.5), as in Lemma 1.5.2. a), b), c).

One can use 5.24) ......5.28) in 5.22) and obtains information about the sign of X'_{1t} and X'_{2t} : of course a little care is needed for taking in account the sign of $A(t)$.

The study of the functions X'_{1t} and X'_{2t}, together with Lemma 1.5.2., enables to draw the relevant conclusions about H-N-D, that we now summarize. In all the figures below, H-N-D is the dark area.

First we illustrate the 5 cases corresponding to 5.24).....5.28):

We will write $X^{\pm}$ in place of $\frac{1}{2}\left[(q-1) \pm \sqrt{(q-1)^2 - 4\lambda_2}\right]$.

Figure 1

Case 5.24)

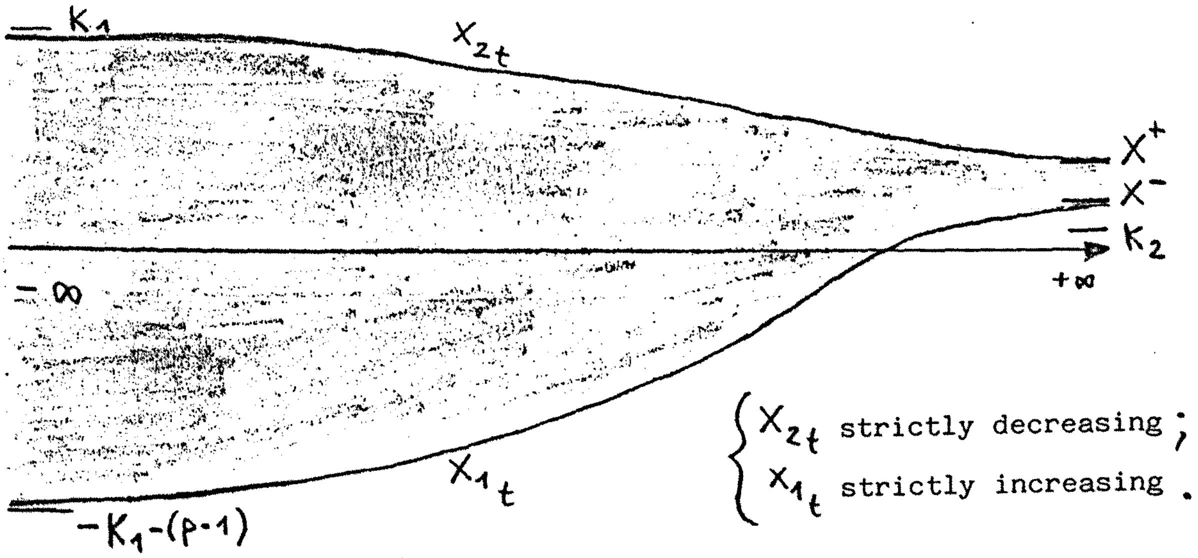


Figure 2

Case 5.25)

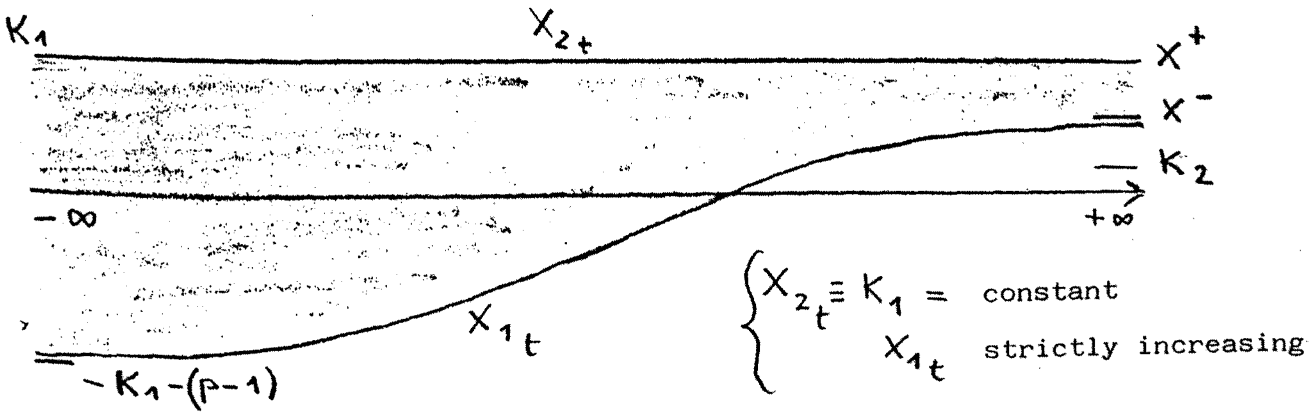
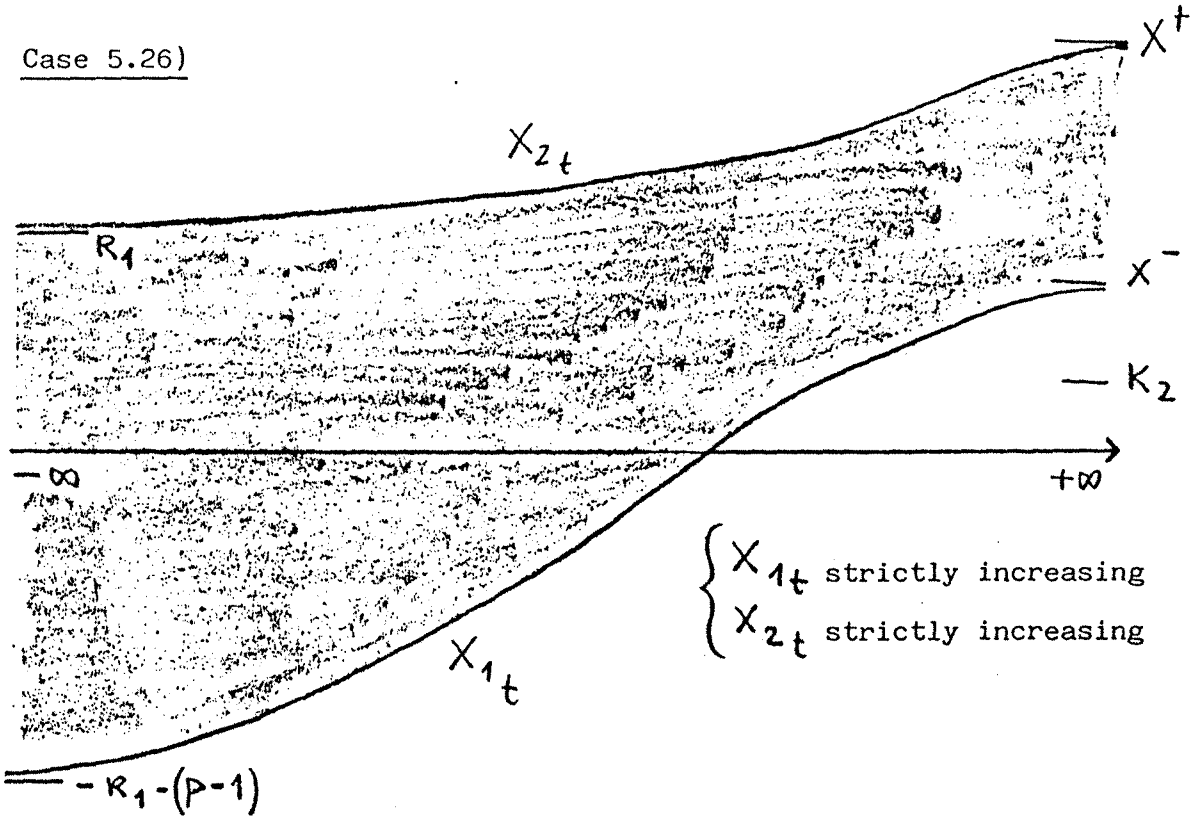


Figure 3

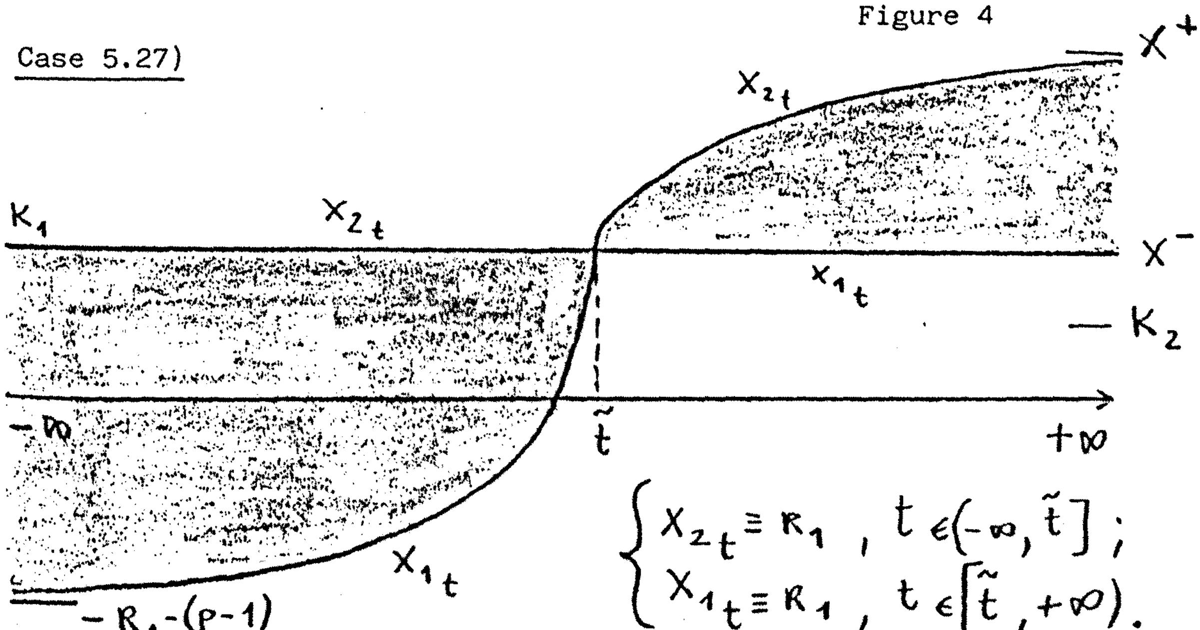
Case 5.26)



$$\begin{cases} X_{1_t} \text{ strictly increasing} \\ X_{2_t} \text{ strictly increasing} \end{cases}$$

Figure 4

Case 5.27)

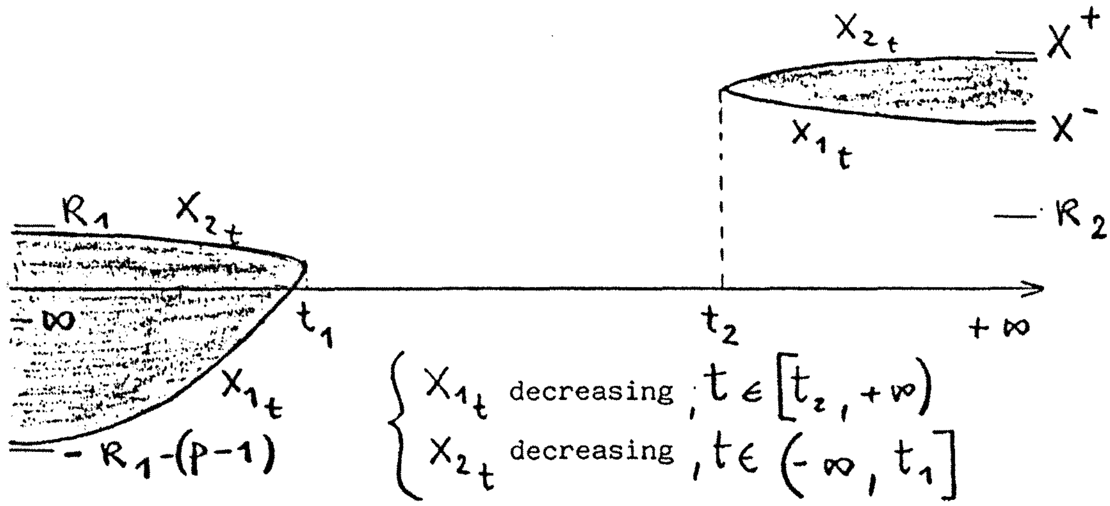


$$\begin{cases} X_{2_t} \equiv K_1 \;, & t \in (-\infty, \tilde{t}\,] \;; \\ X_{1_t} \equiv K_1 \;, & t \in [\,\tilde{t}, +\infty) . \end{cases}$$

$$\begin{cases} X_{1_t} \text{ strictly increasing}, & t \in (-\infty, \tilde{t}\,] \;; \\ X_{2_t} \text{ strictly increasing}, & t \in [\,\tilde{t}, +\infty) \;. \end{cases}$$

Case 5.28)

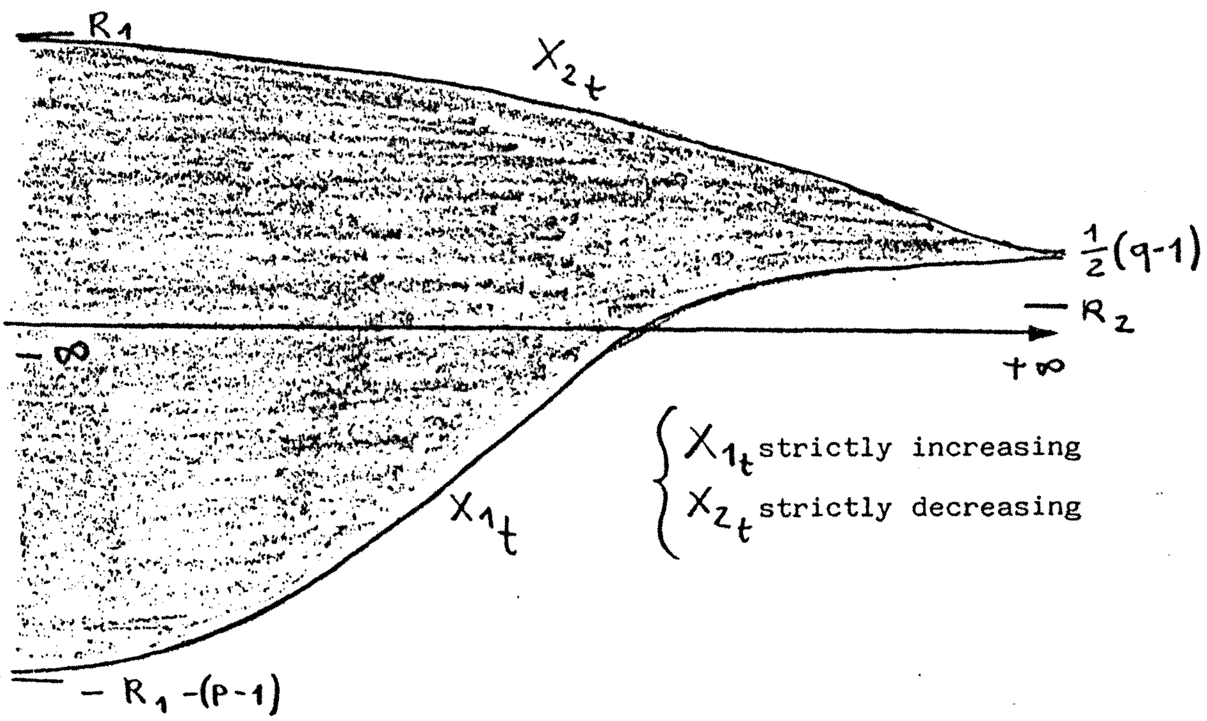
Figure 5



Now we study H-N-D under the hypothesis $(q-1)^2 - 4\lambda_2 = 0$: Case 5.26) disappears and cases 5.25) and 5.27) coalesce into one. The analogous of 5.24) is

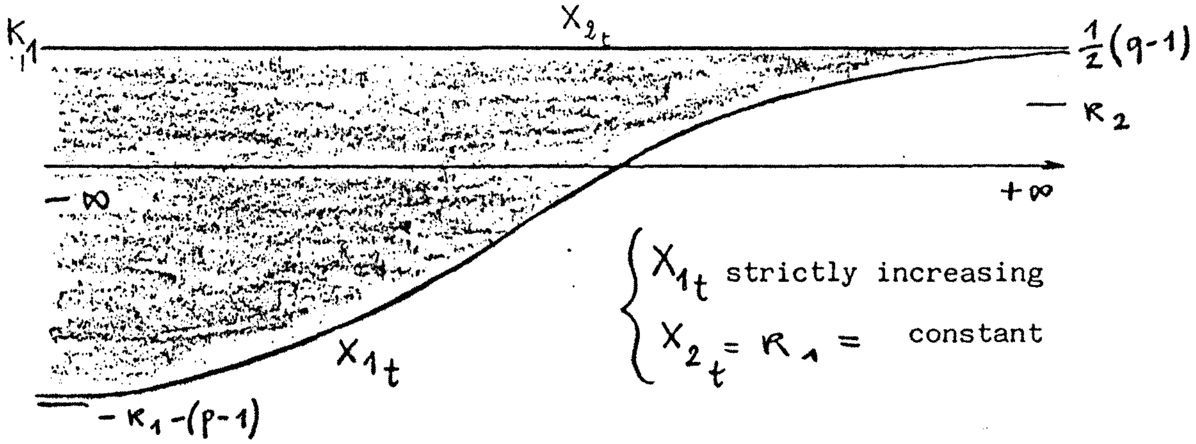
Case 5.29):

Figure 6



Cases 5.25) and 5.27) coalesce into

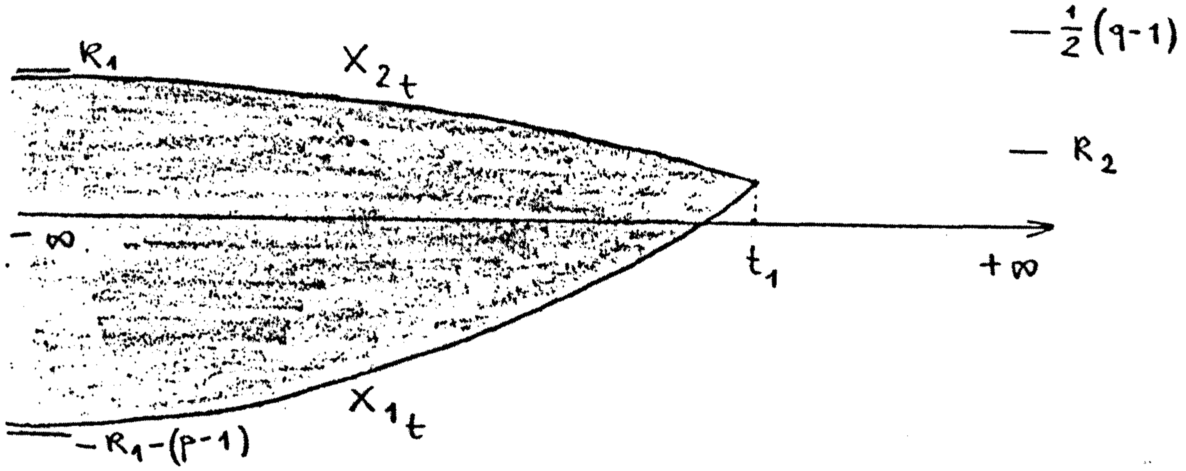
<u>Case 5.30)</u>: Figure 7



Finally case 5.28) becomes

<u>Case 5.31:</u>

Figure 8



$$\begin{cases} X_{2_t} \text{ strictly decreasing}, & t \in (-\infty, t_1] \\ X_{1_t} \text{ strictly increasing}, & t \in (-\infty, t_1] \end{cases}$$

Section 6

The work of section 5 has put us in the right position to prove theorem 1.1.1. , that is to say the statement 2.1).
Following the different cases that we have distinguished in section 5, we have to prove that:

- The cases 5.24), 5.25), 5.26), 5.27), 5.29),5.30) are all non-existence cases.
- The cases 5.28), 5.31) and the case $(q-1)^2-4\lambda_2<0$ give i) as in 2.1).

We start with non-existence: let us suppose that there exist a solution $\alpha(t)$ of 1.3) as in 1.4); let $f(t)$ be associated to $\alpha(t)$ as in 4.5): we recall from section 4 that we have

6.1) $$\mathrm{cond}(f)(t) = 0 \qquad , t \in \mathbb{R}$$

or equivalently

6.2) $$\mathrm{cond}_L(f)(t) - \left[\frac{2 f(t) f'^2(t)}{1+f^2(t)}\right] = 0$$

the function $f(t)$ is positive valued and then 4.9) holds; in terms of the corresponding $H(t)$ as in 4.10) we have that 4.11) holds.

Moreover, by Lemma 1.4.1. we know that

6.3) $$\begin{cases} a) \quad \lim\limits_{t\to-\infty} H(t) = k_1 \\ b) \quad \lim\limits_{t\to+\infty} H(t) = k_2 \end{cases}$$

and from 5.6) we have

$$6.4) \qquad R_2 < \frac{1}{2}\left[(q-1) - \sqrt{(q-1)^2 - 4\lambda_2}\right] \qquad \left((q-1)^2 - 4\lambda_2 \geq 0\right)$$

Now suppose that we are in one of the cases 5.24), 5.25), 5.26), 5.29), 5.30): it is clear that 6.3) and 6.4), together with the conclusions summarized at the end of section 5, force $H'(t) < 0$ somewhere in H-N-D: this is not allowed because it contradicts 4.11) and gives the non-existence.

In the case 5.27), the only possibility for 4.11) to hold globally is that

$$6.5) \qquad H(\tilde{t}) = R_1 \qquad \tilde{t} \text{ as in figure 4.}$$

But if one evaluates cond(f) in $\tilde{t}$ finds

$$6.6) \qquad 0 - \left[\frac{2 f(\tilde{t}) f'^2(\tilde{t})}{1 + f^2(\tilde{t})}\right] < 0$$

6.6) contradicts 6.1) and ends the case 5.27): therefore we have shown that, if G.D.C. a) does not hold, then equation 1.3) does not have special solutions as in 1.4).

Now we generalise this non-existence result to special solutions of the form 1.5):

Let us suppose that a special solution $\alpha(t)$ as in 1.5) exists: then take T to be the point such that

$$\begin{cases} \alpha(T) = m\pi \\ \alpha(t) \in \left(m\pi, \dfrac{\pi}{2} + m\pi\right), \quad t \in (T, +\infty). \end{cases}$$

Then we can write

$$6.7)\quad \alpha(t) = m\pi + \tan^{-1}\left[\exp\left(\int_{t_0}^{t} H(s)\,ds + c\right)\right],\quad t\in(T,+\infty).$$

Because $\dot{\alpha}(T) > 0$ (otherwise $\alpha(t) \equiv m\pi$), we have

$$6.8)\qquad \lim_{t\to T^+} H(t) = +\infty$$

Therefore, if G.D.C. a) does not hold, we can substitute 6.3) a) with 6.8) and apply the same non-existence techniques that we used above, ending the non-existence part.

As for the existence part, we have to prove that in the cases 5.28), 5.31), and in the case $(q-1)^2-4\lambda_2<0$, we can build up a differential equation as in the hypothesis of Lemma 1.3.1.

According to the preliminary work of section 4, this reduces to proving that there exists a function $f:\mathbb{R}\to(0,+\infty)$ that satisfies 4.6) and 4.7): we show how to construct such a function.

First we produce a function $H(t)$ such that the associated function $f_H(t)$ as in 4.10) satisfies

$$6.9)\qquad \mathrm{cond}(f_H)(t) > 0 \quad,\ t\in(-\infty,\tilde{t}_1]\cup[\tilde{t}_2,+\infty)$$

and for some $\tilde{t}_1, \tilde{t}_2 \in\mathbb{R}$.

$$6.10)\qquad \mathrm{cond}_L(f_H)(t) > 0,\quad t\in\mathbb{R}.$$

We observe that if $H(t)$ is of the form

$$6.11)\quad \begin{cases} H(t) = h_1, & t\in(-\infty,T_1],\ h_1\in\mathbb{R},\ h_1>R_1 \\ H(t) = h_2, & t\in[T_2,+\infty),\ h_2\in\mathbb{R},\ 0<h_2<R_2 \end{cases}$$

for some $T_1, T_2 \in\mathbb{R}$.

then 6.9) is always satisfied:

infact the associated $f_H(t)$ satisfies

$$\begin{cases} f_H(t) = a e^{h_1 t}, & t \in (-\infty, T_1] \quad \text{for some } a > 0 \\ f_H(t) = b e^{h_2 t}, & t \in [T_2, +\infty) \quad \text{for some } b > 0 \end{cases}.$$

Then the direct substitution of $f_H(t)$ in the explicit expression of cond(f_H) leads easyly to 6.9).

Now we recall that, in terms of $H(t)$, the explicit expression of 6.10) is

$$6.10') \quad H'(t) + H^2(t) + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right] H(t) + \left[\frac{\lambda_2 e^{t} - \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right] > 0$$

$$t \in \mathbb{R}$$

and distinguish two cases

C. I) $\quad k_1 < k_2$

C.II) $\quad k_1 \geq k_2$.

For C.I), we choose $h_1 = h_2$ in 6.11) and $H(t) = h_1$, $t \in \mathbb{R}$: then it is easy to see that also 6.10') holds.

As for C.II), we deal first with cases 5.28) and 5.31): let us take $H(t)$ to be any function such that

$$6.12) \begin{cases} H(t) = \bar{h}_1 & t \leq t_1 + \delta, \quad \delta > 0 \quad \delta \text{ small} \\ H(t) = \bar{h}_2 & 0 < \bar{h}_2 < k_2, \quad t \geq T, \quad T \text{ large} \\ |H'(t)| < \varepsilon & \\ H(t) \in [\bar{h}_2, \bar{h}_1], & t \in \mathbb{R}. \end{cases}$$

where

t_1 is as in figures 5 and 8 and

6.13) $$K_1 < \bar{h}_1 < \frac{1}{2}\left[(q-1) - \sqrt{(q-1)^2 - 4\lambda_2}\right]$$

6.14) $$\left[\bar{h}_1^2 - (q-1)\bar{h}_1 + \lambda_2\right] e^{t} + \left[\bar{h}_1^2 + (p-1)\bar{h}_1 - \lambda_1\right] e^{-t} > 0$$
$$t \in \left(-\infty, t_1 + \delta\right].$$

6.14) is actually the restriction of 6.10') to $\left(-\infty, t_1+\delta\right]$,with $H(t) = \bar{h}_1$: a short calculation shows that 6.13) and 6.14) are compatible by taking

$\bar{h}_1$ close enough to $\frac{1}{2}\left[(q-1) - \sqrt{(q-1)^2 - 4\lambda_2}\right]$.

Finally ε in 6.12) is defined by

6.15) $$\varepsilon \overset{\text{def.}}{=} \inf_{t \geq t_1+\delta,\ x \in \left[\bar{h}_2, \bar{h}_1\right]} \left\{V_t(x)\right\}, \quad V_t(x) \text{ as in 4.13).}$$

We have to check that $\varepsilon > 0$:

by construction we have that $V_t(x) > 0$ on the whole strip $t \geq t_1 + \delta$, $x \in \left[\bar{h}_2, \bar{h}_1\right]$ as in figures 5, 8.

Moreover it is clear that

$$V_t(x) \geq V_t(\bar{h}_1) \qquad \begin{array}{l} t \geq t_1 + \delta \\ x \in \left[\bar{h}_2, \bar{h}_1\right] \end{array}$$

Then it is sufficient to show that

6.16) $$\lim_{t \to +\infty} V_t(\bar{h}_1) > 0$$

Let us call $\bar{X} = \frac{1}{2}\left[(q-1) - \sqrt{(q-1)^2 - 4\lambda_2}\right]$ and write

6.17) $\bar{h}_1 = \bar{X} - \gamma \ , \quad \gamma > 0.$

From the explicit expression of $V_t(x)$, the limit in 6.16) is

6.18) $$\lim_{t \to +\infty} (\bar{X} - \gamma)^2 + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right](\bar{X} - \gamma) + \left[\frac{\lambda_2 e^{t} - \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right]$$

Now we observe that

6.19) $$\bar{X}^2 + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right]\bar{X} + \left[\frac{\lambda_2 e^{t} - \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right] \geq 0 \qquad t \geq t_1 + \delta .$$

because $\bar{X}$ is below H-N-D- on $[t_1 + \delta, +\infty)$ in the case 5.28); and H-N-D is empty for $t \geq t_1 + \delta$ in the case 5.31).

By using 6.19) and the expression of $\bar{X}$ it follows that the limit in 6.18) is greater or equal than

6.20) $$\gamma^2 + \left[\sqrt{(q-1)^2 - 4\lambda_2}\right]\gamma > 0$$

that gives 6.16) and proves that a function $H(t)$ as in 6.12) exists. By construction $f_H(t)$ associated to $H(t)$ as in 6.12) satisfies 6.9) and 6.10).

The case $(q-1)^2 - 4\lambda_2 < 0$ represents the most evident area of existence and in our context can be easyly treated just on the basis of the elementary Lemma 1.5.1.: just take H(t) to be a function that decreases, if necessary, slowly for t large: this is possible because in this case $\Delta(t) < -\varepsilon$, $\varepsilon > 0$, for t large. Details in this direction are omitted because in this case the conclusion of 2.1) was already well-known by (S2): however the meaning of the restriction $(q-1)^2 - 4\lambda_2 < 0$ is to be confined to a method of investigation based on comparisons with linear equations with constant coefficients.

As a final step, it remains only to see that from $f_H(t)$ satisfying 6.9) and 6.10) we can construct the function $f(t)$ that satisfies 4.6) and 4.7).

Let $a \in (0,1)$, a so small as to have

6.21) $$\mathrm{cond}_L(f_H)(t) - \left[\frac{2\, a^2 f_H(t) f_H'^2(t)}{1 + a^2 f_H^2(t)}\right] > 0\,, \quad t \in [\tilde{t}_1, \tilde{t}_2]$$

Now let

6.22) $$f(t) = a\, f_H(t) \qquad , a \text{ as in 6.21).}$$

Then we are: infact, just reading 6.21), we have

6.23) $$\mathrm{cond}(f)(t) > 0 \quad , t \in [\tilde{t}_1, \hat{t}_2]$$

We observe that

$$6.24)\qquad \frac{a^2 f_H f_H'^2}{\left[1+a^2 f_H^2\right]} < \frac{f_H f_H'^2}{\left[1+f_H^2\right]}$$

Now 6.24) and 6.9) give immediately

$$6.25)\qquad \text{cond}(f)(t) > 0 \quad , \; t \in \left(-\infty, \tilde{t}_1\right] \cup \left[\tilde{t}_2, +\infty\right).$$

Finally 6.25) and 6.23) tell us that our $f(t)$ satisfies cond$(f)(t) > 0$, $t \in \mathbb{R}$, that is to say 4.6).
Condition 4.7) is immediate from the explicit expression of $f(t)$ and this ends theorem 1.1.1.

Remark 3:

Despite the fact that we had to distinguish many different cases in section 5, we can conclude that the area H-N-D depends upon the 4 parameters $p, q, \lambda_1, \lambda_2$ in an extremely regular way: this regularity has been vital to find the exact combination of the parameters that separates existence from non-existence. We think that this great regularity is related to the fact that equation 1.3) comes out from a problem of geometrical and variational interest.
Our method can be applied to investigate the existence of special solutions also in the case of more general gravities and damping: but in the general case even a quite explicit knowledge about H-N-D may leave some uncertainty between existence and non-existence : in particular, it can occur the following case:

$$\Delta(t) > 0 \quad , \quad t \in \mathbb{R}$$

$$\inf_{t \in \mathbb{R}} \{ x_{2_t} \} < \sup_{t \in \mathbb{R}} \{ x_{1_t} \}$$

This case may be quite pathological, because H-N-D would be a strip, but $H(t)$ may cross it without decreasing.

A useful tool that has helped our thinking has been the computer: we have used it in the following elementary way: we were looking for a function $f(t)$ that satisfies cond(f) > 0 or $\text{cond}_L(f) > 0$; we took many different types of functions $f(t)$ that satisfy the boundary conditions at $\pm\infty$ and made the computer print the values of cond(f) or $\text{cond}_L(f)$ in sufficiently many points: an elementary program works for this purpose. After some trials we were able to deduce which class of functions was more suitable and in which compact set these functions still needed to be modified. In our opinion, this kind of method can be valuable especially in the pathological cases mentioned above; more generally, whenever it is possible to reduce the key point of a comparison argument to a no matter how complicated inequality, as for instance in 4.6).

This seems to us to be of interest for the applications to concrete problems, because it reflects the following general idea: let us suppose that we have an ordinary differential equation, and we need to perform a qualitative investigation on the existence of special solutions (also more general boundary values problems can be included); in order to apply comparison arguments, one may use, if such exist, differential equations similar to the given one, and having solutions that behave similarly to the required special solutions.

But it is likely that this approach give just rough information, and

one may lose quite many interesting solutions.
On the contrary, it is much more precise to compare the given equation with equations of the same form and having solutions of the same type as the special solutions that we seek: this was actually the essence of the comparison argument of sections 3-4.
In general, such equations do not exist naturally, but they may be constructed; of course such a construction depends upon the specific problem under consideration, but the starting point is always imposing a suitably chosen function to be a special solution. Working in the spirit of our section 4 leads quite naturally to reducing the study of the given differential equation to the study of an associated differential inequality, in which case the computer can give relevant help. In simple words, the computer can help our understanding of the most suitable form for the function that must play the role of the required special solution.

Remark 4:

The discussion of regularity across the focal varieties of S^{p+q+1} has been omitted, because one can repeat exactly the arguments of (S1); we just limit ourselves to point out that Smith's treatment of regularity can be shortened (see R 3) by showing that the maps of theorem 1.1.1. are globally weakly harmonic (EP) and belong to $\mathcal{L}^2_1(S^{p+q+1}, S^{r+s+1}) \cap \mathcal{C}^0(S^{p+q+1}, S^{r+s+1})$: then they are smooth by a regularity theorem of Hildebrandt (EL1 p. 10).

Section 7

In this section we discuss the consequences of theorem 1.1.1.: every non-existence statement is to be understood to hold for the class of maps of theorem 1.1.1., that is to say for equivariant maps.

Let the map ϕ_2 of Theorem 1.1.1. be the identity id: $S^q \longrightarrow S^q$: then the join map $\phi_1 * \phi_2$ is the (q+1)-suspension of ϕ_1; in this case $\lambda_2 = q$, and an elementary substitution in G.D.C. enables us to state:

Corollary 1.7.1.

Let $\phi_1 : S^p \longrightarrow S^r$ be any harmonic homogeneous polynomial of degree greater or equal than two, and let ϕ_2 be the identity map id: $S^q \longrightarrow S^q$. Then the (q+1)-suspension of ϕ_1 is harmonically representable by an equivariant map of the form $\phi_1 * \phi_2$ if and only if q=o ....5.

Remark 5:

The existence statement in corollary 1.7.1. was announced in (R1): in that context, the key role in the comparison argument was played by the differential equation of the identity map id.: $S^{p+q+1} \longrightarrow S^{p+q+1}$; in this final version, this equation has been substituted by the differential equations that we have built up in order to apply Lemma 1.3.1.: in fact, the much more elementary comparison of (R1) had required also Smith's estimate, and led only to a much weaker understanding of equation 1.3).

Remark 6:

It is worth noticing that not even polynomials of degree $k_1 = 2$ can be suspended 7 times:

In fact, the substitution in G.D.C. a) gives

$$2 \cdot 2 \not< (6-1) - \sqrt{(6-1)^2 - 4 \cdot 6} = 4$$

Then the values $k_1 = 2$, $q = 6 = \lambda_2$, $p \geq 1$ are precisely on the boundary of the non-existence area.

Now we give some applications of theorem 1.1.1. in the direction of existence. The reference for homotopy theory is Toda (T); harmonic polynomials of spheres and their properties are listed and discussed in sec. 8 of (EL1). We are going to use the following examples of harmonic homogeneous polynomial maps:

7.1)

$$\begin{array}{ll}
i_2 : S^1 \longrightarrow S^1 & f_1 : S^4 \longrightarrow S^4 \\
i_3 : S^1 \longrightarrow S^1 & f_2 : S^7 \longrightarrow S^7 \\
h_1 : S^3 \longrightarrow S^2 & f_3 : S^{13} \longrightarrow S^{13} \\
h_2 : S^7 \longrightarrow S^4 & f_4 : S^{25} \longrightarrow S^{25} \\
h_3 : S^{15} \longrightarrow S^8 & g_1 : S^5 \longrightarrow S^5 \\
m_1 : S^{19} \longrightarrow S^{16} & g_2 : S^9 \longrightarrow S^9 \\
m_2 : S^{33} \longrightarrow S^{32} &
\end{array}$$

Where deg $(i_2)=2$, deg $(i_3)=3$, h_i are the Hopf fibrations, deg $(h_i)=2$; f_i, g_i arise as gradients of isoparametric functions, deg $(f_i)=2$, deg $(g_i)=3$; m_i arise from orthogonal multiplications via the Hopf construction, deg $(m_i)=2$.

Among the above maps, we notice that

7.2) $\quad h_3, f_3, f_4, m_1, m_2$

do not satisfy Smith's damping conditions D.C. of section 1, but we have

Examples 1:

Corollary 1.7.1. can be applied to each of the maps in 7.2): harmonic suspensions of h_3 give a harmonic representative for the generator of $\pi_{n+7}(S^n) = \mathbb{Z}_{240}$ n=9 ...... 14.

m_1 represents twice the generator of $\pi_{19}(S^{16}) = \mathbb{Z}_{24}$: then we have a harmonic representative for twice the generator of $\pi_{n+3}(S^n) = \mathbb{Z}_{24}$, n=17 .... 22.

Suspensions of f_3, f_4 give harmonic maps $\phi : S^n \longrightarrow S^n$ of Brouwer degree ± 2, n=14 .....19, 26 ..... 31.

Suspensions of m_2 give harmonic maps $\phi : S^{n+1} \longrightarrow S^n$, n=33 .....38.

A simple calculation in G.D.C. allows to state a second nice corollary:

Corollary 1.7.2.

Let $\phi_1: S^p \longrightarrow S^r$, $\phi_2: S^q \longrightarrow S^s$ be two harmonic homogeneous polynomials of the same degree ($K_1 = K_2$). Then $\phi_1 * \phi_2$ can be harmonically represented.

This corollary has interesting applications in the case $K_1 = K_2 = 2$.

Examples 2: take any map in 7.2) and join it with any map among

$$i_2, h_1, h_2, f_1, f_2.$$

For instance, this gives maps $\phi : S^n \longrightarrow S^n$ of Brouwer degree ± 4, $n=15, 21, 27, 33$ and surjective harmonic maps $\phi : S^n \longrightarrow S^n$ of Brouwer degree 0, $n=18, 30$.

The harmonic join $h_2 * h_3$ gives a harmonic representative for the generator of $\pi_{23}(S^{13}) = \mathbb{Z}_6$.

Examples 3:

One can join any two different maps among those in 7.2): for instance, we have harmonic maps $\phi : S^{39} \longrightarrow S^{39}$ of Brouwer degree ± 4.

Examples 1, 2, 3 do not exhaust all the new harmonic maps produced by theorem 1.1.1.: for instance, the map f_3 does not satisfy D.C., but, according to G.D.C., it can be joined with all maps of degree 3, such as i_3, g_1, g_2 ; in particular, $i_3 * f_3 : S^{15} \longrightarrow S^{15}$ gives a harmonic map of Brouwer degree 6.

Moreover, we notice that all the maps in examples 2, 3 are even; then they factorise to harmonic maps from real projective spaces to spheres.

The Hopf fibrations are also invariant under multiplication by $e^{i\vartheta}$:

then the harmonic maps $h_3 * h_2$, $h_3 * h_1$ above factorise to new harmonic maps from complex projective spaces to spheres.

Because of the composition properties of harmonic maps, combining our maps with totally geodesic spheres into manifolds gives further harmonic maps: examples in this direction can be produced easily by using the work of Fomenko (FO).

Finally we point out that the role of Φ_1, Φ_2 in our construction can also be played by the well-known polynomial minimal immersions described by Do-Carmo-Wallach in (DW):
in this case the equivariant harmonic join yields non-polynomial maps $\Phi_1 * \Phi_2 : S^p \to S^{p+q}$ which are harmonic but not minimal.

^^^^^^^^

Section 8

Now we occupy ourselves with the Hopf construction.

Let $F: S^p \times S^q \longrightarrow S^r$ be any continuous map.

Then the Hopf construction distinguishes a map $H: S^{p+q+1} \longrightarrow S^{r+1}$ as follows:

we write every point $z \in S^{p+q+1} \subseteq \mathbb{R}^{p+1} \times \mathbb{R}^{q+1}$ as

$z = \sin s\, x + \cos s\, y$, $x \in S^p$, $y \in S^q$, $s \in \left[0, \frac{\pi}{2}\right]$.

and every point $u \in S^{r+1} \subseteq \mathbb{R}^{r+1} \times \mathbb{R}$ as

$u = \sin s\, w + \cos s$, $w \in S^r$, $s \in [0, \pi]$.

Now, let $\alpha : \left[0, \frac{\pi}{2}\right] \longrightarrow [0, \pi]$ be any continuous function such that

i) $\alpha(0) = 0$, $\alpha(\frac{\pi}{2}) = \pi$

ii) $\alpha(s) \in (0, \pi)$, $s \in (0, \frac{\pi}{2})$.

Then $H: S^{p+q+1} \longrightarrow S^{r+1}$ is distinguished by

$(\sin s\, x + \cos s\, y) \rightsquigarrow \sin \alpha(s)\ F\ (x, y) + \cos \alpha(s)$

Properly speaking, H is a family of homotopic maps that depend on the choice of the function α :the problem will consist in determining a special function α in such a way that the corresponding H be a harmonic map.

In order that this problem makes sense, we have to impose conditions on the map F: precisely, we require that F be harmonic of constant energy density with respect to each variable separately (i.e. for each $y \in S^q$, the map $x \longrightarrow F(x, y)$ is harmonic of constant energy density $\frac{\lambda_1}{2}$; and, similarly, for each $x \in S^p$, the map $y \longrightarrow F(x, y)$ is harmonic of constant energy density $\frac{\lambda_2}{2}$.).

For example, orthogonal multiplications supply with a large number of such F, as illustrated in section 11.

Under this hypothesis on F, the equivariant theory can be applied and a similar procedure to that for the join leads to the following pendulum equation

$$8.1)\quad \alpha''(t)+\left[\frac{(p-1)e^{-t}-(q-1)e^{t}}{e^{t}+e^{-t}}\right]\alpha'(t)-\left[\frac{\lambda_2 e^{t}+\lambda_1 e^{-t}}{e^{t}+e^{-t}}\right]\sin\alpha(t)\cos\alpha(t)=0 \qquad t\in\mathbb{R}$$

Conditions i), ii) above are now expressed by

$$8.2)\quad \begin{array}{l} \lim_{t\to+\infty}\alpha(t)=\pi\,, \qquad \lim_{t\to-\infty}\alpha(t)=0 \\ \\ \alpha(t)\in(0,\pi)\,, \qquad t\in\mathbb{R} \end{array}$$

We are going to prove the following theorem:

Theorem 1.8.1.:

Let $F:S^p\times S^q\longrightarrow S^r$ be a map which is harmonic with constant energy density $\frac{\lambda_i}{2}$, $i = 1, 2$, in each variable separately.

Then its Hopf construction $H:S^{p+q+1}\longrightarrow S^{r+1}$ can be represented harmonically provided that the following Hopf damping conditions hold

$$\text{H.D.C.}\quad \begin{cases} (p-1)^2>4\lambda_1 \\ (q-1)^2>4\lambda_2 \end{cases} \qquad \text{or} \qquad \begin{array}{c} p=q \\ \lambda_1=\lambda_2 \end{array}$$

Theorem 1.8.1. enables us to produce harmonic maps between spheres of large dimension, as we will show in section 11.

In order to prove the theorem, we introduce in section 9 two functions $\dot{\beta}_0^+$, $\dot{\beta}_0^-$ that play a similar role to that of $\dot{\alpha}_0^+$, $\dot{\alpha}_0^-$ of section 2 for the join.

The crucial estimates for $\dot{\beta}_0^+$, $\dot{\beta}_0^-$ willbe proved in section 10.

Section 9:

The proof of theorem 1.8.1. consists in showing that, if H.D.C. hold, then equation 8.1) admits a solution as in 8.2).

We deal first with the non-symmetric case; the symmetric case is elementary and treated separately at the end of section 10.

First it is convenient to make the substitution $\beta(t) = \alpha(t) - \frac{\pi}{2}$; so 8.1) and 8.2) become respectively

9.1)
$$\beta''(t) + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right]\beta'(t) + \left[\frac{\lambda_2 e^{t} + \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right]\sin\beta(t)\cos\beta(t) = 0 \qquad t \in \mathbb{R}$$

and

9.2)
$$\lim_{t \to +\infty} \beta(t) = \frac{\pi}{2}, \quad \lim_{t \to -\infty} \beta(t) = -\frac{\pi}{2}$$

$$\beta(t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad t \in \mathbb{R}$$

We indicate with $G(t) = \left[\dfrac{\lambda_2 e^{t} + \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right]$ the gravity force

and with $D(t) = \left[\dfrac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right]$ the damping force

It may be helpful to visualize on figure 9 how the two forces influence the motion of the pendulum : in particular, we notice that, if the motion is clockwise, D(t) increases the speed of the pendulum when it is negative;
and it reduces the speed when positive.

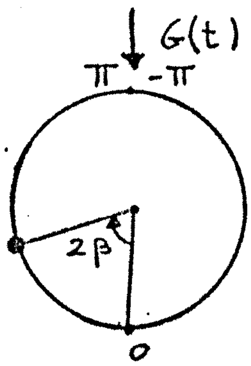


Figure 9.

Let us fix $s \in \mathbb{R}$. We investigate the following question:
Does it exist a positive number $\dot{\beta}_o^+(s)$ such that the solution $\beta(t)$ of 9.1) distinguished by

$$9.3)\quad \begin{cases} \beta(s) = 0 \\ \dot{\beta}(s) = \dot{\beta}_o^+(s) \end{cases}$$

increases asymptotically to $\frac{\pi}{2}$ as t increases from s?

This would correspond to throwing the pendulum, that initially is heading straight down, just hard enough as to make it reach the upright position at time equals to $+\infty$.

We remark that the gravity G(t) is limited, positive and bounded away from zero; so, if there was no damping in 9.1), the answer to the above question would always be affirmative.

But, no matter how softly you have pushed the pendulum, the damping force $D(t)$ may make it reach the upright position in finite time.

A first useful thing that we can say is the following: let $\{T_n\}_{n\in\mathbb{N}}$ be an increasing sequence of real numbers

such that $\lim\limits_{n\to+\infty} T_n = +\infty$, $T_0 > s$.

Then a standard Nagumo theorem (see 5.2) in (S1)) ensures that equation 9.1) has solutions $\beta_n(t)$ such that on $[s, T_n]$ we have:

$$
9.4) \quad \begin{cases} \beta_n(t) \in \left(0, \frac{\pi}{2}\right), & t \in (s, T_n) \\ \dot{\beta}_n(t) > 0, & t \in [s, T_n] \\ \beta_n(s) = 0, & \beta_n(T_n) = \frac{\pi}{2} \end{cases}
$$

The sequence of solutions $\beta_n(t)$ subconverges in C^2 on compact sets to a solution $\beta(t)$.

If $\beta(t)$ is not the trivial solution (i.e. $\beta(t) \not\equiv 0$), then it is easy to see that it has the required properties and in this case $\dot{\beta}_0^+(s)$ is defined (its uniqueness is straightforward).

Now we determine a condition on s that guarantees that $\dot{\beta}_0^+(s)$ is defined.

Let t_0 be the point in which the damping force $D(t)$ vanishes (i.e. $t_o = \frac{1}{2} \lg\left[\frac{p-1}{q-1}\right]$; this makes sense because H.D.C. hold and we are not in the symmetric case).

In order to prove that the limit solution $\beta(t)$ is not trivial, it is clearly enough to know that the sequence $\dot\beta_n(s)$ is bounded away from zero: and this is what happens if $s < t_o$.

Infact, if $s < t_o$, the effect of the damping force on $[s, t_o)$ is to reduce the speed of the pendulum: so, clearly, if the initial push is not hard enough, let us say if $\dot\beta_n(s) \not> \varepsilon$, for some $\varepsilon > 0$, then $\beta_n(t)$ would turn down before t_o. That is to say, $\dot\beta_n(s)$ is bounded away from zero if $s < t_o$: a detailed proof can be easyly performed by comparing, on $[s, t_o]$, the given equation 9.1) with the equation of a pendulum with no damping and constant gravity $\lambda = \min.\{\lambda_1, \lambda_2\}$.

By summarizing, $\dot\beta_o^+(s)$ is well-defined for every $s \in (-\infty, t_o)$; we call t_+, if there exists, the first point in which $\dot\beta_o^+(s)$ is not defined; otherwise, we set $t_+ = +\infty$.

In conclusion, $\dot\beta_o^+(s)$ is a well-defined, positive-valued function on $(-\infty, t_+)$, where $t_+ \in [t_o, +\infty) \cup \{+\infty\}$.

Standard arguments ensures the continuity of $\dot\beta_o^+(s)$ on $(-\infty, t_+)$ and moreover, if $t_+ < +\infty$, then $\lim_{s \to t_+^-} \dot\beta_o^+(s) = 0$.

This last fact is also obvious if one thinks of the physical situation described by equation 9.1: infact, $\dot\beta_o^+(s)$ can cease to be defined in t_+ only if the initial push that makes the pendulum stay vertical at $+\infty$ tends to zero when s approaches to t_+.

Analogously, we introduce the function $\dot\beta_o^-(s)$ in such a way that a solution $\beta(t)$ of 9.1) with initial data

$$\begin{cases} \beta(s) = 0 \\ \dot{\beta}(s) = \dot{\beta}_0^-(s) \end{cases}$$

decreases asymptotically to $-\frac{\pi}{2}$ as t decreases from S .

$\dot{\beta}_0^-(s)$ is a well-defined, continuous, positive-valued function on an open interval $(t_-, +\infty)$, where $t_- \in (-\infty, t_0] \cup \{-\infty\}$.

Moreover, if $t_- > -\infty$, then $\lim_{s \to t_-^+} \dot{\beta}_0^-(s) = 0$.

The theorem is proved if we show that there exists a point $\bar{s}$ in which $\dot{\beta}_0^-$ and $\dot{\beta}_0^+$ are both defined and $\dot{\beta}_0^+(\bar{s}) = \dot{\beta}_0^-(\bar{s})$.

Section 10

In this section we end the proof of theorem 1.8.1., that is to say we show that the conditions H.D.C. are sufficient to the existence of $\bar{S}$ as at the end of section 9.

More precisely, we will prove:

i) $\begin{matrix} \lambda_2 > (q-1) \\ \lambda_1 > (p-1) \end{matrix}$ implies $t_- < t_0 < t_+$

ii) $(q-1)^2 > 4\lambda_2$ implies $t_+ < +\infty$

iii) $(p-1)^2 > 4\lambda_1$ implies $t_- > -\infty$

Clearly the validity of i), ii) and iii) forces the existence of $\bar{s}$, as shown in figure 10.

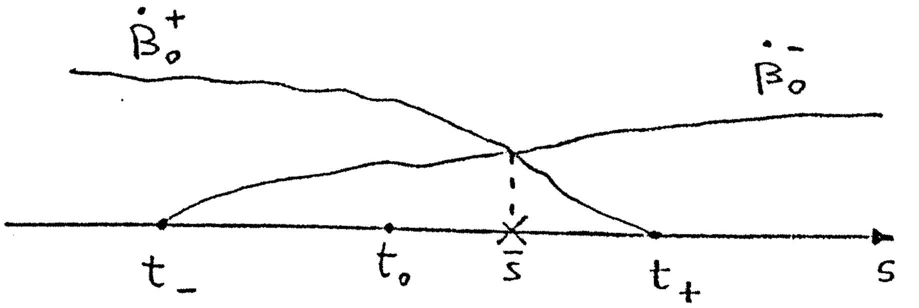


Figure 10.

<u>Remark 7:</u>

The hypothesis in i) has not been required explicitly in theorem 1.8.1. because it is well-known that λ_1, λ_2 are eigenvalues of the laplacian on spheres, and consequently $\lambda_1 \geqslant p$, $\lambda_2 \geqslant q$.

But our method of investigation of equation 9.1) requires the hypothesis in i) to be satisfied.

Roughly, the hypotheses in ii), iii) (i.e. H.D.C.) make sure that a special solution as in 9.2) be not to close respectively to the trivial solution $\beta(t) \equiv -\frac{\pi}{2}$ and $\beta(t) \equiv \frac{\pi}{2}$; on the other hand, the assumption in i) makes the exceptional solution be not to close to the trivial solution $\beta(t) \equiv 0$.

From now on, a little familiarity with the proof of theorem 1.1.1. can help.

We start with i): in order to see that $t_+ > t_0$, it is enough to show that $\dot{\beta}_0^+(t_0)$ is defined:

Lemma 1.10.1.

Let us suppose that there exists a differential equation of the form

$$\beta''(t) + D(t)\,\beta'(t) + A(t)\sin\beta(t)\cos\beta(t) = 0 \tag{10.1}$$

such that

$$A(t) < G(t) \qquad t \in (t_0, +\infty) \tag{10.2}$$

(we recall that $G(t) = \left[\dfrac{\lambda_2 e^{t} + \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right]$

$D(t) = \left[\dfrac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right]$)

and equation 10.1) has a solution F(t) such that

$$\left\{\begin{array}{l} F(t_0) = 0 \\ F(t) \in \left(0, \frac{\pi}{2}\right), \quad t \in (t_0, +\infty) \\ \lim\limits_{t \to +\infty} F(t) = \frac{\pi}{2} \end{array}\right. \tag{10.3}$$

Then $\dot{\beta}_0^{+}(t_0)$ is defined.

Proof:

We take a sequence of solutions $\beta_n(t)$ as in section 9 (with $s=t_o$): then $\dot{\beta}_o^+(t_o)$ is defined provided that $\dot{\beta}_n(t_o) \geqslant \varepsilon \;\; \forall n$, for some $\varepsilon > 0$.

We claim that such ε can be taken to be $\varepsilon = \dot{F}(t_o)$:

Infact, suppose $\dot{\beta}_n(t_o) < \varepsilon = \dot{F}(t_o)$: then there exists T as in figure 11.

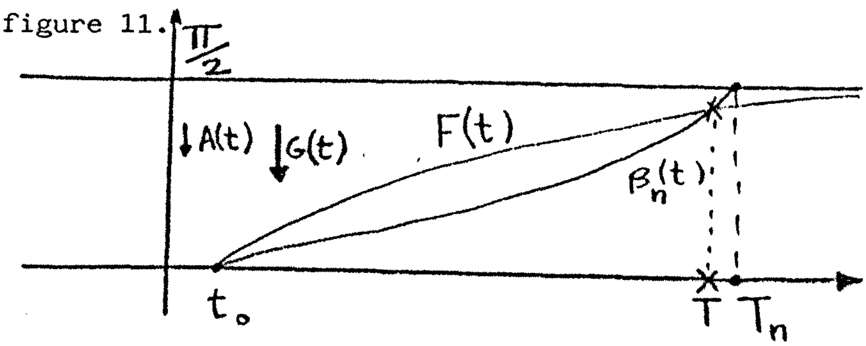


Figure 11.

Because of 10.2), the gravity G(t) pushes $\beta_n(t)$ toward zero stronger than A(t) does for F(t); moreover, the initial push of F(t) is stronger than the one of $\beta_n(t)$. The two equations 9.1), 10.1) have the same damping, thus the existence of T as in figure 11 is physically not acceptable. The detailed mathematical proof follows exactly the arguments of Lemma 1.3.1.

Then, in order to see that $t_+ > t_o$, we only need to build up a differential equation as in the hypothesis of Lemma 1.10.1.

We follow a similar procedure to that one used for the join (see section 4): we impose F(t) to be of the form

$$10.4) \qquad F_a(t) = \tan^{-1}\left[a(t - t_o)\right] \quad , \qquad a > 0$$

We show that, if a is chosen sufficiently small, then

10.5) $F''_a(t) + D(t) \cdot F'_a(t) + G(t) \sin(F_a(t)) \cos(F_a(t)) > 0$

$t \in (t_0, +\infty)$.

Because $F_a(t)$ satisfies 10.3), the fulfilment of 10.5) guarantees that Lemma 1.10.1. can be applied: infact, just take

$$A(t) = \frac{-\left[F''_a(t) + D(t)\ F'_a(t)\right]}{\sin(F_a(t)) \cos(F_a(t))}$$

and 10.5) tells $G(t) > A(t)$, $t \in (t_0, +\infty)$.

The crucial point is to check 10.5), for which we proceed by direct substitution: remembering the identity

$$\sin x \cos x = \frac{\tan x}{1 + \tan^2 x}$$

it is easy to see that 10.5) is equivalent to

$$\text{10.6)}\quad \frac{-2a^2(t-t_0)}{\left[1+a^2(t-t_0)^2\right]} + \left[\frac{(p-1)e^{-t}-(q-1)e^{t}}{e^{t}+e^{-t}}\right] + \left[\frac{\lambda_2 e^{t}+\lambda_1 e^{-t}}{e^{t}+e^{-t}}\right]\cdot(t-t_0) > 0$$

$$t \in (t_0, +\infty)$$

We make the substitution

$y = (t - t_0)$, so that

$$e^{t} = e^{(t-t_0)} \cdot e^{t_0} = e^{y}\cdot\sqrt{\frac{(p-1)}{(q-1)}}$$

$$e^{-t} = e^{-t+t_0}\cdot e^{-t_0} = e^{-y}\cdot\sqrt{\frac{(q-1)}{(p-1)}}$$

Now 10.6) is equivalent to

$$\text{10.7)}\quad \frac{-2a^2}{\left[1+a^2y^2\right]} + \left[\sqrt{\frac{(p-1)}{(q-1)}}\,e^{y} + \sqrt{\frac{(q-1)}{(p-1)}}\,e^{-y}\right]^{-1}\cdot$$

$$\cdot\left\{\sqrt{(p-1)(q-1)}\cdot\frac{\left[e^{-y}-e^{y}\right]}{y} + \left[\lambda_2\sqrt{\frac{(p-1)}{(q-1)}}\,e^{y} + \lambda_1\sqrt{\frac{(q-1)}{(p-1)}}\cdot e^{-y}\right]\right\} > 0$$

$$y > 0$$

The limit for $y \to 0$ of the left hand-side of 10.7) is

10.8) $$-2a^2 + \left[\sqrt{\frac{(p-1)}{(q-1)}} + \sqrt{\frac{(q-1)}{(p-1)}}\right]^{-1} \cdot$$

$$\cdot \left\{-2\cdot\sqrt{(p-1)(q-1)} + \left[\lambda_2\sqrt{\frac{(p-1)}{(q-1)}} + \lambda_1\sqrt{\frac{(q-1)}{(p-1)}}\right]\right\}$$

Under the hypothesis of i), we have

10.9) $$\left[\lambda_2\sqrt{\frac{(p-1)}{(q-1)}} + \lambda_1\sqrt{\frac{(q-1)}{(p-1)}}\right] > 2\sqrt{(p-1)\cdot(q-1)}$$

Moreover, we notice that, if $a \to 0^+$, the left hand-side of 10.7) increases for every $y > 0$.

By using this fact and substituting 10.9) in 10.8), we can conclude that there exist two small numbers $\gamma, \delta > 0$ such that

10.10) $$\begin{cases} \text{Condition 10.7) is satisfied on } (0, \delta] \\ \text{whenever } a \in (0, \gamma) \end{cases}$$

Now we show that

10.11) $$\left\{\sqrt{(p-1)(q-1)}\cdot\left[\frac{e^{-y}-e^{y}}{y}\right]+\left[\lambda_2\sqrt{\frac{(p-1)}{(q-1)}}\,e^{y}+\lambda_1\sqrt{\frac{(q-1)}{(p-1)}}\,e^{-y}\right]\right\}\cdot$$

$$\cdot\left[\sqrt{\frac{(p-1)}{(q-1)}}\,e^{y}+\sqrt{\frac{(q-1)}{(p-1)}}\,e^{-y}\right]^{-1}>0 \qquad y>0$$

From 10.11) the thesis follows: infact, let us denote with $R(y)$ the left hand-side of 10.11): it is elementary that

10.12) $$\lim_{y\to+\infty} R(y)=\lambda_2$$

Then, from 10.12) and 10.11) it follows that $R(y)$ has positive minimum, that we denote with m, on $[\delta,+\infty)$

Now take $a\in(0,\gamma)$ so small as to have

$$\frac{2a^2}{1+a^2y^2}<m \quad , \quad y>0$$

Then $R(y)-\dfrac{2a^2}{1+a^2y^2}>0$, $y>0$, i.e. 10.7) holds globally.

In order to prove 10.11), we notice that the numerator of $R(y)$ is greater than

$$10.13)\quad \sqrt{(p-1)(q-1)}\left[\frac{e^{-y}-e^{y}}{y}+e^{y}+e^{-y}\right]$$

Now we recall that

$$10.14)\qquad \tanh(y) < y \quad , y > 0 .$$

By using 10.14) in 10.13) we have immediately 10.11).

An analogous argument in back-ward time shows that $t_{-} < t_{0}$, and this ends the proof of i).

Now we prove ii).

We have to show that there exists $\bar{t} \in \mathbb{R}$ such that $\dot{\beta}_0^+$ is not defined in $\bar{t}$: this follows essentially the same non-existence argument as the case of the join.

In fact, let $\beta(t)$ be a solution of 9.1) such that

$$10.14)\quad \begin{cases} \beta(\bar{t}) = 0 \\ \beta(t) \in \left(0, \frac{\pi}{2}\right) & t \in (\bar{t}, +\infty) \\ \lim\limits_{t \to +\infty} \beta(t) = \frac{\pi}{2} \end{cases}$$

We show that, if $\bar{t}$ is large, such a solution can not exist.

A solution as in 10.14) can be written as

10.15) $$\beta(t) = \tan^{-1}\left[\exp\left(\int_{t_1}^{t} H(s)\,ds + c\right)\right], \quad t \in (\bar{t}, +\infty)$$

where $t_1 \in (\bar{t}, +\infty)$, $c \in \mathbb{R}$, ,and the function $H(s)$ is uniquely determined by $\beta(t)$.

Exactly as in 6.8) and 4.15) b) we have

10.16) $$\begin{cases} \lim\limits_{t \to \bar{t}} H(t) = +\infty \\ \\ \lim\limits_{t \to +\infty} H(t) = k_2 \qquad k_2 > 0 \end{cases}$$

where, as usual, k_2 is related to λ_2 by

10.17) $$\lambda_2 = k_2 \cdot (k_2 + q - 1)$$

As in 4.11), we have that the following inequality must hold:

10.18) $$H'(t) + H^2(t) + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right] \cdot H(t) + \\ + \left[\frac{\lambda_2 e^{t} + \lambda_1 e^{-t}}{e^{t} + e^{-t}}\right] > 0 \qquad t \in (\bar{t}, +\infty)$$

From 10.18) we have an area H-N-D in which H(t) can not decrease, exactly as in section 4.

Because of the hypothesis ii), for t large the area H-N-D is thick, because the function $\Delta(t) \simeq (q-1)^2 - 4\lambda_2 > 0$.

Then, if $\bar{t}$ is large, the function H(t) would be forced to cross H-N-D decreasing somewhere: this gives the contradiction that we were after (see figure 12).

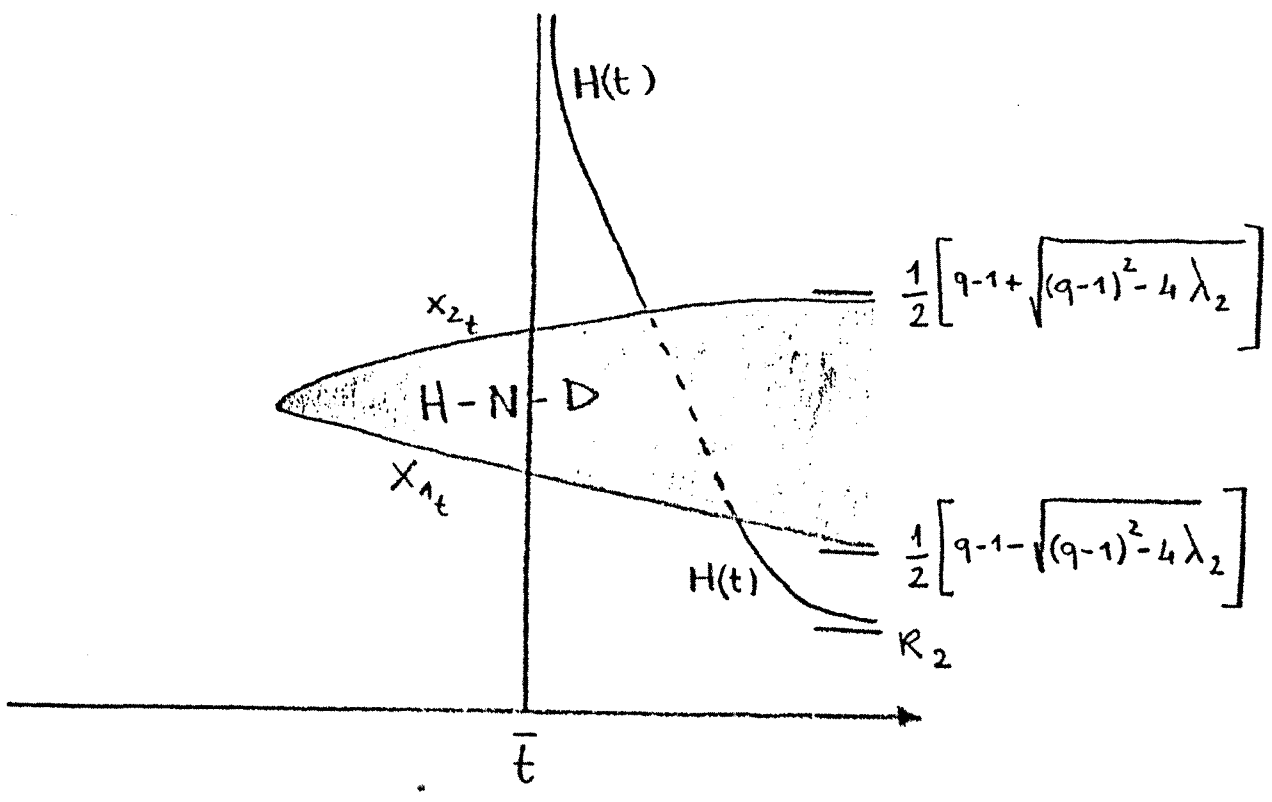


figure 12.

Analogously, one proves iii) and this ends the non-symmetric case.

Symmetric case:

In the symmetric case $p=q$, $\lambda_1 = \lambda_2$, ii) and iii) are not necessary: we just need to use $\lambda_1 > p-1$, $\lambda_2 > q-1$ in order to make sure that $\dot{\beta}_0^{\pm}(t_0)$ are defined.

Then the symmetry forces $\dot{B}_o^+(t_o) = \dot{B}_o^-(t_o)$.

Moreover, in the case p=q= $\lambda_1 = \lambda_2$, the exceptional solution B (t) can be determined explicitely.

Section 11

In this section we show that theorem 1.8.1. enables us to produce interesting harmonic maps between spheres of large dimensions.

Let $\bar{F} : \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} \longrightarrow \mathbb{R}^{r+1}$ be a bilinear map such that $|\bar{F}(x, y)| = |x| \cdot |y|$;

Then $\bar{F}$ is called an orthogonal multiplication.

The induced map

11.1) $$F : S^p \times S^q \longrightarrow S^r$$

is a totally geodesic embedding in each variable separately, and belongs to the class of maps of theorem 1.8.1. with $\lambda_1 = p$, $\lambda_2 = q$.

In particular, if p = q, then we are in the symmetric case and the exceptional solution of equation 8.1) is determined explicitly : actually, in this case it is well-known that the Hopf construction produces harmonic homogeneous polynomial maps of degree 2. For example, the three Hopf fibrations arise in this way, by taking $\bar{F}$ to be respectively the multiplication of complex, quaternionic and Cayley numbers. Other examples of this form have been indicated in 7.1).

Rather more interesting are the applications of theorem 1.8.1.

in the non-symmetric case.

As an immediate consequence of theorem 1.8.1., we have that the Hopf construction $H(F) : S^{p+q+1} \longrightarrow S^{r+1}$ of a map F as in 11.1) can be represented by a harmonic map provided that $p,q \geqslant 6$.

It is not known for which values of p, q, r there exists an orthogonal multiplication as in 11.1); but important results have been obtained in the case $q = r$:

<u>Theorem</u> (Hu):

There exists an orthogonal multiplication $F : S^p \times S^q \longrightarrow S^q$ if and only if p and q are of the following form

11.2)
$$p \leqslant 2^{\alpha} + 8\beta - 1 \qquad 0 \leqslant \alpha \leqslant 3$$
$$q = 2^{\alpha+4\beta} \cdot (2\gamma+1) - 1 \qquad \beta, \gamma \geqslant 0$$

The maps in the previous theorem have relevant importance in the theory of fibre bundles because it can be shown (A) that $p = 2^{\alpha} + 8\beta - 1$ is the greatest possible number of linearly independent vector fields on S^q ; moreover, such p vector fields can be explicitly constructed by using orthogonal multiplications. (Hus).

Now we need to recall something about the (stable) J-homomorphism.

Let $f : S^p \times S^q \longrightarrow S^q$ be a continuous map.

We have already seen (sec. 8) that the Hopf construction of f gives a map $H(f) : S^{p+q+1} \longrightarrow S^{q+1}$, i.e. $H(f) \in \pi_{p+q+1}(S^{q+1})$. Now we identify the continuous functions $S^p \longrightarrow \Theta(q+1)$ with a subset of the continuous functions $f : S^p \times S^q \longrightarrow S^q$: it is clearly enough to require that the map $y \longrightarrow f(x,y)$ be an orthogonal trasformation of S^q, for every $x \in S^p$.

The standard J-homomorphism is the map

$$11.3) \qquad J \; : \pi_p(\Theta(q+1)) \longrightarrow \pi_{p+q+1}(S^{q+1})$$

obtained by restricting the Hopf construction to $\pi_p(\Theta(q+1))$.

The map J is a homomorphism (WH); if $q > p$, then J determines a map between the stable groups, that we denote by $J_p : \pi_p(\Theta) \longrightarrow \pi_p^s$. The map J_p is called the stable J-homomorphism.

In the stable case $q > p$, we have the following facts:

$$11.4) \qquad \pi_p(\Theta(q+1)) = \pi_{p+8}(\Theta(q+1)) \qquad \text{(see ABS)}$$

11.5) J_p is injective for $p = o, 1 \bmod. 8$ (see KM)

11.6) Every class of π_p^s in the image of J_p can be represented as the image (via J_p) of an orthogonal multiplication $F : S^p \times S^q \longrightarrow S^q$ (see Corollary 1.6 (Ba)); moreover, every such a map is homotopic to a quadratic homogeneous polynomial map (Ba).

By using 11.6) and theorem 1.8.1., we obtain harmonic maps between spheres of arbitrarily large dimensions (not prescribed):

Corollary 1.11.1.

Let $[f] \in \pi_p^S$ be a stable class in the image of the stable J-homomorphism $J_p : \pi_p(\mathcal{O}) \longrightarrow \pi_p^S$, $p \geqslant 6$.

Then there exists $q > p$ such that $[f]$ can be represented by a harmonic map $\phi : S^{p+q+1} \longrightarrow S^{q+1}$.

Moreover, ϕ is "even" and therefore gives rise to a harmonic map $\bar{\phi} : \mathbb{P}_{\mathbb{R}}^{p+q+1} \longrightarrow S^{q+1}$.

The values of p in 11.5) give examples in which the maps of Corollary 1.11.1. are homotopically non-trivial; other non-trivial examples can be obtained for p=3, 7 mod. 8 (KM).

We also point out that all our harmonic maps arising from orthogonal multiplications as in 11.2) are <u>surjective</u> and homotopic to quadratic homogeneous polynomials.

Finally, we remark that a map $F : S^p \times S^q \longrightarrow S^2$ which is harmonic in each variable separately is <u>not</u> necessarily an orthogonal multiplication; also such maps have not been classified. An example in this direction, which does not satisfy H.D.C. of theorem 1.8.1., is given in Chapter II, sec. 6.

CHAPTER II

In section 1 we establish a general setting for equivariant theory; this is of substantial importance for our purposes, because all the ordinary differential equations that we study in this thesis arise from equivariant methods.

In section 2 we take up a program of C^{∞}-deformations of metrics and from this point of view study again the join and the Hopf construction of Chapter I: as a result, we obtain that the restrictions G.D.C. and H.D.C. in theorems 1.1.1. and 1.8.1. can be completely removed, provided that suitable riemannian metrics are introduced (see section 3 for statements of the results); moreover, the same holds for maps into ellipsoids (sec. 7).

Section 1

We start with illustrating a simple but instructive example:

Let (M, g) and (N, h) be two riemannian manifolds, respectively of dimension m, n.

Consider the warped products

$$1.1)\qquad \begin{matrix} M \times (0,\pi) \\ N \times (0,\pi) \end{matrix} \qquad \text{with metrics} \qquad \begin{matrix} A^2(s)\, g + ds^2 \\ B^2(s)\, h + ds^2 , \end{matrix}$$

where $s \in (0,\pi)$ and $A(s)$, $B(s)$ are smooth positive functions on $(0,\pi)$.

We will be interested in maps of the following type:

$$1.2)\qquad \begin{array}{ccc} M \times (o,\pi) & \xrightarrow{\Phi} & N \times (o,\pi) \\ (x,\ s) & & (\phi(x),\ \alpha(s)) \end{array}$$

where $\phi : M \to N$

$\alpha : (o,\pi) \to (o,\pi)$ are smooth maps.

Now it comes the crucial point:

If <u>ϕ is harmonic with constant energy density $e(\phi)$</u>, then Φ is harmonic if and only if the function α satisfies the following ordinary differential equation:

$$1.3)\qquad \alpha''(s) + \frac{mA'(s)\ \alpha'(s)}{A(s)} - \frac{2\,e(\phi)\ B(\alpha(s))\ B'(\alpha(s))}{A^2(s)} = o$$

$$s \in (o,\pi)$$

Let us analyse the following particular case:

$M = S^m$, $N = S^n$,

$A(s) = B(s) = \sin s.$

In this case, the two warped products in 1.1) are respectively S^{m+1} and S^{n+1} minus the two poles.

If the function $\alpha(s)$ satisfies

$$1.4)\qquad \lim_{s \to o^+} \alpha(s) = o \quad , \quad \lim_{s \to \pi^-} \alpha(s) = \pi$$

then the map in 1.2) can be extended to a <u>continuous</u> map

$\Phi^{*}: S^{m+1} \longrightarrow S^{n+1}$: just send the south pole to the south pole and the north pole to the north pole.

The map Φ^{*} is called the first suspension of ϕ : if equation 1.3) is satisfied, then Φ^{*} is harmonic on S^{m+1} minus the two poles: the question whether Φ^{*} is globally harmonic is usually investigated on the basis of regularity arguments.

If M and N are not spheres, the suspension can be performed as well: but, in this case, the standard completions of the two warped products in 1.1) obtained by adding two points, are not homeomorphic to manifolds any more: actually, they are spaces with conical singularities; this aspect of the theory will be treated separately in section 3 of Chapter III.

In conclusion, the emphasis with these examples lays on the following points:

a) Reduction to an ordinary differential equation occurs when the symmetries of Φ enable to make analysis by separation of variables: the harmonicity of ϕ controls the variables on the cross-section M and the equation 1.3) controls the suspension parameter.
   In the language of geometric optics, Φ is said to be wave-front preserving.

b) The study of the relevant equation corresponds to doing analysis on an open manifold: the most interesting cases arise when such an open manifold is a dense open set in a compact manifold; in this case, in order to give a global meaning to the analysis, one must impose boundary conditions as in 1.4).

- Other cases occur when the completions of our open manifold is a manifold with boundary or a singular space.

c) The case $B'(s) \equiv o$ is elementary and of little interest; in the relevant cases we have $B'(s) \not\equiv o$, then it is clear that equation 1.3) doesn't make sense if the requirement $\ell(\phi) =$ constant is not fulfilled; geometrically, this means that the harmonic suspension requires that all points of the cross-section M look alike: this homogeneity is expressed precisely by $\ell(\phi) =$ constant.

Now we are in the right position to give the general definitions.

Let (M, g) be a m-dimensional riemannian manifold on which there exist distributions S_J, $j=1 \ldots\ldots p$, such that:

i) $(\bigoplus_{J=1}^{p} S_J)_x = T_x M \qquad x \in M$

ii) $S_i \perp S_J \qquad i \neq J$

iii) S_J is locally integrable, $j=1 \ldots\ldots p$.

For example, M can be a product of riemannian manifolds; but a twisted product as well.

Let M_J be the integral submanifold corresponding to the distribution S_J and g_J be the induced metric on M_J.
$(M_J, g_J$ in general makes sense only locally).

We will develop the equivariant theory on the following class of riemannian manifolds:

$$1.5)\quad \begin{cases} M \times (a,\ b) & \text{with riemannian metric} \\ \sum_{J=1}^{p} A_J^2(s)\, g_J + h^2(s)\, ds^2 \end{cases}$$

where $s \in (a,\ b)$ and $A_J(s)$, $h(s)$ are smooth positive functions on $(a,\ b)$; M and g_J as above.

We call a manifold as in 1.5) an <u>equivariant manifold.</u>

In order to study maps between two equivariant manifolds, we use the above notations for the domain; for the range, we denote our equivariant manifold as follows:

$$N \times (c,\ d) \qquad \text{with riemannian metric}$$

$$1.6)\qquad \sum_{i=1}^{q} B_i^2(s)\, h_i + K^2(s)\, ds^2$$

where $s \in (c,\ d)$ and $B_i(s)$, $K(s)$ are smooth positive functions on $(c,\ d)$; N plays the role of M, and h_i are the metrics corresponding to distributions T_i, $i=1 \ldots\ldots q$.

Let $\phi : M \longrightarrow N$ and $\alpha : (a,\ b) \longrightarrow (c,\ d)$ be smooth maps. A map Φ of the form

$$1.7)\qquad \begin{aligned} \Phi : M \times (a,\ b) &\longrightarrow N \times (c,\ d) \\ (x,\ s) &\longmapsto (\phi(x),\ \alpha(s)) \end{aligned}$$

is said to be an <u>equivariant map</u> if $\phi : M \longrightarrow N$ satisfies

$$1.8)\quad \begin{cases} \text{i) } d\phi(S_j) \subseteq T_{i_j} \text{ for some } i_j\ ,\ j=1\ \ldots\ldots p. \\ \\ \text{ii) } \phi/_{S_j} \text{ is harmonic with constant energy density } e(\phi)_j, \\ \qquad j=1\ \ldots\ldots p. \end{cases}$$

where by condition ii) we mean:

write $x \in M$ as $(x_1, \ldots, x_p)$, with $x_k \in M_k$, $k = 1\ \ldots\ldots p$.

(This is possible because locally M looks like a product : and both i) and ii) are local conditions).

Let us fix $\bar{x}_k \in M_k$, $k \neq j$, and consider

$$x_j \longrightarrow \phi(\bar{x}_1\ \ldots,\ x_j\ \ldots,\ \bar{x}_p\ ).$$

Condition ii) requires that such a map be harmonic, with constant energy density $e(\phi)_j$ that does <u>not</u> depend upon the choice of $\bar{x}_k \in M_k$, $k \neq j$.

<u>Remark 1:</u>

Roughly speaking, the class of equivariant manifolds is large: but the difficulty is to construct equivariant maps when M is not a product of manifolds. A nice example in this direction is given in (B1: example 5.3.5.)

A straightforward computation leads to

<u>**Reduction theorem:**</u> If Φ as in 1.7) is equivariant, then Φ is harmonic <u>if</u> and <u>only if</u> $\alpha(s)$ satisfies the following second order differential equation:

$$1.9)\qquad \alpha''(s) + \left[\sum_{J=1}^{p} \frac{A'_J(s)}{A_J(s)} k_J - \frac{h'(s)}{h(s)}\right] \alpha'(s) +$$

$$-\left[\frac{h^2(s)}{R^2(\alpha(s))} \sum_{J=1}^{p} \frac{2\, e(\phi)_J\, B_{i_J}(\alpha(s))\, B'_{i_J}(\alpha(s))}{A_J^2(s)}\right] +$$

$$+ \quad \alpha'^2(s) \cdot \left[\frac{R'(\alpha(s))}{R(\alpha(s))}\right] = 0$$

$s \in (a, b)$

where $k_J = \dim (S_J)$ and i_J is as in 1.8).

Moreover, the energy density of Φ is expressed by

$$1.10)\quad e(\Phi)(s) = \sum_{J=1}^{p} \frac{B_{i_J}^2(\alpha(s))}{A_J^2(s)} e(\phi)_J + \frac{1}{2}\left[\frac{R^2(\alpha(s))}{h^2(s)} \cdot \alpha'^2(s)\right]$$

We point out that all the differential equations that occur in this thesis are particular cases of equation 1.9).

In order to make the reader more familiar with the concepts that we have introduced so far, we describe how the above equivariant theory applies to the join of two harmonic homogeneous polynomials of spheres.

Take $M = S^p \times S^q$

$N = S^r \times S^s$ with the standard product metrics $g_1 + g_2$

$(a, b) = (c, d) = (o, \frac{\pi}{2})$

$A_1(s) = B_1(s) = \sin s$

$A_2(s) = B_2(s) = \cos s$

$h(s) \equiv K(s) \equiv 1$

Then

$S^p \times S^q \times (o, \frac{\pi}{2})$ with metric

$\sin^2 s \; g_1 + \cos^2 s \; g_2 + ds^2$ is isometric to S^{p+q+1} minus the two focal varieties; and, analogously, the range is dense in S^{r+s+1}.

Let $\phi : M \longrightarrow N$ be of the form

$$\phi : S^p \times S^q \longrightarrow S^r \times S^s$$
$$(x_1, x_2) \qquad (\phi_1(x_1), \phi_2(x_2))$$

where ϕ_i, $i = 1, 2$, is harmonic with constant energy density $e(\phi)_i = \frac{\lambda_i}{2}$

<u>Remark 2:</u>

In the case of maps between spheres, the requirements ϕ harmonic with constant energy density and ϕ harmonic homogeneous polynomial in the sense of Chapter 1, sec. 1, are equivalent.

Let us consider

$$\Phi : S^p \times S^q \times (0,\tfrac{\pi}{2}) \longrightarrow S^r \times S^s \times (0,\tfrac{\pi}{2})$$

1.10')

$$(x,s) \rightsquigarrow (\phi(x), \alpha(s))$$

where ϕ is as above.

It is elementary to see that Φ is equivariant, with

$$e(\phi)_i = \frac{\lambda_i}{2} \qquad i=1, 2.$$

If $\alpha(s)$ satisfies the boundary conditions

1.11) $$\lim_{s \to 0^+} \alpha(s) = 0 , \qquad \lim_{s \to \frac{\pi}{2}^-} \alpha(s) = \frac{\pi}{2}$$

then Φ has a continuous extension to a map

$$\Phi^* : S^{p+q+1} \longrightarrow S^{r+s+1}$$

The map Φ^* is exactly the join map $\phi_1 * \phi_2$ introduced in Chapter I, section 1.

By following equation 1.9), Φ^* is harmonic (on S^{p+q+1} minus the focal varieties) if and only if

1.12) $$\alpha''(s) + \left[\frac{\cos s}{\sin s} p - \frac{\sin s}{\cos s} q\right] \alpha'(s) + - \left[\frac{\lambda_1}{\sin^2 s} - \frac{\lambda_2}{\cos^2 s}\right] \sin(\alpha(s)) \cos(\alpha(s)) = 0$$

$$s \in (0,\tfrac{\pi}{2})$$

After the substitution $e^t = \tan s$, $t \in \mathbb{R}$, the equation 1.12) becomes the equation 1.3) of Chapter I.

Remark 3:

Equivariant theory is widely illustrated in P. Baird's book (B1), where calculations are performed extensively and also connections between this theory and isoparametric functions are pointed out.

In order to follow the notations of (B1), let us call

$$\Delta_s = \sum_{J=1}^{p} \frac{A'_J(s)}{A_J(s)} K_J$$

1.13) $$\gamma_J(s,\alpha(s)) = \frac{2\, e(\phi)_J\, B^2_{i_J}(\alpha(s))}{A^2_J(s)}$$

$$\mu_{i_J}(\alpha(s)) = - \frac{B'_{i_J}(\alpha(s))}{B_{i_J}(\alpha(s))}$$

In these notations, the equation 1.9) becomes

1.14) $$\alpha''(s) + \left[\Delta_s - \frac{h'(s)}{h(s)}\right] \alpha'(s) +$$

$$+ \frac{h^2(s)}{K^2(\alpha(s))} \left[\sum_{J=1}^{p} \gamma_J \mu_{i_J}\right] + \alpha'^2(s) \cdot \left[\frac{K'(\alpha(s))}{K(\alpha(s))}\right] = 0$$

However, we remark that our approach to equivariant theory has essentially two new features:

a) The open manifolds on which we perform our equivariant analysis may be dense on spaces with cone-like singularities (see Chapter III, sec. 3).

b) The integral manifolds M_J above are not required to have constant curvature: just considering elementary warped products shows that b) actually increases the generality of the theory (see Chapter III, sec. 2).

Section 2

First we introduce the concept of equivariant deformation of a metric.

In section 1 we wrote (an open dense subset of) S^{p+q+1} as

$$2.1)\qquad (S^p \times S^q \times (0,\tfrac{\pi}{2}),\ \sin^2 s\, g_1 + \cos^2 s\, g_2 + ds^2).$$

Suppose that we replace the metric in 2.1) with a metric of the form

$$2.2)\qquad A_1^2(s)\, g_1 + A_2^2(s)\, g_2 + h^2(s)\, ds^2$$

where $A_1(s)$, $A_2(s)$, $h(s)$ are smooth positive functions on $(0,\frac{\pi}{2})$.

Then equivariance is clearly preserved.

Of course, if we want to continue to consider the manifold $S^p \times S^q \times (0,\frac{\pi}{2})$ as a dense open set into S^{p+q+1}, we must make sure that

a) The functions $A_1(s)$, $A_2(s)$, $h(s)$ behave qualitatively respectively as sin s, cos s and 1: this ensures that the completion of $S^p \times S^q (o, \frac{\pi}{2})$ can be realised topologically as a $(p+q+1)$-dimensional sphere .

b) The metric in 2.2) extends to a smooth riemannian metric on the whole S^{p+q+1} .

Example 1

Take the metric in 2.2) to be of the form

2.3) $\sin^2 s \; g_1 + \cos^2 s \; g_2 + h^2(s)\, ds^2$,

where

2.4) $h(s) = 1 \quad , \quad s \in (o, \varepsilon) \cup (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2})$,

for some small $\varepsilon > o$.

The metric 2.3) coincides with the euclidean around the focal varieties; then it is globally smooth and gives a first example of a non-standard riemannian metric on S^{p+q+1} .

Example 2

Equivariant deformations of the euclidean metric can give rise also to very well-known objects: for instance, if 2.3) is replaced by

2.5)

$$a^2 \sin^2 s \; g_1 + b^2 \cos^2 s \; g_2 + \left[a^2 \cos^2 s + b^2 \sin^2 s \right] ds^2 \qquad , a, b > o$$

then we have a standard ellipsoid.

Example 3

Of course, equivariant deformations of metrics can be performed on manifolds other than spheres.

Let (S^m, g_1) be the standard m-sphere . Then

2.6) $(S^m \times (0,1]\ ,\ s^2 g_1 + ds^2)$ $s \in (0,1]$ and

2.7) $(S^m \times (0,+\infty),\ \sinh^2 s\ g_1 + ds^2)$ $s \in (0,+\infty)$.

are respectively $B^{m+1}(1) - \{0\}$, i.e. the (m+1)-dimensional closed flat ball of radius 1 minus the origin and

$H^{m+1} - \{0\}$, i.e. the (m+1)-dimensional hyperbolic space minus a point.

Equivariant deformations can be obtained, for instance, by replacing the metrics in 2.6), 2.7) by respectively

2.8) $s^2 g_1 + h^2(s)\ ds^2$

2.9) $\sinh^2 s\ g_1 + h^2(s)\ ds^2$

$h(s) = 1,\ s \in (0,\varepsilon)$.

But of course other deformations can be performed as well.

The equation 1.9) of the reduction theorem changes according to the equivariant deformations of metrics that we performe: this fact makes it very natural to investigate the following problem: is it possible to produce equivariant deformations that guarantee that the corresponding reduction equation admits a solution with prescribed asymptotic behaviour?

Because of the restrictions G.D.C. and H.D.C. in theorems 1.1.1. and 1.8.1. this problem is particulary interesting in the case of equivariant maps between spheres. We will show that G.D.C. and H.D.C. can be totally removed by performing suitable equivariant deformations: this will be the main object of the remaining part of this Chapter II; section 7 is devoted to study maps into ellipsoids.

The investigation of the previous question and related problems on manifolds other than spheres will be done in Chapter III.

But, before beginning the study of equivariant maps of spheres, we make a few more comments on the reduction equation 1.9).

Remark 4:

Suppose that the function $\alpha(s)$ that must play the role of the solution in 1.9) be preassigned. Then, one can look for metrics that make such an $\alpha(s)$ actually be a solution of 1.9): this search is always a matter of first order differential equations.

In particular, suppose that everything is preassigned in 1.9) except for the function h(s): then, in terms of h(s), equation 1.9) takes the form

$$2.10)\qquad h'(s) - \frac{P_\alpha(s)}{2}\, h(s) = - \frac{Q_\alpha(s)}{2}\, h^3(s)$$

where $P_\alpha(s)$, $Q_\alpha(s)$ are smooth functions that, in particular, depend upon the choice of the function $\alpha(s)$.

Equation 2.10) is very special: in fact, after the substitution

2.11) $$y(s) = \frac{1}{h^2(s)}$$

equation 2.10) becomes

2.12) $$y'(s) + P_\alpha(s)\, y(s) = Q_\alpha(s)$$

The solution of the problem is now reduced to see whether $\alpha(s)$ can be chosen in such a way that

i) $\alpha(s)$ has the required asymptotic behaviour.

ii) the first order linear equation 2.12) has a solution y(s) that gives rise to a riemannian metric on the domain.

The fact that 2.12) is linear, and then explicitely solvable, will be vital to investigate condition ii).

For instance, if one performs the same type of deformation, but in the range, he encounters a complicated first order equation that, contrary to 2.12), is not explicitly solvable in general.

Remark 5:

Suppose that the reduction equation 1.9) be satisfied with h(s)=constant, $K(s)$=constant, and that $\alpha(s)$ be bijective.

Then it is easy to check that, if we deform the domain and the range by using

$$2.13)\quad \begin{cases} h(s) & \text{any function} \\ \mathcal{K}(\alpha(s)) = h(s) & \end{cases}$$

then 1.9) is still satisfied.

This means that, in this particular case, certain deformations on the domain can be controled explicitly by introducing suitable deformations on the range.

This applies, for instance, to the join and Hopf construction of Chapter I.

In fact, in those cases, we produced many harmonic maps with $h(s) \equiv 1$, $\mathcal{K}(s) \equiv 1$: then, in each of these cases, we can introduce h(s) as in 2.4) and control such deformation with a deformation on the range as in 2.13).

Section 3

In this section we state the theorems that we are going to prove:

Theorem 2.3.1.:

Let $\phi_1 : S^p \longrightarrow S^r$, $\phi_2 : S^q \longrightarrow S^s$, be any two harmonic homogeneous polynomials.

Then the join map $\phi_1 * \phi_2 : S^{p+q+1} \longrightarrow S^{r+s+1}$ can be rendered harmonic, i.e. the homotopy class of $\phi_1 * \phi_2$ contains a harmonic re-

presentative provided that S^{p+q+1} is given a suitable rieman-
nian metric.

Theorem 2.3.2.:

Let $F : S^p \times S^q \longrightarrow S^r$ be a map which is harmonic with constant energy density $\frac{\lambda_i}{2}$, i=1, 2, in each variable separately; and $H : S^{p+q+1} \longrightarrow S^{r+1}$ be its Hopf construction (sec. 8, Chapter I).

Then $H : S^{p+q+1} \longrightarrow S^{r+1}$ can be harmonically represented provided that S^{p+q+1} is given a suitable riemannian metric.

Theorem 2.3.3.: (non-existence for the Hopf construction):

Let $H : S^{p+q+1} \longrightarrow S^{r+1}$ be as in theorem 2.3.2.; and suppose that

$$p = q = 1, \quad \lambda_1 \neq \lambda_2$$

or

$$p = 1,\ q > 1 \quad , \quad q \cdot \lambda_1 \geqslant \lambda_2 .$$

Then $H : S^{p+q+1} \longrightarrow S^{r+1}$ is never harmonic with respect to the standard metrics.

More generally, the same holds if S^{p+q+1} is given a metric as in example 1 of sec. 2.

The consequences of theorems 2.3.1, 2.3.2., 2.3.3. will be discussed in sec. 6; the proofs are very much in the spirit of Remark 4 of sec. 2 and will be given in section 4 for theorem 2.3.1.; in section 5 for theorems 2.3.2. and 2.3.3.

Section 4

Proof of Theorem 2.3.1.:

We have to study the join map $\phi_1 * \phi_2 : S^{p+q+1} \longrightarrow S^{r+s+1}$ as defined by 1.1) of Chapter I: we will use the standard metric on the range, and the non-standard metric introduced in example 1 of sec. 2 on the domain.

By following equation 1.9), we have that the condition of harmonicity for $\phi_1 * \phi_2$ is given by

$$4.1)\qquad \alpha''(s) + \left[\frac{\cos s}{\sin s}\,p - \frac{\sin s}{\cos s}\,q - \frac{h'(s)}{h(s)}\right]\alpha'(s) + $$

$$- h^2(s)\left[\frac{\lambda_1}{\sin s} - \frac{\lambda_2}{\cos s}\right] \sin\,(\alpha(s))\cos\,(\alpha(s)) = o$$

$$s \in (o,\tfrac{\pi}{2})$$

with boundary conditions for $\alpha : (o,\frac{\pi}{2}) \longrightarrow (o,\frac{\pi}{2})$

$$4.2)\qquad \begin{cases} \lim\limits_{s\to o^+} \alpha(s) = o \\[2ex] \lim\limits_{s\to \frac{\pi}{2}^-} \alpha(s) = \frac{\pi}{2} \end{cases}$$

We also recall that we require that the function $h(s)$ satisfy

$$4.3)\quad \begin{cases} h(s) > o & s \in (o, \frac{\pi}{2}) \\ h(s) = 1 & s \in (o, \varepsilon)\ (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}), \text{for some small } \varepsilon > o. \end{cases}$$

The conditions in 4.3) are sufficient to make sure that the metric distinguished by $h(s)$ is actually a smooth metric on the whole S^{p+q+1}.

Only if $h(s) = 1$, $s \in (o, \frac{\pi}{2})$, we have the standard metric on S^{p+q+1} and 4.1) coincides with 1.12).

By making the substitution $e^t = \tan s$, $t \in \mathbb{R}$, the equation 4.1) becomes:

$$4.4)\quad \begin{aligned} &\alpha''(t) + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}} - \frac{h'(t)}{h(t)}\right]\alpha'(t) + \\ &- h^2(t)\left[\frac{\lambda_1 e^{-t} - \lambda_2 e^{t}}{e^{t} + e^{-t}}\right]\sin(\alpha(t))\cos(\alpha(t)) = o \\ &\qquad t \in \mathbb{R}. \end{aligned}$$

Now conditions 4.2) and 4.3) are expressed respectively by:

$$4.5)\quad \begin{cases} \lim_{t \to -\infty} \alpha(t) = o \\ \lim_{t \to +\infty} \alpha(t) = \frac{\pi}{2} \end{cases}$$

and

$$4.6)\quad \begin{cases} h(t) > o\,, & t \in \mathbb{R} \\ h(t) = 1\,, & t \in (-\infty, t_1) \cup (t_2, +\infty) \text{ for some } t_1,\ t_2 \in \mathbb{R}. \end{cases}$$

Now we proceed as described in Remark 4 of sec. 2.

Let $\alpha : \mathbb{R} \longrightarrow (o, \frac{\pi}{2})$ be any strictly increasing function satisfying the conditions 4.5).

If we use such an $\alpha(t)$ in 4.4), we can consider 4.4) as a first order differential equation, in which the unknown function is h(t); 4.4) can be rewritten as

$$4.7)\quad h'(t) - \frac{P_\alpha(t)}{2}\, h(t) = -\frac{Q_\alpha(t)}{2}\, h^3(t)$$

where

$$4.8)\quad P_\alpha(t) = 2\left[\frac{\alpha''(t)}{\alpha'(t)} + \frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}}\right]$$

$$4.9)\quad Q_\alpha(t) = \left[\frac{\lambda_1 e^{-t} - \lambda_2 e^{t}}{e^{t} + e^{-t}}\right] \cdot \frac{\sin(2\alpha(t))}{\alpha'(t)}$$

By making the substitution

$$4.10)\quad y(t) = \frac{1}{h^2(t)}$$

equation 4.7) becomes

4.11) $$y'(t) + P_\alpha(t)\, y(t) = Q_\alpha(t)$$

And 4.6) in terms of $y(t)$ is

4.12) $$\begin{cases} y(t) > 0 & , \quad t \in \mathbb{R} \\ y(t) = 1 , & t \in (-\infty, t_1) \cup (t_2, +\infty) \text{ for some } t_1, t_2 \in \mathbb{R} . \end{cases}$$

To prove theorem 2.3.1., we have to show that the strictly increasing function $\alpha : \mathbb{R} \to (0, \frac{\pi}{2})$, satisfying 4.5), can be chosen in such a way that equation 4.11) has a solution $y(t)$ that satisfies the conditions 4.12).

First of all, we establish conditions under which the equation 4.11) has a solution $\bar{y}(t)$ such that $\bar{y}(t) = 1$ if $|t|$ is large.

Lemma 2.4.1.

Let us suppose that P_α, Q_α in equation 4.11) satisfy

i) $P_\alpha(t) = Q_\alpha(t) \quad t \in (-\infty, t_1] \cup [t_2, +\infty)$ for some $t_1 < 0$, $t_2 > 0$.

ii) $$\lim_{t \to +\infty} e^{\int_{a_1}^{t} P_\alpha(s)\,ds} = \lim_{t \to -\infty} e^{\int_{a_1}^{t} P_\alpha(s)\,ds} = 0 \qquad a_1 \in \mathbb{R} .$$

Then equation 4.11) has a solution $\bar{y}(t)$ such that

$$\bar{y}(t) = 1 \qquad t \in (-\infty, t_1] \cup [t_2, +\infty)$$

if and only if

iii) $$\int_{-\infty}^{+\infty} Q_\alpha(u)\, e^{\int_{a_1}^{u} P_\alpha(s)ds}\, du = 0$$

<u>Proof</u>: in this proof we write $P(s)$, $Q(s)$ instead of $P_\alpha(s)$, $Q_\alpha(s)$.

Let $a_0, a_1 \in \mathbb{R}$ be arbitrarily fixed: the solutions of 4.11) are given by

$$y_c(t) = \frac{\int_{a_0}^{t} Q(u)\, e^{\int_{a_1}^{u} P(s)ds}\, du + c}{e^{\int_{a_1}^{t} P(u)du}} \qquad c \in \mathbb{R} \tag{4.12'}$$

Let us suppose that iii) does not hold: because of ii), one can hope to have $y_c(t) = 1$ for $|t|$ large only if

$$c = -\int_{a_0}^{-\infty} Q(u)\, e^{\int_{a_1}^{u} P(s)ds}\, du \quad \text{and} \quad c = -\int_{a_0}^{+\infty} Q(u)\, e^{\int_{a_1}^{u} P(s)ds}\, du$$

This would imply the validity of condition iii).

Conversely, we can say:

condition i) implies that there must be a solution $y_{\bar{c}}(t)$

such that $y_{\bar{c}}(t) = 1$, $t \in (-\infty, t_1]$; but, because of ii), the only possible choice for $\bar{c}$ is

$$\bar{c} = - \int_{a_0}^{-\infty} Q(u)\, e^{\int_{a_1}^{u} P(s)\,ds}\, du$$

Analogously, one proves that there exists a solution $y_{\underline{c}}(t)$ such that

$$y_{\underline{c}}(t) = 1 \qquad t \in [t_2, +\infty)$$

where $$\underline{c} = - \int_{a_0}^{+\infty} Q(u)\, e^{\int_{a_1}^{u} P(s)\,ds}\, du$$

But, if iii) holds, $\bar{c} = \underline{c}$ and the proof of Lemma 2.4.1. is completed.

Now we show how to construct functions $\alpha(t)$ such that the corresponding $P_\alpha(t)$, $Q_\alpha(t)$ fulfill the conditions of Lemma 2.4.1; after this the proof of the theorem ends easyly.

Step 1 - : we study condition i) of Lemma 2.4.1.:

The explicit expression for condition i) is

4.13)

$$\alpha''(t)+\left[\frac{(p-1)e^{-t}-(q-1)e^{t}}{e^{t}+e^{-t}}\right]\alpha'(t)-\left[\frac{\lambda_1 e^{-t}-\lambda_2 e^{t}}{e^{t}+e^{-t}}\right]\sin(\alpha(t))\cos(\alpha(t))=0$$

$$t\in(-\infty,\ t_1]\cup[t_2,+\infty).$$

Let us fix $t_1 < 0$ and a small $\varepsilon_1 > 0$. Then equation 4.13) has a smooth, strictly increasing solution $\alpha(t)$ on $(-\infty,\ t_1]$ such that

$$\lim_{t\to-\infty}\alpha(t)=0 \quad \text{and} \quad \alpha(t_1)=\varepsilon_1 .$$

This fact is well-known (see, for example, Lemma 6.1.4. page 100 (B1)): in fact, from (Ha) page 433, one has the existence of solutions $\alpha_n(t), n\in\mathbb{N}$, such that, on $[T_n,\ t_1]$,

$$\alpha_n(T_n)=0 \quad , \alpha_n(t_1)=\varepsilon_1,\ \alpha_n'(t)>0$$

$$T_n\to-\infty \text{ when } n\to+\infty .$$

It is easy to check that the solutions $\alpha_n(t)$ subconverge to a solution $\alpha(t)$ with the required properties.

Analogously, if $t_2>0$, we have a solution $\alpha(t)$ on $[t_2,+\infty)$ such that $\alpha(t_2)=\frac{\pi}{2}-\varepsilon_2$, and $\alpha(t)$ increases asymptotically to $\frac{\pi}{2}$.

If $\alpha(t)$, $t\in\mathbb{R}$, is now any smooth, strictly increasing function that coincides with the solutions determined above on $(-\infty, t_1]\cup[t_2, +\infty)$, then by construction the corresponding $P_\alpha(t)$, $Q_\alpha(t)$ fulfill condition i) of the Lemma 2.4.1.

Remark 6:

The equation 4.13) is the usual condition of harmonicity with respect to the euclidean metrics : in fact, our deformed metric coincides with the euclidean around the focal varieties:

then it is natural that, if we want a function $\alpha(t)$ to give rise to a harmonic map with respect to our deformed metric, we must require $\alpha(t)$ to be a solution in the usual sense around the focal varieties.

Step 2 - :

We study condition ii) of the Lemma 2.4.1. we have

4.14) $$e^{\int_{a_1}^{t} P_\alpha(u)\,du} \doteq e^{\int_{a_1}^{t}\left[2\frac{\alpha''(u)}{\alpha'(u)} + \frac{(2p-2)e^{-u}-(2q-2)e^{u}}{e^{u}+e^{-u}}\right]du}$$

$$= \alpha'^{2}(t)\cdot\left[1+e^{-2t}\right]^{1-p}\cdot\left[1+e^{2t}\right]^{1-q}\cdot K_{a_1}$$

where Ka_1 is a positive constant depending on the choice of a_1 .

If $\alpha(t)$ belongs to the class of functions distinguished in Step 1, also condition ii) of the Lemma 2.4.1. is fulfilled: in fact, just notice that $\lim_{t \to +\infty} \alpha'(t) = \lim_{t \to -\infty} \alpha'(t) = 0$.

Step 3 -:

Now we have to prove that, among the functions as in Step 1, it is possible to determine $\bar{\alpha}(t)$ that satisfies also condition iii) of the Lemma 2.4.1.

By using 4.8) and 4.9), we have that condition iii) is

4.15)
$$\int_{-\infty}^{+\infty} \left[\frac{\lambda_1 e^{-t} - \lambda_2 e^{t}}{e^{t} + e^{-t}}\right] \cdot \left[1 + e^{-2t}\right]^{1-p} \cdot \left[1+e^{2t}\right]^{1-q} \sin(2\alpha(t))\,\alpha'(t)\,dt = 0$$

By standard continuity arguments, the existence of our $\bar{\alpha}(t)$ follows from the following Lemma:

Lemma 2.4.2.

a) Among the functions $\alpha(t)$ of Step 1 one can choose $\bar{\alpha}_+(t)$ in such a way that the integral in 4.15) is greater than zero.

b) Among the functions $\alpha(t)$ of Step 1 one can choose $\bar{\alpha}_-(t)$ in such a way that the integral in 4.15) is less than zero.

Proof:

We call

$$4.16)\quad G(t) = \left[\frac{\lambda_1 e^{-t} - \lambda_2 e^{t}}{e^{t} + e^{-t}}\right] \cdot \left[1 + e^{-2t}\right]^{1-p} \cdot \left[1 + e^{2t}\right]^{1-q}$$

Let t_o be the point such that $G(t_o) = 0$.

$G(t)$ is limited, positive on $(-\infty, t_o)$ and negative on $(t_o, +\infty)$.

By making the substitution $\alpha(t) = z$, the integral in 4.15) becomes

$$4.17)\quad \int_0^{\frac{\pi}{2}} G(\alpha^{-1}(z)) \sin(2z)\, dz$$

Now we follow the notations of Step 1:

First we fix $\alpha(t)$ on $(-\infty, t_1]$, $t_1 < t_o$. We have

$$4.18)\qquad \int_0^{\varepsilon_1} G(\alpha^{-1}(z))\sin(2z)\,dz = \delta > 0$$

We choose $t_2 > t_o$ and ε_2 small enough to make the corresponding $\alpha(t)$ on $[t_2, +\infty)$ such that

$$4.19)\qquad \int_{\frac{\pi}{2}-\varepsilon_2}^{\frac{\pi}{2}} G(\alpha^{-1}(z))\sin(2z)\,dz > -\frac{\delta}{2}$$

Being $G(t)$ limited, this is of course possible.

Now we can obtain the function of Lemma 2.4.2. a): we take a smooth, strictly increasing function $\bar{\alpha}_+(t)$ that extends to $\mathbb{R}$ the functions determined above on $(-\infty, t_1]$ and $[t_2, +\infty)$, in such a way that $\bar{\alpha}_+(t_o)$ is so close to $\frac{\pi}{2} - \varepsilon_2$ as to have

$$4.20)\qquad \int_{\bar{\alpha}_+(t_o)}^{\frac{\pi}{2}-\varepsilon_2} G(\bar{\alpha}_+^{-1}(z))\sin(2z)\,dz > -\frac{\delta}{2}$$

Thus, by using 4.18), 4.19), 4.20), we have

$$4.21)\qquad \int_0^{\frac{\pi}{2}} G(\bar{\alpha}_+^{-1}(z))\sin(2z)\,dz > \int_{\varepsilon_1}^{\bar{\alpha}_+(t_o)} G(\bar{\alpha}_+^{-1}(z))\sin(2z)\,dz > 0$$

Therefore $\bar{\alpha}_+(t)$ fulfills condition a) of Lemma 2.4.2.

Analogously, by shifting $\alpha(t_o)$ toward o, one can construct the function $\bar{\alpha}_-(t)$ of Lemma 2.4.2. b).

Then Lemma 2.4.2, is proved and the existence of $\bar{\alpha}(t)$ obtained.

Final Step - :

We associate to $\bar{\alpha}(t)$ of Step 3 the solution $\bar{y}(t)$ of 4.11) as in Lemma 2.4.1.; we have to check that $\bar{y}(t) > o$, $t \in \mathbb{R}$: this follows from direct inspection.

In fact, take the explicit expression 4.12') for the solutions, and choose $a_o = t_o$, where t_o is as in step 3.

Let us call $N(\bar{y}(t))$ the numerator of $\bar{y}(t)$;
then we have

4.22)
$$\lim_{t \to +\infty} N(\bar{y}(t)) = o^+$$
$$\lim_{t \to -\infty} N(\bar{y}(t)) = o^+$$

Moreover, it is easy to see that $N(\bar{y}(t))$ is increasing on $(-\infty, t_o)$, and decreasing on $(t_o , +\infty)$.

This last fact, together with 4.22), tells that $N(\bar{y}(t)) > o$, $t \in \mathbb{R}$.

Then $\bar{y}(t)$ satisfies 4.12) and so distinguishes a riemannian metric on S^{p+q+1} ; with respect to this metric, the join map $\phi_1 * \phi_2$ defined via $\bar{\alpha}(t)$ is harmonic on S^{p+q+1} minus the two focal varieties.

But, by construction, $\bar{\alpha}(t)$ behaves asymptotically like the maps of Chapter I: thus regularity across the focal varieties follows from the usual arguments and theorem 2.3.1. is proved.

Remark 7

A different solution of the rendering problem for $\dot{\varphi}_1 * \dot{\varphi}_2$ is given by Theorem 9.4.6. pag. 176 (B1): the proof of that Theorem turned out to be not correct because it refers to a wrong harmonicity equation.

In fact equation 9.4.5. pag. 176 (B1) should be substituted by

$$\alpha''(u) + \left[\frac{(q-2)e^{-u} - (p-2)e^{u}}{e^{u} + e^{-u}}\right]\alpha'(u) +$$

$$+ \left\{\left[\frac{a^2\lambda_1 e^{u} - b^2\lambda_2 e^{-u}}{e^{u} + e^{-u}}\right] + (a^2 - b^2)\,\alpha'^2(u)\right\} \frac{\sin\alpha(u)\cos\alpha(u)}{\left[a^2\sin^2\alpha(u) + b^2\cos^2\alpha(u)\right]} = 0$$

More generally, in the notations of (B1), the correct version of 9.3.13 pag. 166 from which 9.4.5 is derived is

$$\alpha''(s) + \Delta s\,\alpha'(s) + \nu^2(\alpha(s)) \sum_{K=1}^{p} \gamma_K \mu_{J_K} - \alpha'^2(s)\,\frac{\nu'(\alpha(s))}{\nu(\alpha(s))} = 0 \tag{4.23}$$

In our equation 1.14), we wrote

4.24) $$K(s) = \frac{1}{\nu(s)}$$

By using 4.24), one sees that 4.23) is a particular case of 1.14), with $h(s) \equiv 1$.

Section 5

Now we prove theorem 2.3.2. and 2.3.3.

The range is the euclidean (r+1)-sphere, that we write as

5.1) $$(S^r \times (0,\pi)\ ,\quad \sin^2 s\, h_1 + ds^2)$$

where h_1 is the standard metric on S^r and $s \in (0,\pi)$.

The domain is given a metric as in theorem 2.3.1., i.e. we write S^{p+q+1} as

5.2) $$S^p \times S^q \times (0,\tfrac{\pi}{2}), \text{ with metric}$$
$$\sin^2 s\, g_1 + \cos^2 s\, g_2 + h^2(s) ds^2$$

as in example 1, sec. 2.

The map $H : S^{p+q+1} \longrightarrow S^{r+1}$ of our theorems can be expressed by

5.3) $$H: S^p \times S^q \times (0,\tfrac{\pi}{2}) \longrightarrow S^r \times (0,\pi)$$
$$(x, y, s) \rightsquigarrow (F(x, y), \alpha(s))$$

with boundary conditions for $\alpha(s)$

$$5.4)\quad \begin{cases} \lim\limits_{s \to 0^+} \alpha(s) = 0 \\ \\ \lim\limits_{s \to \frac{\pi}{2}^-} \alpha(s) = \pi \end{cases}$$

One follows equation 1.9), and finds that H is harmonic with respect to the metrics in 5.1), 5.2) if and only if

$$5.5)\quad \alpha''(s) + \left[\frac{\cos s}{\sin s}\, p - \frac{\sin s}{\cos s}\, q - \frac{h'(s)}{h(s)}\right]\alpha'(s) +$$

$$- h^2(s)\left[\frac{\lambda_1}{\sin^2 s} + \frac{\lambda_2}{\cos^2 s}\right] \sin\alpha(s)\cos\alpha(s) = 0$$

After the substitution $e^t = \tan s$, $t \in \mathbb{R}$, 5.5) becomes

$$5.6)\quad \alpha''(t) + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}} - \frac{h'(t)}{h(t)}\right]\alpha'(t) +$$

$$- h^2(t)\left[\frac{\lambda_1 e^{-t} + \lambda_2 e^{t}}{e^{t} + e^{-t}}\right]\sin\alpha(t)\cos\alpha(t) = 0$$

$$t \in \mathbb{R}$$

And the boundary conditions 5.4) are now

$$5.7)\quad \begin{cases} \lim\limits_{t \to -\infty} \alpha(t) = 0 \\ \lim\limits_{t \to +\infty} \alpha(t) = \pi \end{cases}$$

Step 1, Step 2, and the final step are identical to those for $\phi_1 * \phi_2$; by contrast , step 3 must be studied in detail.

Let us introduce

$$5.8)\quad E(t) = \left[\frac{\lambda_1 e^{-t} + \lambda_2 e^{t}}{e^{t} + e^{-t}}\right] \cdot \left[1 + e^{-2t}\right]^{1-p} \cdot \left[1 + e^{2t}\right]^{1-q}$$

Similarly to the case of the join, it is easy to check that, in this case, condition iii) of Lemma 2.4.1. is given by

$$5.9)\quad \int_{-\infty}^{+\infty} E(t) \sin(2\alpha(t))\, \alpha'(t)\, dt = 0$$

If $\alpha(t) = z$, 5.9) becomes

$$5.10)\quad \int_{0}^{\pi} E(\alpha^{-1}(z)) \sin(2z)\, dz = 0$$

Now we distinguish three cases

a) $p > 1, \quad q > 1$

b) $p = 1, \quad q = 1$

c) $p = 1, \quad q > 1$

First we consider the case a):

Let $\bar{t}$ be the point such that $\alpha(\bar{t}) = \frac{\pi}{2}$.

It is easy to prove the existence of a function $\bar{\alpha}_+(t)$ as in Lemma 2.4.2. a): just make $\bar{t}$ be sufficiently close to $+\infty$.

Analogously, if $\bar{t}$ shifts toward $-\infty$, one obtains $\bar{\alpha}_-(t)$ as in Lemma 2.4.2. b); thus, as in the case of $\phi_1 * \phi_2$, there exists $\bar{\bar{\alpha}}(t)$ that makes 5.9) fulfilled and this gives theorem 2.3.2. in the case a).

Before dealing with the two remaining cases, assume

5.11)
$$p = q = 1, \quad \lambda_1 \neq \lambda_2$$
or
$$p = 1, \quad q > 1, \quad q \cdot \lambda_1 \geq \lambda_2$$

It is straightforward to check that, if 5.11) holds, then the function $E(t)$ is strictly monotone.

Then it is obvious that 5.10) can <u>not</u> be fulfilled and so theorem 2.3.3. follows immediately from Lemma 2.4.1.

This fact also shows that handling the cases b), c) above requires a slightly different approach.

We start with case b):

We put on S^3 the following metric

5.12)
$$S^3 = S^1 \times S^1 \times (o, \frac{\pi}{2})$$

$$\lambda_1 \sin^2 s \, d\vartheta^2 + \lambda_2 \cos^2 s \, d\varphi^2 + h^2(s) \, ds^2$$

where $h(s)$ is a smooth positive function on $(o, \frac{\pi}{2})$ such that

5.13)
$$\begin{cases} h^2(s) = \lambda_1 & s \in (o, \varepsilon) \\ h^2(s) = \lambda_2 & s \in (\frac{\pi}{2} - \varepsilon, \frac{\pi}{2}) \quad \text{for some small } \varepsilon > o \end{cases}$$

We call g the metric in 5.12).

To see that g is actually a riemannian metric on the whole S^3, we have to express it across the two focal varieties, i.e. the loci $s = o$, $s = \frac{\pi}{2}$.

Let $(x, y, \bar{\varphi})$ be coordinates across $s = o$ defined by

5.14)
$$\begin{cases} x = \sin s \sin \vartheta & \\ y = \sin s \cos \vartheta & o \leqslant s \leqslant \varepsilon \\ \bar{\varphi} = \varphi & o \leqslant \vartheta, \varphi < 2\pi \end{cases}$$

In such coordinates, the focal variety $s = o$ is the locus

$$(o, o, \bar{\varphi}), \quad o \leq \bar{\varphi} < 2\pi .$$

The expression for g with respect to x, y, $\bar{\varphi}$ is:

5.15)

$$\lambda_1 \left[\frac{1-y^2}{1-x^2-y^2} \, dx^2 + \frac{1-x^2}{1-x^2-y^2} \, dy^2 + \frac{2xy}{1-X^2-y^2} \, d\,x\,dy \right] +$$

$$+ \lambda_2 (1 - x^2 - y^2) \; d\bar{\varphi}^2 .$$

Then g is smooth across the locus $s = o$ and analogously across $s=\frac{\pi}{2}$.

Now the relevant equation can be obtained as usual by using 1.9): it is

5.16)
$$\alpha''(s) + \left[\frac{\cos s}{\sin s} - \frac{\sin s}{\cos s} - \frac{h'(s)}{h(s)} \right] \alpha'(s) +$$

$$- h^2(s) \left[\frac{1}{\sin^2 s} + \frac{1}{\cos^2 s} \right] \sin \alpha(s) \cdot \cos\alpha(s) = o$$

After the substitution $e^t = \tan s$, $t \in \mathbb{R}$, 5.16) is

5.17)

$$\alpha''(t) - \frac{h'(t)}{h(t)} \alpha'(t) - h^2(t) \; \sin\alpha(t) \cos\alpha(t) = o$$

$$t \in \mathbb{R} .$$

And $\alpha(t)$ must satisfy boundary conditions as in 5.7).

One could now follow the same steps as in theorems 2.3.1., 2.3.2. case a): but we suggest here a more explicit procedure to handle with this particular case.

We distinguish a class of functions $\alpha(t)$ as follows:

i) $\alpha : \mathbb{R} \to (0,\pi)$ smooth and strictly increasing;

ii) $\alpha(t)$ satisfies 5.7);

iii) $\alpha''(t) = \lambda_1 \sin(\alpha(t)) \cos(\alpha(t)) \quad t \in (-\infty, t_0]$, for some t_0

iv) $\alpha''(t) = \lambda_2 \sin(\alpha(t)) \cos(\alpha(t)) \quad t \in [t_1, +\infty)$ for some $t_1 > t_0$.

Because of well-known properties of the differential equation of a pendulum with constant gravity and no damping, the above class of functions is large.

We associate to any such $\alpha(t)$ the function $h_\alpha(t)$ defined by

5.18) $$h_\alpha(t) = \frac{\alpha'(t)}{\sin(\alpha(t))}$$

Now we prove that, if $H : S^3 \to S^{2+1}$ is defined by any function $\alpha(t)$ as in i) ....iv), then H is harmonic with respect to the metric g distinguished by $h_\alpha(t)$.

The fulfillment of conditions 5.13) is not immediate: to see it, we show that $h^2_\alpha(t) = \lambda_1$ on $(-\infty, t_0]$, $h^2_\alpha(t) = \lambda_2$ on $[t_1, +\infty)$.

From assumption iv) for $\alpha(t)$ we have

5.19)

$$2\alpha'(t)\alpha''(t) = 2\lambda_2 \sin(\alpha(t)) \cos(\alpha(t)) \alpha'(t) \quad t \in [t_1, +\infty).$$

we integrate and obtain

$$\int_{t_1}^{t} 2\alpha'(u)\alpha''(u)\,du = \int_{t_1}^{t} 2\lambda_2 \sin(\alpha(u)) \cos(\alpha(u))\alpha'(u)\,du,$$

and from this

5.20)

$$\alpha'^2(t) - \alpha'^2(t_1) = \lambda_2\left[\sin^2(\alpha(t)) - \sin^2(\alpha(t_1))\right] \qquad t \in [t_1, +\infty).$$

Now we pass to the limit for $t \to +\infty$ in 5.20): assumption ii) and $\alpha''(t) < o$ for t large give

5.21) $$-\alpha'^2(t_1) = \lambda_2\left[-\sin^2(\alpha(t_1))\right] .$$

By substituting 5.21) in 5.20), we have

$$\alpha'^2(t) = \lambda_2 \sin^2(\alpha(t)), \quad \text{i.e. } h_\alpha^2(t) = \lambda_2 \,, t \in [t_1, +\infty).$$

Analogously, $h_\alpha^2(t) = \lambda_1 \quad , \quad t \in (-\infty, t_o]$.

Finally, it is easy to check by direct substitution that a pair α, h_α as in 5.18) satisfies the condition of harmonicity 5.17).

Then, we have seen that, in this particular case, the metric that renders $\alpha(t)$ harmonic is given explicitly by 5.18).

This fact is interesting because, as shown by Corollary 2.6.2., such maps render harmonic the group $\pi_3(S^2) = \mathbb{Z}$.

In order to achieve regularity across the focal varieties, one can use the usual arguments; but we give here a more efficient method.

First we establish a proposition of interest in itself:

Proposition 2.5.1.: If (α, h_α) is any pair as above, the map H defined via (α, h_α) has energy $E(H) = 8\pi^2\sqrt{\lambda_1 \cdot \lambda_2}$

Proof

By definition, $\big((EL)$ pag. $10\big)$, $E(H) = \int_{S^3} e(H)\, dv$

where $e(H)$ denote the energy density of H.

Elementary calculations give

a) $$dv = \frac{\sqrt{\lambda_1 \cdot \lambda_2}}{4} \quad \frac{\sin^2(2s)\alpha'(s)}{\sin(\alpha(s))} \quad d\vartheta \; d\varphi \; ds \qquad s \in (0, \tfrac{\pi}{2})$$

b) $$e(H) = 4 \, \frac{\sin^2(\alpha(s))}{\sin^2(2s)}$$

Therefore $E(H) = \int_{S^3} \sqrt{\lambda_1 \cdot \lambda_2} \sin(\alpha(s))\alpha'(s) \; d\vartheta \; d\varphi \; ds =$

$$= \sqrt{\lambda_1 \lambda_2} \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin(\alpha(s))\alpha'(s) \; ds = 8\pi^2 \sqrt{\lambda_1 \lambda_2}\,.$$

Now global regularity follows from the arguments of Remark 4, sec. 6 of Chapter I; proposition 2.5.1. shows in fact that $H \in \mathcal{L}^2(S^3, S^{q+1}) \cap \mathcal{C}^0(S^3, S^{q+1})$.

Now we conclude theorem 2.3.2. by studing case c), i.e. $p = 1$, $q > 1$.

We use on the domain S^{p+q+1} a metric as in 5.12), 5.13), with λ_1, λ_2 replaced by two arbitrary positive numbers a, b.

By following similar arguments to those of Theorem 2.3.2. case a), the proof reduces to the study of the condition

$$5.23) \qquad \int_0^{\pi} E_{ab}\,(\alpha^{-1}(z))\ \sin(2z)\ dz \ = o$$

where

$$5.24) \qquad E_{ab}\,(t) = \left[\frac{\frac{\lambda_1}{a} e^{-t} + \frac{\lambda_2}{b} e^{t}}{e^{t} + e^{-t}}\right] \cdot \left[1+e^{2t}\right]^{1-q}$$

If a is large, b small, then $E_{ab}(t)$ is not monotone any longer and 5.23) can be fulfilled by similar arguments to Lemma 2.4.2.

Section 6

In this section we give some applications of the theorems stated in sec. 3.

Theorem 2.3.1. produces many new harmonic maps, because, in particular, it enables us to remove any restriction on suspensions: we can state

Corollary 2.6.1.

Let $\phi_1 : S^p \longrightarrow S^r$ be a harmonic homogeneous polynomial. Then the homotopy class of the (q+1)-suspension of ϕ_1 can be harmonically represented for every $q \in \mathbb{N}$, provided that the domain is given a suitable riemannian metric.

As an immediate application of Corollary 2.6.1., we can render harmonic the groups:

$$\pi_n(S^n) = \mathbb{Z} \qquad n \geq 1, \text{ and}$$

$$\pi_{n+1}(S^n) = \mathbb{Z}_2 \; , \quad n \geq 3.$$

In fact, we can obtain these groups by iterated suspensions of respectively $i_k : S^1 \longrightarrow S^1$, $h_1 : S^3 \longrightarrow S^2$.

Many further examples of this kind can be easily produced by suspending any other harmonic homogeneous polynomial.

The fact that we have given the range sphere the euclidean metric has relevant consequences because of the composition properties of harmonic maps.

For instance, let $T : S^{n+1} \longrightarrow G$ be a totally geodesic element of $\pi_{n+1}(G)$, G a Lie group.

Then the composition of T with our harmonic maps $id_{n-1} * i_k : S^{n+1} \longrightarrow S^{n+1}$ gives harmonic representatives of all the subgroup of $\pi_{n+1}(G)$

generated by T.

Example: there exist totally geodesic maps

$$T : S^{2k+1} \longrightarrow SU(n) \qquad k \geq 1, \quad n \geq 2^k .$$

Such maps T are the generators of $\pi_{2k+1}(SU(n)) = \mathbb{Z}$:

Thus, each class of all such groups can be rendered harmonic. This was known for $1 \leq k \leq 4$.

Examples in this direction can be obtained by combining our work with (FO).

Corollary 2.6.2.

Each element of the group $\pi_3(S^2) = \mathbb{Z}$ can be rendered harmonic.

Proof:

Just let $F : S^1 \times S^1 \longrightarrow S^1$ be the map induced by the complex multiplication $(w, z) \longrightarrow w^k \cdot z^\ell$.

Then F satisfies the hypothesis of theorem 2.3.2. with $\lambda_1 = k^2$, $\lambda_2 = \ell^2$.

The associated map $H : S^3 \longrightarrow S^2$ has Hopf invariant $k \cdot \ell$, i. e. corresponds to the element $k \cdot \ell \in \mathbb{Z} = \pi_3(S^2)$.

We point out that, if F had been induced by the quaternionic or Cayley numbers multiplication, then it would not have fulfilled any longer the requirements of theorem 2.3.2.

Smith proved non-existence for the Hopf construction in the case $p = q = 1$, $\lambda_1 \neq \lambda_2$, with respect to the standard metrics only; Theorem 2.3.3. gives an extension of this non-existence result.

If one compares the non-existence result of theorem 2.3.3. with the existence result expressed by Theorem 1.8.1., finds that there is a gap between them, that is to say an area of uncertainty.

It is interesting to notice that actually there is a map that falls into this area of uncertainty:

Let S^3 be the unit quaternions and S^2 be the unit quaternions with vanishing real part.

Define $F : S^2 \times S^3 \longrightarrow S^2$ by $F(x, y) = y \cdot x \cdot \bar{y}$.

It is easy to check that F is harmonic with constant energy density in each variable separately; and $\lambda_1 = 2$, $\lambda_2 = 4$.

The Hopf construction applied to F yields (topologically) the generator of $\pi_6(S^3) = \mathbb{Z}_{12}$ (see BS).

In this case, H.D.C. does not hold, and theorem 2.3.3. does not apply; we do not know whether the generator of $\pi_6(S^3)$ can be represented harmonically by an equivariant map.

But theorem 2.3.2. tells us that such a generator can be rendered harmonic by introducing a suitable metric on S^6.

We have seen that the choice of the metric in the domain does influence the existence of equivariant harmonic maps in certain homotopy classes: for instance, the classes of $\pi_3(S^2)$ of Hopf invariant $k \cdot 1$, $k \neq 1$; or the (q+1)-suspension of a harmonic homogeneous polynomial, $q \geqslant 6$.

Then a natural question is: can the choice of the metrics influence the existence of a harmonic representative in a given homotopy class ?

Just looking at equivariant maps, one might be tempted to say yes. However, we remark that there are quite many interesting examples of homotopy classes in $\pi_n(S^n)$ that cannot be represented via suspensions of the map $i_{\mathbb{R}} : S^1 \longrightarrow S^1$; but they can be represented either by suspending other polynomial maps or directly by a polynomial map.

Another natural point to make is the following: our methods to solve the rendering problem are strongly <u>not</u> real-analytic in nature.

But, on the other hand, we could have had more flexibility in the choice of the deformations: then, it would be interesting to know in which cases the suitable deformation can be made real-analytic.

Section 7

In this section we apply the methods of the previous sections to study maps into ellipsoids.

Let $E^{r+s+1}(a, b) = \left\{ (x, y) \in \mathbb{R}^{r+1} \times \mathbb{R}^{s+1} \,/\, \frac{|x|^2}{a^2} + \frac{|y|^2}{b^2} = 1 \right\}$

Smith (S1) obtained the following non-existence result:

Proposition (S1):

If $r \geq 2$, and b large, there is no-harmonic 1-parameter suspension

$$S^{r+1} \longrightarrow E^{r+0+1}(1, b)$$

of the identity on S^r.

The ellipsoid $E^{r+s+1}(a, b)$ is obtained from S^{r+s+1} by an equivariant deformation of metric; in fact, as in example 2 of sec. 2,

7.1) $$E^{r+s+1}(a, b) = S^r \times S^s \times (o, \frac{\pi}{2}), \text{ with metric}$$
$$a^2 \sin^2 s \; g_1 + b^2 \cos^2 s \; g_2 + \left[a^2 \cos^2 s + b^2 \sin^2 s\right] ds^2$$

Then, it is possible to apply equivariant theory to maps

$$\phi_1 * \phi_2 : S^{p+q+1} \longrightarrow E^{r+s+1}(a, b)$$

where $\phi_1 : S^p \longrightarrow S^r$, $\phi_2 : S^q \longrightarrow S^s$ are harmonic homogeneous polynomials and the join map is defined as in 1.10').

To work out the harmonicity equation, one must use the metric 7.1) in equation 1.9); i.e., in the notations of 1.9),

$$B_1(s) = a \sin s, \quad B_2(s) = b \cos s, \quad R(s) = \left[b^2 \sin^2 s + a^2 \cos^2 s\right]^{1/2}.$$

We will prove

Proposition 2.7.1.

Let $\phi_1 : S^p \longrightarrow S^r$, $\phi_2 : S^q \longrightarrow S^s$, be two harmonic homogeneous polynomials; and let $\phi_1 * \phi_2 : S^{p+q+1} \longrightarrow E^{r+s+1}(a, b)$ be the join map.

Then, for any choice of a, b > o, the map $\phi_1 * \phi_2$ can be rendered harmonic by using a suitable riemannian metric on the domain.

<u>Proof:</u>

The proof follows very much the arguments of theorem 2.3.1.): in fact, precisely as in theorem 2.3.1.); we put on the domain a metric as in example 1 of sec. 2.

With respect to such a metric, the relevant equation is given by

7.2)
$$\alpha''(t) + \left[\frac{(p-1)e^{-t} - (q-1)e^{t}}{e^{t} + e^{-t}} - \frac{h'(t)}{h(t)}\right]\alpha'(t) +$$
$$+\left\{h^2(t)\left[\frac{b^2\lambda_2 e^{t} - a^2\lambda_1 e^{-t}}{e^{t} + e^{-t}}\right] + \left(b^2 - a^2\right)\alpha'^2(t)\right\} \cdot$$
$$\cdot\left\{\frac{\sin\alpha(t)\ \cos\alpha(t)}{\left[a^2\cos^2(\alpha(t)) + b^2\sin^2(\alpha(t))\right]}\right\} = o$$
$$t \in \mathbb{R}.$$

where $\alpha : \mathbb{R} \longrightarrow (o, \frac{\pi}{2})$ satisfies boundary conditions

7.3)
$$\begin{cases} \lim\limits_{t \to +\infty} \alpha(t) = \dfrac{\pi}{2} \\ \lim\limits_{t \to -\infty} \alpha(t) = o \end{cases}$$

One makes the substitution

7.4) $$y(t) = \frac{1}{h^2(t)}$$

and rewrite 7.2) in the form

7.5) $$y'(t) + P_\alpha(t)\; y(t) = Q_\alpha(t)$$

with

7.6)
$$P_\alpha(t) = 2\left[\frac{\alpha''(t)}{\alpha'(t)} + \frac{(p-1)e^{-t}-(q-1)e^{t}}{e^{t}+e^{-t}}\right] + \left[\frac{\sin(2\alpha(t))\,\alpha'(t)\,(b^2-a^2)}{a^2\cos^2\alpha(t) + b^2\sin^2\alpha(t)}\right]$$

7.7)
$$Q_\alpha(t) = \left[\frac{a^2\lambda_1 e^{-t} - b^2\lambda_2 e^{t}}{e^{t}+e^{-t}}\right]\frac{\sin(2\alpha(t))}{\alpha'(t)\cdot\left[a^2\cos^2\alpha(t) + b^2\sin^2\alpha(t)\right]}$$

Now one just follows Step 1 - 2 - 3 of section 4:

In order to make sure that the conclusions of Step 1 hold also in this case, just notice that, for small initial pushes and position (i.e. $\alpha'(t_1)$, $\alpha(t_1)$ small), the term $(b^2 - a^2)\,\alpha'^2(t)$ in the gravity of 7.2) is negligible and7.2) is qualitatively like 4.4).

As for Step 2, a computation shows

7.8)
$$e^{a_1 \int^t P_\alpha(s)\,ds} = \alpha'^2(t) \left[1 + e^{-2t}\right]^{1-p} \left[1 + e^{2t}\right]^{1-q} \cdot$$
$$\cdot\, K_{a_1} \cdot \left[a^2 \cos^2\alpha(t) + b^2 \sin^2\alpha(t)\right]$$

Then we can draw the same conclusions as in section 4.

Finally, in Step 3, one has to check condition iii) of Lemma 2.4.1.

This condition is now expressed by

7.9)
$$\int_{-\infty}^{+\infty} \left[\frac{a^2\lambda_1 e^{-t} - b^2\lambda_2 e^{t}}{e^{t} + e^{-t}}\right]\left[1 + e^{-2t}\right]^{1-p} \cdot \left[1 + e^{2t}\right]^{1-q} \sin\,(2\alpha(t))\,\alpha'(t)\;dt = 0$$

If one compares 7.9) with 4.15), he sees that the only difference is given by the coefficients $a, b > o$.

But, because of the fact that in section 4 we did not have any restriction on λ_1, λ_2, we can conclude that also in this case condition iii) can be fulfilled.

The final Step is precisely has in section 4 and this ends proposition 2.7.1.

It is worth emphasising the following technical point: suppose that we write the general reduction equation 1.9) in the form

$$y'(t) + P_{\alpha}(t)\, y(t) = Q_{\alpha}(t)$$

with $y(t) = \dfrac{1}{h^2(t)}$

This is possible provided that $\alpha'(t) > 0$.

Then it is always possible to integrate explicitly (i.e. in terms of A_i, $i = 1, \ldots\ldots p, R(s)$) the function

$$\int_{a_1}^{t} P_{\alpha}(u)\, du.$$

This feature, that we have just applied in 7.8), is vital to study effectively the condition iii) of Lemma 2.4.1.; and enables us to have qualitative informations on equations that otherwise would be quite difficult to be approached.

We will see other applications of this fact in Chapter III: at the moment, we end this section by pointing out that an analogous result to proposition 2.7.1. holds for the Hopf construction; the proof follows the arguments of proposition 2.7.1. and it is omitted.

CHAPTER III

In this chapter we discuss some further applications of equivariant theory. The chapter is divided in three parts.

Part 1: Dirichlet problems and warped products.

Part 2: Harmonic maps and equivariant theory on singular spaces.

Part 3: Further possible developments of equivariant theory.

Each of these parts is self-contained together with the equivariant theory illustrated in Chapter II.

Part 1: Dirichlet problems and warped products.

Section 1:

Let (S^m, g_1), (S^n, h_1) be the standard m and n spheres and let $\phi : S^m \longrightarrow S^n$ be a harmonic map with constant energy density $2\,e(\phi) = \lambda$.

In order to state the Dirichlet problems that we are going to study, we introduce maps Φ as follows:

1.1) $$\begin{aligned} \Phi : (S^m \times [o, 1], g) &\longrightarrow (S^n \times [o, \pi], h) \\ (x, r) &\rightsquigarrow (\phi(x), \alpha(r)) \end{aligned}$$

where the metrics g, h have form

1.2) $$\begin{aligned} g &= f^2(r)\, g_1 + h^2(r)\, dr^2 \\ h &= \sin^2 r\, h_1 + dr^2 \end{aligned}$$

The functions h(r), f(r) are smooth positive functions on $(o, 1]$ such that

1.3) $$\begin{aligned} f(r) &= \mathcal{O}(r) \qquad r \text{ small} \\ h(r) &= 1, \quad r \in (o, \varepsilon] \quad \text{for some small } \varepsilon > o . \end{aligned}$$

Moreover, we require that the metric g be smooth across r = o . This is guaranteed if, for instance,

1.4) $f(r) = r \quad , \quad r \in [o, \varepsilon]$

or

1.5) $\quad f(r) = \sin r \qquad r \in [o, \varepsilon]$

or

1.6) $\quad f(r) = \sin h \ r \qquad r \in [o, \varepsilon]$

Under our assumptions, the range is the standard S^{n+1}.
The domain is a (m+1)-dimensional riemannian manifold with boundary B^{m+1} which is homeomorphic to the standard closed ball. In particular, if one out of 1.4), 1.5), 1.6) holds, then B^{m+1} has constant curvature respectively o, 1, -1, around the origin.

Now our Dirichlet problem can be stated: we impose boundary values in 1.1) of the form

1.7) $\quad (x, 1) \rightsquigarrow (\phi(x), \alpha(1)), \ \alpha(1) \in (o, \pi]$.

and look for harmonic extensions with $\alpha'(o) = o$.

This Dirichlet problem has been previously studied and we now recall the main known results:

Let us assume that

1.8) $\quad \begin{cases} f(r) & \text{strictly increasing} \\ h(r) = 1 & r \in [o, 1] \end{cases}$

Then (KW) the above Dirichlet problem has a solution if and only if $\alpha(1)$ is <u>not too close</u> to π.

Estimates for the previous not too close have been given in (JK), (EL3) in the case when $f(r) = r$, and in (KW) in the general case.

In particular, under 1.8), no solution occurs if $\alpha(1) = \pi$. More generally, as far as constant boundary values, we recall the following results due to Lemaire and Karcher-Wood:

Theorem (LE):

Let $(M, \partial M)$ be a contractible 2-dimensional manifold with boundary, and let N be any riemannian manifold.

Let $\phi : M \to N$ be a harmonic map such that

$$\phi_{/\partial M} = \text{constant}$$

Then ϕ is constant.

Theorem (KW):

Let $(M, \partial M)$ be the closed, m-dimensional, flat ball, $m \geqslant 2$; and let N be any riemannian manifold.

Let $\phi : M \longrightarrow N$ be a harmonic map such that

$$\phi_{/\partial M} = \text{constant}$$

Then ϕ is constant.

We are going to prove what follows

Proposition 3.1.1:

If the function h(r) in 1.8) is replaced by a suitable function, then, for any boundary value $\alpha(1) \in (o,\pi)$, we have a solution of the Dirichlet problem with $\alpha(o) = o$. However, no matter how h(r) is chosen, no non-constant solution occurs in the case $\alpha(1) = \pi$.

Theorem 3.1.1.

If $m \geq 2$, then B^{m+1} can be given a riemannian metric g in such a way that the Dirichlet problem with constant boundary value $\alpha(1) = \pi$ admits a solution with $\alpha(o) = o$.

As an immediate consequence of Theorem 3.1.1., we have that the above Theorem (LE) fails whenever the dimension of the domain is greater or equal than three; and the above theorem (KW) fails if the metric on B^{m+1} does not satisfy prescribed requirements.

Proofs:

By following, as usual, equation 1.9), sec. 1, Chapter II, we obtain the condition of harmonicity for Φ as in 1.1):

1.9)

$$\alpha''(r) + \frac{f'(r)}{f(r)} m\alpha'(r) - \frac{h'(r)}{h(r)}\alpha'(r) - h^2(r)\frac{\lambda \sin\alpha(r)\cos\alpha(r)}{f^2(r)} = o$$

Now we suppose $\alpha(r)$ strictly increasing, and make the substitution

1.10) $$y(r) = \frac{1}{h^2(r)}$$

In terms of $y(r)$, 1.9) becomes

1.11) $\quad y'(r) + P_\alpha(r) \quad y(r) = Q_\alpha(r)$

where

1.12)
$$\begin{cases} P_\alpha(r) = 2\,\dfrac{\alpha''(r)}{\alpha'(r)} + 2m\,\dfrac{f'(r)}{f(r)} \\[2ex] Q_\alpha(r) = \dfrac{\lambda \sin(2\alpha(r))}{f^2(r)\,\alpha'(r)} \end{cases}$$

From now on, we assume that our function $\alpha(r)$ satisfies

1.13)
$$\begin{cases} P_\alpha(r) = Q_\alpha(r) \quad , \quad r \in (o, \varepsilon] \quad , \ \varepsilon \text{ small} \\[2ex] \lim\limits_{r \to o} \alpha(r) = o \quad , \alpha(\varepsilon) = \delta_1 \quad , \ \delta_1 \in (o, \frac{\pi}{2}) \end{cases}$$

As in Step 1, sec. 4, Chapter II, one sees that, for ε, δ_1 as in 1.13), such a function $\alpha(r)$ exists.

Under these assumptions on $\alpha(r)$, it is clear that equation 1.11) has a solution $y_o(r)$ such that

1.14) $\qquad y_o(r) = 1 \quad , \quad r \in (o, \varepsilon]$

It is easy to see that $y_o(r)$ is given by

1.15) $\quad y_o(r) = \dfrac{\int_o^r \lambda f^{2(m-1)}(s) \ \sin(2\alpha(s)) \ \alpha'(s)\, ds}{f^{2m}(r) \quad \alpha'^2(r)}$

Suppose now that $f(r)$ is strictly increasing:

In order to prove proposition 3.1.1., it is enough to show that

i) if $\alpha(1) = \pi$, then $y_o(\bar{r}) = o$, for some $\bar{r} \in (o, 1]$;

ii) if $\alpha(1) = \pi - \delta$, $\delta \in (o, \frac{\pi}{2})$, then $f(r)$ and $\alpha(r)$ can be chosen in such a way that $y_o(r)$ is positive on $[o, 1]$.

In fact, in the case i), the metric g as in 1.2) blows up in $\bar{r}$, i.e. $\lim_{r \to \bar{r}} h(r) = +\infty$

On the contrary, in the case ii) $y_o(r)$ defines a riemannian metric on B^{m+1} as in 1.2) via 1.10).

To investigate the positivity of $y_o(r)$, we need to study the integral

1.16) $$\int_o^{\tau} \lambda f^{2(m-1)}(s) \quad \sin(2\alpha(s))\, \alpha'(s)\, ds$$

After the substitution $\alpha(s) = u$, 1.16) becomes

1.17) $$\int_o^{\alpha(\tau)} \lambda f^{2(m-1)}(\alpha^{-1}(u)) \quad \sin 2u \quad du.$$

Now, being $f(r)$ strictly increasing, it is clear that $\alpha(1) = \pi$ forces the integral in 1.17) to assume the value zero for some $\tau \in (o, 1]$; then i) above is proved.

In order to prove ii), we have to choose the function $\alpha(r)$ carefully.

First we note what follows: let ε_1 be the point such that $\alpha(\varepsilon_1) = \delta$, and assume $\alpha(1) = \pi - \delta$ as in ii).

Then

$$1.18) \quad \int_{\delta}^{\pi-\delta} \lambda f^{2(m-1)}(\alpha^{-1}(u)) \sin 2u \; du \to o \quad \text{if } \varepsilon_1 \to 1.$$

The statement in 1.18) follows from the fact that the integrand function in 1.18) converges uniformly to $\lambda \sin 2u$ as $\varepsilon_1 \to 1$.

Now we write the integral in 1.17), for r = 1, as

1.19)

$$\int_{o}^{\delta} \lambda f^{2(m-1)}(\alpha^{-1}(u)) \sin(2u) \; du + \int_{\delta}^{\pi-\delta} \lambda \; f^{2(m-1)}(\alpha^{-1}(u) \sin(2u) \; du.$$

Roughly speaking, the first part of the sum in 1.19) is positive; if ε_1 tends to 1, then the second part of the sum in 1.19) tends to zero by 1.18).

Then, if ε_1 is close enough to 1, it is not difficult to construct a function $\alpha(r)$ in such a way that the integral in 1.17) is greater than zero for every $r \in (o,1]$.

Now we drop the assumption f(r) increasing and occupy ourselves with Theorem 3.1.1.

In this case we assume $\alpha(1) = \pi$ and $m \geq 2$.

Under these hypotheses, $\alpha(r)$ and f(r) can be chosen in such a way that the integral in 1.17) is strictly positive for every $r \in (o,1]$: roughly, just take f(r) and $\alpha(r)$ qualitatively as in the figure 1 below.

Now the thesis follows as in ii) of the previous proof.

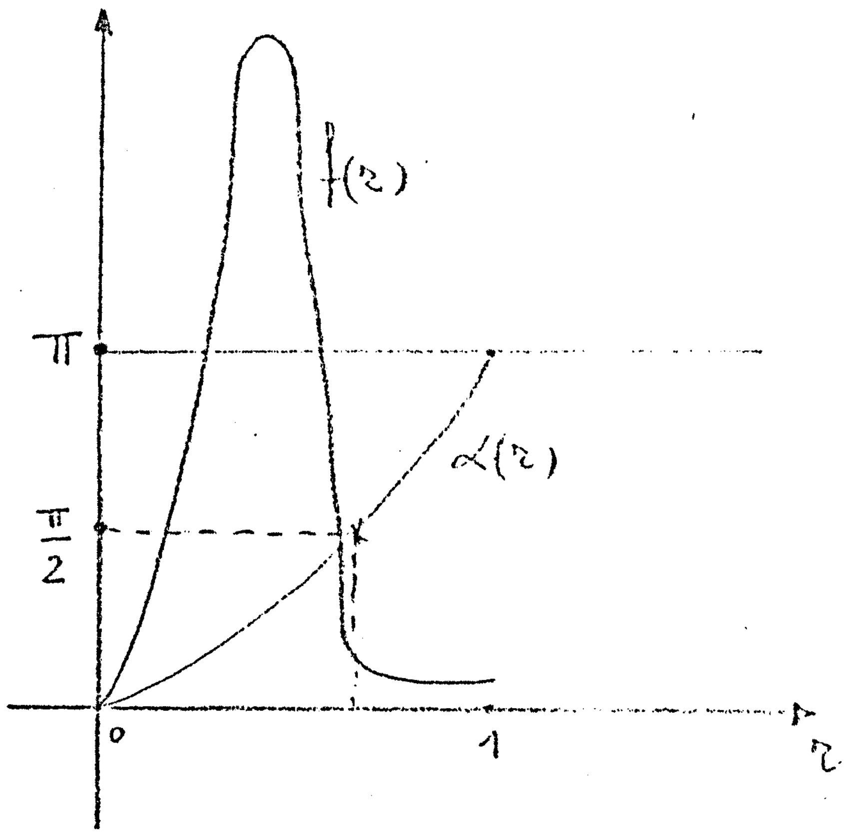


Figure 1

Section 2:

In this section we give some applications of equivariant theory to the study of warped products.

Let (M x (a,b),g) and (N x (c,d),h) be two equivariant manifolds as in section 1, Chapter II; g and h denote the riemannian metrics on M x (a,b), N x (c,d).

Let Φ :M x (a,b) $\longrightarrow$ N x (c,d)

2.1) $(x, s) \rightsquigarrow (\phi(x), \alpha(s))$

be an equivariant map.

Let (M_1, g_1) be any riemannian manifold of dimension m_1.

We will consider maps of the form

$$F : M_1 \times (M \times (a, b) \longrightarrow N \times (c,d)$$

2.2) $(y, x, s) \rightsquigarrow (\phi(x), \alpha(s))$

We give the domain a warped metric of the form

2.3) $f^2(s)g_1 + g$

where f(s) is a smooth positive function such that

2.4) $$\begin{cases} f(s) = c_1 = \text{constant} & s \in (a, a+\varepsilon) \\ f(s) = c_2 = \text{constant} & s \in (b-\varepsilon, b) \end{cases}$$

for some small $\varepsilon > o$.

The conditions in 2.4) make sure that the riemannian metric in 2.3) extends smoothly across the completion of M x (a,b).

<u>**Proposition 3.2.1.**</u>

Let F be a map as in 2.2).

Let us call $G_\alpha(s) = o$ the condition of harmonicity for Φ as in 2.1), and assume that

i) there exists a strictly increasing function $\alpha_1(s)$ such that

$$\begin{cases} G_{\alpha_1}(s) = o, & s \in (a, a+\mathcal{E}) \\ \lim\limits_{s \to a} \alpha_1(s) = c \end{cases}$$

ii) there exists a strictly increasing function $\alpha_2(s)$ such that

$$\begin{cases} G_{\alpha_2}(s) = o, & s \in (b-\mathcal{E}, b) \\ \lim\limits_{s \to b} \alpha_2(s) = d \end{cases}$$

Then it is possible to choose $f(s)$ as in 2.4) and a surjective strictly increasing $\alpha : (a,b) \to (c,d)$ in such a way that the map F is harmonic with respect to the metric in 2.3).

Proof:

The condition of harmonicity for a map F as in 2.2) is given by

2.5) $$m_1 \frac{f'(s)}{f(s)} \alpha'(s) + G_{\alpha}(s) = o$$

Suppose that $\alpha(s)$ is any smooth strictly increasing function which coincides with α_1, α_2 as in the hypothesis on $(a, a+\mathcal{E})$ and $(b-\mathcal{E}, b)$.

By solving equation 2.5) with respect to f(s), we have

2.6) $$f(s) = c\, e^{-\int_{\bar{a}}^{s} \frac{G_{\alpha}(u)}{\alpha'(u) m_1} du} \qquad c > o, \quad \bar{a} \in (a,b)$$

But $G_\alpha(u) = o$ on $(a, a+\varepsilon) \cup (b-\varepsilon, b)$: then clearly $f(s)$ fulfills 2.4) and the proposition is proved.

Examples:

Let (S^m, g) be the standard m-sphere, and (B^m, h) be the flat closed m-ball.

As an application of Proposition 3.2.1., we obtain, for instance, harmonic maps as follows:

2.7)
$$\begin{aligned} F : M_1 \times S^m &\longrightarrow S^m \\ (y, w) &\rightsquigarrow \Phi(w) \end{aligned}$$

$\deg(\Phi) = K$

$m \geqslant 1$

2.8)
$$\begin{aligned} F : M_1 \times S^3 &\longrightarrow S^2 \\ (y, w) &\rightsquigarrow \Phi(w) \end{aligned}$$

Φ with Hopf invariant $K \cdot \ell$

$K, \ell \in \mathbb{Z}$.

2.9)
$$\begin{aligned} F : M_1 \times B^m &\longrightarrow S^m \\ (y, w) &\rightsquigarrow \Phi(w) \end{aligned}$$

$\Phi(o) =$ North pole

$\Phi/\partial B^m =$ South pole

$m \geqslant 2$

In all such examples, (M_1, g_1) can be any manifold, and the domain is given a warped metric as in 2.3).

In particular, the examples in 2.9) give further cases in which Theorem (LE) of section 1 fails if the assumption of dimension 2 for the domain is dropped

<u>Remark 1:</u>

Let $\phi_1 : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a harmonic map with constant energy density.

Let us substitute the map F in 2.2) with

$$2.10) \qquad \begin{aligned} \tilde{F} : M_1 \times (M \times (a,b)) &\longrightarrow M_2 \times (N \times (c,d)) \\ (y, x, s) &\rightsquigarrow (\phi_1(y), \phi(x), \alpha(s)). \end{aligned}$$

We give the domain a metric as in 2.3) and the range the product metric $g_2 + h$.

Then Proposition 3.2.1. holds with $\tilde{F}$ in place of F.

The proof is immediate because $\tilde{F}$ and F have the same harmonicity equation.

As an application, the examples 2.7), 2.8), 2.9) can be replaced by harmonic maps as follows:

$$2.11) \qquad \begin{aligned} \tilde{F} : M_1 \times S^m &\longrightarrow M_1 \times S^m \\ (y, w) &\rightsquigarrow (y, \Phi(w)) \end{aligned} \qquad \deg(\Phi) = k, \quad m \geq 1$$

$$2.12) \qquad \begin{aligned} \tilde{F} : M_1 \times S^3 &\longrightarrow M_1 \times S^2 \\ (y, w) &\rightsquigarrow (y, \Phi(w)) \end{aligned} \qquad \Phi \text{ with Hopf invariant } k \cdot \ell, \quad k, \ell \in \mathbb{Z}.$$

2.13) $\tilde{F} : M_1 \times B^m \longrightarrow M_1 \times S^m$ $\quad \Phi(o) =$ North pole

$(y, w) \rightsquigarrow (y, \Phi(w))$ $\quad \Phi /_{\partial B^m} =$ South pole

$m \geqslant 2.$

Remark 2:

It follows from 1.10) sec. 1, Chapter II, that in the case of maps F as above, the energy density $e(F)$ does not depend on the choice of the function f(s) in 2.3); by using this fact, one can produce harmonic maps as in examples 2.7), 2.8) with arbitrarily small energy: just choose the function $\alpha(s)$ in such a way that nearly all of the manifold is sent close to one of the two focal varieties of the range; maps of this type, but not harmonic, where first described in (ES), pag. 131.

Moreover, we notice that our remark could be related to the following fact (see S2 pag. 111): let π be the projection

$$\pi : (S^m \times S^n, g_1 + g_2) \longrightarrow (S^m, g_1)$$

Then index (π) can be made arbitrarily large by substituting the product metric with a metric of the form

$$g_1 + cg_2 \qquad c \in \mathbb{R}.$$

So far we have used warped metrics on the domain; now we show that similar results can be obtained also by using warped metrics on the range.

Let us consider maps F^* which are defined precisely as $\tilde{F}$ in 2.10); but now we use a product metric $g_1 + g$ on the domain, and a warped metric

2.14) $f^2(s)\; g_2 + h$

on the range, f(s) as in 2.4).

We have

Proposition 3.2.2.

Proposition 3.2.1. holds with F^* in place of F, and 2.14) in place of 2.3).

Proof:

The condition of harmonicity for F^* is given by

2.15) $G_\alpha(s) - 2e(\phi_1)\; f(\alpha(s))\; f'(\alpha(s)) = o$

By solving 2.15) with respect to f, we have

2.16) $$f^2(\alpha(s)) = \frac{1}{e(\phi_1)}\left\{\int_{\bar{a}}^{s} G_{\alpha'}(u)\,\alpha'(u)\,du + c\right\} \qquad c > o$$

c large.

where $\alpha(s)$ is strictly increasing and coincides with $\alpha_1(s), \alpha_2(s)$ on $(a, a+\varepsilon) \cup (b-\varepsilon, b)$ as in the hypothesis of Proposition 3.2.1.

Under these assumptions, the conditions in 2.4) are satisfied and so the proof is ended.

As an application of this last proposition, we have, for instance, maps F^{*} defined as in examples 2.11), 2.12), 2.13). Now these maps are harmonic with respect to a suitable warped metric on the range and the product metric on the domain.

PART 2: Harmonic maps and equivariant theory on singular spaces.

Introduction:

Roughly speaking, a closed riemannian pseudomanifold X is a compact metric space with a closed subset Σ (possibly empty) such that $X-\Sigma$ is a riemannian manifold dense in X.

J. Cheeger set forward an extension of the standard Hodge theory for such pseudomanifolds (see (C1)); L^2- cohomology and Intersection homology turned out to be suitable for this purpose. The main feature of his work was to prove properties of X by making analysis on $X-\Sigma$: in this way of thinking, we define harmonicity on riemannian pseudomanifolds as follows:
let $F: X_1 \longrightarrow X_2$, $F(X_1 - \Sigma_1) \subseteq (X_2 - \Sigma_2)$, be a continuous map; then F will be said to be harmonic if $F_{/X_1 - \Sigma_1}$ is harmonic in the usual sense.

The main motivation of this definition lays on the fact that in many cases it is possible to identify the above $X-\Sigma$ with the open manifold on which we built up the equivariant theory of Chapter II, sec. 1.

In section 3 we recall some definitions and properties of singular spaces and give examples of harmonic maps between them.

In section 5, in a slightly different spirit, we suggest a generalisation of the Hopf construction.

Section 3:

We recall some facts about singular spaces: references on the subject are (C1, C2, CGM, GM) and attached bibliography.

A closed n-dimensional pseudomanifold X^n is a finite simplicial complex with the following properties:

i) the highest dimension of a simplex of X^n is n.

ii) each (n-1)-dimensional simplex belongs to two n-dimensional simplexes.

X^n can be canonically realised as a compact topological space with a triangulation T.

Let T^i denote the i-dimensional skeleton of T. Standard arguments ensure that $X^n - T^{n-2}$ is a differentiable manifold dense in X^n.

We will be interested on riemannian metrics on $X^n - T^{n-2}$ that are conical around T^{n-2}.

To explain what a conical metric is meant to be we need to recall a few concepts:

Let (M^m,h) be a m-dimensional riemannian manifold.

The metric cone $C_{0,1}(M^m)$ is defined by

$$3.1) \qquad C_{o,1}(M^m) = (M^m \times (o,1),\ r^2 h + dr^2\ ) \qquad r \in (o,1)$$

Also, let us denote the cone with included vertex P by

$$3.2) \qquad C^*_{o,1}(M^m) = C_{o,1}(M^m) \cup \{P\}.$$

The distance of a point $(\underline{x}, r) \in C_{o,1}(M^m)$ from the vertex P is r. Moreover, we recall that two riemannian metrics g_1, g_2 are said to be quasi-isometric if there exists a constant $c > o$ such that

$$3.3) \qquad \frac{1}{c} g_1 \leqslant g_2 \leqslant c g_1$$

Now, let us pick up a point x in the interior of a j-dimensional simplex σ_J of T: it is known that there exists a compact manifold M^{n-J-1} such that x has a neighbourhood homeomorphic to

$$3.4) \qquad C^*_{o,1}(M^{n-J-1}) \times \overbrace{(o,1) \times (o,1) \ldots\ldots \times (o,1)}^{j\text{-times}}$$

Around x, the simplex σ_J is identified with $\{P\} \times (o,1) \times \ldots\ldots \times (o,1)$.

Let

$$3.5) \qquad h^J = C_{o,1}(M^{n-J-1}) \times \overbrace{(o,1) \times \ldots\ldots \times (o,1)}^{j\text{-times}}$$

with metric $r^2 h + dr^2 + dr_1^2 + \ldots\ldots + dr_J^2$

We call h^j a j-dimensional handle.

$X^n - T^{n-2}$ can be covered by a finite number of such handles, i.e.

$$3.6) \qquad X^n - T^{n-2} = \bigcup_{j=o}^{n-2} \bigcup_{r=1}^{r_j} h_r^j$$

where h_r^j is a j-dimensional handle.

Let g be a riemannian metric on $X^n - T^{n-2}$:
then <u>g is conical</u> if its restriction to each h_r^j is quasi-iso metric to the metric in 3.5).

A closed pseudomanifold X^n endowed with a conical metric g on $X^n - T^{n-2}$ is called a <u>riemannian pseudomanifold or space with conical singularities.</u>

A handle as in 3.5) is called admissible if $\left[n-j-1\right]$ is odd or

$$3.7) \qquad \dim H^{\frac{n-j-1}{2}}(M^{n-j-1}, \mathbb{R}) = o$$

where $H^p(M^m, \mathbb{R})$ denotes the p-th cohomology group of M^m with real coefficients.

A riemannian pseudomanifold X^n is said to be <u>admissible</u> if it admits a decomposition as in 3.6) by admissible handles.

The condition of a metric g being conical around T^{n-2} is a <u>local</u> condition; in many relevant cases, the topologically singular locus of X^n, that we call Σ, is much smaller than T^{n-2}, i.e. $\Sigma \subsetneq T^{n-2}$.

For instance, this situation occurs in the case of complex algebraic varieties: Σ is the locus of algebraic singularity.

In all such cases, it is sufficient to check that a metric g is conical around Σ.

In particular, let us suppose that Σ can be covered by neighbouroods of the form

3.8) $\quad C^{*}_{o,1}(M_1) \times M_2$

where

3.8') $\begin{cases} \text{i)} & (M_1,\ h_1),\ (M_2, h_2) \text{ are riemannian manifolds;} \\ \text{ii)} & \text{the restriction of the metric } g \text{ to a neighboorood} \\ & \text{as in 3.8) is quasi-isometric to } r^2 h_1 + dr^2 + h_2\ ; \\ \text{iii)} & \Sigma \text{ is identified with } \{P\} \times M_2 \\ \text{iv)} & \dim M_1 \text{ odd or } \quad \dim H^{\frac{\dim M}{2}}(M_1, \mathbb{R}) = 0 \end{cases}$

Then g is conical and the pseudomanifold is admissible; we will use these considerations in the sequel.

Now we recall some more properties of riemannian pseudomanifolds.

Let $\mathcal{H}^{i}_{(2)}(X^n)$, $IH_i(X^n)$ be respectively the i-th group of L^2-cohomology of X^n and the i-th group of intersection homology of X^n.

The L^2-cohomology of X^n is defined to be the cohomology of the L^2-differential forms on $X^n - T^{n-2}$.

Roughly, the intersection homology of X^n is the homology of a suitable subcomplex of simplicial chains.

L^2-cohomology and intersection homology provide a singular version of Hodge's and De-Rham's theorems:

<u>Hodge's theorem:</u>

Let X^n be a closed admissible riemannian pseudomanifold.

Let $H^i(X^n)$ be the group of L^2-harmonic differential i-forms on $X^n - T^{n-2}$.

Then

$$H^i(X^n) \simeq \mathcal{H}^i_{(2)}(X^n)$$

<u>De Rham's theorem:</u>

Let X^n be a closed admissible riemannian pseudomanifold .

Then

$$\mathcal{H}^i_{(2)}(X^n) \simeq \left[IH_i(X^n)\right]^*$$

Moreover, the pairing can be realised by integration of L^2-differential forms on intersection homology chains.

Now we actually start our program of identification between $X^n - T^{n-2}$ and an equivariant manifold in the sense of 1.5), sec. 1, Chapter II.

For the sake of clarity, we limit ourselves to describe the case when the equivariant manifold has the form

3.9) $\quad (M_1 \times M_2 \times (o, \frac{\pi}{2}),\ A_1^2(s)g_1 + A^2(s)g_2 + ds^2)$

where (M_i, g_i), i=1, 2, is a compact riemannian manifold of dimension m_i; $A_i(s)$, i=1, 2, are smooth positive functions on $(o, \frac{\pi}{2})$ such that

$$3.10)\quad \begin{cases} \lim\limits_{s\to o} A_1(s) = \lim\limits_{s\to\frac{\pi}{2}} A_2(s) = o \\ A_1(s) \text{ bounded and bounded away from zero as } s\to\frac{\pi}{2} \\ A_2(s) \text{ bounded and bounded away from zero as } s\to o \end{cases}$$

In fact this class of equivariant manifolds gives rise to the most interesting examples; however, the general case can be treated by similar methods.

The completion X of a manifold as in 3.9) can be realised in a standard way by adding a copy of M_2 for $s=o$, and a copy of M_1 for $s=\frac{\pi}{2}$.

Analogously to the case of the join of two spheres, we call focal varieties the added copies of M_1, M_2.

X can be given a triangulation T in such a way that the union of the two focal varieties, $M_1 \cup M_2$, is contained in T^{n-2}, $n = m_1 + m_2 + 1$.

The singular locus Σ is contained in $M_1 \cup M_2$.

Then, according to 3.8'), X is a riemannian pseudomanifold provided that

$$3.11)\quad \begin{cases} A_1(s) = \mathcal{O}(s) \\ A_2(s) = \mathcal{O}(\frac{\pi}{2}-s) \end{cases}$$

And X is admissible provided that

3.12) $\dim \ H^{\frac{m_i}{2}}(M_i, \mathbb{R}) = 0 \qquad i = 1, 2 .$

Let

$$(N_1 \times N_2 \times (0,\tfrac{\pi}{2}),\ B_1^2(s)\ h_1 + B_2^2(s)\ h_2 + ds^2)$$

be an equivariant manifold as in 3.9), 3.10), and let Y be its completion as above.

We will be interested in maps Φ of the form

3.13)
$$\begin{aligned} \Phi : M_1 \times M_2 \times (0,\tfrac{\pi}{2}) &\longrightarrow N_1 \times N_2 \times (0,\tfrac{\pi}{2}) \\ (x_1,\ x_2,\ s) &\rightsquigarrow (\phi_1(x_1), \phi_2(x_2),\ s) \end{aligned}$$

where ϕ_i is harmonic with constant energy density $e(\phi_i)$, $i = 1, 2$.

By following the reduction equation 1.9), sec. 1, Chapter II, Φ is harmonic if and only if

3.14)

$$\frac{A_1'(s)}{A_1(s)}\ m_1 + \frac{A_2'(s)}{A_2(s)}\ m_2 - \frac{2\,e(\phi_1)\ B_1'(s)\ B_1(s)}{A_1^2(s)} - \frac{2\,e(\phi_2)\ B_2'(s)\ B_2(s)}{A_2^2(s)} = 0$$

In particular, if we choose

$$3.15)\qquad \begin{aligned} A_1(s) &= \sqrt{\frac{2e(\phi_1)}{m_1}}\quad B_1(s) \\ A_2(s) &= \sqrt{\frac{2e(\phi_2)}{m_2}}\quad B_2(s) \end{aligned}$$

then 3.14) is satisfied.

And, as a consequence of 1.10), sec. 1, Chapter II, we have

$$3.16)\qquad e(\Phi) = \frac{m_1 + m_2 + 1}{2}$$

Moreover, it is clear that Φ can be extended to a continuous map $\Phi^*: X \to Y$: just send the focal variety $M_i^{\cdot}$ into the focal variety $N_i^{\cdot}$ by using ϕ_i, $i = 1, 2$.

If we fix

$$3.17)\qquad \begin{aligned} B_1(s) &= \sin s \\ B_2(s) &= \cos s \end{aligned}$$

then conditions i), ii), iii) in 3.8) are fulfilled: thus X, Y are closed riemannian pseudomanifolds, with singular locus Σ contained in the union of the two focal varieties. Each of these spaces is admissible if and only if condition 3.12) is satisfied; in particular, this always happens when m_i, n_i, are odd.

In conclusion, we have used equivariant theory to construct continuous maps $\Phi^*: X \to Y$ between riemannian pseudomanifolds which are harmonic where the notion makes sense, i.e. where the riemannian metrics are defined.

Moreover, our harmonic maps are given explicitely in terms of data on the cross sections and have constant energy density.

In general, both X and Y in the above construction are singular spaces.

However, it is worth pointing out the following particular cases:

Case I:

$$N_1 = S^{n_1}$$

$$N_2 = S^{n_2} \qquad M_2 = S^{m_2} \qquad B_i(s) \text{ as in 3.17)}$$

Case II:

$$N_1 = S^{n_1} \qquad M_1 = S^{m_1}$$

$$N_2 = S^{n_2} \qquad M_2 = S^{m_2} \qquad B_i(s) \text{ as in 3.17)}$$

In the case I, the range is the standard $S^{n_1+n_2+1}$; the domain is homeomorphic to the $(m_2 + 1)$-suspension of M_1.

For instance, it is known that, if M_1 is a compact homogeneous space, then there exists a map $\phi_1 : M_1 \longrightarrow S^{n_1}$ to which our construction apply; moreover, if $n_2 = m_2$ and $\phi_2 = \text{id.}$, then Φ^* is the $(m_2 + 1)$-suspension of the map ϕ_1.

In the case II, we have our usual join of maps between spheres: in this case, the singularity of the domain on the focal varieties concerns only the metric, not the topology.

In the particular case

$$\phi_1 = i_K : S^1 \longrightarrow S^1 , \qquad \phi_2 = \text{id.}: S^m \longrightarrow S^m$$

the metric has a conical singularity on the domain S^{m+2} on a locus which is homeomorphic to S^m ; outside of the singular locus S^m, our maps $\Phi^* : S^{m+2} \longrightarrow S^{m+2}$ are riemannian coverings of degree K.

<u>**Remark:**</u>

i) In the case of maps between spheres, the singularity of the metric can be shifted to the range by replacing 3.17) with

3.17')
$$B_1(s) = \sqrt{\frac{m_1}{2e(\phi_1)}}\ \sin s$$
$$B_2(s) = \sqrt{\frac{m_2}{2e(\phi_2)}}\ \cos s$$

ii) For the purposes of this section, we could have admitted horn-like metrics as well, i.e. in 3.11) we could have had

3.11')
$$A_1(s) = \mathcal{O}(s^c) \qquad c > 0$$
$$A_2(s) = \mathcal{O}\left(\left(\frac{\pi}{2} - s\right)^c\right) \qquad c > 0.$$

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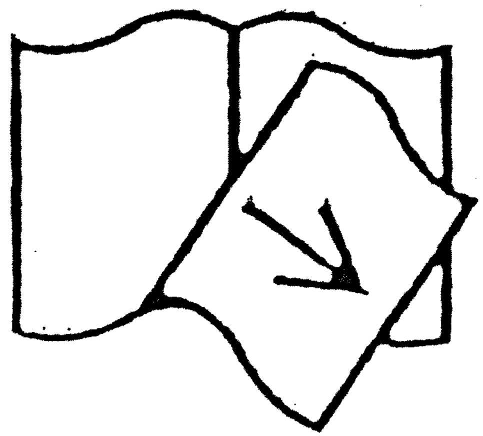

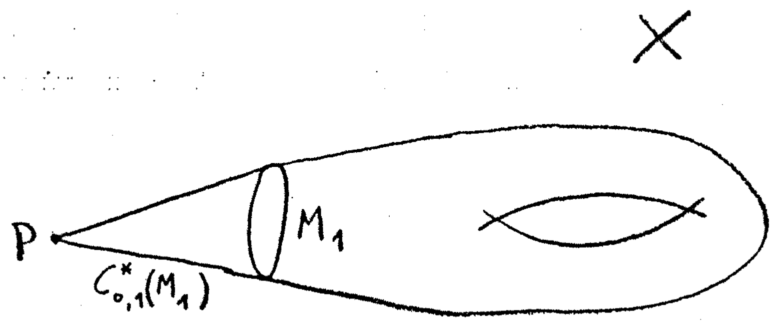

Section 4:

Metrics with cone-like singularities are not the only suitable ones for constructing harmonic maps:

Example: it is well-known that every harmonic map

$\Phi : S^n \longrightarrow S^1$, $n \geq 2$, is constant.

Let us write $S^n - \{\text{two poles}\}$ as $S^{n-1} \times (0,\pi)$, and consider

$$\phi : S^{n-1} \times (0,\pi) \longrightarrow S^1$$
$$(x \ , \ s) \rightsquigarrow 2s$$

Then ϕ extends to a continuous map $\Phi : S^n \longrightarrow S^1$ that maps the two poles into the same point of S^1.

It is easy to check that Φ is harmonic with respect to the metric

4.1) $\quad \sin^2 s\, g_1 + \sin^{2(n-1)} s \quad ds^2$

The metric in 4.1) has a singularity in the two poles of S^n that ∴ is not of conical type.

Section 5:

We will use the following trigonometrical identities: let $p \in \mathbb{N}$, $p \geq 2$. We have

5.1) $$\sum_{i=1}^{p} \sin^2\left(s + (i-1)\frac{\pi}{p}\right) = \frac{p}{2} \qquad s \in \mathbb{R}$$

5.2) $$\sin(ps) = \sqrt{2p} \prod_{i=1}^{p} \sin\left(s + (i-1)\frac{\pi}{p}\right) \qquad s \in \mathbb{R}$$

5.3) $$\operatorname{cotg}(ps) = \frac{1}{p} \sum_{i=1}^{p} \operatorname{cotg}\left(s + (i-1)\frac{\pi}{p}\right) \qquad s \in \left(0, \frac{\pi}{p}\right).$$

Let $S^m(a)$ be the standard m-sphere of radius a, $S^m = S^m(1)$.

Let $M^* \subseteq \mathbb{R}^{(n+1)p}$ be the subset parametrized as follows:

5.4) $$M^* \subset \overbrace{\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \times \cdots\cdots \times \mathbb{R}^{n+1}}^{\text{p-times}}$$
$$\cup$$
$$x = \sin s\, x_1 + \sin\left(s + \frac{\pi}{p}\right) x_2 + \cdots\cdots + \sin\left(s + (p-1)\frac{\pi}{p}\right) x_p$$

where $x_i \in S^n$, $i = 1 \dots p$, $s \in \left[o, \frac{\pi}{p} \right]$

It follows immediately from 5.1) that

5.5) $$M^* \subseteq S^{(n+1)p-1} \left(\sqrt{\frac{p}{2}}\right)$$

Let $M \subset M^*$ be the subset of M^* corresponding to the points for which $s \neq o$, $s \neq \frac{\pi}{p}$.

Then M is isometric to

$$\overbrace{S^n \times S^n \times \dots \dots \times S^n}^{\text{p-times}} \times \left(o, \frac{\pi}{p}\right) \quad \text{with metric}$$

5.6)

$$\sin^2 s\, g_1 + \sin^2\left(s+\frac{\pi}{p}\right)g_2 + \dots\dots + \sin^2\left(s+(p-1)\frac{\pi}{p}\right)g_p + ds^2,$$

where g_i is the standard metric on the i-th factor S^n.

A simple change of coordinates in 5.4) around $s=o$ (or $s=\frac{\pi}{p}$) enables us to state:

i) if $p = 2$, then $M^* = S^{2(n+1)-1}$

ii) if $p > 2$, then the metric in 5.6) is of class C^o, <u>but not C^1</u>, across the loci $s = o$, $s = \frac{\pi}{p}$.

The codimension of M^* in $S^{(n+1)p-1}\left(\sqrt{\frac{p}{2}}\right)$ is $(p - 2)$.

Now we can propose a generalisation of the standard Hopf construction.

$$\text{Let} \qquad F_p : \overbrace{S^n \times S^n \times \dots \times S^n}^{\text{p-times}} \longrightarrow S^r \text{ be a map}$$

which is harmonic with constant energy density $\frac{n}{2}$ in each variable separately.

If $p = 2$, then orthogonal multiplications provide many examples of such maps (see sec. 11 Chapter II).

If $p > 2$, examples can be constructed as follows:

Let $F : S^n \times S^n \longrightarrow S^q$, $G : S^q \times S^n \longrightarrow S^r$ be two orthogonal multiplications. Then define

$$F_3 : S^n \times S^n \times S^n \longrightarrow S^r$$
$$(x_1, x_2, x_3) \rightsquigarrow G(F(x_1, x_2), x_3)$$

Then F_3 has the required properties, and the construction can be iterated. For instance, by using complex, quaternionic and Cayley numbers multiplications, we have examples of F_p as above for $n = r = 1, 3, 7$ and every $p \geq 2$.

We write the standard S^{r+1} as $\left(S^r \times (0,\pi), \sin^2 s\, g_1 + ds^2\right)$.

Then we define

5.7) $$\phi : M \longrightarrow S^{r+1}$$
$$(x, s) \rightsquigarrow (F_p(x), ps)$$

where M is the manifold in 5.6).

We show that

i) the map ϕ has a continuous extension $\phi^{*}: M^{*} \longrightarrow S^{2+1}$

ii) ϕ is harmonic with constant energy density $e(\phi) = \frac{p^2(n+1)}{2}$

The statement i) is obvious: just send the locus $s = o$ of M^{*} into the north pole of S^{2+1}, and the locus $s = \frac{\pi}{p}$ into the south pole.

As for ii), we write down the condition of harmonicity and the expression of $e(\phi)$ according to 1.9), 1.10) of sec. 1, Chapter II.

The above mentioned equation 1.9) gives

5.8)

$$\left[\sum_{i=1}^{p} \operatorname{cotg}\left(s+(i-1)\frac{\pi}{p}\right)\right] np - \sum_{i=1}^{p} \frac{n \sin(ps)\cos(ps)}{\sin^2\left(s+(i-1)\frac{\pi}{p}\right)} = o$$

$$s \in \left(o, \frac{\pi}{p}\right)$$

And for the energy density we have

5.9)

$$e(\phi) = \left[\sum_{i=1}^{p} \frac{n \sin^2(ps)}{\sin^2\left(s+(i-1)\frac{\pi}{p}\right)}\right] + \frac{1}{2} p^2$$

Now 5.8) can be checked easyly by using 5.1), 5.2), 5.3).

And from 5.1), 5.2) follows that $e(\phi) = \frac{p^2(n+1)}{2}$.

We notice that, in the case p=2, we have just reproduced the well-known polynomial maps of spheres arising from the standard Hopf construction.

In the general case $p > 2$, we have again continuous maps from a singular space to a sphere; such maps are harmonic with constant energy density where these notions make sense.

The emphasis with this example lays on the non-trivial trigonometrical identities that rule harmonicity and energy density.

One could substitute the metric in 5.6) by a metric of the form

5.10) $$\sum_{i=1}^{p} A_i^2(s)\, g_i + h^2(s)\, ds^2$$

If the function $A_i(s)$, $h(s)$ are suitable chosen, M^* is a smooth manifold.

Moreover, a suitable choice of $\alpha(s) \neq ps$ would lead to harmonic maps $\phi^*: M^* \to S^{q+1}$ of <u>non</u>-constant energy density: this can be technically realised by using the equivariant methods of Chapter II (Theorem 2.3.1., 2.3.2.). In this case everything would be smooth, but M^* would not be any longer an explicit subset of an euclidean sphere.

PART 3: Further possible developments of equivariant theory.

There are many problems which are naturally related to the construction of harmonic maps of spheres and the equivariant theory of Chapter I, II: we mention

i) classification of harmonic homogeneous polynomials of spheres;

ii) classification of maps $F : S^p \times S^q \to S^r$ which are harmonic with constant energy density in each variable separately; and homotopy classification of maps $H : S^{p+q+1} \to S^{r+1}$ obtained from such F via the Hopf construction;

iii) computation of the Morse index of the non-polynomial harmonic maps of spheres that we produced in Chapters I, II;

iv) construction of non-polynomial harmonic maps of spheres without using equivariant theory;

v) applicability of equivariant theory to construct maps between complex projective spaces.

Another very natural question is the following: which is the right position of equivariant methods into a general setting of calculus of variations ? In connection with this question, in the following section we indicate three possible directions of further development.

Section 6

I): Let $f: [0,+\infty) \to [0, +\infty)$ be a smooth function, that for simplicity we assume to be strictly increasing.

Define a functional on maps $\phi : M \longrightarrow N$ by

$$6.1) \qquad E_f(\phi) = \int_M f(e(\phi)) \, dv_M$$

This functional coincides with the energy functional in the case when $f(x) = x$; it is a conformal invariant of M in the case when $f(x) = x^{\frac{\dim M}{2}}$.

Equivariant theory applies to the class of functionals 6.1): in fact, the Euler-lagrange system of equations associated with the functional 6.1) is given by:

$$6.2) \qquad d^* \left[f' \, (e(\phi)) \, d\phi \right] = o$$

As an immediate consequence of 6.2), we have

<u>**Proposition 3.6.1.**</u>

Suppose that $e(\phi)$ = constant: then ϕ is an extremal of the functional 6.1) if and only if ϕ is a harmonic map.

In local charts, 6.2) is given by

6.3)

$$f' \, (e(\phi)) \, \tau^J(\phi) + f'' \, (e(\phi)) \, g^{st} \frac{\partial e(\phi)}{\partial x_t} \, \frac{\partial \phi^J}{\partial x_s} = o$$

$j = 1 \, \ldots\ldots \, \dim N.$

where x_i are coordinates on M and $\tau^J(\phi)$ is the usual tension field of ϕ.

Now we have

<u>Proposition 3.6.2.:</u>

Let $\Phi : M \times (a,b) \longrightarrow N \times (c,d)$

$(x\ ,\ s) \rightsquigarrow (\phi(x), \alpha(s))$

be an equivariant map as in Chapter II, sec. 1.

Then Φ is an extremal of the functional 6.1) if and only if the following ordinary differential equation is satisfied:

6.4)

$$f'(e(\Phi)(s))\,G_{\alpha}(s) + \frac{f''(e(\Phi)(s))}{h^{2}(s)}\,\frac{\partial e(\Phi)(s)}{\partial s}\,\alpha'(s) = 0$$

where h(s) is as in equation 1.9) sec. 1, Chapter II; $G_{\alpha}(s)$ is the left-hand side member of the above mentioned equation 1.9); and $e(\Phi)(s)$ is given by 1.10), sec. 1, Chapter II.

The proof is just a straight forward computation based on the formula 6.3) and Proposition 3.6.1.

As an application of proposition 3.6.2., one could study, for instance, the join and Hopf construction for functionals as in 6.1): the problem is that the appearance of the term $\frac{\partial e(\Phi)}{\partial s}(s)$ in the relevant equation makes things more difficult: in fact, the relevant equation doesn't appear to have a clear physical interpretation any longer.

Another straightforward computation leads to the formula for the second variation for extremals of the functional 6.1):

6,5) $$H^{f}_{\phi}(v, w) = \int_M \langle T^{f}_{\phi} v, w \rangle \, dv_M$$

where

6,6) $$T^{f}_{\phi} v = f'(e(\Phi)) J_{\phi} v + d^{*}\left[(f''(e(\phi))\langle dv, d\phi\rangle) d\phi\right]$$

where J_{ϕ} is the usual second variation operator.

In the case of extremals of 6.1) as in Proposition 3.6.1., the term $d^{*}\left[f''(e(\phi))\langle dv, d\phi\rangle) d\phi\right]$ vanishes; thus, in this case, the spectrum of T^{f}_{ϕ} equals the spectrum of J_{ϕ} up to a constant factor.

In the general case, the choice of a convex function f(x) seems to increase the stability of extremals.

<u>II):</u>

Equivariant theory can be naturally extended to include manifolds with metric of (m,n)-signature .

We follow the notations of Chapter II, sec. 1: let M x (a,b) be an equivariant manifold with metric given by

6.7) $$\sum_{j=1}^{P} A^{2}_{J}(s)\, g_{J} + h^{2}(s)\, ds^{2}$$

We now consider the more general case in which 6.7) is replaced by

6.8) $$\sum_{j=1}^{p_1} A_J^2(s)\, g_J - \sum_{j=p_1+1}^{p} A_J^2(s)\, g_J \pm h^2(s)\, ds^2$$

The metric in 6.8) has clearly signature of type (m,n).

After the introduction of metrics as in 6.8), one can define equivariant maps precisely as in Chapter II, sec. 1; we simply admit that the functions $A_J(s)$, $B_J(s)$, $h(s)$, $k(s)$ in the above mentioned equation 1.9) may be multiplied by a constant factor i, with $i^2 = -1$. Now the condition of harmonicity can be obtained from the reduction equation 1.9), sec. 1, Chapter II: the presence of the constant factors i influences in an obvious way the sum

6.9) $$\frac{h^2(s)}{K^2(\alpha(s))} \sum_{j=1}^{p} \frac{2e(\phi_j)\, B_{i_J}(\alpha(s))\, B'_{i_J}(\alpha(s))}{A_j^2(s)}$$

which appears in the reduction equation; the remaining terms of the reduction equation are unaffected.

<u>Example:</u>

Let $(\mathbb{R}^{p+1,q+1}, g)$ be the (p+q+2)-dimensional euclidean space with metric

6.10) $$g = \sum_{i=1}^{p+1} dx_i^2 - \sum_{i=p+2}^{p+q+2} dx_i^2$$

Then

$$S^{p,q+1} \overset{\text{def.}}{=} \left\{ x \in \mathbb{R}^{p+1,q+1} \;/\; \langle x, x\rangle = 1 \right\}$$

$S^{p,q+1}$, $q > o$, is isometric to

6.11) $$(S^p \times S^q \times [o,+\infty), \; h)$$

where the metric is h given by

6.12) $$\cosh^2 s \, g_1 - \sinh^2 s \, g_2 - ds^2$$

with $s \in [o,+\infty)$, g_1, g_2 standard metrics on S^p, S^q.

The metric h has signature of type $(p, q+1)$.

The locus $s=o$ is a focal variety homeomorphic to S^p.

$S^{p,1}$ is isometric to

6.13) $$(S^p \times \mathbb{R}, \; \cosh^2 s \; g_1 - ds^2)$$

By using homogeneous harmonic polynomials of spheres, it is easy to produce examples of equivariant maps between equivariant manifolds of type $(p,q+1)$ as above.

For instance, let $\Phi : S^{p,1} \longrightarrow S^{r,1}$ be of the form

6.14) $$\begin{array}{ccc} S^p \times \mathbb{R} & \longrightarrow & S^r \times \mathbb{R} \\ (x, \; s) & \rightsquigarrow & (\phi(x), \alpha(s)) \end{array}$$

Where $\phi : S^p \longrightarrow S^r$ is a harmonic homogeneous polynomial.

Then Φ is harmonic if and only if

$$6.15)\qquad \alpha''(s) + \frac{\sinh s}{\cosh s}\, p\,\alpha'(s) - \frac{e(\phi)\,\sinh\,(2\alpha(s))}{\cosh^2 s} = 0 \qquad s \in \mathbb{R}.$$

In general, by comparing the differential equations arising in this context with the pendulum equations of maps between standard spheres, it appears that, roughly, terms of the form sinh s, cosh s replace terms of the form sin s, cos s; we also notice that these new equations have a strong similarity with equations arising by equivariant maps between hyperbolic spaces as described in (B1).

One of the most natural problems in this context is to investigate the existence of global solutions (G1), (G2): in this direction, also equivariant deformations of metrics can be used: let us suppose that the metric 6.13) on $S^{p,1}$ is replaced by

$$6.16)\qquad \cosh^2 s\; g_1 - h^2(s)\; ds^2 .$$

It is not difficult to show that a suitable choice of the function h(s) insures the existence of global solutions.

III):

The most substantial feature of equivariant theory is to distinguish an horizontal section, which is filled with homogeneous data, and a vertical section, i.e. the join parameter upon which the relevant ordinary differential equation depends.

It is possible to carry out any number n of simultaneous joins: in this case the condition of harmonicity reduces to an elliptic system compound by n equations.

However, when applied to concrete problems, the previous statement meets severe computational difficulties; therefore, rather than developping the general theory, we limit ourselves to give an example.

Let $\phi_i : S^{p_i} \longrightarrow S^{q_i}$ be a harmonic homogeneous polynomial of spheres, and $\lambda_i = 2\,e(\phi_i)$, i=1 .... 4.

One can define

$$6.17) \qquad \Phi : S^{p_1+p_2+p_3+p_4+3} \longrightarrow S^{q_1+q_2+q_3+q_4+3}$$

as follows:

Let g_i, h_i, be the standard metrics on S^{p_i}, S^{q_i}, i=1 .....4.

We write $S^{p_1+p_2+p_3+p_4+3}$ as

$$6.18) \quad \underset{(x_1,\ x_2,\ s)}{\left\{S^{p_1} \times S^{p_2} \times \left[0,\frac{\pi}{2}\right]\right\}} \times \underset{(x_3,\ x_4,\ t)}{\left\{S^{p_3} \times S^{p_4} \times \left[0,\frac{\pi}{2}\right]\right\}} \times \underset{u}{\left[0,\frac{\pi}{2}\right]}$$

where $x_i \in S^{p_i}$, $0 \leq s, t, u \leq \dfrac{\pi}{2}$.

with metric

6.19)

$$\sin^2 u\left[\sin^2 s\ g_1+\cos^2 s\ g_2+ds^2\right]+\cos^2 u\left[\sin^2 t\ g_3+\cos^2 t\ g_4+dt^2\right]+du^2.$$

Analogously, one writes the range $S^{q_1+q_2+q_3+q_4+3}$ as

6.20) $$\left\{S^{q_1}\times S^{q_2}\times\left[0,\frac{\pi}{2}\right]\right\}\times\left\{S^{q_3}\times S^{q_4}\times\left[0,\frac{\pi}{2}\right]\right\}\times\left[0,\frac{\pi}{2}\right]$$

and expresses Φ in 6.17) by

6.21)

$$\left[\left(\phi_1(x_1),\phi_2(x_2),\alpha_1(s,t,u)\right),\left(\phi_3(x_3),\phi_4(x_4),\alpha_2(s,t,u)\right),\alpha_3(s,t,u)\right]$$

Φ is harmonic if and only if the functions $\alpha_1,\alpha_2,\alpha_3$ satisfy an elliptic system compound by 3 equations, with boundary conditions as follows:

6.22) $$\begin{cases}\lim\limits_{s\to 0}\alpha_1(s,t,u)=0\\[2ex]\lim\limits_{s\to\frac{\pi}{2}}\alpha_1(s,t,u)=\dfrac{\pi}{2}\end{cases}\qquad 0<t,u<\frac{\pi}{2}$$

6.23) $$\begin{cases}\lim\limits_{t\to 0}\alpha_2(s,t,u)=0\\[2ex]\lim\limits_{t\to\frac{\pi}{2}}\alpha_2(s,t,u)=\dfrac{\pi}{2}\end{cases}\qquad 0<s,u<\frac{\pi}{2}$$

$$6.24)\quad \begin{cases} \lim\limits_{u \to 0} \alpha_3(s,t,u) = 0 \\ \\ \lim\limits_{u \to \frac{\pi}{2}} \alpha_3(s,t,u) = \frac{\pi}{2} \end{cases} \qquad 0 < s,t < \frac{\pi}{2}$$

The boundary conditions in 6.22), 6.23), 6.24) make sure that Φ in 6.17) extends continuously across the loci $s=0$, $t=0$, $u=0$, $s=\frac{\pi}{2}$, $t=\frac{\pi}{2}$, $u=\frac{\pi}{2}$.

The complement of these loci is an open dense subset on which one should study the relevant system of equations.

The system is of second order, elliptic, semilinear, compound by 3 equations for the functions $\alpha_1, \alpha_2, \alpha_3$.

A direct computation leads to the explicit expression for these three equations: they are given in the next page.

At the moment we do not know how to study effectively such systems.

First equation:

$$\left[\frac{\partial^2\alpha_1}{\partial s^2}\,\frac{1}{\sin^2 u}+\frac{\partial^2\alpha_1}{\partial t^2}\,\frac{1}{\cos^2 u}+\frac{\partial^2\alpha_1}{\partial u^2}\right]+\left\{\frac{1}{\sin^2 u}\,\frac{\partial\alpha_1}{\partial s}\cdot\right.$$

$$\cdot\left[\frac{\cos s}{\sin s}\,P_1-\frac{\sin s}{\cos s}\,P_2\right]+\frac{1}{\cos^2 u}\cdot\frac{\partial\alpha_1}{\partial t}\cdot\left[\frac{\cos t}{\sin t}\,P_1-\frac{\sin t}{\cos t}\,P_2\right]+$$

$$\left.\frac{\partial\alpha_1}{\partial u}\cdot\left(P_1+P_2+1\right)\cdot\left[\frac{\cos u}{\sin u}-\frac{\sin u}{\cos u}\right]\right\}-\frac{1}{\sin^2 u}\cdot\left[\frac{\lambda_1\cos\alpha_1\,\sin\alpha_1}{\sin^2 s}+\right.$$

$$\left.-\frac{\lambda_2\cos\alpha_1\sin\alpha_1}{\cos^2 s}\right]+\frac{\cos\alpha_3}{\sin\alpha_3}\left[\frac{1}{\sin^2 u}\cdot\frac{\partial\alpha_3}{\partial s}\cdot\frac{\partial\alpha_1}{\partial s}+\frac{1}{\cos^2 u}\,\frac{\partial\alpha_3}{\partial t}\,\frac{\partial\alpha_1}{\partial t}+\right.$$

$$\left.\frac{\partial\alpha_3}{\partial u}\,\frac{\partial\alpha_1}{\partial u}\right]=0$$

Second equation:

Just replace $\alpha_1, P_1, P_2, \dfrac{\lambda_1}{\sin^2 s}, \dfrac{\lambda_2}{\cos^2 s}$ in the previous equation with respectively

$$\alpha_2, P_3, P_4, \frac{\lambda_3}{\sin^2 t}, \frac{\lambda_4}{\cos^2 t}$$

Third equation:

$$\left[\frac{\partial^2\alpha_3}{\partial s^2}\frac{1}{\sin^2 u}+\frac{\partial^2\alpha_3}{\partial t^2}\frac{1}{\cos^2 u}+\frac{\partial^2\alpha_3}{\partial u^2}\right]+\left\{\frac{1}{\sin^2 u}\,\frac{\partial\alpha_3}{\partial s}\cdot\left[\frac{\cos s}{\sin s}p_1+\right.\right.$$

$$\left.-\frac{\sin s}{\cos s}p_2\right]+\frac{1}{\cos^2 u}\,\frac{\partial\alpha_3}{\partial t}\left[\frac{\operatorname{cost}}{\operatorname{sint}}p_3-\frac{\operatorname{sint}}{\operatorname{cost}}p_4\right]+\frac{\partial\alpha_3}{\partial u}\left[(p_1+p_2+1)\,\frac{\cos u}{\sin u}+\right.$$

$$\left.\left.-(p_3+p_4+1)\,\frac{\sin u}{\cos u}\right]\right\}-\cos\alpha_3.\ \sin\alpha_3\left\{\left[\frac{\lambda_1\sin^2\alpha_1}{\sin^2 u\,\sin^2 s}+\frac{\lambda_2\cos^2\alpha_1}{\sin^2 u\,\cos^2 s}+\right.\right.$$

$$\left.-\frac{\lambda_3\sin^2\alpha_2}{\cos^2 u\,\sin^2 t}-\frac{\lambda_4\cos^2\alpha_2}{\cos^2 u\,\cos^2 t}\right]+\left[\left(\frac{\partial\alpha_1}{\partial s}\right)^2\frac{1}{\sin^2 u}+\left(\frac{\partial\alpha_1}{\partial t}\right)^2\frac{1}{\cos^2 u}+\right.$$

$$\left.\left.+\left(\frac{\partial\alpha_1}{\partial u}\right)^2-\left(\frac{\partial\alpha_2}{\partial s}\right)^2\frac{1}{\sin^2 u}-\left(\frac{\partial\alpha_2}{\partial t}\right)^2\frac{1}{\cos^2 u}-\left(\frac{\partial\alpha_2}{\partial u}\right)^2\right]\right\}=0$$

We notice that, in the case $\lambda_i = p_i$, $i=1 \;.....\; 4$, the solution is given explicitely by

$$\begin{cases}\alpha_1\,(s,t,u) = s\\ \alpha_2\,(s,t,u) = t\\ \alpha_3\,(s,t,u) = u\end{cases}$$

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