Convergence to agreement and the role of public information [1]

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Abstract

When states of the world are normally distributed, the sequential exchange and revision of beliefs converges to agreement in finitely many rounds of communication. Public information may reduce the information shared by individuals after the revision of beliefs.

Key words: information; revision of beliefs; common knowledge.

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Publicly provided information impacts the information exchanged and shared by individuals.

Better informed individuals, indeed, make better decisions. This is the case at least as long as individuals conform to Savage (1954) and expected utility maximization; Blackwell (1951) made the point. But, the argument does not extend to collective decision making either in competitive markets or in strategic games. In Hirshleifer (1971), absent insurance against adverse contingencies, individuals would be better off trading before information had been revealed. In Morris and Shin (2002), privately informed agents with complementary actions and kin to co-ordinate might have been better off interacting without public disclosures of information about fundamentals. Even if counter-intuitive at first, these results are well known and understood.

The point here is different: public information may affect adversely the information available to individuals following the exchange of beliefs; importantly, this may occur in the absence of strategic or equilibrium considerations. It may do so by limiting if not eliminating the ability of individuals to learn from the beliefs of others.

The communication of beliefs without reference to optimization or strategic considerations was the focus of Aumann (1976). There, under the simplifying assumption of countable information partitions whose join consists of non-null events, common knowledge was formalized and characterized : if the beliefs of individuals are common knowledge, they must coincide, and individuals cannot "agree to disagree."

The argument Aumann (1976) made no reference to the revision of beliefs following the announced beliefs of their interlocutors. This was the focus of Geanakoplos and Polemarchakis (1982): the sequential communication and revision of beliefs converges to common knowledge and agreement.

Concerning the pattern of beliefs on the path to common knowledge and agreement, Geanakoplos and Polemarchakis (1982) provided an example of a path in which beliefs remain ostensibly unchanged until, "suddenly" they jump to coincide. Indeed, Polemarchakis (2012) demonstrated that the path of convergence to agreement is arbitrary: any dialog between bayesian interlocutors is rational – as long as it leads, eventually, to agreement.

Concerning the information available to individuals at common knowledge and agreement, Geanakoplos and Polemarchakis (1982) made a point, through an example: though common knowledge beliefs must coincide, they need not coincide with the beliefs supported by the pooled information of individuals.

Here, first, we consider the case, common in a wide literature, of states of the world that are normally distributed, and we demonstrate convergence to agreement in finitely many rounds of communication. Subsequently, we consider the role of public information on the information shared at common knowledge. As in Dutta and Polemarchakis (2012), we show that public information may eliminate the ability of individuals to learn from others: at common knowledge, following the exchange and revision of beliefs, each individual knows less than he would have known had there been no public information.

The set up Uncertainty is described by a probability space, $\{\Omega, \Sigma, \mu\}$, where Ω is a set of states of the world, Σ is a σ-field of events, $S \in \Sigma$, and μ is a probability measure.

Individual $i = 1$ observes the realization of a random variable x^1, his private information, while individual $i = 2$ observes the realization of a random variable x^2. Of concern is the realization of a random variable y. The distribution of the random variable $w = (x^1, x^2, y)$ is common knowledge; in particular, individuals share common prior beliefs.

Stages of communication are $t = 1, \ldots$; individual 1 speaks at $t = 3, 5, \ldots$, while individual 2 speaks at $t = 2, 4, \ldots$. At $t = 1$, individual 1, informed of the realization of x^1, announces $y_1 = \mathrm{E}(\mathrm{y}|\mathrm{x}^1)$. Then, at $t = 2$, individual 2 announces $y_2 = \mathrm{E}(\mathrm{y}|\mathrm{x}^2, \mathrm{E}(\mathrm{y}|\mathrm{x}^1))$. Subsequently, at $t = 3, 5, \ldots$, individual 1 announces $y_t = \mathrm{E}(\mathrm{y}|\mathrm{x}^1, \mathrm{y}_2, \ldots \mathrm{y}_{\mathrm{t}-2})$, while, at $t = 4, 6, \ldots$, individual 2 announces $y_t = \mathrm{E}(\mathrm{y}|\mathrm{x}^2, \mathrm{y}_1, \ldots \mathrm{y}_{\mathrm{t}-2})$. Communication ends when no further revision would occur: the beliefs of individuals are common knowledge.

Remark (Aumann)*.* The private information of an individual is described by the random variable x^i, whose realization he observes; or by the σ-field, Σ^i, that x^i generates: the coarsest σ-field with respect to which the function $x^i : \Omega \to \mathbb{R}^{\dim x^i}$ is measurable. As we mentioned earlier, Aumann (1976) restricted attention to countable information partitions, and, in particular, with the property that the join (the coarsest common refinement) of the information partitions that describe the private information of individuals, $\vee_i \Sigma^i$, consists of non-null events. At a state of the world $\omega^* \in \Omega$ or, equivalently, at an event $S^* \in \vee_i \Sigma^i(\omega^*)$, an individual knows an event, $E \in \Sigma$, if it contains the element of his partition of which he is informed: $\Sigma^i(\omega^*) \subset E$. The event E is common knowledge at ω^* if it contains the associated element of the meet (finest common coarsening) of the individual partitions: $\wedge \Sigma^i(\omega^*) \subset E$.

The formal definition captures the situation in which individual i knows that $(3-i)$ knows that i knows that ... that $i/(3-i)$ knows E. It suffices to observe that any event in the meet can be reached from any element of the join through a sequence of alternating intersecting elements of the partitions of the individuals. In this case, $\mathrm{E}(\mathrm{y}|\mathrm{S}^1(\omega)) = \mathrm{E}(\mathrm{y}|\mathrm{S}^1(\omega^*))$ for every $\omega \in (\vee_i \Sigma^i)(\omega^*)$, and, as a consequence, $\mathrm{E}(\mathrm{y}|\mathrm{S}^1(\omega *)) = \mathrm{E}(\mathrm{y}|(\vee_\mathrm{i}\Sigma^\mathrm{i})(\omega^*))$; since the same holds for $i = 2$, agreement follows. Monderer and Samet (1989) maintained the restriction to countable information partitions and joins with non-null elements and extended the argument to common p-beliefs, for $p \in (0, 1]$. Brandenburger and Dekel (1987) extended the definition to information structures that do not satisfy the restriction to joins with non-null elements. Last, but not least, common knowledge is employed casually in the statement that the structure of the situation is common knowledge: Mertens and Zamir (1985), also Brandenburger and Dekel (1993), addressed this issue. Geanakoplos and Polemarchakis (1982) considered the sequential announcement and revision of beliefs in the framework of Aumann (1976), and they demonstrated convergence to common knowledge and agreement.

Convergence Random variable are normally distributed [1] and centered:

$$w = (y, x^1, x^2) \sim \mathcal{N}(0, V), \quad V = \begin{pmatrix} V_{0,0} & V_{0,1} & V_{0,2} \\ V_{1,0} & V_{1,1} & V_{1,2} \\ V_{2,0} & V_{2,1} & V_{2,2} \end{pmatrix}.$$

"Normal Bayesian dialogues" were considered in Bacharach (1985), but, only for the case of independent private information between individuals: $x^1 \perp x^2$ or $V_{1,2} = 0$. In this case, $\mathrm{E}(y|x^1, x^2) = \mathrm{E}(y|x^1) + \mathrm{E}(y|x^2)$ and, here, convergence is immediate; in Bacharach (1985), convergence was delayed because individuals did not communicate fully their conditional expectations.

If $\dim y = \dim x^1 = \dim x^2 = 1$, convergence is immediate. If $V_{1,1}$ or $V_{2,2} = 0$ or if $V_{0,1} = V_{0,2} = 0$, then, communication is pointless. If $V_{0,1} \neq 0$, then, y^1 identifies x^1, and, as a consequence, $y^2 = \mathrm{E}(y|x^1, x^2)$, at which point communication terminates with full exchange of information. If $V_{0,1} = 0$, while $V_{0,2} \neq 0$, then, communication terminates at $t = 3$, again with full exchange of information.

The case of interest obtains when $\dim x^1 \geq 2$ and/or $\dim x^2 \geq 2$; for simplicity and without loss of generality, $\dim y = 1$. Communication commences with

$$y_1 = a_1 x^1,$$

$$y_2 = a_2 x^2 + b_2 a_1 x^1;$$

[1] If a random variable, a, is normally distributed, $a \sim \mathcal{N}(\bar{a}, C)$, then, any linear transformation of the random variable, Aa, is normally distributed as well, and

$$Aa \sim \mathcal{N}(A\bar{a}, ACA').$$

If the random variables a^1 and a^2 are jointly normally distributed,

$$(a^1, a^2) \sim \mathcal{N}\left(\begin{pmatrix} \bar{a}^1 \\ \bar{a}^2 \end{pmatrix}, \begin{pmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{pmatrix}\right),$$

then, the conditional distribution $a^1|a^2$ is normal as well, and

$$(a^1|a^2) \sim \mathcal{N}((\bar{a}^1 + C_{1,2}C_{2,2}^{-1}(a_2 - \bar{a}^2)), (C_{1,1} - C_{1,2}C_{2,2}^{-1}C_{2,1})).$$

If the random variables a^1, a^2 and a^3 are jointly normally distributed, then, $\mathrm{E}(a^1|a^2, a^3) = \mathrm{E}(a^1|a^2) + \mathrm{E}(a^1|a^3)$ if and only if $a_2 \perp a^3$ or $\mathrm{Cov}(a^2, a^3) = 0$.

subsequently, at $t = 3, 5, \dots,$

$$y_t = a_t x^1 + b_t H_t x^2$$

where H_t is the matrix with rows $\{a_{t-1}, a_{t-2}, \dots, a_4, a_2\}$, and, at $t = 2, 4, \dots,$

$$y_t = a_t x^2 + b_t H_t x^1$$

where H_t is the matrix with rows $\{a_{t-1}, a_{t-2}, \dots, a_3, a_1\}$.

For $t = 1, 3, \dots,$ if $a_t \in [a_{t-2}, \dots, a_3, a_1]$, then $a_{t+1} = a_{t-1} \in [a_{t-1}, \dots, a_4, a_2]$, and, in turn, $a_{t+2} = a_t \in [a_{t-2}, \dots, a_3, a_1]$: no further revision of beliefs occurs.

Since, at $t = 1, 3, \dots$ communication terminates unless $a_t \notin [a_{t-2}, \dots, a_3, a_1]$, while $\dim[a_t, a_{t-2} \dots, a, a_3, a_1] \leq \dim x^1$, convergence to common knowledge occurs in finitely many steps.

Agreement At common knowledge, when communication ceases, the beliefs of individual 1 are

$$y^1 = a^1 x^1 + b^1 H^2 x^2,$$

while the beliefs of individual 2 are

$$y^2 = a^2 x^2 + b^2 H^1 x^1$$

and, importantly, $a^2 \in [H^2]$, and $a^1 \in [H^1]$.

If

$$\epsilon^1 = y - a^1 x^1 + b^1 H^2 x^2, \quad \text{and} \quad \epsilon^2 = y - a^2 x^2 + b^2 H^1 x^1,$$

then,

$$\epsilon^1 - \epsilon^2 = (a^1 - b^2 H^1) x^1 + (a^2 - b^1 H^2) x^2,$$

and

$$\text{Cov}(\epsilon^1, \epsilon^1 - \epsilon^2) = \text{Cov}(\epsilon^1, \epsilon^1 - \epsilon^2) = 0,$$

since

$$\epsilon^1 \perp \{x^1, H^2 x^2\} \quad \text{and} \quad \epsilon^2 \perp \{x^2, H^1 x^1\}.$$

As a consequence,

$$\text{Var}(\epsilon^1 - \epsilon^2) = 0$$

or

$$(a^1 - b^1 H^2) = 0 \quad \text{and} \quad (a^2 - b^2 H^1) = 0,$$

and agreement follows:

$$y^1 = y^2.$$

Obfuscation States of nature are $\omega \in \{1, \ldots, 5\}$, events are subsets of states of the world, and, according to the prior beliefs common to all individuals, states of the world occur with equal probability, $\mu(\{\omega\}) = 1/5$. Individual 1 observes the realization of the random variable $x^1 = (1, -1, 1, -1, -1)$, while individual 2 observes the realization of the random variable $x^2 = (1, 1, -1, -1, -1)$; equivalently, the information partition of individual 1 is $\{\{1, 3\}, \{2, 4, 5\}\}$, while the information partition of individual 2 is $\{\{1, 2\}, \{3, 4, 5\}\}$, and $\mathrm{E}(y|x^1) = (0, -1/3, 0, -1/3, -1/3)$, while $\mathrm{E}(y|x^2) = (0, 0, -1/3, -1/3, -1/3)$. Of concern is the realization of the random variable $y = (1, -1, -1, 1, -1)$.

At $x^1 = x^2 = y = 1$, the communication that ensues is $y^1 = \mathrm{E}(y|x^1 = 1) = 0, y^2 = \mathrm{E}(y|x^2 = 1, \mathrm{E}(y|x^1 = 1) = 0) = 0$ at which point the communication ends; in particular $\mathrm{E}(y|x^2 = 1, \mathrm{E}(y|x^1 = 1) = \mathrm{E}(y|x^1 = 1, x^2 = 1) = 0 :$ there is full exchange of information.

Alternatively, in addition to the information privately available to individuals, there is public information described by the random variable $z = (1, 1, 1, 1, 1, 0)$, and the associated information partition $\{\{1, 2, 3,\ 4\}, \{5\}\}$. For both individuals, $\mathrm{E}(y|x^i, z) = (0, 0, 0, 0, -1)$, while $\mathrm{E}(y|z) = (0, 0, 0, 0, -1)$. At $x^1 = x^2 = z = y = 1$, the communication that ensues is $y^1 = \mathrm{E}(y|x^1 = 1, \mathrm{E}(y|z = 1) = 0, y^2 = \mathrm{E}(y|x^2 = 1, \mathrm{E}(y|z = 1) = 0, y^1 = 0) = 0$, at which point communication ends: with agreement, but not with full exchange of information.

Remark (Refined private information). Public information can be interpreted as a refinement of the information of individuals. In this case, better informed individuals are less well informed following the communication of beliefs than they would have been had they exchanged coarser information.

Order matters States of nature are $\omega \in \{1, \ldots, 6\}$, events are subsets of states of the world, and, according to the prior beliefs common to all individuals, states of the world occur with equal probability, $\mu(\{\omega\}) = 1/6$.

Individual 1 observes the realization of the random variable $x^1 = (1, 1, 1, -1, -1, -1)$, while individual 2 observes the realization of the random variable $x^2 = (1, 0, -1, 1, 0, -1)$; equivalently, the information partition of individual 1 is $\{\{1, 2, 3\}, \{4, 5, 6\}\}$, while the information partition of individual 2 is $\{\{1, 2\}, \{3, 4\}, \{5, 6\}\}$. Of concern is the realization of the random variable $y = (1, 0, 0, 0, 1, a), a \neq 0$.

At $x^1 = x^2 = y = 1$, the communication that ensues is $y^1 = \mathrm{E}(y|x^1 = 1)$

$= 1/3, y^2 = \mathrm{E}(y|x^2 = 1, \mathrm{E}(y|x^1 = 1) = 1/3) = 1, y^3 = \mathrm{E}(y|x^1 = 1, \mathrm{E}(y|x^2 = 1, \mathrm{E}(y|x^1 = 1) = 1/3) = 1) = 1$, at which point communication ends: with agreement and full exchange of information.

Alternatively, the order of communication is reversed: equivalently, individual 1 observes the realization of the random variable $\tilde{x}^1 = (1, 0, -1, 1, 0, -1)$, while individual 2 observes the realization of the random variable $\tilde{x}^2 = (1, 1, 1, -1, -1 - 1)$. The communication that ensues at $x^1 = x^2 = y = 1$ is $\tilde{y}^1 = \mathrm{E}(y|\tilde{x}^1 = 1) = 1/2, \tilde{y}^2 = \mathrm{E}(y|\tilde{x}^2 = 1, \mathrm{E}(y|\tilde{x}^1 = 1) = 1/2) = 1/2$, at which point communication end: with agreement, but, not with full communication of information.

Remark (Silence)*.* Evidently, individual 1 may choose to remain silent at the first stage of communication, which effectively reverses the order of communication; this may enhance the information exchanged subsequently.

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