Formation Theoretic Properties of Certain Locally Finite Groups

by

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Chapter One. Preliminaries.

1.1. Introduction (Abstract).

The theory of saturated formations introduced by Gaschütz in 1963 is now an integral part of the study of finite soluble groups. Extensions of this theory have since been obtained by Stonehewer (32), for the class of periodic locally soluble groups with a normal locally nilpotent subgroup of finite index, and Tomkinson (34), for the class of periodic locally soluble FC-groups. Wehrfritz (35) also developed a theory of basis normalizers and Carter subgroups for the class of all homomorphic images of periodic soluble linear groups. Much of this work was unified in a recent paper of Gardiner, Hartley and Tomkinson (7). They introduced a class $\mathfrak{U}$ of periodic locally soluble groups and showed that it is possible to obtain a theory of saturated formations in any subclass of $\mathfrak{U}$ which is closed under taking subgroups and homomorphic images. Their work covers all the previous theories except that for periodic locally soluble FC-groups, the situation there being somewhat different.

The class $\mathfrak{U}$ in many ways resembles the class of finite soluble groups. Indeed a result for finite soluble groups which makes sense in the wider context usually holds for $\mathfrak{U}$-groups. We shall show later in this thesis that this is the case with much of the work of Alperin (1), Chambers (5), Mann (21) and Rose (28).

This thesis is divided into three chapters and is organised as follows. In the next section of this chapter we describe the results of Gardiner, Hartley and Tomkinson (7) and also some more recent work of Hartley (12),(13),(14).

We also use section 1.2 to introduce our notation and terminology. Chapter one ends with a section in which we further investigate the Sylow structure of $\mathfrak{U}$-groups. In particular we consider the relation between the set of Sylow bases and the set of p-complement systems of a $\mathfrak{U}$-group. Unlike in finite soluble groups these sets do not, in general, correspond in a one-to-one fashion. However, as we shall show, the situation is far more satisfactory than one might expect.

In chapter two, which forms the bulk of the thesis, we extend our previous work on $\mathfrak{F}$-reducers and $\mathfrak{F}$-subnormalizers in finite soluble groups. We shall consider a fixed, but arbitrary, $\mathfrak{S}$-closed subclass $\mathfrak{K}$ of $\mathfrak{U}$ and a saturated $\mathfrak{K}$-formation $\mathfrak{F}$ satisfying certain conditions. We shall define what is meant by an $\mathfrak{F}$-system of a $\mathfrak{K}$-group $\mathfrak{F}$-reducing into a subgroup and from this we shall obtain $\mathfrak{F}$-reducers in the usual way. Under certain fairly weak conditions we shall characterize the $\mathfrak{F}$-abnormal subgroups as the self $\mathfrak{F}$-reducing subgroups and the $\mathfrak{F}$-projectors as the self $\mathfrak{F}$-reducing $\mathfrak{F}$-subgroups. We then introduce the concept of an $\mathfrak{F}$-serial subgroup of a $\mathfrak{K}$-group and show that, with certain restrictions, a subgroup H is $\mathfrak{F}$-serial in a $\mathfrak{K}$-group G if and only if every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H. In the usual way we then go on to consider $\mathfrak{F}$-serializers and in particular the $\mathfrak{F}$-reducers and $\mathfrak{F}$-serializers of $\mathfrak{F}$-normalizers.

In the final chapter we discuss extensions of the work of Alperin, Chambers and Rose. Section 3.1 deals with that of Alperin (1) and we show for example that if D_1, D_2 are basis normalizers of a $\mathfrak{U}$-group G contained in a Carter subgroup E of G then D_1, D_2 are conjugate in E (cf. (Theorem 2, 1)). This result does not extend in general

to $\mathfrak{F}$-normalizers and $\mathfrak{F}$-projectors. However in section 3.2 we shall show, generalizing Chambers (5), that the $\mathfrak{F}$-normalizers are pronormal in $\mathfrak{K}_A$-groups (i.e. $\mathfrak{K}$-groups with abelian Sylow p-subgroups for each prime p). This result yields a partial extension of Alperin's Theorem for $\mathfrak{F}$-normalizers and $\mathfrak{F}$-projectors of $\mathfrak{K}_A$-groups. We shall also show that the $\mathfrak{F}$-normalizers of $\mathfrak{K}_A$-groups are characterized as those subgroups which cover the $\mathfrak{F}$-central and avoid the $\mathfrak{F}$-eccentric chief factors. In the final section (3.3) we extend Rose's work (28) and consider the class $\mathfrak{B}$ of $\mathfrak{U}$-groups with pronormal basis normalizers. We shall show that $\mathfrak{B}$ is a $\mathfrak{U}$-formation and derive many of its properties from our work in chapter two.

The results in sections 1.3 and 3.1 form the basis of a paper (10) written jointly by Dr. B. Hartley and myself. Apart from these and the results attributed to others, the work in this thesis is to the best of my knowledge original.

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1.2. Notation and Terminology.

In this section we introduce the notation and terminology which we shall use throughout this thesis. We also give a brief resume of the results on the class $\mathfrak{U}$ obtained by Gardiner, Hartley and Tomkinson (7) and Hartley (12), (13), (14).

We shall use capital Roman letters to denote groups and small Roman letters for elements of groups. As usual $\leq$, $<$ and $\lhd$ respectively denote "is a subgroup of ", "is a proper subgroup of" and "is a normal subgroup of".

If π is a set of primes then π' denotes the complementary set, and if p is a prime then p' denotes the set of primes different from p. Suppose π is a set of primes. An element x of finite order in a group is said to be a <u>π-element</u> if the prime divisors of the order of x all lie in the set π. A group G is a <u>π-group</u> if every element of G is a π-element.

If $\mathfrak{X}$ is a class of groups (by which we mean that $\mathfrak{X}$ contains all groups of order 1 and is closed under isomorphisms) and π is a set of primes, we denote by $\mathfrak{X}_\pi$ the class of π-groups in $\mathfrak{X}$ and by $\mathfrak{X}^*$ the class of finite $\mathfrak{X}$-groups. $\mathfrak{S}$ will denote the class of periodic locally soluble groups (that is groups in which every element has finite order and every finitely generated subgroup is soluble), and $\mathfrak{N}$ the class of nilpotent groups. Thus $\mathfrak{S}^*$ and $\mathfrak{N}^*$ are respectively the classes of finite soluble and finite nilpotent groups. We shall use, and assume, the notation of group classes and closure operations developed by P. Hall and set out for example in (11) and (27).

A group is said to be <u>locally finite</u> if every finite set of elements generates a finite subgroup. $\mathfrak{S}$-groups are well known to be locally finite. In this thesis the word "group" will always mean "locally finite group" unless the contrary is explicitly stated.

If G is a group and π a set of primes, by a <u>Sylow π-subgroup</u> of G we mean a maximal π-subgroup of G; Zorn's Lemma shows that every group has Sylow π-subgroups. We shall denote by $\mathrm{Syl}_\pi(G)$ the collection of Sylow π-subgroups of G. The class $\mathfrak{D}_\pi$ consists of all (locally finite) groups G in which $\mathrm{Syl}_\pi(G)$ is a single conjugacy class.

If G is a group we shall denote by $\varrho(G)$ the <u>Hirsch-Plotkin radical</u> of G, that is the unique maximal normal locally nilpotent subgroup of G.

The class $\mathfrak{U}$ consists of all groups G satisfying the following two conditions:-

U1 : G has a finite series

$$1 = G_0 \triangleleft G_1 \triangleleft \ldots\ldots\ldots\ldots \triangleleft G_n = G$$

with locally nilpotent factors.

U2 : Every subgroup of G lies in $\bigcap_\pi \mathfrak{D}_\pi$.

Hartley (14) has shown that condition U2 implies condition U1. Thus $\mathfrak{U}$ essentially consists of all groups which have well-behaved Sylow structure in their subgroups. The upper $\mathfrak{N}$-series

$$1 = R_0 \leq R_1 \leq \ldots\ldots\ldots\ldots \leq R_n = G$$

of a $\mathfrak{U}$-group G is defined inductively by the rules $R_0 = 1$, $R_{i+1}/R_i = \varrho(G/R_i)$; the $\mathfrak{N}$-length of G, denoted by $l(G)$, is the least integer n such that $R_n = G$.

The class $\mathfrak{U}$ is clearly subgroup-closed, and it has also been shown that $\mathfrak{U}$ is closed under taking homomorphic images and finite soluble extensions, (2.2, 7), (6.6, 13). Clearly

$\mathfrak{U}$ is a subclass of $\mathfrak{S}$.

By a <u>Sylow basis</u> of a group G we mean a complete set $\underset{\sim}{S} = \{S_p\}$ of Sylow p-subgroups S_p of G, one for each prime p, such that $\langle S_p \; ; \; p \in \pi \rangle$ is a π-group for each set of primes π. The Sylow structure of $\mathfrak{U}$-groups was examined in ($\underset{\sim}{7}$) where, in particular, it was shown that every $\mathfrak{U}$-group possesses a unique conjugacy class of Sylow bases, (2.10, $\underset{\sim}{7}$). Moreover, (2.6, $\underset{\sim}{7}$), the complete set $\underset{\sim}{S} = \{S_p\}$ of Sylow p-subgroups is a Sylow basis of a $\mathfrak{U}$-group G if and only if the subgroups S_p are pairwise permutable. In such cases we frequently write

$$S_\pi = \langle S_p \; ; \; p \in \pi \rangle \quad , \quad S_{p'} = \langle S_q \; ; \; p \neq q \rangle .$$

In fact S_π is a Sylow π-subgroup of G (2.7, $\underset{\sim}{7}$) and in particular $S_{p'}$ is a Sylow p'-subgroup, or <u>p-complement</u>, of G. Clearly $S_p = \bigcap_{q \neq p} S_{q'}$. Thus each Sylow basis of G determines, and is determined by, a complete system of p-complements of G. However the complete p-complement systems of a $\mathfrak{U}$-group need not be conjugate and so a given complete p-complement system need not arise from a Sylow basis (this latter being the case in finite soluble groups).

If $T \in \mathrm{Syl}_\pi(G)$ we say T <u>reduces</u> into a subgroup H of G, and write $T \searrow H$, if $T \cap H \in \mathrm{Syl}_\pi(H)$. If $\underset{\sim}{S} = \{S_p\}$ is a Sylow basis of G we say that $\underset{\sim}{S}$ <u>reduces</u> into H, and write $\underset{\sim}{S} \searrow H$, if $\underset{\sim}{S} \cap H = \{S_p \cap H\}$ is a Sylow basis of H. It is clear that $\underset{\sim}{S} \searrow H$ if and only if $S_p \searrow H$ for all primes p. We shall show later (Lemma 1.3.4) that if G is a $\mathfrak{U}$-group then $\underset{\sim}{S} \searrow H$ if and only if $S_{p'} \searrow H$ for all primes p. If H is a subgroup of a $\mathfrak{U}$-group G then, by a result of Hartley (2.1, $\underset{\sim}{12}$), there exists at least one Sylow basis of G which reduces into H.

Let $\underset{\sim}{S}$ be a Sylow basis of the $\mathfrak{U}$-group G with associated p-complement system $\{S_{p'}\}$. Then

$$D = \bigcap_p N_G(S_p) = \bigcap_p N_G(S_{p'})$$

is the <u>basis normalizer</u> $N_G(\underset{\sim}{S})$ of $\underset{\sim}{S}$. The basis normalizers of G evidently form a complete conjugacy class of subgroups of G and in fact behave very much like the system normalizers of a finite soluble group (4.6, $\underset{\sim}{7}$). In particular they are locally nilpotent.

In the following lemma we collect the results we shall require on the Sylow structure of $\mathfrak{U}$-groups. Proofs of these and similar results can be found in section two of ($\underset{\sim}{7}$).

<u>Lemma 1.2.1.</u>

Let $G \in \mathfrak{U}$, $N \triangleleft G$, $H,K \leq G$ with $H \leq N_G(K)$. Let π be a set of primes, S a π-subgroup of G, $S_\pi \in Syl_\pi(G)$, $S_{\pi'} \in Syl_{\pi'}(G)$ and $\underset{\sim}{S}$ a Sylow basis of G. Then

(i) (a) $S_\pi \cap N \in Syl_\pi(N)$,

(b) $S_\pi N/N \in Syl_\pi(G/N)$,

(c) $S_\pi S_{\pi'} = G$.

(ii) If $S \cap N \in Syl_\pi(N)$ and $SN/N \in Syl_\pi(G/N)$ then $S \in Syl_\pi(G)$.

(iii) If $S \cap H \in Syl_\pi(H)$ and $S \cap K \in Syl_\pi(K)$ then

$$S \cap HK = (S \cap H)(S \cap K) \in Syl_\pi(HK).$$

(iv) (a) $\underset{\sim}{S} \searrow N$,

(b) $\underset{\sim}{S}N/N = \{S_pN/N\}$ is a Sylow basis of G/N,

(c) If $\underset{\sim}{S} \searrow H,K$ then $\underset{\sim}{S} \searrow HK$.

Let Ω be a totally ordered set and G an arbitrary group. By a <u>series of type Ω of G</u> we mean a set $(\Lambda_\sigma, V_\sigma ; \sigma \in \Omega)$ of pairs of subgroups of G indexed by Ω and satisfying

(i) $V_\sigma \triangleleft \Lambda_\sigma$ for all $\sigma \in \Omega$

(ii) $\Lambda_\alpha \leq V_\sigma$ if $\alpha < \sigma$

(iii) $G - 1 = \bigcup_{\sigma \in \Omega} (\Lambda_\sigma - V_\sigma)$

where $G - 1$ denotes the set of non-identity elements of G and $\Lambda_\sigma - V_\sigma$ the set of elements of Λ_σ which do not belong to V_σ. Such a series is called a <u>normal series</u> if the subgroups Λ_σ and V_σ are all normal in G, and is a <u>chief series</u> if in addition Λ_σ/V_σ is a minimal normal subgroup of G/V_σ for each $\sigma \in \Omega$. Every normal series can be refined to a chief series, but in general Jordan-Hölder theorems do not hold for series of this kind. Every chief factor of an $\mathfrak{S}$- (and hence $\mathfrak{U}$-)group is an elementary abelian p-group (possibly infinite) for some prime p, (25), (4.31, 27). If $G \in \mathfrak{S}$, then the intersection of the centralizers of the p-chief factors of G is $O_{p'p}(G)$, the largest normal $\mathfrak{S}_{p'}\mathfrak{S}_p$-subgroup of G (3.8, 7). Hence $\mathfrak{C}(G)$ is the intersection of the centralizers of the chief factors of G.

If H/K is a chief factor of an arbitrary group G we denote by $A_G(H/K)$ the group of automorphisms induced by G on H/K. Thus $A_G(H/K) \cong G/C_G(H/K)$. If $G \in \mathfrak{S}$, p is a prime and $\mathfrak{Y}$ is some class of groups we denote by $C_G(\mathfrak{Y}, p)$ the intersection of the centralizers in G of those p-chief factors H/K of G for which $A_G(H/K) \in \mathfrak{Y}$. This group is called the $(\mathfrak{Y}, p)$-centralizer of G.

Let $\mathfrak{D}$ be a $\mathfrak{a}$-closed subclass of $\mathfrak{S}$. A subclass $\mathfrak{X}$ of $\mathfrak{D}$ is called a <u>$(\mathfrak{D}, p)$-preformation</u> if the following two conditions are satisfied :

P1 : $\mathfrak{X} = \mathfrak{a}\mathfrak{X}$

P2 : If $G \in \mathfrak{D}$ then $G/C_G(\mathfrak{X}, p) \in \mathfrak{X}$.

These conditions are automatically satisfied if $\mathfrak{X}$ is a <u>$\mathfrak{D}$-formation</u>, that is a $\mathfrak{Q}$-closed subclass of $\mathfrak{D}$ such that $\mathfrak{D} \cap \mathfrak{R}\mathfrak{X} \leq \mathfrak{X}$.

Let $\mathfrak{K}$ be a $\mathfrak{QS}$-closed subclass of $\mathfrak{U}$; we obtain saturated $\mathfrak{K}$-formations as follows. If π is a non-empty set of primes, a <u>$\mathfrak{K}$-preformation function $\mathfrak{f}$ on π</u> associates with each $p \in \pi$ a $(\mathfrak{K},p)$-preformation $\mathfrak{f}(p)$. The <u>saturated $\mathfrak{K}$-formation</u> defined by $\mathfrak{f}$ is

$$\mathfrak{F} = \mathfrak{F}(\mathfrak{f}) = \mathfrak{K} \cap \mathfrak{S}_\pi \cap \bigcap_{p \in \pi} \mathfrak{S}_{p'}\mathfrak{S}_p\mathfrak{f}(p).$$

$\mathfrak{F}$ is in fact a $\mathfrak{K}$-formation (4.2, $\underset{\sim}{7}$) and consists of all π-groups G in $\mathfrak{K}$ such that for all $p \in \pi$ and p-chief factors H/K of G, $A_G(H/K) \in \mathfrak{f}(p)$. As usual, every saturated $\mathfrak{K}$-formation $\mathfrak{F}(\mathfrak{f})$ may be defined by a $\mathfrak{K}$-preformation function which is integrated, that is which satisfies $\mathfrak{f}(p) \leq \mathfrak{F}(\mathfrak{f})$ for all $p \in \pi$ (4.3, $\underset{\sim}{7}$). All the preformation functions we consider will be assumed to have this property.

If $G \in \mathfrak{K}$ and H/K is a p-chief factor of G we say H/K is <u>$\mathfrak{F}$-central</u> if $p \in \pi$ and $A_G(H/K) \in \mathfrak{f}(p)$, and <u>$\mathfrak{F}$-eccentric</u> otherwise. These definitions are independent of the $\mathfrak{K}$-preformation function $\mathfrak{f}$ which defines $\mathfrak{F}$ locally (4.4, $\underset{\sim}{7}$). For short we call the $(\mathfrak{f}(p),p)$-centralizer of a $\mathfrak{K}$-group G the $\mathfrak{f}(p)$-centralizer and denote it by $C_p(G)$. $C_p(G)$ is just the intersection of the centralizers of the $\mathfrak{F}$-central p-chief factors of G, and in view of (4.4, $\underset{\sim}{7}$) is independent of $\mathfrak{f}$. From condition P2 we also have $G/C_p(G) \in \mathfrak{f}(p)$.

If $\underset{\sim}{S} = \{S_p\}$ is a Sylow basis of a $\mathfrak{K}$-group G then the <u>$\mathfrak{F}$-normalizer of G associated with $\underset{\sim}{S}$</u> is the subgroup

$$D = S_\pi \cap \bigcap_{p \in \pi} N_G(S_{p'} \cap C_p(G)).$$

The $\mathfrak{F}$-normalizers of G form a characteristic conjugacy class of subgroups of G and moreover belong to $\mathfrak{F}$ (4.6, 7).

A subgroup H of a group G is said to <u>cover</u> the section U/V of G ($V \triangleleft U \leq G$) if $(H \cap U)V = U$, and to <u>avoid</u> U/V if $H \cap U = H \cap V$. If D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}$-group G then D covers each $\mathfrak{F}$-central chief factor of G and avoids every $\mathfrak{F}$-eccentric chief factor (4.6, 7). However the $\mathfrak{F}$-normalizers are not in general characterized by this covering/avoiding property. D also covers every $\mathfrak{F}$-factor group of G.

We now quote three useful lemmas on $\mathfrak{F}$-normalizers. Proofs of these and similar results can be found in section four of (7).

<u>Lemma 1.2.2.</u>

Suppose $G = RH \in \mathfrak{K}$ where $H \leq G$ and R is a normal $\mathfrak{N}$-subgroup of G. Let $\underset{\sim}{T}$ be a Sylow basis of H, $\underset{\sim}{R}$ the unique Sylow basis of R, and let $\underset{\sim}{S} = \{R_pT_p\}$. Then

(i) $C_p(G) \cap H \leq C_p(H)$,

(ii) $C_p(G) = R(C_p(G) \cap H) \leq RC_p(H)$,

(iii) $\underset{\sim}{S}$ is a Sylow basis of G and if D and D_1 are, respectively, the $\mathfrak{F}$-normalizers of G associated with $\underset{\sim}{S}$ and of H associated with $\underset{\sim}{T}$, then $D \cap H = D_1$.

<u>Lemma 1.2.3.</u>

Let D be the $\mathfrak{F}$-normalizer of the $\mathfrak{K}$-group G associated with the Sylow basis $\underset{\sim}{S}$ of G and suppose $D \leq H \leq G$. Then $C_p(H) \leq C_p(G) \cap H$ and if $\underset{\sim}{S}$ reduces into H then D is contained in the $\mathfrak{F}$-normalizer of H associated with $\underset{\sim}{S} \cap H$.

Lemma 1.2.4.

Let $G \in \mathfrak{K}$ and suppose that the $\mathfrak{F}$-residual (that is the unique smallest normal subgroup with $\mathfrak{F}$-factor group) $G^{\mathfrak{F}}$ of G is abelian. Then $G^{\mathfrak{F}}$ is complemented in G and the complements are precisely the $\mathfrak{F}$-normalizers of G.

If $\mathfrak{X}$ is any class of groups an $\mathfrak{X}$-projector of a group G is an $\mathfrak{X}$-subgroup X of G such that whenever $X \leq H \leq G$, $K \triangleleft H$ and $H/K \in \mathfrak{X}$ then $H = KX$. The main properties of $\mathfrak{F}$-projectors of $\mathfrak{K}$-groups are summarized in our next lemma; as usual proofs may be found in (7).

Lemma 1.2.5.

Let $\mathfrak{K}$ be a $\mathfrak{QS}$-closed subclass of $\mathfrak{U}$ and $\mathfrak{F}$ a saturated $\mathfrak{K}$-formation. Then

(1) Every $\mathfrak{K}$-group possesses a unique conjugacy class of $\mathfrak{F}$-projectors.

(2) If $G \in \mathfrak{K}$ then each $\mathfrak{F}$-projector of G contains an $\mathfrak{F}$-normalizer of G and each $\mathfrak{F}$-normalizer of G lies in an $\mathfrak{F}$-projector of G.

(3) If $G \in \mathfrak{K} \cap (\mathfrak{N})\mathfrak{F}$ then the $\mathfrak{F}$-normalizers and $\mathfrak{F}$-projectors of G coincide.

(4) If $G \in \mathfrak{K} \cap (\mathfrak{N})^2\mathfrak{F}$ then each $\mathfrak{F}$-normalizer of G lies in a unique $\mathfrak{F}$-projector of G.

(5) If $N \triangleleft G \in \mathfrak{K}$ and D is an $\mathfrak{F}$-normalizer (resp. $\mathfrak{F}$-projector) of G then DN/N is an $\mathfrak{F}$-normalizer (resp. $\mathfrak{F}$-projector) of G/N.

(6) If $G = RH$ where $H \leq G$ and R is a normal $\mathfrak{N}$-subgroup of G, then every $\mathfrak{F}$-projector of H has the form $E \cap H$ for some $\mathfrak{F}$-projector E of G.

(7) (Gaschütz) Let $N \triangleleft G \in \mathfrak{K}$. If H/N is an $\mathfrak{F}$-projector of G/N and X is an $\mathfrak{F}$-projector of H then X is an $\mathfrak{F}$-projector of G.

In chapter 2 we shall show that if G is a $\mathfrak{K} \cap (\mathfrak{M})^2\mathfrak{F}$-group then there is a natural relation between an $\mathfrak{F}$-normalizer of G and the unique $\mathfrak{F}$-projector of G containing it.

$\mathfrak{M}$ is the saturated $\mathfrak{U}$-formation defined by the $\mathfrak{U}$-formation function $f(p) = 1$, on the set of all primes. The $\mathfrak{M}$-normalizers of $\mathfrak{U}$-groups are precisely the basis normalizers. The $\mathfrak{M}$-projectors will be called the <u>Carter subgroups</u>, but in $\mathfrak{U}$-groups generally (unlike finite soluble groups) they are not characterized as the self-normalizing $\mathfrak{M}$-subgroups. In chapter 3 we will investigate basis normalizers, Carter subgroups and the relation between them in $\mathfrak{U}$-groups.

In (12) Hartley studied $\mathfrak{F}$-abnormality and related concepts. The following lemma (2.2, 12) is the starting point for his results.

<u>Lemma 1.2.6.</u>

Let M be a maximal subgroup of a $\mathfrak{U}$-group G and let $K = \mathrm{Core}_G(M) = \bigcap_{x \in G} M^x$. Then G/K has a unique minimal normal subgroup H/K which M/K complements. H/K is an elementary abelian p-group for some prime p, and $H/K = O_p(G/K) = \varrho(G/K)$.

A maximal subgroup M of a $\mathfrak{U}$-group G is said to be <u>p-maximal</u> if M complements a p-chief factor of G. If $\mathfrak{F}$ is a saturated $\mathfrak{K}$-formation and $G \in \mathfrak{K}$, M is said to be an <u>$\mathfrak{F}$-normal</u> maximal subgroup of G if $p \in \pi$

and $M/\mathrm{Core}_G(M) \in \mathfrak{f}(p)$; we write $M \lhd_{\mathfrak{F}} G$. Otherwise M is said to be <u>$\mathfrak{F}$-abnormal</u> in G. It is clear that a maximal subgroup of a $\mathfrak{K}$-group G is $\mathfrak{F}$-normal in G if and only if it complements an $\mathfrak{F}$-central chief factor of G; hence, by (4.4, 7), this definition is independent of the $\mathfrak{K}$-preformation function which defines $\mathfrak{F}$.

An arbitrary subgroup H of a $\mathfrak{K}$-group G is said to be <u>$\mathfrak{F}$-abnormal</u> in G, and we write $H \rtimes_{\mathfrak{F}} G$, if whenever $H \leq M < L \leq G$ and M is a maximal subgroup of L then M is $\mathfrak{F}$-abnormal in L.

If G is an arbitrary group, a subgroup H of G is said to be <u>abnormal</u> in G if $x \in \langle H, H^x \rangle$ for each element x in G; we write $H \rtimes G$. H is said to be <u>quasi-abnormal</u> in G if every subgroup of G containing H is self-normalizing in G. H is <u>pronormal</u> in G if H and H^x are conjugate in $\langle H, H^x \rangle$ for each element x in G. Hartley (12) showed that for subgroups of $\mathfrak{U}$-groups the concepts "abnormal", "quasi-abnormal", and "$\mathfrak{M}$-abnormal" coincide.

Hartley's first major result (3.5, 12) on $\mathfrak{F}$-abnormality is the following

<u>Lemma 1.2.7.</u>

The $\mathfrak{F}$-projectors of a $\mathfrak{K}$-group G are precisely the $\mathfrak{F}$-abnormal $\mathfrak{F}$-subgroups of G.

In particular, notice that the Carter subgroups of a $\mathfrak{U}$-group are precisely the abnormal $\mathfrak{M}$-subgroups.

A subgroup H of a $\mathfrak{K}$-group G is said to be <u>$\mathfrak{F}$-ascendabnormal</u> in G if there exists an ordinal σ and a chain $(H_\beta \; ; \; \beta \leq \sigma)$ of subgroups of G such that

$H_o = H$, $H_\beta \rtimes_{\mathfrak{F}} H_{\beta+1}$ for all $\beta < \sigma$, $H_\lambda = \bigcup_{\beta<\lambda} H_\beta$ for limit ordinals $\lambda \leq \sigma$, and $H_\sigma = G$. Such a chain is called an <u>ascending $\mathfrak{F}$-abnormal chain</u> from H to G. If H can be joined to G by such a chain of finite length we shall say that H is <u>$\mathfrak{F}$-subabnormal</u> in G.

Hartley's second major result (4.1, 12) is

<u>Lemma 1.2.8.</u>

Suppose $G \in \mathfrak{K}$. Then every $\mathfrak{F}$-normalizer of G is $\mathfrak{F}$-subabnormal in G and every $\mathfrak{F}$-ascendabnormal subgroup of G contains an $\mathfrak{F}$-normalizer of G.

In particular

<u>Corollary 1.2.9.</u>

If $G \in \mathfrak{U}$, then every basis normalizer of G is subabnormal in G and every ascendabnormal subgroup of G contains a basis normalizer of G.

For $\mathfrak{F}$-projectors a parallel situation occurs. A subgroup H of a $\mathfrak{K}$-group G is <u>$\mathfrak{F}$-crucial</u> in G if $H \rtimes_{\mathfrak{F}} G$ and $H/\mathrm{Core}_G(H) \in \mathfrak{F}$. H is <u>$\mathfrak{F}$-subcrucial</u> in G if there exists a finite chain

$$H = H_o \leq H_1 \leq \cdots\cdots\cdots \leq H_n = G$$

with H_i $\mathfrak{F}$-crucial in H_{i+1} for $0 \leq i \leq n-1$.

<u>Lemma 1.2.10.</u> (4.5, 4.7, 12).

Suppose $G \in \mathfrak{K}$. Then every $\mathfrak{F}$-projector of G is $\mathfrak{F}$-crucial in G and if L is an $\mathfrak{F}$-subcrucial subgroup of G then every $\mathfrak{F}$-projector of L is an $\mathfrak{F}$-projector of G.

Let H be a subgroup of an arbitrary group G and Ω a totally ordered set. By a <u>series of type Ω from H to G</u> we mean a set $(\Lambda_\sigma , V_\sigma ; \sigma \in \Omega)$ of pairs of subgroups of G containing H and satisfying

(1) $V_\sigma \triangleleft \Lambda_\sigma$ for each $\sigma \in \Omega$

(2) $\Lambda_\alpha \leq V_\sigma$ if $\alpha < \sigma$

(3) $G - H = \bigcup_{\sigma \in \Omega} \Lambda_\sigma - V_\sigma$.

We shall say that H is <u>serial</u> in G, and write H <u>ser</u> G, if there is some series from H to G. This concept is a rather far reaching generalization of subnormality. It turns out that if $G \in \mathfrak{U}$, then a subgroup H of G is serial in G if and only if every Sylow basis of G reduces into H (Theorem 2.3.12); this result is of course well known for finite soluble groups. We shall later define the concept of an $\mathfrak{F}$-serial subgroup and show that, under certain conditions, a similar result holds for $\mathfrak{F}$-serial subgroups of $\mathfrak{K}$-groups.

Again let H be a subgroup of an arbitrary group G. By an <u>H-composition series</u> of G we mean a composition series of G considered as a group with operator domain H, the elements of H of course acting by conjugation. Thus an H-composition series of G is a series $(\Lambda_\sigma , V_\sigma ; \sigma \in \Omega)$ from 1 to G such that H normalizes each Λ_σ and V_σ but normalizes no subgroup strictly between them. A factor of such a series is called an <u>H-composition factor</u> of G. If X/Y is an H-composition factor of the $\mathfrak{S}$-group G then X/Y is a chief factor of the group HX so is an elementary abelian p-group for some prime p; in this case we say X/Y is an <u>H-composition p-factor of G</u>.

We say X/Y is an <u>$\mathfrak{F}$-central</u> H-composition p-factor of the $\mathfrak{K}$-group G ($H \leq G$) if $p \in \pi$ and $A_H(X/Y) \in \mathfrak{f}(p)$. Otherwise we say X/Y is <u>$\mathfrak{F}$-eccentric</u>. Clearly X/Y is an $\mathfrak{F}$-central H-composition factor of G if and only if it is an $\mathfrak{F}$-central chief factor of HX. Thus, by (4.4, 7), this definition is independent of the preformation function which defines $\mathfrak{F}$. For H-composition factors of a $\mathfrak{U}$-group G the concepts "$\mathfrak{N}$-central" and "H-central" coincide; an H-composition factor X/Y being <u>H-central</u> if $[H,X] \leq Y$, and <u>H-eccentric</u> otherwise.

To end this section we briefly mention some of the more interesting subclasses of $\mathfrak{U}$ and state a further result of Hartley (Theorem E, 13) which shows the structure of $\mathfrak{U}$-groups.

$\mathfrak{U}$ contains the class $(\mathfrak{N}^*)\mathfrak{S}^*$ of periodic locally soluble groups having a normal locally nilpotent subgroup of finite index (32). A theory of saturated formations, which can be obtained by taking $\mathfrak{K}$ to be $(\mathfrak{N}^*)\mathfrak{S}^*$ in the Gardiner, Hartley, Tomkinson theory, was developed for this class by Stonehewer (32). $\mathfrak{U}$ also contains the class $\mathfrak{C}$ of homomorphic images of periodic soluble linear groups (Theorem A1, 35). We therefore obtain a theory of saturated formations in the class $\mathfrak{C}$ which extends the theory of basis normalizers and Carter subgroups developed by Wehrfritz (35). A recent result of McDougall (24) shows that $\mathfrak{U}$ also contains every metabelian group with the minimal condition on normal subgroups.

An arbitrary group H has finite (Mal'cev special) rank if there exists an integer $n \geq 0$ such that every finitely generated subgroup of H can be generated by n elements, and the rank of H is the least integer n with this property.

Using $\mathfrak{A}$ to denote the class of abelian groups, Hartley's structure theorem is the following

Theorem 1.2.11.

Let G be a $\mathfrak{U}$-group with Hirsch-Plotkin radical R. Then G/R is a countable $\mathfrak{A}^2\mathfrak{S}^*$-group of finite rank. Thus $\mathfrak{U} \leq (\mathfrak{M})\mathfrak{A}^2\mathfrak{S}^*$.

We remark that Hartley also gives an example (14) that shows $\mathfrak{U}$ is not contained in the class $(\mathfrak{M})^2\mathfrak{S}^*$.

1.3. Further results on Sylow theory.

In this section we prove some further results on the Sylow structure of $\mathfrak{U}$-groups. We begin by investigating the relationship between Sylow bases and complete p-complement systems.

Lemma 1.3.1.

Let $\underset{\sim}{S}$ be a Sylow basis of a $\mathfrak{U}$-group G and let $\{T_{p'}\}$ be a complete system of p-complements of G. Then $\{T_{p'}\}$ is the p-complement system associated with a Sylow basis of G if and only if $S_{p'} = T_{p'}$ for all but finitely many primes p.

Proof.

Suppose first that $\{T_{p'}\}$ is the p-complement system associated with the Sylow basis $\underset{\sim}{T}$ of G. Then $\underset{\sim}{S} = \underset{\sim}{T}^x$ for some x in G. Since the subgroups T_p generate G there exists a finite set π such that $x \in T_\pi$. If $p \notin \pi$ then $x \in T_{p'}$ so that $T_{p'} = T_{p'}^x = S_{p'}$, as claimed.

Conversely suppose that π is a finite set of primes such that $T_{p'} = S_{p'}$ for all $p \notin \pi$, and let $n = |\pi|$. We show by induction on n that there exists an element x in G such that $S_{p'}^x = T_{p'}$ for all p. This will show that $\{T_{p'}\}$ is the p-complement system associated with the Sylow basis $\underset{\sim}{S}^x$.

If $n = 0$ there is nothing to prove so assume that $n > 0$ and let q be a prime such that $T_{q'} \neq S_{q'}$. Then $T_{q'} = S_{q'}^y$ for some element y in G and since $G = S_{q'}S_q$ we may suppose that $y \in S_q$. If p is a prime such that $T_{p'} = S_{p'}$ then $p \neq q$ so $y \in S_{p'}$. Consequently $T_{p'} = S_{p'}^y$, and so $T_{r'} \neq S_{r'}^y$ for at most n-1 primes r. By the

inductive hypothesis there is an element z in G such that $T_{p'} = S_{p'}^{yz}$ for all p, and the proof is complete.

Corollary 1.3.2.

Every p-complement system of the $\mathfrak{U}$-group G is associated with some Sylow basis of G if and only if G has a normal Sylow p'-subgroup for all but finitely many primes p.

Proof.

Suppose that there is an infinite set π of primes such that for $p \in \pi$ a Sylow p'-subgroup of G is not normal in G. Then G possesses two distinct Sylow p'-subgroups for each $p \in \pi$, and hence two complete p-complement systems which differ at infinitely many primes. By 1.3.1 one of these cannot be associated with a Sylow basis of G.

The converse is immediate from 1.3.1 since any two complete q-complement systems of G agree at the prime p if G has a normal Sylow p'-subgroup.

The following lemma is fundamental for much of what follows:

Lemma 1.3.3.

Let $\underset{\sim}{S}$ be a Sylow basis of a $\mathfrak{U}$-group G, H a subgroup of G and π the set of all primes p such that $S_{p'}$ does not reduce into H. For each $p \in \pi$ let $T_{p'}$ be a Sylow p'-subgroup of H. Then

(i) π is finite

and (ii) $\{S_{p'} \cap H \; ; \; p \notin \pi\} \cup \{T_{p'} \; ; \; p \in \pi\}$ is the p-complement system associated with some Sylow basis of H.

Proof.

By (2.1, $\underset{\sim}{12}$) there exists a Sylow basis $\underset{\sim}{U}$ of G reducing into H. Let σ be the set of all primes p such that $S_{p'} \neq U_{p'}$. Then $\pi \leq \sigma$ and σ is finite by 1.3.1. Hence π is finite. Furthermore $\{U_{p'} \cap H\}$ is the p-complement system of H associated with the Sylow basis $\underset{\sim}{U} \cap H$; since $S_{p'} \cap H = U_{p'} \cap H$ if $p \notin \sigma$, (ii) is immediate from 1.3.1.

Corollary 1.3.4.

Let $\underset{\sim}{S}$ be a Sylow basis of a $\mathfrak{U}$-group G and H a subgroup of G. Then $\underset{\sim}{S} \searrow H$ if and only if $S_{p'} \searrow H$ for all primes p.

Proof.

It is clear that if $\underset{\sim}{S} \searrow H$ then $S_{p'} \searrow H$ for all p.

The converse is immediate from 1.3.3(ii). For in this case $\pi = \emptyset$ so that $\{S_{p'} \cap H\}$ is the p-complement system associated with some Sylow basis $\underset{\sim}{T}$ of H. Thus $T_p = \bigcap_{q \neq p} (S_{q'} \cap H) = S_p \cap H$, and $S_p \searrow H$ for each prime p.

Our next result is analogous to Theorem B of Alperin ($\underset{\sim}{1}$) and provides, together with Lemma 1.3.3, the basis for the proofs in section 3.1 of our extensions of Alperin's results ($\underset{\sim}{1}$).

Lemma 1.3.5.

Let π be a set of primes and P a π-subgroup of a $\mathfrak{U}$-group G. Suppose that P normalizes some Sylow π'-subgroup of G and that X is a π'-subgroup of G centralized by P. Then there is a Sylow π'-subgroup of G normalized by P and containing X.

In the terminology of Alperin (1), X is <u>extendible</u>. We shall deduce Lemma 1.3.5 from the following result:

<u>Lemma 1.3.6.</u>

Let P be a π-subgroup of a $\mathfrak{U}$-group G and suppose that P normalizes the Sylow π'-subgroup S of G. Then $S \searrow C_G(P)$.

<u>Proof.</u>

Suppose that R is a normal subgroup of G such that the lemma holds in G/R. We shall deduce that the lemma holds in G provided that either (a) R is a π-group or (b) R is a π'-group. The lemma then follows immediately by induction on the $\mathfrak{M}$-length of G.

Let $C = C_G(P)$ and $x \to \bar{x}$ the natural homomorphism of G onto $\bar{G} = G/R$. Then $\bar{P}$ is a π-subgroup of $\bar{G}$ normalizing the Sylow π'-subgroup $\bar{S}$ of $\bar{G}$ and so since the lemma holds in $\bar{G}$ we have $\bar{S} \searrow C_{\bar{G}}(\bar{P}) = C_1/R$ say. Clearly $C \leq C_1$. Now $SR/R \cap C_1/R = (S \cap C_1)R/R$, and this is a Sylow π'-subgroup of C_1/R. Furthermore $S \cap C_1 \cap R = S \cap R$ is a Sylow π'-subgroup of R since $R \triangleleft G$: consequently $S \cap C_1$ is a Sylow π'-subgroup of C_1 by 1.2.1(ii). Since P normalizes S we have

$$[P, S \cap C_1] \leq S \cap R. \qquad (1)$$

<u>Case (a). R is a π-group.</u> Equation (1) gives $[P, S \cap C_1] = 1$ and so $S \cap C_1 \leq C \leq C_1$. It is now clear that $S \cap C = S \cap C_1$ is a Sylow π'-subgroup of C.

<u>Case (b). R is a π'-group.</u> We show first that $S \cap C_1 \leq C_2$ where $C_2 = CR$. In fact let $x \in S \cap C_1$. Then $RP = RP^x$ and so, since P and P^x are Sylow π-subgroups of RP, we have $P = P^{xy}$ for some $y \in R$. Since $R \leq S$ this gives $xy \in N_S(P)$. Since P already normalizes S, $N_S(P) = C_S(P)$, and so

$xy \in C_S(P) \leq C$. Therefore $x \in CR = C_2$, as claimed.

Since $C_2 \leq C_1$ it follows that $S_2 = S \cap C_2$ is a Sylow π'-subgroup of C_2. Now $R \leq S_2$ so $S_2 = R(S_2 \cap C)$. It now follows easily that $S_2 \cap C$ is a Sylow π'-subgroup of C and since $S_2 \cap C = S \cap C$ the proof of the lemma is complete.

Proof of 1.3.5.

Let S be a Sylow π'-subgroup of G normalized by P and let $C = C_G(P)$. Then $X \leq C$ and by 1.3.6, $S \cap C$ is a Sylow π'-subgroup of C. Hence $X \leq (S \cap C)^y = S^y \cap C$ for some $y \in C$, and clearly P normalizes S^y.

The final result in this section generalizes a theorem of Shamash (31). The proof is due to B. Hartley and we are grateful for his permission to include it here.

Theorem 1.3.7. (Hartley).

Let $\underset{\sim}{S}$ be a Sylow basis of the $\mathfrak{U}$-group G and suppose that for each $\lambda \in \Lambda$, $\underset{\sim}{S}$ reduces into the subgroup H_λ of G. Then $\underset{\sim}{S}$ reduces into $\bigcap_{\lambda \in \Lambda} H_\lambda$.

It is easy to see that 1.3.7 is an immediate consequence of the following

Lemma 1.3.8.

Suppose $H_\lambda \leq G \in \mathfrak{U}$ $(\lambda \in \Lambda)$, $S \in Syl_\pi(G)$, $T \in Syl_{\pi'}(G)$, and $S,T \searrow H_\lambda$ for each $\lambda \in \Lambda$. Then $S,T \searrow \bigcap_{\lambda \in \Lambda} H_\lambda$.

Remark.

In contrast to 1.3.8, if $S_p \in Syl_p(G)$, $H,K \leq G$ and $S_p \searrow H,K$ then it does not follow that $S_p \searrow H \cap K$ even if $G \in \mathfrak{S}^*$. For let G be the symmetric group on 4 letters,

$H = \langle (12), (123) \rangle$, $K = \langle (12), (124) \rangle$ and $S_2 = \langle (13), (1234) \rangle$. Then S_2 reduces into both H and K but not into $H \cap K = \langle (12) \rangle$.

Proof of 1.3.8.

Suppose that R is a normal subgroup of G such that the lemma holds in G/R. We shall deduce that the lemma holds in G provided either (a) R is a π-group or (b) R is a π'-group. The lemma then follows by induction on the $\mathfrak{M}$-length of G.

From the symmetry in the statement of the lemma it is clearly sufficient to consider only case (a). Now SR/R, TR/R are, respectively, Sylow π- and Sylow π'-subgroups of G/R reducing into $H_\lambda R/R$ for each $\lambda \in \Lambda$, so since the lemma holds in G/R we have

$$SR/R \ , \ TR/R \searrow \left(\bigcap_{\lambda \in \Lambda} H_\lambda R \right)/R.$$

It is clear that 1.3.8 now follows from

Lemma 1.3.9.

Let N be a normal π-subgroup of the $\mathfrak{U}$-group G, S a Sylow π- and T a Sylow π'-subgroup of G. Suppose that S, T reduce into subgroups H_λ of G $(\lambda \in \Lambda)$ and S/N, TN/N reduce into X/N where $X = \bigcap_{\lambda \in \Lambda} (H_\lambda N)$. Then S, T reduce into $H = \bigcap_{\lambda \in \Lambda} H_\lambda$.

Proof.

By hypothesis S/N, $TN/N \searrow X/N$ so that $(S \cap X)/N \in \mathrm{Syl}_\pi(X/N)$ and $(T \cap X)N/N \in \mathrm{Syl}_{\pi'}(X/N)$. Since N is a normal π-subgroup of X it follows easily that $S \cap X \in \mathrm{Syl}_\pi(X)$ and $T \cap X \in \mathrm{Syl}_{\pi'}(X)$.

Let $\lambda \in \Lambda$. By hypothesis $T \cap H_\lambda \in \mathrm{Syl}_{\pi'}(H_\lambda)$, so $T \cap H_\lambda \in \mathrm{Syl}_{\pi'}(H_\lambda N)$ since N is a π-group. Hence

$T \cap H_\lambda = T \cap H_\lambda N$ and $T \cap X \leq T \cap H_\lambda$. Since λ is an arbitrary member of Λ we therefore have $T \cap X \leq T \cap H$. But $T \searrow X$ so it follows that $T \cap H = T \cap X$ is a Sylow π'-subgroup of H, i.e. $T \searrow H$.

By 1.2.1(i), $X = (T \cap X)(S \cap X) = (T \cap H)(S \cap X)$, so the modular law gives $H = (T \cap H)(S \cap H)$. Since S is a π-group and T is a π'-group it is now clear that $S \cap H$ is a Sylow π-subgroup of H, which completes the proof.

Chapter Two. $\mathfrak{F}$-reducers and $\mathfrak{F}$-serializers.

In this chapter, which forms the bulk of this thesis, we extend the theory of $\mathfrak{F}$-reducers and $\mathfrak{F}$-subnormalizers which was developed in (8),(9) for finite soluble groups.

2.1. Basic definitions and elementary consequences.

We shall assume throughout Chapter 2, except where the contrary is explicitly stated, that the following hypothesis holds.

Hypothesis 2.1.1.

$\mathfrak{K}$ is a $\mathfrak{QS}$-closed subclass of $\mathfrak{U}$, $\mathfrak{F} = \mathfrak{F}(f)$ is the saturated $\mathfrak{K}$-formation defined by the $\mathfrak{K}$-preformation function f on the set of primes π, and f satisfies

(2.1.2) $\mathfrak{K} \cap R_0 f(p) = f(p)$ for all $p \in \pi$.

A preformation function satisfying 2.1.2 will be said to be R_0-closed.

It is obviously desirable that "$\mathfrak{F}$-reducers" (when defined) depend only on $\mathfrak{F}$ and in no way on the preformation function which defines $\mathfrak{F}$. This does not appear to be the case without 2.1.2.

Before giving the basic definitions we prove

Lemma 2.1.3.

Suppose $H \leq G \in \mathfrak{K}$, $p \in \pi$ and $S \in Syl_{p'}(G)$. Then the following three statements are equivalent:-

(i) there exists a normal subgroup K_p of H such that $H/K_p \in \mathfrak{f}(p)$ and $S \cap C_p(G) \cap K_p \in Syl_{p'}(K_p)$.

(ii) there exists a normal subgroup K_p of H such that $K_p \leq C_p(H)$, $H/K_p \in \mathfrak{f}(p)$ and $S \cap C_p(G) \cap K_p \in Syl_{p'}(K_p)$.

(iii) $H/(H \cap C_p(G)) \in \mathfrak{S}_p\mathfrak{f}(p)$ and there exists a normal subgroup L_p of H such that $H/L_p \in \mathfrak{f}(p)$ and $S \cap L_p$ is a Sylow p'-subgroup of L_p.

<u>Proof</u>.

(i) => (ii). Suppose that $K_p \triangleleft H$, $H/K_p \in \mathfrak{f}(p)$ and $S \cap C_p(G) \cap K_p \in Syl_{p'}(K_p)$. Let $K_p^* = K_p \cap C_p(H)$. Clearly K_p^* is a normal subgroup of H contained in $C_p(H)$. Also $H/K_p^* \in \mathfrak{K} \cap R_o\mathfrak{f}(p) = \mathfrak{f}(p)$ and $S \cap C_p(G) \cap K_p^* \in Syl_{p'}(K_p^*)$ so (ii) follows.

(ii) => (iii). Again suppose that $K_p \triangleleft H$, $H/K_p \in \mathfrak{f}(p)$ and $S \cap C_p(G) \cap K_p \in Syl_{p'}(K_p)$. Let $T \in Syl_p(K_p)$. Then $K_p = T(S \cap C_p(G) \cap K_p)$ and hence $K_p(H \cap C_p(G)) = T(H \cap C_p(G))$. Therefore $K_p(H \cap C_p(G))/(H \cap C_p(G)) \in \mathfrak{S}_p$. Since $H/K_p(H \cap C_p(G)) \in \mathfrak{Q}\mathfrak{f}(p) = \mathfrak{f}(p)$ it follows that $H/(H \cap C_p(G)) \in \mathfrak{S}_p\mathfrak{f}(p)$. If we now take L_p to be K_p then (iii) follows.

(iii) => (i). If $H/(H \cap C_p(G)) \in \mathfrak{S}_p\mathfrak{f}(p)$ there exists a normal subgroup U_p of H such that $H/U_p \in \mathfrak{f}(p)$ and $U_p/(H \cap C_p(G)) \in \mathfrak{S}_p$. Set $K_p = U_p \cap L_p$. Then $H/K_p \in \mathfrak{f}(p)$ since $\mathfrak{f}$ is R_o-closed , and $S \cap K_p \in Syl_{p'}(K_p)$ since $K_p \triangleleft L_p$. Now $U_p/(H \cap C_p(G))$ is a p-group so $S \cap U_p = S \cap C_p(G) \cap H$. Thus $S \cap C_p(G) \cap K_p = S \cap K_p \in Syl_{p'}(K_p)$. Hence (iii) implies (i), completing the proof.

Let $\underset{\sim}{S} = \{S_p\}$ be a Sylow basis of the $\mathfrak{K}$-group G with associated p-complement system $\{S_{p'}\}$. Let

$\underset{\sim}{S}^{\mathfrak{F}} = \{S_{p'} \cap C_p(G);\ p \in \pi\}$; $\underset{\sim}{S}^{\mathfrak{F}}$ is called the <u>$\mathfrak{F}$-system of G associated with $\underset{\sim}{S}$</u>.

Since the Sylow bases of G are conjugate so are the $\mathfrak{F}$-systems of G. The $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\underset{\sim}{S}$ clearly normalizes the $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of G.

Let H be a subgroup of G. We say the $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of G <u>strongly reduces</u> into H if $\{S_{p'} \cap C_p(G) \cap C_p(H); p \in \pi\}$ is an $\mathfrak{F}$-system of H.

We say $\underset{\sim}{S}^{\mathfrak{F}}$ <u>$\mathfrak{F}$-reduces</u> into H, and write $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$, if there exist normal subgroups K_p of H ($p \in \pi$) and an $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of H such that $K_p \leq C_p(H)$, $H/K_p \in \mathfrak{f}(p)$ and $S_{p'} \cap C_p(G) \cap K_p = T_{p'} \cap C_p(H) \cap K_p$ for each $p \in \pi$. We say that $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H to the $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of H.

Using the results in section 1.3, Hartley (5.2, 12) proved the following simplifications of these definitions:-

<u>Lemma 2.1.4.</u>

Suppose $H \leq G \in \mathfrak{K}$ and $\underset{\sim}{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G. Then

(i) $\underset{\sim}{S}^{\mathfrak{F}}$ strongly reduces into H if and only if $S_{p'} \cap C_p(G) \cap C_p(H) \in \mathrm{Syl}_{p'}(C_p(H))$ for each $p \in \pi$.

(ii) $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H if and only if there exist normal subgroups K_p of H ($p \in \pi$) such that for each $p \in \pi$, $K_p \leq C_p(H)$, $H/K_p \in \mathfrak{f}(p)$ and $S_{p'} \cap C_p(G) \cap K_p \in \mathrm{Syl}_{p'}(K_p)$.

From this result and Lemma 2.1.3 we have

<u>Corollary 2.1.5</u>

Suppose $H \leq G \in \mathfrak{K}$ and $\underset{\sim}{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G. Then the following are equivalent:-

(i) $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H,

(ii) for each prime $p \in \pi$ there exists a normal subgroup K_p of H such that $H/K_p \in \mathfrak{f}(p)$ and $S_{p'} \cap C_p(G) \cap K_p$ is a Sylow p'-subgroup of K_p.

(iii) for each prime $p \in \pi$, $H/(H \cap C_p(G)) \in \mathfrak{S}_p\mathfrak{f}(p)$ and there exists a normal subgroup L_p of H such that $H/L_p \in \mathfrak{f}(p)$ and $S_{p'} \cap L_p \in Syl_{p'}(L_p)$.

Remarks.

1. Since the $\mathfrak{f}(p)$-centralizers of a $\mathfrak{K}$-group are independent of the $\mathfrak{K}$-preformation function which defines $\mathfrak{F}$, the notion of strong reducibility is also independent of such an $\mathfrak{f}$.

2. It would appear from the definition that $\mathfrak{F}$-reducibility depends on the preformation function $\mathfrak{f}$. However, by a recent unpublished result of B. Hartley, if $\mathfrak{F}$ is also defined by the R_0-closed $\mathfrak{K}$-preformation function $\mathfrak{f}^*$ then weak $\mathfrak{f}$- and weak $\mathfrak{f}^*$-reducibility coincide (in the terminology of (12)). Since our definition "$\mathfrak{F}$-reduces" coincides with Hartley's "weakly $\mathfrak{f}$-reduces" the concept of $\mathfrak{F}$-reducibility is also independent of the R_0-closed preformation function which defines $\mathfrak{F}$.

3. It is clear that "strongly reduces" implies "$\mathfrak{F}$-reduces".

4. If $\underset{\sim}{S}$ is a Sylow basis of a $\mathfrak{U}$-group G then the $\mathfrak{M}$-system $\underset{\sim}{S}^{\mathfrak{M}}$ is just the p-complement system of G associated with $\underset{\sim}{S}$. In view of 1.3.4, the concepts of strong reducibility and $\mathfrak{M}$-reducibility coincide with the usual definition of reducibility for Sylow bases.

If H is a subgroup of a $\mathfrak{K}$-group G there may exist no $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H (Example 2.1.13). We therefore make the following definition:-

A subgroup H of a $\mathfrak{K}$-group G is <u>$\mathfrak{F}$-connected to G</u> if there is some $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H.

<u>Lemma 2.1.6.</u>

Suppose $H \leq G \in \mathfrak{K}$ and $\underset{\sim}{S}$ is a Sylow basis of G.

(i) If $H/(H \cap C_p(G)) \in \mathfrak{S}_p f(p)$ for each $p \in \pi$ and $\underset{\sim}{S} \searrow H$ then $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$.

(ii) H is $\mathfrak{F}$-connected to G if and only if for each prime $p \in \pi$, $H/(H \cap C_p(G)) \in \mathfrak{S}_p f(p)$.

(iii) If H is $\mathfrak{F}$-connected to G and $\underset{\sim}{S} \searrow H$ then $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$.

(iv) If H is $\mathfrak{F}$-connected to G and $G \in \mathfrak{F}$ then $H \in \mathfrak{F}$.

<u>Proof.</u>

(i) is immediate from 2.1.5 if we take L_p to be H for each $p \in \pi$.

(ii) is immediate from 2.1.5, (i) and the fact that there is some Sylow basis of G which reduces into H (2.1, $\underset{\sim}{12}$).

(iii) is immediate from (i) and (ii).

(iv) If $G \in \mathfrak{F}$ then every chief factor of G is $\mathfrak{F}$-central, so $C_p(G) = O_{p'p}(G)$ by (3.8, $\underset{\sim}{7}$). Furthermore G, and hence H, is a π-group. Thus if H is $\mathfrak{F}$-connected to G then from (ii) we deduce that $H/(H \cap C_p(G))$ is an $\mathfrak{S}_p f(p)$-group for each $p \in \pi$ and hence that H belongs to $\mathfrak{K}_\pi \cap \bigcap_{p \in \pi} \mathfrak{S}_{p'} \mathfrak{S}_p f(p) = \mathfrak{F}$, as required.

The following lemma is another unpublished result of B. Hartley.

<u>Lemma 2.1.7.</u>

Suppose $H \leq G \in \mathfrak{K}$. Then H is $\mathfrak{F}$-connected to G if and only if for each $p \in \pi$ there exists a normal subgroup K_p of H and a Sylow p'-subgroup $S_{p'}$ of G such that the following two conditions hold:-

(i) every H-composition p-factor of G centralized by K_p is $\mathfrak{F}$-central,

(ii) $S_{p'} \cap C_p(G) \cap K_p \in Syl_{p'}(K_p)$.

Proof.

If H is $\mathfrak{F}$-connected to G and $\mathfrak{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H then there exist normal subgroups K_p of H ($p \in \pi$) such that $H/K_p \in \mathfrak{f}(p)$ and $S_{p'} \cap C_p(G) \cap K_p \in Syl_{p'}(K_p)$ for each $p \in \pi$. Clearly every H-composition p-factor of G centralized by K_p is $\mathfrak{F}$-central, so the two conditions are necessary.

To prove that the conditions are sufficient we show by induction on the $\mathfrak{M}$-length of G that $H/(H \cap C_p(G))$ is an $\mathfrak{S}_p\mathfrak{f}(p)$-group for each $p \in \pi$; the result is then an immediate consequence of 2.1.6.

We may clearly assume that the $\mathfrak{M}$-length of G is greater than zero. Let $R = \mathfrak{Q}(G)$, and let p be an arbitrary member of π. Set $C_1/R = C_p(G/R)$. If $(A/R)/(B/R)$ is an HR/R-composition factor of G/R then A/B is an H-composition factor of G. If furthermore $(A/R)/(B/R)$ is a p-factor of G/R centralized by K_pR/R then A/B is centralized by K_p and by hypothesis is therefore $\mathfrak{F}$-central; hence $(A/R)/(B/R)$ is an $\mathfrak{F}$-central HR/R-composition factor of G/R. Thus every HR/R-composition p-factor of G/R centralized by K_pR/R is $\mathfrak{F}$-central.

Now $C_p(G)$ centralizes every $\mathfrak{F}$-central p-chief factor of G above R, so $C_p(G)R/R \leq C_1/R$. From condition (ii) it follows that $(S_{p'} \cap C_p(G) \cap K_p)R/R \in Syl_{p'}(K_pR/R)$ and since this subgroup is contained in $S_{p'}R/R \cap C_1/R \cap K_pR/R$ we have $S_{p'}R/R \cap C_1/R \cap K_pR/R \in Syl_{p'}(K_pR/R)$. Thus conditions (i) and (ii) are satisfied for HR/R by the normal subgroup K_pR/R.

Applying our inductive hypothesis to G/R we have $(HR/R)/(HR/R \cap C_1/R) \in \mathfrak{S}_p\mathfrak{f}(p)$ and hence $HR/(H \cap C_1)R \in \mathfrak{S}_p\mathfrak{f}(p)$. However $HR/(H \cap C_1)R \cong H/H \cap C_1$, so therefore $H/H \cap C_1 \in \mathfrak{S}_p\mathfrak{f}(p)$.

Let D be the intersection of the centralizers of the $\mathfrak{F}$-central p-chief factors of G below R. By (4.1, 7) $C_p(G) = D \cap C_1$. Since $\mathfrak{K} \cap R_0\mathfrak{S}_p\mathfrak{f}(p) \leq \mathfrak{S}_p\mathfrak{f}(p)$, it suffices, to complete the proof, to show that $H/H \cap D \in \mathfrak{S}_p\mathfrak{f}(p)$.

Now D contains $C_p(G)$ so from condition (ii) we have $S_{p'} \cap D \cap K_p \in Syl_{p'}(K_p)$ and hence $K_p/K_p \cap D \in \mathfrak{S}_p$. Let U/V be an $\mathfrak{F}$-central p-chief factor of G below R. Then $K_p \cap D$ centralizes U/V by definition of D, and R centralizes U/V by (3.8, 7). Refine the series $1 \triangleleft V \triangleleft U \triangleleft G$ to an H-composition series of G and let L/M be an H-composition factor in this refinement lying between V and U. Since $K_p/K_p \cap D$ is a p-group and $K_p \cap D$ centralizes L/M it follows from (3.2, 7) that K_p also centralizes L/M. By hypothesis therefore, L/M is an $\mathfrak{F}$-central H-composition factor of G. Since R centralizes U/V, and therefore L/M also, it follows that L/M is an $\mathfrak{F}$-central p-chief factor of HR and hence that $C_p(HR)$ centralizes L/M. Thus $C_p(HR)$ centralizes every H-composition factor of G in this refinement which lies between V and U. From (4.11, 7) we now deduce that $O^p(C_p(HR))$, the $\mathfrak{S}_p$-residual of $C_p(HR)$, centralizes U/V. Since this holds for all such U/V we have $X = O^p(C_p(HR)) \leq D$. Hence $H \cap X \leq H \cap D$.

Now $R \leq C_p(HR)$ by (3.8, 7) so that $HR = HC_p(HR)$. Thus $H/(H \cap C_p(HR)) \cong HR/C_p(HR) \in \mathfrak{f}(p)$. Since $(H \cap C_p(HR))/(H \cap X)$ is isomorphic to a subgroup of the

p-group $C_p(HR)/X$, it follows that $H/H \cap X \in \mathfrak{S}_p\mathfrak{f}(p)$. But $H \cap X \leq H \cap D$ so we finally have $H/H \cap D \in \mathfrak{S}_p\mathfrak{f}(p)$, which as above completes the proof.

The property of being $\mathfrak{F}$-connected does not depend on the R_0-closed $\mathfrak{K}$-preformation function $\mathfrak{f}$ which defines $\mathfrak{F}$. This follows either from the fact that the concept "$\mathfrak{F}$-reduces" does not depend on such an $\mathfrak{f}$ or alternatively from 2.1.7 once we remark that whether an H-composition factor is $\mathfrak{F}$-central or not depends only on $\mathfrak{F}$.

The next four lemmas are due to B. Hartley and proofs can be found im (12).

Lemma 2.1.8.

Let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of the $\mathfrak{K}$-group G.

(i) If $p \in \pi$ then $S_{p'} \cap C_p(G) = O_{p'}(N_G(S_{p'} \cap C_p(G)))$.

(ii) If $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into the subgroup H of G to the $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of H, then $\underset{\sim}{T}^{\mathfrak{F}}$ is uniquely determined by $\underset{\sim}{S}^{\mathfrak{F}}$ and is moreover independent of $\mathfrak{f}$.

Lemma 2.1.9.

Let H be a subgroup of a $\mathfrak{K}$-group G and suppose that H contains an $\mathfrak{F}$-normalizer D of G. Let $\underset{\sim}{T}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of H and $\underset{\sim}{S}$ a Sylow basis of G extending $\underset{\sim}{T}$. Then $\underset{\sim}{S}^{\mathfrak{F}}$ strongly reduces into H to $\underset{\sim}{T}^{\mathfrak{F}}$. In particular H is $\mathfrak{F}$-connected to G.

Remark.

The fact that H is $\mathfrak{F}$-connected to G in 2.1.9 follows also from 2.1.6. For $G/C_p(G) \in \mathfrak{F}$ since $\mathfrak{f}$ is integrated, and therefore H covers $G/C_p(G)$ since the $\mathfrak{F}$-normalizer D does. Hence $H/H \cap C_p(G)$ is isomorphic to the $\mathfrak{f}(p)$-group $G/C_p(G)$ and, by 2.1.6, H is $\mathfrak{F}$-connected to G.

Lemma 2.1.10.

Suppose $G \in \mathfrak{K}$ and each $\mathfrak{K}$-preformation $\mathfrak{f}(p)$ is closed under taking normal subgroups ($p \in \pi$). Then every $\mathfrak{F}$-system of G strongly reduces into every normal subgroup of G.

Lemma 2.1.11.

Suppose that $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for each $p \in \pi$.

(i) If $H \leq G \in \mathfrak{K}$, $\underset{\sim}{T}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of H and $\underset{\sim}{S}$ is a Sylow basis of G which extends $\underset{\sim}{T}$ then $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H to $\underset{\sim}{T}^{\mathfrak{F}}$.

(ii) If $L \leq H \leq G \in \mathfrak{K}$, and $\underset{\sim}{S}^{\mathfrak{F}}$, $\underset{\sim}{T}^{\mathfrak{F}}$, $\underset{\sim}{U}^{\mathfrak{F}}$ are $\mathfrak{F}$-systems of G, H, and L respectively such that $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H to $\underset{\sim}{T}^{\mathfrak{F}}$ and $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into L to $\underset{\sim}{U}^{\mathfrak{F}}$ then $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into L to $\underset{\sim}{U}^{\mathfrak{F}}$.

Corollary 2.1.12.

Suppose $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for each $p \in \pi$ and $G \in \mathfrak{K}$. Then every subgroup of G is $\mathfrak{F}$-connected to G.

This corollary follows more readily from 2.1.6. For if $H \leq G$ then $H/H \cap C_p(G)$ is isomorphic to a subgroup of the $\mathfrak{f}(p)$-group $G/C_p(G)$, and therefore lies in the subgroup-closed class $\mathfrak{f}(p)$. Thus by 2.1.6(ii) H is $\mathfrak{F}$-connected to G.

At this point we give an easy example to show that the situation in general is far from satisfactory.

Example 2.1.13.

We take $\mathfrak{K}$ to be the class $\mathfrak{S}^*$ of finite soluble groups. Let $\mathfrak{X}$ be the class of $\mathfrak{S}^*$-groups in which the basis normalizers are 2-groups. It is easy to check that $\mathfrak{X}$ is an $\mathfrak{S}^*$-formation. Notice that the symmetric group

on three letters shows that $\mathfrak{X}$ is not closed under taking normal subgroups. Let $\mathfrak{f}$ be the $\mathfrak{S}^*$-formation function on the set of all primes defined by $\mathfrak{f}(p) = \mathfrak{X}$ for all p, and $\mathfrak{F}$ the saturated $\mathfrak{S}^*$-formation defined by $\mathfrak{f}$, i.e. $\mathfrak{F} = \mathfrak{N}^*\mathfrak{X}$.

Consider the wreath product $C_5 \wr \Sigma_3$ of a cyclic group C_5 of order 5 by the symmetric group on 3 letters, the wreath product being taken with respect to the natural representation of Σ_3. Let Z be the centre of this group, i.e. the "diagonal" of the base group, and set $G = (C_5 \wr \Sigma_3)/Z$. G is a group of order $5^2.3.2$, and is the semidirect product of an elementary abelian group N of order 5^2 by Σ_3. Let H be the unique Sylow 3- and K any Sylow 2-subgroup of Σ_3. It is easy to see that G, HK and H all lie in $\mathfrak{F}$ but that M = NH does not belong to $\mathfrak{F}$. Now $M \triangleleft G$ but by 2.1.6(iv) M is not $\mathfrak{F}$-connected to G. Thus no $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into M, showing the necessity of the condition in 2.1.10.

Since $G \in \mathfrak{F}$, $C_p(G) = O_{p'p}(G)$ for each prime p. Hence $C_3(G) = M$, $C_5(G) = N$ and $C_p(G) = G$ for $p \neq 3,5$. Since HK supplements $C_p(G)$ in G for each prime p, it follows, from 2.1.6(ii), that HK is $\mathfrak{F}$-connected to G. Thus the unique $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of the $\mathfrak{F}$-group G must $\mathfrak{F}$-reduce into HK to the unique $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of HK.

Since HK also belongs to $\mathfrak{F}$ the same reasoning shows that $C_3(HK) = H$ and $C_p(HK) = HK$ for $p \neq 3$. Hence H is $\mathfrak{F}$-connected to HK, by 2.1.6, and the unique $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of HK must therefore $\mathfrak{F}$-reduce into H.

However $H \cap C_5(G) = H \cap N = 1$ so that $H/H \cap C_5(G) \cong H \notin \mathfrak{S}_5^*\mathfrak{f}(5)$. Therefore H is not $\mathfrak{F}$-connected to G and since $H \in \mathfrak{F}$ this shows that the converse of 2.1.6(iv) is false in general. Moreover $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into HK to $\underset{\sim}{T}^{\mathfrak{F}}$

and $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H but $\underset{\sim}{S}^{\mathfrak{F}}$ does not $\mathfrak{F}$-reduce into H. Thus $\mathfrak{F}$-reducibility in general is not "transitive" (cf. 2.1.11(ii)). Finally H is $\mathfrak{F}$-connected to HK and HK is $\mathfrak{F}$-connected to G but H is not $\mathfrak{F}$-connected to G, so $\mathfrak{F}$-connectedness is also not transitive in general. It is easy to see that this example also shows that the next two lemmas are false in general.

Lemma 2.1.14.

Suppose that $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for each $p \in \pi$, $U \leq E \leq G \in \mathfrak{K}$, and $E \in \mathfrak{F}$. If $\underset{\sim}{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into E then $\underset{\sim}{S}^{\mathfrak{F}}$ also $\mathfrak{F}$-reduces into U.

Proof.

$\underset{\sim}{S}^{\mathfrak{F}}$ must $\mathfrak{F}$-reduce into E to the unique $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of E. But U is $\mathfrak{F}$-connected to E by 2.1.12, so $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into U. The result is now immediate from 2.1.11(ii).

Lemma 2.1.15.

Suppose that $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for each $p \in \pi$ and $H \leq L \leq G \in \mathfrak{K}$. If $\underset{\sim}{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into L then there is an element y in L such that $\underset{\sim}{S}^{y\mathfrak{F}}$ $\mathfrak{F}$-reduces into H.

Proof.

Suppose that $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into L to the $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of L. Now H is $\mathfrak{F}$-connected to L by 2.1.12, so there is some $\mathfrak{F}$-system $\underset{\sim}{T}^{y\mathfrak{F}}$ of L ($y \in L$) which $\mathfrak{F}$-reduces into H. Since $S^{y\mathfrak{F}}$ clearly $\mathfrak{F}$-reduces into L to $\underset{\sim}{T}^{y\mathfrak{F}}$ the result follows immediately from 2.1.11(ii).

Lemma 2.1.16.

Suppose that $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for each $p \in \pi$,

$H \leq K \leq G \in \mathfrak{K}$ and $\underset{\sim}{T}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of K into which the $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of G $\mathfrak{F}$-reduces. If $\underset{\sim}{S}^{\mathfrak{F}}$ also $\mathfrak{F}$-reduces into H then $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H.

<u>Proof</u>.

Since $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into K to $\underset{\sim}{T}^{\mathfrak{F}}$, there exists for each $p \in \pi$ a normal subgroup L_p of K such that $L_p \leq C_p(K)$, $K/L_p \in \mathfrak{f}(p)$ and

$$S_{p'} \cap C_p(G) \cap L_p = T_{p'} \cap C_p(K) \cap L_p.$$

If $\underset{\sim}{S}^{\mathfrak{F}}$ also $\mathfrak{F}$-reduces into H then there exists for each prime $p \in \pi$ a normal subgroup V_p of H such that $H/V_p \in \mathfrak{f}(p)$ and $S_{p'} \cap C_p(G) \cap V_p \in Syl_{p'}(V_p)$.

Now $H/H \cap L_p \in \mathfrak{f}(p)$ since it is isomorphic to a subgroup of the $\mathfrak{f}(p)$-group K/L_p. Hence $H/V_p \cap L_p \in \mathfrak{f}(p)$ since $\mathfrak{f}$ is R_o-closed. Since $V_p \cap L_p$ is a normal subgroup of V_p we also have $S_{p'} \cap C_p(G) \cap (V_p \cap L_p) \in Syl_{p'}(V_p \cap L_p)$. Thus $T_{p'} \cap C_p(K) \cap (V_p \cap L_p) = S_{p'} \cap C_p(G) \cap (V_p \cap L_p)$ is a Sylow p'-subgroup of $(V_p \cap L_p)$ and $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H, as required.

The following lemma is our induction weapon and is used crucially in the proofs of most of our major results.

<u>Lemma 2.1.17</u>.

Suppose H is a subgroup of a $\mathfrak{K}$-group G and R is a subgroup of the Hirsch-Plotkin radical of G normalized by H. Let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G. Then

(1) H is $\mathfrak{F}$-connected to G <=> HR is $\mathfrak{F}$-connected to G.

(2) If $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H then $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into HR.

(3) If $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H and (by (2)) $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into HR to the $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of HR then $\underset{\sim}{T}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$. Furthermore if $x \in HR$ and $\underset{\sim}{T}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H$ then $\underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H$.

Proof.

(1) By (3.8, 7) $R \leq C_p(G)$ for each $p \in \pi$. Thus $HR \cap C_p(G) = (H \cap C_p(G))R$ and $HR/HR \cap C_p(G) \cong H/H \cap C_p(G)$. We now obtain (1) as an immediate consequence of 2.1.6(ii).

(2) If $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H then HR is $\mathfrak{F}$-connected to G by (1). Also, by 2.1.5, there exists for each $p \in \pi$ a normal subgroup L_p of H such that $H/L_p \in \mathfrak{f}(p)$ and $S_{p'} \cap L_p \in Syl_{p'}(L_p)$. Then $L_pR \triangleleft HR$ and HR/L_pR is isomorphic to the $\mathfrak{f}(p)$-group $H/L_p(H \cap R)$. Since H normalizes R and $S_{p'} \nsubseteq R$ we have, by 1.2.1(iii), $S_{p'} \cap L_pR = (S_{p'} \cap L_p)(S_{p'} \cap R) \in Syl_{p'}(L_pR)$. Thus, by 2.1.5, $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into HR.

(3) Suppose $\underset{\sim}{S}^{\mathfrak{F}} \nsubseteq_{\mathfrak{F}} H$ and (by (2)) $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into HR to the $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of HR. Then there exist normal subgroups K_p of H such that $K_p \leq C_p(H)$, $H/K_p \in \mathfrak{f}(p)$ and $S_{p'} \cap C_p(G) \cap K_p \in Syl_{p'}(K_p)$ for each $p \in \pi$. There also exist normal subgroups L_p of HR such that $L_p \leq C_p(HR)$, $HR/L_p \in \mathfrak{f}(p)$ and $S_{p'} \cap C_p(G) \cap L_p = T_{p'} \cap C_p(HR) \cap L_p$ for each prime $p \in \pi$.

Since $R \leq \mathfrak{Q}(G)$, $S_{p'} \cap R = T_{p'} \cap R$ is the unique Sylow p'-subgroup of R. But $R \leq C_p(G) \cap C_p(HR)$, so from 1.2.1(iii) we deduce that

$$\begin{aligned} S_{p'} \cap C_p(G) \cap L_pR &= (S_{p'} \cap C_p(G) \cap L_p)(S_{p'} \cap C_p(G) \cap R) \\ &= (T_{p'} \cap C_p(HR) \cap L_p)(T_{p'} \cap C_p(HR) \cap R) \\ &= T_{p'} \cap C_p(HR) \cap L_pR \in Syl_{p'}(L_pR). \end{aligned}$$

Now $HR/L_pR \in \mathfrak{f}(p)$ so $H/H \cap L_pR \in \mathfrak{f}(p)$. Let $V_p = K_p \cap L_pR$. Clearly V_p is a normal subgroup of H contained in $C_p(H)$, and since $\mathfrak{f}$ is $\mathfrak{R}_0$-closed we further have $H/V_p \in \mathfrak{f}(p)$. Finally $T_{p'} \cap C_p(HR) \cap V_p = S_{p'} \cap C_p(G) \cap V_p$ is a Sylow p'-subgroup of V_p since $V_p \triangleleft K_p$. Thus $\underset{\sim}{T}^{\mathfrak{F}} \nsubseteq_{\mathfrak{F}} H$, as claimed.

To complete the proof it remains to show that if $x \in HR$ and $\underset{\sim}{T}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H$ then $\underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H$. Indeed if $\underset{\sim}{T}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H$ $(x \in HR)$ then there exists for each $p \in \pi$ a normal subgroup W_p of H such that $W_p \leq C_p(H)$, $H/W_p \in \mathfrak{f}(p)$ and $T_{p'}^x \cap C_p(HR) \cap W_p$ is a Sylow p'-subgroup of W_p. Let $X_p = W_p \cap V_p$. Then X_p is a normal subgroup of H such that $X_p \leq C_p(H)$ and $H/X_p \subset \mathfrak{f}(p)$. Since

$$\begin{aligned} S_{p'}^x \cap C_p(G) \cap X_p &= (S_{p'}^x \cap C_p(G) \cap L_pR) \cap X_p \\ &= (S_{p'} \cap C_p(G) \cap L_pR)^x \cap X_p \\ &= (T_{p'} \cap C_p(HR) \cap L_pR)^x \cap X_p \\ &= T_{p'}^x \cap C_p(HR) \cap X_p \in \mathrm{Syl}_{p'}(X_p), \end{aligned}$$

we have $\underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H$, as required.

Our next result is a generalization of 1.3.7. The proof is again due to B. Hartley and we are grateful for his permission to include it here.

<u>Theorem 2.1.18</u>. (Hartley).

Suppose that (in 2.1.1) π is the set of all primes and $\mathfrak{f}(p) = \mathfrak{S}\mathfrak{f}(p)$ for all p. Suppose further that $H_\lambda \leq G$ $(\lambda \in \Lambda)$ and $\underset{\sim}{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H_λ for each $\lambda \in \Lambda$. Then $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} \bigcap_{\lambda \in \Lambda} H_\lambda$.

The proof of this theorem is, as usual, by induction on the $\mathfrak{N}$-length of G, so before giving the proof we have to briefly discuss the behaviour of $\mathfrak{F}$-systems under homomorphisms.

If N is a normal subgroup of a $\mathfrak{K}$-group G and $\underset{\sim}{S}$ is a Sylow basis of G then $\underset{\sim}{S}N/N$ is a Sylow basis of G/N by 1.2.1(iv). However the $\mathfrak{f}(p)$-centralizers of G are not homomorphism invariant (in general we only have $C_p(G)N/N \leq C_p(G/N)$) so that $\underset{\sim}{S}^{\mathfrak{F}}N/N$ is, in general, not

an $\mathfrak{F}$-system of G/N. Hartley (12) showed that there is however a reasonably natural way of obtaining a corresponding $\mathfrak{F}$-system $\mathcal{V}_N(\underset{\sim}{S}^{\mathfrak{F}})$ of G/N. We reproduce Hartley's argument to clarify the terminology we shall adopt in this situation.

Let Q be any Sylow p'-subgroup of $C_p(G)$ and let S be a Sylow p'-subgroup of G which reduces into Q. Consider the subgroup

$$\mathcal{V}_N(Q) = SN/N \cap C_p(G/N).$$

$C_p(G)$ centralizes every $\mathfrak{F}$-central p-chief factor of G above N, so $C_p(G)N/N \leq C_p(G/N)$. Since $G/C_p(G)N \in \mathfrak{f}(p)$, $C_p(G/N)$ and $C_p(G)N/N$ centralize the same set of p-chief factors of G/N. Therefore by (3.7, 7) the normalizer in G/N of $\mathcal{V}_N(Q)$ is the same as that of $SN/N \cap C_p(G)N/N = (S \cap C_p(G))N/N = QN/N$. Since Q is pronormal in G it follows that $N_G(Q)N/N$ is the normalizer in G/N of $\mathcal{V}_N(Q)$. Thus, by 2.1.8(i), the subgroup $\mathcal{V}_N(Q)$ is uniquely determined by Q and independent of S.

Therefore given an $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of G there is a uniquely determined $\mathfrak{F}$-system $\mathcal{V}_N(\underset{\sim}{S}^{\mathfrak{F}}) = \{ \mathcal{V}_N(S_{p'} \cap C_p(G)) \}$ of G/N. Hartley goes on (5.7, 12) to prove the following

<u>Lemma 2.1.19.</u>

Let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of the $\mathfrak{K}$-group G and $N \triangleleft G$. Then $\mathcal{V}_N(\underset{\sim}{S}^{\mathfrak{F}})$ is an $\mathfrak{F}$-system of G/N and if $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into the subgroup H of G then $\mathcal{V}_N(\underset{\sim}{S}^{\mathfrak{F}})$ $\mathfrak{F}$-reduces into HN/N.

<u>Corollary 2.1.20.</u>

If H is $\mathfrak{F}$-connected to $G \in \mathfrak{K}$ and $N \triangleleft G$ then HN/N is $\mathfrak{F}$-connected to G/N.

We will require the following partial converse to 2.1.19.

Lemma 2.1.21.

Suppose $N \leq H \leq G \in \mathfrak{K}$, $N \triangleleft G$ and H is $\mathfrak{F}$-connected to G. If $\underset{\sim}{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G such that $\nu_N(\underset{\sim}{S}^{\mathfrak{F}})$ $\mathfrak{F}$-reduces into H/N then $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H.

Proof.

Suppose $\nu_N(\underset{\sim}{S}^{\mathfrak{F}}) \searrow_{\mathfrak{F}} H/N$. Then, by 2.1.5, for each $p \in \pi$ there exists a normal subgroup L_p/N of H/N such that $(H/N)/(L_p/N) \in \mathfrak{F}(p)$ and $S_{p'}N/N \cap L_p/N \in Syl_{p'}(L_p/N)$. From this and 1.2.1(ii) it is clear that L_p is a normal subgroup of H such that $H/L_p \in \mathfrak{F}(p)$ and $S_{p'} \cap L_p \in Syl_{p'}(L_p)$, for each $p \in \pi$. Since H is, by hypothesis, $\mathfrak{F}$-connected to G it follows from 2.1.5 and 2.1.6(ii) that $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$, as required.

Remark.

Lemma 2.1.21 is false if we remove the hypothesis that H is $\mathfrak{F}$-connected to G. For in example 2.1.13 M/N is $\mathfrak{F}$-connected to G/N but M is not $\mathfrak{F}$-connected to G.

Before proving Hartley's theorem we require a further lemma.

Lemma 2.1.22.

Suppose $G = RH \in \mathfrak{K}$ where R is a normal $\mathfrak{N}$-subgroup of G and $H \leq G$. If the $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of G $\mathfrak{F}$-reduces into H then $S_{p'} \cap H \cap C_p(G) \in Syl_{p'}(H \cap C_p(G))$ for each $p \in \pi$.

Proof.

Suppose that $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$ and let $p \in \pi$. Then there is a normal subgroup K_p of H such that $H/K_p \in \mathfrak{F}(p)$, $K_p \leq C_p(H)$ and $S_{p'} \cap C_p(G) \cap K_p \in Syl_{p'}(K_p)$. Let $R_{p'}$ be the unique Sylow p'-subgroup of R. Then RK_p is a normal subgroup of $G = RH$ and $R_{p'}S \in Syl_{p'}(RK_p)$

where $S = S_{p'} \cap C_p(G) \cap K_p$. Set $N = N_G(R_{p'}S)$ and let T be a Sylow p'-subgroup of H containing S. Then $S = T \cap K_p$ so that $T \leq N_H(S)$. Hence $R_{p'}T \leq N$. Now $R_{p'}T$ is a Sylow p'-subgroup of G so $R_{p'}T \in Syl_{p'}(N)$, and hence

$$O_{p'}(N) \leq R_{p'}T \leq R_{p'}H.$$

Since $G/RK_p \cong H/K_p(H \cap R) \in \mathfrak{F}(p)$, every p-chief factor of G centralized by RK_p is $\mathfrak{F}$-central and hence centralized by $C_p(G)$. Suppose conversely that U/V is an $\mathfrak{F}$-central p-chief factor of G and let $C = C_G(U/V)$. Then $R \leq C$ by (3.8, $\mathfrak{Z}$) so that $G = HC$. Now S is a Sylow p'-subgroup of K_p and $S \leq C$ so K_pC/C is a normal p-subgroup of $HC/C = G/C$. From (3.3, $\mathfrak{Z}$) we deduce that $K_p \leq C$ and hence that K_pR centralizes U/V. Thus K_pR and $C_p(G)$ centralize the same set of p-chief factors of G, and so, by (3.7, $\mathfrak{Z}$) $N_G(S_{p'} \cap C_p(G)) = N_G(S_{p'} \cap K_pR)$. Since $R_{p'}S$ is a Sylow p'-subgroup of RK_p contained in $S_{p'} \cap RK_p$ we have $N_G(S_{p'} \cap C_p(G)) = N_G(R_{p'}S) = N$. Lemma 2.1.8(i) now gives

$$S_{p'} \cap C_p(G) = O_{p'}(N) \leq R_{p'}H.$$

Let $X = S_{p'} \cap C_p(G)$. Then $X = R_{p'}(S_{p'} \cap C_p(G) \cap H)$ by the modular law. Since X is a Sylow p'-subgroup of $C_p(G)$ it follows that X is also a Sylow p'-subgroup of $R_{p'}(H \cap C_p(G))$. Thus if Y is a Sylow p'-subgroup of $H \cap C_p(G)$ containing $S_{p'} \cap H \cap C_p(G)$ then $X = R_{p'}Y$. Therefore $Y \leq X \cap H = S_{p'} \cap H \cap C_p(G)$, and hence $S_{p'} \cap H \cap C_p(G) = Y \in Syl_{p'}(H \cap C_p(G))$, as claimed.

<u>Proof of 2.1.18.</u>

We distinguish two cases:-

Case (1). $|\Lambda|$ is finite.

Let $\lambda \in \Lambda$. Since $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H_\lambda$ there exists for each prime p a normal subgroup L_p^λ of H_λ such that $L_p^\lambda \leq C_p(H_\lambda)$, $H_\lambda/L_p^\lambda \in \mathfrak{f}(p)$ and $S_{p'} \cap C_p(G) \cap L_p^\lambda \in Syl_{p'}(L_p^\lambda)$. Since $\mathfrak{F}$ is a $\mathfrak{K}$-formation and $\mathfrak{f}$ is integrated, the $\mathfrak{F}$-residual $H_\lambda^{\mathfrak{F}}$ of H_λ is contained in L_p^λ for each p. Clearly $S_{p'} \searrow L_p^\lambda$, so $S_{p'} \searrow H_\lambda^{\mathfrak{F}}$ for each p. Thus by 1.3.4, $\underset{\sim}{S} \searrow H_\lambda^{\mathfrak{F}}$ and in particular $S_p \searrow H_\lambda^{\mathfrak{F}}$ for each p.

Since every p-chief factor of H_λ above $H_\lambda^{\mathfrak{F}}$ is $\mathfrak{F}$-central, (3.8, $\underset{\sim}{7}$) implies that $C_p(H_\lambda)/H_\lambda^{\mathfrak{F}} \leq O_{p'p}(H_\lambda/H_\lambda^{\mathfrak{F}})$. Therefore $L_p^\lambda/H_\lambda^{\mathfrak{F}} \in \mathfrak{S}_{p'}\mathfrak{S}_p$ for each prime p. Let $M_p^\lambda/H_\lambda^{\mathfrak{F}} = O_{p'}(L_p^\lambda/H_\lambda^{\mathfrak{F}})$; then $M_p^\lambda \triangleleft L_p^\lambda$ so $S_{p'} \searrow M_p^\lambda$. Since S_p reduces into $H_\lambda^{\mathfrak{F}}$ and $M_p^\lambda/H_\lambda^{\mathfrak{F}}$ is a p'-group it follows that S_p also reduces into M_p^λ. Thus for each $\lambda \in \Lambda$ we have S_p, $S_{p'} \searrow M_p^\lambda$. Therefore S_p, $S_{p'} \searrow \bigcap_{\lambda \in \Lambda} M_p^\lambda$, by 1.3.8.

Set $M_p = \bigcap_{\lambda \in \Lambda} M_p^\lambda$, $L_p = \bigcap_{\lambda \in \Lambda} L_p^\lambda$. If $\alpha \in \Lambda$ then $L_p/(L_p \cap M_p^\alpha)$ is isomorphic to a subgroup of the p-group L_p^α/M_p^α. Thus $L_p/M_p \in \mathfrak{U} \cap R\mathfrak{S}_p = \mathfrak{S}_p$. But we have already shown that $S_{p'}$ reduces into M_p so it follows that $S_{p'} \searrow L_p$.

Set $H = \bigcap_{\lambda \in \Lambda} H_\lambda$. Again let $\alpha \in \Lambda$. Then $H/H \cap L_p^\alpha$ is isomorphic to a subgroup of the $\mathfrak{f}(p)$-group H_α/L_p^α, so by hypothesis belongs to $\mathfrak{f}(p)$. Since $|\Lambda|$ is finite it follows that $H/L_p \in \mathfrak{K} \cap R_0\mathfrak{f}(p) = \mathfrak{f}(p)$. Thus for each prime p we have a normal subgroup L_p of H such that $H/L_p \in \mathfrak{f}(p)$ and $S_{p'} \cap L_p \in Syl_{p'}(L_p)$. Since H is $\mathfrak{F}$-connected to G (by 2.1.12) it now follows, from 2.1.5 and 2.1.6(ii), that $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H, as required.

Case (2). $|\Lambda|$ arbitrary.

The proof in this case is by induction on the $\mathfrak{M}$-length of G. Since there is nothing to prove when $l(G) = 0$ we assume that $l(G) > 0$ and set $R = \varrho(G)$.

Then, by 2.1.19, $\nu_R(\underset{\sim}{S}^{\mathfrak{F}})$ is an $\mathfrak{F}$-system of G/R which $\mathfrak{F}$-reduces into $H_\lambda R/R$ for each $\lambda \in \Lambda$, so by induction $\nu_R(\underset{\sim}{S}^{\mathfrak{F}}) \searrow_{\mathfrak{F}} L/R$ where $L = \bigcap_{\lambda \in \Lambda}(H_\lambda R)$. By 2.1.12 and 2.1.21, $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into L, say to the $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of L. By case (1) $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into $L \cap H_\lambda = M_\lambda$ for each $\lambda \in \Lambda$. Therefore by 2.1.16, $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into M_λ for each $\lambda \in \Lambda$. Now $H = \bigcap_{\lambda \in \Lambda} H_\lambda = \bigcap_{\lambda \in \Lambda} M_\lambda$, so by 2.1.11(ii), it suffices to prove that $\underset{\sim}{T}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$.

Let $\lambda \in \Lambda$. Since $R \leq L \leq RH_\lambda$ we have $L = R(L \cap H_\lambda) = RM_\lambda$. Therefore, applying Lemma 2.1.22, $T_{p'} \cap C_p(L) \cap M_\lambda$ is a Sylow p'-subgroup of $M_\lambda \cap C_p(L)$ for each prime p.

Let $Y = \bigcap_p C_p(L)$ and $Z_p/Y = O_{p'}(C_p(L)/Y)$. Since $\mathfrak{F}$ is a $\mathfrak{K}$-formation and $\mathfrak{f}$ is integrated it follows that the $\mathfrak{F}$-residual $L^{\mathfrak{F}}$ of L is contained in Y. Therefore $L/Y \in \mathfrak{F}$ and for each prime p, $C_p(L)/Y \leq C_p(L/Y) = O_{p'p}(L/Y)$. Hence $C_p(L)/Z_p \in \mathfrak{S}_p$ for each prime p.

Since $Z_p \cap M_\lambda$ and $Y \cap M_\lambda$ are normal subgroups of $C_p(L) \cap M_\lambda$ and $T_{p'}$ reduces into $C_p(L) \cap M_\lambda$, we have $T_{p'} \searrow Z_p \cap M_\lambda$, $Y \cap M_\lambda$. Therefore $T_{q'}$ reduces into $Y \cap M_\lambda$ for each prime q, and so, by 1.3.4, $\underset{\sim}{T} \searrow Y \cap M_\lambda$. In particular therefore $T_p \searrow Y \cap M_\lambda$. Now $(Z_p \cap M_\lambda)/(Y \cap M_\lambda)$ is isomorphic to a subgroup of the p'-group Z_p/Y so it follows that $T_p \searrow Z_p \cap M_\lambda$. Thus for each prime p and $\lambda \in \Lambda$, T_p, $T_{p'} \searrow Z_p \cap M_\lambda$. Applying Lemma 1.3.8 we have T_p, $T_{p'} \searrow \bigcap_{\lambda \in \Lambda}(Z_p \cap M_\lambda) = Z_p \cap H$. Since $(H \cap C_p(L))/(H \cap Z_p)$ is isomorphic to a subgroup of the p-group $C_p(L)/Z_p$ it follows that $T_{p'}$ also reduces into $H \cap C_p(L)$.

Now $H/H \cap C_p(L)$ is isomorphic to a subgroup of the $\mathfrak{f}(p)$-group $L/C_p(L)$. Thus for each prime p we have a normal subgroup $H \cap C_p(L)$ of H such that $H/H \cap C_p(L) \in \mathfrak{f}(p)$ and $T_{p'} \cap H \cap C_p(L) \in \mathrm{Syl}_{p'}(H \cap C_p(L))$. Since H is $\mathfrak{F}$-connected to L by 2.1.12, it follows from 2.1.5 and 2.1.6 that $\underset{\sim}{T}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$

which, as above, completes the proof.

To end this section we show the connection between the present work and the theory originally developed in (8),(9) for finite soluble groups. For the remainder of this section only we shall assume that each $\mathfrak{f}(p)$ is a $\mathfrak{K}$-formation ($p \in \pi$), i.e. $\mathfrak{K} \cap R\,\mathfrak{f}(p) = \mathfrak{f}(p)$ for each $p \in \pi$.

In each $\mathfrak{K}$-group G there is a unique smallest normal subgroup with $\mathfrak{f}(p)$-factor group in G. This subgroup, the $\mathfrak{f}(p)$-residual of G, we shall denote by G^p. If $\underset{\sim}{S}$ is a Sylow basis of G let $\mathfrak{F}(\underset{\sim}{S}) = \{S_{p'} \cap G^p;\ p \in \pi\}$. $\mathfrak{F}(\underset{\sim}{S})$ is called the <u>$\mathfrak{F}$-basis of G associated with $\underset{\sim}{S}$</u>. The $\mathfrak{F}$-bases of G evidently form a single conjugacy class of subgroups of G.

If $H \leq G$ we say that the $\mathfrak{F}$-basis $\mathfrak{F}(\underset{\sim}{S})$ <u>reduces</u> into H if $\{S_{p'} \cap G^p \cap H^p;\ p \in \pi\}$ is an $\mathfrak{F}$-basis of H. Thus $\mathfrak{F}(\underset{\sim}{S})$ reduces into H if and only if there is a Sylow basis $\underset{\sim}{T}$ of H such that $S_{p'} \cap G^p \cap H^p = T_{p'} \cap H^p$ for each $p \in \pi$.

The methods used to prove (5.2, 12) can clearly be adapted to prove

<u>Lemma 2.1.23</u>.

Suppose $H \leq G \in \mathfrak{K}$ and $\mathfrak{F}(S)$ is an $\mathfrak{F}$-basis of G. Then $\mathfrak{F}(S)$ reduces into H if and only if $S_{p'} \cap G^p \cap H^p$ is a Sylow p'-subgroup of H^p for each $p \in \pi$.

Of course there need not be an $\mathfrak{F}$-basis of G reducing into H, but if there is we shall say that H is <u>residually $\mathfrak{F}$-connected</u> to G.

Lemma 2.1.24.

Suppose $H \leq G \in \mathfrak{K}$ and $\mathfrak{F}(\underset{\sim}{S})$ is an $\mathfrak{F}$-basis of G.

(i) If $\mathfrak{F}(\underset{\sim}{S})$ reduces into H then $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H.

(ii) If H is residually $\mathfrak{F}$-connected to G then H is $\mathfrak{F}$-connected to G.

(iii) If H is residually $\mathfrak{F}$-connected to G then $\mathfrak{F}(\underset{\sim}{S})$ reduces into H if and only if $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H.

Proof.

(i) Suppose that $\mathfrak{F}(\underset{\sim}{S})$ reduces into H. Then $S_{p'} \cap G^p \cap H^p$ is a Sylow p'-subgroup of H^p for each prime $p \in \pi$. Now $G/C_p(G) \in \mathfrak{f}(p)$ so $G^p \leq C_p(G)$ for each $p \in \pi$. Therefore $S_{p'} \cap C_p(G) \cap H^p \in Syl_{p'}(H^p)$ for each $p \in \pi$. Taking K_p to be H^p in 2.1.5(ii) it is clear that $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$, as claimed.

(ii) is immediate from (i) and the definitions.

(iii) Suppose H is residually $\mathfrak{F}$-connected to G and let $\mathfrak{F}(\underset{\sim}{S})$ be an $\mathfrak{F}$-basis of G which reduces into H. To prove (iii) it is sufficient (by (i)) to prove that if $\underset{\sim}{T}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H then $\mathfrak{F}(\underset{\sim}{T})$ reduces into H.

Let $p \in \pi$. Since $\mathfrak{F}(\underset{\sim}{S})$ reduces into H, $S_{p'} \cap G^p \cap H^p$ is a Sylow p'-subgroup of H^p. Therefore $H^p/H^p \cap G^p \in \mathfrak{S}_p$. It follows that the $\mathfrak{S}_p$-residual, $O^p(H^p)$, of H^p is contained in $H^p \cap G^p$.

Suppose $\underset{\sim}{T}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H. Then there exists a normal subgroup K_p of H such that $H/K_p \in \mathfrak{f}(p)$ and $T_{p'} \cap C_p(G) \cap K_p \in Syl_{p'}(K_p)$ Since $H^p \leq K_p$ and $T_{p'}$ reduces into K_p it follows that $T_{p'}$ reduces into H^p. But $H^p/O^p(H^p)$ is a p-group, so we must have $T_{p'} \cap O^p(H^p) = T_{p'} \cap H^p \in Syl_{p'}(H^p)$. Since $O^p(H^p)$ is contained in $H^p \cap G^p$ it is clear that

$T_{p'} \cap G^p \cap H^p = T_{p'} \cap H^p \in Syl_{p'}(H^p)$. But p is an arbitrary prime in π, so by 2.1.23, $\mathfrak{F}(\underset{\sim}{T})$ reduces into H and the proof is complete.

Corollary 2.1.25.

Suppose $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for each $p \in \pi$ and $G \in \mathfrak{K}$. Then every subgroup of G is residually $\mathfrak{F}$-connected to G. Moreover if $\underset{\sim}{S}$ is a Sylow basis of G and $H \leq G$ then $\mathfrak{F}(\underset{\sim}{S})$ reduces into H if and only if $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H.

Proof.

Suppose $H \leq G$ and $p \in \pi$. Then $H/H \cap G^p$ is isomorphic to a subgroup of G/G^p so by hypothesis $H/H \cap G^p \in \mathfrak{f}(p)$. Thus $H^p \leq H \cap G^p$. It is now clear that if $\underset{\sim}{T}$ is a Sylow basis of G reducing into H then $\mathfrak{F}(\underset{\sim}{T})$ reduces into H. Therefore H is residually $\mathfrak{F}$-connected to G and we have the first statement of the corollary. The second statement is now immediate from 2.1.24(iii).

From 2.1.24 and 2.1.25 it is clear that our theory does indeed generalize the original, and that when the $\mathfrak{f}(p)$ are subgroup-closed $\mathfrak{S}^*$-formations the two theories coincide.

2.2. $\mathfrak{F}$-reducers.

Suppose H is an $\mathfrak{F}$-connected subgroup of the $\mathfrak{K}$-group G and let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H. Let

$$R_G(H\ ;\mathfrak{F}) \ = \ \langle\, x \in G\ ;\ \underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H \,\rangle.$$

It is easy to see that this subgroup is independent of the $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of G chosen to define it. Since $\mathfrak{F}$-reducibility is independent of the R_0-closed $\mathfrak{K}$-preformation function $\mathfrak{f}$ which defines $\mathfrak{F}$, it follows that $R_G(H\ ;\mathfrak{F})$ is also independent of such an $\mathfrak{f}$. The subgroup $R_G(H\ ;\mathfrak{F})$ is called the <u>$\mathfrak{F}$-reducer of H in G</u>.

We have already seen that the $\mathfrak{M}$-systems of a $\mathfrak{U}$-group G are the p-complement systems associated with the Sylow bases of G. From 1.3.4 it follows that if $H \leq G$ and $\underset{\sim}{S}$ is a Sylow basis of G which reduces into H then $R_G(H\ ;\mathfrak{M}) = \langle\, x \in G\ ;\ \underset{\sim}{S}^{x} \searrow H \,\rangle$. This subgroup, usually denoted $R_G(H)$, is called the <u>reducer of H in G</u>. As an immediate consequence of 2.1.6(iii) we have

<u>Lemma 2.2.1.</u>

If H is an $\mathfrak{F}$-connected subgroup of the $\mathfrak{K}$-group G then $H \leq R_G(H) \leq R_G(H\ ;\mathfrak{F})$.

From the results of the previous section we have several immediate consequences.

<u>Lemma 2.2.2.</u>

Suppose that H is an $\mathfrak{F}$-connected subgroup of the $\mathfrak{K}$-group G. Then

(i) $R_G(H\ ;\mathfrak{F})$ is $\mathfrak{F}$-connected to G,

(ii) If $\underset{\sim}{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H then $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into every subgroup of G which contains $R_G(H\ ;\mathfrak{F})$.

Proof.

(i) Since H is $\mathfrak{F}$-connected to G there is some $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of G which $\mathfrak{F}$-reduces into H. Let D be the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\underset{\sim}{T}$. Then D normalizes $\underset{\sim}{T}^{\mathfrak{F}}$ so that $\underset{\sim}{T}^{d\mathfrak{F}} = \underset{\sim}{T}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$ for each element $d \in D$. Hence $D \leq R_G(H\ ;\mathfrak{F})$ and, by 2.1.9, $R_G(H\ ;\mathfrak{F})$ is $\mathfrak{F}$-connected to G.

(ii) Suppose $R_G(H\ ;\mathfrak{F}) \leq K \leq G$. Now $R_G(H\ ;\mathfrak{F})$ contains an $\mathfrak{F}$-normalizer of G so, by 2.1.9, K is $\mathfrak{F}$-connected to G. Let $\underset{\sim}{T}$ be a Sylow basis of G which reduces into both H and K; such a $\underset{\sim}{T}$ exists by 2.2.1 and (2.1, $\underset{\sim}{12}$). Then $R_G(H\ ;\mathfrak{F}) = \langle x \in G;\ \underset{\sim}{T}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H \rangle$ by 2.1.6(iii). If $\underset{\sim}{T}^{x\mathfrak{F}}$ is an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H then $x \in R_G(H\ ;\mathfrak{F})$. Hence $x \in K$ and $\underset{\sim}{T}^{x} \searrow K^{x} = K$. Since K is $\mathfrak{F}$-connected to G it follows that $\underset{\sim}{T}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} K$, by 2.1.6(iii), and this establishes (ii).

Corollary 2.2.3.

If $\underset{\sim}{S}$ is a Sylow basis of the $\mathfrak{U}$-group G reducing into the subgroup H of G then $\underset{\sim}{S}$ reduces into every subgroup of G containing $R_G(H)$.

Lemma 2.2.4.

Suppose $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for each $p \in \pi$, and $H \leq G \in \mathfrak{K}$. Let $X = R_G(H\ ;\mathfrak{F})$. Then X is self $\mathfrak{F}$-reducing in G, i.e. $X = R_G(X\ ;\mathfrak{F})$.

Proof.

Under the given hypothesis H is $\mathfrak{F}$-connected to G so there is an $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of G which $\mathfrak{F}$-reduces into H. From 2.2.2(ii) we have $R_G(H\ ;\mathfrak{F}) = \langle x \in G;\ \underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H \rangle$ and $R_G(X\ ;\mathfrak{F}) = \langle y \in G;\ \underset{\sim}{S}^{y\mathfrak{F}} \searrow_{\mathfrak{F}} X \rangle$. If $y \in G$ and $\underset{\sim}{S}^{y\mathfrak{F}} \searrow_{\mathfrak{F}} X$ then, by 2.2.1 and 2.1.15, there is an element $x \in X$ such

that $\underset{\sim}{S}^{yx^{\mathfrak{F}}} \searrow_{\mathfrak{F}} H$. Therefore $yx \in X$ and hence $y \in X$. Since y is a typical generator of $R_G(X ; \mathfrak{F})$ we have $R_G(X ; \mathfrak{F}) \leq X$ and the result now follows from 2.2.1.

Remark.

We have been unable to decide whether Lemma 2.2.4 holds if the assumption that each $\mathfrak{f}(p)$ is subgroup-closed is replaced by the assumption that H is $\mathfrak{F}$-connected to G.

Lemma 2.2.5.

Suppose that $\mathfrak{f}(p) = \mathfrak{S}\mathfrak{f}(p)$ for each prime $p \in \pi$, and $H \leq K \leq G \in \mathfrak{K}$. Then $R_K(H ; \mathfrak{F}) \leq R_G(H ; \mathfrak{F})$.

Proof.

Let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into both H and K. Then $R_G(H ; \mathfrak{F}) = \langle x \in G ; \underset{\sim}{S}^{x^{\mathfrak{F}}} \searrow_{\mathfrak{F}} H \rangle$ and $R_K(H ; \mathfrak{F}) = \langle y \in K ; (\underset{\sim}{S} \cap K)^{y^{\mathfrak{F}}} \searrow_{\mathfrak{F}} H \rangle$. If $y \in K$ then $\underset{\sim}{S}^{y^{\mathfrak{F}}}$ clearly $\mathfrak{F}$-reduces into K to $(\underset{\sim}{S} \cap K)^{y^{\mathfrak{F}}}$ so if $(\underset{\sim}{S} \cap K)^{y^{\mathfrak{F}}}$ also $\mathfrak{F}$-reduces into H then $\underset{\sim}{S}^{y^{\mathfrak{F}}}$ $\mathfrak{F}$-reduces into H by 2.1.11(ii). It is now clear that $R_K(H ; \mathfrak{F}) \leq R_G(H ; \mathfrak{F})$.

Lemma 2.2.6.

Suppose that $\mathfrak{f}(p) = \mathfrak{S}\mathfrak{f}(p)$ for each prime $p \in \pi$.

(i) If $U \leq E \leq G \in \mathfrak{K}$ and $E \in \mathfrak{F}$ then $E \leq R_G(E ; \mathfrak{F}) \leq R_G(U ; \mathfrak{F})$.

(ii) If $H \leq G \in \mathfrak{F}$ then $G = R_G(H : \mathfrak{F})$.

Proof.

The first statement is an immediate consequence of 2.2.1 and 2.1.14, and the second clearly follows from the first.

Our first major task in this section is to prove the "homomorphism invariance" of $\mathfrak{F}$-reducers.

Theorem 2.2.7.

Suppose H is an $\mathfrak{F}$-connected subgroup of the $\mathfrak{K}$-group G and $L \triangleleft G$. Then $R_{G/L}(HL/L\ ;\mathfrak{F}) = R_G(H\ ;\mathfrak{F})L/L$.

Proof.

We first remark that HL/L is $\mathfrak{F}$-connected to G/L by 2.1.20, so the $\mathfrak{F}$-reducer of HL/L in G/L is defined.

Let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into H. Then, by 2.1.6(iii) and 2.1.19,

$$R_G(H\ ;\mathfrak{F}) = \langle x \in G\ ;\ \underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H \rangle$$

$$R_{G/L}(HL/L\ ;\mathfrak{F}) = \langle xL \in G/L\ ;\ \nu_L(\underset{\sim}{S}^{x\mathfrak{F}}) \searrow_{\mathfrak{F}} HL/L \rangle.$$

It is now clear from 2.1.19 that

$$R_G(H\ ;\mathfrak{F})L/L \leq R_{G/L}(HL/L\ ;\mathfrak{F}) \ldots\ldots\ldots\ldots (1)$$

Suppose that $\nu_L(\underset{\sim}{S}^{x\mathfrak{F}}) \searrow_{\mathfrak{F}} HL/L$. Then for each prime $p \in \pi$ there exists a normal subgroup V_p/L of HL/L such that $(HL/L)/(V_p/L) \in \mathfrak{F}(p)$ and $S^x_{p'}L/L \cap V_p/L \in Syl_{p'}(V_p/L)$. Now $H/H \cap V_p \cong HV_p/V_p = HL/V_p \in \mathfrak{F}(p)$ and, using 1.2.1(ii), we also have $S^x_{p'} \cap V_p \in Syl_{p'}(V_p)$. Therefore

$$\left.\begin{array}{r} H/H \cap V_p \in \mathfrak{F}(p) \\ \text{and } S^x_{p'} \cap V_p \in Syl_{p'}(V_p) \end{array}\right\} \text{ for each } p \in \pi \ldots (2)$$

Let σ be the set of all primes q such that $S^x_{q'}$ does not reduce into HL. Then σ is a finite set by 1.3.3(i). We define Sylow q'-subgroups $U_{q'}$ of HL as follows:-

$U_{q'} = S^x_{q'} \cap HL$ if $q \notin \sigma$,

$U_{q'}$ = any Sylow q'-subgroup of HL containing $S^x_{q'} \cap V_q$ if $q \in \pi \cap \sigma$,

$U_{q'}$ = any Sylow q'-subgroup of HL if $q \in \sigma - \pi$.

By 1.3.3(ii), $\{U_{q'}\}$ is the q-complement system associated with some Sylow basis of HL. Now $\underset{\sim}{S}$ reduces into HL by 1.2.1(iv), so there is an element z in HL such that

$U_{q'} = S_{q'}^{z} \cap HL$ for all q. Thus by equation (2) and the choice of the subgroups $U_{q'}$ we have $S_{p'}^{z} \cap V_p = S_{p'}^{x} \cap V_p$ for each $p \in \pi$. Since V_p is a normal subgroup of HL this implies that $S_{p'} \cap V_p = S_{p'}^{xz^{-1}} \cap V_p$ for each $p \in \pi$. But $H \cap V_p$ is a normal subgroup of H and $\underset{\sim}{S}$ reduces into H. Therefore $S_{p'}^{xz^{-1}} \cap (H \cap V_p) = S_{p'} \cap (H \cap V_p)$ is a Sylow p'-subgroup of $H \cap V_p$ for each $p \in \pi$. Thus H is $\mathfrak{F}$-connected to G, and for each prime $p \in \pi$ we have a normal subgroup $H \cap V_p$ of H such that $H/H \cap V_p \in \mathfrak{F}(p)$ and $S_{p'}^{xz^{-1}} \cap H \cap V_p \in Syl_{p'}(H \cap V_p)$. Therefore, by 2.1.5 and 2.1.6(ii), $\underset{\sim}{S}^{xz^{-1}} \underset{\mathfrak{F}}{\rightarrow} H$. Hence $xz^{-1} \in R_G(H\ ;\mathfrak{F})$. Since $z \in HL$ and $H \leq R_G(H\ ;\mathfrak{F})$ it follows that $x \in R_G(H\ ;\mathfrak{F})L$.

Now xL is a typical generator of $R_{G/L}(HL/L\ ;\mathfrak{F})$ so we have shown that $R_{G/L}(HL/L\ ;\mathfrak{F}) \leq R_G(H\ ;\mathfrak{F})L/L$. This, together with equation (1), completes the proof of the theorem.

<u>Corollary 2.2.8.</u>

If $H \leq G \in \mathfrak{U}$ and $L \triangleleft G$ then $R_{G/L}(HL/L) = R_G(H)L/L$.

Our next result is the following generalization of (2.14, $\underset{\sim}{8}$).

<u>Theorem 2.2.9.</u>

(i) If π (in 2.1.1) is the set of all primes and H is an $\mathfrak{F}$-abnormal subgroup of the $\mathfrak{K}$-group G then $H = R_G(H\ ;\mathfrak{F})$.

(ii) If $H = R_G(H\ ;\mathfrak{F}) \leq G \in \mathfrak{K}$ then H is $\mathfrak{F}$-abnormal in G.

Before proving this theorem we require three lemmas.

Lemma 2.2.10.

If M is a p-maximal subgroup of a $\mathfrak{U}$-group G then M contains every Sylow p'-subgroup of G which reduces into it.

Proof.

Suppose M is a p-maximal subgroup of $G \in \mathfrak{U}$, and S is a Sylow p'-subgroup of G which reduces into M. Now M complements some p-chief factor H/K of G, i.e. $G = HM$ and $H \cap M = K$. Since S also reduces into H we have $S = (S \cap H)(S \cap M)$ by 1.2.1(iii). But H/K is a p-factor so $S \cap H = S \cap K \leq M$. Hence $S \leq M$ as required.

Lemma 2.2.11.

Suppose $H = R_G(H ; \mathfrak{F}) \leq G \in \mathfrak{K}$ and R is a subgroup of the Hirsch-Plotkin radical of G normalized by H. Then $R_{HR}(H ; \mathfrak{F}) = H$.

Proof.

This result is immediate from 2.1.17.

Lemma 2.2.12.

Suppose $H = R_G(H ; \mathfrak{F}) \leq G \in \mathfrak{K}$ and H is $\mathfrak{F}$-connected to every subgroup of G containing H. Then H is $\mathfrak{F}$-abnormal in G.

Proof.

Suppose for a contradiction that H is not $\mathfrak{F}$-abnormal in G. Then there are subgroups M, L of G such that $H \leq M < L \leq G$ and M is an $\mathfrak{F}$-normal maximal subgroup of L. Suppose M is a p-maximal subgroup of L ($p \in \pi$) and let $K = \mathrm{Core}_L(M)$. By 1.2.6, L/K has a unique minimal normal subgroup N/K which is complemented by M/K. Since M is $\mathfrak{F}$-normal in L it follows that N/K

is an $\mathfrak{F}$-central chief factor of L. Hence $C_p(L)$ is contained in the centralizer N of N/K. Since N/K is a p-factor, the $\mathfrak{S}_p$-residual, $O^p(C_p(L))$, of $C_p(L)$ is contained in K. Thus $H \cap O^p(C_p(L)) \leq H \cap K$.

Now H is $\mathfrak{F}$-connected to L by hypothesis, so, by 2.1.6(ii), $H/H \cap C_p(L) \in \mathfrak{S}_p\mathfrak{f}(p)$. Since $H/H \cap K$ is a quotient of $H/(H \cap O^p(C_p(L)))$ and $(H \cap C_p(L))/(H \cap O^p(C_p(L)))$ is a p-group it follows that $H/H \cap K \in \mathfrak{S}_p\mathfrak{f}(p)$.

Let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into H, M and L; such an $\underset{\sim}{S}$ exists by (2.1, 12). Then $L = (S_p \cap L)(S_{p'} \cap L)$ and $S_{p'} \cap L$ reduces into M so by 2.2.10, $S_{p'} \cap L \leq M$.

Let $x \in S_p \cap L$. Since K is a normal subgroup of L, $S_{p'} \cap K$ and $S_{p'}^x \cap K$ are Sylow p'-subgroups of K and there is therefore an element k in K such that $S_{p'} \cap K = S_{p'}^{xk} \cap K$; since $K = (S_p \cap K)(S_{p'} \cap K)$ we may suppose that $k \in S_p \cap K$. Let $L_p/(H \cap K) = O_p(H/H \cap K)$. Then $H/L_p \in \mathfrak{f}(p)$ and $S_{p'}^{xk} \cap L_p = S_{p'}^{xk} \cap (H \cap K) = S_{p'} \cap (H \cap K) = S_{p'} \cap L_p$ is a Sylow p'-subgroup of L_p since $L_p \triangleleft H$ and $\underset{\sim}{S}$ reduces into H.

For $q \neq p$ let $L_q = H$. Now $xk \in S_p$ so, for $q \neq p$, $S_{q'}^{xk} \cap L_q = S_{q'} \cap L_q \in \mathrm{Syl}_{q'}(L_q)$. Therefore for each prime $q \in \pi$ we have a normal subgroup L_q of H such that $H/L_q \in \mathfrak{f}(q)$ and $S_{q'}^{xk} \cap L_q \in \mathrm{Syl}_{q'}(L_q)$. Since H is $\mathfrak{F}$-connected to G it follows, from 2.1.5, that $\underset{\sim}{S}^{xk\mathfrak{F}} \searrow_{\mathfrak{F}} H$. We certainly also have $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$ by 2.1.6(iii), so $xk \in R_G(H;\mathfrak{F}) = H$. Now $H \leq M$ and $K = \mathrm{Core}_L(M)$ so $x \in M$. Since x is an arbitrary element of $S_p \cap L$ this shows that $S_p \cap L \leq M$. Hence $L = (S_p \cap L)(S_{p'} \cap L) \leq M$, a contradiction. Having obtained this contradiction the proof of the lemma is complete.

Proof of Theorem 2.2.9.

(i) Suppose H is an $\mathfrak{F}$-abnormal subgroup of the $\mathfrak{K}$-group G. Then H is $\mathfrak{F}$-connected to G, by 1.2.8 and 2.1.9. Hartley (6.3, 12) showed that if $\underset{\sim}{S}^{\mathfrak{F}}$ and $\underset{\sim}{S}^{x\mathfrak{F}}$ are $\mathfrak{F}$-systems of G which $\mathfrak{F}$-reduce into H then $x \in H$. It follows immediately from this and 2.2.1 that $H = R_G(H\ ;\mathfrak{F})$.

(ii) Conversely suppose that $H = R_G(H\ ;\mathfrak{F}) \leq G \in \mathfrak{K}$. We prove that H is $\mathfrak{F}$-abnormal in G by induction on the $\mathfrak{M}$-length of G, which, as usual, we may suppose to be greater than zero. Let $R = \mathfrak{C}(G)$. Then HR/R is $\mathfrak{F}$-abnormal in G/R by induction and 2.2.7. Hence $HR \trianglelefteq_{\mathfrak{F}} G$.

Suppose that $H \leq K \leq HR$. Then $K = H(K \cap R)$ and Lemma 2.1.17(iii) shows that H is $\mathfrak{F}$-connected to K. From Lemmas 2.2.11 and 2.2.12 we now conclude that $H \trianglelefteq_{\mathfrak{F}} HR$. Therefore $H \trianglelefteq_{\mathfrak{F}} HR \trianglelefteq_{\mathfrak{F}} G$.

If $H \leq L \leq G$ then $HR \trianglelefteq_{\mathfrak{F}} LR$ since $HR \trianglelefteq_{\mathfrak{F}} G$. Thus H is $\mathfrak{F}$-subabnormal in LR so by 1.2.8 and 2.1.9, H is $\mathfrak{F}$-connected to LR. Therefore for each $p \in \pi$, $H/H \cap C_p(LR) \in \mathfrak{S}_p\mathfrak{f}(p)$. But by 1.2.2(i), $H \cap C_p(LR) = H \cap (L \cap C_p(LR)) \leq H \cap C_p(L)$, so $H/H \cap C_p(L) \in \mathfrak{S}_p\mathfrak{f}(p)$ for each $p \in \pi$. In particular H is $\mathfrak{F}$-connected to L, by 2.1.6.

Thus H is $\mathfrak{F}$-connected to every subgroup of G containing H, and Lemma 2.2.12 now yields the result.

As we remarked earlier abnormality and $\mathfrak{M}$-abnormality coincide for $\mathfrak{U}$-groups. We therefore have the following

Corollary 2.2.13.

Suppose $H \leq G \in \mathfrak{U}$. Then H is abnormal in G if and only if $H = R_G(H)$.

The following theorem generalizes (2.15, 8). The proof is immediate from 1.2.7 and the corresponding section of 2.2.9. However we include an alternative proof of part (ii) which does not require the machinery needed to prove 2.2.9(ii).

Theorem 2.2.14.

(i) If π is the set of all primes and H is an $\mathfrak{F}$-projector of the $\mathfrak{K}$-group G then $H = R_G(H;\mathfrak{F}) \in \mathfrak{F}$.

(ii) If $H = R_G(H;\mathfrak{F})$ is an $\mathfrak{F}$-subgroup of the $\mathfrak{K}$-group G then H is an $\mathfrak{F}$-projector of G.

Alternative Proof of (ii).

The proof here is by induction on the $\mathfrak{M}$-length of G which we may clearly suppose to be greater than zero. Let $R = \mathfrak{Q}(G)$. Then HR/R is an $\mathfrak{F}$-projector of G/R by 2.2.7 and our inductive hypothesis. By 1.2.5(6) there is an $\mathfrak{F}$-projector E of HR containing H, and E is an $\mathfrak{F}$-projector of G by 1.2.5(7). Thus to complete the proof we need only show that E = H.

Let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H. Since $E = H(E \cap R) \in \mathfrak{F}$, it follows from 2.1.17 that $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into E to the unique $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of E and $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H. If $e \in E$ then $\underset{\sim}{T}^{\mathfrak{F}} = \underset{\sim}{T}^{e\mathfrak{F}}$ so by 2.1.17(iii) $\underset{\sim}{S}^{e\mathfrak{F}} \searrow_{\mathfrak{F}} H$. Hence $e \in R_G(H;\mathfrak{F}) = H$. Thus $E \leq H$ and hence E = H, as required.

Corollary 2.2.15.

Suppose $H \leq G \in \mathfrak{U}$. Then H is a Carter subgroup of G if and only if $H = R_G(H) \in \mathfrak{M}$.

We now prove a useful generalization of a result of A. Mann (Theorem 1, 22). This result extends (4.1, 9).

Theorem 2.2.16.

Suppose π is the set of all primes, $\mathfrak{f}(p) = \mathfrak{S}\mathfrak{f}(p)$ for all p, and $H \leq K \leq G \in \mathfrak{K}$. Then $K \geq R_G(H ; \mathfrak{F})$ if and only if K covers every $\mathfrak{F}$-central H-composition factor of G.

Corollary 2.2.17.

Suppose $H \leq K \leq G \in \mathfrak{U}$. Then $K \geq R_G(H)$ if and only if K covers every H-central H-composition factor of G.

Remark.

In his paper, (22), Mann showed that if G is finite in 2.2.17 then to obtain $K \geq R_G(H)$ it suffices to assume that K covers every H-central H-composition factor of G in some H-composition series of G. However this is not the case in all $\mathfrak{U}$-groups. For in (16) Hartley gives an example of a locally finite p-group P with a proper subgroup Q which (i) covers every chief factor of P, and (ii) covers <u>every</u> Q-composition factor in <u>some</u> Q-composition series of P. Since $Q < P = R_P(Q)$ we cannot hope to obtain the same result as Mann for all $\mathfrak{U}$-groups.

Proof of 2.2.16.

Suppose firstly that $K \geq R_G(H ; \mathfrak{F})$ and let A/B be an $\mathfrak{F}$-central H-composition factor of G. Let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into both H and HA, and let D be the $\mathfrak{F}$-normalizer of HA associated with the Sylow basis $\underset{\sim}{T} = \underset{\sim}{S} \cap HA$. Now $\underset{\sim}{T}^{\mathfrak{F}} \trianglelefteq_{\mathfrak{F}} H$ by 2.1.6(iii) and D normalizes $\underset{\sim}{T}^{\mathfrak{F}}$ so $D \leq R_{HA}(H ; \mathfrak{F})$. Therefore $D \leq K$ by 2.2.5. But D covers A/B since the latter is an

$\mathfrak{F}$-central chief factor of HA. Therefore K covers A/B, as required.

Conversely suppose that K covers every $\mathfrak{F}$-central H-composition factor of G. We show by induction on the $\mathfrak{M}$-length of G that $X = R_G(H\ ;\mathfrak{F}) \leq K$. There is nothing to prove if $l(G) = 0$ so we assume that $l(G) > 0$ and set $R = \Phi(G)$. Clearly $HR/R \leq KR/R \leq G/R$. If $(A/R)/(B/R)$ is an $\mathfrak{F}$-central HR/R-composition factor of G/R then A/B is an $\mathfrak{F}$-central H-composition factor of G. By hypothesis therefore K covers A/B, and so KR/R covers $(A/R)/(B/R)$. Thus KR/R covers every $\mathfrak{F}$-central HR/R-composition factor of G/R. Therefore $XR \leq KR$ by 2.2.7 and our inductive hypothesis. We claim that if $Y = KR$ then

$$X = R_Y(H\ ;\mathfrak{F}) \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

For let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into both Y and H, and let $\underset{\sim}{T} = \underset{\sim}{S} \cap Y$. Then by 2.1.6,

$$X = R_G(H\ ;\mathfrak{F}) = \langle\, x \in G\ ;\ \underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H \,\rangle$$

$$R_Y(H\ ;\mathfrak{F}) = \langle\, y \in Y\ ;\ \underset{\sim}{T}^{y\mathfrak{F}} \searrow_{\mathfrak{F}} H \,\rangle$$

If $x \in G$ and $\underset{\sim}{S}^{x\mathfrak{F}}$ $\mathfrak{F}$-reduces into H then $x \in Y$ since $X \leq Y$. Since $\underset{\sim}{S}^{x\mathfrak{F}}$ clearly $\mathfrak{F}$-reduces into Y to $\underset{\sim}{T}^{x\mathfrak{F}}$ it follows from 2.1.16, that $\underset{\sim}{T}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H$. Thus $X \leq R_Y(H\ ;\mathfrak{F})$ and we now obtain (3) from 2.2.5.

We show next that

$$K = R_Y(K\ ;\mathfrak{F}) \quad \ldots\ldots\ldots\ldots\ldots \quad (4)$$

To show this it suffices, by 2.2.9(i), to prove that $K \rtimes_{\mathfrak{F}} Y = KR$. Suppose that this is not the case. Then there are subgroups M, L of Y with $K \leq M < L \leq Y$ and M an $\mathfrak{F}$-normal maximal subgroup of L. Now $M = K(M \cap R)$ and $L = K(L \cap R)$ so that $M \cap R < L \cap R$. Since $H \leq K$,

the subgroups $M \cap R$ and $L \cap R$ are H-invariant. Regarding R as a group with H acting (by conjugation) as an operator group, we can refine the series $1 \leq M \cap R < L \cap R \leq R$ to a maximal H-invariant chain of subgroups of R. By (2.3, 12) every link in this refinement is normal so that the refinement forms part of an H-composition series of G. Let A/B be an H-composition factor in this series with $M \cap R \leq B < A \leq L \cap R$. Now the $\mathfrak{F}$-normal maximal subgroup M of L complements the chief factor $L \cap R/M \cap R$ of L, so this factor is $\mathfrak{F}$-central (as a chief factor of L). Thus if M is a p-maximal subgroup of L then $L/C_L(L \cap R/M \cap R) \in \mathfrak{F}(p)$. Since $L \cap R$ centralizes $L \cap R/M \cap R$ and $L = K(L \cap R)$ it follows that $K/C_K(L \cap R/M \cap R) \in \mathfrak{F}(p)$. Now $H/C_H(A/B)$ is a section of $K/C_K(L \cap R/M \cap R)$ so by hypothesis $H/C_H(A/B) \in \mathfrak{F}(p)$, i.e. A/B is an $\mathfrak{F}$-central H-composition factor of G. By hypothesis therefore K must cover A/B. But $K \cap A$ is contained in $M \cap R$ and hence in B, so K in fact avoids A/B. This contradiction shows that K is $\mathfrak{F}$-abnormal in L and establishes (4).

Our next step is to show

$$H(K \cap R) \rtimes_{\mathfrak{F}} HR \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

We again suppose that this is not the case and obtain subgroups M,L with $H(K \cap R) \leq M < L \leq HR$ and M an $\mathfrak{F}$-normal maximal subgroup of L. Now $M = H(M \cap R)$ and $L = H(L \cap R)$ so that $M \cap R$ is a maximal H-invariant subgroup of $L \cap R$. By refining the series $1 \leq M \cap R < L \cap R \leq R$ as before, we find that $L \cap R/M \cap R$

is an H-composition factor of G. Since M is an $\mathfrak{F}$-normal maximal subgroup of L, $L \cap R/M \cap R$ is an $\mathfrak{F}$-central H-composition factor of G. By hypothesis therefore K covers $L \cap R/M \cap R$. But $K \cap L \cap R = K \cap R \leq M \cap R$ so that K in fact avoids $L \cap R/M \cap R$. This contradiction establishes (5).

Let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of Y which $\mathfrak{F}$-reduces into H. Since $\underset{\sim}{S}$ is a Sylow basis of $Y = KR$ there is an element $r \in R$ such that $\underset{\sim}{S}^{r}$ reduces into K. Then $\underset{\sim}{S}^{r\mathfrak{F}} \searrow_{\mathfrak{F}} K$ by 2.1.6. Since $R \leq \mathcal{C}(Y)$ we have, by Lemma 2.1.17, $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} HR$. If $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into HR to $\underset{\sim}{T}^{\mathfrak{F}}$ then $\underset{\sim}{S}^{r\mathfrak{F}}$ $\mathfrak{F}$-reduces into HR to $\underset{\sim}{T}^{r\mathfrak{F}}$. Thus $\underset{\sim}{S}^{r\mathfrak{F}}$ $\mathfrak{F}$-reduces into both K and HR so by 2.1.18, $\underset{\sim}{S}^{r\mathfrak{F}} \searrow_{\mathfrak{F}} K \cap HR = H(K \cap R)$. Since $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H we also have $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H(K \cap R)$ by 2.1.17. Thus, by 2.1.16, we have $\mathfrak{F}$-systems $\underset{\sim}{T}^{\mathfrak{F}}$, $\underset{\sim}{T}^{r\mathfrak{F}}$ of HR which $\mathfrak{F}$-reduce into $H(K \cap R)$. From (5) and 2.2.9(i) we deduce that $r \in H(K \cap R)$. In particular therefore $r \in K$ and $\underset{\sim}{S}$ reduces into $K^{r^{-1}} = K$. Hence $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} K$ by 2.1.6.

The above argument shows that every $\mathfrak{F}$-system of Y which $\mathfrak{F}$-reduces into H also $\mathfrak{F}$-reduces into K. Therefore $X = R_Y(H;\mathfrak{F}) \leq R_Y(K;\mathfrak{F}) = K$, by (3), (4). This completes the proof of the theorem

Our next result gives a useful characterization of pronormal subgroups of $\mathfrak{U}$-groups in terms of their reducers. We recall that a subgroup H of an arbitrary group G is pronormal in G if H and H^x are conjugate in $\langle H, H^x \rangle$ for each element x in G.

Theorem 2.2.18.

Suppose $H \leq G \in \mathfrak{U}$. Then the following three conditions are equivalent:-

(i) H is pronormal in G,

(ii) If $\underset{\sim}{S}$ is a Sylow basis of G, $x \in G$ and $\underset{\sim}{S} \rightsquigarrow H, H^x$ then $x \in N_G(H)$,

(iii) $R_G(H) = N_G(H)$.

Proof.

(i) => (ii). As usual we argue by induction on the $\mathfrak{U}$-length of G which we may suppose to be greater than zero. Let $R = \mathfrak{C}(G)$ and suppose that $\underset{\sim}{S}$ is a Sylow basis of G, $x \in G$ and $\underset{\sim}{S} \rightsquigarrow H, H^x$. The subgroup HR/R is pronormal in G/R and the Sylow basis $\underset{\sim}{S}R/R$ of G/R reduces into both HR/R , H^xR/R , so by induction $xR \in N_{G/R}(HR/R)$. Since H is pronormal in G, $N_{G/R}(HR/R) = N_G(H)R/R$ so that $x = nr$ for some $r \in R$, $n \in N_G(H)$. Suppose for a contradiction that $x \notin N_G(H)$. Then $r \in HR - N_{HR}(H)$.

By considering a maximal chain of subgroups from $N_{HR}(H)$ to HR we obtain subgroups V,U such that

$$N_{HR}(H) \leq V < U \leq HR \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

V is a maximal subgroup of U and $r \in U - V$. Now H is pronormal in G and hence in HR. Therefore $N_{HR}(H) \rtimes HR$ and from (6) it follows that $V \rtimes U$.

Now $\underset{\sim}{S} \rightsquigarrow H, H^x$ and $H^x = H^{nr} = H^r$ so $\underset{\sim}{S}, \underset{\sim}{S}^{r^{-1}} \rightsquigarrow H$. Since $\underset{\sim}{S}, \underset{\sim}{S}^{r^{-1}}$ reduce into every subgroup of R it follows that $\underset{\sim}{S}, \underset{\sim}{S}^{r^{-1}}$ reduce into $H(U \cap R) = U$ and $H(V \cap R) = V$. Since $r \in U$ we have Sylow bases $(\underset{\sim}{S} \cap U)$, $(\underset{\sim}{S} \cap U)^{r^{-1}}$ of U which reduce into the abnormal subgroup V of U, and so, by 2.2.13, $r \in V$. But this contradicts the choice of U,V so we must have $x \in N_G(H)$, as claimed.

(ii) => (iii). Since the reducer of H always contains the normalizer of H, (iii) follows immediately from (ii).

(iii) => (i). Let x be any element of G and let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into both $\langle H,H^x\rangle$ and H; such a Sylow basis exists by (2.1, 12). There is an element $y \in \langle H,H^x\rangle$ such that $(\underset{\sim}{S} \cap \langle H,H^x\rangle)^y$ reduces into H^x. Therefore $\underset{\sim}{S}$, $\underset{\sim}{S}^{yx^{-1}}$ reduce into H and $yx^{-1} \in R_G(H)$. From condition (iii) we now have $H^x = H^y$. This shows that H is pronormal in G, and the proof is complete.

To end this section we give an alternative characterization of the $\mathfrak{F}$-reducer in the case where the preformations $\mathfrak{F}(p)$ are subgroup-closed.

Suppose H is an $\mathfrak{F}$-connected subgroup of the $\mathfrak{K}$-group G. Let Ω be the collection of all $\mathfrak{F}$-systems of G ; G permutes the elements of Ω by conjugation. Let Δ be the collection of $\mathfrak{F}$-systems of G which $\mathfrak{F}$-reduce into H ; by assumption Δ is not empty. Denote by Δ_o the block generated by Δ in Ω, i.e. the smallest subset of Ω containing Δ such that for each element $g \in G$ either $\Delta_o \cap \Delta_o^g = \emptyset$ or $\Delta_o = \Delta_o^g$. Define $Q_G(H\ ;\mathfrak{F})$ to be the stabilizer of Δ_o in G, i.e.

$$Q_G(H\ ;\mathfrak{F}) = \{\ g \in G\ ;\ \Delta_o = \Delta_o^g\ \}.$$

<u>Lemma 2.2.19</u>.

If H is a self $\mathfrak{F}$-reducing subgroup of the $\mathfrak{K}$-group G then Δ is a block and $Q_G(H\ ;\mathfrak{F}) = H$.

<u>Proof</u>.

Let x be an element of G and suppose that $\Delta \cap \Delta^x$ is not empty. If $\underset{\sim}{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G in $\Delta \cap \Delta^x$ then $\underset{\sim}{S}^{\mathfrak{F}}$, $\underset{\sim}{S}^{x^{-1}\mathfrak{F}}$ both $\mathfrak{F}$-reduce into H so that $x \in R_G(H\ ;\mathfrak{F}) = H$. Since H clearly stabilizes Δ we have $\Delta = \Delta^x$. This shows that Δ is a block, so in the previous notation $\Delta = \Delta_o$ and $Q_G(H\ ;\mathfrak{F})$ is the

G-stabilizer of Δ. However the above argument shows that H both contains and is contained in the G-stabilizer of Δ. Therefore $H = Q_G(H\ ;\mathfrak{F})$ as required.

Lemma 2.2.20.

Suppose $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for each $p \in \pi$, and $H \leq G \in \mathfrak{K}$. Then $R_G(H\ ;\mathfrak{F}) = Q_G(H\ ;\mathfrak{F})$ and Δ_o is the set of $\mathfrak{F}$-systems of G which $\mathfrak{F}$-reduce into $R_G(H\ ;\mathfrak{F})$.

Proof.

By 2.2.2(ii), Δ is contained in the set Δ_1 of $\mathfrak{F}$-systems of G which $\mathfrak{F}$-reduce into $R_G(H\ ;\mathfrak{F})$. Now Δ_1 is a block by 2.2.4 and 2.2.19, so $\Delta_o \leq \Delta_1$. Therefore the G-stabilizer $Q_G(H\ ;\mathfrak{F})$ of Δ_o is contained in the G-stabilizer of Δ_1, i.e. (by 2.2.19) $Q_G(H\ ;\mathfrak{F}) \leq R_G(H\ ;\mathfrak{F})$.

On the other hand let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H and let x be a generator of $R_G(H\ ;\mathfrak{F})$, i.e. suppose that $\underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H$. Then $\underset{\sim}{S}^{\mathfrak{F}} \in \Delta \cap \Delta^{x^{-1}}$ so that in particular $\Delta_o \cap \Delta_o^{x^{-1}}$ is not empty. Since Δ_o is a block it follows that x belongs to the stabilizer $Q_G(H\ ;\mathfrak{F})$ of Δ_o. Hence $R_G(H\ ;\mathfrak{F}) \leq Q_G(H\ ;\mathfrak{F})$. This, together with our previous inequality, gives the first statement of the lemma.

We have already seen that $\Delta_o \leq \Delta_1$. Again let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H. If $x \in G$ and $\underset{\sim}{S}^{x\mathfrak{F}} \in \Delta_1$ then $x \in R_G(H\ ;\mathfrak{F}) = Q_G(H\ ;\mathfrak{F})$ by 2.2.2(ii) and 2.2.4. Therefore $\underset{\sim}{S}^{x\mathfrak{F}} \in \Delta^x \leq \Delta_o^x = \Delta_o$. Hence $\Delta_1 \leq \Delta_o$, and the proof is complete.

Remark.

If 2.2.4 held under the weaker condition "H is $\mathfrak{F}$-connected to G", it is clear, from the proof, that 2.2.20 would also hold under this condition.

2.3. $\mathfrak{F}$-serial subgroups.

Let H be a subgroup of an arbitrary group G and Ω a totally ordered set. By a chain of type Ω from H to G we mean a set $(\Lambda_\sigma, V_\sigma; \sigma \in \Omega)$ of pairs of subgroups of G containing H and satisfying

(1) $V_\sigma \leq \Lambda_\sigma$ for each $\sigma \in \Omega$,

(2) $\Lambda_\beta \leq V_\sigma$ if $\beta < \sigma$,

(3) $G - H = \bigcup_{\sigma \in \Omega} (\Lambda_\sigma - V_\sigma)$.

It is clear from these conditions that $\Lambda_\beta \leq \Lambda_\sigma$ and $V_\beta \leq V_\sigma$ if $\beta \leq \sigma$ $(\beta, \sigma \in \Omega)$.

We say the chain $(\Lambda_\sigma, V_\sigma; \sigma \in \Omega)$ is a maximal chain from H to G if it satisfies the further requirement

(4) V_σ is a maximal subgroup of Λ_σ for each $\sigma \in \Omega$.

If $H \leq G \in \mathfrak{K}$ an $\mathfrak{F}$-normal maximal chain from H to G is a chain $(\Lambda_\sigma, V_\sigma; \sigma \in \Omega)$ from H to G satisfying

(5) V_σ is an $\mathfrak{F}$-normal maximal subgroup of Λ_σ for each $\sigma \in \Omega$.

We say H is an $\mathfrak{F}$-serial subgroup of the $\mathfrak{K}$-group G, and write H $\mathfrak{F}$-ser G, if there is some $\mathfrak{F}$-normal maximal chain from H to G.

Let M be a p-maximal subgroup of a $\mathfrak{K}$-group G and set $K = \mathrm{Core}_G(M) = \bigcap_{x \in G} M^x$. By 1.2.6, M complements the unique minimal normal subgroup H/K of G/K and $H/K = O_p(G/K)$. If $M \lhd_{\mathfrak{F}} G$ then $M/K \in \mathfrak{f}(p)$ so that G/K is an $\mathfrak{S}_p\mathfrak{f}(p)$-group. Since $\mathfrak{f}$ is integrated it follows that $G/K \in \mathfrak{F}$. Conversely if $G/K \in \mathfrak{F}$ then H/K is an $\mathfrak{F}$-central chief factor of G/K and $M \lhd_{\mathfrak{F}} G$. Therefore

(6) $M \lhd_{\mathfrak{F}} G$ if and only if $G/K \in \mathfrak{F}$.

As an immediate consequence of (6) we have

Lemma 2.3.1.

Suppose $H \leq G \in \mathfrak{K}$. Then H is $\mathfrak{F}$-serial in G if and only if there is a maximal chain $(\Lambda_\sigma, V_\sigma ; \sigma \in \Omega)$ from H to G such that $\Lambda_\sigma/\mathrm{Core}_{\Lambda_\sigma}(V_\sigma) \in \mathfrak{F}$ for all $\sigma \in \Omega$.

Lemma 2.3.2.

If $\mathfrak{F} = s\mathfrak{F}$ and $H \leq G \in \mathfrak{K}$ then H is $\mathfrak{F}$-serial in G if and only if there is a chain $(\Lambda_\sigma, V_\sigma ; \sigma \in \Omega)$ from H to G such that $\Lambda_\sigma/\mathrm{Core}_{\Lambda_\sigma}(V_\sigma) \in \mathfrak{F}$ for all $\sigma \in \Omega$.

Proof.

By 2.3.1 it is enough to show that H is $\mathfrak{F}$-serial in G if there is a chain $(\Lambda_\sigma, V_\sigma ; \sigma \in \Omega)$ from H to G satisfying $\Lambda_\sigma/\mathrm{Core}_{\Lambda_\sigma}(V_\sigma) \in \mathfrak{F}$ for all $\sigma \in \Omega$.

Given such a chain we can certainly refine it to a maximal chain $(\Lambda^*_\beta, V^*_\beta ; \beta \in \Omega^*)$ from H to G. If $\beta \in \Omega^*$ then there is some element $\sigma \in \Omega$ such that $V_\sigma \leq V^*_\beta < \Lambda^*_\beta \leq \Lambda_\sigma$. Now $\Lambda^*_\beta/\mathrm{Core}_{\Lambda^*_\beta}(V^*_\beta)$ is isomorphic to a section of the $\mathfrak{F}$-group $\Lambda_\sigma/\mathrm{Core}_{\Lambda_\sigma}(V_\sigma)$ so by hypothesis belongs to $\mathfrak{F}$. Since this holds for each $\beta \in \Omega^*$ it follows, from 2.3.1, that H is $\mathfrak{F}$-serial in G.

Corollary 2.3.3.

Suppose $\mathfrak{F} = s\mathfrak{F}$ and H is $\mathfrak{F}$-serial in the $\mathfrak{K}$-group G.

(i) If $L \leq G$ then $H \cap L$ is $\mathfrak{F}$-serial in L.

(ii) If K $\mathfrak{F}$-ser G then $H \cap K$ $\mathfrak{F}$-ser G.

Proof.

(i) Let $(\Lambda_\sigma, V_\sigma ; \sigma \in \Omega)$ be a chain from H to G such that $\Lambda_\sigma/\mathrm{Core}_{\Lambda_\sigma}(V_\sigma) \in \mathfrak{F}$ for all $\sigma \in \Omega$. Let $\sigma \in \Omega$. Then $(\Lambda_\sigma \cap L)/\mathrm{Core}_{(\Lambda_\sigma \cap L)}(V_\sigma \cap L)$ is isomorphic to

a section of the $\mathfrak{F}$-group $\Lambda_\sigma/\mathrm{Core}_{\Lambda_\sigma}(V_\sigma)$, so by hypothesis belongs to $\mathfrak{F}$. Therefore $(\Lambda_\sigma \cap L\ ,\ V_\sigma \cap L\ ;\ \sigma \in \Omega)$ is a chain from $H \cap L$ to L satisfying the condition in 2.3.2. Thus, by that result, $H \cap L$ $\mathfrak{F}$-<u>ser</u> L.

(ii) From (i) we have $H \cap K$ $\mathfrak{F}$-<u>ser</u> K $\mathfrak{F}$-<u>ser</u> G so clearly $H \cap K$ $\mathfrak{F}$-<u>ser</u> G.

<u>Remarks</u>.

1. If $\mathfrak{F}$ is not subgroup-closed then neither of the two previous results holds in general. For in example 2.1.13 we have $H \lhd_{\mathfrak{F}} HK \lhd_{\mathfrak{F}} G$ so H is $\mathfrak{F}$-serial in G but $H = H \cap M$ is an $\mathfrak{F}$-abnormal maximal subgroup of M. Thus 2.3.3(i) is false in general and examples can be found to illustrate the other cases.

2. The saturated $\mathfrak{U}$-formation $\mathfrak{M}$ is defined by the $\mathfrak{U}$-formation function $f(p) = 1$ on the set of all primes, so a maximal subgroup of a $\mathfrak{U}$-group is $\mathfrak{M}$-normal if and only if it is normal. From this it is clear that in $\mathfrak{U}$-groups the concepts "$\mathfrak{M}$-serial" and "serial" (as defined in section 1.2) coincide.

Our first major result in this section is

<u>Theorem 2.3.4</u>.

Suppose π (in 2.1.1) is the set of all primes and $H \leq G \in \mathfrak{K}$. If every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H then H is $\mathfrak{F}$-serial in G.

<u>Proof</u>.

As usual we argue by induction on the $\mathfrak{M}$-length of G, there being nothing to prove when this is zero. Suppose then that $l(G) > 0$ and that every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H. Let $R = \varrho(G)$. Then every $\mathfrak{F}$-system

of G/R $\mathfrak{F}$-reduces into HR/R by 2.1.19, so by our inductive hypothesis HR/R $\mathfrak{F}$-<u>ser</u> G/R. It follows that HR $\mathfrak{F}$-<u>ser</u> G so to complete the proof we need only show that H $\mathfrak{F}$-<u>ser</u> HR.

Let Σ be a totally ordered set of subgroups of HR, containing H, into which no further subgroups can be inserted; the total ordering being of course inclusion. The existence of Σ follows from Zorn's Lemma. If $x \notin H$ then the intersection Λ_x of the members of Σ which contain x and the union V_x of the members of Σ which do not contain x both belong to Σ and it is clear that V_x is a maximal subgroup of Λ_x. Therefore the subgroups V_x and Λ_x form a maximal chain from H to HR. We shall show that this is an $\mathfrak{F}$-normal maximal chain by proving that if M is a maximal subgroup of L with $H \leq M < L \leq HR$ then $M \lhd_{\mathfrak{F}} L$. This will show that H $\mathfrak{F}$-<u>ser</u> HR and complete the proof.

Suppose then that M is a maximal subgroup of L with $H \leq M < L \leq HR$. Let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G; by hypothesis $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$. Now $L = H(L \cap R)$ so by Lemma 2.1.17 $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} L$. Suppose that $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into L to the $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of L, then Lemma 2.1.17 also gives $\underset{\sim}{T}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$. If $x \in L$ then $\underset{\sim}{S}^{x\mathfrak{F}}$ $\mathfrak{F}$-reduces into H by hypothesis so that, by 2.1.17, $\underset{\sim}{T}^{x\mathfrak{F}}$ $\mathfrak{F}$-reduces into H since $\underset{\sim}{S}^{x\mathfrak{F}}$ $\mathfrak{F}$-reduces into L to $\underset{\sim}{T}^{x\mathfrak{F}}$. Thus every $\mathfrak{F}$-system of L $\mathfrak{F}$-reduces into H. Now $M = H(M \cap R)$ and $M \cap R \leq L \cap R \leq \mathcal{C}(L)$. We now apply 2.1.17 to L and deduce that every $\mathfrak{F}$-system of L $\mathfrak{F}$-reduces into M. It is now clear from 2.2.9(i) that $M \lhd_{\mathfrak{F}} L$, as required.

We use a similar argument to prove

Theorem 2.3.5.

Suppose $H \leq G \in \mathfrak{K}$ and every H-composition factor of G avoided by H is $\mathfrak{F}$-central. Then H is $\mathfrak{F}$-serial in G.

Proof.

We again argue by induction on the $\mathfrak{M}$-length of G, which we may assume to be greater than zero. Let $R = \varrho(G)$. If $(A/R)/(B/R)$ is an HR/R-composition factor of G/R avoided by HR/R then A/B is an H-composition factor of G avoided by H so by hypothesis A/B is $\mathfrak{F}$-central. Hence $(A/R)/(B/R)$ is an $\mathfrak{F}$-central HR/R-composition factor of G/R. Therefore our induction hypothesis gives HR/R $\mathfrak{F}$-ser G/R and as in the proof of 2.3.4 it suffices to prove that H $\mathfrak{F}$-ser HR. The argument used in the proof of 2.3.4 shows that to complete the proof it is enough to show that if M is a maximal subgroup of L with $H \leq M < L \leq HR$ then $M \triangleleft_{\mathfrak{F}} L$.

Suppose then that M is a maximal subgroup of L with $H \leq M < L \leq HR$. Now $M = H(M \cap R)$ and $L = H(L \cap R)$ so $M \cap R$ is a maximal H-invariant subgroup of $L \cap R$. Refine the series $1 \leq M \cap R < L \cap R \leq R$ to a maximal H-invariant chain of subgroups of R. By (2.3, 12) every link in this refinement is normal so that the refinement forms part of an H-composition series of G. In particular $L \cap R/M \cap R$ is an H-composition factor of G. Since H avoids this factor it follows that $L \cap R/M \cap R$ is an $\mathfrak{F}$-central H-composition factor of G and hence an $\mathfrak{F}$-central chief factor of $H(L \cap R) = L$. But the maximal subgroup M of L complements $L \cap R/M \cap R$ so finally $M \triangleleft_{\mathfrak{F}} L$, as required.

Remark.

The converse of each of the two previous theorems

is false in general. For in example 2.1.13 the subgroup M is a maximal normal subgroup of G but M is not $\mathfrak{F}$-connected to G, and H is $\mathfrak{F}$-serial in G but the H-composition factor N/1 is avoided by H and $\mathfrak{F}$-eccentric.

We shall show later that the situation is far more satisfactory when $\mathfrak{f}(p)$ is subgroup-closed for each prime p. However our immediate goal is the following

Theorem 2.3.6.

Suppose π is the set of all primes and $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for all p. If H is $\mathfrak{F}$-serial in the $\mathfrak{K}$-group G then every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H.

We shall deduce this result from

Lemma 2.3.7.

Suppose $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for each prime $p \in \pi$ and H is a proper $\mathfrak{F}$-serial subgroup of the $\mathfrak{K}$-group G. Let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G. Then $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into some proper subgroup of G containing H.

To prove 2.3.7 we require

Lemma 2.3.8.

Suppose $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for each prime $p \in \pi$ and M is an $\mathfrak{F}$-normal maximal subgroup of the $\mathfrak{K}$-group G. Then every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into M.

Proof.

Let $K = \mathrm{Core}_G(M)$ then $G/K \in \mathfrak{F}$ by (6). If $\underset{\sim}{S}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of G then $\nu_K(\underset{\sim}{S}^{\mathfrak{F}})$ $\mathfrak{F}$-reduces into M/K by 2.1.14, so applying 2.1.12 and 2.1.21 we obtain $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} M$, as required.

Proof of 2.3.7.

Since H is $\mathfrak{F}$-serial in G there is some $\mathfrak{F}$-normal maximal chain $(\Lambda_\sigma , V_\sigma ; \sigma \in \Omega)$ from H to G. Inductively we construct a set $\{H_\alpha ; \alpha \leq \varrho\}$ of subgroups of G containing H and Sylow bases $\underset{\sim}{T}^{(\alpha)}$ of H_α with the following properties:-

(1) $H_\varrho = G$,

(2) $H_\alpha < H_{\alpha+1}$ for each $\alpha < \varrho$,

(3) For each limit ordinal $\gamma \leq \varrho$, $\bigcup_{\alpha<\gamma} H_\alpha \leq H_\gamma$.

(4) For each $\alpha < \varrho$ there exists an element $\sigma \in \Omega$ such that $H_\alpha = \Lambda_\sigma$,

(5) If $\lambda \leq \alpha \leq \varrho$, then $\underset{\sim}{T}^{(\alpha)}$ reduces into H_λ to $\underset{\sim}{T}^{(\lambda)}$.

Since H is by hypothesis a proper subgroup of G there is some element $x \in G - H$. There exists some $\sigma \in \Omega$ such that $x \in \Lambda_\sigma - V_\sigma$. Put $H_0 = \Lambda_\sigma$ and let $\underset{\sim}{T}^{(0)}$ be any Sylow basis of H_0. It is clear that the appropriate conditions among (1),.....,(5) are satisfied, so the construction begins.

Suppose α is not a limit ordinal and we have constructed H_λ, $\underset{\sim}{T}^{(\lambda)}$ for each $\lambda < \alpha$. If $H_{\alpha-1} = G$ we put $\varrho = \alpha-1$ and the construction is complete. If on the other hand $H_{\alpha-1} < G$, there exists some $y \in G - H_{\alpha-1}$. Then $y \in \Lambda_\tau - V_\tau$ for some $\tau \in \Omega$, and we set $H_\alpha = \Lambda_\tau$. By induction $H_{\alpha-1} = \Lambda_\sigma$ for some $\sigma \in \Omega$, and if $\tau < \sigma$ then $y \in \Lambda_\tau \leq \Lambda_\sigma = H_{\alpha-1}$, a contradiction. Thus $\sigma \leq \tau$ and $\Lambda_\sigma \leq \Lambda_\tau$. Since $y \in H_\alpha - H_{\alpha-1}$ we therefore have $H_{\alpha-1} < H_\alpha$. If we now let $\underset{\sim}{T}^{(\alpha)}$ be any Sylow basis of H_α which extends $\underset{\sim}{T}^{(\alpha-1)}$ then it is clear that the construction proceeds.

Suppose γ is a limit ordinal and we have constructed

H_α, $\underset{\sim}{T}^{(\alpha)}$ for each $\alpha < \gamma$. Let $K = \bigcup_{\alpha<\gamma} H_\alpha$ and $T_p^* = \bigcup_{\alpha<\gamma} T_p^{(\alpha)}$. Then $\underset{\sim}{T}^* = \{T_p^*\}$ is a Sylow basis of the subgroup K of G by condition (5). If K = G we put $\varrho = \gamma$ and $\underset{\sim}{T}^{(\varrho)} = \underset{\sim}{T}^*$, and this clearly completes the construction. If K < G there exists some $z \in G - K$ and hence $\delta \in \Omega$ such that $z \in \Lambda_\delta - V_\delta$. Set $H_\gamma = \Lambda_\delta$. If $\alpha < \gamma$ then by induction $H_\alpha = \Lambda_\beta$ for some $\beta \in \Omega$. Now δ cannot be less than or equal to β since this would imply that z belonged to Λ_β and hence to K. Therefore $\beta < \delta$ and $H_\alpha = \Lambda_\beta \leq \Lambda_\delta = H_\gamma$. Thus $\bigcup_{\alpha<\gamma} H_\alpha \leq H_\gamma$ and condition (3) is satisfied. If we now let $\underset{\sim}{T}^{(\gamma)}$ be any Sylow basis of H_γ which extends $\underset{\sim}{T}^*$ then the construction proceeds in this case also.

The above shows that the construction can be carried out. At the final stage only two possibilities can occur. Either (a) $G = \Lambda_\sigma$ for some $\sigma \in \Omega$, or (b) $G = \bigcup_{\alpha<\varrho} H_\alpha$.

<u>Case (a)</u>. If $G = \Lambda_\sigma$ for some $\sigma \in \Omega$ then V_σ is an $\mathfrak{F}$-normal maximal subgroup of G and $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into V_σ by 2.3.8. Since V_σ is a proper subgroup of G containing H the proof is complete in this case.

<u>Case (b)</u>. If $G = \bigcup_{\alpha<\varrho} H_\alpha$ then by condition (1) ϱ is a limit ordinal and $H_\alpha < G$ for each $\alpha < \varrho$. Now $\underset{\sim}{T}^{(\varrho)}$ is a Sylow basis of G so there is some element $g \in G$ such that $\underset{\sim}{S} = \underset{\sim}{T}^{(\varrho)g}$. There exists $\alpha < \varrho$ such that $g \in H_\alpha$, so by condition (5), $\underset{\sim}{S}$ reduces into $H_\alpha^g = H_\alpha$. Thus H_α is a proper subgroup of G containing H such that, by 2.1.6(iii), $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H_\alpha$, so the proof is complete in this case also.

Proof of Theorem 2.3.6.

We have H $\mathfrak{F}$-ser G and have to show that every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H. Suppose that there is an $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of G which does not $\mathfrak{F}$-reduce into H. Let L be the intersection of all those subgroups of G containing H into which $\underset{\sim}{S}^{\mathfrak{F}}$ does $\mathfrak{F}$-reduce. Then $\underset{\sim}{S}^{\mathfrak{F}} \searrow_{\mathfrak{F}} L$ by 2.1.18. Suppose $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into L to $\underset{\sim}{T}^{\mathfrak{F}}$.

Now $\mathfrak{F}$ is subgroup-closed since each $\mathfrak{f}(p)$ is subgroup-closed, so H is $\mathfrak{F}$-serial in L by 2.3.3(i). H is a proper subgroup of L since $\underset{\sim}{S}^{\mathfrak{F}}$ does not $\mathfrak{F}$-reduce into H, so by 2.3.7 there is a proper subgroup K of L containing H into which $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces. By 2.1.11(ii) $\underset{\sim}{S}^{\mathfrak{F}}$ now $\mathfrak{F}$-reduces into K which contradicts the definition of L. With this contradiction the proof of 2.3.6 is complete.

Lemma 2.3.9.

Suppose $\mathfrak{f}(p) = \mathfrak{s}\mathfrak{f}(p)$ for each prime $p \in \pi$ and $H \leq G \in \mathfrak{K}$. If every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H then every H-composition factor of G avoided by H is $\mathfrak{F}$-central.

Proof.

If A/B is an H-composition factor of G avoided by H then HB is a maximal subgroup of HA. Let $\underset{\sim}{T}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of HA and $\underset{\sim}{S}^{\mathfrak{F}}$ an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into HA to $\underset{\sim}{T}^{\mathfrak{F}}$; such an $\mathfrak{F}$-system exists by 2.1.11(i). Suppose that every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H, then $\underset{\sim}{S}^{\mathfrak{F}}$, and hence, by 2.1.16, $\underset{\sim}{T}^{\mathfrak{F}}$, $\mathfrak{F}$-reduces into H. Now $B \triangleleft HA$ so, by 2.1.19, $\nu_B(\underset{\sim}{T}^{\mathfrak{F}})$ $\mathfrak{F}$-reduces into HB/B. Thus, by 2.1.21, $\underset{\sim}{T}^{\mathfrak{F}} \searrow_{\mathfrak{F}} HB$. The above shows that every $\mathfrak{F}$-system of HA $\mathfrak{F}$-reduces into HB, so by 2.2.9(i) $HB \triangleleft_{\mathfrak{F}} HA$. Now HB complements

the chief factor A/B of HA, so this chief factor must be $\mathfrak{F}$-central. Hence A/B is an $\mathfrak{F}$-central H-composition factor of G, proving the result.

Collecting our last few results we have the following

Theorem 2.3.10.

Suppose π is the set of all primes, $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for all p, and $H \leq G \in \mathfrak{K}$. Then the following three conditions are equivalent:-

(1) H is $\mathfrak{F}$-serial in G,

(2) Every H-composition factor of G avoided by H is $\mathfrak{F}$-central,

(3) Every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H.

The proof is immediate from 2.3.5, 2.3.6, and 2.3.9.

Remark.

In general none of the three statements in 2.3.10 are equivalent. For in the remark following the proof of 2.3.5 we saw that (1) implies neither (2) nor (3) when the preformations $\mathfrak{f}(p)$ are not subgroup-closed. The inequivalence of (2) and (3) can also be seen from example 2.1.13. For the group G in that example has a unique M-composition series, namely $1 \triangleleft N \triangleleft M \triangleleft G$. Thus the only M-composition factor of G avoided by M is G/M and this is certainly $\mathfrak{F}$-central. However M is not $\mathfrak{F}$-connected to G, so M satisfies (2) but not (3).

Corollary 2.3.11.

Suppose π is the set of all primes and $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for all p. If H is $\mathfrak{F}$-serial in the $\mathfrak{K}$-group G and $N \triangleleft G$ then HN/N is $\mathfrak{F}$-serial in G/N.

Proof.

It is clear that condition (2) in 2.3.10 holds for HN/N in G/N if it holds for H in G. The proof is therefore immediate from 2.3.10.

Taking $\mathfrak{K} = \mathfrak{U}$ and $\mathfrak{F} = \mathfrak{M}$, we have the following special cases of 2.3.10 and 2.3.11.

Theorem 2.3.12.

Let H be a subgroup of a $\mathfrak{U}$-group G. Then the following three conditions are equivalent:-

(1) H is serial in G,

(2) H centralizes every H-composition factor of G which it avoids,

(3) Every Sylow basis of G reduces into H.

Corollary 2.3.13.

If H is serial in the $\mathfrak{U}$-group G and $N \triangleleft G$ then HN/N is serial in G/N.

Remark.

Theorem 2.3.12 is a generalization of the well-known result that a subgroup H of a finite soluble group G is subnormal in G if and only if every Sylow system of G reduces into H. Corollary 2.3.13 in fact holds in arbitrary locally finite groups (15).

As a further corollary to 2.3.10 we have

Corollary 2.3.14.

Suppose π is the set of all primes and $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for all p. Suppose further that $H\ \mathfrak{F}\text{-}\underline{ser}\ K \leq G \in \mathfrak{K}$. Then every H-composition factor of G covered by K and avoided by H is $\mathfrak{F}$-central.

Proof.

Suppose that A/B is an H-composition factor of G covered by K and avoided by H. Then $A = (K \cap A)B$ so that the factor A/B is H-isomorphic to $K \cap A/K \cap B$. It follows that $K \cap A/K \cap B$ is an H-composition factor of K avoided by H, so, by 2.3.10, it must be $\mathfrak{F}$-central. Therefore A/B is an $\mathfrak{F}$-central H-composition factor of G, as required.

Remark.

Suppose π is the set of all primes and $f(p) = sf(p)$ for all p. If H is an $\mathfrak{F}$-serial subgroup of the $\mathfrak{K}$-group G then, by 2.3.10, every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H and hence $G = R_G(H ; \mathfrak{F})$. It is not true however that H is $\mathfrak{F}$-serial in G if $R_G(H ; \mathfrak{F}) = G$, and we give an example to illustrate this point.

Example.

We let $\mathfrak{K} = \mathfrak{S}^*$ the class of finite soluble groups and $\mathfrak{F} = \mathfrak{N}^*$ the class of finite nilpotent groups. Clearly $\mathfrak{N}^*$ is the saturated $\mathfrak{S}^*$-formation defined by the $\mathfrak{S}^*$-formation function $f(p) = 1$ on the set of all primes. If $H \leq G \in \mathfrak{S}^*$ then $R_G(H ; \mathfrak{N}^*) = R_G(H)$ and H is $\mathfrak{N}^*$-subnormal in G if and only if H is subnormal in G. The following example is of a finite soluble group G with a non-subnormal subgroup whose reducer in G is the whole of G.

Let $H = \langle x,y ; x^4 = y^2 = 1, x^y = x^{-1} \rangle$ be a dihedral group of order 8. Then there is a faithful irreducible representation

$$x \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad y \to \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

of H over $GF(3)$, the field with three elements. Let V be a vector space of dimension 2 over $GF(3)$ on which

H acts in the manner described above. Let $G = VH$ be the semidirect product of V by H. Clearly G is a finite soluble group.

Let $K = \langle y \rangle$, $S_2 = H$ and $S_3 = V$. Then $\underset{\sim}{S} = \{S_2, S_3\}$ is a Sylow basis of G which reduces into K. Since $H \leq N_G(\underset{\sim}{S})$ we have $H \leq R_G(K)$. Now $V = \langle v_1 \rangle \oplus \langle v_2 \rangle$ where $v_1^y = -v_1$ and $v_2^y = v_2$. Therefore $v_2 \in R_G(K)$ since it centralizes K. Since G is generated by H and v_2 it follows that $R_G(K) = G$.

We claim that K is not subnormal in G. For if it were then every Sylow basis of G would have to reduce into K. If $\underset{\sim}{S}^{v_1}$ reduced into K we would have $K \leq H^{v_1}$ and hence $[K, v_1] \leq H^{v_1} \cap V = 1$. Since K does not centralize v_1, it follows that K is not subnormal in G.

Suppose $H \leq G \in \mathfrak{K}$. We shall say that H is an <u>$\mathfrak{F}$-superserial subgroup</u> of G, and write H $\mathfrak{F}$-<u>superser</u> G, if every $\mathfrak{F}$-system of G $\mathfrak{F}$-reduces into H.

By 2.3.4, $\mathfrak{F}$-superserial implies $\mathfrak{F}$-serial. Moreover, by 2.3.10, the two concepts coincide when π is the set of all primes and each preformation $\mathfrak{f}(p)$ is subgroup-closed.

Suppose $H \leq K \leq G \in \mathfrak{K}$. We say K is the <u>$\mathfrak{F}$-serializer</u> of H in G if (i) H $\mathfrak{F}$-<u>ser</u> K, and (ii) H $\mathfrak{F}$-<u>ser</u> $L \leq G$ implies $L \leq K$.

We say K is the <u>$\mathfrak{F}$-superserializer</u> of H in G if (i) H $\mathfrak{F}$-<u>superser</u> K, and (ii) H $\mathfrak{F}$-<u>superser</u> $L \leq G$ implies $L \leq K$.

Of course, by 2.3.10, these concepts coincide when π is the set of all primes and each preformation $\mathfrak{f}(p)$ is subgroup-closed. However, in general they do not. For in example 2.1.13, the $\mathfrak{F}$-serializer of H

in G is clearly G but the $\mathfrak{F}$-superserializer of M in G is M itself.

From 2.3.4, it is immediate that if K is the $\mathfrak{F}$-serializer of H in the $\mathfrak{K}$-group G then K contains every subgroup of G in which H is $\mathfrak{F}$-superserial.

If $H \leq G \in \mathfrak{K}$ we say H has a <u>strong $\mathfrak{F}$-superserializer</u> in G if H is $\mathfrak{F}$-connected to G and $R_G(H;\mathfrak{F})$ is the $\mathfrak{F}$-superserializer of H in G. In this situation we often refer to $R_G(H;\mathfrak{F})$ as the <u>strong $\mathfrak{F}$-superserializer</u> of H in G.

A subgroup may have an $\mathfrak{F}$-superserializer but not a strong one. For example the subgroup M in 2.1.13 is its own $\mathfrak{F}$-superserializer but M is not $\mathfrak{F}$-connected to G so does not possess a strong $\mathfrak{F}$-superserializer in G.

We shall be primarily concerned with the case when π is the set of all primes and the preformations $\mathfrak{f}(p)$ are subgroup-closed. In this case, by 2.2.5 and 2.3.10, a subgroup H of a $\mathfrak{K}$-group G has a strong $\mathfrak{F}$-superserializer in G if and only if H is $\mathfrak{F}$-serial in $R_G(H;\mathfrak{F})$. In this situation we say that H has a <u>strong $\mathfrak{F}$-serializer</u> in G, or alternatively that $R_G(H;\mathfrak{F})$ is the <u>strong $\mathfrak{F}$-serializer</u> of H in G.

When $\mathfrak{K} = \mathfrak{U}$ and $\mathfrak{F} = \mathfrak{N}$ the concepts "$\mathfrak{N}$-serializer" and "strong $\mathfrak{N}$-serializer" reduce to "serializer" and "strong serializer" respectively, these latter being defined in the obvious way (cf.(10)). Even in this case it is possible for a subgroup to have a serializer which is not strong. For if H is a self-normalizing subgroup of a finite soluble group G then the serializer (i.e. subnormalizer) of H in G is H itself. If H is not abnormal in G then H cannot be subnormal in $R_G(H)$ by 2.2.13,

so does not possess a strong subnormalizer in G.

Lemma 2.3.15.

Suppose π is the set of all primes and $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for all p. Suppose further that $H \leq K \leq G \in \mathfrak{K}$. Then the following three conditions are equivalent:-

(i) H $\mathfrak{F}$-ser K and $K = R_G(H ; \mathfrak{F})$,

(ii) K covers every H-composition factor of G which is either $\mathfrak{F}$-central or covered by H and avoids the rest,

(iii) H $\mathfrak{F}$-ser K and K covers every $\mathfrak{F}$-central H-composition factor of G.

Proof.

(i) => (ii) If $K = R_G(H ; \mathfrak{F})$ then K covers every H-composition factor of G which is either $\mathfrak{F}$-central or covered by H, by 2.2.16. On the other hand K certainly covers or avoids every H-composition factor of G, and if H $\mathfrak{F}$-ser K then those that K covers but H avoids are $\mathfrak{F}$-central, by 2.3.14. Thus if K satisfies (i) then K covers every H-composition factor of G which is either $\mathfrak{F}$-central or covered by H and avoids the rest.

(ii) => (iii). Clearly we need only show that if K satisfies condition (ii) then H $\mathfrak{F}$-ser K. This we do by induction on the $\mathfrak{A}$-length of G which we may assume to be greater than zero. Let $R = \mathfrak{C}(G)$ and let $(A/R)/(B/R)$ be an HR/R-composition factor of G/R. Then $(A/R)/(B/R)$ is an $\mathfrak{F}$-central HR/R-composition factor of G/R if and only if A/B is an $\mathfrak{F}$-central H-composition factor of G. Moreover KR/R covers $(A/R)/(B/R)$ if and only if K covers A/B. Thus $HR/R \leq KR/R \leq G/R$ and KR/R covers every HR/R-composition factor of G/R which is either $\mathfrak{F}$-central or covered by HR/R and avoids the rest. Thus by induction, HR/R $\mathfrak{F}$-ser KR/R and hence HR $\mathfrak{F}$-ser KR. Therefore, by 2.3.3(i), $H(K \cap R) = HR \cap K$ $\mathfrak{F}$-ser K, so we need only

show that H $\mathfrak{F}$-<u>ser</u> $H(K \cap R)$.

Suppose M is a maximal subgroup of L with $H \leq M < L \leq H(K \cap R)$. We shall show that $M \lhd_{\mathfrak{F}} L$, and from this it will follow, as in the proof of 2.3.4, that H $\mathfrak{F}$-<u>ser</u> $H(K \cap R)$, as required.

Now $M = H(M \cap R)$ and $L = H(L \cap R)$ so by the usual argument (cf. the proof of 2.3.5) $L \cap R/M \cap R$ is an H-composition factor of G. Clearly H avoids this factor. Now $L = L \cap H(K \cap R) = H(L \cap K \cap R)$ so that $L = M(L \cap K \cap R)$ and hence $L \cap R = (M \cap R)(L \cap K \cap R)$ by the modular law. Thus K covers and H avoids the H-composition factor $L \cap R/M \cap R$ of G which by hypothesis is therefore $\mathfrak{F}$-central. Hence $L \cap R/M \cap R$ is an $\mathfrak{F}$-central chief factor of L complemented by the maximal subgroup M. Thus $M \lhd_{\mathfrak{F}} L$, as required.

(iii) => (i) If K covers every $\mathfrak{F}$-central H-composition factor of G then $R_G(H;\mathfrak{F}) \leq K$ by 2.2.16. If H $\mathfrak{F}$-<u>ser</u> K then, by 2.2.5 and 2.3.10, $K = R_K(H;\mathfrak{F}) \leq R_G(H;\mathfrak{F})$. Thus if condition (iii) holds then $K = R_G(H;\mathfrak{F})$ and (i) follows, completing the proof.

<u>Lemma 2.3.16.</u>

Suppose π is the set of all primes, $\mathfrak{f}(p) = \mathfrak{s}\mathfrak{f}(p)$ for all p and $H \leq G \in \mathfrak{K}$. Then H has a strong $\mathfrak{F}$-serializer in G if and only if the set of $\mathfrak{F}$-systems of G which $\mathfrak{F}$-reduce into H forms a block.

<u>Proof.</u>

As at the end of section 2.2 we let Δ be the set of $\mathfrak{F}$-systems of G which $\mathfrak{F}$-reduce into H and Δ_o the block generated by Δ in the set Ω of all $\mathfrak{F}$-systems of G. Using this notation we have to prove that H has a strong $\mathfrak{F}$-serializer in G if and only if $\Delta = \Delta_o$.

By 2.2.20, $\Delta = \Delta_o$ if and only if the same $\mathfrak{F}$-systems of G $\mathfrak{F}$-reduce into H and $R_G(H\ ;\mathfrak{F})$. Now if $\underset{\sim}{T}^{\mathfrak{F}}$ is an $\mathfrak{F}$-system of $R_G(H\ ;\mathfrak{F})$ and $\Delta = \Delta_o$ then, by 2.1.11(i) and 2.1.16, $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into H. If conversely every $\mathfrak{F}$-system of $R_G(H\ ;\mathfrak{F})$ $\mathfrak{F}$-reduces into H then, by 2.2.2(ii) and 2.1.11(ii), $\Delta = \Delta_o$. Thus $\Delta = \Delta_o$ if and only if every $\mathfrak{F}$-system of $R_G(H\ ;\mathfrak{F})$ $\mathfrak{F}$-reduces into H, i.e. if and only if H $\mathfrak{F}$-<u>superser</u> $R_G(H\ ;\mathfrak{F})$ The result is now immediate from the definition of strong $\mathfrak{F}$-serializer.

Collecting together our previous results and remarks we have

<u>Theorem 2.3.17.</u>

Suppose π is the set of all primes, $\mathfrak{S}(p) = s\mathfrak{S}(p)$ for all p, and $H \leq G \in \mathfrak{K}$. Then H has a strong $\mathfrak{F}$-serializer in G if and only if one (and hence all) of the following equivalent conditions holds:-

(1) H $\mathfrak{F}$-<u>ser</u> $R_G(H\ ;\mathfrak{F})$,

(2) H $\mathfrak{F}$-<u>superser</u> $R_G(H\ ;\mathfrak{F})$,

(3) H has a strong $\mathfrak{F}$-superserializer in G,

(4) $R_G(H\ ;\mathfrak{F})$ covers every H-composition factor of G which is either $\mathfrak{F}$-central or covered by H and avoids the rest,

(5) $R_G(H\ ;\mathfrak{F})$ avoids every $\mathfrak{F}$-eccentric H-composition factor of G which H avoids,

(6) The set of $\mathfrak{F}$-systems of G which $\mathfrak{F}$-reduce into H forms a block in the set of all $\mathfrak{F}$-systems of G.

<u>Proof.</u>

As remarked earlier (1),(2) and (3) are, under the given conditions, equivalent formulations of the definition of strong $\mathfrak{F}$-serializer. (4) and (5) are

equivalent by 2.2.16, so the result follows from 2.3.15 and 2.3.16.

Lemma 2.3.18.

Suppose π is the set of all primes, $f(p) = sf(p)$ for all p, $H \leq G \in \mathfrak{K}$, and $N \triangleleft G$. If X is the strong $\mathfrak{F}$-serializer of H in G then XN/N is the strong $\mathfrak{F}$-serializer of HN/N in G/N.

Proof.

If X is the strong $\mathfrak{F}$-serializer of H in G then by definition H $\mathfrak{F}$-ser X = $R_G(H ; \mathfrak{F})$. Therefore $H(X \cap N)/(X \cap N)$ $\mathfrak{F}$-ser $X/(X \cap N)$ by 2.3.11. By considering the natural isomorphism between XN/N and $X/(X \cap N)$ it follows easily that HN/N $\mathfrak{F}$-ser XN/N. But by 2.2.7 $XN/N = R_{G/N}(HN/N ; \mathfrak{F})$, so by definition XN/N is the strong $\mathfrak{F}$-serializer of HN/N in G/N.

Remark.

In comparison with 2.3.18 we now give an example to show that in general $\mathfrak{F}$-serializers are not "homomorphism invariant".

Let G be the group defined in the example after 2.3.14, i.e. G is the semidirect product VH of a 2-dimensional vector space V over GF(3) by a dihedral group H. With the notation used in that example let $L = \langle v_2 \rangle \times \langle y \rangle$. It is easy to see that the normalizer of L in G is the group $L\langle x^2 \rangle$ and that this subgroup is also the subnormalizer of L in G. However G/V is a 2-group so that G/V is the subnormalizer of LV/V in G/V. Thus subnormalizers are in general not homomorphism invariant.

We obtain the following generalizations of results of Mann (21) if we take $\mathfrak{K} = \mathfrak{U}$ and $\mathfrak{F} = \mathfrak{M}$ in 2.3.17 and 2.3.18.

Theorem 2.3.19.

Suppose $H \leq G \in \mathfrak{U}$. Then H has a strong serializer in G if and only if one (and hence all) of the following equivalent conditions holds:-

(1) H ser $R_G(H)$,

(2) Every Sylow basis of $R_G(H)$ reduces into H,

(3) $R_G(H)$ avoids every H-eccentric H-composition factor of G which H avoids,

(4) The set of Sylow bases of G which reduce into H forms a block in the set of all Sylow bases of G.

Corollary 2.3.20.

Suppose $H \leq G \in \mathfrak{U}$ and $N \triangleleft G$. If X is the strong serializer of H in G then XN/N is the strong serializer of HN/N in G/N.

We close this section with a proof of

Lemma 2.3.21.

Suppose π is the set of all primes and $G = RH \in \mathfrak{K}$ where R is a normal $\mathfrak{M}$-subgroup of G and $H \in \mathfrak{F}$. Then

(i) H is $\mathfrak{F}$-connected to G,

(ii) $R_G(H ; \mathfrak{F})$ is an $\mathfrak{F}$-projector of G,

(iii) $R_G(H ; \mathfrak{F})$ is the strong $\mathfrak{F}$-superserializer of H in G,

(iv) H $\mathfrak{F}$-ser $R_G(H ; \mathfrak{F})$.

Corollary 2.3.22.

Suppose π is the set of all primes and $f(p) = sf(p)$ for all p. If $G = RH \in \mathfrak{K}$ where R is a normal $\mathfrak{M}$-subgroup

of G and $H \in \mathfrak{F}$, then $R_G(H;\mathfrak{F})$ is an $\mathfrak{F}$-projector of G and is moreover the strong $\mathfrak{F}$-serializer of H in G.

As a special case of this result we have

Corollary 2.3.23.

Suppose $G = RH \in \mathfrak{U}$ where $R \triangleleft G$ and $R, H \in \mathfrak{N}$. Then $R_G(H)$ is a Carter subgroup of G and is moreover the strong serializer of H in G.

Remark.

Corollary 2.3.23 extends a well-known result of Rose for finite soluble groups ((Lemma 1, 29) or (Lemma 12, 21)). For if G is finite then $R_G(H) = N_G^{\infty}(H)$, the hypernormalizer of H in G. To see this we notice that H is subnormal in $N_G^{\infty}(H)$ so $N_G^{\infty}(H) \leq R_G(H)$. But now $N_G^{\infty}(H)$ is a self-normalizing subgroup of the nilpotent group $R_G(H)$ (by 2.3.23) so we must have $R_G(H) = N_G^{\infty}(H)$ as claimed.

Proof of 2.3.21.

(i) The fact that H is $\mathfrak{F}$-connected to G is immediate from 2.1.17(i).

(ii) By 1.2.5(6) there is an $\mathfrak{F}$-projector E of G containing H, and since $G = RH$ we have $E = H(R \cap E)$. Thus by 2.1.17(ii) every $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H also $\mathfrak{F}$-reduces into E, and so, by 2.2.14(i), $R_G(H;\mathfrak{F}) \leq R_G(E;\mathfrak{F}) = E$. Let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into H. Then $\underset{\sim}{S}^{\mathfrak{F}}$ must $\mathfrak{F}$-reduce into E to the unique $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of E, and, by 2.1.17(iii), $\underset{\sim}{T}^{\mathfrak{F}} \searrow_{\mathfrak{F}} H$. Since every element of E normalizes $\underset{\sim}{T}^{\mathfrak{F}}$, it follows, from 2.1.17(iii), that $\underset{\sim}{S}^{e\mathfrak{F}}$ $\mathfrak{F}$-reduces into H for each $e \in E$. Hence $E \leq R_G(H;\mathfrak{F})$

and (ii) now follows.

(iii) As shown above the unique $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of $E = R_G(H\ ;\mathfrak{F})$ $\mathfrak{F}$-reduces into H. Thus by definition, H $\mathfrak{F}$-<u>superser</u> E. Suppose that H $\mathfrak{F}$-<u>superser</u> $K \leq G$. Now $K = H(K \cap R)$ so by 2.1.17, $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into K say to the $\mathfrak{F}$-system $\underset{\sim}{U}^{\mathfrak{F}}$ of K. But every $\mathfrak{F}$-system of K $\mathfrak{F}$-reduces into H so $\underset{\sim}{U}^{k\mathfrak{F}}$, and hence, by 2.1.17(iii), $\underset{\sim}{S}^{k\mathfrak{F}}$, $\mathfrak{F}$-reduces into H for each $k \in K$. Thus $K \leq R_G(H\ ;\mathfrak{F})$ and $R_G(H\ ;\mathfrak{F})$ is the $\mathfrak{F}$-superserializer of H in G. (iii) now follows from the definition.

(iv) is an immediate consequence of 2.3.4.

2.4. $\mathfrak{F}$-reducers and $\mathfrak{F}$-serializers of $\mathfrak{F}$-normalizers.

We begin this section be stating a lemma concerning $\mathfrak{F}$-normalizers which does not require the overriding condition 2.1.1. A proof of this result is given by Tomkinson (4.16, 34) for periodic locally soluble FC-groups, and it is clear that his proof carries over to $\mathfrak{U}$-groups.

Lemma 2.4.1.

Suppose $\mathfrak{K} = \mathfrak{Q}\mathfrak{S}\mathfrak{K} \leq \mathfrak{U}$ and $\mathfrak{f}$ is a $\mathfrak{K}$-preformation function on the set of primes π (not necessarily satisfying 2.1.1). Let $\mathfrak{F}$ be the saturated $\mathfrak{K}$-formation defined by $\mathfrak{f}$. If D is an $\mathfrak{F}$-normalizer of the $\mathfrak{K}$-group G and $N_\lambda \triangleleft G$ $(\lambda \in \Lambda)$ then $\bigcap_{\lambda \in \Lambda} (DN_\lambda) = D(\bigcap_{\lambda \in \Lambda} N_\lambda)$.

We now return to the situation governed by the condition 2.1.1.

Theorem 2.4.2.

Suppose π is the set of all primes. Let D be an $\mathfrak{F}$-normalizer of the $\mathfrak{K} \cap (\mathfrak{U})^2\mathfrak{F}$-group G and E the unique $\mathfrak{F}$-projector of G containing D (cf. 1.2.5(4)). Then E is the strong $\mathfrak{F}$-superserializer of D in G. Moreover D is $\mathfrak{F}$-serial in E and E contains every subgroup of G in which D is $\mathfrak{F}$-subnormal.

As usual a subgroup H of a $\mathfrak{K}$-group G is <u>$\mathfrak{F}$-subnormal</u> in G if there is a finite $\mathfrak{F}$-normal maximal chain from H to G, i.e. a chain

$$H = H_0 < H_1 < \ldots\ldots\ldots\ldots < H_n = G$$

in which H_i is an $\mathfrak{F}$-normal maximal subgroup of H_{i+1} for each $i = 0, \ldots\ldots, n-1$.

In 2.4.2 we have been unable to decide whether E is the $\mathfrak{F}$-serializer of D in G, i.e. if E contains every subgroup of G which contains D as an $\mathfrak{F}$-serial subgroup. However we do have the following special cases of 2.4.2.

Corollary 2.4.3.

Suppose $\mathfrak{F}$ is a saturated formation of finite soluble groups which contains the class of finite nilpotent groups. Let D be an $\mathfrak{F}$-normalizer of the $(\mathfrak{N}^*)^2\mathfrak{F}$-group G and E the unique $\mathfrak{F}$-projector of G which contains D. Then E is both the strong $\mathfrak{F}$-supersubnormalizer and the $\mathfrak{F}$-subnormalizer of D in G.

Proof.

The proof of this result is immediate from 2.4.2 once we remark that every saturated formation of the given type can be defined by an $\mathfrak{S}^*$-formation function on the set of all primes.

Remark.

Corollary 2.4.3 is a stronger version of a result of Hawkes (Theorem C, 17).

Corollary 2.4.4.

Suppose π is the set of all primes and $f(p) = \mathfrak{S}f(p)$ for all p. Let D be an $\mathfrak{F}$-normalizer of $G \in \mathfrak{K} \cap (\mathfrak{N})^2\mathfrak{F}$ and E the unique $\mathfrak{F}$-projector of G containing D. Then E is the strong $\mathfrak{F}$-serializer of D in G.

Corollary 2.4.5.

Suppose $G \in \mathfrak{U} \cap (\mathfrak{N})^3$ and D is a basis normalizer of G. Then the unique Carter subgroup of G containing D is the strong serializer of D in G.

Remark.

Corollary 2.4.5 generalizes a well-known result of Carter for finite soluble groups with nilpotent length at most three (Theorem 3, $\underset{\sim}{3}$). Carter's result was later reproved by Alperin (Theorem 5, $\underset{\sim}{1}$).

Proof of Theorem 2.4.2.

Let $R = Q(G)$. By 1.2.5(3,5), $ER = DR$ so that $E = D(E \cap R)$. Now D is $\mathfrak{F}$-connected to G by 2.1.9, and lemma 2.1.17 shows that every $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into D also $\mathfrak{F}$-reduces into E. Thus, by 2.2.14(i), $R_G(D\ ;\mathfrak{F}) \leq E$. Let $\underset{\sim}{S}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into D. Then $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into E to the unique $\mathfrak{F}$-system $\underset{\sim}{T}^{\mathfrak{F}}$ of E and $\underset{\sim}{T}^{\mathfrak{F}} \searrow_{\mathfrak{F}} D$ (2.1.17). Since every element of E normalizes $\underset{\sim}{T}^{\mathfrak{F}}$ we deduce, from 2.1.17(iii), that $E \leq R_G(D\ ;\mathfrak{F})$. Thus by our previous inequality $E = R_G(D\ ;\mathfrak{F})$. Since the unique $\mathfrak{F}$-system of E $\mathfrak{F}$-reduces into D we further have D $\mathfrak{F}$-<u>superser</u> E.

Suppose now that D $\mathfrak{F}$-<u>superser</u> $H \leq G$. Then $H = R_H(D\ ;\mathfrak{F})$ so, by 2.2.7, $HR/R = R_{HR/R}(DR/R\ ;\mathfrak{F})$. Now $DR/R = ER/R$ is an $\mathfrak{F}$-projector of HR/R so, by 2.2.14(i), we must have $DR = HR$ and hence $H = D(H \cap R)$. The usual argument, using 2.1.17 and the fact that every $\mathfrak{F}$-system of H $\mathfrak{F}$-reduces into D, now yields $H \leq R_G(D\ ;\mathfrak{F}) = E$. Thus E contains every subgroup of G in which D is $\mathfrak{F}$-superserial, so by definition E is the strong $\mathfrak{F}$-superserializer of D in G. Since D is $\mathfrak{F}$-serial in E by 2.3.4, there remains only to show that E contains every subgroup of G in which D is $\mathfrak{F}$-subnormal.

Suppose D is $\mathfrak{F}$-subnormal in the subgroup H of G. Then there is a chain $D = D_0 < D_1 < \ldots\ldots < D_n = H$ from D to H in which D_i is an $\mathfrak{F}$-normal maximal subgroup

of D_{i+1} for each $i = 0,\ldots,n-1$. Now DR/R is an $\mathfrak{F}$-projector of G/R so certainly $DR \rtimes_{\mathfrak{F}} G$. If $D_iR < D_{i+1}R$ for some $i \in \{0,\ldots,n-1\}$ then D_iR is an $\mathfrak{F}$-normal maximal subgroup of $D_{i+1}R$ which is impossible since D_iR contains the $\mathfrak{F}$-abnormal subgroup DR of G. Thus $DR = D_oR = D_1R = \ldots\ldots = D_nR = HR$ and hence $D_i = D(D_i \cap R)$ for each $i = 0,\ldots\ldots,n$.

Suppose that $D_{i-1} \leq E$ for some $i \in \{1,\ldots,n\}$. Now the $\mathfrak{F}$-system $\underset{\sim}{S}^{\mathfrak{F}}$ of G $\mathfrak{F}$-reduces into D so, by 2.1.17, $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into D_i, say to the $\mathfrak{F}$-system $\underset{\sim}{U}^{\mathfrak{F}}$ of D_i. Let Q be the $\mathfrak{F}$-normalizer of D_i associated with the Sylow basis $\underset{\sim}{U}$. Then Q normalizes $\underset{\sim}{U}^{\mathfrak{F}}$ so, from 2.1.17(iii), we deduce in the usual way that $Q \leq R_G(D\,;\mathfrak{F}) = E$. Since $D_{i-1} \lhd_{\mathfrak{F}} D_i$ we have $D_i/\mathrm{Core}_{D_i}(D_{i-1}) \in \mathfrak{F}$ and hence, by (4.6, $\underset{\sim}{7}$), $D_i = Q.\mathrm{Core}_{D_i}(D_{i-1})$. But $D_{i-1} \leq E$ by our hypothesis above, so therefore $D_i \leq E$. The above shows that if $D_{i-1} \leq E$ then $D_i \leq E$. Since $D_o = D \leq E$ it follows that $H = D_n \leq E$. Thus E contains every subgroup of G in which D is $\mathfrak{F}$-subnormal, and this completes the proof.

<u>Corollary 2.4.6.</u>

Suppose π is the set of all primes. If D is an $\mathfrak{F}$-normalizer of $G \in \mathfrak{K} \cap (\mathfrak{M})^t\mathfrak{F}$ $(t \geq 2)$ then $R_G(D\,;\mathfrak{F}) \in \mathfrak{K} \cap (\mathfrak{M})^{t-2}\mathfrak{F}$.

<u>Proof.</u>

We argue by induction on t. If $t = 2$ then, by 2.4.2, $R_G(D\,;\mathfrak{F})$ is the unique $\mathfrak{F}$-projector of G containing D, so $R_G(D\,;\mathfrak{F}) \in \mathfrak{K} \cap \mathfrak{F} = \mathfrak{F}$, and the induction begins. If $t > 2$ let $R = \mathcal{R}(G)$. Then, by induction and 2.2.7, $XR/R \in \mathfrak{K} \cap (\mathfrak{M})^{t-3}\mathfrak{F}$ where $X = R_G(D\,;\mathfrak{F})$. Now $X \cap R$ is a normal $\mathfrak{M}$-subgroup of X and $X/X \cap R \cong XR/R \in (\mathfrak{M})^{t-3}\mathfrak{F}$. Thus $X \in (\mathfrak{M})^{t-2}\mathfrak{F}$ and since $\mathfrak{K}$ is subgroup-closed the proof is complete.

Another result on $\mathfrak{K} \cap (\mathfrak{M})^2\mathfrak{F}$-groups is the following which holds for any saturated $\mathfrak{K}$-formation.

Theorem 2.4.7.

Let f be a $\mathfrak{K}$-preformation function on the set of primes π (not necessarily satisfying 2.1.1) and $\mathfrak{F}$ the saturated $\mathfrak{K}$-formation defined by f. If H is a subgroup of $G \in \mathfrak{K} \cap (\mathfrak{M})^2\mathfrak{F}$ which contains an $\mathfrak{F}$-normalizer of G then every $\mathfrak{F}$-projector of H has the form $E \cap H$ for some $\mathfrak{F}$-projector E of G. In particular this result holds when H is $\mathfrak{F}$-ascendabnormal in G.

Proof.

Suppose H contains the $\mathfrak{F}$-normalizer D of G. Let E be the unique $\mathfrak{F}$-projector of G containing D and $R = \mathfrak{R}(G)$. Then DR = ER by 1.2.5(3,5) and in particular therefore $E \leq HR$. Thus E is an $\mathfrak{F}$-projector of HR and the first statement of the theorem follows from 1.2.5(6). The second statement is an immediate consequence of 1.2.8.

Corollary 2.4.8.

If H is an ascendabnormal subgroup of $G \in \mathfrak{U} \cap (\mathfrak{M})^3$ then every Carter subgroup of H has the form $E \cap H$ for some Carter subgroup E of G.

Remark.

Corollary 2.4.8 generalizes a result of Alperin (Theorem 10, 1) since a nonnormal maximal subgroup of any group is certainly an abnormal subgroup.

Our aim now is to give several different descriptions of the $\mathfrak{F}$-reducer of an $\mathfrak{F}$-normalizer. However we first have the following elementary lemma.

Lemma 2.4.9.

(i) If $\mathfrak{F} = S\mathfrak{F}$ and $H \leq G \in \mathfrak{F}$ then H $\mathfrak{F}$-ser G.

(ii) If $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for each $p \in \pi$ and $H \leq G \in \mathfrak{F}$ then H $\mathfrak{F}$-superser G.

Proof.

(i) Since $\mathfrak{F}$ is subgroup-closed the chain (G,H) from H to G shows, by 2.3.2, that H $\mathfrak{F}$-ser G.

(ii) By 2.1.12, H is $\mathfrak{F}$-connected to G so the unique $\mathfrak{F}$-system of G must $\mathfrak{F}$-reduce into H. Thus H $\mathfrak{F}$-superser G, as claimed.

Remark.

Neither of the conclusions in 2.4.9 holds if the subgroup closure hypotheses are removed. For Hawkes (3.8, 17) gives an example of a saturated formation $\mathfrak{F}$ of finite soluble groups and an $\mathfrak{F}$-subgroup X of an $\mathfrak{F}$-group G such that X is not $\mathfrak{F}$-subnormal in G. Thus 2.4.9(i) is false in general and example 2.1.13 gives a counterexample in the case of 2.4.9(ii).

Suppose D is the $\mathfrak{F}$-normalizer of the $\mathfrak{K}$-group G associated with the Sylow basis $\underset{\sim}{S}$ of G, and let A/B be a D-composition factor of G. Then A ser G so $\underset{\sim}{S}$ reduces into A by 2.3.12. Since $\underset{\sim}{S}$ reduces into D (2.13, 7) it follows, from 1.2.1(iv), that $\underset{\sim}{S}$ reduces into DA. Therefore, by 1.2.3, D is contained in the $\mathfrak{F}$-normalizer of DA associated with the Sylow basis $\underset{\sim}{S} \cap DA$. We denote this $\mathfrak{F}$-normalizer by $D(A/B)$.

Theorem 2.4.10.

Suppose π is the set of all primes and $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for all p. Let D be an $\mathfrak{F}$-normalizer of the $\mathfrak{K}$-group G and define the following subgroups of G:-

$Y_o = \langle D(A/B)$; as A/B runs over all $\mathfrak{F}$-central D-composition factors of $G \rangle$

$Y_1 = \langle D^*$; $D \leq D^* \leq G$ and D^* is an $\mathfrak{F}$-normalizer of some subgroup of $G \rangle$

$Y_2 = \langle D^*$; $D \leq D^* \leq G$ and $D^* \in \mathfrak{F} \rangle$

$Y_3 = \langle D^*$; D $\mathfrak{F}$-<u>ser</u> $D^* \leq G \rangle$.

Then $Y_o = Y_1 = Y_2 = Y_3 = R_G(D ; \mathfrak{F})$.

<u>Proof</u>.

Since $\mathfrak{F}$-normalizers always lie in $\mathfrak{F}$ (4.6, 7) it is clear from 2.4.9 that $Y_o \leq Y_1 \leq Y_2 \leq Y_3$. If D is $\mathfrak{F}$-serial in a subgroup D^* of G then, by 2.2.5 and 2.3.10, $D^* \leq R_G(D ; \mathfrak{F})$. Therefore $Y_3 \leq R_G(D ; \mathfrak{F})$, and we complete the proof by showing that $R_G(D ; \mathfrak{F}) \leq Y_o$.

If A/B is an $\mathfrak{F}$-central D-composition factor of G then A/B is an $\mathfrak{F}$-central chief factor of DA, so, by (4.6, 7), $D(A/B)$ covers A/B. Therefore the subgroup Y_o contains D and covers every $\mathfrak{F}$-central D-composition factor of G, so $R_G(D ; \mathfrak{F}) \leq Y_o$ by 2.2.16, as required.

<u>Corollary 2.4.11</u>.

Suppose π is the set of all primes and $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for all p. Let D be an $\mathfrak{F}$-normalizer of the $\mathfrak{K}$-group G. Then D has an $\mathfrak{F}$-serializer in G if and only if D has a strong $\mathfrak{F}$-serializer in G.

<u>Proof</u>.

It is clear that if D has a strong $\mathfrak{F}$-serializer in G then $R_G(D ; \mathfrak{F})$ is the $\mathfrak{F}$-serializer of D in G.

Suppose conversely that K is the $\mathfrak{F}$-serializer of D in G. Then K contains $R_G(D ; \mathfrak{F})$, by 2.4.10, since the latter is generated by subgroups of G in which D is $\mathfrak{F}$-serial. However, D $\mathfrak{F}$-<u>ser</u> K so by the fourth

description in 2.4.10, $K \leq R_G(D\ ;\mathfrak{F})$. Therefore $D\ \mathfrak{F}\text{-}\underline{\text{ser}}\ K = R_G(D\ ;\mathfrak{F})$ and, by 2.3.17, D has a strong $\mathfrak{F}$-serializer in G.

<u>Theorem 2.4.12</u>

Suppose π is the set of all primes and $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for all p. Let $\mathfrak{X}(\mathfrak{K},\mathfrak{F})$ be the class of all $\mathfrak{K}$-groups in which the $\mathfrak{F}$-normalizers possess $\mathfrak{F}$-serializers. Then $\mathfrak{X}(\mathfrak{K},\mathfrak{F})$ is a $\mathfrak{K}$-formation containing the class $\mathfrak{K} \cap (\mathfrak{U})^2\mathfrak{F}$.

<u>Proof</u>.

We show first that $\mathfrak{X}(\mathfrak{K},\mathfrak{F})$ is Q-closed. Indeed let N be a normal subgroup and D an $\mathfrak{F}$-normalizer of the $\mathfrak{X}(\mathfrak{K},\mathfrak{F})$-group G. Then $D\ \mathfrak{F}\text{-}\underline{\text{ser}}\ R_G(D\ ;\mathfrak{F})$ so, by 2.3.11 and 2.2.7, $DN/N\ \mathfrak{F}\text{-}\underline{\text{ser}}\ R_{G/N}(DN/N\ ;\mathfrak{F})$. Since DN/N is an $\mathfrak{F}$-normalizer of G/N it follows, from 2.3.17 and 2.4.11, that $G/N \in \mathfrak{X}(\mathfrak{K},\mathfrak{F})$ and hence that $\mathfrak{X}(\mathfrak{K},\mathfrak{F})$ is Q-closed.

Suppose that $G \in \mathfrak{K}$ and there exist normal subgroups N_λ of G $(\lambda \in \Lambda)$ such that $G/N_\lambda \in \mathfrak{X}(\mathfrak{K},\mathfrak{F})$ for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. Let D be an $\mathfrak{F}$-normalizer of G and $\underset{\sim}{S}^{\mathfrak{F}}$ an $\mathfrak{F}$-system of $R_G(D\ ;\mathfrak{F})$. Let $\underset{\sim}{T}^{\mathfrak{F}}$ be an $\mathfrak{F}$-system of G which $\mathfrak{F}$-reduces into $R_G(D\ ;\mathfrak{F})$ to $\underset{\sim}{S}^{\mathfrak{F}}$; such an $\mathfrak{F}$-system exists by 2.1.11(i). Let $\lambda \in \Lambda$. Then DN_λ/N_λ is an $\mathfrak{F}$-normalizer of the $\mathfrak{X}(\mathfrak{K},\mathfrak{F})$-group G/N_λ so that $DN_\lambda/N_\lambda\ \mathfrak{F}\text{-}\underline{\text{ser}}\ R_{G/N_\lambda}(DN_\lambda/N_\lambda\ ;\mathfrak{F})$. Now $\nu_{N_\lambda}(\underset{\sim}{T}^{\mathfrak{F}})$ $\mathfrak{F}$-reduces into $R_{G/N_\lambda}(DN_\lambda/N_\lambda\ ;\mathfrak{F})$ by 2.2.7 and 2.1.19, so, by 2.1.11(ii) and 2.3.10, $\nu_{N_\lambda}(\underset{\sim}{T}^{\mathfrak{F}})$ $\mathfrak{F}$-reduces into DN_λ/N_λ. Thus by 2.1.21, $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into DN_λ. Since λ was an arbitrary member of Λ it follows, from 2.1.18 and 2.4.1

that $\underset{\sim}{T}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into D. Therefore $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into D by 2.1.16. The above shows that every $\mathfrak{F}$-system of $R_G(D\ ;\mathfrak{F})$ $\mathfrak{F}$-reduces into D, i.e. D $\mathfrak{F}$-superser $R_G(D\ ;\mathfrak{F})$. From 2.3.17 and 2.4.11 we now deduce that $G \in \mathfrak{X}(\mathfrak{K},\mathfrak{F})$ and hence that $\mathfrak{K} \cap R\mathfrak{X}(\mathfrak{K},\mathfrak{F}) = \mathfrak{X}(\mathfrak{K},\mathfrak{F})$.

Thus $\mathfrak{X}(\mathfrak{K},\mathfrak{F})$ is a $\mathfrak{K}$-formation as claimed. That this class contains $\mathfrak{K} \cap (\mathfrak{N})^2\mathfrak{F}$ follows immediately from 2.4.4 and 2.4.11.

We now turn our attention to weak $\mathfrak{F}$-serializers; these generalize the weak subnormalizers introduced by Mann (21) for finite soluble groups.

Suppose $H \leq K \leq G \in \mathfrak{K}$. We say K is a weak $\mathfrak{F}$-serializer of H in G if H is an $\mathfrak{F}$-serial subgroup of K but H is not $\mathfrak{F}$-serial in any subgroup of G which properly contains K.

Theorem 2.4.13.

Suppose π is the set of all primes and $\mathfrak{f}(p) = S\mathfrak{f}(p)$ for all p. Let D be an $\mathfrak{F}$-normalizer and E an $\mathfrak{F}$-projector of the $\mathfrak{K} \cap (\mathfrak{N})^3\mathfrak{F}$-group G. Then

(1) there exists at most one weak $\mathfrak{F}$-serializer of D in G which contains E,

(2) if $D \leq E$ there is exactly one weak $\mathfrak{F}$-serializer of D in G which contains E,

(3) if F is a weak $\mathfrak{F}$-serializer of D in G then F contains an $\mathfrak{F}$-projector of G containing D if and only if $F \rtimes_{\mathfrak{F}} G$ and D normalizes some $\mathfrak{F}$-system of F.

Proof.

Let $R = \mathfrak{C}(G)$. Suppose D $\mathfrak{F}$-ser F where $E \leq F$. Then $F \leq R_G(D\ ;\mathfrak{F})$ by 2.2.5 and 2.3.10, so by 2.2.7, FR/R is contained in $R_{G/R}(DR/R\ ;\mathfrak{F})$. Now FR/R contains the $\mathfrak{F}$-

projector ER/R of G/R and $R_{G/R}(DR/R\ ;\mathfrak{F})$ is an $\mathfrak{F}$-projector of G/R by 2.4.2. Therefore $ER/R = FR/R = R_{G/R}(DR/R\ ;\mathfrak{F})$ and hence $F = E(F \cap R)$.

Let $H = R_{DR}(D\ ;\mathfrak{F})$. Now D $\mathfrak{F}$-ser $F \cap DR = D(F \cap R)$ by 2.3.3(i), so by 2.3.10 and 2.2.5, $D(F \cap R) \leq H$. Let T be the unique maximal E-invariant subgroup of H. Since E is contained in F, $F \cap R$ is E-invariant and therefore contained in T. Thus $F = E(F \cap R) \leq ET$. Since $D \leq F$ we have in particular $D \leq ET$.

Now $T \lhd ET$ and $ET/T \in \mathfrak{F}$, so DT/T $\mathfrak{F}$-ser ET/T by 2.4.9(i). Thus DT $\mathfrak{F}$-ser ET. By 2.3.21, D is $\mathfrak{F}$-serial in H, so by 2.3.3(i), D $\mathfrak{F}$-ser DT. We therefore have D $\mathfrak{F}$-ser DT $\mathfrak{F}$-ser ET, so clearly D $\mathfrak{F}$-ser ET.

(1) Suppose that F is a weak $\mathfrak{F}$-serializer of D in G containing E. Then the above argument shows that $F \leq ET$ and D $\mathfrak{F}$-ser ET. Thus by definition of weak $\mathfrak{F}$-serializer we must have $F = ET$. Since T is "independent" of F it follows that there is at most one weak $\mathfrak{F}$-serializer of D in G containing E. (Of course, in general we need not have $D \leq ET$).

(2) If $D \leq E$ then certainly $D \leq ET$ and the above argument shows that ET is the unique weak $\mathfrak{F}$-serializer of D in G containing E.

(3) Again let $R = \mathcal{Q}(G)$ but now let E be an $\mathfrak{F}$-projector of G containing D. Since D is $\mathfrak{F}$-serial in F, it follows from 2.3.10 and 2.2.5 that $F \leq R_G(D\ ;\mathfrak{F})$. Now ER/R is an $\mathfrak{F}$-projector of G/R containing DR/R, so by 2.4.2, $ER/R = R_{G/R}(DR/R\ ;\mathfrak{F})$. Thus by 2.2.7, $F \leq ER$.

Assume firstly that F contains the $\mathfrak{F}$-projector E^g of G and E^g contains D. Since E^g is $\mathfrak{F}$-abnormal in G

by 1.2.7, we clearly have $F \rtimes_{\mathfrak{F}} G$. Now E^g is an $\mathfrak{F}$-projector of F and $F \in \mathfrak{K} \cap (\mathfrak{M})\mathfrak{F}$ since $F \leq ER$, so by 1.2.5(3), E^g is an $\mathfrak{F}$-normalizer of F. Since $D \leq E^g$ it is now clear that D normalizes some $\mathfrak{F}$-system of F.

Conversely suppose that $F \rtimes_{\mathfrak{F}} G$ and D normalizes some $\mathfrak{F}$-system of F. Then $F \rtimes_{\mathfrak{F}} ER$ so, by 1.2.8, F contains some $\mathfrak{F}$-normalizer of ER. Since ER is a $\mathfrak{K} \cap (\mathfrak{M})\mathfrak{F}$-group it follows, from 1.2.5(3), that F contains some conjugate E^g of E ($g \in ER$). Since F also belongs to $\mathfrak{K} \cap (\mathfrak{M})\mathfrak{F}$, E^g is also an $\mathfrak{F}$-normalizer of F. But D normalizes some $\mathfrak{F}$-system of F, so it follows that D is contained in some conjugate E^{gf} of E^g ($f \in F$). Thus E^{gf} is an $\mathfrak{F}$-projector of G with $D \leq E^{gf} \leq F$, which completes the proof of both (3) and the result.

As a special case of 2.4.13 we have the following generalization of two results of Mann (Theorem 16, Lemma 13, 21).

<u>Theorem 2.4.14.</u>

Suppose D is a basis normalizer and C a Carter subgroup of $G \in \mathfrak{U} \cap (\mathfrak{M})^4$. Then

(1) there is at most one weak serializer of D in G containing C,

(2) if $D \leq C$ there is exactly one weak serializer of D in G which contains C,

(3) if F is a weak serializer of D in G then F contains a Carter subgroup of G containing D if and only if $F \rtimes G$ and D normalizes some Sylow basis of F.

In this case we also have

Theorem 2.4.15.

Let D be a basis normalizer of $G \in \mathfrak{U} \cap (\mathfrak{N})^4$ and C a Carter subgroup of G containing D. Let F be the unique weak serializer of D in G containing C and $N \triangleleft G$. Then FN/N is the unique weak serializer of DN/N in G/N containing CN/N.

Remarks.

1. We have been unable to decide whether, with the hypotheses in 2.4.13, the weak $\mathfrak{F}$-serializers of $\mathfrak{F}$-normalizers are "homomorphism invariant" in $\mathfrak{K} \cap (\mathfrak{N})^3\mathfrak{F}$-groups (in the same sense as 2.4.15).

2. It is possible for a basis normalizer to have a weak serializer but no serializer. For Carter (pages 562-3, 3) gives an example of a finite soluble group G of nilpotent length 4 in which the system normalizers do not possess subnormalizers. It is clear that they have weak subnormalizers. Carter's example also serves to show that, in general, $\mathfrak{K} \cap (\mathfrak{N})^3\mathfrak{F}$ is not contained in the class $\mathfrak{X}(\mathfrak{K}, \mathfrak{F})$ considered in 2.4.12.

Before proving 2.4.15 we require three lemmas:-

Lemma 2.4.16.

Let G be an arbitrary locally finite group and H a serial locally nilpotent subgroup of G. Then H is contained in the Hirsch-Plotkin radical of G.

Proof.

Let $L = H^G$, the normal closure of H in G. If X is any finite subset of L then there is a finite subgroup F of G such that $X \leq (H \cap F)^F$, the normal closure of $H \cap F$ in F. Since H ser G and F is finite

it follows that $H \cap F$ is subnormal in F. Now H is locally nilpotent so $H \cap F$ is a subnormal nilpotent subgroup of F. In particular therefore $H \cap F$ lies in the Fitting subgroup Y of F. Thus $X \leq (H \cap F)^F \leq Y$ and hence $L \in \mathfrak{M}$. Therefore L is a normal $\mathfrak{M}$-subgroup of G, so $H \leq L \leq \mathfrak{Q}(G)$, as required.

<u>Lemma 2.4.17.</u>

Let D be the normalizer $N_G(\underset{\sim}{S})$ of the Sylow basis $\underset{\sim}{S}$ of the $\mathfrak{U}$-group G. Suppose $N \triangleleft G$, DN/N <u>ser</u> H/N and $\underset{\sim}{S}N/N$ reduces into H/N. Then there is a subgroup H_1 of H and a basis normalizer D_2 of H such that D <u>ser</u> H_1, $H = H_1N$, $D \leq D_2 \leq H_1$, and $\underset{\sim}{S}$ reduces into both D_2 and H_1.

<u>Proof.</u>

Let $R/N = \mathfrak{Q}(H/N)$; then $DN/N \leq R/N$ by 2.4.16. Since $\underset{\sim}{S}N/N$ reduces into H/N it follows that $\underset{\sim}{S}$ reduces into H and hence into R since $R \triangleleft H$. Thus $\underset{\sim}{S} \cap R$ is a Sylow basis of R. Set $D_1 = N_R(\underset{\sim}{S} \cap R)$, $D_2 = N_H(\underset{\sim}{S} \cap H)$ and $H_1 = N_H(\underset{\sim}{S} \cap R)$. By the Frattini argument $H = RH_1$. Now D_1 is a basis normalizer of R and R/N is locally nilpotent so $R = D_1N$ by (4.6, $\underset{\sim}{7}$). Thus $H = H_1D_1N = H_1N$ since D_1 is clearly contained in H_1.

Now D normalizes $\underset{\sim}{S}$ and $D \leq R$ so $D \leq D_1$. Hence D <u>ser</u> D_1 by 2.4.9(i). But $D_1 = R \cap H_1 \triangleleft H_1$, so we further have D <u>ser</u> H_1. Since R is a normal subgroup of H the basis normalizer D_2 of H is certainly contained in H_1. Thus $D \leq D_2 \leq H_1$ since D normalizes $\underset{\sim}{S} \cap H$. Finally, $\underset{\sim}{S}$ reduces into both H_1 and D_2 by (2.13(ii), $\underset{\sim}{7}$) so the proof is complete.

Lemma 2.4.18.

Suppose $G \in \mathfrak{U} \cap (\mathfrak{L}\mathfrak{N})^4$ and D is a basis normalizer of G contained in the Carter subgroup C of G. Let F be the unique weak serializer of D in G containing C. If N is a normal subgroup of G contained in the Hirsch-Plotkin radical of G then FN/N is the unique weak serializer of DN/N in G/N containing CN/N.

Proof.

Let X/N be the unique weak serializer of DN/N in G/N containing CN/N, and $R = \mathfrak{C}(G)$. By hypothesis $N \leq R$. Thus, by 2.3.13, DR/R ser XR/R. Now G/R is a $\mathfrak{U} \cap (\mathfrak{L}\mathfrak{N})^3$-group and CR/R is the unique Carter subgroup of G/R containing DR/R, so, by 2.4.5, $XR/R \leq CR/R$. Since $C \leq X$ we therefore have $CR = XR$.

By hypothesis $D \leq C$. In section 3.1 we shall show (Lemma 3.1.2) that there is a Sylow basis $\underset{\sim}{S}$ of G such that $\underset{\sim}{S} \searrow C$ and $D = N_G(\underset{\sim}{S})$. Since $C \leq X$ it follows, from 2.2.3 and 2.2.15, that $\underset{\sim}{S}$ reduces into X. Therefore, by 2.4.17, there is a subgroup H_1 of X and a basis normalizer D_2 of X such that $X = H_1N$, $D \leq D_2 \leq H_1$, D ser H_1, and $\underset{\sim}{S}$ reduces into both H_1 and D_2. Now $X \in \mathfrak{U} \cap (\mathfrak{L}\mathfrak{N})^2$ since $X \leq CR$, so D_2 is a Carter subgroup of X and there exists an element $x \in X$ such that $D_2 = C^x$. Now $\underset{\sim}{S} \searrow C, C^x$ so by 2.2.15, $x \in C$. Therefore $C = D_2$ and we have D ser H_1 with $C \leq H_1$.

Since F is the unique weak serializer of D in G containing C we must have $H_1 \leq F$ and hence $X/N = H_1N/N \leq FN/N$. But X/N is the weak serializer of DN/N in G/N containing CN/N and, by 2.3.13, DN/N ser FN/N. Thus $X/N = FN/N$, as required.

Proof of Theorem 2.4.15.

We have $D \leq C \leq G \in \mathfrak{U} \cap (\mathfrak{A})^4$, D a basis normalizer and C a Carter subgroup of G, F is the unique weak serializer of D in G containing C, and $N \lhd G$. We have to show that FN/N is the (unique) weak serializer of DN/N in G/N containing CN/N.

We argue by induction on the $\mathfrak{A}$-length of N, there being nothing to prove when this is zero. Suppose then that $l(N) > 0$. Let $x \to \bar{x}$ be the natural epimorphism of G onto $\bar{G} = G/\mathfrak{C}(N)$. Since $\mathfrak{C}(N) \leq \mathfrak{C}(G)$, $\bar{F}$ is the weak serializer of $\bar{D}$ in $\bar{G}$ containing $\bar{C}$ by 2.4.18. Now $l(\bar{N}) < l(N)$, so by induction $\overline{FN}/\bar{N}$ is the weak serializer of $\overline{DN}/\bar{N}$ in $\bar{G}/\bar{N}$ containing $\overline{CN}/\bar{N}$. From the natural isomorphism between $\bar{G}/\bar{N}$ and G/N we deduce that FN/N is the weak serializer of DN/N in G/N containing CN/N, and this completes the proof.

2.5. The Convergence Processes.

In this section we describe two ways of "constructing" the $\mathfrak{F}$-projectors of a $\mathfrak{K}$-group G, i.e. we show how to obtain an $\mathfrak{F}$-projector of G as the limiting term in a "converging" series of subgroups of G. The first method is somewhat unsatisfactory in that at each stage one has to "construct" an $\mathfrak{F}$-projector of some subgroup of G. The second approach, which consists of successively taking $\mathfrak{F}$-normalizers and $\mathfrak{F}$-reducers, overcomes the previous objection but is, more often than not, too cumbersome for the actual computation of $\mathfrak{F}$-projectors. The processes we shall describe generalize similar constructions for Carter subgroups of finite soluble groups due to Carter (3), Fischer (unpublished), Mann (21), and Rose (28).

The First Convergence Process.

Theorem 2.5.1.

Suppose $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for all $p \in \pi$. Let D be an arbitrary $\mathfrak{F}$-subgroup of the $\mathfrak{K}$-group G, and define subgroups R_i , D_i of G inductively as follows:-

$R_0 = G$, $D_0 = D$ and for $i \geq 0$, $R_{i+1} = R_G(D_i ; \mathfrak{F})$, and D_{i+1} any $\mathfrak{F}$-projector of R_{i+1} .

Then this process yields an $\mathfrak{F}$-projector of G ; more precisely, if $G \in \mathfrak{K} \cap (\mathfrak{M})^t\mathfrak{F}$ then D_{t+1} is an $\mathfrak{F}$-projector of G.

Proof.

Since every $\mathfrak{K}$-group has finite $\mathfrak{M}$-length it is clearly sufficient to prove the final statement which we do by induction on t.

If $t = 0$ then $G \in \mathfrak{F}$ so by 2.2.6(ii) $G = R_G(D\ ;\mathfrak{F})$, i.e. $G = R_1$. Thus D_1 is an $\mathfrak{F}$-projector of G by construction, so the induction begins.

If $t \geq 1$ let $R = \mathfrak{C}(G)$. It is immediate, from 2.2.7 and the homomorphism invariance of $\mathfrak{F}$-projectors, that the subgroups $D_i(G/R) = D_iR/R$ and $R_i(G/R) = R_iR/R$ are the ith terms in a convergence process for G/R, the first term in this series being the $\mathfrak{F}$-subgroup DR/R of G/R. Now $G/R \in \mathfrak{K} \cap (\mathfrak{N})^{t-1}\mathfrak{F}$ so by induction D_tR/R is an $\mathfrak{F}$-projector of G/R. Since $D_t \in \mathfrak{F}$ by construction, there is an $\mathfrak{F}$-projector E of D_tR containing D_t by 1.2.5(6). Moreover E is an $\mathfrak{F}$-projector of G by Gaschütz Lemma (1.2.5(7)). By 2.2.6(i), $E \leq R_G(D_t\ ;\mathfrak{F}) = R_{t+1}$. Thus E is an $\mathfrak{F}$-projector of R_{t+1} and hence is conjugate in R_{t+1} to D_{t+1}. In particular therefore D_{t+1} is an $\mathfrak{F}$-projector of G, as claimed. Notice that $D_{t+1} = D_i$ for all $i \geq t+1$, and that when π is the set of all primes then $D_{t+1} = R_j$ for all $j \geq t+2$ by 2.2.14(i).

<u>The Second Convergence Process.</u>

<u>Lemma 2.5.2.</u>

Let D be the $\mathfrak{F}$-normalizer associated with the Sylow basis $\underset{\sim}{S}$ of the $\mathfrak{K}$-group G. Then D is contained in the $\mathfrak{F}$-normalizer of $R_G(D\ ;\mathfrak{F})$ associated with the Sylow basis $\underset{\sim}{S} \cap R_G(D\ ;\mathfrak{F})$.

<u>Proof.</u>

By (2.13(ii), $\underset{\sim}{7}$) $\underset{\sim}{S}$ reduces into D, so by 2.2.3 $\underset{\sim}{S}$ reduces into every subgroup of G containing $R_G(D)$. In particular, by 2.2.1, $\underset{\sim}{S}$ reduces into $R_G(D\ ;\mathfrak{F})$. The result is now immediate from 1.2.3.

The second convergence process is defined in the following way. Let $\underset{\sim}{S}$ be a Sylow basis of the $\mathfrak{K}$-group G and let D be the $\mathfrak{F}$-normalizer of G associated with $\underset{\sim}{S}$. Put $D_o = D_1 = D$ and $R_o = G$. Let D_2 be the $\mathfrak{F}$-normalizer of $R_1 = R_G(D\ ;\mathfrak{F})$ associated with the Sylow basis $\underset{\sim}{S} \cap R_1$. Then $D_o = D_1 \leq D_2$ by 2.5.2. The same argument shows that $D_2 \leq D_3$, the $\mathfrak{F}$-normalizer of $R_2 = R_{R_1}(D_2\ ;\mathfrak{F})$ associated with the Sylow basis $(\underset{\sim}{S} \cap R_1) \cap R_2 = \underset{\sim}{S} \cap R_2$. Continuing in this way we obtain two sequences of subgroups of G

$$D = D_o = D_1 \leq D_2 \leq D_3 \leq \cdots\cdots\cdots\cdots \quad (1)$$

$$G = R_o \geq R_1 \geq R_2 \geq R_3 \geq \cdots\cdots\cdots\cdots \quad (2)$$

where for each $i \geq 1$, $R_i = R_{R_{i-1}}(D_i\ ;\mathfrak{F})$ and D_i is the $\mathfrak{F}$-normalizer of R_{i-1} associated with the Sylow basis $\underset{\sim}{S} \cap R_{i-1}$, i.e.

$$D_i = (S_\pi \cap R_{i-1}) \cap \bigcap_{p \in \pi} N_{R_{i-1}}((S_{p'} \cap R_{i-1}) \cap C_p(R_{i-1}))$$

$$= S_\pi \cap \bigcap_{p \in \pi} N_{R_{i-1}}(S_{p'} \cap C_p(R_{i-1})).$$

<u>Lemma 2.5.3.</u>

For each $i \geq 0$, $D_i \leq D_{i+1} \leq R_{i+1} \leq R_i$.

<u>Proof.</u>

We certainly have $D_i \leq D_{i+1} \leq D_{i+2}$ and $R_{i+1} \leq R_i$ for each $i \geq 0$. But by construction D_{i+2} is an $\mathfrak{F}$-normalizer of R_{i+1}, so in particular $D_{i+2} \leq R_{i+1}$. The result is now clear.

We therefore have

$$D = D_o = D_1 \leq D_2 \leq D_3 \leq \cdots\cdots\cdots \leq R_2 \leq R_1 \leq R_o = G \quad (3)$$

and, as in the proof of 2.5.2, $\underset{\sim}{S}$ reduces into each D_i and R_i. Thus by 2.1.9,

$\underset{\sim}{S}^{\mathfrak{F}}$ strongly reduces into D_i, R_i for each $i \geq 0$ (4)

Remark.

Where we want to specify the group G in the above process we shall write $D_i = D_i(G)$ and $R_i = R_i(G)$. It is clear from the conjugacy of Sylow bases that the series obtained above is, in some sense, an invariant of G ; for the corresponding series for the Sylow basis $\underset{\sim}{S}^x$ of G is just the conjugate, by x, of the series (3).

As an immediate consequence of 2.2.7 and the "homomorphism invariance" of $\mathfrak{F}$-normalizers, we have

Lemma 2.5.4.

If $N \triangleleft G$ then $D_i(G/N) = D_i(G)N/N$ and $R_i(G/N) = R_i(G)N/N$, for each $i \geq 0$.

Lemma 2.5.5.

Suppose π is the set of all primes. Then the sequence (3) above converges. More precisely:-

(1) if $G \in \mathfrak{K} \cap (\mathfrak{M})^{2t}\mathfrak{F}$ $(t \geq 0)$ then

$\cdots = D_{t+2} = D_{t+1} = D_t = R_t = R_{t+1} = R_{t+2} = \cdots\cdots$

(2) if $G \in \mathfrak{K} \cap (\mathfrak{M})^{2t+1}\mathfrak{F}$ $(t \geq 0)$ then

$\cdots = D_{t+3} = D_{t+2} = D_{t+1} = R_{t+1} = R_{t+2} = R_{t+3} = \cdots$

Proof.

Since every $\mathfrak{K}$-group has finite $\mathfrak{M}$-length it suffices to prove (1) and (2), which we do by a simultaneous induction on t.

Case (a). $t = 0$.

(1) In this case $G \in \mathfrak{F}$ so that $D = G$ and, by 2.2.1, $R_1 = G$. Hence $\cdots\cdots = R_2 = R_1 = R_0 = G = D = D_0 = D_1 = \cdots$ as required.

(2) Here $G \in \mathfrak{K} \cap (\mathfrak{M})\mathfrak{F}$ so that D is an $\mathfrak{F}$-projector of G by 1.2.5(3). Therefore, by 2.2.14(i), $R_1 = D$ and hence $\cdots\cdots = D_3 = D_2 = D_1 = D_0 = D = R_1 = R_2 = \cdots\cdots$

Case (b). $t > 0$.

(1) By 2.4.6, $R_1 = R_G(D\ ;\mathfrak{F}) \in \mathfrak{K} \cap (\mathfrak{M})^{2(t-1)}\mathfrak{F}$.

Thus, by induction

$$\cdots\cdots = D_{t+1}(R_1) = D_t(R_1) = D_{t-1}(R_1) = R_{t-1}(R_1) = R_t(R_1) = \cdots$$

Now it is clear that, if we begin the construction for R_1 with the Sylow basis $\underset{\sim}{S} \cap R_1$ of R_1 then $D_i(R_1) = D_{i+1}(G)$ and $R_i(R_1) = R_{i+1}(G)$ for each $i \geq 0$. Therefore

$$\cdots\cdot = D_{t+2} = D_{t+1} = D_t = R_t = R_{t+1} = R_{t+2} = \cdots\cdots\cdot$$

as required.

(2) In this case $R_1 = R_G(D\ ;\mathfrak{F}) \in \mathfrak{K} \cap (\mathfrak{M})^{2(t-1)+1}\mathfrak{F}$ and a similar argument to that in case (1) gives

$$\cdots\cdot = D_{t+3} = D_{t+2} = D_{t+1} = R_{t+1} = R_{t+2} = R_{t+3} = \cdots\cdots\cdot$$

which completes the induction argument.

Remark.

We shall show later that the results in 2.5.5 are best possible in the sense that there exists a $\mathfrak{S}$-closed subclass $\mathfrak{K}$ of $\mathfrak{U}$, a saturated $\mathfrak{K}$-formation $\mathfrak{F}$ satisfying the hypotheses in 2.5.5, and groups G_{2t} , G_{2t+1} in $\mathfrak{K}$ such that for each $t \geq 1$

$$G_{2t} \in \mathfrak{K} \cap (\mathfrak{M})^{2t}\mathfrak{F} \text{ but } D_{t-1} \neq R_{t-1}$$
$$G_{2t+1} \in \mathfrak{K} \cap (\mathfrak{M})^{2t+1}\mathfrak{F} \text{ but } D_t \neq R_t.$$

Lemma 2.5.6.

Suppose $\mathfrak{f}(p) = \mathfrak{S}\mathfrak{f}(p)$ for all $p \in \pi$. Then for each integer $i \geq 1$, $R_i = R_G(D_i\ ;\mathfrak{F})$.

Proof.

We again argue by induction on i.

If $i = 1$ then $R_1 = R_G(D_1\ ;\mathfrak{F})$ by definition, so we may assume that $i > 1$ and that, by induction, $R_{i-1} = R_G(D_{i-1}\ ;\mathfrak{F})$. Now $D_{i-1} \leq D_i \in \mathfrak{F}$, so by 2.2.6(i), $R_G(D_i\ ;\mathfrak{F}) \leq R_{i-1}$.

Now $\underset{\sim}{S}$ reduces into R_{i-1} and D_i, so, by 2.1.6,

$$R_G(D_i\ ;\mathfrak{F}) = \langle x \in G\ ;\ \underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} D_i \rangle$$

$$R_i = R_{R_{i-1}}(D_i\ ;\mathfrak{F}) = \langle y \in R_{i-1}\ ;\ (\underset{\sim}{S} \cap R_{i-1})^{y\mathfrak{F}} \searrow_{\mathfrak{F}} D_i \rangle.$$

If $x \in G$ and $\underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} D_i$ then, from above, $x \in R_{i-1}$. Since $\underset{\sim}{S}^{x\mathfrak{F}}$ clearly $\mathfrak{F}$-reduces into R_{i-1} to $(\underset{\sim}{S} \cap R_{i-1})^{x\mathfrak{F}}$ it follows, from 2.1.16, that $(\underset{\sim}{S} \cap R_{i-1})^{x\mathfrak{F}}$ $\mathfrak{F}$-reduces into D_i. Thus $x \in R_i$ and hence $R_G(D_i\ ;\mathfrak{F}) \leq R_i$. From 2.2.5 we now obtain the result.

For the remainder of this section we shall assume that

$$(5) \qquad \begin{cases} \pi \text{ is the set of all primes} \\ \mathfrak{f}(p) = \mathfrak{s}\mathfrak{f}(p) \text{ for all } p. \end{cases}$$

If $G \in \mathfrak{K}$ we let $E(\underset{\sim}{S})$ be the limit of the sequence (3), i.e. if $G \in \mathfrak{K} \cap (\mathfrak{N})^{2t}\mathfrak{F}$ then $E(\underset{\sim}{S}) = R_t = D_t = \cdots$ and if $G \in \mathfrak{K} \cap (\mathfrak{N})^{2t+1}\mathfrak{F}$ then $E(\underset{\sim}{S}) = R_{t+1} = D_{t+1} = \cdots$ Since each subgroup D_i belongs to $\mathfrak{F}$ we have, from 2.5.6,

$$E(\underset{\sim}{S}) = R_G(E(\underset{\sim}{S})\ ;\mathfrak{F}) \in \mathfrak{F}.$$

Thus, by 2.2.14(ii), we have

<u>Theorem 2.5.7.</u>

$E(\underset{\sim}{S})$ is an $\mathfrak{F}$-projector of G.

<u>Corollary 2.5.8.</u>

$E(\underset{\sim}{S})$ is the unique $\mathfrak{F}$-projector of G into which $\underset{\sim}{S}$ reduces and into which $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces. Moreover $\underset{\sim}{S}^{\mathfrak{F}}$ strongly reduces into $E(\underset{\sim}{S})$.

<u>Proof.</u>

By construction $\underset{\sim}{S}$ reduces and $\underset{\sim}{S}^{\mathfrak{F}}$ strongly reduces into each D_i and R_i. In particular therefore $\underset{\sim}{S}$ reduces

and $\underset{\sim}{S}^{\mathfrak{F}}$ strongly reduces into $E(\underset{\sim}{S})$. If $x \in G$ and $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces into $E(\underset{\sim}{S})^x$ then $x \in R_G(E(\underset{\sim}{S}) ; \mathfrak{F}) = E(\underset{\sim}{S})$, so that $E(\underset{\sim}{S})^x = E(\underset{\sim}{S})$. Thus $E(\underset{\sim}{S})$ is the unique $\mathfrak{F}$-projector of G into which $\underset{\sim}{S}^{\mathfrak{F}}$ $\mathfrak{F}$-reduces. Finally, by 2.1.6(iii), $E(\underset{\sim}{S})$ is the unique $\mathfrak{F}$-projector of G into which $\underset{\sim}{S}$ reduces.

If G is a $\mathfrak{K}$-group we define the <u>$\mathfrak{F}$-speed</u> of G to be the least integer $i = i_{\mathfrak{F}}(G)$ such that $R_i = E(\underset{\sim}{S})$. It is immediate, from the conjugacy of Sylow bases, that $i_{\mathfrak{F}}(G)$ is an invariant of G.

If r is a non-negative integer we define $r(\mathfrak{K}, \mathfrak{F})$ to be the class of all $\mathfrak{K}$-groups with $\mathfrak{F}$-speed at most r, i.e. $r(\mathfrak{K}, \mathfrak{F}) = \{ G \in \mathfrak{K} ; i_{\mathfrak{F}}(G) \leq r\}$.

<u>Theorem 2.5.9.</u>

$r(\mathfrak{K}, \mathfrak{F})$ is a $\mathfrak{K}$-formation for each non-negative integer r.

<u>Proof.</u>

Suppose $\underset{\sim}{S}$ is a Sylow basis and N a normal subgroup of a $\mathfrak{K}$-group G. Then, by 2.5.4,

$$E(\underset{\sim}{S}N/N) = E(\underset{\sim}{S})N/N \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Since the sequence (3) becomes stationary when it reaches $E(\underset{\sim}{S})$, it is clear that the $\mathfrak{K}$-group $G \in r(\mathfrak{K}, \mathfrak{F})$ if and only if $R_r(G) = E(\underset{\sim}{S})$.

We show first that $r(\mathfrak{K}, \mathfrak{F})$ is $\mathfrak{Q}$-closed. Indeed let N be a normal subgroup of the $r(\mathfrak{K}, \mathfrak{F})$-group G. Then $R_r(G) = E(\underset{\sim}{S})$ so, by 2.5.4 and (6), $R_r(G/N) = E(\underset{\sim}{S}N/N)$. Thus by our previous remarks, $G/N \in r(\mathfrak{K}, \mathfrak{F})$. This shows that $r(\mathfrak{K}, \mathfrak{F})$ is $\mathfrak{Q}$-closed.

If $G \in \mathfrak{K} \cap R(r(\mathfrak{K}, \mathfrak{F}))$ then there exist normal

subgroups N_λ of G $(\lambda \in \Lambda)$ such that $G/N_\lambda \in r(\mathfrak{K},\mathfrak{F})$ for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. Thus by 2.5.4 and (6), we have $R_r(G)N_\lambda = E(\underset{\sim}{S})N_\lambda$ for each $\lambda \in \Lambda$. Hence, by (3.6(i), $\underset{\sim}{7}$) $R_r(G) \leq \bigcap_{\lambda \in \Lambda} (E(\underset{\sim}{S})N_\lambda) = E(\underset{\sim}{S})$. But $E(\underset{\sim}{S}) \leq R_r(G)$ by construction, so we must have $R_r(G) = E(\underset{\sim}{S})$. Thus $G \in r(\mathfrak{K},\mathfrak{F})$ and hence $\mathfrak{K} \cap R(r(\mathfrak{K},\mathfrak{F})) = r(\mathfrak{K},\mathfrak{F})$. This shows that $r(\mathfrak{K},\mathfrak{F})$ is a $\mathfrak{K}$-formation, as claimed.

Thus for each saturated $\mathfrak{K}$-formation $\mathfrak{F}$ satisfying (5) we obtain a series of $\mathfrak{K}$-formations

$$0(\mathfrak{K},\mathfrak{F}) \leq 1(\mathfrak{K},\mathfrak{F}) \leq 2(\mathfrak{K},\mathfrak{F}) \leq \ldots\ldots\ldots\ldots \tag{7}$$

and it is immediate, from Lemma 2.5.5, that for each $t \geq 0$

$$\left.\begin{aligned} \mathfrak{K} \cap (\mathfrak{M})^{2t}\mathfrak{F} &\leq t(\mathfrak{K},\mathfrak{F}) \\ \mathfrak{K} \cap (\mathfrak{M})^{2t+1}\mathfrak{F} &\leq t+1(\mathfrak{K},\mathfrak{F}) \end{aligned}\right\} \tag{8}$$

Since every $\mathfrak{K}$-group has finite $\mathfrak{M}$-length we also have

$$\mathfrak{K} = \bigcup_{n=0}^{\infty} n(\mathfrak{K},\mathfrak{F}) \tag{9}$$

<u>Lemma 2.5.10.</u>

(1) $0(\mathfrak{K},\mathfrak{F}) = \mathfrak{F}$.

(2) $1(\mathfrak{K},\mathfrak{F})$ contains the class of $\mathfrak{K}$-groups in which the $\mathfrak{F}$-normalizers and $\mathfrak{F}$-projectors coincide.

(3) $\mathfrak{K} \cap (\mathfrak{M})^2 t(\mathfrak{K},\mathfrak{F}) \leq t+1(\mathfrak{K},\mathfrak{F})$ for each $t \geq 0$.

<u>Proof.</u>

(1) $G \in 0(\mathfrak{K},\mathfrak{F}) \iff G = R_0 = E(\underset{\sim}{S}) \iff G \in \mathfrak{F}$. Therefore $0(\mathfrak{K},\mathfrak{F}) = \mathfrak{F}$.

(2) If the $\mathfrak{F}$-normalizers and $\mathfrak{F}$-projectors of the $\mathfrak{K}$-group G coincide, then with the usual notation, $D = E(\underset{\sim}{S})$. Thus $R_1(G) = R_G(D\,;\mathfrak{F}) = D = E(\underset{\sim}{S})$, by 2.2.14(i). Hence $G \in 1(\mathfrak{K},\mathfrak{F})$.

(3) Suppose $G \in \mathfrak{K} \cap (\mathfrak{M})^2 t(\mathfrak{K},\mathfrak{F})$. Let F denote the $(\mathfrak{M})^2$-radical of G, i.e. $F/\mathfrak{Q}(G) = \mathfrak{Q}(G/\mathfrak{Q}(G))$.

Then $G/F \in t(\mathfrak{K},\mathfrak{F})$ since this class is Q-closed by 2.5.9. Thus by 2.5.4 and (6), $R_t(G)F/F = E(\underset{\sim}{S})F/F$. Hence $R_t(G) \in \mathfrak{K} \cap (\mathfrak{M})^2\mathfrak{F}$. Now $D_{t+1}(G)$ is an $\mathfrak{F}$-normalizer of $R_t(G)$ so by Theorem 2.4.2, $R_{t+1}(G) = R_{R_t(G)}(D_{t+1}(G)\ ;\mathfrak{F})$ is an $\mathfrak{F}$-projector of $R_t(G)$. Since $E(\underset{\sim}{S})$ is an $\mathfrak{F}$-projector of $R_t(G)$ contained in $R_{t+1}(G)$ we must have $R_{t+1}(G) = E(\underset{\sim}{S})$. Thus $G \in t+1(\mathfrak{K},\mathfrak{F})$, as required.

<u>Corollary 2.5.11.</u>

If $t \geq 0$ then $t(\mathfrak{K},\mathfrak{F}) < t+1(\mathfrak{K},\mathfrak{F}) \iff t(\mathfrak{K},\mathfrak{F}) < \mathfrak{K}$.

<u>Proof.</u>

It is clear that we need only show that $t(\mathfrak{K},\mathfrak{F})$ is a proper subclass of $t+1(\mathfrak{K},\mathfrak{F})$ if it is a proper subclass of $\mathfrak{K}$. Suppose that this is not the case. Then, by 2.5.10, we have $\mathfrak{K} \cap (\mathfrak{M})^2 t(\mathfrak{K},\mathfrak{F}) \leq t+1(\mathfrak{K},\mathfrak{F}) = t(\mathfrak{K},\mathfrak{F})$. Thus $\mathfrak{K} \cap (\mathfrak{M})^2 t(\mathfrak{K},\mathfrak{F}) = t(\mathfrak{K},\mathfrak{F})$ and an easy induction argument shows that for each $n \geq 0$, $\mathfrak{K} \cap (\mathfrak{M})^n t(\mathfrak{K},\mathfrak{F}) = t(\mathfrak{K},\mathfrak{F})$. Since every $\mathfrak{K}$-group has finite $\mathfrak{M}$-length it follows that $\mathfrak{K} \leq t(\mathfrak{K},\mathfrak{F})$ contradicting the hypothesis that $t(\mathfrak{K},\mathfrak{F})$ is a proper subclass of $\mathfrak{K}$. This contradiction completes the proof.

From 2.5.10 and 2.5.11 we see that the ascending sequence (7) commences at $\mathfrak{F}$ and becomes stationary only when it reaches $\mathfrak{K}$.

<u>Corollary 2.5.12.</u>

$G \in 1(\mathfrak{K},\mathfrak{F})$ if and only if $E(\underset{\sim}{S})$ is the strong $\mathfrak{F}$-serializer of D in G.

<u>Proof.</u>

Suppose firstly that $G \in 1(\mathfrak{K},\mathfrak{F})$. Then

$R_G(D\ ;\mathfrak{F}) = R_1 = E(\underset{\sim}{S})$. Now D $\mathfrak{F}$-ser $E(\underset{\sim}{S})$ by (5) and 2.4.9(i), so in this case D $\mathfrak{F}$-ser $R_G(D\ ;\mathfrak{F})$. Thus, by 2.3.17, $E(\underset{\sim}{S})$ is the strong $\mathfrak{F}$-serializer of D in G.

If on the other hand $E(\underset{\sim}{S})$ is the strong $\mathfrak{F}$-serializer of D in G then $E(\underset{\sim}{S}) = R_G(D\ ;\mathfrak{F})$ by definition. Thus $R_1 = E(\underset{\sim}{S})$ and $G \in 1(\mathfrak{K}, \mathfrak{F})$, completing the proof.

We now give an example to show that the classes $t(\mathfrak{K}, \mathfrak{F})$ are in general not subgroup-closed. This example also shows that the sequence (7) can be strictly ascending and that (1),(2) in 2.5.5 are best possible.

Example 2.5.13.

We take $\mathfrak{K} = \mathfrak{S}^*$ the class of finite soluble groups and $\mathfrak{F} = \mathfrak{N}^*$ the class of finite nilpotent groups. Certainly these satisfy (5) and in this case the $\mathfrak{N}^*$-reducers are exactly the reducers.

We shall show later (Theorem 3.2.20) that if G is a $\mathfrak{U}_A$-group, that is a $\mathfrak{U}$-group with abelian Sylow p-subgroups for each prime p, then the basis normalizers of G are pronormal. Thus if D is a basis normalizer of a $\mathfrak{U}_A$-group G then, by 2.2.18, $R_G(D) = N_G(D)$. Inspection now shows, using (Theorem 6, $\underset{\sim}{3}$), that, for $\mathfrak{U}_A^*$-groups, our second convergence process reduces to that defined defined by Carter (§4, $\underset{\sim}{3}$) ; $\mathfrak{U}_A^*$ being the class of finite soluble A-groups.

In ($\underset{\sim}{3}$) Carter constructs a $\mathfrak{U}_A^*$-group G_j for each $j \geq 1$, in the following way. Let $p_1, p_2, p_3, \ldots\ldots$ be a sequence of distinct primes and inductively define

$$G_1 = C_{p_1}\ , \quad G_k = C_{p_k} \wr G_{k-1} \quad (k > 1)$$

where C_p denote a cyclic group of order p. Carter shows

(Theorem 12, 3) that, for each $n \geq 1$, G_{2n} is a $\mathfrak{U}_A^*$-group of nilpotent length 2n in which $F_{n-1} \neq E$ (i.e. in our notation $R_{n-1} \neq E(\mathfrak{S})$) and G_{2n+1} is a $\mathfrak{U}_A^*$-group of nilpotent length 2n+1 in which $D_n \neq E$ (i.e. in our notation $D_n \neq E(\mathfrak{S})$). In the latter case $R_{n-1} \neq E(\mathfrak{S})$ since D_n is a basis normalizer of R_{n-1}. Thus, by (8), we have

$$\left.\begin{array}{l} G_{2n+1} \in (\mathfrak{N}^*)^{2n+1} \cap (n(\mathfrak{S}^*, \mathfrak{N}^*) - n-1(\mathfrak{S}^*, \mathfrak{N}^*)) \\ G_{2n+2} \in (\mathfrak{N}^*)^{2n+2} \cap (n+1(\mathfrak{S}^*, \mathfrak{N}^*) - n(\mathfrak{S}^*, \mathfrak{N}^*)) \end{array}\right\} (10)$$

Hence equations (1),(2) in 2.5.5 are best possible. Also the sequence (7) is strictly ascending in this case, i.e.

$$\mathfrak{N}^* = 0(\mathfrak{S}^*, \mathfrak{N}^*) < 1(\mathfrak{S}^*, \mathfrak{N}^*) < 2(\mathfrak{S}^*, \mathfrak{N}^*) < \ldots\ldots$$

For each $t \geq 1$, the $\mathfrak{S}^*$-formation $t(\mathfrak{S}^*, \mathfrak{N}^*)$ is not subgroup-closed. For, by 2.5.10(2), $1(\mathfrak{S}^*, \mathfrak{N}^*)$ contains all SC-groups, i.e. finite soluble groups in which the basis normalizers and Carter subgroups coincide. Now the Alperin - Thompson Theorem (page 747, 20) states that every finite soluble group can be embedded in an SC-group. Thus if $t(\mathfrak{S}^*, \mathfrak{N}^*)$ were subgroup-closed for some $t \geq 1$, then we would have to have $\mathfrak{S}^* = t(\mathfrak{S}^*, \mathfrak{N}^*)$, contradicting (10) above. Thus $t(\mathfrak{S}^*, \mathfrak{N}^*)$ is not subgroup-closed for each $t \geq 1$.

Suppose $\mathfrak{f}_i$ $(i = 1,2)$ is a $\mathfrak{K}$-preformation function on the set of all primes satisfying $\mathfrak{f}_i(p) = s\mathfrak{f}_i(p)$ for all p, and $\mathfrak{F}_i$ is the saturated $\mathfrak{K}$-formation defined by $\mathfrak{f}_i$. We close this section with examples to show

(a) $\mathfrak{F}_1 \leq \mathfrak{F}_2$ does not imply $t(\mathfrak{K}, \mathfrak{F}_1) \leq t(\mathfrak{K}, \mathfrak{F}_2)$ $(t > 0)$

(b) $\mathfrak{F}_1 \leq \mathfrak{F}_2$ does not imply $t(\mathfrak{K}, \mathfrak{F}_2) \leq t(\mathfrak{K}, \mathfrak{F}_1)$ $(t > 0)$

In fact (b) follows easily from our previous example. For if we take $\mathfrak{K} = \mathfrak{F}_2 = \mathfrak{S}^*$ and $\mathfrak{F}_1 = \mathfrak{N}^*$ then, as in 2.5.13, $t(\mathfrak{S}^*, \mathfrak{N}^*) < t(\mathfrak{S}^*, \mathfrak{S}^*) = \mathfrak{S}^*$ for all $t \geq 0$.

Our example for (a) is somewhat more complex.

Example 2.5.14.

We in fact consider the group G which Hawkes (19) constructed as follows:-

Let $Q = \langle a,b \; ; \; a^4 = 1, \; a^2 = b^2 = [a,b] \rangle$ be a quaternion group of order 8 and S a subgroup of the automorphism group of Q isomorphic to the symmetric group of degree 3. S is chosen so as to contain an involution x whose action on Q is defined by $a^x = b$, $b^x = a$. Let $R = QS$ be the semidirect product of Q by S. We write $z = [a,b]$ and $Z = \langle z \rangle$; Z is the centre of both Q and R. We let T denote the normal subgroup of index 2 in S.

Now set $K = \langle (12)(35), (12345) \rangle$, a dihedral group of order 10 considered as a subgroup of the alternating group of degree 5, and let $H = \langle (12345) \rangle$ be the normal subgroup of index 2 in K. Let $G = R \wr K$ the wreath product of R by K according to this permutation representation. Let $\sigma_i : R \to R_i$ denote an isomorphism $(i = 1,\ldots,5)$ and let $D = R_1 \times \ldots \times R_5$ be the base group of G. Using the suffix i to denote images under σ_i Hawkes defined the following subgroups of G :-

$$\bar{A} = \langle \bar{z} \rangle \quad ; \; A = Z_1 \times \ldots \times Z_5 \; ;$$

$$B = Q_1 \times \ldots \times Q_5 \quad ; \; C = B(T_1 \times \ldots \times T_5) \; ;$$

$$\bar{D} = C\langle \bar{x} \rangle \quad ; \; \bar{S} = S_1 \times \ldots \times S_5 \; ;$$

$$E_1 = \langle \bar{z} \rangle \times \langle \bar{x} \rangle \times \langle k \rangle \quad ; \; F = (A \times \bar{S})\langle k \rangle \; ;$$

where $\bar{z} = z_1z_2z_3z_4z_5$, $\bar{x} = x_1x_2x_3x_4x_5$, $k = (12)(35)$.

Hawkes also considers the saturated $\mathfrak{S}^*$-formation $\mathfrak{F}$ defined by the $\mathfrak{S}^*$-formation function

$$\mathfrak{f}(p) = \{1\} \text{ for } p \neq 3$$
$$\mathfrak{f}(3) = \mathfrak{S}_2^* \text{ , the class of finite 2-groups.}$$

It is easy to see that the upper nilpotent series of G is $1 < B < C < D < DH < G$ so that G has nilpotent length 5 and belongs to $(\mathfrak{N}^*)^4\mathfrak{F}$ but not to $(\mathfrak{N}^*)^3\mathfrak{F}$.

Hawkes showed in his paper that E_1 is both an $\mathfrak{F}$-normalizer and basis normalizer of G and that F is an $\mathfrak{F}$-projector of G.

Let $S_2 = B(\langle x_1\rangle \times \ldots\ldots \times \langle x_5\rangle)\langle k\rangle$, $S_3 = T_1 \times \ldots \times T_5$, $S_5 = H$. Then $\underset{\sim}{S} = \{S_2, S_3, S_5\}$ is a Sylow basis of G which reduces into both E_1 and F. Thus in our usual terminology $F = E(\underset{\sim}{S})$. The p-complement system of G associated with $\underset{\sim}{S}$ is $\{S_{2'}, S_{3'}, S_{5'}\}$ where $S_{2'} = (T_1 \times \ldots\ldots \times T_5)H$, $S_{3'} = B(\langle x_1\rangle \times \ldots\ldots \times \langle x_5\rangle)K$, $S_{5'} = D\langle k\rangle$.

We calculate the $\mathfrak{F}$-reducer of E_1 in G. Since $\underset{\sim}{S}$ reduces into E_1 we have $R_G(E_1\ ;\mathfrak{F}) = \langle\, y \in G;\ \underset{\sim}{S}^{y\mathfrak{F}} \searrow_{\mathfrak{F}} E_1 \,\rangle$ by 2.1.6. Now the $\mathfrak{f}(p)$-residual $E_1^{\mathfrak{p}}$ of E_1 is E_1 for $p \neq 3$ and 1 if $p = 3$. Also $C_p(G) = G$ for $p \neq 3$. Thus

$$\begin{aligned}
\underset{\sim}{S}^{y\mathfrak{F}} \searrow_{\mathfrak{F}} E_1 \quad &\Leftrightarrow \quad S_{p'}^y \cap E_1 \in \mathrm{Syl}_{p'}(E_1) \text{ for each } p\neq 3 \\
&\Leftrightarrow \quad S_{5'}^y \searrow E_1 \quad \text{(since } E_1 \text{ is a 2-group)} \\
&\Leftrightarrow \quad E_1 \leq (D\langle k\rangle)^y \\
&\Leftrightarrow \quad D\langle k\rangle = (D\langle k\rangle)^y \quad \text{(since } D \lhd G) \\
&\Leftrightarrow \quad y \in N_G(D\langle k\rangle) = D\langle k\rangle.
\end{aligned}$$

Thus $R_G(E_1\ ;\mathfrak{F}) = D\langle k\rangle$. Since $D\langle k\rangle$ is not an $\mathfrak{F}$-projector of G we therefore have $G \notin 1(\mathfrak{S}^*, \mathfrak{F})$.

We now calculate the reducer of E_1 in G. Since $R_G(E_1) \leq R_G(E_1\ ;\mathfrak{F}) = D\langle k\rangle$ by 2.2.1, we have $R_G(E_1) = R_{D\langle k\rangle}(E_1)$. Now $\underset{\sim}{S}$ reduces into $D\langle k\rangle$ so that

$R_{D<k>}(E_1) = \langle y \in D<k> ; (\underset{\sim}{S} \cap D<k>)^y$ reduces into $E_1 \rangle$. But E_1 is a 2-group so it follows that $R_{D<k>}(E_1) = \langle y \in D<k> ; S_2^y \searrow E_1 \rangle$. Hence $S_2 \leq R_{D<k>}(E_1)$.

Suppose $y \in D<k>$ and S_2^y reduces into E_1. Now $D<k> = S_2(T_1 \times \ldots\ldots \times T_5)$ so that $y = uv$ where $u \in S_2$ and $v \in (T_1 \times \ldots\ldots \times T_5)$. Then $S_2^v = S_2^y \searrow E_1$. Therefore $E_1 \leq S_2^v$ and, since $\bar{x}$ normalizes $T_1 \times \ldots\ldots \times T_5$, $[\bar{x}, v] \in S_2^v \cap (T_1 \times \ldots\ldots \times T_5) = 1$. Now the centralizer of x_i in T_i is the identity, so it follows that the centralizer of $\bar{x}$ in $(T_1 \times \ldots\ldots \times T_5)$ is also the identity. Thus $v = 1$ and $y = u \in S_2$. Hence $R_{D<k>}(E_1) \leq S_2$ and from our previous inequality we have $R_G(E_1) = R_{D<k>}(E_1) = S_2$.

It follows, for example from 2.2.4 and 2.2.15, that S_2 is a Carter subgroup of G. Thus the $\mathfrak{N}^*$-convergence process takes one step, i.e. $G \in l(\mathfrak{S}^*, \mathfrak{N}^*)$.

Now it is clear that $\mathfrak{N}^* \leq \mathfrak{F}$, so we have an example to show (a) since $G \in l(\mathfrak{S}^*, \mathfrak{N}^*) - l(\mathfrak{S}^*, \mathfrak{F})$.

2.6. $\mathfrak{F}$-abnormal depth and $(R,\mathfrak{F})$-chains.

Throughout this section we shall assume that

(2.6.1) $f(p) = {}^{s}f(p)$ for each $p \in \pi$.

Suppose $H \leq G \in \mathfrak{K}$. A finite chain $H = H_0 \leq H_1 \leq \ldots\ldots\ldots\ldots \leq H_r = G$ from H to G is said to be $\mathfrak{F}$-balanced if H_i is either $\mathfrak{F}$-abnormal or $\mathfrak{F}$-serial in H_{i+1} for each $i = 0,\ldots\ldots,r-1$.

Remarks.

1. It seems possible that in the most general cases a subgroup H of a $\mathfrak{K}$-group G may not be joined to G by an $\mathfrak{F}$-balanced chain. However if there is such a chain we let $a^{\mathfrak{F}}(G:H)$ denote the minimal number of $\mathfrak{F}$-abnormal links in an $\mathfrak{F}$-balanced chain from H to G. $a^{\mathfrak{F}}(G:H)$ is called the $\mathfrak{F}$-abnormal depth of H in G.

2. If we take $\mathfrak{K} = \mathfrak{S}^*$ and $\mathfrak{F} = \mathfrak{N}^*$ in the above definitions we obtain the concepts "balanced", "abnormal depth" introduced by Rose (29) for finite soluble groups. The first three theorems in this section generalize similar results of Rose (29).

Theorem 2.6.2.

Suppose H is an $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{M})^t\mathfrak{F}$-group G. Then $a^{\mathfrak{F}}(G:H) \leq t$.

Proof.

Since $G \in \mathfrak{K} \cap (\mathfrak{M})^t\mathfrak{F}$ there is a series

$$1 = U_0 \leq U_1 \leq U_2 \leq \ldots\ldots\ldots\ldots \leq U_t \leq U_{t+1} = G$$

of normal subgroups U_i of G such that $U_i/U_{i-1} \in \mathfrak{M}$ for $i = 1,\ldots\ldots,t$ and $G/U_t \in \mathfrak{F}$. Set $H_i = HU_i$. Then $H = H_0 \leq H_1 \leq \ldots\ldots\ldots \leq H_t \leq H_{t+1} = G$. Let $i \in \{0,\ldots\ldots,t-1\}$. Then $H_{i+1}/U_i = U_{i+1}/U_i \cdot H_i/U_i$ and

$H_i/U_i \in \mathfrak{F}$, so, by 1.2.5(6), there is an $\mathfrak{F}$-projector F_i/U_i of H_{i+1}/U_i containing H_i/U_i. Now F_i/U_i is $\mathfrak{F}$-abnormal in H_{i+1}/U_i by 1.2.7, and H_i/U_i is $\mathfrak{F}$-serial in F_i/U_i by 2.6.1 and 2.4.9(i). Thus H_i $\mathfrak{F}$-<u>ser</u> $F_i \rtimes_{\mathfrak{F}} H_{i+1}$ and we have

$$H = H_o \ \mathfrak{F}\text{-}\underline{\text{ser}}\ F_o \rtimes_{\mathfrak{F}} H_1 \ \mathfrak{F}\text{-}\underline{\text{ser}}\ F_1 \rtimes_{\mathfrak{F}} H_2 \ldots\ldots\ldots H_t \leq G \qquad (1).$$

Now $G/U_t \in \mathfrak{F}$ so, by 2.4.9(i), H_t/U_t $\mathfrak{F}$-<u>ser</u> G/U_t. Thus H_t $\mathfrak{F}$-<u>ser</u> G, and the chain (1) is $\mathfrak{F}$-balanced. Since there are (at most) t $\mathfrak{F}$-abnormal links in this chain we have $a^{\mathfrak{F}}(G:H) \leq t$, as required.

<u>Theorem 2.6.3.</u>

If $G \in \mathfrak{K} \cap \mathfrak{A}\mathfrak{F}$ and H is any subgroup of G then $a^{\mathfrak{F}}(G:H) \leq 1$.

<u>Proof.</u>

Let $A = G^{\mathfrak{F}}$, the $\mathfrak{F}$-residual of G. Then A is abelian by hypothesis and it is clear that

$$a^{\mathfrak{F}}(G:H) \leq a^{\mathfrak{F}}(G:AH) + a^{\mathfrak{F}}(AH:H) \qquad (2)$$

Now $G/A \in \mathfrak{F}$ so, by 2.4.9(i), AH $\mathfrak{F}$-<u>ser</u> G. Thus $a^{\mathfrak{F}}(G:AH) = 0$. Since A is an abelian normal subgroup of G, $A \cap H$ is a normal subgroup of AH. It follows that $a^{\mathfrak{F}}(AH:H) = a^{\mathfrak{F}}(AH/A \cap H : H/A \cap H)$. Since $\mathfrak{F}$ is subgroup-closed and $H/A \cap H \cong AH/A$, we have $H/A \cap H \in \mathfrak{F}$. Thus, by Theorem 2.6.2, $a^{\mathfrak{F}}(AH/A \cap H : H/A \cap H) \leq 1$. From (2) and our previous remarks we now deduce $a^{\mathfrak{F}}(G:H) \leq 1$, as required.

<u>Remark.</u>

In his paper, (29), Rose shows that for each integer $n \geq 1$ there is a finite supersoluble group G with a subgroup H such that $a^{\mathfrak{N}^*}(G:H) = a(G:H) = n$. Such a

group G is necessarily metanilpotent so Theorem 2.6.3 cannot be extended to the case where $G \in \mathfrak{K} \cap (\mathfrak{N})\mathfrak{F}$.

Theorem 2.6.4.

Suppose π is the set of all primes and H is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{N})^t\mathfrak{F}$-group G ($t \geq 2$). Then $a^{\mathfrak{F}}(G:H) \leq t-1$.

Proof.

Since $G \in \mathfrak{K} \cap (\mathfrak{N})^t\mathfrak{F}$ there is a series

$$1 = U_0 \leq U_1 \leq \cdots\cdots\cdots\cdots \leq U_t \leq U_{t+1} = G$$

of normal subgroups U_i of G such that $U_{i+1}/U_i \in \mathfrak{N}$ for $i = 0,\ldots\ldots,t-1$ and $G/U_t \in \mathfrak{F}$. Now HU_{t-2} is a $\mathfrak{K} \cap (\mathfrak{N})^{t-2}\mathfrak{F}$-group, so, by Theorem 2.6.2, we have $a^{\mathfrak{F}}(HU_{t-2} : H) \leq t-2$. Since $a^{\mathfrak{F}}(G:H) \leq a^{\mathfrak{F}}(G:HU_{t-2}) + a^{\mathfrak{F}}(HU_{t-2}:H)$ it suffices to prove that $a^{\mathfrak{F}}(G:HU_{t-2}) \leq 1$.

By 1.2.8, H contains an $\mathfrak{F}$-normalizer of G, so HU_{t-2}/U_{t-2} contains some $\mathfrak{F}$-normalizer D/U_{t-2} of G/U_{t-2}. Since $H \in \mathfrak{F}$, $HU_{t-2}/U_{t-2} \leq R_{G/U_{t-2}}(D/U_{t-2};\mathfrak{F}) = X/U_{t-2}$ by 2.2.6(i). Now G/U_{t-2} is a $\mathfrak{K} \cap (\mathfrak{N})^2\mathfrak{F}$-group so, by 2.4.2, X/U_{t-2} is an $\mathfrak{F}$-projector of G/U_{t-2}. In particular therefore $X/U_{t-2} \rtimes_{\mathfrak{F}} G/U_{t-2}$ by 1.2.7. Also HU_{t-2}/U_{t-2} is $\mathfrak{F}$-serial in X/U_{t-2} by 2.4.9(i). Thus HU_{t-2} $\mathfrak{F}$-ser $X \rtimes_{\mathfrak{F}} G$ and we have $a^{\mathfrak{F}}(G:HU_{t-2}) \leq 1$, as required.

Remark.

If H is a subgroup of an $\mathfrak{F}$-projector E of the $\mathfrak{K}$-group G then, by 2.4.9(i) and 1.2.7, H $\mathfrak{F}$-ser $E \rtimes_{\mathfrak{F}} G$ so that $a^{\mathfrak{F}}(G:H) \leq 1$. In particular if D is an $\mathfrak{F}$-normalizer of G then $a^{\mathfrak{F}}(G:D) \leq 1$.

Lemma 2.6.5.

Suppose π is the set of all primes and D is an $\mathfrak{F}$-normalizer of the $\mathfrak{K}$-group G. Then $a^{\mathfrak{F}}(G:D) = 0$ if and only if $G \in \mathfrak{F}$. Thus $a^{\mathfrak{F}}(G:D) = 1$ if and only if $G \notin \mathfrak{F}$.

Proof.

Suppose $a^{\mathfrak{F}}(G:D) = 0$. Then there is a series $D = D_0 \leq D_1 \leq \cdots\cdots\cdots\cdots \leq D_n = G$ with D_i $\mathfrak{F}$-ser D_{i+1} for each $i = 0,\ldots\ldots,n-1$. Clearly this implies that D is $\mathfrak{F}$-serial in G. It now follows, from 2.3.10, that $G = R_G(D\ ;\mathfrak{F})$. From 2.4.6, we now deduce that G belongs to either $\mathfrak{F}$ or $\mathfrak{K} \cap (\mathfrak{M})\mathfrak{F}$. If $G \in \mathfrak{K} \cap (\mathfrak{M})\mathfrak{F}$ then, by 1.2.5(3) and 2.2.14(i), $G = D \in \mathfrak{F}$. Thus in either case we have $G \in \mathfrak{F}$, as required.

If conversely $G \in \mathfrak{F}$ then $D = G$ and clearly $a^{\mathfrak{F}}(G:D) = 0$. We have therefore shown that $a^{\mathfrak{F}}(G:D) = 0$ if and only if $G \in \mathfrak{F}$. The last part of the result now follows from the remark prior to the statement of this lemma.

We now wish to discuss $(R,\mathfrak{F})$-chains, but before doing so we extend the hypothesis (2.6.1) that covers this section. For the remainder of this section we shall assume that, except where the contrary is explicitly stated,

(2.6.6) π is the set of all primes and $\mathfrak{f}(p) = s\mathfrak{f}(p)$ for all primes p.

Lemma 2.6.7.

Suppose H is a subgroup of the $\mathfrak{K}$-group G. Then there is a unique smallest $\mathfrak{F}$-serial subgroup of G containing H.

Proof.

Let $\mathfrak{B}$ be the collection of all $\mathfrak{F}$-serial subgroups of G containing H ; $\mathfrak{B}$ is non-empty since $G \in \mathfrak{B}$. From 2.1.18 and 2.3.10 it follows that the intersection B of all the members of $\mathfrak{B}$ is also $\mathfrak{F}$-serial in G. Clearly B is the unique smallest $\mathfrak{F}$-serial subgroup of G containing H.

If H is a subgroup of a $\mathfrak{K}$-group G we denote by $S^{\mathfrak{F}}(G:H)$ the unique smallest $\mathfrak{F}$-serial subgroup of G containing H.

Lemma 2.6.8.

Suppose $H \leq G \in \mathfrak{K}$ and $N \triangleleft G$. Then

$$S^{\mathfrak{F}}(G/N:HN/N) = S^{\mathfrak{F}}(G:H)N/N.$$

Proof.

Let $S^{\mathfrak{F}}(G/N:HN/N) = X/N$. Then $HN/N \leq X/N$ $\mathfrak{F}$-ser G/N so that $H \leq X$ $\mathfrak{F}$-ser G. Therefore $S^{\mathfrak{F}}(G:H) \leq X$ by definition and hence $S^{\mathfrak{F}}(G:H)N/N \leq X/N$. But $S^{\mathfrak{F}}(G:H)N/N$ $\mathfrak{F}$-ser G/N by 2.3.11, so we must have $S^{\mathfrak{F}}(G:H)N/N = X/N$, as claimed.

Suppose H is a subgroup of the $\mathfrak{K}$-group G. We define subgroups $S_i = S_i(G:H:\mathfrak{F})$ and $R_i = R_i(G:H:\mathfrak{F})$ of G containing H inductively as follows:-

$$S_1 = S^{\mathfrak{F}}(G:H) \quad ; \quad R_1 = R_{S_1}(H\ ;\mathfrak{F})$$

$$S_{i+1} = S^{\mathfrak{F}}(R_i:H) \quad ; \quad R_{i+1} = R_{S_{i+1}}(H\ ;\mathfrak{F}) \quad (i \geq 1)$$

In this way we obtain a chain

$$G \geq S_1 \geq R_1 \geq S_2 \geq R_2 \geq \cdots\cdots\cdots\cdots \qquad (3)$$

of subgroups of G containing H. It seems possible that in the most general cases the series (3) may not reach H after a finite number of steps. However, by 2.2.4 and 2.2.9(ii) we do have

$$H \leq \ldots\ldots\ldots R_2 \rtimes_{\mathfrak{F}} S_2 \ \mathfrak{F}\text{-}\underline{\text{ser}}\ R_1 \rtimes_{\mathfrak{F}} S_1 \ \mathfrak{F}\text{-}\underline{\text{ser}}\ G \qquad (4)$$

Thus when the chain (4) is finite and reaches H it is an $\mathfrak{F}$-balanced chain from H to G. We call (3) the <u>(R, $\mathfrak{F}$)-chain of H in G</u>, and when it reaches H after a finite number of steps we denote by $b^{\mathfrak{F}}(G:H)$ the number of $\mathfrak{F}$-abnormal links in it. It is clear that $a^{\mathfrak{F}}(G:H) \leq b^{\mathfrak{F}}(G:H)$ when defined.

Our (R, $\mathfrak{F}$)-chains generalize Mann's Q-chains (23) and our first aim is to show that at least in finite $\mathfrak{K}$-groups they have some meaning, i.e. if H is a subgroup of a finite $\mathfrak{K}$-group then the (R, $\mathfrak{F}$)-chain of H in G reaches H (after a finite number of steps). To do this we require two lemmas:-

<u>Lemma 2.6.9.</u>

Suppose H is a subgroup of the finite $\mathfrak{K}$-group G and $H < R_G(H\ ;\mathfrak{F}) = G$. Then H lies in an $\mathfrak{F}$-normal maximal subgroup of G. Hence $S^{\mathfrak{F}}(G:H) < G$.

<u>Proof.</u>

We argue by induction on the order of G. Since H is a proper subgroup of G, G is non trivial. Let N be a minimal normal subgroup of G; then $R_{G/N}(HN/N\ ;\mathfrak{F}) = G/N$ by 2.2.7. If HN/N is a proper subgroup of G/N then by induction HN/N lies in an $\mathfrak{F}$-normal maximal subgroup M/N of G/N. Thus H is contained in the $\mathfrak{F}$-normal maximal subgroup M of G, as required. If HN = G then H is a maximal subgroup of G and must be $\mathfrak{F}$-normal by 2.2.9(i). Thus in either case H lies in an $\mathfrak{F}$-normal maximal subgroup M of G. By definition $S^{\mathfrak{F}}(G:H) \leq M$ so the final statement of the lemma is immediate.

Lemma 2.6.10.

Suppose the $\mathfrak{K}$-preformation function f satisfies 2.6.1 but not necessarily 2.6.6. If $H \leq G \in \mathfrak{K}$ and $X = R_G(H\ ;\mathfrak{F})$ then $R_X(H\ ;\mathfrak{F}) = X$.

Proof.

Let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into both H and X. Then, by 2.1.6(iii),

$$X = R_G(H\ ;\mathfrak{F}) = \langle\ x \in G\ ;\ \underset{\sim}{S}^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H\ \rangle$$

$$R_X(H\ ;\mathfrak{F}) = \langle\ y \in X\ ;\ (\underset{\sim}{S} \cap X)^{y\mathfrak{F}} \searrow_{\mathfrak{F}} H\ \rangle.$$

Suppose $x \in G$ and $\underset{\sim}{S}^{x\mathfrak{F}}$ $\mathfrak{F}$-reduces into H. Then $x \in X$ and, since $\underset{\sim}{S}^{x\mathfrak{F}}$ clearly $\mathfrak{F}$-reduces into X to $(\underset{\sim}{S} \cap X)^{x\mathfrak{F}}$, we have $(\underset{\sim}{S} \cap X)^{x\mathfrak{F}} \searrow_{\mathfrak{F}} H$ by 2.1.16. Thus $X \leq R_X(H\ ;\mathfrak{F})$ and the result follows.

Suppose now that H is a subgroup of the finite $\mathfrak{K}$-group G. Then, by 2.6.9 and 2.6.10, every containment in the chain (3) is strict (except possibly $G \geq S_1$) until H is reached. Thus the $(R,\mathfrak{F})$-chain of H in G reaches H and $b^{\mathfrak{F}}(G:H)$ is defined.

We have been unable to decide whether Lemma 2.6.9 holds in general.

Our aim now is to improve Theorems 2.6.2, 2.6.3 and 2.6.4 by showing that $b^{\mathfrak{F}}(G:H)$ may replace $a^{\mathfrak{F}}(G:H)$ in each of the statements. Techniques similar to those used by Mann (23) can be employed to prove these extensions but we give here alternative proofs which use our work on $\mathfrak{F}$-reducers.

We shall require the following

Lemma 2.6.11.

Suppose H is an $\mathfrak{F}$-ascendabnormal subgroup of the $\mathfrak{K}$-group G. Then $S^{\mathfrak{F}}(G:H) = G$.

Proof.

Let $S = S^{\mathfrak{F}}(G:H)$. Since H is $\mathfrak{F}$-ascendabnormal in G there is an ordinal σ and a chain $(H_\beta \; ; \; \beta \leq \sigma)$ of subgroups of G such that $H = H_o$, $H_\beta \rtimes_{\mathfrak{F}} H_{\beta+1}$ for $\beta < \sigma$, $H_\lambda = \bigcup_{\beta<\lambda} H_\beta$ for limit ordinals $\lambda \leq \sigma$, and $H_\sigma = G$. We prove by transfinite induction that $H_\beta \leq S$ for each $\beta \leq \sigma$. This will show that $G = H_\sigma \leq S$, proving the result.

If $\beta = 0$ then $H_o = H \leq S$ by definition ; therefore the induction begins.

Suppose $\beta = \alpha+1$ for some $\alpha < \sigma$ and $H_\alpha \leq S$. Then $H_\alpha \leq S \cap H_{\alpha+1}$. Now $H_\alpha \rtimes_{\mathfrak{F}} H_{\alpha+1}$ so that $S \cap H_{\alpha+1} \rtimes_{\mathfrak{F}} H_{\alpha+1}$. Also $S \cap H_{\alpha+1}$ $\mathfrak{F}$-ser $H_{\alpha+1}$ by 2.3.3(i). A proper subgroup of a $\mathfrak{K}$-group cannot be both $\mathfrak{F}$-abnormal and $\mathfrak{F}$-serial so we must have $S \cap H_{\alpha+1} = H_{\alpha+1}$. Thus $H_\beta = H_{\alpha+1} \leq S$, and the induction goes through in this case.

If $\lambda \leq \sigma$ is a limit ordinal and $H_\beta \leq S$ for each $\beta < \lambda$ then clearly $H_\lambda = \bigcup_{\beta<\lambda} H_\beta \leq S$. This completes the induction argument and the proof.

As an immediate consequence of 1.2.8 and 2.6.11 we have

Corollary 2.6.12.

If D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}$-group G then $S^{\mathfrak{F}}(G:H) = G$.

Lemma 2.6.13.

Suppose H is an $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{N})\mathfrak{F}$-group G. Then $b^{\mathfrak{F}}(G:H) \leq 1$.

Proof.

Let $R = \mathcal{Q}(G)$. Then $G/R \in \mathfrak{F}$ so by 2.4.9(i),

HR $\mathfrak{F}$-<u>ser</u> G. Therefore $S_1 = S_1(G:H:\mathfrak{F}) = S^{\mathfrak{F}}(G:H) \leq HR$. Since $H \leq S_1$ the modular law gives $S_1 = H(S_1 \cap R)$. Now $H \in \mathfrak{F}$, so, by 2.3.21, H $\mathfrak{F}$-<u>ser</u> $R_{S_1}(H;\mathfrak{F}) = R_1(G:H:\mathfrak{F})$. Thus $H = S^{\mathfrak{F}}(R_1:H) = S_2(G:H:\mathfrak{F})$. Hence

$$H = S_2 \ \mathfrak{F}\text{-}\underline{ser}\ R_1 \rtimes_{\mathfrak{F}} S_1 \ \mathfrak{F}\text{-}\underline{ser}\ G$$

and $b^{\mathfrak{F}}(G:H) \leq 1$, as required.

<u>Theorem 2.6.14.</u>

Suppose H is an $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{M})^t\mathfrak{F}$-group G. Then $b^{\mathfrak{F}}(G:H) \leq t$.

<u>Proof.</u>

We argue by induction on t. If $t = 0$ then $G \in \mathfrak{F}$ and, by 2.4.9(i), H $\mathfrak{F}$-<u>ser</u> G. Thus $S_1(G:H:\mathfrak{F}) = H$ and $b^{\mathfrak{F}}(G:H) = 0$.

If $t > 0$ let $R = \mathfrak{Q}(G)$. Then HR/R is an $\mathfrak{F}$-subgroup of G/R so by induction $b^{\mathfrak{F}}(G/R:HR/R) \leq t-1$ and hence $S_t(G/R:HR/R:\mathfrak{F}) = HR/R$. Now it is clear, from 2.2.7 and 2.6.8, that $S_i(G/R:HR/R:\mathfrak{F}) = S_i(G:H:\mathfrak{F})R/R$ and $R_i(G/R:HR/R:\mathfrak{F}) = R_i(G:H:\mathfrak{F})R/R$ for each i, i.e. the $(R,\mathfrak{F})$-chain of HR/R in G/R is the image in G/R of the $(R,\mathfrak{F})$-chain of H in G. Thus $S_t = S_t(G:H:\mathfrak{F}) \leq HR$ and in particular $S_t \in \mathfrak{K} \cap (\mathfrak{M})\mathfrak{F}$. Therefore, by 2.6.13, $b^{\mathfrak{F}}(S_t:H) \leq 1$. Now it is clear from the definitions that $S_1(S_t:H:\mathfrak{F}) = S_t$, $R_1(S_t:H:\mathfrak{F}) = R_t$, $S_2(S_t:H:\mathfrak{F}) = S_{t+1}$. Since $b^{\mathfrak{F}}(S_t:H) \leq 1$ we have $H = S_2(S_t:H:\mathfrak{F}) = S_{t+1}$ and hence $b^{\mathfrak{F}}(G:H) \leq t$, as claimed.

<u>Theorem 2.6.15.</u>

If $G \in \mathfrak{K} \cap \mathfrak{A}\mathfrak{F}$ and $H \leq G$ then $b^{\mathfrak{F}}(G:H) \leq 1$.

<u>Proof.</u>

Let $A = G^{\mathfrak{F}}$, the $\mathfrak{F}$-residual of G; A is abelian by hypothesis. Therefore $H \cap A$ is a normal subgroup of AH

and, as in the proof of 2.6.14, $b^{\mathfrak{F}}(AH/A \cap H : H/A \cap H) = b^{\mathfrak{F}}(AH:H)$. Now $H/A \cap H$ is isomorphic to a subgroup of the $\mathfrak{F}$-group G/A so, by 2.6.6, $H/A \cap H \in \mathfrak{F}$. Thus, by 2.6.14, $b^{\mathfrak{F}}(AH/A \cap H : H/A \cap H) \leq 1$. Hence $b^{\mathfrak{F}}(AH:H) \leq 1$.

Now $G/A \in \mathfrak{F}$ so, by 2.4.9(i), AH $\mathfrak{F}$-<u>ser</u> G. Therefore $S^{\mathfrak{F}}(G:H) \leq AH$ and hence $S^{\mathfrak{F}}(G:H) = S^{\mathfrak{F}}(AH:H)$. Thus the $(R,\mathfrak{F})$-chain of H in G coincides with the $(R,\mathfrak{F})$-chain of H in AH so that $b^{\mathfrak{F}}(G:H) \leq 1$ since $b^{\mathfrak{F}}(AH:H) \leq 1$.

<u>Remark.</u>

Since $a^{\mathfrak{F}}(G:H) \leq b^{\mathfrak{F}}(G:H)$ when the latter is defined, it follows, from the remark after the proof of 2.6.3, that we cannot hope to extend 2.6.15 to the case where G is a $\mathfrak{K} \cap (\mathfrak{N})\mathfrak{F}$-group.

<u>Lemma 2.6.16.</u>

Suppose H is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of $G \in \mathfrak{K} \cap (\mathfrak{N})^2\mathfrak{F}$. Then $b^{\mathfrak{F}}(G:H) \leq 1$.

<u>Proof.</u>

By 1.2.8, H contains an $\mathfrak{F}$-normalizer D of G, and since $H \in \mathfrak{F}$ we have $R_G(H;\mathfrak{F}) \leq R_G(D;\mathfrak{F})$ by 2.2.6. Now $\mathfrak{F}$ is subgroup-closed and $R_G(D;\mathfrak{F})$ is an $\mathfrak{F}$-projector of G by 2.4.2. Therefore $R_G(H;\mathfrak{F}) \in \mathfrak{F}$ and by 2.4.9(i) H $\mathfrak{F}$-<u>ser</u> $R_G(H;\mathfrak{F})$. Now $S_1(G:H:\mathfrak{F}) = G$ since H is $\mathfrak{F}$-ascendabnormal in G (2.6.11), so $R_1(G:H:\mathfrak{F}) = R_G(H;\mathfrak{F})$. Therefore $S_2(G:H:\mathfrak{F}) = S^{\mathfrak{F}}(R_G(H;\mathfrak{F}) : H) = H$. Thus $b^{\mathfrak{F}}(G:H) \leq 1$, as required.

<u>Theorem 2.6.17.</u>

Suppose H is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{N})^t\mathfrak{F}$-group G, where $t \geq 2$. Then $b^{\mathfrak{F}}(G:H) \leq t-1$.

Proof.

We argue by induction on t, the case $t = 2$ being covered by 2.6.16. If $t > 2$ and $R = \mathfrak{R}(G)$ then HR/R is an $\mathfrak{F}$-ascendabnormal $\mathfrak{F}$-subgroup of G/R by (4.5, 12) so by induction $b^{\mathfrak{F}}(G/R:HR/R) \leq t-2$. In particular therefore $S_{t-1}(G/R:HR/R:\mathfrak{F}) = HR/R$. As in the proof of 2.6.14 we now obtain $S_{t-1}(G:H:\mathfrak{F}) \leq HR$, and in particular $S_{t-1}(G:H:\mathfrak{F}) \in \mathfrak{K} \cap (\mathfrak{N})\mathfrak{F}$. The argument used to complete the proof of 2.6.14 now shows that $S_t(G:H:\mathfrak{F}) = H$. Thus $b^{\mathfrak{F}}(G:H) \leq t-1$, as claimed.

Theorem 2.6.18.

Suppose D is an $\mathfrak{F}$-normalizer of the $\mathfrak{K}$-group G. Then $a^{\mathfrak{F}}(G:D) = b^{\mathfrak{F}}(G:D)$ if and only if D has a strong $\mathfrak{F}$-serializer in G.

Proof.

If $G \in \mathfrak{F}$ then $D = G$ and $a^{\mathfrak{F}}(G:D) = b^{\mathfrak{F}}(G:D) = 0$, so there is nothing to prove. We may therefore suppose that $G \notin \mathfrak{F}$. Then $a^{\mathfrak{F}}(G:D) = 1$ by 2.6.5.

By 2.6.12, $S_1(G:D:\mathfrak{F}) = G$ so that $R_1(G:D:\mathfrak{F}) = R_G(D;\mathfrak{F})$ and $S_2(G:D:\mathfrak{F}) = S^{\mathfrak{F}}(R_G(D;\mathfrak{F}) : D)$. Thus

$$\begin{aligned} a^{\mathfrak{F}}(G:D) = b^{\mathfrak{F}}(G:D) &\iff b^{\mathfrak{F}}(G:D) = 1 \\ &\iff S_2(G:D:\mathfrak{F}) = D \\ &\iff D\ \mathfrak{F}\text{-}\underline{\text{ser}}\ R_G(D;\mathfrak{F}) \\ &\iff D \text{ has a strong } \mathfrak{F}\text{-serializer in } G \ (2.3.17) \end{aligned}$$

We close this section by considering briefly a generalization of another of Rose's concepts (30).

If $H \leq G \in \mathfrak{K}$ we say H is <u>$\mathfrak{F}$-contranormal</u> in G if $S^{\mathfrak{F}}(G:H) = G$.

By 2.6.11, every $\mathfrak{F}$-ascendabnormal subgroup is $\mathfrak{F}$-contranormal.

Lemma 2.6.19.

If H is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{M})\mathfrak{F}$-group G then H lies in some $\mathfrak{F}$-projector of G.

Proof.

Let $R = Q(G)$. Then $G/R \in \mathfrak{F}$ so, by 2.4.9(i), HR $\mathfrak{F}$-ser G. Since H is $\mathfrak{F}$-contranormal in G we must have $G = HR$. The result is now immediate from 1.2.5(6).

Lemma 2.6.20.

If $G \in \mathfrak{K} \cap \mathfrak{A}\mathfrak{F}$ then the $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroups of G are precisely the $\mathfrak{F}$-projectors of G.

Proof.

The $\mathfrak{F}$-projectors of G are certainly $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroups of G by 1.2.7 and 2.6.11. On the other hand suppose H is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of G. Let $A = G^{\mathfrak{F}}$, the $\mathfrak{F}$-residual of G; by hypothesis A is abelian. Now $G/A \in \mathfrak{F}$ so, by 2.4.9(i), HA $\mathfrak{F}$-ser G. Since H is $\mathfrak{F}$-contranormal in G we must therefore have $G = HA$. Now H is contained in some $\mathfrak{F}$-projector E of G by 2.6.19, so by the modular law $E = H(E \cap A)$. But E complements A in G by 1.2.4 and 1.2.5(3). Therefore $H = E$, an $\mathfrak{F}$-projector of G, and the proof is complete.

Remark.

If G is a finite soluble group then a subgroup H of G is $\mathfrak{N}^*$-contranormal in G if and only if H lies in no proper normal subgroup of G, i.e. if and only if the normal closure H^G of H in G is G. Thus for $\mathfrak{S}^*$-

groups the concepts " $\mathfrak{N}^*$-contranormal" and "contranormal" (as defined in (30)) coincide.

In his paper, (30), Rose gives an example to show that $(\mathfrak{N}^*)^2$-groups may have nilpotent contranormal subgroups which are not Carter subgroups. We cannot therefore hope to improve 2.6.20 to the case where G is a $\mathfrak{K} \cap (\mathfrak{U})\mathfrak{F}$-group.

Using Lemma 2.6.20 we can sharpen 2.6.19 to give

Lemma 2.6.21.

If H is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{U})\mathfrak{A}\mathfrak{F}$-group G then H lies in some $\mathfrak{F}$-projector of G.

Proof.

Let $R = \varphi(G)$. Then HR/R is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of G/R by 2.6.8. Thus, by 2.6.20, HR/R is an $\mathfrak{F}$-projector of G/R. By 1.2.5(6), H lies in an $\mathfrak{F}$-projector E of HR, and since E is an $\mathfrak{F}$-projector of G (1.2.5(7)) we have the desired result.

The final result in this section sharpens 2.6.14.

Theorem 2.6.22.

If H is an $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{U})^t\mathfrak{A}\mathfrak{F}$-group G $(t \geq 1)$ then $b^{\mathfrak{F}}(G:H) \leq t$.

Proof.

We argue by induction on t.

If $t = 1$, then H is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{U})\mathfrak{A}\mathfrak{F}$-group $S_1 = S_1(G:H:\mathfrak{F}) = S^{\mathfrak{F}}(G:H)$. Thus, by 2.6.21, H lies in an $\mathfrak{F}$-projector E of S_1. Let R be the Hirsch-Plotkin radical of S_1. Then, by 2.6.8, HR/R is an $\mathfrak{F}$-contranormal $\mathfrak{F}$-subgroup of S_1/R, so, by 2.6.20, HR = ER. Hence $E = H(E \cap R)$. Now the usual argument

using Lemma 2.1.17 shows that

$$R_1 = R_1(G:H:\mathfrak{F}) = R_{S_1}(H\ ;\mathfrak{F}) = E.$$

But H $\mathfrak{F}$-<u>ser</u> E by 2.4.9(i), so we have $S_2(G:H:\mathfrak{F}) = H$. Thus $b^{\mathfrak{F}}(G:H) \leq 1$, and the induction begins.

If $t > 1$ and $Y = \mathfrak{Q}(G)$ then HY/Y is an $\mathfrak{F}$-subgroup of the $\mathfrak{K} \cap (\mathfrak{M})^{t-1}\mathfrak{A}\mathfrak{F}$-group G/Y, so by induction $b^{\mathfrak{F}}(G/Y:HY/Y) \leq t-1$. Thus $S_t(G/Y:HY/Y:\mathfrak{F}) = HY/Y$. The argument at the end of the proof of 2.6.14 now shows that $S_{t+1}(G:H:\mathfrak{F}) = H$. Thus $b^{\mathfrak{F}}(G:H) \leq t$, which completes the proof.

Chapter Three. Some Generalizations of Theorems due to Alperin, Chambers and Rose.

In this chapter we show that many of the results obtained by Alperin (1), Chambers (5) and Rose (28) for finite soluble groups can be extended to the appropriate subclasses of $\mathfrak{U}$.

3.1. Basis normalizers and Carter subgroups in $\mathfrak{U}$-groups.

In this section we show that most of Alperin's results on system normalizers, Carter subgroups and the relation between them in finite soluble groups can be extended to the class $\mathfrak{U}$. Some of these results have already been dealt with in previous sections. For example, Lemma 1.3.5 generalizes Alperin's crucial "extendibility" theorem (Theorem B, 1), Corollary 2.4.5 generalizes (Theorem 5, 1), and (Theorem 10, 1) for $\mathfrak{U}$-groups is a special case of Corollary 2.4.8. Our first result extends (Theorem 2, 1):-

Theorem 3.1.1.

Let D_1 , D_2 be basis normalizers of a $\mathfrak{U}$-group G contained in the same Carter subgroup E of G. Then D_1 and D_2 are conjugate by an element of E.

We shall deduce Theorem 3.1.1 from

Lemma 3.1.2.

Let H be a subgroup of a $\mathfrak{U}$-group G containing a basis normalizer D of G. Suppose that for each prime p, the Sylow p-subgroup of D centralizes some p-complement of H. Then D is the normalizer of some Sylow basis of G which reduces into H.

We remark that the hypotheses of 3.1.2 hold in particular if H is locally nilpotent.

Proof of 3.1.2.

Suppose that D is the normalizer $N_G(\underset{\sim}{S})$ of the Sylow basis $\underset{\sim}{S}$ of G, and let π be the set of all primes p such that $S_{p'}$ does not reduce into H. Then π is finite by 1.3.3.

Suppose $p \in \pi$. Let D_p , $D_{p'}$ be the Sylow p- and Sylow p'-subgroups of D respectively and let $C = C_H(D_p)$. By hypothesis, C contains a p-complement of H and since $D_{p'} \leq C$ it follows that $D_{p'}$ is contained in a p-complement $U_{p'}$ of H centralized by D_p. By Lemma 1.3.5, there is a p-complement $T_{p'}$ of G normalized by D_p and containing $U_{p'}$. Since $D_{p'} \leq T_{p'}$, D also normalizes $T_{p'}$.

Lemma 1.3.1 now shows that $\{S_{p'} ; p \notin \pi\} \cup \{T_{p'} ; p \in \pi\}$ is the p-complement system associated with some Sylow basis $\underset{\sim}{T}$ of G. By 1.3.4, $\underset{\sim}{T}$ reduces into H and clearly $D \leq N_G(\underset{\sim}{T}) = D_1$. Now $D_1 = D^x$ for some $x \in G$ and since $x^n = 1$ for some integer $n \geq 0$ we have
$D \leq D_1 = D^x \leq D^{x^2} \leq \ldots\ldots \leq D^{x^n} = D$. Hence $D = D_1$, as required.

Proof of Theorem 3.1.1.

Since E is locally nilpotent, Lemma 3.1.2 shows that $D_i = N_G(\underset{\sim}{S}_i)$ $(i = 1,2)$ where $\underset{\sim}{S}_i$ is a Sylow basis of G reducing into E. By (6.5, 12), or equivalently 2.2.15, we have $\underset{\sim}{S}_2 = \underset{\sim}{S}_1^x$ for some $x \in E$; hence $D_2 = D_1^x$ and 3.1.1 is established.

Corollary 3.1.3 (cf. (Theorem 3, 1)).

Let D be a basis normalizer of a $\mathfrak{U}$-group G

contained in a Carter subgroup E of G. Then the Carter subgroups of G containing D are precisely the conjugates of E by elements of $N_G(D)$.

Proof.

Clearly every conjugate of E by an element of $N_G(D)$ is a Carter subgroup of G containing D. On the other hand, if E^* is a Carter subgroup of G containing D then $E^* = E^x$ for some $x \in G$. Consequently D, $D^{x^{-1}}$ are basis normalizers of G contained in E and so, by 3.1.1, there is an element $y \in E$ such that $D^{x^{-1}y} = D$. Hence $w = x^{-1}y \in N_G(D)$ and $E^* = E^x = E^{yw^{-1}} = E^{w^{-1}}$, as claimed.

If $\mathfrak{X}$ is a subclass of $\mathfrak{U}$ we shall denote by $\mathfrak{X}_A$ the class of $\mathfrak{X}$-groups whose Sylow p-subgroups are abelian for each prime p. In particular we have the class $\mathfrak{U}_A$ previously mentioned. As a further corollary to 3.1.1 we have the following generalization of a result of Carter (§2, 3) which was reproved by Alperin (Theorem 4, 1).

Corollary 3.1.4.

If G is a $\mathfrak{U}_A$-group then each Carter subgroup of G contains a unique basis normalizer of G.

Proof.

In $\mathfrak{U}_A$-groups the Carter subgroups are abelian so the result is immediate from 3.1.1.

Remarks.

1. In (19) Hawkes gives an example of a saturated $\mathfrak{S}^*$-formation $\mathfrak{F}$ and a finite soluble group G with $\mathfrak{F}$-normalizers D_1 and D_2 contained in the same $\mathfrak{F}$-projector E of G but not conjugate in E.

2. In the next section we shall show that if $\mathfrak{K}$ is a

as-closed subclass of $\mathfrak{U}$ and $\mathfrak{F}$ is a saturated $\mathfrak{K}$-formation (not necessarily satisfying 2.1.1) then an extension of 3.1.1 holds for $\mathfrak{K}_A$-groups.

Theorem 3.1.5.

If $\underset{\sim}{S}$ is a Sylow basis of a $\mathfrak{U}$-group G then there is a unique Carter subgroup of G into which $\underset{\sim}{S}$ reduces.

Proof.

This theorem, which generalizes (Theorem 6, $\underset{\sim}{1}$) is an immediate consequence of (6.5, $\underset{\sim}{12}$).

We now turn our attention to the following extension of (Theorem 7, $\underset{\sim}{1}$).

Theorem 3.1.6.

Let $\underset{\sim}{S}$ be a Sylow basis of the $\mathfrak{U}$-group G with normalizer D. Then $\underset{\sim}{S}$ reduces into $N_G(D)$.

Proof.

Let $N = N_G(D)$ and $(\Lambda_\sigma , V_\sigma ; \sigma \in \Omega)$ a chief series of G. Then $(\Lambda_\sigma \cap D , V_\sigma \cap D ; \sigma \in \Omega)$ is a series of D. Now D covers the central chief factors of G and avoids the eccentric ones (4.6, $\underset{\sim}{7}$); thus if $\Lambda_\sigma \cap D/V_\sigma \cap D$ is non-trivial then Λ_σ/V_σ is central in G so that $[N,\Lambda_\sigma \cap D] \leq V_\sigma \cap D$. Therefore D has a series with factors central in N. Let T be any Sylow p'-subgroup of N. Then the Sylow p-subgroup D_p of D has a series with factors centralized by T, consequently $[D_p,T] = 1$ by (4.11, $\underset{\sim}{7}$).

It now follows, from Lemma 3.1.2, that D is the normalizer of some Sylow basis $\underset{\sim}{T}$ of G which reduces into N. Since any two Sylow bases with normalizer D are clearly conjugate by an element of N it follows that $\underset{\sim}{S}$ reduces into N, which completes the proof.

The next result is analogous to Theorems 8,9 of (1).

Theorem 3.1.7.

Let H be a subgroup of a $\mathfrak{U}$-group G. Let $\{\pi_1, \ldots\ldots, \pi_n\}$ be a partition of the set of all primes, where n is a positive integer, and suppose that for each i $(1 \leq i \leq n)$ a Sylow π_i-subgroup of H normalizes some Sylow basis of G. Then H normalizes some Sylow basis of G.

Proof.

Suppose that for each i $(1 \leq i \leq n)$ the Sylow π_i-subgroup H_i of H normalizes the Sylow basis $\underset{\sim}{S}^{(i)}$ of G. We have to show that H normalizes some Sylow basis of G.

We show first that H is locally nilpotent. To see this let U/V be any chief factor of G. Then U/V is a p-chief factor for some prime p which belongs say to π_i. If U/V is eccentric then every basis normalizer of G avoids U/V (4.6, 7), and in particular therefore H_i, and hence also H, avoids U/V. Thus the intersection with H of any chief series of G is a central series of H, and intersecting this series further with an arbitrary finite subgroup F of H provides a central series of F. Hence H is locally nilpotent.

It follows from Lemmas 1.3.1 and 1.3.3 that there is a finite set σ of primes with the property that if $p \notin \sigma$ then $S_{p'}^{(i)} = S_{p'}^{(j)} = T_{p'}$ say, for all i,j and $T_{p'} \trianglelefteq H$. Let $q \in \sigma$. Then the Sylow q-subgroup H_q of H is contained in some H_i and so normalizes some Sylow q'-subgroup of G. Since H_q centralizes the Sylow q'-subgroup $H_{q'}$ of H, Lemma 1.3.5 shows that H_q normalizes some Sylow q'-subgroup $T_{q'}$ of G with

$T_{q'} \geq H_{q'}$. Hence H normalizes $T_{q'}$. By 1.3.1, $\{T_{p'}\}$ is the p-complement system associated with some Sylow basis $\underset{\sim}{T}$ of G and clearly $H \leq N_G(\underset{\sim}{T})$. This establishes the result.

Corollary 3.1.8. (cf. (Theorem 8, $\underset{\sim}{1}$)).

With the notation employed in the proof of Theorem 3.1.7, suppose that H_i is the Sylow π_i-subgroup of some basis normalizer of G $(1 \leq i \leq n)$. Then H is a basis normalizer of G.

Proof.

This result is an immediate consequence of Theorem 3.1.7 since, as we saw in the proof of 3.1.2, a subgroup of a locally finite group cannot be conjugate to a proper subgroup of itself.

If in Theorem 3.1.7 one of the sets π_i happens to be such that the Sylow π_i-subgroups of G are locally nilpotent it suffices to assume that a Sylow π_i-subgroup of H normalizes some Sylow π_i'-subgroup of G. This follows from

Lemma 3.1.9.

Let H be a π-subgroup of a $\mathfrak{U}$-group G normalizing some Sylow π'-subgroup S of G, and suppose that the Sylow π-subgroups of G are locally nilpotent. Then H normalizes some Sylow basis of G.

Proof.

Since the Sylow π'-subgroups of G are conjugate there exists a Sylow basis $\underset{\sim}{S}$ of G such that $S = S_{\pi'}$. Let T be any Sylow π-subgroup of G containing H. Then

$T = S_\pi^x$ for some $x \in G$ and since $G = S_\pi S_{\pi'}$ we may suppose that $x \in S_{\pi'}$, so that $S = S_{\pi'} = S_{\pi'}^x$. If $\underset{\sim}{U} = \underset{\sim}{S}^x$ then $\underset{\sim}{U}$ is a Sylow basis of G such that $H \leq U_\pi$ and H normalizes $U_{\pi'}$. Now $H \leq U_{q'}$ if $q \notin \pi$ so H normalizes $U_{q'}$ if $q \notin \pi$. If $q \in \pi$ then $U_{q'} = U_\sigma U_{\pi'}$ where $\sigma = \pi - \{q\}$. Since U_π is locally nilpotent, H normalizes U_σ and since H normalizes $U_{\pi'}$ by assumption it normalizes $U_{q'}$ in this case also. Thus H normalizes the Sylow basis $\underset{\sim}{U}$, as required.

We end the discussion of Alperin's Theorem 8 by showing that a full generalization is not possible.

<u>Example 3.1.10.</u>

There exists a $\mathfrak{U}$-group G with a locally nilpotent subgroup H such that

(i) For each prime p the Sylow p-subgroup H_p of H is the Sylow p-subgroup of some basis normalizer of G, but

(ii) H is not a basis normalizer of G.

<u>Proof.</u>

To demonstrate the example let A denote the symmetric group on three symbols 1,2,3, let π denote the set of all primes other than 2,3 and let R be any locally nilpotent π-group each of whose Sylow p-subgroups R_p is non-abelian. Let $G = R \wr A$, where A is taken in its natural permutation representation. Then G is an extension of the periodic locally nilpotent group $\bar{R}$, the base group of G, by a finite soluble group. Thus $G \in (\mathfrak{N}^*)\mathfrak{S}^*$ and hence to $\mathfrak{U}$.

Let T_p denote the Sylow p-subgroup of $\bar{R}$ if $p \in \pi$ and put $T_3 = \langle(123)\rangle$ and $T_2 = \langle(12)\rangle$. Then $\underset{\sim}{T} = \{T_p\}$ is a Sylow basis of G with normalizer $D = C \times T_2$

where $C = C_{\bar{R}}(A)$. The unique Carter subgroup of G containing D is $E = B \rtimes T_2$, where $B = C_{\bar{R}}(T_2)$. Now if $p \in \pi$ then the Sylow p-subgroup E_p of E is the Sylow p-subgroup of B and consists of all elements of $\bar{R}$ whose components all come from R_p and whose first two components are equal. The Sylow p-subgroup D_p of D consists of all such elements all of whose components are equal. Since R_p is not abelian it follows easily that D_p is not normal in E_p.

For $p \in \pi$ let x_p be an element of E_p which does not normalize D_p and let $H = \langle D_p^{x_p} ; p \in \pi \rangle \rtimes T_2$. Then H satisfies (i). But H is not conjugate to D in E since any element of E must normalize all but finitely many of the subgroups D_p. Hence, by Theorem 3.1.1, H is not a basis normalizer of G.

There remain only two of Alperin's theorems which we have not covered, viz. Theorems A and 1, and we deal with these now.

<u>Theorem 3.1.11</u> (cf. (Theorem 1, 1).

Suppose the $\mathfrak{U}$-group G has p-length one for all primes p. Let D be a basis normalizer of G and H a subgroup of G. If D is contained in H and normalizes the Sylow basis $\underset{\sim}{T}$ of H then D is the normalizer of some Sylow basis $\underset{\sim}{S}$ of G which reduces into H to $\underset{\sim}{T}$.

To prove 3.1.11 we require the following generalization of Alperin's (Theorem A, 1).

<u>Lemma 3.1.12.</u>

Suppose π is a set of primes and the $\mathfrak{U}$-group G has π-length one. If H is a π-subgroup of G which normalizes the Sylow π'-subgroup S of G and the

π'-subgroup K of G then there is a Sylow π'-subgroup T of G containing K and normalized by H.

Proof.

Let $X = O_{\pi'}(G)$ and $Y = O_{\pi'\pi}(G)$. Then Y/X is the unique Sylow π-subgroup of G/X since G has π-length one, so that $HX/X \leq Y/X$ since H is a π-group. Now K is a π'-subgroup of G normalized by H so $[H,K] \leq Y \cap K \leq X$. Therefore the π-subgroup HX/X of G/X normalizes the Sylow π'-subgroup S/X of G/X and centralizes the π'-subgroup KX/X. Thus, by 1.3.5, there is a Sylow π'-subgroup T/X of G/X normalized by HX/X and containing KX/X. Since X is a π'-group it follows that T is a Sylow π'-subgroup of G normalized by H and containing K, which proves the lemma.

Proof of Theorem 3.1.11.

Suppose D is the normalizer $N_G(\underset{\sim}{S})$ of the Sylow basis $\underset{\sim}{S}$ of G and $D \leq N_H(\underset{\sim}{T})$. Let $\underset{\sim}{V}$ be a Sylow basis of G which extends $\underset{\sim}{T}$ and π the set of primes p such that $V_{p'} \neq S_{p'}$. Then π is finite by 1.3.1, and if $p \notin \pi$ then $T_{p'} \leq V_{p'} = S_{p'}$.

If $p \in \pi$ then the Sylow p-subgroup D_p of D normalizes both $S_{p'}$ and $T_{p'}$ so by Lemma 3.1.12 there is a Sylow p'-subgroup $U_{p'}$ of G normalized by D_p with $T_{p'} \leq U_{p'}$. Now $\underset{\sim}{T}$ reduces into $N_H(\underset{\sim}{T})$ by (2.13(ii), $\underset{\sim}{7}$) so $\underset{\sim}{T}$ also reduces into D. Thus $D_{p'} \leq T_{p'} \leq U_{p'}$ and D normalizes $U_{p'}$.

By 1.3.1, $\{S_{p'} ; p \notin \pi\} \cup \{U_{p'} ; p \in \pi\}$ is the p-complement system associated with some Sylow basis $\underset{\sim}{U}$ of G, and $\underset{\sim}{U}$ reduces into H to $\underset{\sim}{T}$ by construction. Since D clearly normalizes $\underset{\sim}{U}$ the argument used in the proof of 3.1.2 shows that $D = N_G(\underset{\sim}{U})$, completing the proof.

Remark.

An example of Shamash (4.3(1), 31) shows that Theorem 3.1.11 fails to hold if the "p-length one" condition is dropped.

3.2. $\mathfrak{U}$-groups with Abelian Sylow Subgroups.

In this section we consider the class $\mathfrak{U}_A$ in greater detail and show in particular that most of Chambers' results on finite soluble A-groups can be extended to the class $\mathfrak{U}_A$ or appropriate subclasses of it. As before if $\mathfrak{X}$ is a subclass of $\mathfrak{U}$ we let $\mathfrak{X}_A$ denote the class of $\mathfrak{X}$-groups with abelian Sylow p-subgroups for each prime p. It is clear that if $\mathfrak{X}$ is a QS-closed subclass of $\mathfrak{U}$ then so is $\mathfrak{X}_A$.

Lemma 3.2.1.

Every $\mathfrak{U}_A$-group is soluble.

Proof.

If $G \in \mathfrak{U}_A$ then G has a finite normal series with locally nilpotent factors. Since $\mathfrak{U}_A$ is QS-closed and every locally nilpotent $\mathfrak{U}_A$-group is abelian, each of these factors is abelian. Hence G is soluble as claimed.

Lemma 3.2.2.

Suppose the $\mathfrak{U}$-group G has abelian Sylow p-subgroups for some prime p. Then the p-length $l_p(G)$ of G is at most one. In particular $\mathfrak{U}_A$-groups have p-length at most one for all primes p.

Proof.

Let P be a Sylow p-subgroup of G and H/K a p-chief factor of G. Then $H/K \leq PK/K$ and since P is abelian by hypothesis, P centralizes H/K. Therefore $P \leq O_{p'p}(G)$ by (3.8, 7). Hence $G/O_{p'p}(G)$ is a p'-group and G has p-length at most one, as required.

Suppose p is a prime and H_p is a Sylow p-subgroup

of a subgroup H of an (arbitrary) group G. We say H is <u>p-normally embedded</u> in G if H_p is a Sylow p-subgroup of some normal subgroup of G. It is easy to see that H is p-normally embedded in G if and only if H_p is a Sylow p-subgroup of the normal closure of H_p in G.

We now use Theorem 2.2.18 to establish the following lemma which will later yield the pronormality of $\mathfrak{F}$-normalizers in $\mathfrak{K}_A$-groups.

<u>Lemma 3.2.3.</u>

Suppose $V \leq G \in \mathfrak{U}$ and V is p-normally embedded in G for each prime p. Then V is pronormal in G.

<u>Proof.</u>

Suppose $\underset{\sim}{S}$ is a Sylow basis of G and $\underset{\sim}{S}$, $\underset{\sim}{S}^x$ reduce into V for some $x \in G$. Then $\underset{\sim}{S} \cap V$ and $\underset{\sim}{S}^x \cap V$ are Sylow bases of V so there is an element $y \in V$ such that $S_p \cap V = S_p^{xy} \cap V = V_p$ say, for each prime p.

Let p be any prime. Since V is p-normally embedded in G there is a normal subgroup M of G such that V_p is a Sylow p-subgroup of M. Now V_p is contained in both S_p and S_p^{xy} so it follows that $V_p = S_p \cap M = S_p^{xy} \cap M$ and hence that $V_p = V_p^{xy}$.

Since p was an arbitrary prime and $V = \langle V_p \text{ ; all } p \rangle$ we therefore have $V = V^{xy}$ and hence $xy \in N_G(V)$. Hence $x \in N_G(V)$. From 2.2.18 we deduce that V is pronormal in G, as required.

<u>Lemma 3.2.4.</u>

Suppose the $\mathfrak{U}$-group G has abelian Sylow p-subgroups for some prime p, and H is a subgroup of G containing a p-complement of G. Then H is p-normally embedded in G.

<u>Proof.</u>

Suppose H contains the p-complement S of G. Now

G has p-length at most one by 3.2.2, so $G = O_{p'pp'}(G)$.

Suppose firstly that $O_{p'}(G) = 1$. Then G has a unique Sylow p-subgroup P which is abelian by hypothesis. Since $G = PS$ it follows that $G = HP$ and hence that $H \cap P$ is a normal subgroup of G. But P contains the unique Sylow p-subgroup of H so $H \cap P \in Syl_p(H)$.

Now suppose that the genaral case prevails and let H_p be a Sylow p-subgroup of H. Then $H_pO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $HO_{p'}(G)/O_{p'}(G)$ so by the case just considered $H_pO_{p'}(G)/O_{p'}(G)$ is a normal subgroup of G. Clearly H_p is a Sylow p-subgroup of the normal subgroup $H_pO_{p'}(G)$ of G, so H is p-normally embedded in G, as claimed.

We shall show later that in $\mathfrak{K}_\lambda$-groups the $\mathfrak{F}$-normalizers complement the $\mathfrak{F}$-residual. To do this we require the following generalization of a result of Taunt (33), (VI.14.3, 20).

<u>Lemma 3.2.5.</u>

Let p be a prime and suppose the Sylow p-subgroup P of the $\mathfrak{U}$-group G is abelian. Then $P \cap G' \cap Z(G) = 1$.

<u>Proof.</u>

Suppose that there exists a non-trivial element x in $P \cap G' \cap Z(G)$. Then $x = [y_1,z_1]\ldots\ldots[y_n,z_n]$ for some $y_i, z_i \in G$ $(1 \leq i \leq n)$. Let $G_1 = \langle y_i, z_i \; ; \; 1 \leq i \leq n \rangle$. Then G_1 is a finite and if P_1 is a Sylow p-subgroup of G_1 containing $P \cap G_1$ then $P_1 \cap G_1' \cap Z(G_1) = 1$, by the finite case of the lemma (VI.14.3, 20). But this gives a contradiction since x clearly belongs to $P_1 \cap G_1' \cap Z(G_1)$. The result now follows.

Corollary 3.2.6.

If G is a $\mathfrak{U}_A$-group then $G' \cap Z(G) = 1$.

Lemma 3.2.7.

Suppose that G is a (not necessarily finite) soluble group and H is a subgroup of G which covers every chief factor of G. Then H = G.

Proof.

We argue by induction on the derived length of G. If G is abelian then by taking a chief series of G passing through H we see that H = G otherwise H cannot cover the chief factors of G above H.

If G is not abelian we set N to be the last non-trivial term in the derived series of G. Then HN/N covers every chief factor of G/N so by induction G = HN. Since N is abelian $H \cap N$ is a normal subgroup of G and we may take a chief series of G passing through $H \cap N$ and N. If $H \cap N < N$ then H covers no chief factor in this series between $H \cap N$ and N, contrary to hypothesis. Thus $H \cap N = N$ and G = H as claimed.

Remark.

Lemma 3.2.7 fails to hold if the solubility condition is removed. For, as remarked earlier, Hartley (16) gives an example of a locally finite p-group P with a proper subgroup Q which covers every chief factor of P.

For the remainder of this section $\mathfrak{K}$ denotes a $\mathfrak{S}$-closed subclass of $\mathfrak{U}$, f a $\mathfrak{K}$-preformation function on a set of primes π (not necessarily satisfying 2.1.1), and $\mathfrak{F}$ the saturated $\mathfrak{K}$-formation defined by f. If σ

is a set of primes we shall denote by $\mathfrak{K}_{A,\sigma}$ the class of $\mathfrak{K}$-groups with abelian Sylow p-subgroups for each prime $p \in \sigma$. In particular if p is a prime then $\mathfrak{K}_{A,p}$ denotes the class of $\mathfrak{K}$-groups with abelian Sylow p-subgroups.

Lemma 3.2.8.

Suppose $p \in \pi$ and $G \in \mathfrak{K}_{A,p}$. If S is a Sylow p'-subgroup of G and $C_p(G)$ is the $\mathfrak{f}(p)$-centralizer of G then $N_G(S \cap C_p(G))$ is p-normally embedded in G.

Proof.

Since S certainly normalizes $S \cap C_p(G)$ this result is an immediate consequence of Lemma 3.2.4.

Corollary 3.2.9.

Suppose $p \in \pi$ and $G \in \mathfrak{K}_{A,p}$. If D is an $\mathfrak{F}$-normalizer of G then D is p-normally embedded in G.

Proof.

Suppose D is the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\underset{\sim}{S}$ of G. Then $S_p \cap D = S_p \cap N_G(S_{p'} \cap C_p(G))$ is a Sylow p-subgroup of both D and $N_G(S_{p'} \cap C_p(G))$ by (2.13(i), $\underset{\sim}{7}$). The result is now immediate from 3.2.8.

We now establish the following generalization of (3.5, $\underset{\sim}{5}$).

Theorem 3.2.10.

Suppose D is an $\mathfrak{F}$-normalizer of the $\mathfrak{K}_{A,\pi}$-group G. Then D is p-normally embedded in G for all primes p and hence is pronormal in G.

Proof.

If $p \in \pi$ then D is p-normally embedded in G by 3.2.9. If $p \notin \pi$ then $D_p = 1$ since D is a π-group, so D is trivially p-normally embedded in G in this case. Therefore D is p-normally embedded in G for all p, and so, by 3.2.3,

is pronormal in G.

Corollary 3.2.11.

The $\mathfrak{F}$-normalizers of $\mathfrak{K}_A$-groups are pronormal.

Corollary 3.2.12.

If D_1 and D_2 are $\mathfrak{F}$-normalizers of the $\mathfrak{K}_{A,\pi}$-group G contained in the same $\mathfrak{F}$-projector E of G then D_1 and D_2 are conjugate in E.

Proof.

There is an element $x \in G$ such that $D_1 = D_2^x$. Now D_2 is pronormal in G by 3.2.10, so D_2 and D_2^x are conjugate in $\langle D_2, D_2^x \rangle$. The result now follows since $\langle D_2, D_2^x \rangle \leq E$.

Remark.

As we remarked earlier Corollary 3.2.12 is false in general as Hawkes example (19) demonstrates.

Lemma 3.2.13.

Suppose the $\mathfrak{K}$-group G has pronormal $\mathfrak{F}$-normalizers and D is an $\mathfrak{F}$-normalizer of G contained in the subgroup H of G. Then D is contained in some $\mathfrak{F}$-normalizer of H.

Proof.

Suppose D is the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\underset{\sim}{S}$ and let $\underset{\sim}{S}^x$ be a Sylow basis of G which reduces into both H and D. Now $\underset{\sim}{S}$ reduces into D by (2.13(ii), 7) so $x \in R_G(D)$. Since D is by hypothesis pronormal in G we have $x \in N_G(D)$ by 2.2.18. Thus $D = D^x$ is the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\underset{\sim}{S}^x$ and $\underset{\sim}{S}^x$ reduces into H. The result now follows from 1.2.3.

Corollary 3.2.14.

If D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}_{A,\pi}$-group G contained in a subgroup H of G then D is contained in some $\mathfrak{F}$-normalizer of H.

Remark.

Shamash (4.3(2), 31) gives an example of a finite soluble group G with a subgroup H containing a basis normalizer D of G such that D normalizes no Sylow basis of H. In general therefore we cannot hope to improve much upon 3.2.13.

We now discuss Chambers' characterization of $\mathfrak{F}$-normalizers (of finite soluble A-groups) by the covering/avoiding property. We shall prove that a similar characterization holds for the $\mathfrak{F}$-normalizers of $\mathfrak{K}_{A,\pi}$-groups.

Lemma 3.2.15.

Suppose $p \in \pi$ and $G \in \mathfrak{K}_{A,p}$. Let H be a p-subgroup of G which avoids every $\mathfrak{F}$-eccentric p-chief factor of G. Then $H \leq N_G(S \cap C_p(G))$ for every p-complement S of G.

Proof.

Let S be a p-complement of G and set $N = N_G(S \cap C_p(G))$; S is clearly contained in N. Let $P = O_{p'p}(G)$. Then G/P is a p'-group by 3.2.2 so G = PS and hence G = PN. Now $O_{p'}(G) \leq P \cap N \leq P$ and $P/O_{p'}(G)$ is abelian by hypothesis, so $P \cap N$ is a normal subgroup of PN = G. Let $C = P \cap N$.

By (3.1, 7) and the definition of $C_p(G)$, N covers the $\mathfrak{F}$-central and avoids the $\mathfrak{F}$-eccentric p-chief factors of G. Now N covers every chief factor of G between $O_{p'}(G)$ and C so these factors are $\mathfrak{F}$-central; also N avoids every chief factor of G between C and P so such factors

are $\mathfrak{F}$-eccentric. By hypothesis therefore H avoids every chief factor of G between C and P. It follows that $H \leq C$. For let $x \to \bar{x}$ be the natural epimorphism of G onto $\bar{G} = G/O_{p'}(G)$. Then $\bar{H} \leq \bar{P}$ since H is a p-subgroup of G. Suppose $\bar{H}$ is not contained in $\bar{C}$ and let $\bar{x} \in \bar{H} - \bar{C}$. Then $\bar{x} \in \bar{P} - \bar{C}$ so there is a chief factor $\bar{U}/\bar{V}$ of G such that $\bar{C} \leq \bar{V} < \bar{U} \leq \bar{P}$ and $\bar{x} \in \bar{U} - \bar{V}$. Now H avoids U/V so $\bar{H}$ avoids $\bar{U}/\bar{V}$. Thus $\bar{H} \cap \bar{U} = \bar{H} \cap \bar{V}$ and we have a contradiction since $\bar{x} \in (\bar{H} \cap \bar{U}) - (\bar{H} \cap \bar{V})$. In view of this contradiction we have $\bar{H} \leq \bar{C}$ and hence $H \leq C$. Since $C \leq N$ the proof is complete.

<u>Lemma 3.2.16.</u>

Suppose H is a subgroup of a $\mathfrak{K}_{\Lambda,\pi}$-group G and H avoids every $\mathfrak{F}$-eccentric chief factor of G. Then H is contained in some $\mathfrak{F}$-normalizer of G.

<u>Proof.</u>

We show first that H is a π-group. If this is not the case then we can find an element x of order q in H for some prime $q \notin \pi$. Let $(\Lambda_\sigma, V_\sigma \; ; \; \sigma \in \Omega)$ be a chief series of G. Then x lies in some layer $\Lambda_\sigma - V_\sigma$ $(\sigma \in \Omega)$. Since $x \in (H \cap \Lambda_\sigma) - (H \cap V_\sigma)$, H does not avoid Λ_σ/V_σ which by our hypothesis must then be $\mathfrak{F}$-central. Thus if Λ_σ/V_σ is a p-factor then $p \in \pi$. But now xV_σ is a non-trivial element of order q in Λ_σ/V_σ so $q = p \in \pi$, a contradiction. Therefore H is a π-subgroup of G as claimed above.

Let $\underset{\sim}{S}$ be a Sylow basis of G which reduces into H. If $p \in \pi$ then $H_p = H \cap S_p$ is a p-subgroup of G which avoids every $\mathfrak{F}$-eccentric p-chief factor of G, so,

by 3.2.15, $H_p \leq N_G(S_{p'} \cap C_p(G))$. Since $H_{p'} = H \cap S_{p'} \leq S_{p'} \leq N_G(S_{p'} \cap C_p(G))$ it follows that H normalizes $S_{p'} \cap C_p(G)$. But H is a π-group so $H = H \cap S_\pi$ and hence $H \leq S_\pi \cap \bigcap_{p \in \pi} N_G(S_{p'} \cap C_p(G))$, the $\mathfrak{F}$-normalizer of G associated with the Sylow basis $\underset{\sim}{S}$ of G. This establishes the lemma.

Lemma 3.2.17.

Suppose D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}_{A,\pi}$-group G.

(1) Every chief factor of G below the $\mathfrak{F}$-residual $G^{\mathfrak{F}}$ of G is $\mathfrak{F}$-eccentric.

(2) D complements $G^{\mathfrak{F}}$ in G.

Proof.

(1) Suppose that the result is false and let H/K be an $\mathfrak{F}$-central chief factor of G below $G^{\mathfrak{F}}$. Then H/K is an $\mathfrak{F}$-central minimal normal subgroup of G/K contained in $(G/K)^{\mathfrak{F}} = G^{\mathfrak{F}}K/K$, so in obtaining our contradiction we may assume without loss of generality that H is an $\mathfrak{F}$-central minimal normal subgroup of G contained in $G^{\mathfrak{F}}$. Now $G^{\mathfrak{F}} \leq C_G(H)$ since $\mathfrak{F}$ is integrated and H is $\mathfrak{F}$-central, so $H \leq Z(G^{\mathfrak{F}})$. If H is a p-group then $p \in \pi$ since H is $\mathfrak{F}$-central, and $G^{\mathfrak{F}}$ has abelian Sylow p-subgroups by hypothesis. Therefore $H \cap (G^{\mathfrak{F}})' = 1$ by 3.2.5.

Let $L = G^{\mathfrak{F}}$. Then the $\mathfrak{F}$-residual L/L' of G/L' is abelian, so, by 1.2.4, the $\mathfrak{F}$-normalizers of G/L' complement L/L' in G/L'. Thus, by (4.6, 7), every chief factor of G/L' below L/L' is $\mathfrak{F}$-eccentric. But $H \cap L' = 1$ so H is G-isomorphic to HL'/L'. Thus HL'/L' is an $\mathfrak{F}$-central chief factor of G/L' below L/L', which gives the desired contradiction and establishes (1).

(2) Suppose there is a non-trivial element $x \in D \cap G^{\mathfrak{F}}$.

Then taking a chief series of G passing through $G^{\mathfrak{F}}$ we obtain a chief factor X/Y of G below $G^{\mathfrak{F}}$ such that $x \in X - Y$. Since $x \in (D \cap X) - (D \cap Y)$, the chief factor X/Y is $\mathfrak{F}$-central (4.6, 7) which contradicts (1). Therefore $D \cap G^{\mathfrak{F}} = 1$. Since $G/G^{\mathfrak{F}} \in \mathfrak{F}$ we also have $G = DG^{\mathfrak{F}}$ by (4.6, 7) and this establishes (2).

We can now prove our generalization of (3.6, 5).

Theorem 3.2.18.

Suppose $H \leq G \in \mathfrak{K}_{A,\pi}$. Then H is an $\mathfrak{F}$-normalizer of G if and only if H covers every $\mathfrak{F}$-central chief factor of G and avoids every $\mathfrak{F}$-eccentric chief factor of G.

Proof.

$\mathfrak{F}$-normalizers certainly have the required property by (4.6, 7) so we need only show that a subgroup H with the given "covering/avoiding" property is an $\mathfrak{F}$-normalizer of G.

Suppose then that H covers every $\mathfrak{F}$-central chief factor of G and avoids every $\mathfrak{F}$-eccentric chief factor of G. Now $G/G^{\mathfrak{F}} \in \mathfrak{F}$ so in particular $G/G^{\mathfrak{F}}$ is a π-group. Thus, by hypothesis, $G/G^{\mathfrak{F}}$ has abelian Sylow p-subgroups for each prime p, and it follows, as in the proof of 3.2.1, that $G/G^{\mathfrak{F}}$ is soluble. Now $HG^{\mathfrak{F}}/G^{\mathfrak{F}}$ covers every chief factor of $G/G^{\mathfrak{F}}$ since all such factors are $\mathfrak{F}$-central. Therefore $G = HG^{\mathfrak{F}}$ by 3.2.7. Now H is contained in some $\mathfrak{F}$-normalizer D of G by 3.2.16, and D complements $G^{\mathfrak{F}}$ in G, by 3.2.17. Therefore $D = D \cap HG^{\mathfrak{F}} = H(D \cap G^{\mathfrak{F}}) = H$, and the proof is complete.

Remark.

Basis normalizers, and hence $\mathfrak{F}$-normalizers, are

not usually characterized by their covering/avoiding property as an example of Hawkes (18) demonstrates.

If D is an $\mathfrak{F}$-normalizer of a $\mathfrak{K}_A$-group then $R_G(D) = N_G(D)$ by 3.2.11 and 2.2.18. It might be hoped that D satisfies the stronger condition $R_G(D\ ;\mathfrak{F}) = N_G(D)$ when $\mathfrak{F}$ satisfies 2.1.1. However this is not the case as the following example shows:-

Example 3.2.19.

We take $\mathfrak{K}$ to be the class $\mathfrak{S}^*$ of finite soluble groups and $\mathfrak{F}$ the saturated $\mathfrak{S}^*$-formation of finite supersoluble groups. By (6.1, 4) $\mathfrak{F}$ is defined by the $\mathfrak{S}^*$-formation function $\mathfrak{f}$, where $\mathfrak{f}(p)$ is the $\mathfrak{S}^*$-formation of finite abelian groups of exponent dividing p-1, for each prime p. Clearly $\mathfrak{f}$ satisfies 2.1.1 and 2.6.6.

Let A be an alternating group of degree 4, and let $G = C_7 \wr A$, the wreath product of a cyclic group C_7 of order 7 by A, the wreath product being taken with respect to the natural permutation representation of A on 4 symbols. $|G| = 7^4.3.2^2$. G is certainly a $\mathfrak{U}_A^*$- (i.e. finite soluble A-) group and is the semidirect product of an elementary abelian group N of order 7^4 with A. The centre Z of G is the "diagonal" of the base group N of G and has order 7. Let V denote the Sylow 2- and U a Sylow 3-subgroup of A. Then $1 < Z < N < NV < NVU = G$ is a chief series of G. The supersoluble central chief factors are just the cyclic chief factors (6.3, 4) so it follows, for example from 3.2.18, that $D = Z \times U$ is a supersoluble normalizer of G. U is a supersoluble projector of A so UN/N is a supersoluble projector of G/N. Now UN is supersoluble so by Gaschütz Lemma (1.2.5(7)) UN is a supersoluble projector of G. Since U is a

characteristic subgroup of D it follows that D is not normal in UN. However $UN \leq R_G(D\ ;\mathfrak{F})$ by 2.2.6, so D is not normal in $R_G(D\ ;\mathfrak{F})$, giving the desired example.

We conclude this section with a brief discussion of the basis normalizers of $\mathfrak{U}_A$-groups. We have the following special cases of our previous results, the first of which generalizes a result of Rose (2.4, 28)

Theorem 3.2.20.

The basis normalizers of $\mathfrak{U}_A$-groups are pronormal.

Theorem 3.2.21.

If $H \leq G \in \mathfrak{U}_A$ then H is a basis normalizer of G if and only if H covers every central chief factor of G and avoids every eccentric chief factor of G.

Lemma 3.2.22.

Suppose $G \in \mathfrak{K}_{A,\pi}$ and D is an $\mathfrak{F}$-normalizer of G. Then $N_G(D) = D \times (G^{\mathfrak{F}} \cap C_G(D))$.

Proof.

Let $N = N_G(D)$. Now D complements $G^{\mathfrak{F}}$ in G by 3.2.17, so $N = D(N \cap G^{\mathfrak{F}})$. Since $[D, N \cap G^{\mathfrak{F}}] \leq D \cap G^{\mathfrak{F}} = 1$, it is clear that $N \cap G^{\mathfrak{F}} = G^{\mathfrak{F}} \cap C_G(D)$. But $D \cap G^{\mathfrak{F}} = 1$ so we finally have $N = D \times (G^{\mathfrak{F}} \cap C_G(D))$, as claimed.

Corollary 3.2.23.

Let D be a basis normalizer of a $\mathfrak{U}_A$-group G. Then

(1) D complements the derived group of G in G,

(2) $N_G(D) = D \times (G' \cap C_G(D))$,

(3) $N_G(D) = C_G(D)$.

Proof.

Since every locally nilpotent $\mathfrak{U}_A$-group is abelian it follows that the $\mathfrak{U}$-residual of G is in fact the derived group of G. Therefore (1) is immediate from 3.2.17, and (2) follows from 3.2.22. The third statement follows from the second since D, being a locally nilpotent $\mathfrak{U}_A$-group, is abelian.

Lemma 3.2.24.

(1) Suppose D is a basis normalizer and N a normal subgroup of the $\mathfrak{U}$-group G. Then $N = (N \cap G')(N \cap D)$.

(2) If N is an abelian normal subgroup of the $\mathfrak{U}_A$-group G then $N = (N \cap G') \times (N \cap Z(G))$.

Proof.

(1) Since N is a normal subgroup of G, $[N,G] \leq N \cap G'$. Thus every chief factor of G lying between $N \cap G'$ and N is central and so covered by D (4.6, 7). Furthermore $N/N \cap G' \leq Z(G/N \cap G')$ so that $(D \cap N)(N \cap G')$ is a normal subgroup of G. Therefore $N = (D \cap N)(N \cap G')$; for otherwise we can take a chief series of G passing through $(D \cap N)(N \cap G')$ and N, and D will cover no chief factor in this series between $(D \cap N)(N \cap G')$ and N. This establishes (1).

(2) If D is a basis normalizer of G then $N = (N \cap G')(D \cap N)$ by (1) and $D \cap G' = 1$ by 3.2.23. Thus to prove (2) it suffices to show that $N \cap D = N \cap Z(G)$. Now Z(G) normalizes every Sylow basis of G so we certainly have $N \cap Z(G) \leq N \cap D$. It therefore remains to show that $N \cap D \leq Z(G)$.

Let D_p, N_p denote the unique Sylow p-subgroup of D,N respectively. Then $N_p \triangleleft G$ since N is an abelian normal subgroup of G. If D is the normalizer of the

Sylow basis $\underset{\sim}{S}$ of G then $[N_p \cap D_p, S_{p'}] \leq N_p \cap S_{p'} = 1$, so $N_p \cap D_p$ centralizes $S_{p'}$. But $N_p \cap D_p \leq S_p$ and G is a $\mathfrak{U}_A$-group, so S_p is abelian and $N_p \cap D_p$ centralizes S_p also. Since $G = S_{p'}S_p$ it follows that $N_p \cap D_p$ is contained in the centre of G. But $N_p \cap D_p$ is the unique Sylow p-subgroup of $N \cap D$ so we finally have $N \cap D \leq Z(G)$ which, as above, completes the proof.

As we have already seen (3.2.1) every $\mathfrak{U}_A$-group is soluble. This fact gives us the following description of the Hirsch-Plotkin radical of $\mathfrak{U}_A$-groups.

<u>Theorem 3.2.25</u>

Suppose G is a $\mathfrak{U}_A$-group of derived length n. Then $\varrho(G) = Z(G) \times Z(G') \times \ldots\ldots\ldots \times Z(G^{(n-1)})$ where $G^{(i)}$ denotes the ith term in the derived series of G.

<u>Proof.</u>

We argue by induction on the derived length n of G, the result being clear when G is abelian. We may therefore suppose that $n > 1$ and $\varrho(G') = Z(G') \times \ldots\ldots \times Z(G^{(n-1)})$ by induction. Now $\varrho(G)$ is a locally nilpotent subgroup of the $\mathfrak{U}_A$-group G so is in fact abelian. Therefore $\varrho(G) = (\varrho(G) \cap G') \times (\varrho(G) \cap Z(G))$ by 3.2.24. But $Z(G)$ is clearly contained in $\varrho(G)$ and $\varrho(G) \cap G' = \varrho(G')$. Thus $\varrho(G) = Z(G) \times \varrho(G') = Z(G) \times Z(G') \times \ldots\ldots \times Z(G^{(n-1)})$ and the proof is complete.

We close this section with a characterization of the normalizers of basis normalizers of $\mathfrak{U}_A$-groups.

<u>Theorem 3.2.26.</u>

Let D be a basis normalizer of a $\mathfrak{U}_A$-group G and let $N = N_G(D)$. Suppose H is a subgroup of G

containing D. Then $H = N$ if and only if H covers every D-central and avoids every D-eccentric D-composition factor of G.

Proof.

N is the reducer of D in G by 2.2.18 and 3.2.20, so certainly N covers every D-central D-composition factor of G by 2.2.17. Suppose A/B is a D-composition factor of G covered by N. Then $A = (N \cap A)B$ and hence $[A,D] \leq B$ since N centralizes D by 3.2.23. Therefore A/B, and hence every D-composition factor of G covered by N, is D-central. Since N certainly covers or avoids every D-composition factor of G it follows that N covers every D-central and avoids every D-eccentric D-composition factor of G.

Suppose conversely that H covers every D-central and avoids every D-eccentric D-composition factor of G. Then $N \leq H$ by 2.2.17. By 3.2.14, D normalizes some Sylow basis $\underset{\sim}{T}$ of H. Since $\underset{\sim}{T}$ reduces into $N_H(\underset{\sim}{T})$ we certainly have $D_p \leq T_p$ for each prime p. But T_p is abelian since G is a $\mathfrak{U}_A$-group, so D_p centralizes T_p for each prime p.

Let $(A_\sigma , V_\sigma ; \sigma \in \Omega)$ be a D-composition series of G. Then $(A_\sigma \cap H , V_\sigma \cap H ; \sigma \in \Omega)$ is a D-series of H in which every non-trivial factor is D-central. Intersecting this series futher with T_p we obtain a series of T_p in which every non-trivial factor is centralized by $D_{p'}$ so by (4.11, $\underset{\sim}{7}$) $D_{p'}$ centralizes T_p. Hence $D = D_p D_{p'}$ centralizes T_p for each prime p. Since H is generated by the subgroups T_p it follows that

$H \leq C_G(D)$. In particular therefore $H \leq N$ and this, together with our previous inequality, completes the proof.

3.3. $\mathfrak{U}$-groups with pronormal basis normalizers.

In this our final section we extend Rose's results (28) on finite soluble groups with pronormal system normalizers. In many cases our results can be proved using Rose's techniques but we have tried where possible to give different proofs using our work in chapter two.

Using the language of section 2.3 we have the following restatement of 2.2.18:-

Theorem 3.3.1.

Suppose $H \leq G \in \mathfrak{U}$. Then the following three conditions are equivalent:-

(1) H is pronormal in G,

(2) $R_G(H) = N_G(H)$,

(3) $N_G(H)$ is the strong serializer of H in G.

Suppose $\mathfrak{K}$ is a $\mathfrak{S}$-closed subclass of $\mathfrak{U}$ and $\mathfrak{F}$ is a saturated $\mathfrak{K}$-formation not necessarily satisfying 2.1.1. Let $\mathfrak{B}(\mathfrak{K},\mathfrak{F})$ denote the class of all $\mathfrak{K}$-groups with pronormal $\mathfrak{F}$-normalizers. If $\mathfrak{F} \geq \mathfrak{K} \cap \mathfrak{N}$ then, by 1.2.5(3) and (5.7, 7), $\mathfrak{K} \cap (\mathfrak{N})\mathfrak{F} \leq \mathfrak{B}(\mathfrak{K},\mathfrak{F})$.

Lemma 3.3.2.

$\mathfrak{B}(\mathfrak{K},\mathfrak{F})$ is a $\mathfrak{K}$-formation.

Proof.

It is clear that $\mathfrak{B}(\mathfrak{K},\mathfrak{F})$ is $\mathfrak{S}$-closed, so we need only show that $\mathfrak{K} \cap \mathfrak{R}\,\mathfrak{B}(\mathfrak{K},\mathfrak{F}) = \mathfrak{B}(\mathfrak{K},\mathfrak{F})$.

Suppose $G \in \mathfrak{K} \cap \mathfrak{R}\,\mathfrak{B}(\mathfrak{K},\mathfrak{F})$. Then there exist normal subgroups N_λ of G ($\lambda \in \Lambda$) such that $G/N_\lambda \in \mathfrak{B}(\mathfrak{K},\mathfrak{F})$ for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} N_\lambda = 1$. Let D be an $\mathfrak{F}$-normalizer of G and $\lambda \in \Lambda$. Then DN_λ/N_λ is an $\mathfrak{F}$-normalizer of G/N_λ so, by 2.2.8 and 3.3.1, $R_G(D)N_\lambda/N_\lambda = N_{G/N_\lambda}(DN_\lambda/N_\lambda)$.

Therefore $DN_\lambda \triangleleft RN_\lambda$ where $R = R_G(D)$, and hence $[D,R] \leq DN_\lambda$. Since λ was an arbitrary member of Λ we have, using 2.4.1, $[D,R] \leq \bigcap_{\lambda \in \Lambda} (DN_\lambda) = D$. This shows that D is a normal subgroup of R and it follows that $R = N_G(D)$. Therefore $G \in \mathfrak{B}(\mathfrak{K},\mathfrak{F})$ by 3.3.1, and the proof is complete.

Remark.

In general the $\mathfrak{K}$-formations $\mathfrak{B}(\mathfrak{K},\mathfrak{F})$ are neither saturated nor subgroup-closed. For Rose (3.5 and 5.6, 28) shows that the class $\mathfrak{B}(\mathfrak{S}^*,\mathfrak{N}^*)$ has neither of these properties; $\mathfrak{B}(\mathfrak{S}^*,\mathfrak{N}^*)$ is of course the class of finite soluble groups with pronormal system normalizers.

For the remainder of this section we shall consider the class $\mathfrak{B} = \mathfrak{B}(\mathfrak{U},\mathfrak{M})$ which naturally extends $\mathfrak{B}(\mathfrak{S}^*,\mathfrak{N}^*)$; $\mathfrak{B}$ is the class of $\mathfrak{U}$-groups with pronormal basis normalizers. From 3.3.2 and 3.2.20 we have the following generalization of (2.4 and 3.4, 28).

Corollary 3.3.3.

$\mathfrak{B}$ is a $\mathfrak{U}$-formation containing the class $\mathfrak{U}_A$.

Lemma 3.3.4.

Suppose D is a basis normalizer of the $\mathfrak{U}$-group G. Then D is pronormal in G if and only if $N_G(D)$ is the serializer of D in G.

Proof.

This result is an immediate consequence of 3.3.1 and 2.4.11.

From this lemma we deduce the following generalization of (6.1, 28):-

Theorem 3.3.5.

If D is a basis normalizer of the $\mathfrak{U}$-group G then $G \in \mathfrak{D}$ if and only if $N_G(D)$ is the serializer of D in G.

Corollary 3.3.6.

Let D be a basis normalizer of the $\mathfrak{U} \cap (\mathfrak{N})^3$-group G and C the unique Carter subgroup of G containing D. Then $G \in \mathfrak{D}$ if and only if $D \triangleleft C$.

Proof.

This result is immediate from 2.4.5 and 3.3.5.

Remark.

Corollary 3.3.6 extends a similar result (6.5, 28) of Rose. He also shows that the condition that G is a $\mathfrak{U} \cap (\mathfrak{N})^3$-group is necessary in that there exists a finite soluble group G of nilpotent length 4 with basis normalizer D and Carter subgroup C such that $D \triangleleft C$ but $G \notin \mathfrak{D}(\mathfrak{S}^*, \mathfrak{N}^*)$.

From 3.2.13 we have the following extension of (4.1, 28):-

Lemma 3.3.7.

Suppose D is a basis normalizer of the $\mathfrak{D}$-group G contained in a subgroup H of G. Then D is contained in some basis normalizer of H.

We also obtain an extension of (4.2, 28):-

Lemma 3.3.8.

If G is a $\mathfrak{D}$-group then each Carter subgroup of G contains a unique basis normalizer of G.

Proof.

Let C be a Carter subgroup of G and suppose D, D^x

are basis normalizers of G contained in C ($x \in G$). Since C is locally nilpotent we have $C \leq R_G(D) \cap R_G(D^x) = R_G(D) \cap R_G(D)^x$ by 2.2.6. Now C is abnormal in G by 1.2.7, so $x \in R_G(D)$. From 3.3.1 we now obtain $D = D^x$, which completes the proof.

Lemma 3.3.9.

Let D be the normalizer of the Sylow basis $\underset{\sim}{S}$ of the $\mathfrak{U}$-group G. Then $R_G(D) = \langle C_{S_p}(D_{p'})$; all primes p $\rangle$.

Proof.

$\underset{\sim}{S}$ reduces into D by (2.13, $\underset{\sim}{7}$) so $D_p = D \cap S_p \leq C_{S_p}(D_{p'})$ for each prime p. Thus $D_{p'} \times C_{S_p}(D_{p'})$ is a locally nilpotent subgroup of G containing D and so, by 2.4.10, lies in $R_G(D)$. Hence $\langle C_{S_p}(D_{p'})$; all p $\rangle \leq R_G(D)$. We complete the proof by showing that if A/B is a D-central D-composition factor of G then D(A/B) is contained in $\langle C_{S_p}(D_{p'})$; all p $\rangle$; the result is then immediate from 2.4.10.

Indeed let A/B be a D-central D-composition factor of G and set $H = D(A/B) = N_{DA}(\underset{\sim}{S} \cap DA)$. $\underset{\sim}{S}$ reduces into the locally nilpotent subgroup H which contains D so $H_p = H \cap S_p$ centralizes $H_{p'}$ and hence also $D_{p'}$. Thus $H_p \leq C_{S_p}(D_{p'})$ and hence $H \leq \langle C_{S_p}(D_{p'})$; all p $\rangle$ as required.

We use Lemma 3.3.9 to prove the following extension of (5.2, $\underset{\sim}{28}$):-

Theorem 3.3.10.

Suppose the $\mathfrak{U}$-group G has a normal Sylow p-subgroup P such that $G/P \in \mathfrak{B}$, for some prime p. Let D_p, $D_{p'}$ denote the unique Sylow p- and Sylow p'-subgroups of

the basis normalizer D of G respectively. Then $G \in \mathfrak{D}$ if and only if $D_p \triangleleft C_P(D_{p'})$.

Proof.

Suppose firstly that $G \in \mathfrak{D}$. Then $C_P(D_{p'}) \leq N_G(D)$ by 3.3.1 and 3.3.9. Now D_p is a characteristic subgroup of D contained in $C_P(D_{p'})$ so it follows that $D_p \triangleleft C_P(D_{p'})$.

Conversely suppose $D_p \triangleleft C_P(D_{p'})$. Let D be the normalizer of the Sylow basis $\underset{\sim}{S}$ of G. Then $\underset{\sim}{S}$ reduces into D by (2.13(ii), $\underset{\sim}{7}$), so $D_p = D \cap S_p$, $D_{p'} = D \cap S_{p'}$. If $R = R_G(D)$ then $\underset{\sim}{S}$ also reduces into R by 2.2.3. Let $R_{p'} = R \cap S_{p'}$. Now DP/P is a basis normalizer of the $\mathfrak{D}$-group G/P so, by 2.2.8 and 3.3.1, $DP/P \triangleleft RP/P$. Therefore $[D_{p'}, R_{p'}] \leq S_{p'} \cap DP = S_{p'} \cap D_{p'}P = D_{p'}(S_{p'} \cap P) = D_{p'}$. Thus $D_{p'} \triangleleft R_{p'}$. If $q \neq p$ then D_q is a characteristic subgroup of $D_{p'}$ and $C_{S_q}(D_{q'}) \leq R_{p'}$ by 3.3.9, so that $C_{S_q}(D_{q'})$ normalizes D_q and hence D. Thus $C_{S_q}(D_{q'}) \leq N_G(D)$ for each prime $q \neq p$. But $S_p = P$ and $D_p \triangleleft C_P(D_{p'})$ by hypothesis. Therefore $C_P(D_{p'}) \leq N_G(D)$ and hence, by 3.3.9, $R_G(D) \leq N_G(D)$. Thus $R_G(D) = N_G(D)$ and $G \in \mathfrak{D}$, by 3.3.1, as required.

Corollary 3.3.11.

Suppose the $\mathfrak{U}$-group G has a normal abelian Sylow p-subgroup P such that $G/P \in \mathfrak{D}$, for some prime p. Then $G/P \in \mathfrak{D}$.

If $G \in \mathfrak{U}$ and $p_1, p_2, \ldots\ldots, p_r$ are distinct primes, we say G has a Sylow tower of complexion $p_1, p_2, \ldots\ldots, p_r$ if G has a normal series $1 = G_0 \leq G_1 \leq \ldots\ldots \leq G_r = G$ such that $G_i/G_{i-1} \in Syl_{p_i}(G/G_{i-1})$ for each $i = 1, \ldots, r$.

Now $\mathfrak{U} \cap (\mathfrak{M})^2 \leq \mathfrak{D}$ by 1.2.5(3) and 1.2.7. Thus by repeated application of 3.3.11 we obtain

Corollary 3.3.12.

Suppose the $\mathfrak{U}$-group G has a Sylow tower of complexion $p_1, p_2, \ldots\ldots, p_r$ where $r \geq 3$ and abelian Sylow p_i-subgroups for $1 \leq i \leq r-2$. Then $G \in \mathfrak{D}$.

Remark.

Corollaries 3.3.11 and 3.3.12 generalize similar results (5.3 and 5.4, 28) in Rose's paper. He also gives an example (5.6, 28) to show that in general r-2 cannot be replaced by r-3 in 3.3.12.

Our final result is a generalization of (5.5, 28). To prove it we require the following

Lemma 3.3.13.

Suppose the $\mathfrak{U}$-group G has a normal Sylow p-subgroup P such that $G/P \in \mathfrak{D}$, for some prime p. Suppose further that for each locally nilpotent ascendabnormal subgroup H of G there is a basis normalizer D of G such that $D \lhd H$. Then $G \in \mathfrak{D}$.

Proof.

Let D be a basis normalizer of G. Then $D_p = D \cap P \leq C_P(D_{p'}) = P_o$. Now $D^* = D_{p'} \times P_o$ is a locally nilpotent subgroup of DP containing D so, by 2.2.6, is contained in $\bar{D} = R_{DP}(D)$. By 2.3.23, $\bar{D}$ is a Carter subgroup of DP so, by 1.2.7, $\bar{D} \rtimes DP$. Also DP/P is a basis normalizer of G/P so, by 1.2.9, is subabnormal in G/P. Therefore DP, and hence $\bar{D}$, is subabnormal in G. Since $\bar{D}$ is locally nilpotent there is, by hypothesis, a basis normalizer D^x of G such that $D^x \lhd \bar{D}$. Thus $P_o^{x^{-1}} \leq (\bar{D})^{x^{-1}} \leq N_G(D) \leq N_G(D_{p'})$.

Now $P \triangleleft G$, therefore $P_o^{x^{-1}} \leq P \cap N_G(D_{p'}) = C_p(D_{p'}) = P_o$. Hence $P_o^{x^{-1}} = P_o$ since a subgroup of a locally finite group cannot be conjugate to a proper subgroup of itself. Thus $P_o \leq N_G(D) \leq N_G(D_p)$. Since $D_p \leq P_o$ we therefore have $D_p \triangleleft P_o$, which by 3.3.10 is enough to prove the lemma.

<u>Theorem 3.3.14.</u>

Suppose the $\mathfrak{U}$-group G has p-length at most one for each prime p. Then $G \in \mathfrak{D}$ if and only if for each locally nilpotent ascendabnormal subgroup H of G there is a basis normalizer D of G such that $D \triangleleft H$.

<u>Proof.</u>

Suppose firstly that H is a locally nilpotent ascendabnormal subgroup of the $\mathfrak{D}$-group G. Then H contains some basis normalizer D of G by 1.2.9, and since $G \in \mathfrak{D}$, $R_G(D) = N_G(D)$. But $H \leq R_G(D)$ by 2.2.6, so $D \triangleleft H$ which gives the necessity of our condition.

Suppose conversely that for each locally nilpotent ascendabnormal subgroup H of G there is a basis normalizer D of G such that $D \triangleleft H$. Let N be a normal subgroup of G and H/N a locally nilpotent ascendabnormal subgroup of G/N. Then H is ascendabnormal in G and if X is a Carter subgroup of H then $XN/N = H/N$ and $X \rtimes H$. Thus X is a locally nilpotent ascendabnormal subgroup of G, so by hypothesis there is a basis normalizer D of G such that $D \triangleleft X$. Therefore DN/N is a basis normalizer of G/N such that $DN/N \triangleleft XN/N = H/N$. Since G/N certainly has p-length at most one for each prime p, the hypotheses on G carry over to G/N and hence to every factor group of G. We are now in a position to prove the result by induction on the $\mathfrak{N}$-length of G.

If $l(G) \leq 2$ there is nothing to prove since the class $\mathfrak{U} \cap (\mathfrak{N})^2$ is contained in $\mathfrak{F}$. Therefore we may assume without loss of generality that $l(G) > 2$. Let $R = \mathfrak{C}(G)$. Then, as shown above, G/R satisfies the same hyotheses as G so by induction $G/R \in \mathfrak{F}$. Let p be any prime. Then $R \leq O_{p'p}(G)$ so $G/O_{p'p}(G) \in \mathfrak{F}$ since this class is $\mathfrak{a}$-closed. Now G has p-length at most one so $G/O_{p'}(G)$ has a normal Sylow p-subgroup, namely $O_{p'p}(G)/O_{p'}(G)$. Since $G/O_{p'}(G)$ satisfies the same hypotheses as G and $G/O_{p'p}(G) \in \mathfrak{F}$ we have $G/O_{p'}(G) \in \mathfrak{F}$ by 3.3.13. But $\mathfrak{F}$ is a $\mathfrak{U}$-formation (3.3.3) and $\bigcap_p O_{p'}(G) = 1$, so we finally have $G \in \mathfrak{F}$, which completes the proof.

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