Groups from Link Diagrams

by

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Declaration

I declare that the work contained in this thesis is entirely my own, except where otherwise stated.

Summary

This thesis is an attempt to generalise the methods of the Wirtinger presentation for obtaining groups which are invariants of a classical link.

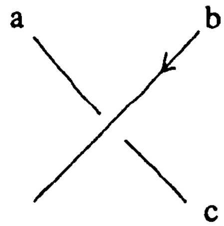
It is closely allied to the ideas of Joyce on quandles.

In it are produced a number of groups which are obtained by associating two generators to each arc of a link projection, and two relations to each crossing. Some properties of these invariants are established

Introduction

We start with three motivational examples.

If L is a picture of an oriented link , we can write down a finite presentation of a group by taking the arcs as generators, and taking relations $c = b^{-1} a b$ for each crossing whose incident arcs are as shown :



This gives the Wirtinger presentation for the fundamental group of the link exterior, and is thus an invariant of not just the picture but of the link.

If instead we had taken the arcs to be generators of a $\mathbb{Z}[t, t^{-1}]$-module, and chosen the equation $c = t a + (1-t) b$ then we would obtain a presentation of a form of the Alexander module of the link. The form we get is the one variable module (associated with an infinite cyclic cover), and moreover we get one more free factor of $\mathbb{Z}[t, t^{-1}]$ than is usual.

Substituting $t = -1$ in a matrix presenting the (usual) one variable Alexander module gives a presentation for the first homology group of the twofold cover of the exterior of the link, so for our form of Alexander module we will get one extra factor of $\mathbb{Z}$. That means that if we regard the arcs as generators of an abelian group, and choose the relation at a crossing as $c = 2b - a$ then we will obtain a presentation for $\mathbb{Z} \oplus H_1$(two-fold cover of the exterior of L).

In all these cases we have started with a simple equation, "$c = b^{-1} a b$", "$c = ta + (1-t) b$" and "$c = 2b - a$" and have obtained an invariant of the link.

In this thesis we are interested in what these equations have in common, and in how the properties of the invariants so obtained are related to the properties of the equations.

(For a reference covering these three motivational examples use [R].)

Recently Jones [Jon] has reawakened interest in link invariants. The Jones polynomial, and the Homfly polynomial [FYHLMO] can (now) be regarded as the result of applying simple combinatorics to a link diagram. Generalisations and consequences of these simple combinatorics abound (see for example [L]).

Joyce and others ([J], [FR]) have adopted another simple combinatoric approach to link diagrams and again, in theory, obtain powerful invariants.

This thesis should be regarded as being part of the approach of applying simple algebra to a link diagram, and of thus obtaining invariants of the link. There are of course many ways to regard a link diagram, and many ways of applying combinatorics, thus this thesis can be regarded as an attempt to systematically approach one of these areas, while giving a nod in some of the other directions.

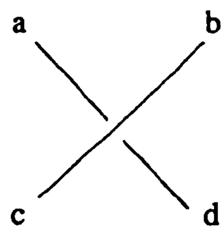
The groups obtained as invariants are also of interest in their own right, being of low deficiency, but also since the underlying topology provides some control on which groups will be similar to each other. Thus the examples have been used in [HR] and other enquiries have been made about them.

The particular approach we centre upon is this. We are looking at finitely presented groups obtained from an oriented link diagram by taking n generators for each arc and writing down n relations at each crossing. In chapter 8 we concentrate on the case n = 2. We are interested in which relations we can take to obtain an invariant of the link.The tools to study the examples of chapter 8 are the bulk of the subject matter for the earlier chapters.

We now give a rough guide to the material in each chapter.

Chapter 2 contains an outline of the basic ideas of link diagrams, and contains the theorems of Reidemeister and Alexander which underpin all of this thesis, as well as much of the recent work on link invariants. This material is probably well known to the reader, and is included as an exercise in "the naming of parts".

Chapter 3 contains the basic definitions of the thesis. The objects we study are called *link-representations* and they are a systematic way of combining simple algebra with link diagrams. The material in this chapter is set out so as to make it easy to generalise. We are primarily interested in group invariants based on one view of link diagrams, the chapter is intended to make it clear what needs to be done for other sorts of invariants, and for other views of the diagrams. The most important result in this chapter is that the relations we write at a crossing



enable us to write a and b in terms of c and d, or any other adjacent pair in terms of the other two. This gives us the idea of labelling a diagram, we can write some function of a and b in place of c, and so on.

Chapter 4 is on examples. We see that it is easy to list all one dimensional link-representations. We also include pointers to obtaining invariants which are not just groups, and pointers to the work in chapter 8.

In chapter 5 we see that we are not so much writing n unrelated objects on each *subarc*, but rather a basis for the free group on n objects. Thus automorphisms of the free group act on the invariants, and we obtain a notion of *conjugacy*. This notion will apply in more general cases, and is also very useful later in the thesis.

In chapter 6 we obtain topological information about the invariants. In particular we look at the question of which examples will be invariant under changes of orientation. Many of the examples, although they use an orientation in their calculation for a link, are then independent of the choice of orientation. Here we use the idea of a *tangle-representation*, where we introduce relations not just at crossings but also at maxima and minima. Although we never state explicitly the conditions that a collection of relations give us a tangle-representation (or an oriented-tangle-representation, or a framed-tangle-representation) these should be clear from the discussion, and would provide alternative ways of looking at link diagrams.

In chapter 7 we prove the only result in the thesis accorded the status of "theorem". It states that the groups obtained from a particular link-representation which uses subarcs as generators can also be obtained from one which uses arcs as generators, a *quandle-representation*. In terms of labelling a diagram this means that the labels on a string either side of an overcrossing are the same, as was the case for the motivational examples. We also give a formula for this representation, and this should be applicable in cases other than the one we are interested in. This also allows us to define an action for the automorphisms of the free group which act as the identity for conjugation on a particular representation.

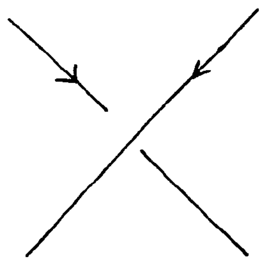
In chapter 8 we look at two dimensional link representations, and in particular at quandle-representations. The examples here are given for the most part in tidy families which can then be studied as a whole. When first looking for examples of two dimensional quandle-representations I found in excess of 500 seemingly distinct examples, including infinite families, and with the prospect of many more link-representations. The question naturally arose of how many different groups will these give for a particular link, and can we classify all the possible examples.

Chapter 7 allows us to ignore all the cases which are not quandle-representations, and chapter 5 allows us to look for a particular representative for each conjugacy class. Thus the small number of classes are the result of applying these chapters to the confusion that we had before to end up with a manageable result.

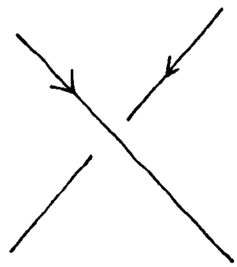
Chapter One - Notation

We mention a few points of notation and some matters arising from it.

For an oriented link diagram, a neighbourhood of a crossing looks like



or



and we call these crossings +ve and -ve respectively.

$<S \mid R>$ denotes the finitely presented group with generators S and relations R.

$G * H$ denotes the free product of the groups G and H. We have also in places used the notation $F_n \times F_n$ rather than $F_n * F_n$.

$G *_K H$ denotes the amalgamated product of the groups G and H over the diagram K which is :

$$\begin{tikzcd} & G \\ F \arrow[ur] \arrow[dr] & \\ & H \end{tikzcd}$$

If G has presentation $< g_1,\dots,g_m \mid r_1,\dots,r_{m'} >$ and H has presentation $< h_1,\dots,h_n \mid s_1,\dots,s_{n'} >$ then $G*H$ has presentation
$< g_1,\dots,g_m , h_1,\dots,h_n \mid r_1,\dots,r_{m'} , s_1,\dots,s_{n'} >$.

If $\theta{:}F \longrightarrow G$ and $\varphi{:}F \longrightarrow H$ are the maps in the diagram K, and if F has generating set $f_1,\dots,f_q$ then $G *_K H$ has a presentation
$< g_1,\dots,g_m , h_1,\dots,h_n \mid r_1,\dots,r_{m'} , s_1,\dots,s_{n'} ; \theta(f_i) = \varphi(f_i), 1 \le i \le q >$.

If W is a collection of elements of a group G then $<W>$ denotes the subgroup generated by the elements of W, and ${}_N<W>$ denotes the normal subgroup generated by W.

For any a, $\underline{a}$ denotes the n-tuple $(a_1,\dots,a_n)$. Similarly $(\underline{a}, \underline{b})$ denotes the 2n-tuple $(a_1,\dots,a_n,b_1,\dots,b_n)$. We also regard the a_i as the elements of $\underline{a}$, and hence we write $< \underline{a} >$

for $< a_1,...,a_n >$.

F_n will denote the free object, usually the free group, on the generators $x_1,...,x_n$. So an element of F_n is a word in the letters $x_1,...,x_n$.

If $\theta:F_n \longrightarrow F_n$ is a morphism then θ_i means the word $\theta(x_i)$, and $\underline{\theta}$ means the n-tuple of words $(\theta_1,...,\theta_n)$.

If w is a word in n letters then $w[a_1,...,a_n]$, (or just $w[\underline{a}]$) means the word w but with a_i written in place of the i^{th} letter. So if $\theta:F_n \longrightarrow F_n$ we have $\underline{\theta} [\underline{x}] = \underline{\theta}$, that is to say $\theta_i[\underline{x}] = \theta(x_i)$ for each i.

Similarly $\underline{\theta}[\underline{a} , \underline{b}]$ will be associated with a map $\theta:F_n \longrightarrow F_{2n}$, and will be an n-tuple whose i^{th} element is the word $\theta(x_i)$, but with a_j substituted for x_j and b_j substituted for x_{j+n} , for each $j=1,...,n$.

Thus if $\underline{v}$ and $\underline{w}$ are n-tuples of words in 2n and 3n letters respectively, an expression such as $\underline{v}[\underline{a} , \underline{w}[\underline{a} , \underline{b} , \underline{a}]]$ is well defined and should make perfect sense to the reader.

Also if F_{mn} is the free group on mn generators, and $a^{(i)}_j$, $1 \leq j \leq n$, $1 \leq i \leq m$ are letters then $\underline{\theta}^{(i)}[\underline{a}^{(1)},...,\underline{a}^{(m)}]$ means the n-tuple whose k^{th} element is $\theta(x_{n(i-1)+k})$ with $a^{(r)}_j$ substituted in place of $x_{n(r-1)+j}$.

Chapter 2 - Diagrams and Reidemeister's theorem.

In this chapter we do 3 things.

First we look at what we mean by a link diagram, also we look at orientations, framings and subdiagrams.

Second we look at Reidemeister's theorem. This we need not only in its usual setting of unoriented links, but also for oriented and framed cases. We also regard the analogous result for tangles to be a version of the basic theorem.

Finally we look at Alexander's theorem that any oriented link may be drawn as a closed braid, and state the result as we shall need it.

As a general reference for this chapter use [B].

(§2.1) Diagrams.

By an oriented link we mean the oriented images of a collection of embeddings of circles into S^3. We also require that the embeddings are in some suitable category (PL usually) and "tame". For a framed link we have not just embeddings of circles but disjoint embeddings of $S^1 \times D^2$.

We are interested in links only up to ambient isotopy, so we call two links "the same" if they are ambient isotopic. A framing is determined by the image of $S^1 \times \{1\}$, and we call this the framing curve.

We think of S^3 as $\mathbb{R}^3 \cup \{\infty\}$, and all links can be (and are) chosen so as to avoid the point ∞.

Let p and q be the projection maps $p(x,y) = y$ and $q(x,y,z) = (x,y)$. We can require that a link be in normal position with respect to the maps p and q by means of a small isotopy. That is, we require there to be no triple points; finitely many crossings, maxima, minima; disjoint maxima and minima; all crossings transverse. (A crossing = a double point of q.)

By a link projection we mean the image of the link under the map q, together with some way of keeping track of the overcrossings/undercrossings (i.e. the local component with the greater/lesser forgotten coordinate).

Of course the same link has many different projections.

We mostly consider link projections only up to ambient plane isotopy. Call these quotient classes link diagrams. These link diagrams will still be in normal position, and will most often be oriented.

For the projection of a framed link we should like the framing to agree with the one obtained from the normal direction to the plane. This means that we can draw the projection of a framing curve as a parallel curve on the plane. This is easily arranged by introducing small 'kinks'

or

into the link components, since these change the framing inherited from the projection by ± 1.

Define a component of a link diagram to be the image of a component of the link.

Define an arc in a link diagram to be the image of a maximal connected subset of a component that contains no undercrossing points. Define a subarc (or occasionally a link-subarc) to be to be the image of a maximal connected subset of a component that contains no undercrossing or overcrossing points.

Define a tangle subarc to be such a subset which also contains no maxima or minima of the map p.

A (0,0)-tangle picture is a class of link projections quotiented by plane isotopies $H:\mathbb{R}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that ($p(\underline{x}) = p(\underline{y}) \Rightarrow (p(H_t(\underline{x})) = p(H_t(\underline{y})) \ \forall\ 0 \leq t \leq 1;\ \underline{x},\underline{y} \in \mathbb{R}^2$). Call these isotopies level isotopies.

Let D^2 be the PL disc $[-1,1]^2$, and ∂D^2 be its boundary. Define a link-in-a-box to be a proper compact 1 dimensional submanifold of $D^2\times\mathbb{R}$, modulo ambient isotopies which fix $\partial D^2\times\mathbb{R}$ (again we require the submanifold to be tame, and we only consider examples which are in normal position with respect to the maps p and q).

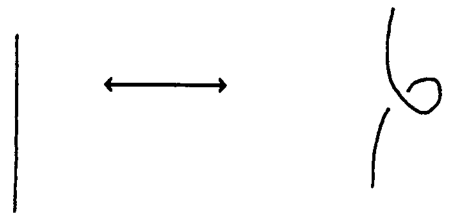
We can define a link-subdiagram to be the image of a link-in-a-box under the map q, keeping track of over and under crossings, modulo isotopies of D^2 fixing ∂D^2; and a tangle picture as the image of a link-in-a-box all of whose endpoints lie on the lines $y=\pm 1$, under the map q, taken modulo level isotopies of D^2 fixing ∂D^2. The number of endpoints on $y=+1$ and $y=-1$ respectively can be counted to give definitions of an (n,m)-tangle. The preimage of a tangle-picture will be called a concrete tangle.

We occasionally refer to specific subdiagrams as diagrams, and sometimes have called a link-in-a-box a sublink.

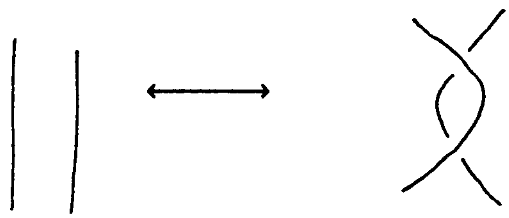
(§2.2) Reidemeister's theorem.

Reidemeister's theorem [R] is usually stated as saying that two (unoriented) link diagrams are projections of the same (i.e. elements of the same ambient isotopy class of) link iff they differ by a sequence of (plane isotopies and) local changes (changes of subdiagrams) of the three following types:

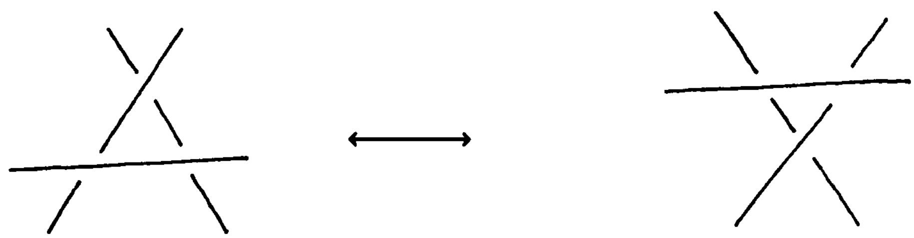
type I



type II



type III



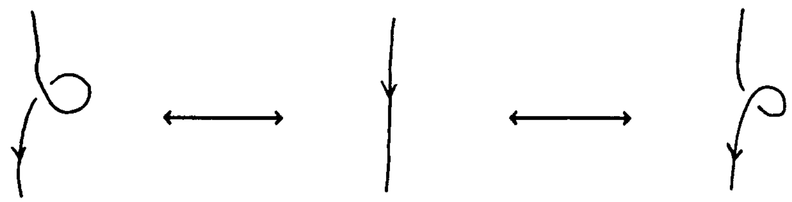
The proof is really a matter of looking at the isotopy between links in general position with respect to the projection q.

We shall also require versions of Reidemeister's theorem for oriented links, for framed links, and for tangle pictures.

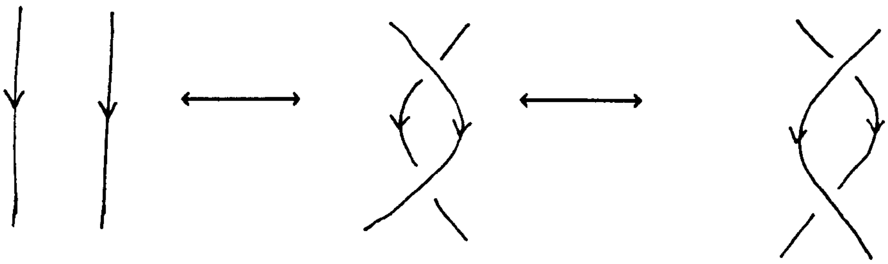
For oriented links we use the same basic moves but with all possible choices of orientation, and then discard any duplication caused by diagrams being rotations of each other. Thus we can easily reduce ourselves to two diagrams of type I, three diagrams of type II, and four diagrams of type III.

We can then reduce further by using modifications of type II on those of type III to see that we only need two diagrams of type III. We call the resulting diagrams as those of types I^o, II^o and III^o.

type I^{o}



type II^{o}



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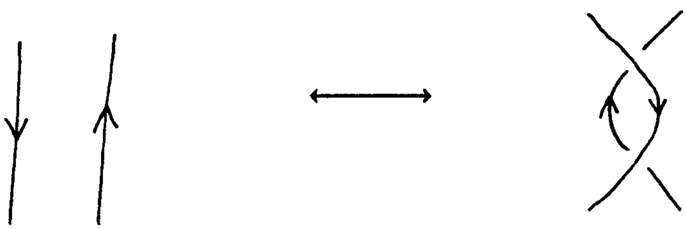
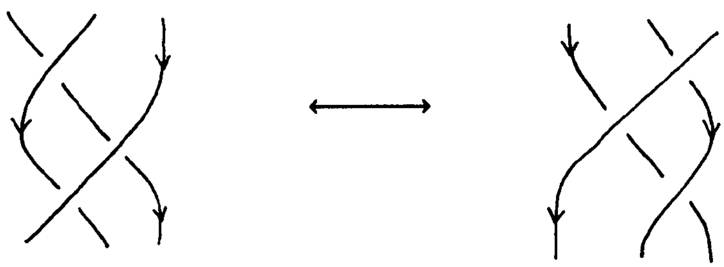
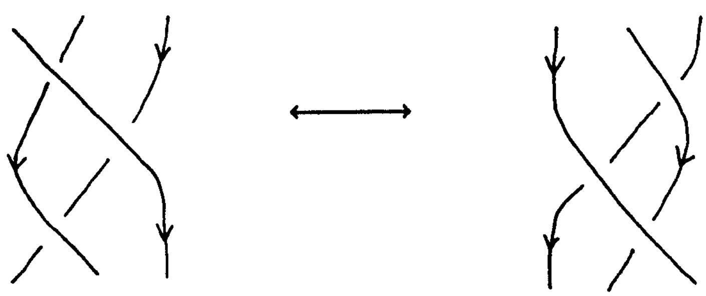


type III^{o}

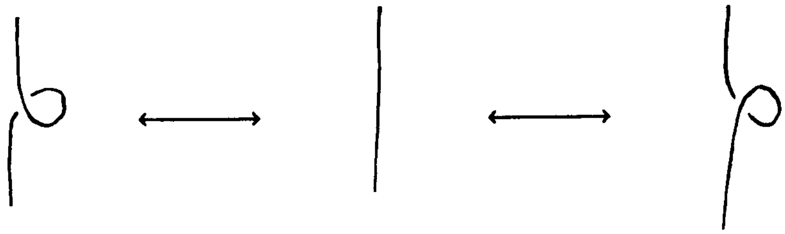


and

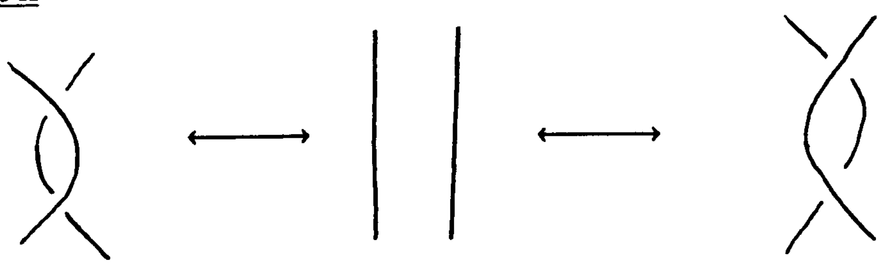


For tangle-pictures we need a slightly different outlook, since although the diagrams are unoriented we must keep track of maxima and minima. Thus we end up with these diagrams:

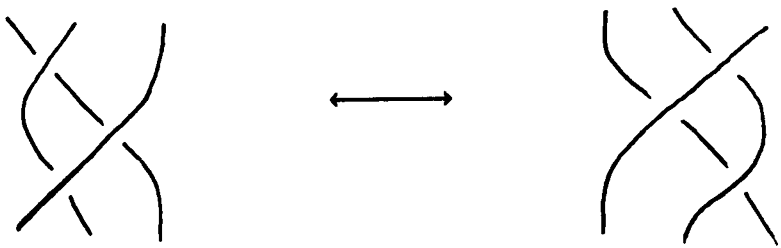
type I^t



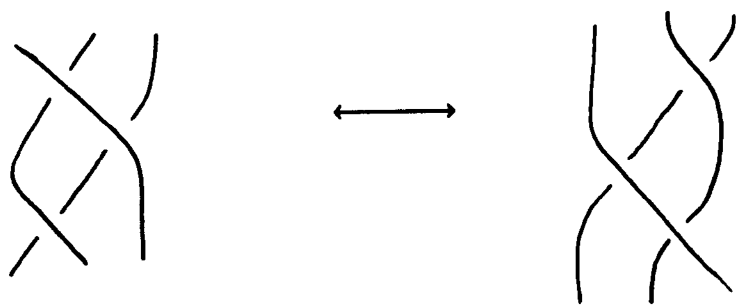
type II^t



type III^t

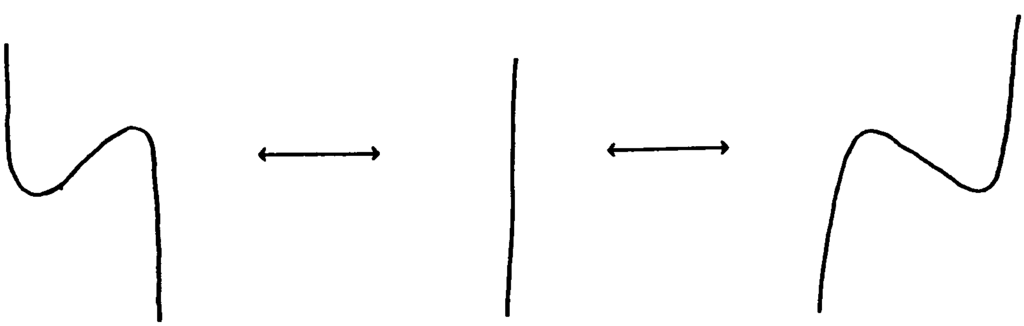


and

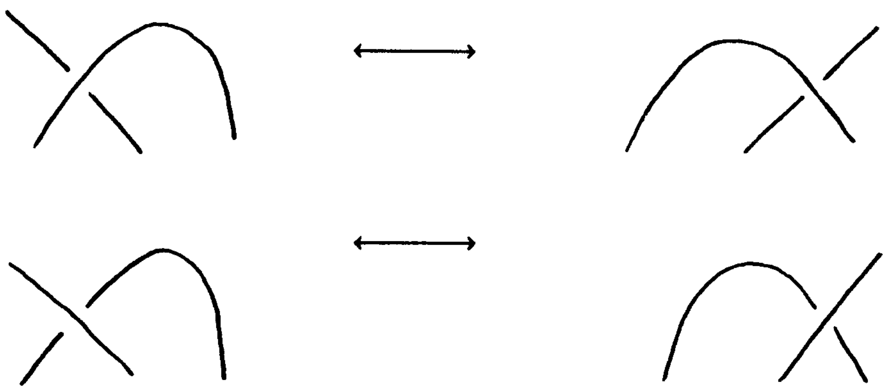


but we also get

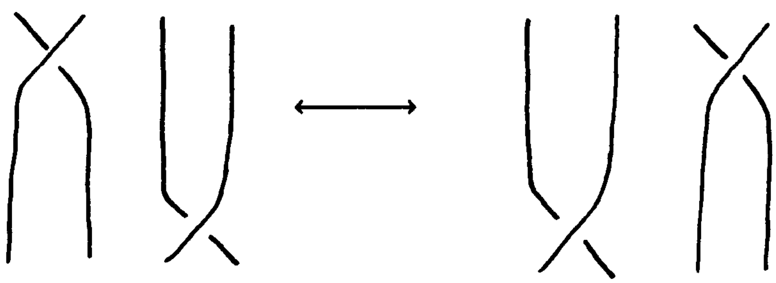
type IV^t

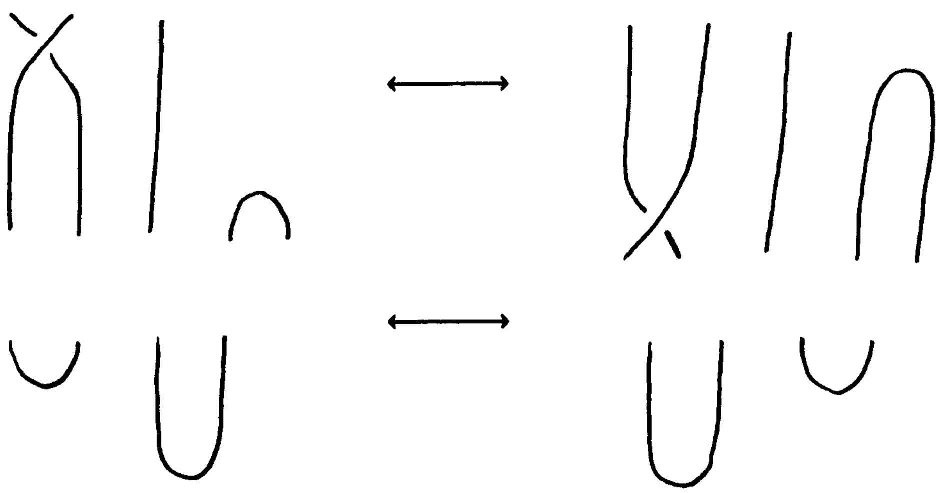


type V^t



The moves I^t–V^t are concerned with how crossings, maxima, minima can interfere with each other. We also must have a collection of moves saying that some critical points do not interfere with each other. This is because we are only allowed level-isotopies for the equivalence of tangle-pictures. A few examples of moves of type VI^t are:

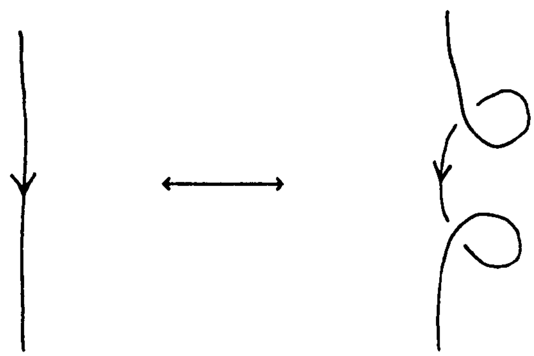




We do not need the list of these in detail since they will not have any effect on our construction.

Tangle picures are the geometric pictures which are being thought of when the tangle category is defined, and the moves of type VI^t correspond to the axioms which say that certain pairs of generators of the tangle category commute. See [T] or [FY] for this, although Freyd and Yetter regard the moves of type VI^t to be trivial. We could have just as well adopted this approach, by changing the definition of level isotopy.

For framed links the moves of type II and III are the same as for oriented links, however move I^o does not preserve the framing. We instead define a move I^f given by:



If we then take move II^f = move II^o and move III^f = move III^o we have a set of moves which generate the equivalence classes of framed link diagrams which are projections of the same framed link. It is a classical result about planar isotopy and Whitney degree that these moves suffice to generate the equivalence we require.

The full version of Reidemeister's theorem which we require is:

<u>Theorem:</u> Let L_1 and L_2 be { link-subdiagrams | oriented link subdiagrams | framed link subdiagrams | tangle pictures } with the same sets of boundary points. Then L_1 and L_2 are projections of the same { link-in-a-box | oriented link-in-a-box | framed link-in-a-box | tangle picture } if and only if they are related by a sequence of moves of type { I - III | I^o - III^o | I^f - III^f | I^t - VI^t }.

(The reader is expected to make the same choice from the curly brackets { } in each case.)

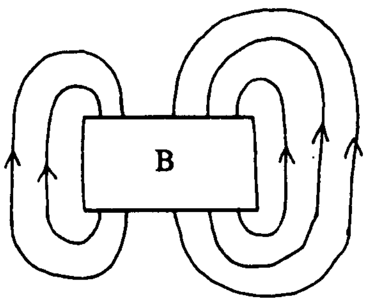
(§2.3) Alexander's theorem and braids.

We require some elementary ideas about braids. Define a braid picture on n strings to be an (n,n)-tangle picture with no local maxima or minima of the y-coordinate.

When we want the braid picture to be oriented we shall take the orientation to run from $y=+1$ to $y=-1$.

The relationship between a braid picture and an element of the braid groups is then the same as the relationship between a tangle picture and an element of the tangle category.

We say an oriented link projection L is a closed braid if it has a subdiagram containing all the crossing points which is an (oriented) braid, and if moreover the arcs outside the braid contain just one maximum and minimum each. That is if L looks like



where B is a braid picture.

As generators of the braid groups we take the usual σ_i , a braid with only one crossing, which is between the i^{th} and the $(i+1)^{st}$ string. A braid picture on n strings is then described as a word on the generators $\sigma_1,...,\sigma_{n-1}$.

By the inverse of a braid picture we mean a braid picture described by the inverse word. Thus if B is a braid picture we are quite happy with B^{-1} as the inverse braid picture. Also if B_1 and B_2 are braids we are quite happy with B_1B_2 the composite braid picture, described by the product of the words describing B_1 and B_2 .

We shall use Alexander's theorem that any oriented link has a projection as a closed braid. We also use a form of Alexander's theorem [A] to say that any framed link has a projection as a closed braid. This is true since we can draw the framed link as a braid except that the framings might be wrong. We can then correct the framings by introducing small kinks which can then be enlarged to give us a closed braid again. This is really just using the fact that the Markov move (replacing a braid picture B on n strings by the braid picture $B\sigma_n^{\pm 1}$ on (n+1) strings) changes the framing of one component by ± 1.

Chapter 3 - Formalities.

In this chapter we begin the original material contained in this thesis.

It is an important chapter since it contains the basic definitions and ideas which will be used in the remainder of the thesis.

The chapter is in five sections.

The first contains the definition of a link-representation, and of some variants on the same idea.

The second contains the basic algebra that allows us to use Reidemeister's theorem.

The next reduces the definitions to a more familiar and tractable form.

Section four contains some remarks about braids.

Section five contains results which make it easier to search for examples of the invariants.

(§3.1) Definitions.

Let L be an oriented link diagram. Let n be a fixed +ve integer.

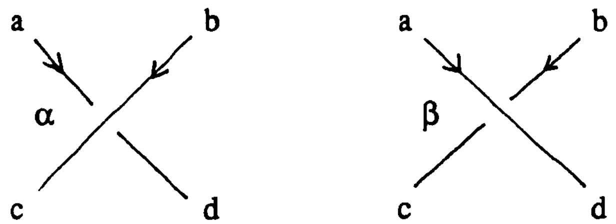
Let S be the set of subarcs.

Let C be the set of crossings, with $C = C^+ \cup C^-$, where C^+ are the +ve crossings and C^- are the -ve ones.

Suppose R^+ and R^- are finite sets whose elements are words in 4n letters. Write R_i^+ and R_i^- for the elements of R^+ and R^- respectively. We can now form a finitely presented group associated with the link diagram in the following way:

Generators are $S \times \{1,2,\dots,n\}$. So for each subarc a we have generators $a_1, a_2, \dots, a_n$.

Relations are introduced for each crossing. Suppose α is a +ve crossing, with incident subarcs a,b,c,d as pictured:



then introduce the relations $R_i^+[\,\underline{a}, \underline{b}, \underline{c}, \underline{d}\,]$ for each $R_i^+ \in R^+$.

Similarly if given the -ve crossing β as pictured above we introduce the relations $R_i^-[\,\underline{a}, \underline{b}, \underline{c}, \underline{d}\,]$ for each $R_i^- \in R^-$.

The reader is reminded that here $\underline{a}$ means the n-tuple $(a_1,\dots,a_n)$ and $R_i^{+/-}[\,\dots\,]$ means "the word $R_i^{+/-}$ with the 4n-tuple in brackets substitute for the standard letters".

<u>Defn.</u> Let $R = (R^+, R^-)$, and call R a link-diagram-representation of dimension n.

Write $G_R(L)$ for the group whose presentation has been described.

Now let L be a link subdiagram. We can still form $G_R(L)$ as above.

Let E be the set of endpoints of L, and let F(L) be the free group on $E \times \{1,\dots,n\}$. There is an obvious map, $inc_* : F(L) \longrightarrow G_R(L)$, defined by saying if $e \in E$ is an endpoint of the subarc $a \in S$ then $inc_*(e_i) = a_i$ for each i.

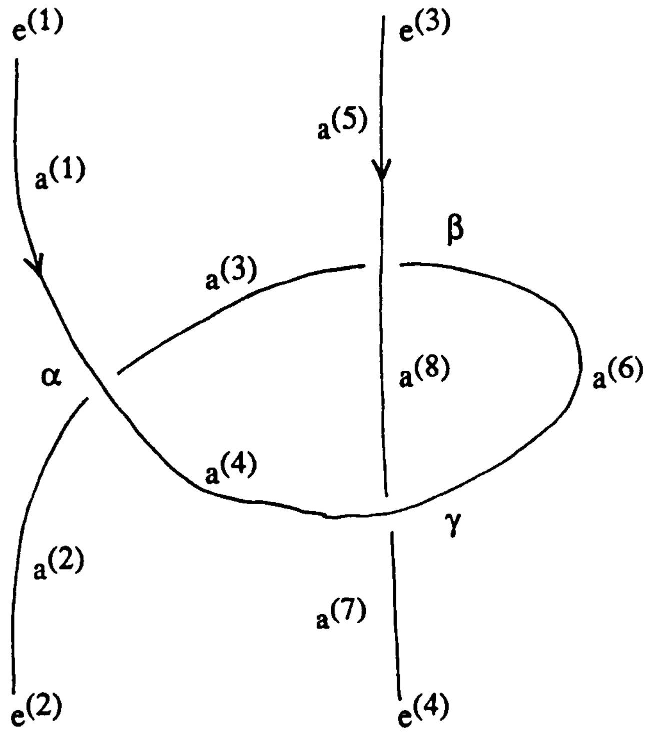
<u>Defn.</u> Call $\{\, inc_* : F(L) \longrightarrow G_R(L) \,\}$ the R-system associated to L, and write it $sys_R(L)$. (We will omit the symbol 'R' when it is clear which link-diagram-representation we mean.)

A Pointless example (by way of illustration)

$n = 1.$

$R^+ = \emptyset\, ;\, R^- = \{\, x_1 x_3^{-1}, x_2 x_4^{-1} \}$

L is given by



$A = \{\, a^{(1)}, \ldots, a^{(8)} \},\ E = \{\, e^{(1)}, e^{(2)}, e^{(3)}, e^{(4)} \}$

$C = \{\, \alpha, \beta, \gamma \},\ C^+ = \emptyset,\ C^- = \{\, \alpha, \beta, \gamma \}$

$F(L) = \langle\, e_1^{(1)},\ e_1^{(2)},\ e_1^{(3)},\ e_1^{(4)} \rangle$

$G_R(L) = \langle\ a_i \text{ for } 1 \leq i \leq 8 \mid a_1^{(1)}(a_1^{(2)})^{-1}, a_1^{(3)}(a_1^{(4)})^{-1}, a_1^{(5)}(a_1^{(3)})^{-1},$

$a_1^{(6)}(a_1^{(8)})^{-1}, a_1^{(4)}(a_1^{(7)})^{-1}, a_1^{(8)}(a_1^{(6)})^{-1} \rangle$

and inc_* :

$e_1^{(1)} \longrightarrow a_1^{(1)}$

$e_1^{(2)} \longrightarrow a_1^{(2)}$

$e_1^{(3)} \longrightarrow a_1^{(5)}$

$e_1^{(4)} \longrightarrow a_1^{(7)}$

Then after some simplification we get $\mathrm{sys}_R(L)$ is given by

$\langle e_1^{(1)},\ e_1^{(2)},\ e_1^{(3)},\ e_1^{(4)} \rangle \longrightarrow \langle\ a_1^{(1)}, a_1^{(3)}, a_1^{(6)} \rangle$

with $\mathrm{inc}_*(e_1^{(1)}) = \mathrm{inc}_*(e_1^{(2)}) = a_1^{(1)}$,

and $\mathrm{inc}_*(e_1^{(3)}) = \mathrm{inc}_*(e_1^{(4)}) = a_1^{(3)}$. □

Defn. Suppose L_1 and L_2 are link-subdiagrams with the same set of endpoints, and such that there is a commutative diagram

$$\begin{array}{llcl} \mathrm{inc}_*: & F(L_1) & \longrightarrow & G_R(L_1) \\ & \| & & \downarrow h \\ \mathrm{inc}_*: & F(L_2) & \longrightarrow & G_R(L_2) \end{array}$$

with h an isomorphism, then we say L_1 and L_2 have isomorphic R-systems and write $\mathrm{sys}_R(L_1) \cong \mathrm{sys}_R(L_2)$.

If L_1 and L_2 are in fact link diagrams (i.e. no endpoints) then this is just saying that $G_R(L_1) \cong G_R(L_2)$.

We can now obtain a link invariant from R if whenever L_1 and L_2 are projections of the same link, $G_R(L_1) \cong G_R(L_2)$. So

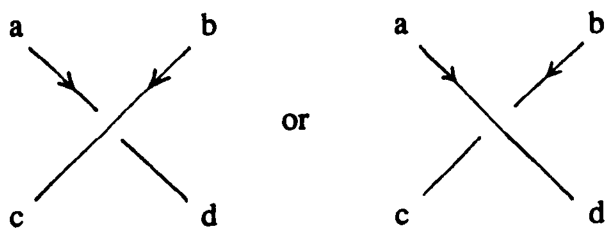
Defn. We call R a link-representation if whenever L_1 and L_2 are projections of the same link-in-a-box, then $\mathrm{sys}_R(L_1) \cong \mathrm{sys}_R(L_2)$. □

Defn. We call R a framed-link-representation if whenever L_1 and L_2 are projections of the same framed-link-in-a-box, then $\mathrm{sys}_R(L_1) \cong \mathrm{sys}_R(L_2)$. □

These are the basic definitions which will be used through the rest of the thesis.

If we had started with the generators being taken as n for each arc rather than as n for each subarc then these definitions would have given us notion of "a representation of a link quandle in finitely presented groups " and "a representation of a geometric rack in finitely presented groups " using the definitions of quandle in [J] and of rack in [FR].

We can make the transfer to arcs rather than subarcs easily by adding the extra condition (*). Suppose we are given a crossing labelled as



then

Condition (*): The relations $\{ R_i^+[\underline{a}, \underline{b}, \underline{c}, \underline{d}] : R_i^+ \in R^+ \}$ must imply $\underline{b} = \underline{c}$, and the relations $\{ R_i^-[\underline{a}, \underline{b}, \underline{c}, \underline{d}] : R_i^- \in R^- \}$ must imply $\underline{a} = \underline{d}$.

Defn. We will call a link-representation satisfying condition (*) a quandle-representation, and a framed-link-representation satisfying condition (*) a rack-representation.

(§3.2) Reidemeister's Theorem.

We want to be able to say "this invariant of link diagrams is unchanged by the Reidemeister moves and hence is an invariant of the link". In order to do this we must make some remarks to justify the local to global implication.

Let R be a link-diagram-representation of dimension n.

Let L be a link diagram.

Let D be a subdisc of the plane such that $\partial D \cap \partial L = \emptyset$, and such that the intersections of L and ∂D are in normal position.

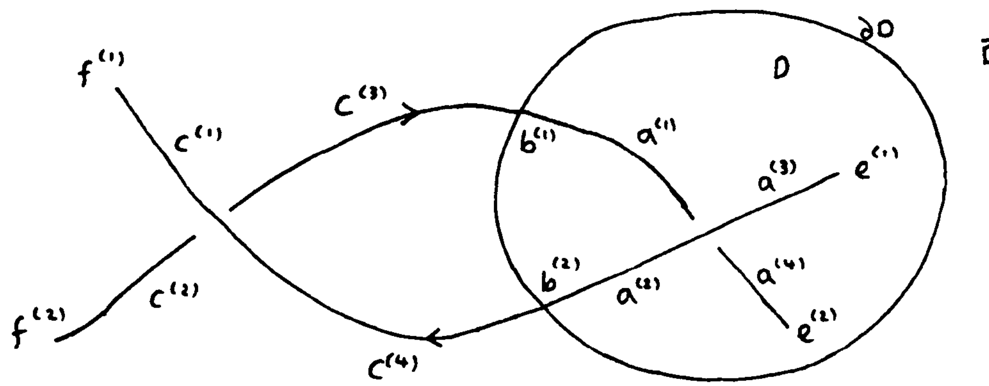
Let $E = \partial L \cap D = \{ e^{(1)},...,e^{(rE)}\}$;

$F = \partial L \cap \bar{D} = \{ f^{(1)},...,f^{(rF)} \}$, i.e. F is the set of endpoints of L which lie in the complement of the disc D.

Let $B = L \cap \partial D = \{ b^{(1)},...,b^{(rB)} \}$.

Let $A = \{ a^{(1)},...,a^{(rA)}\}$ be the subarcs of $L \cap D$ and

$C = \{ c^{(1)},...,c^{(rC)} \}$ be the subarcs of $L \cap \bar{D}$.



For any of the sets A, B, C, E, F, write (for example) $\underline{B}$ to mean the collection $\{ B_j^{(i)} : 1 \le j \le n, 1 \le i \le rB \}$. Write $S(L \cap D)$ and $S(L \cap \bar{D})$ for the collections of relations (words in $\underline{A}$ and $\underline{C}$ respectively) arising from the relations in the regions $(L \cap D)$ and $(L \cap \bar{D})$ respectively.

Then $sys(L \cap D)$ and $sys(L \cap \bar{D})$ are given by

$\theta: \langle \underline{B}, \underline{E} \rangle \longrightarrow \langle \underline{A} \mid S(L \cap D) \rangle$

and $\varphi: \langle \underline{B}, \underline{F} \rangle \longrightarrow \langle \underline{C} \mid S(L \cap \bar{D}) \rangle$.

We thus have a commutative diagram

$$
\begin{tikzcd}
 & \langle \underline{B},\underline{F} \rangle \arrow[rr, "\theta"] & & \langle \underline{C} \mid S(L \cap D) \rangle \arrow[dd] \\
 & & \langle \underline{F} \rangle \arrow[ul] \arrow[d] & \\
\langle \underline{B} \rangle \arrow[uur] \arrow[ddr] & & \langle \underline{E},\underline{F} \rangle \arrow[r, "\psi"] & \langle \underline{C},\underline{A} \mid S(L \cap D), S(L \cap \bar{D});\ \theta(\underline{B}) = \varphi(\underline{B}) \rangle \\
 & & \langle \underline{E} \rangle \arrow[u] \arrow[dl] & \\
 & \langle \underline{B},\underline{E} \rangle \arrow[rr, "\varphi"] & & \langle \underline{A} \mid S(L \cap \bar{D}) \rangle \arrow[uu]
\end{tikzcd}
$$

where all of the unnamed maps are defined by the inclusion of generators, and ψ defines sys(L).

That is to say that $G_R(L) = G_R(L \cap D) *_K G_R(L \cap \bar{D})$ where K is the diagram consisting of the two maps $\langle \underline{B} \rangle \longrightarrow G_R(L \cap D)$ and $\langle \underline{B} \rangle \longrightarrow G_R(L \cap \bar{D})$.

If you prefer category theory , you might want to rewrite this as saying that sys(L) is the push forward of sys $(L \cap D)$ and sys$(L \cap \bar{D})$ over sys$(L \cap \partial D)$, and then go on to deal with an open cover $\{ U_i \}$ of the plane which has all intersections connected, and then say that sys(L) is the colimit of the sys$(L \cap U_i)$. We don't do this.

We only want the above remarks to be able to say

<u>Proposition(3.2.1)</u> Let L_1 and L_2 be link subdiagrams, and D be a disc such that $L_1 \cap \partial D = L_2 \cap \partial D$

Then if sys$(L_1 \cap D) \cong$ sys$(L_2 \cap D)$ and sys$(L_1 \cap \bar{D}) \cong$ sys$(L_2 \cap \bar{D})$ then sys$(L_1) \cong$ sys(L_2) $\square$

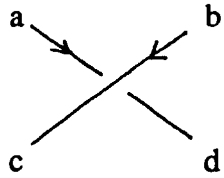
The case of the proposition we are interested in is where L_1 and L_2 are links which differ only by a Reidemeister move inside a disc D. If L_1 and L_2 are two diagrams which appear in a Reidemeister move of type M, and $\text{sys}_R(L_1) \cong \text{sys}_R(L_2)$ then say that R is invariant under moves of type M. We can now conclude from the proposition :

<u>Corol(3.2.2)</u> (i) A link-diagram-representation is a link representation iff it is invariant under moves of types I^o, II^o and III^o.

(ii) A link-diagram-representation is a framed-link-representation iff it is invariant under moves of types I^f, II^f and III^f.

(§3.3) Associated words.

In the motivational examples in the introduction, the relations associated to a crossing

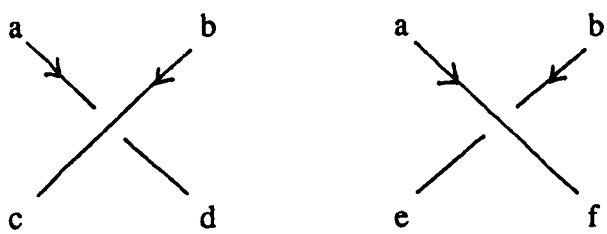


were of the form $\underline{c} = \underline{f}[\underline{a}, \underline{b}]$, $\underline{d} = g[\underline{a}, \underline{b}]$; for some functions $\underline{f}$ and g. In this section we show that this is the general situation and define certain standard words associated to a link-representation.

We defined a link-representation as two collections of words R^+ and R^-, to be used as generators of relations in a group. If we have two other collections S^+ and S^- such that $_N\langle R^+ \rangle = {}_N\langle S^+ \rangle$ and $_N\langle R^- \rangle = {}_N\langle S^- \rangle$, then R and S will define the same invariants of link-subdiagrams, and we are not interested in distinguishing between such R and S, thus

Defn. If R and S are link-diagram-representations of the same dimension such that $_N\langle R^+ \rangle = {}_N\langle S^+ \rangle$ and $_N\langle R^- \rangle = {}_N\langle S^- \rangle$, then R and S will be called equivalent.

Lemma(3.3.1) Let R be a link-diagram-representation which is invariant under moves of type Π^o, then R is equivalent to one of the form

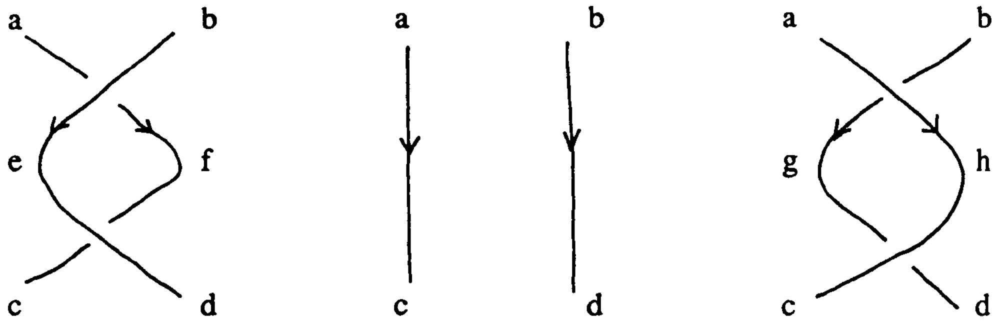


$\underline{c} = \underline{v}^+[\underline{a}, \underline{b}]$ $\qquad$ $\underline{e} = \underline{w}^-[\underline{b}, \underline{a}]$

$\underline{d} = \underline{w}^+[\underline{a}, \underline{b}]$ $\qquad$ $\underline{f} = \underline{v}^-[\underline{b}, \underline{a}]$

Remark: This means that we can take R^+ as containing 2n words, of the form $c_i = v_i^+[\underline{a}, \underline{b}]$ or $d_i = w_i^+[\underline{a}, \underline{b}]$. Invariance under moves of type Π^o will also have other consequences for the forms of these words and these will be included as corollaries.

Proof: Consider the diagrams (from those of move II^o).



L_1 $\qquad$ L_2 $\qquad$ L_3

So invariance under moves of type II^o implies that $sys_R(L_1) \cong sys_R(L_2) \cong sys_R(L_3)$. Let us look at the diagrams for these isomorphisms.

$$\begin{tikzcd}
sys_R(L_1): & \langle \underline{a},\underline{b},\underline{c},\underline{d} \rangle \arrow[r] \arrow[d, "id"'] & \langle \underline{a},\underline{b},\underline{c},\underline{d},\underline{e},\underline{f} \mid \underline{R}^+[\underline{a},\underline{b},\underline{e},\underline{f}], \underline{R}^-[\underline{e},\underline{f},\underline{c},\underline{d}] \rangle \arrow[d, "h"] \\
sys_R(L_2): & \langle \underline{a},\underline{b},\underline{c},\underline{d} \rangle \arrow[r] & \langle \underline{a},\underline{b},\underline{c},\underline{d} \mid \underline{a}=\underline{c}, \underline{b}=\underline{d} \rangle
\end{tikzcd}$$

Where h is an isomorphism making the square commute.

For each i, $h(e_i)$ and $h(f_i)$ are equal to some words in $\underline{a}$ and $\underline{b}$ (and also in $\underline{c}$ and $\underline{d}$). Let $h(e_i) = v_i^+[\underline{a},\underline{b}]$, and $h(f_i) = w_i^+[\underline{a},\underline{b}]$. Hence the words $(e_i)^{-1}v_i^+[\underline{a},\underline{b}]$ and $(f_i)^{-1}w_i^+[\underline{a},\underline{b}]$ lie in ker(h), as do the words $a_i(c_i)^{-1}$ and $b_i(d_i)^{-1}$.

So $N\langle \underline{R}^+[\underline{a},\underline{b},\underline{e},\underline{f}], \underline{R}^-[\underline{e},\underline{f},\underline{c},\underline{d}] \rangle$

$= N\langle \underline{R}^+[\underline{a},\underline{b},\underline{e},\underline{f}], \underline{R}^-[\underline{e},\underline{f},\underline{c},\underline{d}], \underline{a}=\underline{c}, \underline{b}=\underline{d}, \underline{e}=\underline{v}^+[\underline{a},\underline{b}], \underline{f}=\underline{w}^+[\underline{a},\underline{b}] \rangle$

Now $\underline{R}_j^+[\underline{a},\underline{b},\underline{e},\underline{f}] = \underline{R}_j^+[\underline{a},\underline{b},\underline{v}^+[\underline{a},\underline{b}],\underline{w}^+[\underline{a},\underline{b}]]$ and this is some relation in the $\underline{a}$'s and $\underline{b}$'s, but h is an isomorphism hence any such relation must be trivial. Similarly for $\underline{R}^-[\underline{e},\underline{f},\underline{c},\underline{d}]$. Hence

$N\langle \underline{R}^+[\underline{a},\underline{b},\underline{e},\underline{f}], \underline{R}^-[\underline{e},\underline{f},\underline{c},\underline{d}] \rangle = N\langle \underline{a}=\underline{c}, \underline{b}=\underline{d}, \underline{e}=\underline{v}^+[\underline{a},\underline{b}], \underline{f}=\underline{w}^+[\underline{a},\underline{b}] \rangle$

Similarly

$N\langle \underline{R}^+[\underline{a},\underline{b},\underline{e},\underline{f}], \underline{R}^-[\underline{e},\underline{f},\underline{c},\underline{d}] \rangle = N\langle \underline{a}=\underline{c}, \underline{b}=\underline{d}, \underline{e}=\underline{x}^+[\underline{c},\underline{d}], \underline{f}=\underline{y}^+[\underline{c},\underline{d}] \rangle$

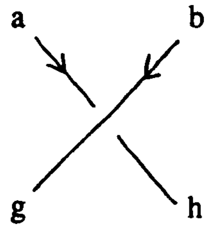
and

$N\langle \underline{R}^-[\underline{a},\underline{b},\underline{g},\underline{h}], \underline{R}^+[\underline{g},\underline{h},\underline{c},\underline{d}] \rangle = N\langle \underline{a}=\underline{c}, \underline{b}=\underline{d}, \underline{h}=\underline{v}^-[\underline{b},\underline{a}], \underline{g}=\underline{w}^-[\underline{b},\underline{a}] \rangle$

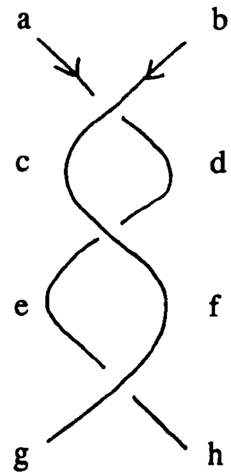
$= N\langle \underline{a}=\underline{c}, \underline{b}=\underline{d}, \underline{h}=\underline{x}^-[\underline{d},\underline{c}], \underline{g}=\underline{y}^-[\underline{d},\underline{c}] \rangle$

for suitable choices of $\underline{x}^+$, $\underline{y}^+$, $\underline{w}^-$, $\underline{v}^-$, $\underline{x}^-$, $\underline{y}^-$.

Now consider



L_4



L_5

Then $sys_R(L_4) \cong sys_R(L_5)$ by a move of type II^o, and the relations in $G_R(L_5)$ are

$$N\langle \underline{R}^{+}[\underline{a},\underline{b},\underline{c},\underline{d}],\ \underline{R}^{-}[\underline{c},\underline{d},\underline{e},\underline{f}],\ \underline{R}^{+}[\underline{e},\underline{f},\underline{g},\underline{h}]\rangle$$

$$= N\langle \underline{R}^{+}[\underline{a},\underline{b},\underline{c},\underline{d}],\ \underline{R}^{-}[\underline{c},\underline{d},\underline{e},\underline{f}],\ \underline{R}^{-}[\underline{c},\underline{d},\underline{e},\underline{f}],\ \underline{R}^{+}[\underline{e},\underline{f},\underline{g},\underline{h}]\rangle$$

$$= N\langle \underline{c} = \underline{v}^{+}[\underline{a},\underline{b}],\ \underline{d} = \underline{w}^{+}[\underline{a},\underline{b}],\ \underline{e} = \underline{a},\ \underline{f} = \underline{b},$$
$$\underline{e} = \underline{x}^{+}[\underline{g},\underline{h}],\ \underline{f} = \underline{y}^{+}[\underline{g},\underline{h}],\ \underline{c} = \underline{g},\ \underline{d} = \underline{h}\rangle$$

$$= N\langle \underline{g} = \underline{v}^{+}[\underline{a},\underline{b}],\ \underline{h} = \underline{w}^{+}[\underline{a},\underline{b}],\ \underline{e} = \underline{a},\ \underline{f} = \underline{b},\ \underline{c} = \underline{g},\ \underline{d} = \underline{h},$$
$$\underline{a} = \underline{x}^{+}[\underline{v}^{+}[\underline{a},\underline{b}],\ \underline{w}^{+}[\underline{a},\underline{b}]],\ \underline{b} = \underline{y}^{+}[\underline{v}^{+}[\underline{a},\underline{b}],\ \underline{w}^{+}[\underline{a},\underline{b}]]\rangle$$

The last two (collections of) relations are purely in $\underline{a}$ and $\underline{b}$ and hence must be trivial, since otherwise they would give rise to nontrivial relations between $\underline{a}$ and $\underline{b}$ in $G_R(L_1)$.

Now if we look at

$$\begin{tikzcd} & G_R(L_4) \arrow[d] \\ \langle \underline{a},\underline{b},\underline{g},\underline{h}\rangle \arrow[ur] \arrow[r] & G_R(L_5) \end{tikzcd}$$

we can see that $sys_R(L_4)$ is given by

$$\langle \underline{a},\underline{b},\underline{g},\underline{h}\rangle \longrightarrow \langle \underline{a},\underline{b},\underline{g},\underline{h} \mid \underline{g} = \underline{v}^{+}[\underline{a},\underline{b}],\ \underline{h} = \underline{w}^{+}[\underline{a},\underline{b}]\rangle$$

and we have written the relations introduced at a positive crossing in the form required by the lemma.

For a negative crossing we would simply look at diagrams L_6 and L_7 corresponding to L_4 and L_5 with L_6 a negative crossing, and proceed as for a positive crossing. □

Corol(3.3.2) Suppose we have a link-diagram-representation R, invariant under moves of type II^o, given by collections of words $\underline{v}^+$, $\underline{w}^+$, $\underline{v}^-$, $\underline{w}^-$, then

(i) $\underline{w}^-[\underline{w}^+[\underline{a},\underline{b}],\underline{v}^+[\underline{a},\underline{b}]] = \underline{a}$

(ii) $\underline{v}^-[\underline{w}^+[\underline{a},\underline{b}],\underline{v}^+[\underline{a},\underline{b}]] = \underline{b}$

(iii) $\underline{v}^+[\underline{w}^-[\underline{b},\underline{a}],\underline{v}^-[\underline{b},\underline{a}]] = \underline{a}$

(iv) $\underline{w}^+[\underline{w}^-[\underline{b},\underline{a}],\underline{v}^-[\underline{b},\underline{a}]] = \underline{b}$.

Proof

(i) and (ii) follow from $sys_R(L_1) \cong sys_R(L_2)$, since in $G_R(L_1)$ we have

$\underline{e} = \underline{v}^+[\underline{a},\underline{b}]$; $\underline{f} = \underline{w}^+[\underline{a},\underline{b}]$; $\underline{c} = \underline{w}^-[\underline{f},\underline{e}]$; $\underline{d} = \underline{v}^-[\underline{f},\underline{e}]$;

so $\underline{c} = \underline{a} \Rightarrow$ (i) and $\underline{d} = \underline{b} \Rightarrow$ (ii).

The proofs of (iii) and (iv) follow in the same way from $sys_R(L_3) \cong sys_R(L_2)$ □

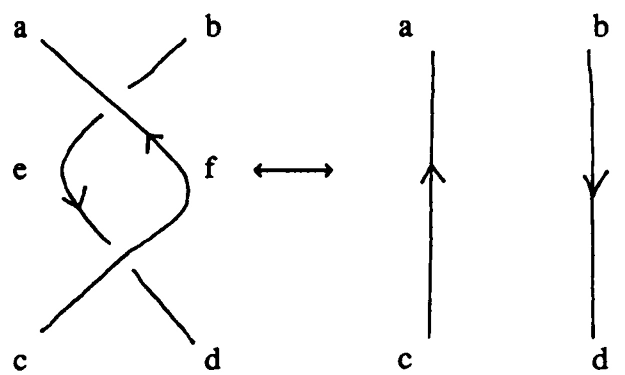
Corol(3.3.3) For R, $\underline{v}^+$,$\underline{w}^+$ as in the lemma, there exists a collection of words $\underline{x}_R$ such that

$$\underline{c} = \underline{v}^+[\underline{a},\underline{b}], \underline{d} = \underline{w}^+[\underline{a},\underline{b}];$$

is equivalent to

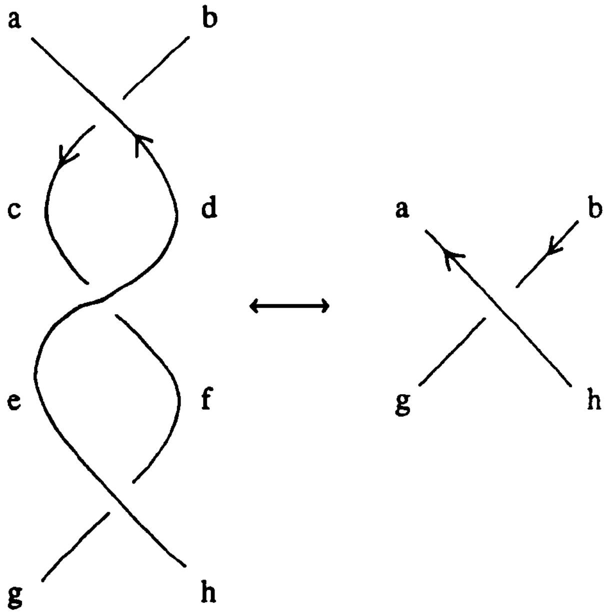
$$\underline{b} = \underline{x}_R[\underline{a},\underline{c}], \underline{d} = \underline{w}^+[\underline{a},\underline{x}_R[\underline{a},\underline{c}]];$$

Proof In the proof of the lemma we only used one form of the move II^o. There is also



which we will use to prove this corollary.

Exactly as in the proof of the lemma we can write $\underline{e}$ and $\underline{f}$ as words in $\underline{a}$ and $\underline{b}$, and then consider the diagrams



and again copy the proof of the lemma to obtain g and $\underline{h}$ as words in $\underline{a}$ and b.

The form of d as words in $\underline{a}$ and $\underline{c}$ follows by substituting $x_R[\,\underline{a}\,,\,\underline{c}\,]$ for $\underline{b}$ □

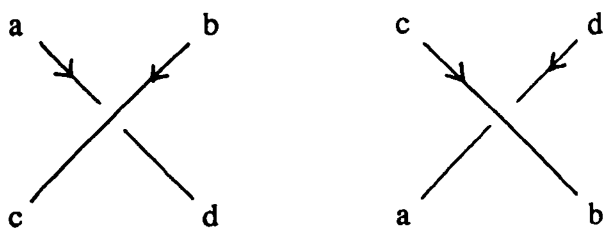
<u>Remark</u> In the proof of the corollary we will also obtain words expressing $\underline{a}$ and $\underline{c}$ in terms of $\underline{b}$ and $\underline{d}$, and thus we could also define words $\underline{y}_R$ such that the relations at $\underline{a}$ crossing also imply

$$\underline{a} = \underline{y}_R[\,\underline{b}\,,\,\underline{d}\,] \text{ and } \underline{c} = \underline{v}^+[\,\underline{y}_R[\,\underline{b}\,,\,\underline{d}\,]\,,\,\underline{b}\,].$$

However these words are not required elsewhere in the thesis so we don't bother to do so.

We can now combine the results of the lemma and of the corollaries to define the associated words for a link-diagram-representation which is invariant under moves of type II^o.

Consider either a positive or a negative crossing with arcs labelled as follows



then we can write the relations associated with either crossing as

(i) $\underline{c} = \underline{v}_R^{+}[\underline{a}, \underline{b}]$, $\underline{d} = \underline{w}_R^{+}[\underline{a}, \underline{b}]$;

or

(ii) $\underline{a} = \underline{v}_R^{-}[\underline{d}, \underline{c}]$, $\underline{b} = \underline{w}_R^{-}[\underline{d}, \underline{c}]$;

or

(iii) $\underline{b} = \underline{x}_R[\underline{a}, \underline{c}]$, $\underline{d} = \underline{w}_R^{+}[\underline{a}, \underline{x}_R[\underline{a}, \underline{c}]]$;

and any of the pairs (i) or (ii) or (iii) is equivalent to the other two.

Also notice that

Corol(3.3.4) $\underline{v}_R^{+}[\underline{a}, \underline{x}_R[\underline{a}, \underline{b}]] = \underline{b}$

and $\underline{x}_R[\underline{a}, \underline{v}^{+}{}_R[\underline{a}, \underline{b}]] = \underline{b}$ □

Defn. We call $\underline{v}_R^{+}$, $\underline{w}_R^{+}$, $\underline{v}_R^{-}$, $\underline{w}_R^{-}$, and $\underline{x}_R$ the words associated with the link-diagram-representation.

The subscript R is often omitted when it is clear from context.

It is clear that the result of this section apply to both link-representations and to framed-link-representations, since both are invariant under moves of type Π^{o} (or equivalently Π^{f}).

(§3.4) R-systems of braids

Associated with link-representations and framed-link-representations there are representations of the braid groups, so in this section we make some remarks about the R-systems of braids.

Let R be a link-diagram-representation which is invariant under moves of types II^o and III^o.

Let C be a braid on m strings. Label the strings at y=+1 by $a^{(1)},...,a^{(m)}$ and those at y=-1 by $b^{(1)},...,b^{(m)}$. Then there are words $\gamma^{(i)}{}_j$ in (m×n) letters such that

$G_R(C) \cong < \underline{a}^{(i)},\underline{b}^{(i)}; 1 \leq i \leq m \mid \underline{b}^{(i)} = \underline{\gamma}^{(i)}[\underline{a}^{(1)},...,\underline{a}^{(m)}] >.$

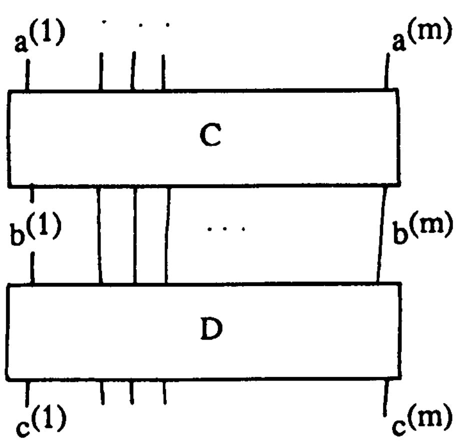
To obtain the words we simply start by labelling at the top of the braid with $a^{(i)}{}_j$'s and work our way down labelling subarcs by words in the $a^{(i)}{}_j$'s as we go. The fact that R is invariant under moves of types II^o and III^o means that the result is determined by C, and is independent of the particular choice of picture for C.

Thus we have defined a map, ξ, from B_m to $Aut(F_{mn})$, the automorphism group of the free object on (m×n) generators. It is easy to see that

Lemma(3.4.1) The map $\xi: B_m \longrightarrow Aut(F_{mn})$ is a representation.

Proof First notice that the identity in B_n can be drawn as a braid with no crossings, and so $\xi(id)$ is the identity automorphism.

Now let C and D be braids on m strings. Label the subarcs in a braid picture of CD as follows:



Then $G_R(CD)$ is given by

$<\underline{a}^{(i)},\underline{b}^{(i)},\underline{c}^{(i)};\ 1\leq i\leq m \mid \underline{b}^{(i)} = \underline{\xi(C)}^{(i)}[\underline{a}^{(1)},...,\underline{a}^{(m)}]\ ,\ \underline{c}^{(i)} = \underline{\xi(D)}^{(i)}[\underline{b}^{(1)},...,\underline{b}^{(m)}]>$

(where $\underline{\xi(C)}^{(i)}$ is explained in chapter 1)

Eliminating the $b^{(i)}$'s in the group $G_R(CD)$ gives

$<\underline{a}^{(i)},\underline{c}^{(i)};\ 1\leq i\leq m \mid \underline{c}^{(i)} = \underline{\eta}^{(i)}[\underline{a}^{(1)},...,\underline{a}^{(m)}]>$

where $\eta = (\xi(C))\ (\xi(D))$.

But also, by the definition of ξ , $\eta = \xi(CD)$, hence $\xi(CD) = (\xi(C))\ (\xi(D))$. □

In the next section we will want the following result

Lemma(3.4.2) Let C and D be braid pictures, and let R be invariant under II^o, then

$sys_R(B) \cong sys_R(C) \Rightarrow sys_R(B^{-1}) \cong sys_R(C^{-1})$.

Proof The cancellation of braid $C.C^{-1}$ and $B.B^{-1}$ only require moves of type II^o, hence we can deduce from (§3.2) that

$sys_R(B^{-1}) \cong sys_R(B^{-1}.C.C^{-1}) \cong sys_R(B^{-1}.B.C^{-1}) \cong sys_R(C^{-1})$ □

(§3.5) Looking for representations

The object of all this formalism is to produce and to calculate new invariants of links and of framed links. The procedure we use is to start by picking candidates for $\underline{v}^+$ and $\underline{w}^+$. Then we check whether we can find words $\underline{v}^-$, $\underline{w}^-$ and $\underline{x}$ so as to satisfy invariance under move of type II^o. After this there is a finite amount of checking to be done, in the case of oriented links we want to check moves I^o and III^o, and for framed links we want to check I^f and III^f. In this section we produce some results to make this easier.

So, given words $\underline{v}^+$ and $\underline{w}^+$ we define a link-diagram-representation as in the remarks before corol (3.3.4).

Lemma (3.5.1) Suppose we are given $\underline{v}^+$ and $\underline{w}^+$ two collections of n words in 2n variables, then given

(i) $\langle \underline{a}, \underline{b} \rangle = \langle \underline{v}^+[\underline{a}, \underline{b}], \underline{w}^+[\underline{a}, \underline{b}] \rangle$

(ii) $\langle \underline{b} \rangle \subseteq \langle \underline{a}, \underline{v}^+[\underline{a}, \underline{b}] \rangle$

(iii) $\langle \underline{a} \rangle \subseteq \langle \underline{b}, \underline{w}^+[\underline{a}, \underline{b}] \rangle$,

we can define a link diagram representation invariant under moves of type II^o.

Proof

The conditions (i), (ii) and (iii) are just those that allow us to find the words $\underline{v}^-$, $\underline{w}^-$, $\underline{x}$ and $\underline{y}$ as described in (§3.3) and satisfying (3.3.2), (3.3.3) and (3.3.4). In particular (i) $\Rightarrow \exists\, \underline{v}^-$ and $\underline{w}^-$, (ii) $\Rightarrow \exists\, \underline{x}$, and (iii) $\Rightarrow \exists\, \underline{y}$; since for example,

$$a_i \in \langle \underline{v}^+[\underline{a}, \underline{b}], \underline{w}^+[\underline{a}, \underline{b}] \rangle$$

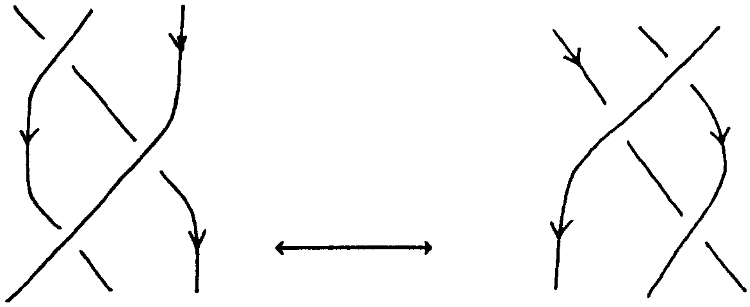
$\Rightarrow \exists$ word w_i^- such that

$$a_i = w_i^-[\underline{v}^+[\underline{a}, \underline{b}], \underline{w}^+[\underline{a}, \underline{b}]].$$

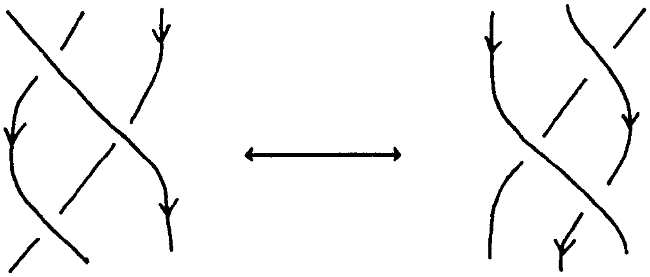
The other words are defined in a similar way and invariance under moves of type II^o follows immediately from the definitions □

If we look back at the moves of types I^o and III^o then we see that there are two moves in each case. We now show that we only need to check invariance under one of the moves, and then invariance under the other will follow.

Lemma(3.5.2) Let R be a link-diagram representation which is invariant under moves of type II^o, then invariance under



is equivalent to invariance under



Proof The diagrams in the second version of the move are the braid picture inverses of those in the first, and hence we can apply lemma(3.4.2) □

Corol(3.5.3) If R is invariant under moves of type II^o then

R is invariant under moves of type III^o

iff

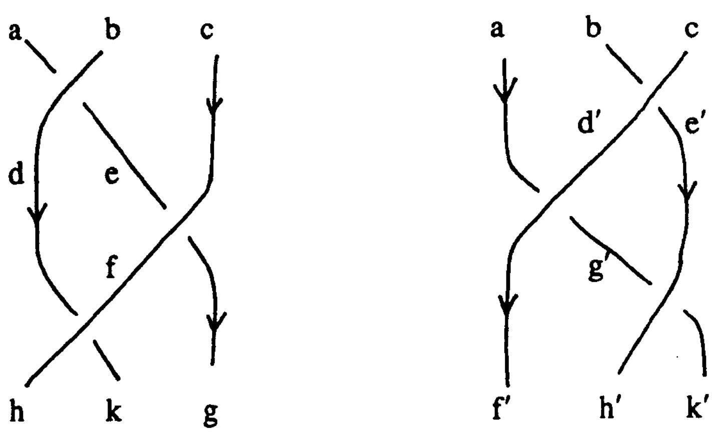
(i) $\underline{v}^+[\underline{v}^+[\underline{a},\underline{b}],\underline{v}^+[\underline{w}^+[\underline{a},\underline{b}],\underline{c}]] = \underline{v}^+[\underline{a},\underline{v}^+[\underline{b},\underline{c}]]$

(ii) $\underline{w}^+[\underline{v}^+[\underline{a},\underline{b}],\underline{v}^+[\underline{w}^+[\underline{a},\underline{b}],\underline{c}]] = \underline{v}^+[\underline{w}^+[\underline{a},\underline{v}^+[\underline{b},\underline{c}]],\underline{w}^+[\underline{b},\underline{c}]]$

(iii) $\underline{w}^+[\underline{w}^+[\underline{a},\underline{b}],\underline{c}] = \underline{w}^+[\underline{w}^+[\underline{a},\underline{v}^+[\underline{b},\underline{c}]],\underline{w}^+[\underline{b},\underline{c}]]$

Proof

consider the diagrams



L_1 L_2

Then in sys(L_1) we have

$$\underline{d} = \underline{v}^+[\underline{a}, \underline{b}], \underline{e} = \underline{w}^+[\underline{a}, \underline{b}]$$

$$\underline{f} = \underline{v}^+[\underline{e}, \underline{c}] = \underline{v}^+[\underline{w}^+[\underline{a}, \underline{b}], \underline{c}]$$

$$\underline{g} = \underline{w}^+[\underline{e}, \underline{c}] = \underline{w}^+[\underline{w}^+[\underline{a}, \underline{b}], \underline{c}]$$

$$\underline{h} = \underline{v}^+[\underline{d}, \underline{f}] = \underline{v}^+[\underline{v}^+[\underline{a}, \underline{b}], \underline{v}^+[\underline{w}^+[\underline{a}, \underline{b}], \underline{c}]]$$

$$\underline{k} = \underline{w}^+[\underline{d}, \underline{f}] = \underline{w}^+[\underline{v}^+[\underline{a}, \underline{b}], \underline{v}^+[\underline{w}^+[\underline{a}, \underline{b}], \underline{c}]]$$

Similarly in sys(L_2) we have

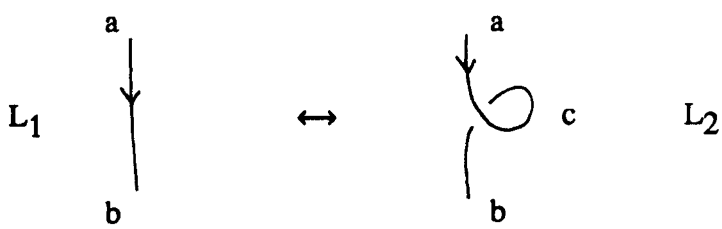
$$\underline{f}' = \underline{v}^+[\underline{a}, \underline{v}^+[\underline{b}, \underline{c}]]$$

$$\underline{h}' = \underline{v}^+[\underline{w}^+[\underline{a}, \underline{v}^+[\underline{b}, \underline{c}]], \underline{w}^+[\underline{b}, \underline{c}]]$$

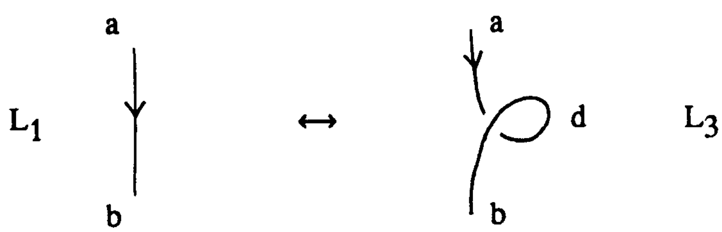
$$\underline{k}' = \underline{w}^+[\underline{w}^+[\underline{a}, \underline{v}^+[\underline{b}, \underline{c}]], \underline{w}^+[\underline{b}, \underline{c}]],$$

and sys(L_1) $\cong$ sys(L_2) iff equations (i), (ii), and (iii) hold. □

Lemma(3.5.4) Let R be a link diagram representation which is invariant under moves of type II$^{\text{O}}$. Then invariance under



is equivalent to invariance under



Proof.

sys$_R$(L_1) is $<\underline{a}, \underline{b}> \longrightarrow <\underline{a}, \underline{b} \mid \underline{a} = \underline{b}>$

sys$_R$(L_2) is $<\underline{a}, \underline{b}> \longrightarrow <\underline{a}, \underline{b}, \underline{c} \mid \underline{a} = \underline{v}^+[\underline{b}, \underline{c}], \underline{c} = \underline{w}^+[\underline{b}, \underline{c}]>$

sys$_R$(L_3) is $<\underline{a}, \underline{b}> \longrightarrow <\underline{a}, \underline{b}, \underline{d} \mid \underline{b} = \underline{v}^+[\underline{a}, \underline{d}], \underline{d} = \underline{w}^+[\underline{a}, \underline{d}]>$

so sys$_R$(L_1) $\cong$ sys$_R$ (L_2) implies the relations $\underline{a} = \underline{v}^+[\underline{b}, \underline{c}]$ and $\underline{c} = \underline{w}^+[\underline{b}, \underline{c}]$ are equivalent to ones $\underline{a} = \underline{b}$, $\underline{c} = \underline{h}[\underline{a}, \underline{b}]$ for some words $\underline{h}$.

Hence $\underline{b} = \underline{v}^+[\underline{a}, \underline{d}]$ and $\underline{d} = \underline{w}^+[\underline{a}, \underline{d}]$ are equivalent to $\underline{b} = \underline{a}$, $\underline{d} = \underline{h}[\underline{b}, \underline{a}]$, thus sys$_R$($L_1$) $\cong$ sys$_R$(L_3).

This proves the implication in one direction, the reverse implication is proved similarly. □

<u>Corol(3.5.5)</u> If R is invariant under moves of type II^o then

R is invariant under moves of type I^o

iff

$\underline{x}_R[\underline{a}, \underline{b}] = \underline{w}_R^{+}[\underline{a}, \underline{x}_R[\underline{a}, \underline{b}]]$.

<u>Proof</u>

Let L_1 and L_3 be as in the lemma. Then $sys_R(L_3)$ can be written as

$< \underline{a}, \underline{b} > \longrightarrow < \underline{a}, \underline{b}, \underline{d} \mid \underline{d} = \underline{x}_R[\underline{a}, \underline{b}], \underline{d} = \underline{w}_R^{+}[\underline{a}, \underline{x}_R[\underline{a}, \underline{b}]] >$

hence $sys_R(L_1) \cong sys_R(L_3)$

iff $\quad \underline{x}_R[\underline{a}, \underline{b}] = \underline{w}_R^{+}[\underline{a}, \underline{x}_R[\underline{a}, \underline{b}]]$ □

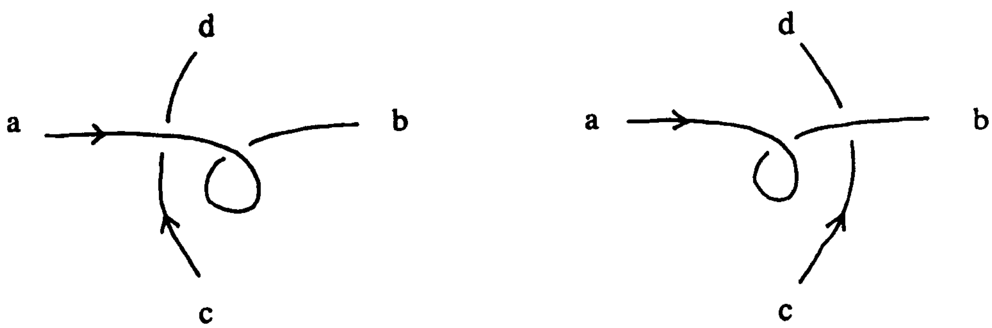
The final simplification in this section is for framed link representations.

<u>Lemma(3.5.6)</u> If R is invariant under moves of types II^o and III^o then it is also invariant under I^f.

<u>Remark</u> We will only prove the lemma here in the case that R is a rack-representation, i.e. when $\underline{v}^{+}[\underline{a}, \underline{b}] = \underline{b}$. The proof in the more general case is left until chapter 7. Although the proof is elementary, it would require the introduction of certain expressions seemingly pulled out of thin air; by the time we have proved the theorem in chapter 7 the proof will seem easy, and the expressions will be the obvious ones.

<u>Proof</u>

First we consider the diagrams



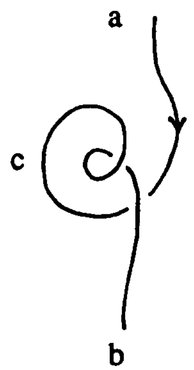
L_1 $\qquad\qquad$ L_2

then in $G_R(L_1)$ the relations are $\underline{b} = \underline{w}^{+}[\underline{a}, \underline{a}]$ and $\underline{d} = \underline{w}^{+}[\underline{c}, \underline{a}]$

whereas in $G_R(L_2)$ they are $\underline{b} = \underline{w}^{+}[\underline{a}, \underline{a}]$ and $\underline{d} = \underline{w}^{+}[\underline{c}, \underline{w}^{+}[\underline{a}, \underline{a}]]$.

But L_1 and L_2 are related by moves of types II^o and III^o hence $sys_R(L_1) \cong sys_R(L_2)$, and thus we can conclude that $\underline{w}^{+}[\underline{c}, \underline{a}] = \underline{w}^{+}[\underline{c}, \underline{w}^{+}[\underline{a}, \underline{a}]]$, $\forall \underline{a}$ and $\underline{c}$.

Now we can use moves II^o and III^o to redraw the link diagrams in move I^f as



L_3

L_4

Then $sys_R(L_3)$ is given by

$$<\underline{a}, \underline{b}> \longrightarrow <\underline{a}, \underline{b}, \underline{c} \mid \underline{b} = \underline{w}^{+}[\underline{c}, \underline{c}] , \underline{a} = \underline{w}^{+}[\underline{c}, \underline{b}]>,$$

then $\underline{w}^{+}[\underline{c}, \underline{b}] = \underline{w}^{+}[\underline{c}, \underline{w}^{+}[\underline{c}, \underline{c}]] = \underline{w}^{+}[\underline{c}, \underline{c}]$ by the calculation above.

Hence$\underline{b} = \underline{w}^{+}[\underline{c}, \underline{c}] = \underline{a}$, but also we can write $\underline{c}$ in terms of $\underline{a}$ and $\underline{b}$, so $sys_R(L_3)$ is given by

$$<\underline{a}, \underline{b}> \longrightarrow <\underline{a}, \underline{b}, \underline{c} \mid \underline{c} = \underline{w}^{-}[\underline{a}, \underline{b}] , \underline{a} = \underline{b}>$$

hence $sys_R(L_3) \cong sys_R(L_4)$ and R is invariant under moves I^f as required □

All of this section simplifies for quandle representations and rack representations. Condition (*) in section (3.1) now says $\underline{v}^{+}[\underline{a}, \underline{b}] = \underline{b}$.

(Which also implies that $\underline{v}^{-}[\underline{a}, \underline{b}] = \underline{b}$ and $\underline{x}[\underline{a}, \underline{c}] = \underline{c}$.)

Invariance under moves I^o says $\underline{w}^{+}[\underline{c}, \underline{c}] = \underline{c}$.

Invariance under moves II^o says that $(\underline{a}, \underline{b}) \longrightarrow (\underline{w}^{+}[\underline{a}, \underline{b}], \underline{b})$ defines an automorphism of the free group $<\underline{a}, \underline{b}>$.

Invariance under moves III^o says

$$\underline{w}^{+}[\underline{w}^{+}[\underline{a}, \underline{b}], \underline{c}]] = \underline{w}^{+}[\underline{w}^{+}[\underline{a}, \underline{c}], \underline{w}^{+}[\underline{b}, \underline{c}]].$$

Chapter 4 - Examples

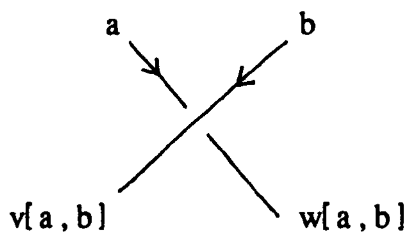
This chapter contains a few examples of link-diagram-representations which are invariant under moves of type II^{O} and III^{O}.

Section one is an exhaustive search in the 1 dimensional case.

Section two includes extensions of some of the definitions from chapter three, and gives some more examples, as does section three.

(§4.1) 1 dimensional examples

We have now done enough theory for it to be easy to find a complete list of the 1 dimensional examples, which will be written as



The first observation to make is that we only need to consider
$v[a, b] = a^r b^\varepsilon a^s$, $w[a, b] = b^p a^\eta b^q$; with $p,q,r,s \in \mathbb{Z}$; $\varepsilon,\eta \in \{\pm 1\}$.

This follows from lemma(3.5.1); in particular the form of v[a,b] follows from the fact that we can write b as some word in a and v[a,b].

From now on we only need to consider move III^o, and in particular the equations in corol(3.5.3). Write A, B, C respectively for the left hand sides of the equations, and A′, B′, C′ for their right hand sides.

First we compare the degrees of a, b, c in A, B, C and A′, B′, C′; That is we look at the image of $A = A'$ etc under the abelianisation map $\langle a, b, c \rangle \longrightarrow \mathbb{Z}^3$.
Write $\deg_a(A)$ to mean the degree of a in A, and we obtain

$\deg_a(A) = \varepsilon\eta(r+s) + (r+s)^2$	$\deg_a(A') = (r+s)$
$\deg_b(A) = \varepsilon(r+s)(p+q+1)$	$\deg_b(A') = \varepsilon(r+s)$
$\deg_c(A) = \varepsilon^2$	$\deg_c(A') = \varepsilon^2$
$\deg_a(B) = \eta(r+s)(p+q+1)$	$\deg_a(B') = \eta(r+s)$
$\deg_b(B) = \varepsilon\eta + (r+s)(p+q)^2$	$\deg_b(B') = \varepsilon\eta + (r+s)^2(p+q)$
$\deg_c(B) = \varepsilon(p+q)$	$\deg_c(B') = \varepsilon(p+q)(r+s+1)$
$\deg_a(C) = \eta^2$	$\deg_a(C') = \eta^2$
$\deg_b(C) = \eta(p+q)$	$\deg_b(C') = \eta(p+q)(r+s+1)$
$\deg_c(C) = (p+q)$	$\deg_c(C') = \varepsilon\eta(p+q) + (p+q)^2$

thus we obtain

$$(r+s)(p+q) = 0$$
$$(r+s)(r+s+\varepsilon\eta-1) = 0$$
$$(p+q)(p+q+\varepsilon\eta-1) = 0$$

Hence we get three cases:

case 1: $p+q = r+s = 0$

case 2: $r+s = 0,\ p+q = 2,\ \varepsilon\eta = -1$

case 3: $p+q = 0,\ r+s = 2,\ \varepsilon\eta = -1$

<u>case 1:</u> $v[a, b] = a^t b^{\varepsilon} a^{-t}$

$w[a, b] = b^s a^{\eta} b^{-s}$

$s,t \in \mathbb{Z};\ \varepsilon,\eta \in \{\pm 1\}$

$C'[a, b, c] = w[w[a, v[b, c]], w[b, c]]$

$= w[w[a, b^t c^{\varepsilon} b^{-t}], c^s b^{\eta} c^{-s}]$

$= c^s b^{s\eta} c^{-s} b^t c^{\varepsilon s} b^{-t} a\, b^t c^{-s\varepsilon} b^{-t} c^s b^{-s\eta} c^{-s}$

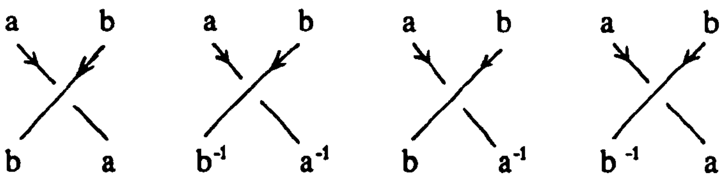
$C[a, b, c] = c^s b^s a\, b^{-s} c^{-s}$

$\therefore c^s b^s = c^s b^{s\eta} c^{-s} b^t c^{\varepsilon s} b^{-t}$

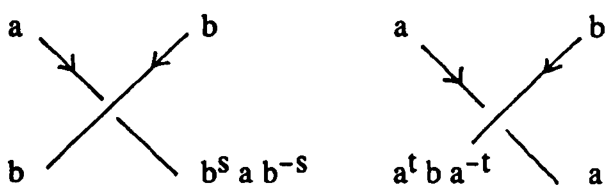
$\therefore$ either $s=0$, or ($\varepsilon = \eta = 1$ and $t=0$)

Similarly from $A = A'$ we obtain either $t=0$, or ($\varepsilon = \eta = 1$ and $s=0$)

Thus we obtain either $s=t=0$ andthe examples



or $\varepsilon = \eta = 1$ and $st = 0$, when we get the examples



<u>Case 2</u>: $r+s = 0,\ p+q = 2,\ \varepsilon\eta = -1$

We split this into two subcases, in (2a) $\varepsilon = +1$, and in (2b) $\varepsilon = -1$.

<u>case 2a</u>: $v[a, b] = a^r b\, a^{-r}$

$w[a, b] = b^p a^{-1} b^{2-p}$

$A[a, b, c] = a^r b^r a^{-r} (b^p a^{-1} b^{2-p})^r c\, (b^p a^{-1} b^{2-p})^{-r} a^r b^{-r} a^{-r}$

$A'[a, b, c] = a^r b^r c\, b^{-r} a^{-r}$

$\therefore a^r b^r a^{-r} (b^p a^{-1} b^{2-p})^r = a^r b^r$

$\therefore r=0$ from the coefficient of b

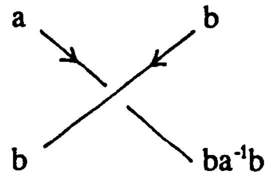
$\therefore$ $v[a, b] = b$

$\therefore$ $C[a, b, c] = c^p\, b^{2-p}\, a\, b^p\, c^{2-p}$

and $C'[a, b, c] = (c^p b^{-1} c^{2-p})^p\, c^{p-2}\, a\, c^{-p}\, (c^p b^{-1} c^{2-p})^{2-p}$

$\therefore$ from the coefficient of b before the a we get $-p=p-2$ i.e. $p=1$

which gives us an example



case 2b $\qquad v[a, b] = a^r\, b^{-1}\, a^{-r}$

$\qquad\qquad w[a, b] = b^p\, a\, b^{2-p}$

$A[a, b, c] = a^r\, b^{-r}\, a^{-r}\, (b^p\, a\, b^{2-p})^r\, c\, (b^p\, a\, b^{2-p})^{-r}\, a^r\, b^r\, a^{-r}$

$A'[a, b, c] = a^r\, b^r\, c\, b^{-r}\, a^{-r}$

$\therefore$ $a^r\, b^r = a^r\, b^{-r}\, a^{-r}\, (b^p\, a\, b^{2-p})^r$

$\therefore$ $b^{2r} = a^{-r}\, (b^p\, a\, b^{2-p})^r$

$\therefore$ $r=0$ or $(r=1, p=0)$ or $(r=-1, p=2)$

$(r=1, p=0)$ gives

$\qquad v[a, b] = ab^{-1} a^{-1}$, $w[a, b] = ab^2$;

$(r=-1, p=2)$ gives

$\qquad v[a, b] = a^{-1} b^{-1} a$, $w[a, b] = b^2 a$;

and both of these are solutions.

If $r=0$ then $A=A'=c$,

$B[a, b, c] = c^{-p}\, b^{-1}\, c^{p-2}$

$B'[a, b, c] = c^{p-2}\, b^{-1}\, c^{-p}$

$\therefore$ $p=1$ and we get

$\qquad v[a, b] = b^{-1}$, $w[a, b] = bab$;

which is again a solution.

<u>Case 3</u> is just the mirror image of case 2, if v,w is a solution in case 2 then

$\qquad v'[a, b] = w[b, a]$, $w'[a, b] = v[b, a]$

gives a solution in case 3.

Thus we can draw up a full list of examples:

	Example number (#)	v[a, b]	w[a, b]
case 1	1	b	a
	2	b^{-1}	a^{-1}
	3	b	a^{-1}
	4	b^{-1}	a
	5_t	b	$b^t a b^{-t}$
	6_t	$a^t b a^{-t}$	a
case2	7	b	$ba^{-1}b$
	8	$ab^{-1}a^{-1}$	ab^2
	9	$a^{-1}b^{-1}a$	b^2a
	10	b^{-1}	bab
case3	11	$ab^{-1}a$	a
	12	ba^2	$ba^{-1}b^{-1}$
	13	a^2b	$b^{-1}a^{-1}b$
	14	aba	a^{-1}

Of these only #3 and #4 are not invariant under moves of type I^o, which is to say that only #3 and #4 care about the framing of the link - and they only notice the framing modulo 2.

There is much duplication of invariants of links on this list. For a given link L, the groups obtained from $\#5_t$ and $\#6_t$ are the same, also $\#5_t$ and $\#5_{-t}$. #1 and #2 just give the same as $\#5_0$.

Moreover all the examples #7 through to #14 all give the same groups as each other for any particular L. These duplicationsare one of the subjects for chapter 7.

(§4.2) Not just Groups

All we have said so far has been in the context of finitely presented groups. However it should be clear that the situation is more general than this.

What we require is a class of objects containing products, free objects and finite presentations. Suppose that Λ is such a class, and let F_n be the free object in Λ on n generators. Then

Defn. A Λ-link-diagram-representation consists of two collections R^+ and R^- of equations in $F_n \times F_n \times F_n \times F_n$. These are the relations which (after substitution) we introduce at a positive or negative crossing.

We can then follow through (§3.1) to define R-systems, and then Λ-link-representations and Λ-framed-link-representations (also Λ-quandle-representations and Λ-rack-representations). After repeating sections 3.2 and 3.3 we think of n dimensional Λ-framed-link-representations as being given by automorphisms θ of $F_n \times F_n$ satisfying

(i) $F_n \times F_n = \langle F_n \times \mathrm{id}, \theta(F_n \times \mathrm{id}) \rangle = \langle \theta(\mathrm{id} \times F_n), \mathrm{id} \times F_n \rangle$

and

(ii) the compositions

$$(\mathrm{id}\times\theta)\circ(\theta\times\mathrm{id})\circ(\mathrm{id}\times\theta) : F_n\times F_n\times F_n \longrightarrow F_n\times F_n\times F_n$$

and $$(\theta\times\mathrm{id})\circ(\mathrm{id}\times\theta)\circ(\theta\times\mathrm{id}) : F_n\times F_n\times F_n \longrightarrow F_n\times F_n\times F_n$$

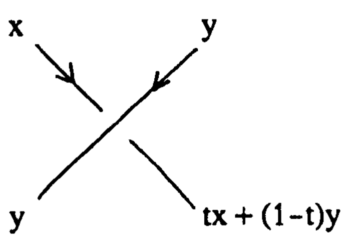
are equal.

(For an invariant which is not changed by framing, we would also require invariance under move I^0. To obtain this we would define the analog of the words $\underline{x}_R$, and then use corol(3.5.5).)

Example(4.2.1) Λ consists of (finitely presented) $\mathbb{Z}[t, t^{-1}]$-modules; n=1;

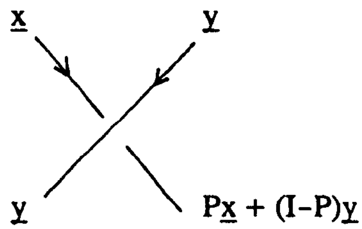
$\theta(x, y) = (y, tx + (1-t)y)$.

ie



This example is just the (1 variable) Alexander module as described in the introduction.

Example(4.2.2) Λ consists of (finitely presented) abelian groups. If P is an invertible n×n integer matrix (i.e. $P \in GL(n,\mathbb{Z})$) then we obtain an n dimensional abelian-group-link-representation by



ie $\theta(\underline{x}, \underline{y}) = (\underline{y}, P\underline{x} + (I-P)\underline{y})$.

It is easy to check that this is a link-representation, and that it is easily calculated from the Alexander module. Given a matrix presenting the Alexander module of a link L as a $\mathbb{Z}[t,t^{-1}]$-module, substitute P for t and replace integers m by mI and you will have a matrix presenting the abelian group invariant for L.

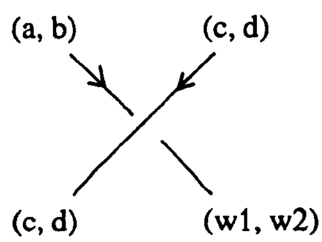
This example is used when looking for group-link-representations, since the given automorphism of $F_n \times F_n$ defining a link-representation induces an automorphism of $\mathbb{Z}^n \times \mathbb{Z}^n$ which is an abelian group link representation.

An explicit example of this is given as a part of the next set of examples.

(§4.3) misc.

Chapter 8 is concerned with 2 dimensional (group) quandle representations, and much of the material in the coming chapters was motivated by those examples. For that reason we include a few examples here from chapter 8 as a sampler.

We will consider crossings as being labelled by



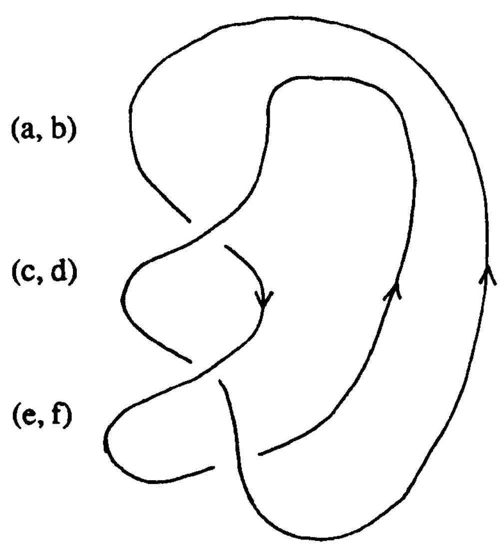
where w1 and w2 are words in a,b,c and d.

Examples (4.3.1) $w1 = abd^{-1}$, $w2 = db^{-1}d$.

$(4.3.2)_r$ $w1 = ab^r d^{-r}$, $w2 = d^r b d^{-r}$.

(4.3.3) $w1 = cbd^{-1}$, $w2 = db^{-1}a^{-1}cd$.

As a demonstration we calculate $G_R(L)$ where R is example (4.3.3) and where L is the trefoil



Then

$$G_R(L) = \langle a, b, c, d, e, f \mid e= cbd^{-1}, f= db^{-1}a^{-1}cd, a= edf^{-1}, b= fd^{-1}c^{-1}ef,$$
$$c= afb^{-1}, d=bf^{-1}e^{-1}ab \rangle$$
$$=\langle a, b, c, d \mid a= cbd^{-1}c^{-1}abd^{-1}, b= db^{-1}a^{-1}ca^{-1}cd \rangle$$

If we abelianise $G_R(L)$ we get

$$\langle a, b, c, d \mid a = a + 2b - 2d, b = -2a - b + 2c + 2d \rangle$$
$$\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

If we abelianise w1,w2 we get

$$w1 = c + b - d, \quad w2 = -b + c - a + 2d.$$

i.e. in the terms of the previous section we have

$$P = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

The Alexander module of L is presented by the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 1-t+t^2 \end{bmatrix}$$

so the abelian group obtained from P and L is presented by

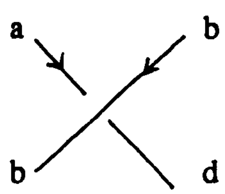
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$$\text{since } I - P + P^2 = \begin{bmatrix} 0 & -2 \\ 2 & 2 \end{bmatrix}$$

and this group is of course $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ again

(end of example) $\square$

As in [J] we can, given a rack or quandle representation R, given by



$\underline{d} = \underline{w}[\,\underline{a}\,,\underline{b}\,],$

define $\underline{w}^{(1)}[\,\underline{a}\,,\underline{b}\,] = \underline{w}[\,\underline{a}\,,\underline{b}\,]$
and $\underline{w}^{(r+1)}[\,\underline{a}\,,\underline{b}\,] = \underline{w}[\,\underline{w}^{(r)}[\,\underline{a}\,,\underline{b}\,]\,,\underline{b}\,]$ for each $r \in \mathbb{N}$.

Then $\underline{d} = \underline{w}^{(r)}[\,\underline{a}\,,\underline{b}\,]$ also defines a rack or quandle representation.
We write R^r for this representation.
We also write R^0 for the representation given by $\underline{w}[\,\underline{a}\,,\underline{b}\,] = \underline{a}$, and R^{-r} for $\bar{R}^r$.

If $\underline{w}^{(r)}[\,\underline{a}\,,\underline{b}\,] = \underline{a}$, and r is the least value for which this is true then say that R has order r.

Then for the two dimensional quandle representations defined above we have
(4.3.1) is of order 2.
$(4.3.2)_r$ is an infinite family; i.e. if we let $\underline{w}^{(1)}[\, a, b, c, d\,] = (\, abd^{-1}, dbd^{-1})$, then by the above construction we obtain $\underline{w}^{(r)}[\, a, b, c, d\,] = (\, ab^r d^{-r}, d^r b d^{-r})$.
(4.3.3) is of order 3.

For an abelian group quandle representation defined from a matrix P we have

$\underline{w}^{(1)}[\,\underline{x}\,,\underline{y}\,] = P\underline{x} + (I-P)y$

$\underline{w}^{(2)}[\,\underline{x}\,,\underline{y}\,] = P(\,P\underline{x} + (I-P)y\,) + (I-P)y$

$= P^2\underline{x} + (I-P^2)y$

and $\underline{w}^{(r)}[\,\underline{x}\,,\underline{y}\,] = P^r\underline{x} + (I-P^r)y.$

Hence the order of the quandle representation is equal to the order of the matrix (as an element of $GL(n, \mathbb{Z})$).

Chapter 5 - Action by automorphisms

If P and Q are conjugate in $GL(n,\mathbb{Z})$, then the abelian group invariants for a link L that we obtain from them will be isomorphic. If $P = A^{-1}QA$, we regard A as an automorphism of $\mathbb{Z}^n$, the free object in the class Λ. Then A allows us to define the isomorphism $G_P(L) \to G_Q(L)$.

In this chapter we see that this is the general situation. If R is a link-diagram-representation of dimension n and θ is an automorphim of F_n then we describe another link-diagram representation $S = R^\theta$, such that $G_R(L) \cong G_S(L)$ $\forall$ links L.

This chapter can also be thought of as the first part of chapter 7 in which we look at some more general cases of pairs R and S such that $G_R(L) \cong G_S(L)$ $\forall$ links L.

(§5.1) Definition of the action.

Let R be a link-diagram-representation invariant under moves of type II^o and III^o. Let R have dimension n, let F_n be the free object on n generators and let θ be an automorphism of F_n.

We define another link-diagram-representation $S = R^\theta$ by

$\underline{v}_S{}^+[\underline{a},\underline{b}] = \underline{\theta}^{-1}[\underline{v}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]]$

$\underline{w}_S{}^+[\underline{a},\underline{b}] = \underline{\theta}^{-1}[\underline{w}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]]$

Lemma(5.1.1) $S = R^\theta$ is invariant under moves of types II^o and III^o.

Proof

To check invariance under II^o we must (by lemma 3.5.1) check 3 things. The basic facts to be used are: first that for any choice of an n-tuple $\underline{x}$, since θ is an automorphism $\langle \underline{x} \rangle = \langle \underline{\theta}[\underline{x}] \rangle = \langle \underline{\theta}^{-1}[\underline{x}] \rangle$; and secondly we use the conclusions of lemma (3.5.1) for R.

(i) $\langle \underline{v}_S{}^+[\underline{a},\underline{b}], \underline{w}_S{}^+[\underline{a},\underline{b}] \rangle$

$= \langle \underline{\theta}^{-1}[\underline{v}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]], \underline{\theta}^{-1}[\underline{w}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]] \rangle$

$= \langle [\underline{v}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]], [\underline{w}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]] \rangle$

by the first basic fact

$= \langle \underline{\theta}[\underline{a}], \underline{\theta}[\underline{b}] \rangle$

by the second

$= \langle \underline{a}, \underline{b} \rangle$ as required, again by the first.

The arguments for the other two are much the same.

(ii) $\langle \underline{b} \rangle \subseteq \langle \underline{a}, \underline{v}_R{}^+[\underline{a},\underline{b}] \rangle$

$\Rightarrow$ $\langle \underline{b} \rangle = \langle \underline{\theta}[\underline{b}] \rangle \subseteq \langle \underline{\theta}[\underline{a}], \underline{v}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]] \rangle$

$= \langle \underline{a}, \underline{\theta}^{-1}[\underline{v}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]] \rangle$

$= \langle \underline{a}, \underline{v}_S{}^+[\underline{a},\underline{b}] \rangle$

(iii) $\langle \underline{a} \rangle \subseteq \langle \underline{b}, \underline{w}_R{}^+[\underline{a},\underline{b}] \rangle$

$\Rightarrow$ $\langle \underline{a} \rangle = \langle \underline{\theta}[\underline{a}] \rangle \subseteq \langle \underline{\theta}[\underline{b}], \underline{w}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]] \rangle$

$= \langle \underline{b}, \underline{\theta}^{-1}[\underline{w}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]] \rangle$

$= \langle \underline{b}, \underline{w}_S{}^+[\underline{a},\underline{b}] \rangle$

To check invariance under III^o we use lemma (3.5.3), and thus we must check 3 equations. Since they all work the same way we only check one here:

$\underline{v}_S{}^+[\underline{v}_S{}^+[\underline{a},\underline{b}],\underline{v}_S{}^+[\underline{w}_S{}^+[\underline{a},\underline{b}],\underline{c}]]$

$$= \underline{\theta}^{-1}[\underline{v}_R{}^+[\underline{\theta}[\underline{\theta}^{-1}[\underline{v}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]]],\ \underline{\theta}[\underline{\theta}^{-1}[\underline{v}_R{}^+[\underline{\theta}[\underline{\theta}^{-1}[\underline{w}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]]],\underline{\theta}[\underline{c}]]]]]]$$

by defn of S

$$= \underline{\theta}^{-1}[\underline{v}_R{}^+[\underline{v}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]],\underline{v}_R{}^+[\underline{w}_R{}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]],\underline{\theta}[\underline{c}]]]]$$

by cancelling pairs $\theta\theta^{-1}$

$$= \underline{\theta}^{-1}[\underline{v}_R{}^+[\underline{\theta}[\underline{a}],\underline{v}_R{}^+[\underline{\theta}[\underline{b}],\underline{\theta}[\underline{c}]]]]$$

since R satisfies lemma (3.5.3)

$= \underline{v}_S{}^+[\underline{a},\underline{v}_S{}^+[\underline{b},\underline{c}]]$ by defn of S.

That completes the proof of the lemma. □

Prop. (5.1.2) With R and S as in the lemma

$$\underline{v}_S{}^-[\underline{a},\underline{b}] = \underline{\theta}^{-1}[\underline{v}_R{}^-[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]]$$
$$\underline{w}_S{}^-[\underline{a},\underline{b}] = \underline{\theta}^{-1}[\underline{w}_R{}^-[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]]$$
$$\underline{x}_S[\underline{a},\underline{b}] = \underline{\theta}^{-1}[\underline{x}_R[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]]$$

Proof It is easy to see that the words on the right hand side satisfy the equations in corol (3.3.2) and corol (3.3.4) and hence must be $\underline{v}_S{}^-$, $\underline{w}_S{}^-$ and $\underline{x}_S$ by uniqueness of inverses. □

Corol. (5.1.3) With R and S as in the lemma, R is invariant under moves of type I^o iff S is.

Proof If we use corol (3.5.5) then this becomes another equation which checks the same way as before. □

We have defined the action of θ by defining the words associated with R^θ. We could have started by defining the maps θ^* induced on the R and S systems as described in the next lemma, and then have deduced the associated words.

Thus the following lemma could have been the starting point for the chapter, and the material in it could have been used to define R^θ. It also shows how this chapter and chapter 7 are related.

Lemma (5.1.4) Let $S = R^\theta$.

Let L be a link subdiagram. Let $e^{(1)},\dots,e^{(r)}$ be the endpoints of L, and let $a^{(1)},\dots,a^{(s)}$ be the subarcs of L.

We define

$$\theta^* : \langle \underline{e}^{(i)} : 1 \le i \le r \rangle \longrightarrow \langle \underline{e}^{(i)} : 1 \le i \le r \rangle$$

by $\theta^*(e^{(i)}{}_j) = \theta_j[\underline{e}^{(i)}]$

and $\theta^* : G_R(L) \longrightarrow G_S(L)$

by $\theta^*(a^{(i)}{}_j) = \theta_j[\underline{a}^{(i)}]$.

These maps θ^* allow us to define a map $\theta^* : sys_R(L) \longrightarrow sys_S(L)$ given by the commutative diagram

$$\begin{tikzcd}
\langle \underline{e}^{(i)} : 1 \leq i \leq r \rangle \arrow[r] \arrow[d, "\theta^*"] & G_R(L) \arrow[d, "\theta^*"'] \\
\langle \underline{e}^{(i)} : 1 \leq i \leq r \rangle \arrow[r] & G_S(L)
\end{tikzcd}$$

and the vertical maps θ^* are both isomorphisms.

<u>Proof</u> The only nontrivial part of the lemma is that the map $\theta^* : G_R(L) \longrightarrow G_S(L)$ is well defined and is an isomorphism. $\theta^* : \langle \underline{a}^{(i)} : 1 \leq i \leq s \rangle \longrightarrow \langle \underline{a}^{(i)} : 1 \leq i \leq s \rangle$ is certainly defined and $\theta^* : G_R(L) \longrightarrow G_S(L)$ is well defined and is an isomorphism because θ^* on the free group maps the relations of $G_R(L)$ to those of $G_S(L)$. For suppose a, b, c, d are the subarcs incident at a crossing, and the relations in $G_R(L)$ are

$$\underline{c} = \underline{v}_R{}^{+}[\underline{a}, \underline{b}], \quad \underline{d} = \underline{w}_R{}^{+}[\underline{a}, \underline{b}].$$

Applying θ^* to these gives

$$\underline{\theta}[\underline{c}] = \underline{v}_R{}^{+}[\underline{\theta}[\underline{a}], \underline{\theta}[\underline{b}]]$$

and $\quad \underline{\theta}[\underline{d}] = \underline{w}_R{}^{+}[\underline{\theta}[\underline{a}], \underline{\theta}[\underline{b}]]$

which are equivalent to

$$\underline{c} = \underline{\theta}^{-1}[\underline{v}_R{}^{+}[\underline{\theta}[\underline{a}], \underline{\theta}[\underline{b}]]]$$

and $\quad \underline{d} = \underline{\theta}^{-1}[\underline{w}_R{}^{+}[\underline{\theta}[\underline{a}], \underline{\theta}[\underline{b}]]]$

i.e. $\quad \underline{c} = \underline{v}_S{}^{+}[\underline{a}, \underline{b}]$ and $\underline{d} = \underline{w}_S{}^{+}[\underline{a}, \underline{b}]$. □

<u>Defn.</u> If $S = R^{\theta}$ then say that R and S are conjugate.

<u>Prop.(5.1.5)</u> $R^{id} = R$

$$R^{(\theta\varphi)} = (R^{\theta})^{\varphi}$$

<u>Proof</u> Trivial, so long as $\theta\varphi$ means apply θ and then apply φ. □

Corol (5.1.6) Conjugacy is an equivalence relation on link-diagram-representations, link-representations, framed-link-representations, rack-representations and on quandle-representations.

$\mathrm{Aut}(F_n)$ acts on those elements which have dimension n.

Proof To define the action on all link-diagram-representations we just define $S = R^{\theta}$ by

$$S_i^{+}[\underline{a}, \underline{b}, \underline{c}, \underline{d}] = R_i^{+}[\underline{\theta}[\underline{a}], \underline{\theta}[\underline{b}], \underline{\theta}[\underline{c}], \underline{\theta}[\underline{d}]]$$

$$S_i^{-}[\underline{a}, \underline{b}, \underline{c}, \underline{d}] = R_i^{-}[\underline{\theta}[\underline{a}], \underline{\theta}[\underline{b}], \underline{\theta}[\underline{c}], \underline{\theta}[\underline{d}]].$$

The only thing that remains to be checked is that if R is a rack-representation then so is S. That just involves checking that $(\underline{v}_R{}^{+}[\underline{a}, \underline{b}] = \underline{b}) \Rightarrow (\underline{v}_S{}^{+}[\underline{a}, \underline{b}] = \underline{b})$, which is obvious.

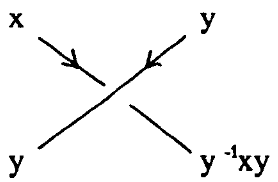
□

(§5.2) Examples

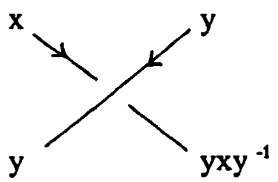
Example (5.2.1) Let $P \in GL(n, \mathbb{Z})$, and write P also for the abelian-group-quandle-representation given by $\underline{w}[\,\underline{x}\,,\underline{y}\,] = P\underline{x} + (I-P)\underline{y}$.

In this case the free object is $\mathbb{Z}^n$, so an element of $\mathrm{Aut}(F_n)$, θ, is also an element of $GL(n, \mathbb{Z})$. Let $Q = \theta^{-1} P\, \theta$, then $P^{\theta} = Q$. (Hence the choice of the name.) For abelian-group-quandle-representations conjugacy is the same as conjugacy of matrices.

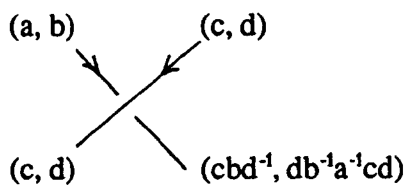
Example (5.2.2) Let R be a (group)-link-representation. Let $\theta(a_i) = a_i^{-1}$ $\forall 1 \le i \le \dim(R)$. Then R^{θ} is given by replacing the words v_i and w_i by their reverses. For example if R is given by



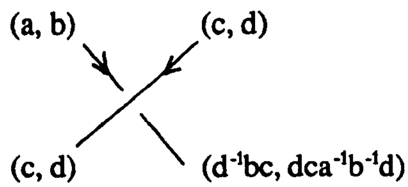
then R^{θ} is given by



or if R is given by



then R^{θ} is given by



Example (5.2.3) Let R and S be two dimensional quandle-representations given by

$\underline{w}_R{}^{+}[\,a, b, c, d\,] = (\,a\, b^{r}\, d^{-r}\,, d^{r}\, b\, d^{-r}\,)$

and $\underline{w}_S{}^{+}[\,a, b, c, d\,] = (\,a\,, a^{r}\, b\, d^{-1}\, c^{-r}\, d\,)$, for the same fixed integer r.

Let $\theta : F_2 \longrightarrow F_2$ be given by

$$\theta(x_1) = x_2$$
$$\theta(x_2) = x_2^{-1} x_1 x_2 \ .$$

Then $\theta^{-1}(x_1) = x_1 x_2 x_1^{-1}$

$$\theta^{-1}(x_2) = x_1 \ .$$

So $\underline{\theta}[a, b] = (b, b^{-1} a b)$; $\underline{\theta}[c, d] = (d, d^{-1} c d)$;

$$\underline{w}_R{}^{+}[\underline{\theta}[a, b], \underline{\theta}[c, d]] = (b(b^{-1} a b)^r (d^{-1} c d)^{-r}, (d^{-1} c d)^r (b^{-1} a b)(d^{-1} c d)^{-r})$$
$$= (a^r b d^{-1} c^{-r} d, d^{-1} c^r d b^{-1} a b d^{-1} c^{-r} d)$$

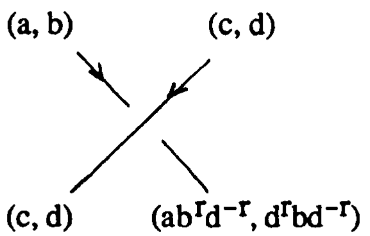
so

$$(\theta^{-1})_1 [\underline{w}_R{}^{+}[\underline{\theta}[a, b], \underline{\theta}[c, d]]]$$
$$= (a^r b d^{-1} c^{-r} d) d^{-1} c^r d b^{-1} a b d^{-1} c^{-r} d (a^r b d^{-1} c^{-r} d)^{-1}$$
$$= a$$

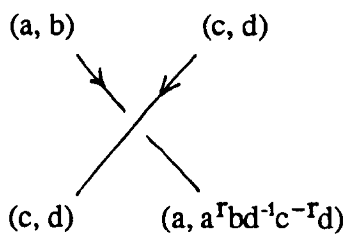
and $(\theta^{-1})_2 [\underline{w}_R{}^{+}[\underline{\theta}[a, b], \underline{\theta}[c, d]]] = a^r b d^{-1} c^{-r} d$

ie $S = R^{\theta}$.

To reiterate the important content of proposition (5.1.4) this means that the groups obtained from a link L by using the labellings



and



are isomorphic.

<u>Example (5.2.4)</u> Let R be as in example (5.2.3) and θ be defined by

$\underline{\theta}[a, b] = (ab^t, b)$; for some integer t.

Then $\underline{\theta}^{-1}[a, b] = (ab^{-t}, b)$

$$\underline{\theta}^{-1}[\underline{w}_R{}^{+}[\underline{\theta}[a, b], \underline{\theta}[c, d]]]$$
$$= \underline{\theta}^{-1}[\underline{w}_R{}^{+}[ab^t, b, cd^t, d]]$$
$$= \underline{\theta}^{-1}[a b^t b^r d^{-r}, d^r b d^{-r}]$$
$$= (a b^r d^{-r}, d^r b d^{-r})$$

hence $R^{\theta} = R$.

This example is included to show that the action of $\mathrm{Aut}(F_n)$ is not free. Also because this example will reappear in chapter 7 where there is given a different action for those θ such that $R^{\theta} = R$.

Chapter 6 - Orientation invariance.

So far everything has been said in the context of oriented links, and hence the invariants obtained can so far only be thought of as invariants of oriented links.

However some of the examples are in fact invariants of unoriented links. An example of this is the fundamental group of the link complement, as obtained from the Wirtinger presentation. Another, also drawn from the motivational examples in the introduction is the first homology group of the two-fold cover of the complement.

This chapter is concerned with such examples.

(§6.1) Mirror Images

It is well known that if $A = A(t)$ is a square matrix with elements in $\mathbb{Z}[t, t^{-1}]$ which presents the Alexander module of a knot, and if B is the matrix obtained from A by transposition , followed by replacing t by t^{-1}, i.e. $B = A^t(t^{-1})$, then the matrix B presents the same module. We interpret this as saying that if K is a knot and A_K is its Alexander module; and if K' is the mirror image knot taken with reversed orientation then A_K and $A_{K'}$ are isomorphic.

This is also true for the (1 variable) Alexander module of a link, and more generally. Let L be a link representation.

Lemma (6.1.1) Let L be a link and let L' be the link obtained from L by taking its mirror image and then reversing the orientations of each of its components.

Then $G_R(L) \cong G_R(L')$.

Proof

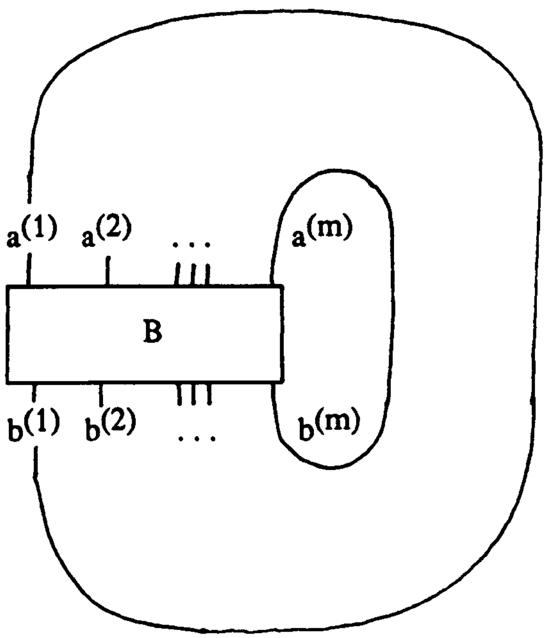
Use Alexander's theorem to draw L as the closure of a braid B on m strings. Then by (§3.4), $sys_R(B)$ is given by

$$< \underline{a}^{(i)}, \underline{b}^{(i)} ; 1 \le i \le m > \longrightarrow < \underline{a}^{(i)}, \underline{b}^{(i)} ; 1 \le i \le m \mid \underline{b}^{(i)} = \underline{\xi(B)}^{(i)}[\underline{a}^{(1)}, \ldots, \underline{a}^{(m)}] ; \text{ for } 1 \le i \le m >$$

hence

$$G_R(L) = < \underline{a}^{(i)}, \underline{b}^{(i)} ; 1 \le i \le m \mid \underline{a}^{(i)} = \underline{b}^{(i)} = \underline{\xi(B)}^{(i)}[\underline{a}^{(1)}, \ldots, \underline{a}^{(m)}] ; \text{ for } 1 \le i \le m >$$

where $a^{(i)}$ are the labels on the strings at y=+1, and $b^{(i)}$ are those at y=-1.



We can now obtain a picture of L′ from one of L by changing all undercrossings to overcrossings (and vice versa), and then changing all orientations.

If we apply this to the picture of L as a closed braid, and then rotate the link by 180^o about the x-axis we obtain a picture of L′ as the closure of the braid B^{-1}.

Thus

$G_R(L') = \langle \underline{a}^{(i)}, \underline{b}^{(i)} ; 1 \leq i \leq m \mid \underline{a}^{(i)} = \underline{b}^{(i)} = \underline{\xi(B^{-1})}^{(i)}[\underline{a}^{(1)},..., \underline{a}^{(m)}] ; \text{ for } 1 \leq i \leq m \rangle$

but ξ is a representation so $\xi(B^{-1}) = (\xi(B))^{-1}$.

Hence $G_R(L) \cong G_R(L')$, by an isomorphism sending

$a^{(i)}_j \longrightarrow b^{(i)}_j$

and $b^{(i)}_j \longrightarrow a^{(i)}_j$ □

Now let L^{op} be the link obtained from L by reversing the orientation of each component of L. Let L^m be the mirror image of L.

Corol (6.1.2) $G_R(L^{op}) \cong G_R(L^m)$.

R satisfies $G_R(L) \cong G_R(L^m)\ \forall L$

$\Leftrightarrow$ R satisfies $G_R(L) \cong G_R(L^{op})\ \forall L$

Proof trivial □

We next aim for a condition on R which will imply that $G_R(L) \cong G_R(L^{op}) \cong G_R(L^m)$. If L is a knot this will imply that R gives an invariant of the unoriented knot (and that the invariant will also be the same for the mirror image knot).

Defn If R is a link-representation then we define another link-representation $\bar{R}$ by

$\underline{v}_{\bar{R}}^{+}[\underline{a}, \underline{b}] = \underline{v}_R^{-}[\underline{a}, \underline{b}]$

$\underline{w}_{\bar{R}}^{+}[\underline{a}, \underline{b}] = \underline{w}_R^{-}[\underline{a}, \underline{b}]$

Remark It is easy to see that $\bar{R}$ is a link-representation by looking at the mirror images of the diagrams used to show invariance under the moves of various types. It is also clear that

$\underline{v}_{\bar{R}}^{-}[\underline{a}, \underline{b}] = \underline{v}_R^{+}[\underline{a}, \underline{b}]$

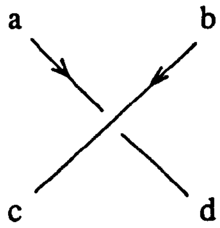
$\underline{w}_{\bar{R}}^{-}[\underline{a}, \underline{b}] = \underline{w}_R^{+}[\underline{a}, \underline{b}]$

Lemma (6.1.3) R is conjugate to $\bar{R}$

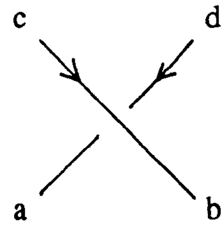
$\Rightarrow$ $G_R(L) \cong G_R(L^{op})\ \forall L$.

Proof R conjugate to $\bar{R}$ implies $G_R(L) \cong G_{\bar{R}}(L)\ \forall L$, thus the lemma will follow if we prove that $G_{\bar{R}}(L) \cong G_R(L^{op})\ \forall L$.

Use the same labels for the subarc of L^{op} as for the corresponding subarc of L. The isomorphism $G_{\bar{R}}(L) \cong G_R(L^{op})$ is induced by the identity map on the free group generated by the {subarcs of L}×{1,...,n}, since given a crossing in L



or



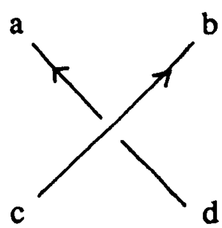
in $G_{\bar{R}}(L)$ we introduce the relations

$$\underline{a} = \underline{w}_{\bar{R}}{}^{-}[\underline{d}, \underline{c}]\ ,\ \underline{b} = \underline{v}_{\bar{R}}{}^{-}[\underline{d}, \underline{c}]$$

which by the definition of $\bar{R}$ is just

$$\underline{a} = \underline{w}_{R}{}^{+}[\underline{d}, \underline{c}]\ ,\ \underline{b} = \underline{v}_{R}{}^{+}[\underline{d}, \underline{c}].$$

In L^{op} the corresponding crossing is



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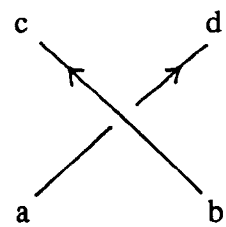


so in $G_R(L^{op})$ we introduce the relations

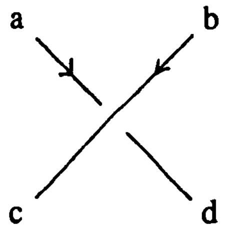
$$\underline{a} = \underline{w}_{R}{}^{+}[\underline{d}, \underline{c}]\ ,\ \underline{b} = \underline{v}_{R}{}^{+}[\underline{d}, \underline{c}]$$

Thus $G_{\bar{R}}(L) \cong G_R(L^{op})$ as claimed. □

(§6.2) Orientation of components

In this section we produce a condition that will imply that a link-representation is independent of the choices of orientations for individual components.

First we recap from (§3.3) the various forms for the relations that we associate with a crossing. Given a crossing



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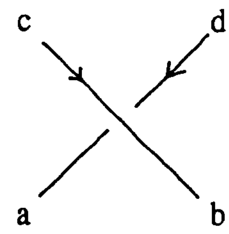


we can write the associated relations as

(i) $\underline{c} = \underline{v}^{+}[\,\underline{a}\,,\underline{b}\,]\,,\underline{d} = \underline{w}^{+}[\,\underline{a}\,,\underline{b}\,]$;

or (ii) $\underline{a} = \underline{v}^{-}[\,\underline{d}\,,\underline{c}\,]\,,\underline{b} = \underline{w}^{-}[\,\underline{d}\,,\underline{c}\,]$;

or (iii) $\underline{b} = \underline{x}[\,\underline{a}\,,\underline{c}\,]\,,\underline{d} = \underline{w}^{+}[\,\underline{a}\,,\underline{x}[\,\underline{a}\,,\underline{c}\,]\,]$;

The condition that we prove in this section is

Lemma (6.2.1) Let L be an unoriented link. Let L_1 and L_2 be oriented links each obtained from L by make choices for the orientation of each component. Let R be a link-representation of dimension n with associated words $\underline{v}^{+}$, $\underline{w}^{+}$,$\underline{v}^{-}$, $\underline{w}^{-}$, $\underline{x}$.
Suppose there exists an involution f on F_n such that

(i) $R^f = \bar{R}$

(ii) $\underline{f}[\,\underline{x}[\,\underline{a}\,,\underline{b}\,]\,] = \underline{v}^{-}[\,\underline{a}\,,\underline{f}[\,\underline{b}\,]\,]$

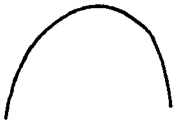
(iii) $\underline{w}^{+}[\,\underline{a}\,,\underline{x}[\,\underline{a}\,,\underline{b}\,]\,] = \underline{w}^{-}[\,\underline{a}\,,\underline{f}[\,\underline{b}\,]\,]$

then $G_R(L_1) \cong G_R(L_2)$. □

It is possible to prove this lemma by constructing an isomorphism $f^{*}: G_R(L_1) \longrightarrow G_R(L_2)$ where f^{*} is the identity on generators whose subarcs have the same orientation in L_1 and L_2 and f^{*} acts as f on the generators associated to a subarc whose orientations differ.

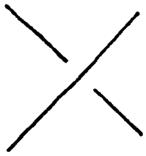
However it is more pleasing to proceed via the ideas of a tangle-representation.

Recall that a tangle-subarc was defined as a maximal section of curve containing no crossing points, or maxima or minima (with respect to the y coordinate). We can define a tangle-picture-representation of dimension n by associating n generators to each tangle-subarc, and associating a collection of relations at maxima, minima and crossings. We still have two types of crossings, positive and negative since the increasing y direction (up the page) is preserved by level-isotopies.

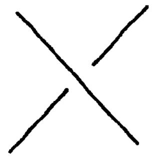


maximum

minimum



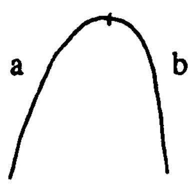
positive crossing



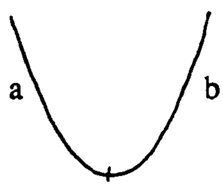
negative crossing

Define a tangle-representation to be a tangle-picture-representation which gives isomorphic systems for different projections of the same concrete tangle. It is easy to see, as in section (3.2) that a tangle picture representation is a tangle representation iff it is invariant under moves of types $I^t, II^t, III^t, IV^t, V^t$, and VI^t (as defined in section 2.2).

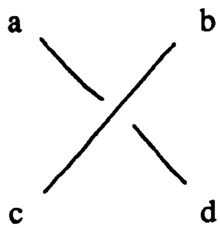
<u>Proposition (6.2.2)</u> Given R and f as in the lemma we can define a tangle-representation by the relations



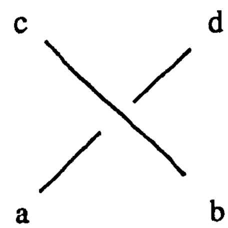
$\underline{a} = \underline{f}[\underline{b}]$



$\underline{b} = \underline{f}[\underline{a}]$



or

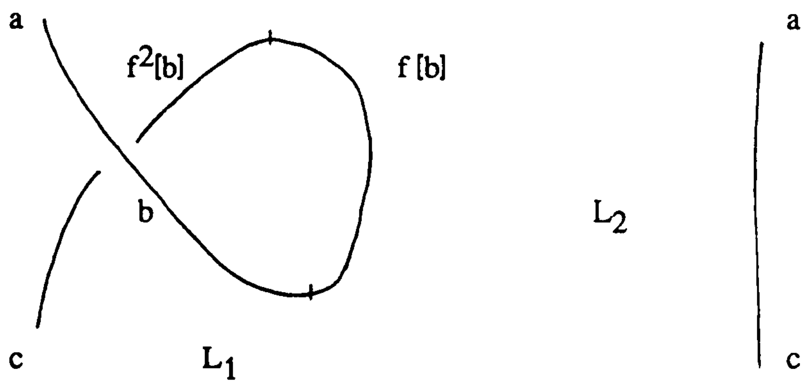


$\underline{c} = \underline{v}^+[\underline{a}, \underline{b}]$

$\underline{d} = \underline{w}^+[\underline{a}, \underline{b}]$

Proof We just check moves I^t to VI^t.

move I^t:



consider the oriented diagrams

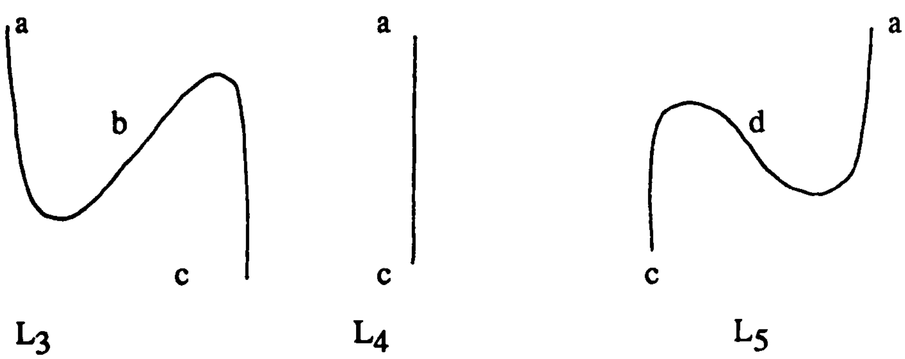
Then since f^2 is the identity,

$$sys_{(R,f)}(L_1) \cong sys_R(L_1{}^o) \cong sys_R(L_2{}^o) \cong sys_{(R,f)}(L_2)\,.$$

So R is invariant under moves of type I^o iff (R,f) is invariant under moves of type I^t.

Moves II^t and III^t contain no maxima or minima, hence orienting the components down the page gives us identical R-systems and (R,f)-systems.

Move IV^t



$sys_{(R,f)}(L_3) : \langle \underline{a}\,, \underline{c} \rangle \longrightarrow \langle \underline{a}, \underline{b}, \underline{c} \mid \underline{b} = \underline{f}[\,\underline{a}\,], \underline{b} = \underline{f}[\,\underline{c}\,] \rangle$

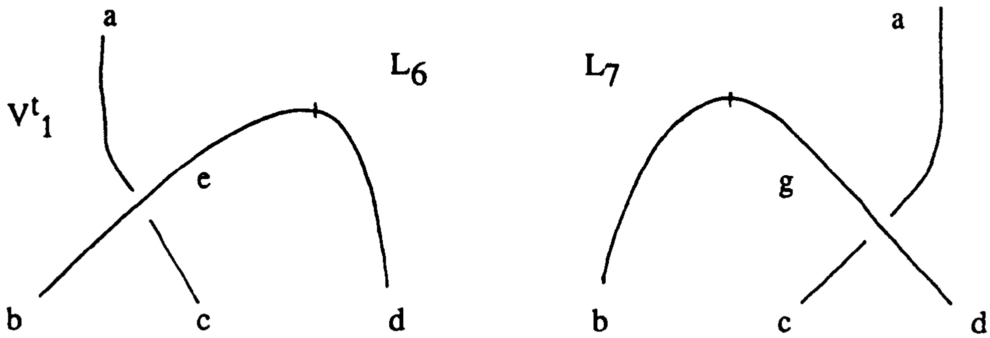
$sys_{(R,f)}(L_4) : \langle \underline{a}\,, \underline{c} \rangle \longrightarrow \langle \underline{a}, \underline{c} \mid \underline{a} = \underline{c} \rangle$

$sys_{(R,f)}(L_5) : \langle \underline{a}\,, \underline{c} \rangle \longrightarrow \langle \underline{a}, \underline{b}, \underline{c} \mid \underline{a} = \underline{f}[\,\underline{d}\,], \underline{c} = \underline{f}[\,\underline{d}\,] \rangle$

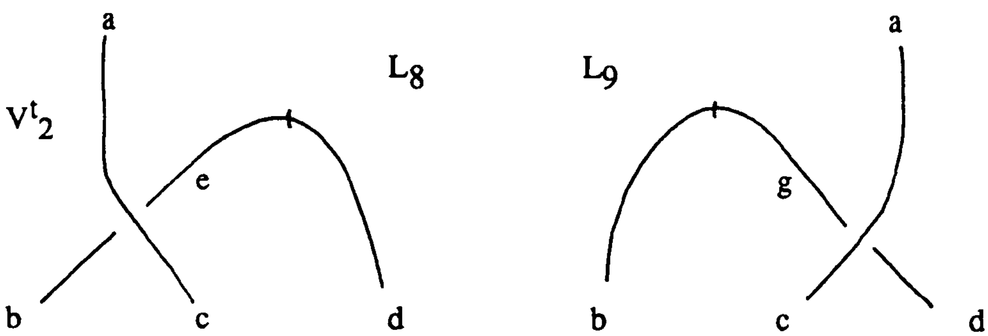
and these three (R,f)-systems are isomorphic since f is an isomorphism.

Move V^t Comes in two parts, $V^t{}_1$ and $V^t{}_2$:

$V^t{}_1$



$V^t{}_2$



In $G_{(R,f)}(L_6)$, $\underline{e} = \underline{f}[\underline{d}]$ and in $G_{(R,f)}(L_7)$ $\underline{g} = \underline{f}[\underline{b}]$ so we have (R,f)-systems

L_6: $\langle \underline{a}, \underline{b}, \underline{c}, \underline{d} \rangle \longrightarrow \langle \underline{a}, \underline{b}, \underline{c}, \underline{d} \mid \underline{c} = \underline{w}^+[\underline{a}, \underline{x}[\underline{a}, \underline{b}]], \underline{d} = \underline{f}[\underline{x}[\underline{a}, \underline{b}]] \rangle$

L_7: $\langle \underline{a}, \underline{b}, \underline{c}, \underline{d} \rangle \longrightarrow \langle \underline{a}, \underline{b}, \underline{c}, \underline{d} \mid \underline{c} = \underline{w}^-[\underline{a}, \underline{f}[\underline{b}]], \underline{d} = \underline{v}^-[\underline{a}, \underline{f}[\underline{b}]] \rangle$

Hence $sys_{(R,f)}(L_6) \cong sys_{(R,f)}(L_7)$ by the conditions (ii) and (iii).

Similarly the (R,f)-systems for L_8 and L_9 are given by

L_8: $\langle \underline{a}, \underline{b}, \underline{c}, \underline{d} \rangle \longrightarrow \langle \underline{a}, \underline{b}, \underline{c}, \underline{d} \mid \underline{a} = \underline{v}^+[\underline{b}, \underline{c}], \underline{d} = \underline{f}[\underline{w}^+[\underline{b}, \underline{c}]] \rangle$

L_9: $\langle \underline{a}, \underline{b}, \underline{c}, \underline{d} \rangle \longrightarrow \langle \underline{a}, \underline{b}, \underline{c}, \underline{d} \mid \underline{d} = \underline{w}^+[\underline{f}[\underline{b}], \underline{x}[\underline{f}[\underline{b}], \underline{c}]], \underline{a} = \underline{x}[\underline{f}[\underline{b}], \underline{c}] \rangle$

but $\underline{x}[\underline{f}[\underline{b}], \underline{c}] = \underline{f}[\underline{v}^-[\underline{f}[\underline{b}], \underline{f}[\underline{c}]]]$ (by condition(ii))

$= \underline{v}^+[\underline{b}, \underline{c}]$ (by (i))

and $\underline{w}^+[\underline{f}[\underline{b}], \underline{x}[\underline{f}[\underline{b}], \underline{c}]] = \underline{w}^-[\underline{f}[\underline{b}], \underline{f}[\underline{c}]]$ (by condition (iii))

$= \underline{f}[\underline{w}^+[\underline{b}, \underline{c}]]$ (by (i))

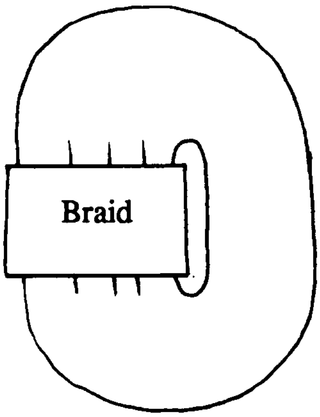
thus (R,f) is invariant under moves of type V^t.

None of the moves of type VI^t change the way we write down the (R,f)-system, and hence (R,f) is invariant under these moves also. That completes the proof of the proposition □

Proof of Lemma (6.2.1)

We prove the lemma by proving that $G_R(L_1) \cong G_{(R,f)}(L)$, since then also $G_R(L_2) \cong G_{(R,f)}(L)$ and hence $G_R(L_1) \cong G_R(L_2)$, and thus we have an invariant of the unoriented link.

To show that $G_R(L_1) \cong G_{(R,f)}(L)$ we find a projection of the link where it is obviously true. Draw L_1 as a closed braid (by Alexander's theorem)

L_1 

then the notions of positive crossing and of negative crossing agree for on one hand the oriented link, and on the other hand for the unoriented tangle. Also the maxima and minima occur in pairs so that f^2 = id means that for this projection of L_1 and L the R-system and the (R,f)-system are the same. □

If R is in fact a quandle-representation, i.e. $\underline{v}^+[\,\underline{a}\,,\underline{b}\,] = \underline{b}$ then the conditions simplify:

Corol (6.2.3) If R is a quandle-representation, and f is an involution such that

(i) $R^f = \bar{R}$

and (ii) $\underline{w}^+[\,\underline{a}\,,\underline{f}\,[\,\underline{b}\,]\,] = \underline{w}^-[\,\underline{a}\,,\underline{b}\,]$

then R defines an invariant of unoriented links. □

(§6.3) Examples

First we look at some of the one dimensional examples

Example (6.3.1) $v^+[a, b] = b$; $w^+[a, b] = b^{-r} a b^r$.

Let $f[y] = y^{-1}$ be the nontrivial involution on F_1

then $f[w^+[f[a], f[b]]] = f[w^+[a^{-1}, b^{-1}]] = f[b^r a^{-1} b^{-r}] = b^r a b^{-r} = w^-[a, b]$

so $R^f = \overline{R}$.

Also $w^+[a, f[b]] = b^r a b^{-r} = w^-[a, b]$.

Hence this link-representation is invariant of the link's orientation.

Example (6.3.2) Example #13 from section (4.1)

$v^+[a, b] = a^2 b$, $w^+[a, b] = b^{-1} a^{-1} b$

then $v^-[a, b] = b a^2$, $w^-[a, b] = b a^{-1} b^{-1}$

and $x[a, b] = a^{-2} b$.

Let $f[y] = y^{-1}$ then

(i) $f[v^+[f[a], f[b]]] = f[a^{-2} b^{-1}] = b a^2 = v^-[a, b]$

$f[w^+[f[a], f[b]]] = b a^{-1} b^{-1} = w^-[a, b]$

(ii) $f[x[a, b]] = f[a^{-2} b] = b^{-1} a^2$

$v^-[a, f[b]] = v^-[a, b^{-1}] = b^{-1} a^2$

(iii) $w^+[a, x[a, b]] = w^+[a, a^{-2} b] = b^{-1} a^{-1} b = w^-[a, f[b]]$

Hence this example satisfies the three conditions of lemma (6.2.1), and is an invariant of the unoriented link.

Example (6.3.3) Any quandle-representation of order 2 satisfies $\underline{w}^+[\underline{a}, \underline{b}] = \underline{w}^-[\underline{a}, \underline{b}]$, and hence is shown to be orientation invariant by taking f = identity map.

Next we look at a couple of examples drawn from chapter 8. These examples are quandle-reresentations, and hence we only need to define $\underline{w}^+$ and $\underline{w}^-$.

Example (6.3.4) $\underline{w}^+[a, b, c, d] = (a, a b d^{-1} c^{-1} d)$

then $\underline{w}^-[a, b, c, d] = (a, a^{-1} b d^{-1} c d)$.

If we take $\underline{f}[a, b] = (a^{-1}, b)$ then we find that $R^f = \overline{R}$.

However we cannot find any involution to satisfy the conditions of corol (6.2.3). In chapter 8 we show by considering examples of links that although this link is invariant under mirror image (and hence *is* independent of the orientation of a knot), it is

dependent
on the choice of orientations for the components of a link.

Example (6.3.5) $\underline{w}^+[a, b, c, d] = (d b^{-1} a b d^{-1}, d b^{-1} a^2 c^{-2} d)$
then $\underline{w}^-[a, b, c, d] = (c^{-2} d b^{-1} a b d^{-1} c^2, c^{-2} d b^{-1} a^{-2} d)$.

If we take $f : F_2 \longrightarrow F_2$ to be given by $\underline{f}[a, b] = (a^{-1}, a^{-2} b)$ then we satisfy the conditions of corol (6.2.3), showing that this example gives us an invariant of unoriented links. □

There are several more examples like this in chapter 8.

Next some remarks on abelian-group-quandle-representations. As described in section (4.2) these are defined by $A \in GL(n, \mathbb{Z})$,
and given by $\underline{w}^+[\underline{x}, \underline{y}] = A\underline{x} + (I - A)\underline{y}$
and $\underline{w}^-[\underline{x}, \underline{y}] = A^{-1}\underline{x} + (I - A^{-1})\underline{y}$
In this context lemma (6.1.3) and corol (6.2.3) say

$A^2 = I \Rightarrow$ the quandle-representation is orientation invariant and

A conjugate to $A^{-1} \Rightarrow$ the quandle-representation is reflection invariant.

As examples take

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} ; B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} ; C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

A is conjugate to its inverse , but $A^2 \neq I$, and we can show that the choice of orientations for components does make a difference. This example is the abelianisation of example (4.3.3) which is of order 3 (also $A^3 = I$). In chapter 8 we give an example of 2 choices for orientation of a link such that A gives different abelian groups, and hence such that example (4.3.3) gives different groups..

B is not conjugate to B^{-1}, and hence we would not expect the invariants obtained to necessarily be unchanged by reflecting the link. However, I have not been able to find an example where it does change.

C is conjugate to its inverse , but $C^2 \neq I$, so we would not necessarily expect the abelian groups obtained to be independent of orientation. However I have not been able to find an example where the orientation does make a difference.

(§6.4) Action by automorphisms

Finally in this chapter we check the interaction between the involution f of lemma (6.2.1) and the action of automorphisms as described in chapter 5.

We already know that the conjugate representations have the same invariance properties with respect to orientation changes and reflection since they give the same invariants, here what we show is that the map f is also conjugated by the action of the automorphism.

Proposition (6.4.1) Let R and S be link-representations of dimension n, and suppose that $R^{\theta} = S$.

Let f be an automorphism of F_n , and let $g = \theta^{-1} f \theta$ (hence if f is an involution then so is g)

Then (i) $R^f = \bar{R}$ implies $S^g = \bar{S}$

(ii) $\underline{f}[\underline{x}_R[\underline{a},\underline{b}]] = \underline{v}_R^-[\underline{a},\underline{f}[\underline{b}]]$

$\Rightarrow \quad \underline{g}[\underline{x}_S[\underline{a},\underline{b}]] = \underline{v}_S^-[\underline{a},\underline{g}[\underline{b}]]$

(iii) $\underline{w}_R^+[\underline{a},\underline{x}_R[\underline{a},\underline{b}]] = \underline{w}_R^-[\underline{a},\underline{f}[\underline{b}]]$

$\Rightarrow \quad \underline{w}_S^+[\underline{a},\underline{x}_S[\underline{a},\underline{b}]] = \underline{w}_S^-[\underline{a},\underline{g}[\underline{b}]]$

Proof

First notice that is proposition (5.1.5), by (f θ) we mean apply f and then apply θ. This also means that $\underline{(f\theta)}[\underline{x}] = \underline{f}[\underline{\theta}[\underline{x}]]$. Hence we don't need to bracket such products of elements of F_n.

(i) $S^g = S^{\theta^{-1} f \theta} = R^{f\theta} = \bar{R}^{\theta}$.

Thus what we have to show is that $\bar{R}^{\theta} = \bar{S}$.

$\underline{\theta}^{-1}[\underline{v}_{\bar{R}}^+[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]] = \underline{\theta}^{-1}[\underline{v}_R^-[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]] = \underline{v}_S^-[\underline{a},\underline{b}] = \underline{v}_{\bar{S}}^+[\underline{a},\underline{b}]$

and similarly for $\underline{w}_{\bar{R}}^+$.

(ii) $\underline{g}[\underline{x}_S[\underline{a},\underline{b}]] = \underline{(\theta^{-1} f \theta \theta^{-1})}[\underline{x}_R[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]]$

$= \underline{\theta}^{-1}[\underline{v}_R^-[\underline{\theta}[\underline{a}],\underline{f\theta}[\underline{b}]]]$

$= \underline{\theta}^{-1}[\underline{v}_R^-[\underline{\theta}[\underline{a}],\underline{\theta g}[\underline{b}]]]$

$= \underline{v}_S^-[\underline{a},\underline{g}[\underline{b}]]$

(iii) $\underline{w}_S^+[\underline{a},\underline{x}_S[\underline{a},\underline{b}]] = \underline{\theta}^{-1}[\underline{w}_R^+[\underline{\theta}[\underline{a}],\underline{x}_R[\underline{\theta}[\underline{a}],\underline{\theta}[\underline{b}]]]]$

$= \underline{\theta}^{-1}[\underline{w}_R^-[\underline{\theta}[\underline{a}],\underline{f\theta}[\underline{b}]]]$

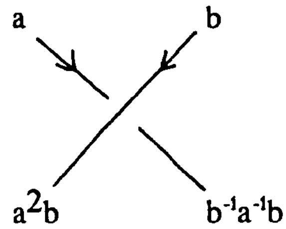
$= \underline{\theta}^{-1}[\underline{w}_R^-[\underline{\theta}[\underline{a}],\underline{\theta g}[\underline{b}]]]$

$= \underline{w}_S^-[\underline{a},\underline{g}[\underline{b}]]$ □

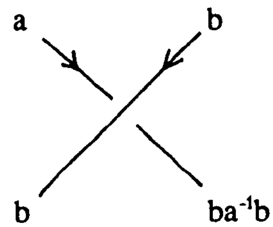
Chapter 7 - Some more isomorphisms.

First some preliminary motivation.

Let R be given by

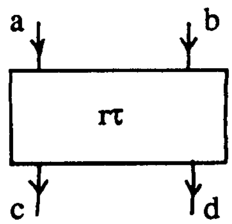


and S be given by



.

Let $r\tau$ mean r right handed half twists, and let L_r be



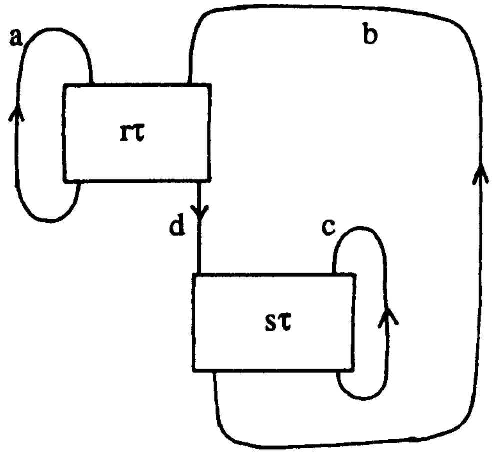
Then it is easy to see that

$sys_R(L_r)$ is $\langle a, b, c, d \rangle \longrightarrow \langle a, b, c, d \mid c = a\,(ab)^r,\ d = (b^{-1}a^{-1})^r\, b \rangle$

and

$sys_S(L_r)$ is $\langle a, b, c, d \rangle \longrightarrow \langle a, b, c, d \mid c = a\,(a^{-1}b)^r,\ d = b(a^{-1}b)^r \rangle$

so if L is given by



then $\quad G_R(L) = \langle\, a, b, c, d \mid a = a\,(ab)^r,\ d = (b^{-1}a^{-1})^r\, b,\ b = d\,(dc)^s,\ c = (c^{-1}\, d^{-1})^s\, c \,\rangle$

$\qquad\qquad = \langle\, a, b, c \mid (ab)^r = (bc)^s = 1 \,\rangle$

and $\quad G_S(L) = \langle\, a, b, c, d \mid a = a\,(a^{-1}b)^r,\ d = b\,(a^{-1}b)^r,\ b = d\,(d^{-1}c)^s,\ c = c\,(d^{-1}\, c)^s \,\rangle$

$\qquad\qquad = \langle\, a, b, c \mid (a^{-1}b)^r = (b^{-1}c)^s = 1 \,\rangle$

so $G_R(L) \cong G_S(L)$ for these links L.

In fact $G_R(L)$ and $G_S(L)$ will be isomorphic whichever link L we choose, and in this chapter we prove this and other results.

The main result of the chapter is that to obtain invariants of links we don't need to change the label on the top string.

To be more precise, we have been constructing link-diagram-representations satisfying certain conditions by starting with a pair of collections of words $\underline{v}_R{}^+$ and $\underline{w}_R{}^+$. The main conclusion of this chapter is that given such an R, there exists an S with $\underline{v}_S[\,\underline{a}\,,\,\underline{b}\,] = \underline{b}$ and such that $G_R(L) \cong G_S(L)$ for all links L.

The first part of the chapter will describe the general approach, and define the isomorphism in general terms.

The second part will produce the specific maps we will use, and will state and prove the theorem.

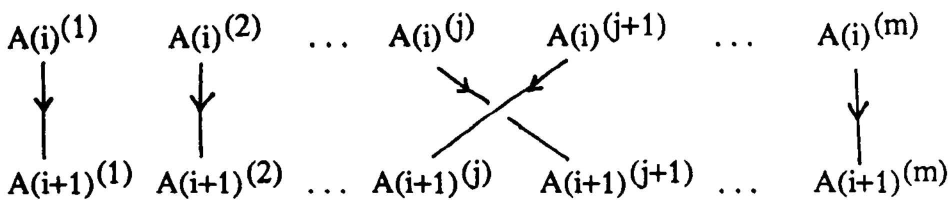
The third part will contain the examples.

(§7.1) Basic ideas

Let R be a link-diagram-representation of dimension n, which is invariant under moves of types II^o and III^o.

Let L be a link (or framed link), drawn as the closure of a braid B on m strings, which we think of as a word of length l in the generators $\sigma_1,\dots\sigma_{m-1}$.

We can write down $G_R(L)$ by regarding the plane as consisting of l+1 parts, l containing 1 crossing each, and 1 part containing the arcs which close the braid. Suppose the i^{th} letter in the word for B is σ_j, then we can label the subarcs and obtain a diagram, D_i,



and $sys_R(D_i)$ is given by

$$F_{mn}\times F_{mn} \longrightarrow < \underline{A(i)}^{(t)}, \underline{A(i+1)}^{(t)} ; 1\le t\le m \mid \underline{A(i)}^{(t)} = \underline{A(i+1)}^{(t)} \text{ if } t \neq j,j+1;$$
$$\underline{A(i+1)}^{(j)} = \underline{v}_R{}^{+}[\,\underline{A(i)}^{(j)}, \underline{A(i)}^{(j+1)}\,],$$
$$\underline{A(i+1)}^{(j+1)} = \underline{w}_R{}^{+}[\,\underline{A(i)}^{(j)}, \underline{A(i)}^{(j+1)}\,] >$$

where the first F_{mn} is generated by the $\underline{A(i)}^{(t)}$ and the second by the $\underline{A(i+1)}^{(t)}$.

So $G_R(L)$ will have the relations from $D_1,\dots,D_l$, together with the relations $\underline{A(1)}^{(t)} = \underline{A(l+1)}^{(t)}$ for $1\le t\le m$ which come from the arcs used to close the braid.

Now let $\varphi \in \mathrm{Aut}(F_{mn})$, then we obtain a commutative diagram

$$\begin{tikzcd}
F_{mn} \times F_{mn} \arrow[r] \arrow[d, "\varphi\times\varphi"] & G_R(D_i) \arrow[d, "\varphi^*"] \\
F_{mn} \times F_{mn} \arrow[r] & G
\end{tikzcd}$$

where $\varphi^*(A(k)_s{}^{(t)}) = \varphi_s{}^{(t)}[\,\underline{A(k)}^{(1)},\dots,\underline{A(k)}^{(m)}\,]$, for $k = i, i+1$;
and where G has generators $\underline{A(i)}^{(t)}$, $\underline{A(i+1)}^{(t)}$, for $1\le t\le m$; and relations obtained by applying $(\varphi\times\varphi)$ to the relations of $G_R(D_i)$.

$(\varphi\times\varphi)$ and φ^* are then both isomorphisms.

We try to find maps φ such that the bottom horizontal map in the commutative diagram is $sys_S(D_i)$ for some link-diagram-representation S. Suppose that for each i,

$1 \leq i \leq l$ we have produced an isomorphism in this way $\varphi^*(i): G_R(D_i) \longrightarrow G_S(D_i)$, then

Claim: we have in fact produced an isomorphism $\Phi^*: G_R(L) \longrightarrow G_S(L)$.

Proof of claim: The maps φ^* carry the relations of $G_R(L)$ arising from crossings to those of $G_S(L)$. The remaining set of relations in $G_R(L)$, arising from the arcs closing the braid, are $\underline{A(1)}^{(t)} = \underline{A(l+1)}^{(t)}$ for $1 \leq t \leq m$. The R-system of this region is given by $F_{mn} \times F_{mn} \longrightarrow \langle \underline{A(1)}^{(t)}, \underline{A(l+1)}^{(t)} ; 1 \leq t \leq m \mid \underline{A(1)}^{(t)} = \underline{A(l+1)}^{(t)} ; 1 \leq t \leq m \rangle$.
If we apply $\varphi \times \varphi$ to this system as we did for the D_i then it is unchanged since φ is an automorphism. □

Thus we have

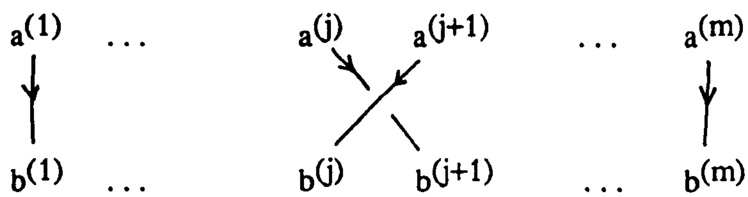
Basic Idea (7.1.1) Suppose we can produce a family of maps $\Phi = \{\varphi_m, m \in \mathbb{N}\}$, such that $\forall$ m $\forall$ braids B on m strings,

the map $\{ F_{mn} \times F_{mn} \longrightarrow G \}$ is $sys_S(B)$, with the same S in every case

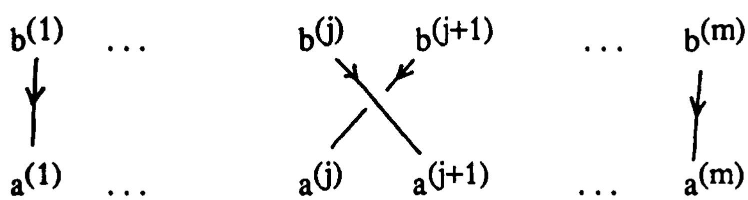
then we have shown that

$G_R(L) \cong G_S(L)$ for all links L. □

Of course we don't need to look at all braids, we only need to look at the braid generators σ_j. We don't need to worry about the inverses of generators since



and



give the same set of relations in $G_R(L)$.

We have already looked at some examples of such isomorphisms in chapter 5 (Action by automorphisms). In that case we started with $\varphi \in \mathrm{Aut}(F_n)$, and then looked at $\varphi \times \varphi \times \ldots \times \varphi$ as an automorphism of F_{mn}, and thus produced $S = R^{\varphi}$.

To see what properties the φ_m must have let us look at the case of $B = \sigma_j$ labelled by $a^{(i)}$ and $b^{(i)}$ as in the last diagram. We write φ for φ_m.

$sys_R(B)$ is

$$< \underline{a}^{(i)}, \underline{b}^{(i)} ; 1 \le i \le m > \longrightarrow < \underline{a}^{(i)}, \underline{b}^{(i)} ; 1 \le i \le m \mid \underline{b}^{(j)} = \underline{v}_R{}^{+}[\, \underline{a}^{(j)}, \underline{a}^{(j+1)} \,],$$
$$\underline{b}^{(j+1)} = \underline{w}_R{}^{+}[\, \underline{a}^{(j)}, \underline{a}^{(j+1)} \,],$$
$$\underline{b}^{(i)} = \underline{a}^{(i)} \text{ for } i \ne j, j+1 >.$$

Writing $\underline{\varphi}^*[\,\underline{a}^{(i)}\,]$ for $\underline{\varphi}^{(i)}[\,\underline{a}^{(1)},\dots,\underline{a}^{(m)}\,]$, and $\underline{\varphi}^*[\,\underline{b}^{(i)}\,]$ for $\underline{\varphi}^{(i)}[\,\underline{b}^{(1)},\dots,\underline{b}^{(m)}\,]$ G is given by

$$< \underline{a}^{(i)}, \underline{b}^{(i)} ; 1 \le i \le m \mid \underline{\varphi}^*[\,\underline{b}^{(j)}\,] = \underline{v}_R{}^{+}[\, \underline{\varphi}^*[\,\underline{a}^{(j)}\,], \underline{\varphi}^*[\,\underline{a}^{(j+1)}\,] \,],$$
$$\underline{\varphi}^*[\,\underline{b}^{(j+1)}\,] = \underline{w}_R{}^{+}[\, \underline{\varphi}^*[\,\underline{a}^{(j)}\,], \underline{\varphi}^*[\,\underline{a}^{(j+1)}\,] \,],$$
$$\underline{\varphi}^*[\,\underline{b}^{(i)}\,] = \underline{\varphi}^*[\,\underline{a}^{(i)}\,] \text{ for } i \ne j, j+1 >$$

Since φ is an automorphism we also have a map φ^{-1} which satisfies

$$\underline{(\varphi^{-1})}^{(i)}[\, \underline{\varphi}^*[\,\underline{b}^{(1)}\,], \underline{\varphi}^*[\,\underline{b}^{(2)}\,], \dots, \underline{\varphi}^*[\,\underline{b}^{(m)}\,] \,] = \underline{b}^{(i)}$$

Hence we can transform the relations of G to

$$\underline{b}^{(i)} = \underline{(\varphi^{-1})}^{(i)}[\, \underline{\varphi}^*[\,\underline{a}^{(1)}\,], \dots, \underline{V}, \underline{W}, \dots, \underline{\varphi}^*[\,\underline{a}^{(m)}\,] \,]$$
$$\text{for } 1 \le i \le m \qquad \text{—— equations}(**)$$

where

$$\underline{V} = \underline{v}_R{}^{+}[\, \underline{\varphi}^*[\,\underline{a}^{(j)}\,], \underline{\varphi}^*[\,\underline{a}^{(j+1)}\,] \,],$$
$$\underline{W} = \underline{w}_R{}^{+}[\, \underline{\varphi}^*[\,\underline{a}^{(j)}\,], \underline{\varphi}^*[\,\underline{a}^{(j+1)}\,] \,],$$

and $\underline{V}$ and $\underline{W}$ are in the j^{th} and $j+1^{st}$ places in the bracket.

Write $\underline{A}^{(i)}$ for the right hand side of equation (**).

Then for the equations (**) to arise from the S-system of a link-diagram-representation S we require

<u>Conditions (7.1.2)</u>

$$\underline{A}^{(j)} = \underline{v}_S{}^{+}[\, \underline{a}^{(j)}, \underline{a}^{(j+1)} \,]$$
$$\underline{A}^{(j+1)} = \underline{w}_S{}^{+}[\, \underline{a}^{(j)}, \underline{a}^{(j+1)} \,]$$

and $\quad \underline{A}^{(i)} = \underline{a}^{(i)}$ if $i \ne j, j+1$ □

(§7.2) Theorem:

Let R be a link-diagram-representation which is invariant under moves of types II^o and III^o.

We can define another link-diagram-representation S given by

$$\underline{v}_S^+[\underline{a},\underline{b}] = \underline{b}$$

$$\underline{w}_S^+[\underline{a},\underline{b}] = \underline{v}_R^+[\underline{b},\underline{w}_R^+[\underline{a},\underline{x}_R[\underline{a},\underline{b}]]].$$

Then (i) S is invariant under moves of types II^o and III^o.

(ii) S is invariant under moves of type I^o iff R is.

(iii) If L is any link then $G_R(L) \cong G_S(L)$ □

The bulk of the proof will be the construction of the automorphisms Φ, and verifying that they satisfy conditions (7.1.2). That will also prove part (iii). Parts (i) and (ii) will then follow very easily.

So, suppose that R is as in the statement of the theorem.

Construction of the maps φ

As usual we define the maps in terms of the images of the words. We construct the maps φ_m and their inverses which we call θ_m together.

We can think of $F_{mn} \subset F_{(m+1)n}$ as the inclusion of

$$< \underline{x}^{(j)} ; 1 \leq j \leq m > \subset < \underline{x}^{(j)} ; 1 \leq j \leq m+1 >.$$

The maps φ_m and $\theta_m \in F_{mn}$ will be compatible with these inclusions.

That is to say

$$\varphi_m(x^{(j)}{}_i) = \varphi_{m'}(x^{(j)}{}_i) \text{ for all } m, m' \geq j \text{ (modulo an inclusion map)}$$

Thus we need to define $\varphi(x^{(j)}{}_i)$ as a word in the $x^{(r)}{}_k$ with $r \leq j$, and $1 \leq k \leq n$.

Also we can ignore the subscript m on φ_m.

$X^{(i)}$ and $V^{(i)}$ will be n-tuples of words in $(n+1)\times i$ letters.

Define $X^{(0)}[\underline{a}^{(1)}] = \underline{a}^{(1)}$.

For $i > 0$ define

$$X^{(i)}[\underline{a}^{(1)},\dots,\underline{a}^{(i+1)}] = \underline{x}_R[X^{(i-1)}[\underline{a}^{(1)},\dots,\underline{a}^{(i)}], X^{(i-1)}[\underline{a}^{(1)},\dots,\underline{a}^{(i-1)},\underline{a}^{(i+1)}]]$$

Define $V^{(0)}[\underline{a}^{(1)}] = \underline{a}^{(1)}$.

For $i > 0$ define

$$V^{(i)}[\underline{a}^{(1)},\dots,\underline{a}^{(i+1)}] = \underline{v}_R^+[\underline{a}^{(1)}, V^{(i-1)}[\underline{a}^{(2)},\dots,\underline{a}^{(i+1)}]].$$

Now we can define $\underline{\varphi}^{(i)} = X^{(i-1)}$ and $\underline{\theta}^{(i)} = V^{(i-1)}$.

For convenience write

$$\varphi^*[\underline{a}^{(i)}] = X^{(i-1)}[\underline{a}^{(1)},\dots,\underline{a}^{(i)}]$$

and $\quad \theta^*[\,\underline{a}^{(i)}\,]$ for $V^{(i-1)}[\,\underline{a}^{(1)},\ldots,\underline{a}^{(i)}]$.

First we show algebraically that φ and θ are inverses. This will involve repeated uses of the identities

$$\underline{b} = \underline{v}_R^{+}[\,\underline{a}\,,\underline{x}_R[\,\underline{a}\,,\underline{b}\,]\,]$$

and

$$\underline{b} = \underline{x}_R[\,\underline{a}\,,\underline{v}_R^{+}[\,\underline{a}\,,\underline{b}\,]\,]$$

(as introduced in chapter 3).

(If the reader wants to follow through a particular value of i in what follows, I can recommend $i = 3$).

To avoid confusion of the various x's around we regard $a^{(i)}{}_j$ as the standard generators of F_{mn}.

<u>Proposition</u> $\theta\,\varphi = \mathrm{id}$

<u>Proof</u> If $i = 1$ then $\theta\,\varphi\,(a^{(i)}{}_j) = \varphi\,(a^{(i)}{}_j) = a^{(i)}{}_j$

(Remember $\theta\,\varphi$ means apply θ then apply φ.)

Now let $i+1 > 1$.

Suppose $\theta\,\varphi\,(a^{(i)}{}_j) = a^{(i)}{}_j \;\forall\; 1 \le j \le n$

so $\quad a^{(i)}{}_j = \varphi\,(\,V^{(i-1)}{}_j\,)$

i.e. $\quad \underline{a}^{(i)} = V^{(i-1)}[\,\varphi^*[\,\underline{a}^{(1)}\,],\ldots,\ \varphi^*[\,\underline{a}^{(i)}\,]\,]$

$$= \underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(1)}\,],V^{(i-2)}[\,\varphi^*[\,\underline{a}^{(2)}\,],\ldots,\ \varphi^*[\,\underline{a}^{(i)}\,]\,]\,]$$

$$= \underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(1)}\,],\underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(2)}\,],\ldots,\underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(i-1)}\,],\varphi^*[\,\underline{a}^{(i)}\,]\,]\ldots]$$

$$\therefore\ \underline{a}^{(i)} = \underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(1)}\,],\underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(2)}\,],$$
$$\ldots,\underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(i-1)}\,],X^{(i-1)}[\,\underline{a}^{(1)},\ldots,\underline{a}^{(i-1)},\underline{a}^{(i)}]\,]\ldots]$$

We regard this as a word identity with $\underline{a}^{(i)}$ as the variable and as $\underline{a}^{(1)},\ldots,\underline{a}^{(i-1)}$ as fixed. In particular it will remain true if we replace $\underline{a}^{(i)}$ by $\underline{a}^{(i+1)}$.

Also $\quad \underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(i)}\,],\varphi^*[\,\underline{a}^{(i+1)}\,]\,] = \underline{v}_R^{+}[\,X^{(i-1)},X^{(i)}\,]$

$$= \underline{v}_R^{+}[\,X^{(i-1)},\underline{x}_R[\,X^{(i-1)},X^{(i-1)}[\,\underline{a}^{(1)},\ldots,\underline{a}^{(i-1)},\underline{a}^{(i+1)}]\,]\,]$$

$$= X^{(i-1)}[\,\underline{a}^{(1)},\ldots,\underline{a}^{(i-1)},\underline{a}^{(i+1)}].$$

Hence $\theta\,\varphi\,(a^{(i+1)}{}_j\,) = \varphi\,(\,V^{(i)}{}_j\,)$

$$= V^{(i)}{}_j[\,\varphi^*[\,\underline{a}^{(1)}\,],\ldots,\ \varphi^*[\,\underline{a}^{(i+1)}\,]\,]$$

$$= \underline{v}_R{}^{+}{}_j[\,\varphi^*[\,\underline{a}^{(1)}\,],\underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(2)}\,],\ldots,$$
$$\underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(i-1)}\,],\underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(i)}\,],\varphi^*[\,\underline{a}^{(i+1)}\,]\,]\,]\ldots]\,]$$

$$= \underline{v}_R{}^{+}{}_j[\,\varphi^*[\,\underline{a}^{(1)}\,],\underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(2)}\,],\ldots$$
$$\ldots,\underline{v}_R^{+}[\,\varphi^*[\,\underline{a}^{(i-1)}\,],X^{(i-1)}[\,\underline{a}^{(1)},\ldots,\underline{a}^{(i-1)},\underline{a}^{(i+1)}]\,]\ldots]\,]$$

$$= a^{(i+1)}{}_j$$

Hence we have proved the proposition by induction on i. $\square$

Proposition $\varphi\,\theta = id$

Proof If $i = 1$ then $\varphi\,\theta\,(a^{(i)}_j) = \theta\,(a^{(i)}_j) = a^{(i)}_j$.

Now let $i+1 > 1$

Suppose $\varphi\,\theta\,(a^{(i)}_j) = a^{(i)}_j$ for $1 \le j \le n$.

then $\underline{a}^{(i)} = X^{(i-1)}[\,\theta^*[\underline{a}^{(1)}], \ldots, \theta^*[\underline{a}^{(i)}]\,]$

$=X^{(i-1)}[\,\theta^*[\underline{a}^{(1)}], \ldots, \theta^*[\underline{a}^{(i-1)}], V^{(i-1)}[\underline{a}^{(1)}, \ldots, \underline{a}^{(i-1)}, \underline{a}^{(i)}]\,]$.

Regard this as an equation in $\underline{a}^{(i)}$ with $\underline{a}^{(1)}, \ldots, \underline{a}^{(i-1)}$ as constants. Thus it will remain true if we replace $\underline{a}^{(i)}$ by $\underline{v}_R{}^{+}[\underline{a}^{(i)}, \underline{a}^{(i+1)}]$.

Also notice that $V^{(i)} = V^{(i)}[\underline{a}^{(1)}, \ldots, \underline{a}^{(i)}, \underline{a}^{(i+1)}]$

$= V^{(i-1)}[\underline{a}^{(1)}, \ldots, \underline{a}^{(i-1)}, \underline{v}_R{}^{+}[\underline{a}^{(i)}, \underline{a}^{(i+1)}]\,]$

Hence $\varphi\,\theta\,(a^{(i+1)}_j) = \theta\,(X^{(i)}_j)$

$= X^{(i)}[\,\theta^*[\underline{a}^{(1)}], \ldots, \theta^*[\underline{a}^{(i+1)}]\,]$

$= x_{R\,j}[\,X^{(i-1)}[\,\theta^*[\underline{a}^{(1)}], \ldots, \theta^*[\underline{a}^{(i)}]\,],$
$X^{(i-1)}[\,\theta^*[\underline{a}^{(1)}], \ldots, \theta^*[\underline{a}^{(i-1)}], \theta^*[\underline{a}^{(i+1)}]\,]\,]$

$= x_{R\,j}[\,\underline{a}^{(i)},$
$X^{(i-1)}[\,\theta^*[\underline{a}^{(1)}], \ldots, \theta^*[\underline{a}^{(i-1)}], V^{(i-1)}[\underline{a}^{(1)}, \ldots, \underline{a}^{(i-1)}, \underline{v}_R{}^{+}[\underline{a}^{(i)}, \underline{a}^{(i+1)}]\,]\,]\,]$

$= x_{R\,j}[\,\underline{a}^{(i)}, \underline{v}_R{}^{+}[\underline{a}^{(i)}, \underline{a}^{(i+1)}]\,]$

$= a^{(i+1)}_j$

Hence we have proved the proposition by induction on i. $\square$

Next we want to check the conditions (7.1.2). Because of the form of the map φ we have chosen, the conditions are

(a) $V^{(i-1)}[\,\varphi^*[\underline{a}^{(1)}], \ldots, \varphi^*[\underline{a}^{(i)}]\,] = \underline{a}^{(i)}$ if $i < j$

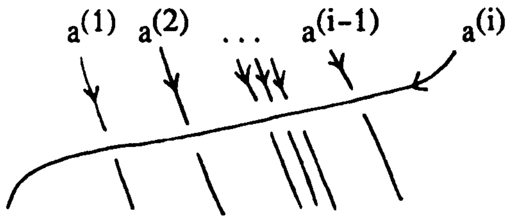
(b) $V^{(j-1)}[\,\varphi^*[\underline{a}^{(1)}], \ldots, \varphi^*[\underline{a}^{(j-1)}], \underline{v}_R{}^{+}[\,\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]\,]\,] = \underline{a}^{(j+1)}$

(c) $V^{(j)}[\,\varphi^*[\underline{a}^{(1)}], \ldots, \varphi^*[\underline{a}^{(j-1)}],$
$\underline{v}_R{}^{+}[\,\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]\,], \underline{w}_R{}^{+}[\,\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]\,]\,]$
$= \underline{v}_R{}^{+}[\,\underline{a}^{(j+1)}, \underline{w}_R{}^{+}[\,\underline{a}^{(j)}, \underline{x}_R[\underline{a}^{(j)}, \underline{a}^{(j+1)}]\,]\,]$

(d) $V^{(i-1)}[\,\varphi^*[\underline{a}^{(1)}], \ldots, \varphi^*[\underline{a}^{(j-1)}], \underline{v}_R{}^{+}[\,\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]\,],$
$\underline{w}_R{}^{+}[\,\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]\,], \varphi^*[\underline{a}^{(j+2)}], \ldots, \varphi^*[\underline{a}^{(i)}]\,]$
$= \underline{a}^{(i)}$ if $i > j+1$

We will not prove these equations algebraically, but rather diagramatically.

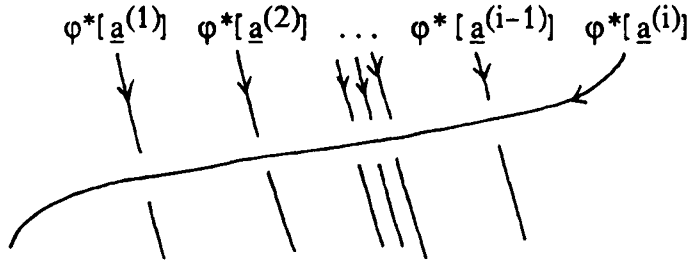
Consider the diagram L



Then in $G_R(L)$ we clearly have

$$\underline{A} = \underline{v}_R{}^+[\,\underline{a}^{(1)}, \underline{v}_R{}^+[\,\underline{a}^{(2)}, \underline{v}_R{}^+[\,\underline{a}^{(3)}, \ldots, \underline{v}_R{}^+[\,\underline{a}^{(i-1)}, \underline{a}^{(i)}\,] \ldots]]]$$
$$= V^{(i-1)}[\,\underline{a}^{(1)}, \underline{a}^{(2)}, \ldots, \underline{a}^{(i)}\,]$$
$$= \theta^*[\,\underline{a}^{(i)}\,]$$

Now by using the fact that θ and φ are inverses we can see that we could also label the diagram as



and now we find that $\underline{A} = \underline{a}^{(i)}$.

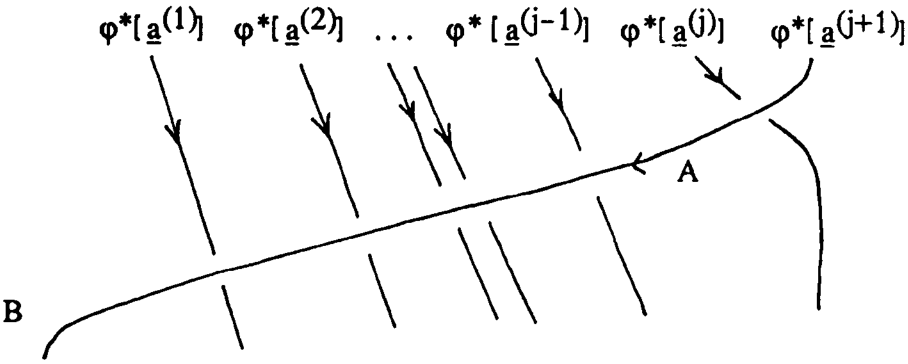
We shall use such labellings to prove (a), (b), (c) and (d)

(a) $\quad V^{(i-1)}[\,\varphi^*[\,\underline{a}^{(1)}\,], \ldots, \varphi^*[\,\underline{a}^{(i)}\,]\,] = \underline{a}^{(i)}$

This is in fact obvious from the definitions of θ and φ. Alternatively we can just read it off the diagrams above. The upper diagram says that to evaluate $V^{(i-1)}$, take the diagram and label the strings across the top by the arguments you wish to evaluate it on , and the answer can be read off as the label on A. The lower diagram is then the calculation, where all of the hard work has been done in the proof of the propositions.

(b) $\quad V^{(j-1)}[\,\varphi^*[\,\underline{a}^{(1)}\,], \ldots, \varphi^*[\,\underline{a}^{(j-1)}\,], \underline{v}_R{}^+[\,\varphi^*[\,\underline{a}^{(j)}\,], \varphi^*[\,\underline{a}^{(j+1)}\,]\,]\,] = \underline{a}^{(j+1)}$

Consider the diagram, again labelled in accordance with R



Now $\underline{A} = \underline{v}_R{}^+[\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]]$

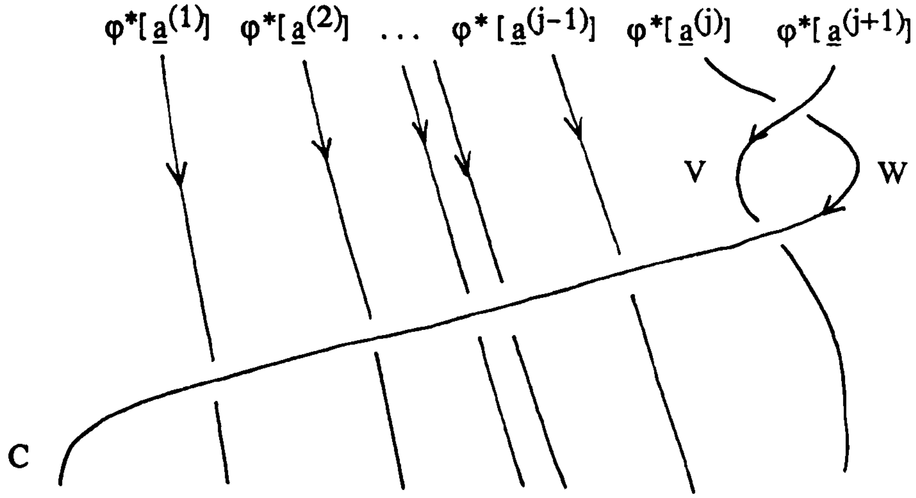
and $\underline{B} = V^{(j-1)}[\varphi^*[\underline{a}^{(1)}], \ldots, \varphi^*[\underline{a}^{(j-1)}], \underline{A}]$, thus the label on B is equal to the left hand side of the equation. But we can also look at it as

$\underline{B} = V^{(j)}[\varphi^*[\underline{a}^{(1)}], \ldots, \varphi^*[\underline{a}^{(j+1)}]] = \underline{a}^{(j+1)}$,

hence (b) follows from the diagram.

(c) $V^{(j)}[\varphi^*[\underline{a}^{(1)}], \ldots, \varphi^*[\underline{a}^{(j-1)}],$
$\underline{v}_R{}^+[\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]], \underline{w}_R{}^+[\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]]]$
$= \underline{v}_R{}^+[\underline{a}^{(j+1)}, \underline{w}_R{}^+[\underline{a}^{(j)}, \underline{x}_R[\underline{a}^{(j)}, \underline{a}^{(j+1)}]]]$

Consider the diagram L_c



Then $G_R(L_c)$ is freely generated by $\underline{a}^{(1)}, \ldots, \underline{a}^{(j+1)}$

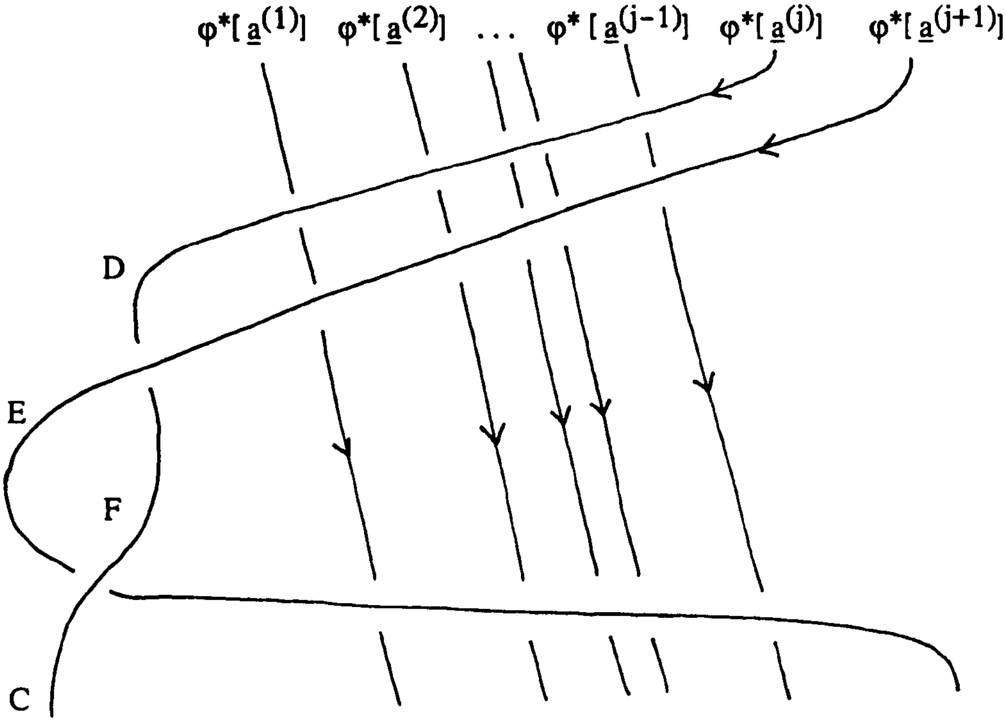
and $\underline{V} = \underline{v}_R{}^+[\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]]$

$\underline{W} = \underline{w}_R{}^+[\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]]$

so $\underline{C} = V^{(j)}[\varphi^*[\underline{a}^{(1)}], \ldots, \varphi^*[\underline{a}^{(j-1)}], \underline{V}, \underline{W}]$

which is just the left hand side of the equation (c).

Now we can use a sequence of moves of types II^o and III^o on L_c to obtain the diagram L_c'



Since R is invariant under moves of types II^o and III^o, and $G_R(L_c')$ is also freely generated by $\underline{a}^{(1)},\ldots,\underline{a}^{(j+1)}$, if we write $\underline{C}$ as words in $\underline{a}^{(1)},\ldots,\underline{a}^{(j+1)}$ these words must be the same as they were in L_c.

$\underline{D} = \underline{a}^{(j)}$

$\underline{E} = \underline{a}^{(j+1)}$

$\underline{F} = \underline{w}_R^{+}[\underline{D}, \underline{x}_R[\underline{D}, \underline{E}]]$

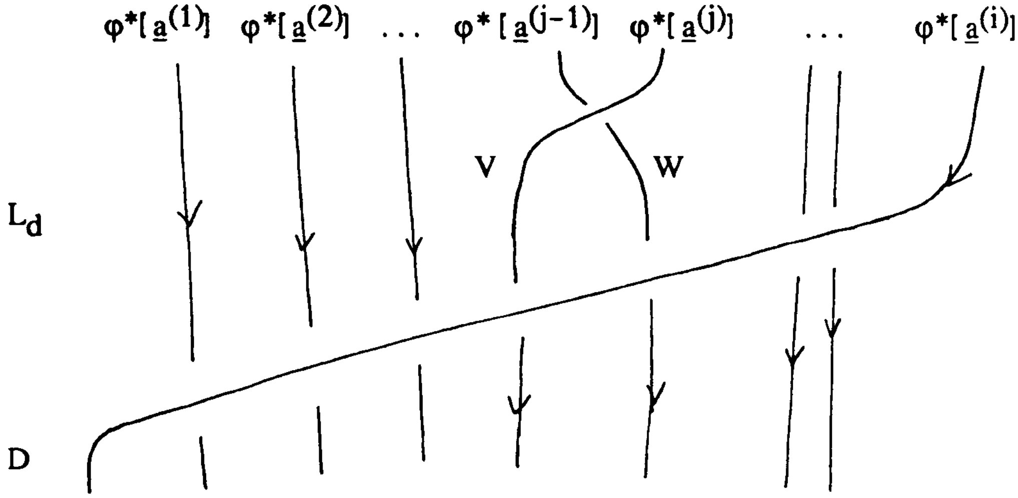
and $\underline{C} = \underline{v}_R^{+}[\underline{E}, \underline{F}]$

hence $\underline{C} = \underline{v}_R^{+}[\underline{a}^{(j+1)}, \underline{w}_R^{+}[\underline{a}^{(j)}, \underline{x}_R[\underline{a}^{(j)}, \underline{a}^{(j+1)}]]]$

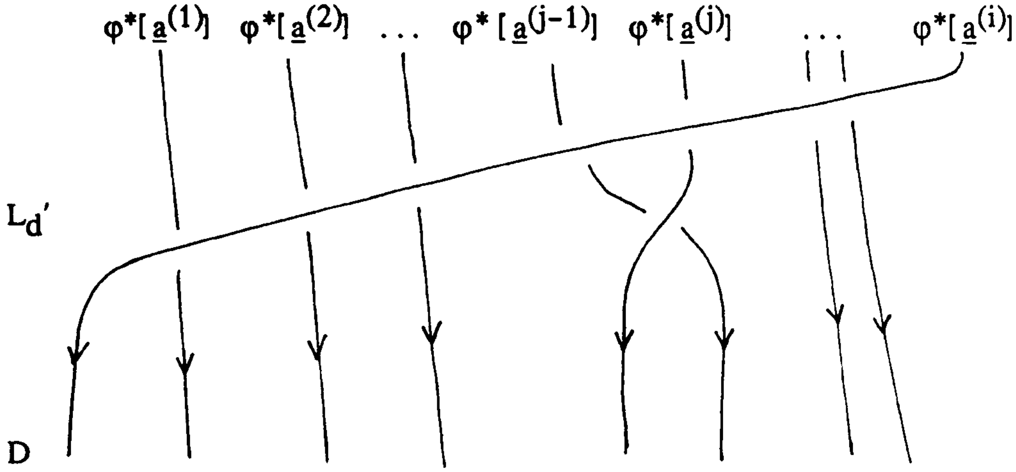
which is as required and we have proved (c)

(d) $$V^{(i-1)}[\varphi^*[\underline{a}^{(1)}],\ldots,\varphi^*[\underline{a}^{(j-1)}], \underline{v}_R^{+}[\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]], \underline{w}_R^{+}[\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]], \varphi^*[\underline{a}^{(j+2)}],\ldots,\varphi^*[\underline{a}^{(i)}]] = \underline{a}^{(i)} \text{ if } i > j+1$$

Consider the two diagrams



and



Then L_d and L_d' are related by a single move of type III^o hence have isomorphic R- systems, and $G_R(L_d)$ and $G_R(L_d')$ are freely generated by $\underline{a}^{(1)},\dots,\underline{a}^{(i)}$ so, as in the proof of (c), from diagram L_d

$$\underline{D} = V^{(i-1)}[\varphi^*[\underline{a}^{(1)}], \dots, \varphi^*[\underline{a}^{(j-1)}], \underline{v}_R{}^+[\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]], \underline{w}_R{}^+[\varphi^*[\underline{a}^{(j)}], \varphi^*[\underline{a}^{(j+1)}]], \varphi^*[\underline{a}^{(j+2)}], \dots, \varphi^*[\underline{a}^{(i)}]]$$

and from diagram L_d'

$$\underline{D} = \underline{a}^{(i)}$$

This proves (d).

Thus we have shown φ satisfies conditions (7.1.2) for S given as in the statement of the theorem by

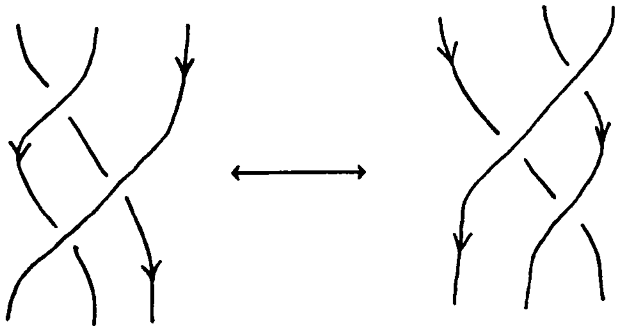
$$\underline{v}_S{}^+[\underline{a},\underline{b}] = \underline{b}$$

$$\underline{w}_S{}^+[\underline{a},\underline{b}] = \underline{v}_R{}^+[\underline{b},\underline{w}_R{}^+[\underline{a},\underline{x}_R[\underline{a},\underline{b}]]]$$

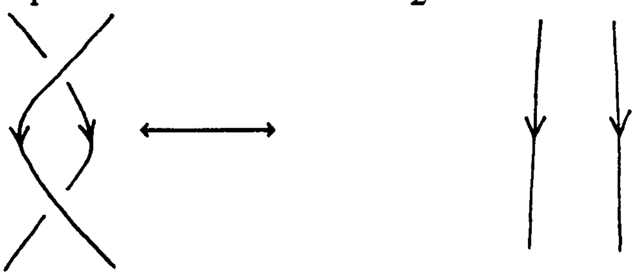
To prove the theorem it now remains to check results (i), (ii), and (iii).

(i) Invariance of S under

L_1 L_2



or under L_1 L_2



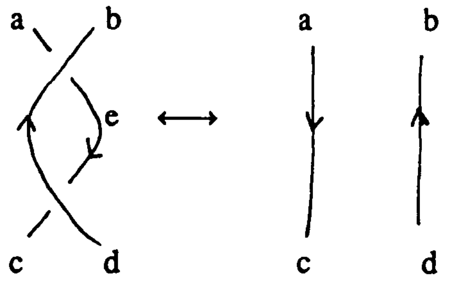
follows from the commutative diagram

$$\begin{tikzcd}
F_{mn} \times F_{mn} \arrow[r] & G_S(L_1) \\
F_{mn} \times F_{mn} \arrow[r] \arrow[u, "\varphi\times\varphi"'] \arrow[d, equal] & G_R(L_1) \arrow[u] \arrow[d, leftrightarrow] \\
F_{mn} \times F_{mn} \arrow[r] \arrow[d, "\varphi\times\varphi"'] & G_R(L_2) \arrow[d] \\
F_{mn} \times F_{mn} \arrow[r] & G_S(L_2)
\end{tikzcd}$$

where the right hand vertical maps we have just shown to be isomorphisms. Hence S is invariant under these moves.

That leaves invariance under

L_1



L_2

which is easy since

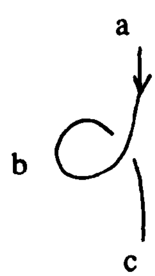
$\text{sys}_S(L_2)$ is $< \underline{a}, \underline{b}, \underline{c}, \underline{d} > \longrightarrow < \underline{a}, \underline{b}, \underline{c}, \underline{d} \mid \underline{a} = \underline{c}, \underline{b} = \underline{d} >$

and $\text{sys}_S(L_1)$ is $< \underline{a}, \underline{b}, \underline{c}, \underline{d} > \longrightarrow < \underline{a}, \underline{b}, \underline{c}, \underline{d}, \underline{e} \mid \underline{b} = \underline{d},\ \underline{e} = \underline{w_S}^{-}[\underline{a}, \underline{b}], \underline{c} = \underline{w_S}^{+}[\underline{e}, \underline{b}] >$
which is $< \underline{a}, \underline{b}, \underline{c}, \underline{d} > \longrightarrow < \underline{a}, \underline{b}, \underline{c}, \underline{d} \mid \underline{a} = \underline{c}, \underline{b} = \underline{d} >$

(ii) Consider

L_1

and L_2



Then $\text{sys}_R(L_1) \cong \text{sys}_S(L_1)$ and is given by

$< \underline{a}, \underline{c} > \longrightarrow < \underline{a}, \underline{c} \mid \underline{a} = \underline{c} >$

$\text{sys}_S(L_2)$ is

$< \underline{a}, \underline{b}, \underline{c} > \longrightarrow < \underline{a}, \underline{b}, \underline{c} \mid \underline{a} = \underline{b}, \underline{c} = \underline{w_S}^{+}[\underline{b}, \underline{b}] >$

$\text{sys}_R(L_2)$ is

$< \underline{a}, \underline{b}, \underline{c} > \longrightarrow < \underline{a}, \underline{b}, \underline{c} \mid \underline{a} = \underline{x_R}[\underline{b}, \underline{b}], \underline{c} = \underline{w_R}^{+}[\underline{b}, \underline{x_R}[\underline{b}, \underline{b}]] >$

so $\text{sys}_R(L_1) \cong \text{sys}_R(L_2)$

$\Leftrightarrow$ $\quad \underline{x_R}[\underline{b}, \underline{b}] = \underline{w_R}^{+}[\underline{b}, \underline{x_R}[\underline{b}, \underline{b}]]$

$\Leftrightarrow$ $\quad \underline{b} = \underline{v_R}^{+}[\underline{b}, \underline{w_R}^{+}[\underline{b}, \underline{x_R}[\underline{b}, \underline{b}]]]$

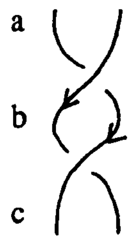
$\Leftrightarrow$ $\quad \underline{b} = \underline{w_S}^{+}[\underline{b}, \underline{b}]$

$\Leftrightarrow$ $\quad \text{sys}_S(L_1) \cong \text{sys}_S(L_2)$

(iii) follows from basic idea (7.1.1), hence that completes the proof of the theorem. □

Remark 1

Let L be given by

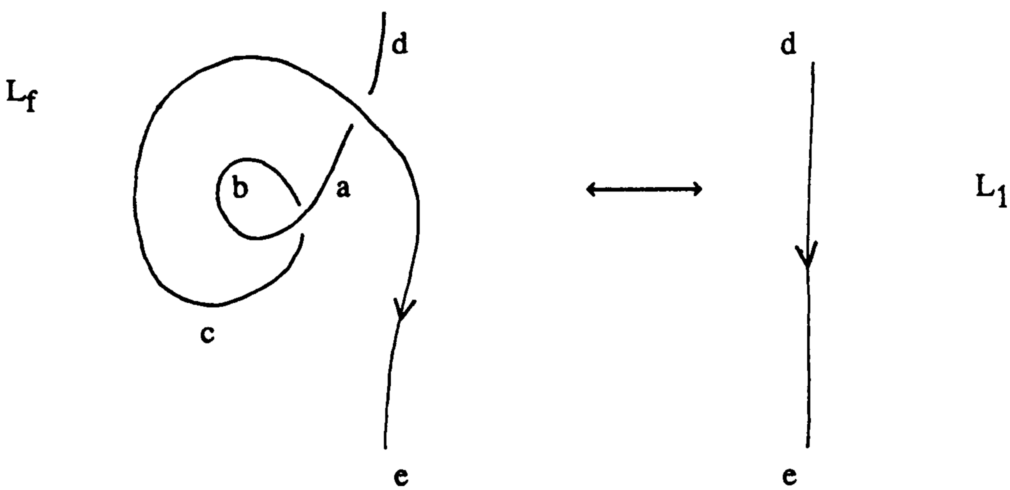


Then $G_R(L)$ is freely generated by $\underline{a}$ and $\underline{b}$ and in $G_R(L)$ we have

$$\underline{c} = \underline{v}_R{}^{+}[\,\underline{b}\,,\ \underline{w}_R{}^{+}[\,\underline{a}\,,\underline{x}_R[\,\underline{a}\,,\underline{b}\,]\,]\,]$$

i.e. $\underline{c} = \underline{w}_S{}^{+}[\,\underline{a}\,,\underline{b}\,]$

Remark 2 In chapter 3 we deferred the proof of lemma (3.5.6), that any link-diagram-representation which is invariant under moves of types II^o and III^o is also invariant under moves of type I^f, although we did prove it in the case where $\underline{v}_R[\,\underline{a}\,,\underline{b}\,] = \underline{b}$. We now prove the more general case. Draw move I^f as



then $sys_R(L_f)$ is given by

$$\langle\,\underline{d}\,,\underline{e}\,\rangle \longrightarrow \langle\,\underline{a}, \underline{b}, \underline{c}, \underline{d}, \underline{e} \mid \underline{b} = \underline{v}_R{}^{+}[\,\underline{b}\,,\underline{a}\,]\,,\ \underline{c} = \underline{w}_R{}^{+}[\,\underline{b}\,,\underline{a}\,]\,,$$
$$\underline{c} = \underline{v}_R{}^{+}[\,\underline{a}\,,\underline{e}\,]\,,\ \underline{d} = \underline{w}_R{}^{+}[\,\underline{a}\,,\underline{e}\,]\,\rangle$$

Now let $\underline{B} = \underline{b}$, $\underline{C} = \underline{v}_R{}^{+}[\,\underline{b}\,,\underline{c}\,]$, $\underline{A} = \underline{v}_R{}^{+}[\,\underline{b}\,,\underline{a}\,]$,

$\underline{D} = \underline{v}_R{}^{+}[\,\underline{b}\,,\underline{v}_R{}^{+}[\,\underline{c}\,,\underline{d}\,]\,]$ and

$\underline{E} = \underline{v}_R{}^{+}[\,\underline{b}\,,\underline{v}_R{}^{+}[\,\underline{a}\,,\underline{e}\,]\,] = \underline{v}_R{}^{+}[\,\underline{b}\,,\underline{v}_R{}^{+}[\,\underline{c}\,,\underline{e}\,]\,]$.

We can, by the proof of the theorem, write the relations in $G_R(L_f)$ as

$$\underline{A} = \underline{B}\ , \underline{C} = \underline{E}\ , \underline{C} = \underline{w}_S{}^+[\underline{B}\ , \underline{B}\ ]\ , \underline{D} = \underline{w}_S{}^+[\underline{A}\ , \underline{E}\ ]\ ,$$

so $\quad \underline{D} = \underline{w}_S{}^+[\underline{B}\ , \underline{w}_S{}^+[\underline{B}\ , \underline{B}\ ]] = \underline{w}_S{}^+[\underline{B}\ , \underline{B}\ ]$ (from the proof of lemma (3.5.6))

so $\quad \underline{D} = \underline{E}$

so $\quad \underline{d} = \underline{e}$

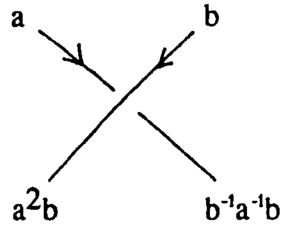
and $sys_R(L_f) \cong sys_R(L_1)$ as claimed.

An "elementary" proof in chapter 3 would have required us to invent the expressions for $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ and $\underline{E}$, prove that the two expressions for $\underline{E}$ are equal and a proof that $\underline{D} = \underline{w}_S{}^+[\underline{A}\ , \underline{E}\ ]$. All of these are embedded in the proof of the theorem.

(§7.3) Examples, etc.

(7.3.1) (From the motivation at the start of the chapter)

R is given by



so $\quad v_R{}^+[a, b] = a^2b$

$w_R{}^+[a, b] = b^{-1}a^{-1}b$

$x_R[a, b] = a^{-2}b$

Then $\quad w_S{}^+[a, b] = v_R{}^+[b, w_R{}^+[a, x_R[a, b]]]$

$= b^2\, w_R{}^+[a, a^{-2}b]$

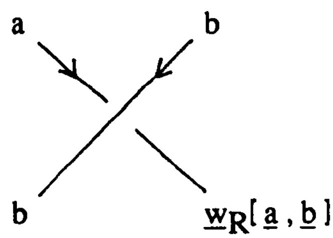
$= b^2\, b^{-1}\, a^2\, a^{-1}\, a^{-2}\, b$

$= b\, a^{-1}\, b$

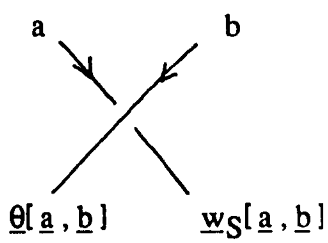
Thus the examples R and S given at the start of the chapter do give $G_R(L) \cong G_S(L)\ \forall$ links L $\quad\square$

Next we produce a converse to the theorem.

Suppose R (invariant under moves II^{o} and III^{o}) is given by



We try to find words $\underline{\theta}$ such that we can find words $\underline{w}_S[\underline{a}, \underline{b}]$ such that if S is given by



then $G_R(L) \cong G_S(L)\ \forall$ links L.

Lemma (7.3.2) Suppose $\underline{\theta}$ is a collection of words such that $< \underline{b} > \subseteq < \underline{\theta}[\underline{a},\underline{b}],\underline{a} >$, and let $\underline{\varphi}$ be the words defined by

$\underline{\theta}[\underline{a},\underline{\varphi}[\underline{a},\underline{b}]] = \underline{b}$

and $\underline{\varphi}[\underline{a},\underline{\theta}[\underline{a},\underline{b}]] = \underline{b}$

Define S by $\underline{v}_S{}^+[\underline{a},\underline{b}] = \underline{\theta}[\underline{a},\underline{b}]$

$\underline{w}_S{}^+[\underline{a},\underline{b}] = \underline{\varphi}[\underline{\theta}[\underline{a},\underline{b}],\underline{w}_R[\underline{a},\underline{\theta}[\underline{a},\underline{b}]]]$

then

(i) $< \underline{a} > \subseteq < \underline{b},\underline{w}_S{}^+[\underline{a},\underline{b}] >$

$\Rightarrow$ S is invariant under moves of type II^o.

(ii) If also $\underline{w}_R[\underline{\theta}[\underline{a},\underline{b}],\underline{\theta}[\underline{a},\underline{c}]] = \underline{\theta}[\underline{a},\underline{w}_R[\underline{b},\underline{c}]]$

and $\underline{\theta}[\underline{a},\underline{\theta}[\underline{b},\underline{c}]] = \underline{\theta}[\underline{\theta}[\underline{a},\underline{b}],\underline{\theta}[\underline{w}_S{}^+[\underline{a},\underline{b}],\underline{c}]]$

then

S is invariant under moves of type III^o

and $G_R(L) \cong G_S(L)\ \forall$ links L.

Proof

For part (i) we use lemma (3.5.1).

Two of the conditions in the lemma are

$< \underline{a} > \subseteq < \underline{b},\underline{w}_S{}^+[\underline{a},\underline{b}] >$

and $< \underline{b} > \subseteq < \underline{a},\underline{v}_S{}^+[\underline{a},\underline{b}] >$

and these are given in the statement of this lemma. The final check is that

$$\begin{aligned}< \underline{w}_S{}^+[\underline{a},\underline{b}],\underline{v}_S{}^+[\underline{a},\underline{b}] > &= < \underline{\varphi}[\underline{\theta}[\underline{a},\underline{b}],\underline{w}_R[\underline{a},\underline{\theta}[\underline{a},\underline{b}]]],\underline{\theta}[\underline{a},\underline{b}] > \\ &= < \underline{\theta}[\underline{a},\underline{b}],\underline{w}_R[\underline{a},\underline{\theta}[\underline{a},\underline{b}]] > \\ &= < \underline{\theta}[\underline{a},\underline{b}],\underline{a} > \text{ by lemma (3.5.1) for R} \\ &= < \underline{a},\underline{b} >.\end{aligned}$$

Part (ii):

If S is invariant under moves of type III^o then we can apply theorem (7.2) to it, and $G_R(L) \cong G_S(L)$ will follow from the fact that $\underline{x}_S[\underline{a},\underline{b}] = \underline{\varphi}[\underline{a},\underline{b}]$, which implies that

$$\begin{aligned}\underline{v}_S{}^+[\underline{b},\underline{w}_S{}^+[\underline{a},\underline{x}_S[\underline{a},\underline{b}]]] &= \underline{\theta}[\underline{b},\underline{\varphi}[\underline{\theta}[\underline{a},\underline{\varphi}[\underline{a},\underline{b}]],\underline{w}_R[\underline{a},\underline{\theta}[\underline{a},\underline{\varphi}[\underline{a},\underline{b}]]]]] \\ &= \underline{\theta}[\underline{b},\underline{\varphi}[\underline{b},\underline{w}_R[\underline{a},\underline{b}]]] \\ &= \underline{w}_R[\underline{a},\underline{b}].\end{aligned}$$

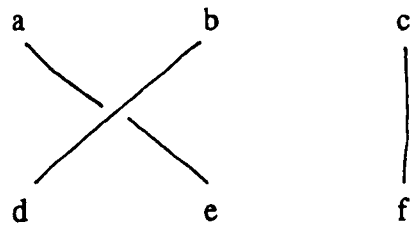
Define a map $T : F_{3n} \longrightarrow F_{3n}$ by

$\underline{T}^{(1)}[\underline{a},\underline{b},\underline{c}] = \underline{a}$

$\underline{T}^{(2)}[\underline{a},\underline{b},\underline{c}] = \underline{\theta}[\underline{a},\underline{b}]$

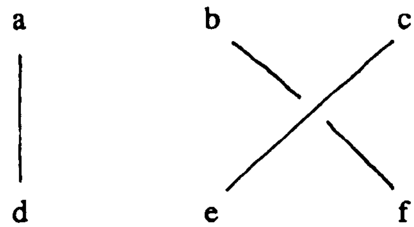
$\underline{T}^{(3)}[\underline{a},\underline{b},\underline{c}] = \underline{\theta}[\underline{a},\underline{\theta}[\underline{b},\underline{c}]]$

Let σ_1 be



and

Let σ_2 be



Then we can follow the proof of the theorem, and in particular of part (i), to say that to prove invariance of S under moves of type $\mathrm{III}^{\mathrm{o}}$ we need only check that T carries the relations of $G_R(\sigma_1)$ to those of $G_S(\sigma_1)$ and those of $G_R(\sigma_2)$ to those of $G_S(\sigma_2)$. If we can do this then we can construct the commutative diagram

$$\begin{tikzcd}
F_{3n} \times F_{3n} \arrow[r] & G_S(L_1) \\
F_{3n} \times F_{3n} \arrow[r] \arrow[u, "T \times T"] \arrow[d, equal] & G_R(L_1) \arrow[u] \arrow[d, leftrightarrow] \\
F_{3n} \times F_{3n} \arrow[r] \arrow[d, "T \times T"] & G_R(L_2) \arrow[d] \\
F_{3n} \times F_{3n} \arrow[r] & G_S(L_2)
\end{tikzcd}$$

where $L_1 = \sigma_1\sigma_2\sigma_1$ and $L_2 = \sigma_2\sigma_1\sigma_2$.

The relations in $G_R(\sigma_1)$ are $_N\langle\, \underline{d} = \underline{b} \,,\, \underline{e} = \underline{w}_R[\,\underline{a}\,,\underline{b}\,] \,,\, \underline{c} = \underline{f}\,\rangle$.

Apply TxT and we obtain

$$_N\langle\, \underline{d} = \underline{\theta}[\,\underline{a}\,,\underline{b}\,]\,,\,\underline{\theta}[\,\underline{d}\,,\underline{e}\,] = \underline{w}_R[\,\underline{a}\,,\underline{\theta}[\,\underline{a}\,,\underline{b}\,]]\,,$$
$$\underline{\theta}[\,\underline{a}\,,\underline{\theta}[\,\underline{b}\,,\underline{c}\,]] = \underline{\theta}[\,\underline{d}\,,\underline{\theta}[\,\underline{e}\,,\underline{f}\,]]\,\rangle$$

$$= N\langle \underline{d} = \underline{\theta}[\underline{a},\underline{b}], \underline{e} = \underline{\varphi}[\underline{d}, \underline{w}_R[\underline{a}, \underline{\theta}[\underline{a},\underline{b}]]], \\ \underline{\theta}[\underline{a},\underline{\theta}[\underline{b},\underline{c}]] = \underline{\theta}[\underline{d},\underline{\theta}[\underline{e},\underline{f}]]\rangle$$

(by the definition of $\underline{\varphi}$)

$$= N\langle \underline{d} = \underline{\theta}[\underline{a},\underline{b}], \underline{e} = \underline{\varphi}[\underline{\theta}[\underline{a},\underline{b}], \underline{w}_R[\underline{a}, \underline{\theta}[\underline{a},\underline{b}]]], \\ \underline{\theta}[\underline{a},\underline{\theta}[\underline{b},\underline{c}]] = \underline{\theta}[\underline{\theta}[\underline{a},\underline{b}],\underline{\theta}[\underline{e},\underline{f}]]\rangle$$

(substitute for $\underline{d}$)

$$= N\langle \underline{d} = \underline{\theta}[\underline{a},\underline{b}], \underline{e} = \underline{w}_S[\underline{a},\underline{b}], \\ \underline{\theta}[\underline{a},\underline{\theta}[\underline{b},\underline{c}]] = \underline{\theta}[\underline{\theta}[\underline{a},\underline{b}],\underline{\theta}[\underline{w}_S[\underline{a},\underline{b}],\underline{f}]]\rangle$$

(substitute for $\underline{e}$ and definition of S)

$$= N\langle \underline{d} = \underline{\theta}[\underline{a},\underline{b}], \underline{e} = \underline{w}_S[\underline{a},\underline{b}], \\ \underline{\theta}[\underline{a},\underline{\theta}[\underline{b},\underline{c}]] = \underline{\theta}[\underline{a},\underline{\theta}[\underline{b},\underline{f}]]\rangle$$

(by assumption for the lemma)

$$= N\langle \underline{d} = \underline{\theta}[\underline{a},\underline{b}], \underline{e} = \underline{w}_S[\underline{a},\underline{b}],\ \underline{c} = \underline{f}\rangle$$

which are the relations in $G_S(\sigma_1)$.

The relations in $G_R(\sigma_2)$ are $N\langle \underline{d} = \underline{a}, \underline{e} = \underline{c}, \underline{f} = \underline{w}_R[\underline{b},\underline{c}]\rangle$.

Apply T×T and we obtain

$$N\langle \underline{d} = \underline{a}, \underline{\theta}[\underline{d},\underline{e}] = \underline{\theta}[\underline{a},\underline{\theta}[\underline{b},\underline{c}]], \\ \underline{\theta}[\underline{d},\underline{\theta}[\underline{e},\underline{f}]] = \underline{w}_R[\underline{\theta}[\underline{a},\underline{b}],\underline{\theta}[\underline{a},\underline{\theta}[\underline{b},\underline{c}]]\rangle$$

$$= N\langle \underline{d} = \underline{a},\ \underline{e} = \underline{\theta}[\underline{b},\underline{c}], \\ \underline{\theta}[\underline{a},\underline{\theta}[\underline{e},\underline{f}]] = \underline{\theta}[\underline{a},\underline{w}_R[\underline{b},\underline{\theta}[\underline{b},\underline{c}]]]\rangle$$

(by assumption for the lemma)

$$= N\langle \underline{d} = \underline{a},\ \underline{e} = \underline{\theta}[\underline{b},\underline{c}], \\ \underline{\theta}[\underline{e},\underline{f}] = \underline{w}_R[\underline{b},\underline{\theta}[\underline{b},\underline{c}]]\rangle$$

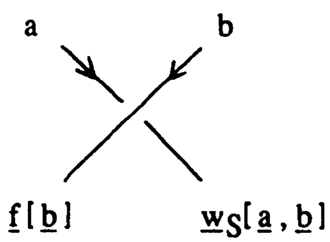
$$= = N\langle \underline{d} = \underline{a},\ \underline{e} = \underline{\theta}[\underline{b},\underline{c}],\ \underline{f} = \underline{w}_S[\underline{b},\underline{c}]\rangle$$

(by definition of S)

which are the relations of $G_S(\sigma_2)$.

That completes the proof of the lemma. □

If we look for examples



then the conditions in the lemma simplify:

Corol (7.3.3) Let $f \in Aut(F_n)$, and let R be a link diagram representation of dimension n which is invariant under moves of types II^o and III^o.

Suppose $R^f = R$.

Define $\underline{v}_S{}^{+}[\underline{a}, \underline{b}] = \underline{f}[\underline{b}]$

$\underline{w}_S{}^{+}[\underline{a}, \underline{b}] = \underline{f}^{-1}[\underline{w}_R{}^{+}[\underline{a}, \underline{f}[\underline{b}]]]$

then

S is invariant under moves of types II^o and III^o and $G_R(L) \cong G_S(L)\ \forall$ links L.

Proof

Just replace $\underline{\theta}[\underline{a}, \underline{b}]$ by $\underline{f}[\underline{b}]$ in lemma (7.3.2) □

Recall that at the end of chapter 5 we indicated that those f such that $R^f = R$ would take part in another action. Corol(7.3.3) defines this action. Thus we can recall example (5.2.4):

Example(7.3.4) Define R, a two dimensional quandle representation by

$\underline{w}^{+}[a, b, c, d] = (a\, b^r d^{-r}, d^r b\, d^{-r})$

and $f \in Aut(F_2)$ by

$\underline{f}[a, b] = (a\, b^t, b)$

then $R^f = R$.

Hence if we define S by

$\underline{v}_S{}^{+}[a, b, c, d] = (c\, d^t, d)$

$\underline{w}_S{}^{+}[a, b, c, d] = (a\, b^{r-t} d^{-r}, d^r b\, d^{-r})$

then S is a link representation and $G_R(L) \cong G_S(L)\ \forall$ links L.

Example(7.3.5) Let R be a one dimensional quandle representation given by

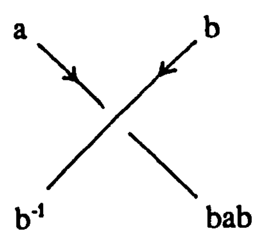
$w_R[a, b] = b\, a^{-1} b$.

Let $\theta[x] = x^{-1}$.

Then $R^\theta = R$, and we get

$w_S[a, b] = (b^{-1} a^{-1} b^{-1})^{-1} = b\, a\, b$

i.e. S is given by



Thus we finally have enough isomorphisms to prove the claim in §4.1. All the groups (for a particular link) which can be obtained from a one dimensional link representation can be obtained from

$\#5_t, t \geq 0:$ $\quad w[a, b] = b^{-t} a b^t$

$\#7$ $\quad w[a, b] = b a^{-1} b.$

(In both cases $v[a, b] = b$, of course.)

Chapter 8 - Two dimensional quandle representations

The examples in this chapter are the main motivation for the rest of the thesis. The results on distinguishing the invariants and on their properties have been chosen to be sufficient for the examples in this chapter.

First we look at some obvious examples which arise from the one dimensional examples.

Then we look at examples produced from a computer search, identifying those which occurred in the previous section, and ensuring that they behave as one would expect from the previous chapters by calculating some invariants for a few links.

(§8.1) ... from one dimensional examples.

The most obvious two dimensional quandle representations are those we can write down from one dimensional examples. We give two collections of examples of these, and in each case it is easy to see which examples are conjugate.

The examples are for the most part the ones which we wish to get out of the way before getting on with the more interesting ones in the next section.

The first of the collections of examples is obtained just by taking the free product of two one dimensional examples, in the following sense:

Let R and S be (group)-quandle-representations, of dimensions m and n respectively. Define R∗S, a quandle representation of dimension m+n by

$$\underline{w}_{R*S}[\underline{a}, \underline{b}] = (\underline{w}_R[a_1,...,a_n, b_1,...,b_n], \underline{w}_S[a_{n+1},...,a_{n+m}, b_{n+1},...,b_{n+m}])$$

where $\underline{a}$ and $\underline{b}$ are (m+n)-tuples. It is easy to see that R∗S is a quandle representation, and that $G_{R*S}(L) \cong G_R(L) * G_S(L)$

We can alternatively think of R∗S as arising from a group invariant of a two coloured link.

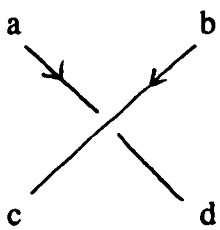
Let a two coloured link be a link whose components are divided into two sets, called Red and Blue. Project to obtain a two-coloured-link-diagram.

Define a two-coloured-link-diagram-representation by taking generators associated with subarcs (or with arcs in the case of a two coloured quandle representation), taking (say) m generators for a Red subarc, and n generators for each Blue subarc; and relations associated with crossings. We will be prepared to recognise (up to) 8 types of crossings, according to whether the crossing is positive or negative, and according to the colours of the two components involved.

We can then pass through two coloured Reidemeister's theorem to define a two coloured link representation.

R∗S arises as a two coloured representation as follows:

Given a crossing



if the components are of different colours then take the relations $\underline{c} = \underline{b}$ and $\underline{a} = \underline{d}$;

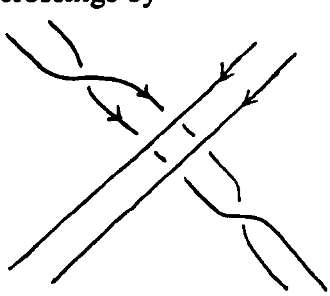
if the components are both Red then take the relations on $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ as for R, and if the components are both Blue then take the relationships arising as for S. It is easy to check from the two-coloured-Reidemeister that this gives a two-coloured-link-representation.

Given a link L, we can define a two coloured link L^*, consisting of L and a parallel to each component of L, where we colour the two components differently.

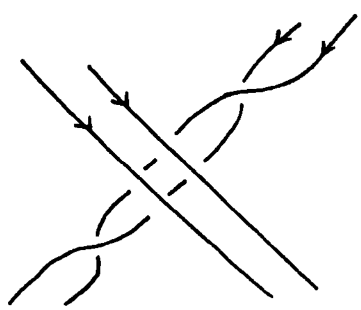
If we evaluate the two coloured link representation above on L^*, this is the same as evaluating $R*S$ on L. We did not specify a framing for the parallel copies to L which give us L^* since they make no difference to the calculation of the invariant, as changing the framing can be done just by introducing extra crossings between the red and blue copies.

The other construction given in this section is also obtained as an invariant for the two coloured parallel link L^*, although this time the framing will be specified. Also this time R and S will be of the same dimension. (Dimension 1 in the cases we are interested in .)

To specify the framing on L^*, we draw a projection of L^* from one of L by replacing positive crossings by



and negative crossings by

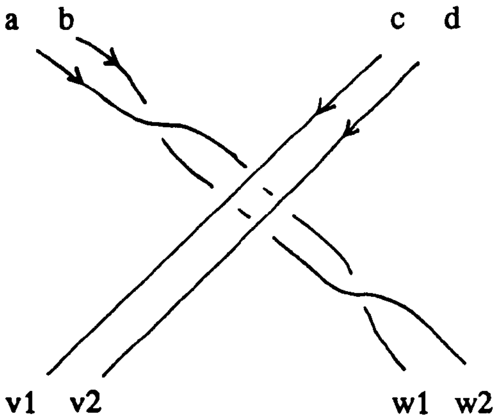


and colouring the left hand copy of an arc Blue, and the right hand one Red.

For a link L this defines a particular two coloured parallel L^* . The framing on the chosen parallel for a component l of L is then equal to the sum of the linking numbers of other components of L with l.

Given R and S a pair of link diagram representations we try to define a two coloured link representation as follows. Take n generators for each subarc. Think of there being only four types of crossings by ignoring the colour of the string passing underneath. Then introduce the relations as for R if the top string is Red and as for S if the top string is Blue.

This will not always give a two coloured link representation, but when it does call it R×S. We can then also define a link representation of dimension 2n by looking at the following diagram, L_+ in which the components at a and c are Red, and those at b and d are Blue.



L_+

Then we can write $\underline{v1}$, $\underline{v2}$, $\underline{w1}$, $\underline{w2}$ as words in $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ in $G_{R\times S}(L_+)$.

Take $\underline{v}_{R\times S}[\,\underline{a}\,,\underline{b}\,,\underline{c}\,,\underline{d}\,] = (\,v1\,,v2\,)$ and

$\underline{w}_{R\times S}[\,\underline{a}\,,\underline{b}\,,\underline{c}\,,\underline{d}\,] = (\,w1\,,w2\,)$

where we are writing for R×S for the 2n-dimensional link representation as well as for the 2-coloured-link-representation. Then it is clear that

$G_{R\times S}(L) \cong G_{R\times S}(L^*)$.

When n=1 we have a list of the quandle representations (in §4.1)

Let $R(\,t\,)$ be given by $w[\,a, b\,] = b^t\, a\, b^{-t}$

and S be given by $w[\,a, b\,] = b\, a^{-1}\, b$

It is easy to check that R(t)×R(x) defines a quandle representation $\forall$ t, x $\in \mathbb{Z}$, and so do R(0)×S and S×S. (Also S×R(0), but this is conjugate to R(0)×S via a⟶b, b⟶a, so we ignore it.)

Explicitly these examples are given by words as follows:

<u>S×S</u> $w1 = d\,c^{-1}\,a\,b^{-1}\,a\,b^{-1}\,a\,c^{-1}\,d$

$w2 = d\,c^{-1}\,a\,b^{-1}\,a\,c^{-1}\,d$

$\underline{R(0)\times S}$ $\quad w1 = c\,a^{-1}\,c$

$\quad w2 = c\,a^{-1}\,b\,a^{-1}\,c$

$\underline{R(t)\times R(x)}$ $\quad w1 = d^t\,c^x\,a^{-x}\,b^{-t}\,a\,b^t\,a^x\,c^{-x}\,d^{-t}$

$\quad w2 = d^t\,c^x\,a^{-x}\,b\,a^x\,c^{-x}\,d^{-t}$

We now consider the question of conjugacy amongst the elements of R(t)×R(x). It will be easiest to look at this question in a larger class of examples:

Example (8.1.1)

Let $\alpha \in F_n$.

We define an n-dimensional quandle representation A = conj(α) by

$$w_i[\underline{a}, \underline{b}] = \alpha[\underline{b}](\alpha[\underline{a}])^{-1} a_i\ \alpha[\underline{a}](\alpha[\underline{b}])^{-1} \qquad \text{for each } 1 \leq i \leq n$$

It is easy to see that this is a quandle representation since

$$w_i[\underline{a}, \underline{a}] = a_i$$

and $w_i[\underline{w}[\underline{a}, \underline{b}], \underline{c}] = w_i[\underline{w}[\underline{a}, \underline{c}], \underline{w}[\underline{b}, \underline{c}]]$

$$= \alpha[\underline{c}]\,\alpha[\underline{b}]\,(\alpha[\underline{a}])^{-2}\, a_i\,(\alpha[\underline{a}])^2\,(\alpha[\underline{b}])^{-1}\,(\alpha[\underline{c}])^{-1}$$

and $\langle \underline{a}, \underline{b} \rangle = \langle \underline{b}, \underline{w}[\underline{a}, \underline{b}] \rangle$

since $(\alpha[\underline{b}])^{-1}\,\alpha[\underline{w}[\underline{a}, \underline{b}]]\,\alpha[\underline{b}] = \alpha[\underline{a}]$, by the definition of $\underline{w}$, and thus we can use $\alpha[\underline{b}]$ and $\alpha[\underline{a}]$ to recover a_i from $w_i[\underline{a}, \underline{b}]$.

If $\theta \in \mathrm{Aut}(F_n)$, let $\beta = \theta(\alpha)$, and B = conj(β).

Then $A^\theta = B$ since $\alpha[\underline{\theta}[\underline{a}]] = (\theta(\alpha))[\underline{a}] = \beta[\underline{a}]$ means that

$$w_i[\underline{\theta}[\underline{a}],\ \underline{\theta}[\underline{b}]] = \beta[\underline{b}](\beta[\underline{a}])^{-1}\theta_i[\underline{a}]\ \beta[\underline{a}](\beta[\underline{b}])^{-1}$$

and observing that we are conjugating $\theta_i[\underline{a}]$ by the same word for each i we can conclude that

$$(\theta^{-1})_i[\underline{w}[\underline{\theta}[\underline{a}],\ \underline{\theta}[\underline{b}]]] = \beta[\underline{b}](\beta[\underline{a}])^{-1}\theta_i[\underline{\theta}[\underline{a}]]\ \beta[\underline{a}](\beta[\underline{b}])^{-1}$$

$$= \beta[\underline{b}](\beta[\underline{a}])^{-1} a_i\ \ \beta[\underline{a}](\beta[\underline{b}])^{-1}$$

So writing $\alpha \sim \beta$ if $\exists\, \theta \in \mathrm{Aut}(F_n)$ such that $\theta(\alpha) = \beta$ then we have the conjugacy classes of the examples conj(F_n) are given by $F_n/\mathrm{Aut}(F_n) = F_n/\sim$ □

For R(t)×R(x) we have $\alpha[a, b] = b^t a^x$.

It is easy to see from the above example that

$$R(t)\times R(x) \sim R(x)\times R(t) \sim R(-t)\times R(x)$$

and that $\quad R(1)\times R(x) \sim R(1)\times R(0)\ \forall\, x \in \mathbb{Z}$,

and that (for example) $R(1)\times R(0) \not\sim R(2)\times R(0)$.

Finally in this section we make some remarks about orientations.

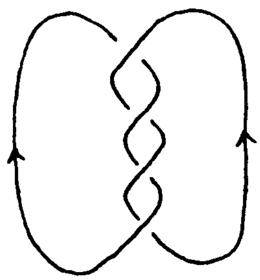
Recall that S and R(x) $\forall x \in \mathbb{Z}$ are orientation invariant (via 6.3.3 and 6.3.1 respectively). It is easy to see that $A*B$ is orientation invariant if A and B both are since

$$G_A(L_1) \cong G_A(L_2) \text{ and } G_B(L_1) \cong G_B(L_2)$$
$$\Rightarrow \quad G_{A*B}(L_1) \cong G_{A*B}(L_2).$$

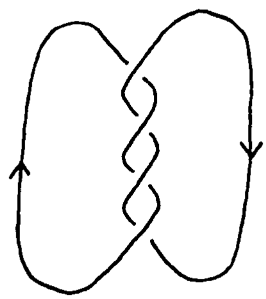
For $A \times B$ the situation is not so easy.

It is true that if A and B are both orientation invariant then $A \times B$ is orientation invariant considered as an invariant of 2 coloured links. However the formation of L^* from L is not invariant under change of orientations, since the framings of the parallels chosen will be changed.

For example if L_1 and L_2 are given by



and



then L_1^* and L_2^*, considered as unoriented links, will differ, and we find (in the next section) that for the representations $R(0) \times R(1)$ or $S \times S$ or $R(0) \times R(2)$ (for example) that $G_R(L_1) \not\cong G_R(L_2)$

(§8.2) ... by computer

This short section is a guide to the work which was done on two dimensional quandle representations with the help of a computer, which was used both to find examples, and in the calculation of invariants.

The finding of examples is in theory very amenable to solution by computer. A program

" For θ an automorphism of F_4 such that $\theta_1 = x_3$ and $\theta_2 = x_4$,

If ($\underline{\theta}[\underline{\theta}[\underline{a},\underline{c}],\underline{\theta}[\underline{b},\underline{c}]] = \underline{\theta}[\underline{\theta}[\underline{a},\underline{b}],\underline{c}]$)

then θ is invariant under moves of type III^o.

If ($\underline{\theta}[\underline{a},\underline{a}] = \underline{a}$)

then θ is invariant under moves of type I^o.

If θ is invariant under moves of type III^o and I^o

then θ is a quandle representation."

would in theory give us a list of all possible examples, in due course. In practice it would be very unlikely to give us many examples which do not arise as R$*$S where R and S are one dimensional quandle representations, at least for the foreseeable future.

Nevertheless, the basic principle is that we can automate the search for solutions to such word problems in free groups, and obtain results by applying a little intelligence.

For one dimensional examples of (group) link representations computers are certainly sufficiently fast to search for examples in this way, and to look for conjugacy (since $\mathrm{Aut}(F_1) \cong \mathbb{Z}_2$), and for the actions and isomorphisms in chapter 7. However, since the theory is also simple enough to be carried out by hand, the only point is to check that no simple cases have been overlooked.

For three dimensional examples, the number of triples of words of small length in six generators grows so fast that the computer is far too slow to find anything of interest.

For two dimensional examples however the situation is between these two positions and, once we have directed the search somewhat, we are able to obtain a number of unexpected examples.

Various ideas were used to direct the search for solutions. The most successful was to search for solutions of the form

$$w1[a, b, c, d] = (\alpha 1[c, d])^{-1}\, \alpha 1[a, b]\, a\, \alpha 2[a, b]\, (\alpha 2[c, d])^{-1}$$

$$w2[a, b, c, d] = (\alpha 3[c, d])^{-1}\, \alpha 3[a, b]\, b\, \alpha 4[a, b]\, (\alpha 4[c, d])^{-1}$$

where $\alpha 1$, $\alpha 2$, $\alpha 3$, $\alpha 4$ are words in two generators. In such a form (w1, w2) is certain to be invariant under moves of type I^o. Although not all solutions are of this form, all those found so far have been conjugate to one of this form. We have no reason to believe that this is true in general, but we have no counterexample either.

There were other ways to speed the search as well. Rather than checking whether

$$(\underline{\theta}[\underline{\theta}[\underline{a},\underline{c}],\underline{\theta}[\underline{b},\underline{c}]]=\underline{\theta}[\underline{\theta}[\underline{a},\underline{b}],\underline{c}])$$

in the general case, it is quicker to check in the case when $\underline{b}$ = (id, id). Moreover this check eliminates the overwhelming majority of cases, to the point that it is more efficient to run this check before the full one.

So far as conjugacy is concerned, it is easy enough to remove duplications of pairs which are conjugate by maps $f \in \mathrm{Aut}(F_2)$ where

$$\underline{f}[a,b]=(a^{\pm 1},b^{\pm 1}) \text{ or } \underline{f}[a,b]=(b^{\pm 1},a^{\pm 1}).$$

In order to deal with conjugation by other elements of $\mathrm{Aut}(F_2)$, it was simply a matter of evaluating a few invariants of the groups obtained for a link and, if they coincided for two quandle representations, trying to find a map which conjugated one to the other. This proved to be quite easy to do, in the many such cases which arose.

When it comes to calculating the invariants from a quandle representation and a link, we do not generally use the entire group, since problems such as recognising groups, and finding isomorphisms are notoriously difficult. Thus we deal instead with certain quotient groups.

The most obvious of these is the abelianisation, which although very easy to calculate (as in §4.2) is only of limited use for distinguishing between the examples. It is of more use when looking for conjugacy, since conjugation between the matrices associated with the abelianisations two dimensional quandle representations is just conjugacy in $GL(2, \mathbb{Z})$.

Of more use are the subabelian quotients, which we define here for those unfamiliar with them.

Let G be a group, and Ab(G) be its abelianisation. Let p be a prime number. Then Ab(G) has a (unique) maximal quotient group which is a direct sum of copies of $\mathbb{Z}_p$. (This is also the maximal quotient group of the group G which is a direct sum of copies of $\mathbb{Z}_p$, and is $H_1(G;\mathbb{Z}_p)$.)

Thus we have a map $G \longrightarrow Ab(G) \longrightarrow \mathbb{Z}_p{}^k$ for some maximal k with the maps onto. Let $OG^{(p)}$ be the kernel of this composition, and $AG^{(p)} = Ab(OG^{(p)})$.

The groups $AG^{(p)}$ are the ones which we most often used in place of G, and which appear in the tables in (§8.3).

There arose pairs of groups which were not distinguished by $AG^{(p)}$, but which instead are distinguished by their maps onto some finite group. In this case we counted the number of classes of surjections, modulo conjugation in the image group.

These invariants of the groups were chosen partly because they can be calculated by computer using the Reidemeister-Schreier method (see [MKS] for example), but more importantly because there were ready to use programs available from S Rees and D Holt. In exchange they obtained a large collection of groups to use for their purposes, for which see [HS].

(§8.3) ... the examples.

In this section we present the examples obtained (by whatever method), some remarks on them, and some calculations on links to verify the properties.

The examples will be given as w1, w2 where (w1, w2) = $\underline{w}^{+}$[a, b, c, d]. Where the inverse has been given we write ($w1^{-}$, $w2^{-}$) = $\underline{w}^{-}$[a, b, c, d].

Most of the examples lie in infinite families, although these do intersect, and there are a few odd examples. The examples from (§8.1) are included for completeness, and because there are calculations for some of them.

All the examples are invariant under reflection, so we do not always to say this explicitly.

Notation: R(x) and S are the one dimensional examples as defined in (§8.1), and the operations × and ∗ are as in that section.

If R is a quandle representation and n is a positive integer then R^n is as defined in (§4.3); R^1 = R, and R^{n+1} is defined by

$$\underline{w}_{n+1} = \underline{w}[\, \underline{w}_n[\, \underline{a}, \underline{b}\,], \underline{b}\,].$$

We will write ($w1_n$, $w2_n$) for $\underline{w}_n$[a, b, c, d].

(8.3.1) R = A∗B where A and B are the 1-dimensional examples R(x) or S. These are all orientation invariant. Some of these are included amongst the calculations for comparison. □

(8.3.2) R = conj(α) for $\alpha \in F_n$.
This example is described in (8.1.1). This includes the examples R(x)×R(t), some of which are included in the tables. □

(8.3.3) A(n)

If A is given by $w1 = c\,b\,d^{-1}$, $w2 = d\,a\,c^{-1}$;
(so ($w1^{-}$, $w2^{-}$) = ($d^{-1}bc$, $c^{-1}ad$)) then A generates an infinite family, i.e. A^n is never trivial for n>0. Take A(n) = A^n.

A(2) is given by $w1_2 = c\,d\,a\,c^{-1}\,d^{-1}$, $w2_2 = d\,c\,b\,c^{-1}\,d^{-1}$, which is conjugate to $w1' = c\,a\,c^{-1}$, $w2' = c\,b\,d^{-1}\,c^{-1}\,d$ which is a member of the next class of examples.

A(3) is given by $w1_3 = c\,d\,c\,b\,d^{-1}\,c^{-1}\,d^{-1}$, $w2_3 = d\,c\,d\,a\,c^{-1}\,d^{-1}\,c^{-1}$, and after this the pattern for A(n) should be clear.

All the examples can be shown to be orientation invariant using corol (6.2.3) for the map $f \in Aut(F_2)$ with $\underline{f}$[a , b] = (b^{-1} , a^{-1}). □

(8.3.4) B(r, s, t)

Given by $w1 = c^r a c^{-r}$, $w2 = c^r a^t b d^{-1} c^s d$;

where r, s and t are integers with r+s+t = 0.

If r+s+t $\neq$ 0 then B(r, s, t) is still invariant under moves of type III^o, hence gives an invariant of framed links, but is not invariant under moves of type I^o.

These examples (with r+s+t = 0) are not in general orientation invariant, unless t = 0. If t = 0 we get

$w1 = c^r a c^{-r}$, $w2 = c^r b d^{-1} c^{-r} d$,

which is conjugate to A(2r) (as defined above), and these examples are indeed orientation invariant.

If r = 0 we get

$w1 = a$, $w2 = a^t b d^{-1} c^{-t} d$,

which is the example used in (5.2.3), i.e. these examples are conjugate to ones given by

$w1' = a b^r d^{-r}$, $w2' = d^r b d^{-r}$.

If r = s = 1, t = -2 we get

$w1 = c a c^{-1}$, $w2 = c a^{-2} b d^{-1} c d$

which is conjugate to S×S (given by $w1' = d c^{-1} a b^{-1} a b^{-1} a c^{-1} d$, $w2' = d c^{-1} a b^{-1} a c^{-1} d$) via the map f where $\underline{f}$ [a , b] = (b a^{-1} , b) □

(8.3.5) C

Given by $w1 = c b d^{-1}$, $w2 = d b^{-1} a^{-1} c d$

and $w1^- = c d b^{-1} a^{-1} c$, $w2^- = c^{-1} a d$.

This is the example of order 3 given as example (4.3.3). It is invariant under reflection via $\underline{f}$ [a , b] = (b^{-1} , a^{-1}), but it is not invariant under changes of orientation, since if LS and LD are the links to be defined shortly we have

Ab(G_C(LS)) $\not\cong$ Ab(G_C(LD)) □

(8.3.6) D

Given by $w1 = a b d^{-1}$, $w2 = d b^{-1} d$.

This example is an involution (i.e. of order two) and hence must be orientation invariant. This example doesn't seem to fit into any of the patterns. □

(8.3.7) E(t)

Given by $w1 = d\, b^{-1} a\, b\, d^{-1}$, $w2 = d\, b^{-1} a^t c^{-t} d$.

These are all shown to be orientation invariant by $\underline{f}[a, b] = (a^{-1}, a^{-t} b)$.

E(0) is conjugate to S×R(0).

E(1) is conjugate to A.

$E(t)^2$ is conjugate to A(2t), and although $E(2)^2$ is conjugate to $A(4) = A(2)^2$ we do not have E(2) conjugate to A(2) (since they give different invariants of links).

If we define $E(t, r) = E(t)^r$ then we would have a family containing the E(t) and the A(t), or at least conjugates of them. As yet I cannot find any nice form for the words defining such a class E(t , r). □

Summary

If we were to follow the last remark and define the class E(t, r), then we would be able to write all the examples obtained so far as

B(r , s , t)	for $r+s+t=0$
E(t , r)	for $r, t \in \mathbb{Z}$
R∗S	for R and S 1-dimensional examples
conj(α)	for $\alpha \in F_2$

plus the two odd case

C

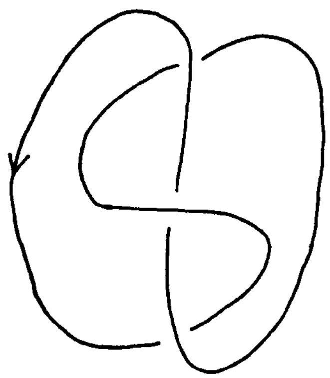
D □

We will now give calculations of $AG^{(2)}$ and $AG^{(3)}$ for 21 examples of two dimensional quandle representations on 4 links.

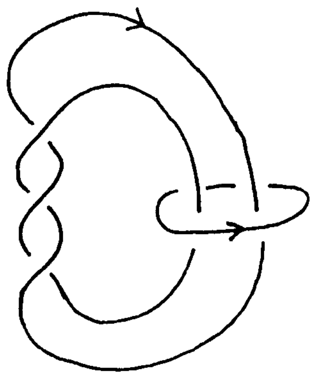
The examples are

Number	Name (and ~ conjugates)	words
#1	$R(1)*R(0)$	$w1 = a$, $w2 = d^{-1} b d$
#2	$R(1)*R(1)$	$w1 = c^{-1} a c$, $w2 = d^{-1} b d$
#3	$R(0)*S$	$w1 = a$, $w2 = d b^{-1} d$
#4	$S*S$	$w1 = c a^{-1} c$, $w2 = d b^{-1} d$
#5	$S*R(1)$	$w1 = c a^{-1} c$, $w2 = d^{-1} b d$
#6	$R(0) \times R(1) \sim R(1) \times R(t)\ \forall t$ $\sim conj(x_1) \sim conj(x_1 x_2^t)$	$w1 = c a c^{-1}$, $w2 = c a^{-1} b a c^{-1}$
#7	$R(0) \times R(2) \sim conj(x_1^2)$	$w1 = c^2 a c^{-2}$ $w2 = c^2 a^{-2} b a^2 c^{-2}$
#8	$R(0) \times R(3) \sim conj(x_1^3)$	$w1 = c^3 a c^{-3}$ $w2 = c^3 a^{-3} b a^3 c^{-3}$
#9	$A = A(1) \sim E(1)$	$w1 = c b d^{-1}$, $w2 = d a c^{-1}$
#10	$A(2) \sim B(1, -1, 0)$	$w1 = c a c^{-1}$, $w2 = c b d^{-1} c^{-1} d$
#11	$A(3)$	$w1 = c d c b d^{-1} c^{-1} d^{-1}$ $w2 = d c d a c^{-1} d^{-1} c^{-1}$
#12	$B(0, -1, 1)$	$w1 = a$, $w2 = a b d^{-1} c^{-1} d$
#13	$B(0, -2, 2)$	$w1 = a$, $w2 = a^2 b d^{-1} c^{-2} d$
#14	$B(1, 1, -2) \sim S \times S$	$w1 = c a c^{-1}$ $w2 = c a^{-2} b d^{-1} c d$
#15	$B(0, -3, 3)$	$w1 = a$, $w2 = a^3 b d^{-1} c^{-3} d$
#16	$B(2, -3, 1)$	$w1 = c^2 a c^{-2}$ $w2 = c^2 a^{-3} b d^{-1} c d$
#17	$B(2, -1, -1)$	$w1 = c^2 a c^{-2}$ $w2 = c^2 a^{-1} b d^{-1} c^{-1} d$
#18	C	$w1 = c b d^{-1}$, $w2 = d b^{-1} a^{-1} c d$
#19	D	$w1 = a b d^{-1}$, $w2 = d b^{-1} d$
#20	$E(0) \sim S \times R(0)$	$w1 = c a^{-1} c$, $w2 = c a^{-1} b a c^{-1}$
#21	$E(2)$	$w1 = d b^{-1} a b d^{-1}$ $w2 = d b^{-1} a^2 c^{-2} d$

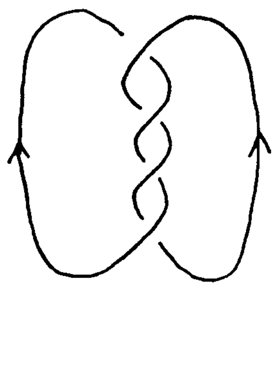
The four links we use are



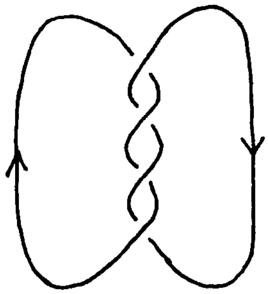
Trefoil



Homeob



LS



LD

Then LS and LD are the same unoriented link but with different choices for orientation.

Homeob has the same exterior as LD, and is obtained by a Dehn twist along one component. If we form Homeob* and LD* the two coloured parallels as in (§8.1) then the resulting links again have the same exteriors.

Table of calculations for Trefoil

Example number	$AG^{(2)}$	$AG^{(3)}$
#1	$2\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$6\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#2	$4\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#3	$4\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$46\mathbb{Z}$
#4	$8\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$190\mathbb{Z}$
#5	$6\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$18\mathbb{Z}_2 \oplus 46\mathbb{Z}$
#6	$2\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$6\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#7	$5\mathbb{Z}$	$6\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#8	$2\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$10\mathbb{Z}$
#9	$2\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$40\mathbb{Z}$
#10	$4\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#11	$2\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$46\mathbb{Z}$
#12	$2\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$6\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#13	$5\mathbb{Z}$	$6\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#14	$4\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#15	$2\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$10\mathbb{Z}$
#16	$2\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#17	$2\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#18	$4\mathbb{Z}_2 \oplus 29\mathbb{Z}$	$18\mathbb{Z}_2 \oplus 10\mathbb{Z}$
#19	$4\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$46\mathbb{Z}$
#20	$4\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$46\mathbb{Z}$
#21	$4\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$39\mathbb{Z}$

Remark: On this table and on the others to follow we have used

$n\mathbb{Z}$ to mean $\mathbb{Z}^n$

and $n\mathbb{Z}_m$ to mean $\oplus^n \mathbb{Z}_m$ (i.e. ${\mathbb{Z}_m}^n$)

Table of calculations for LD and LS with p = 2

Example #	$AG^{(2)}(LD)$	$AG^{(2)}(LS)$
#1	$45\mathbb{Z}$	$45\mathbb{Z}$
#2	$41\mathbb{Z}$	$41\mathbb{Z}$
#3	$8\mathbb{Z}_2 \oplus 41\mathbb{Z}$	$8\mathbb{Z}_2 \oplus 45\mathbb{Z}$
#4	$16\mathbb{Z}_2 \oplus 33\mathbb{Z}$	$16\mathbb{Z}_2 \oplus 33\mathbb{Z}$
#5	$8\mathbb{Z}_2 \oplus 37\mathbb{Z}$	$8\mathbb{Z}_2 \oplus 37\mathbb{Z}$
#6	$42\mathbb{Z}$	$4\mathbb{Z}_2 \oplus 38\mathbb{Z}$
#7	$11\mathbb{Z}_2 \oplus 38\mathbb{Z}$	$11\mathbb{Z}_2 \oplus 38\mathbb{Z}$
#8	$3\mathbb{Z}_3 \oplus 42\mathbb{Z}$	$3\mathbb{Z}_3 \oplus 4\mathbb{Z}_6 \oplus 38\mathbb{Z}$
#9	$35\mathbb{Z}$	$35\mathbb{Z}$
#10	$37\mathbb{Z}$	$37\mathbb{Z}$
#11	$35\mathbb{Z}$	$35\mathbb{Z}$
#12	$41\mathbb{Z}$	$8\mathbb{Z}_2 \oplus 33\mathbb{Z}$
#13	$12\mathbb{Z}_2 \oplus 37\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 37\mathbb{Z}$
#14	$2\mathbb{Z}_8 \oplus 35\mathbb{Z}$	$4\mathbb{Z}_4 \oplus 8\mathbb{Z}_8 \oplus 27\mathbb{Z}$
#15	$4\mathbb{Z}_3 \oplus 41\mathbb{Z}$	$4\mathbb{Z}_3 \oplus 8\mathbb{Z}_6 \oplus 33\mathbb{Z}$
#16	$6\mathbb{Z}_2 \oplus 2\mathbb{Z}_4 \oplus 2\mathbb{Z}_{24} \oplus 31\mathbb{Z}$	$8\mathbb{Z}_2 \oplus 2\mathbb{Z}_{24} \oplus 31\mathbb{Z}$
#17	$6\mathbb{Z}_2 \oplus 2\mathbb{Z}_4 \oplus 2\mathbb{Z}_8 \oplus 31\mathbb{Z}$	$8\mathbb{Z}_2 \oplus 2\mathbb{Z}_8 \oplus 31\mathbb{Z}$
#18	$4\mathbb{Z}_6 \oplus 29\mathbb{Z}$	$4\mathbb{Z}_3 \oplus 5\mathbb{Z}$
#19	$41\mathbb{Z}$	$41\mathbb{Z}$
#20	$2\mathbb{Z}_2 \oplus 41\mathbb{Z}$	$2\mathbb{Z}_2 \oplus 41\mathbb{Z}$
#21	$37\mathbb{Z}$	$37\mathbb{Z}$

Table of calculations for LD and LS with $p = 3$

Example #	$AG^{(3)}(LD)$	$AG^{(3)}(LS)$
#1	$18\mathbb{Z}_2 \oplus 172\mathbb{Z}$	$18\mathbb{Z}_2 \oplus 172\mathbb{Z}$
#2	$36\mathbb{Z}_2 \oplus 100\mathbb{Z}$	$36\mathbb{Z}_2 \oplus 100\mathbb{Z}$
#3	$27\mathbb{Z}_4 \oplus 55\mathbb{Z}$	$27\mathbb{Z}_4 \oplus 55\mathbb{Z}$
#4	$18\mathbb{Z}_4 \oplus 10\mathbb{Z}$	$18\mathbb{Z}_4 \oplus 10\mathbb{Z}$
#5	$6\mathbb{Z}_2 \oplus 27\mathbb{Z}_4 \oplus 31\mathbb{Z}$	$6\mathbb{Z}_2 \oplus 27\mathbb{Z}_4 \oplus 31\mathbb{Z}$
#6	$26\mathbb{Z}_2 \oplus 56\mathbb{Z}$	$26\mathbb{Z}_2 \oplus 56\mathbb{Z}$
#7	$18\mathbb{Z}_2 \oplus 8\mathbb{Z}_4 \oplus 56\mathbb{Z}$	$18\mathbb{Z}_2 \oplus 8\mathbb{Z}_4 \oplus 56\mathbb{Z}$
#8	$44\mathbb{Z}_2 \oplus 200\mathbb{Z}$	$44\mathbb{Z}_2 \oplus 200\mathbb{Z}$
#9	$15\mathbb{Z}_2 \oplus 31\mathbb{Z}$	$15\mathbb{Z}_2 \oplus 31\mathbb{Z}$
#10	$48\mathbb{Z}_2 \oplus 88\mathbb{Z}$	$48\mathbb{Z}_2 \oplus 88\mathbb{Z}$
#11	$39\mathbb{Z}_2 \oplus 43\mathbb{Z}$	$39\mathbb{Z}_2 \oplus 43\mathbb{Z}$
#12	$27\mathbb{Z}_2 \oplus 55\mathbb{Z}$	$9\mathbb{Z}_2 \oplus 55\mathbb{Z}$
#13	$18\mathbb{Z}_2 \oplus 9\mathbb{Z}_4 \oplus 55\mathbb{Z}$	$9\mathbb{Z}_4 \oplus 55\mathbb{Z}$
#14	$48\mathbb{Z}_2 \oplus 3\mathbb{Z}_{36} \oplus 55\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 3\mathbb{Z}_{36} \oplus 55\mathbb{Z}$
#15	$45\mathbb{Z}_2 \oplus 199\mathbb{Z}$	$45\mathbb{Z}_2 \oplus 199\mathbb{Z}$
#16	$45\mathbb{Z}_2 \oplus 3\mathbb{Z}_{36} \oplus 85\mathbb{Z}$	$45\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 3\mathbb{Z}_{36} \oplus 85\mathbb{Z}$
#17	$45\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 3\mathbb{Z}_{36} \oplus 31\mathbb{Z}$	$9\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 3\mathbb{Z}_{36} \oplus 31\mathbb{Z}$
#18	$18\mathbb{Z}_2 \oplus 46\mathbb{Z}$	$18\mathbb{Z}_2 \oplus 46\mathbb{Z}$
#19	$27\mathbb{Z}_2 \oplus 55\mathbb{Z}$	$27\mathbb{Z}_2 \oplus 55\mathbb{Z}$
#20	$24\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 46\mathbb{Z}$	$24\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 46\mathbb{Z}$
#21	$12\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 31\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 31\mathbb{Z}$

Notes:

(1) #18 (which is example C) has

$$Ab(\, G_C(\, LS\,)\,) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$$
$$Ab(\, G_C(\, LD\,)\,) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_6.$$

(2) If you were to try to guess purely on the basis of this table whether one of the examples was orientation invariant, you would only be wrong in one case. The case would be example #7, which is R(0)×R(2), where $G_{\#7}(\, LS\,) \ncong G_{\#7}(\, LD\,)$.

We can prove this by counting the classes of epimorphisms to the permutation group S_4, modulo inner automorphisms of S_4. For $G_{\#7}(\, LS\,)$ and $G_{\#7}(\, LD\,)$ there are

5616 and 6048 classes respectively.

Table of calculations for Homeob

Example number	$AG^{(2)}$	$AG^{(3)}$
#1	$45\mathbb{Z}$	$18\mathbb{Z}_2 \oplus 172\mathbb{Z}$
#2	$41\mathbb{Z}$	$36\mathbb{Z}_2 \oplus 100\mathbb{Z}$
#3	$8\mathbb{Z}_6 \oplus 41\mathbb{Z}$	$27\mathbb{Z}_4 \oplus 55\mathbb{Z}$
#4	$16\mathbb{Z}_6 \oplus 33\mathbb{Z}$	$18\mathbb{Z}_4 \oplus 10\mathbb{Z}$
#5	$8\mathbb{Z}_6 \oplus 37\mathbb{Z}$	$6\mathbb{Z}_2 \oplus 27\mathbb{Z}_4 \oplus 31\mathbb{Z}$
#6	$4\mathbb{Z}_4 \oplus 38\mathbb{Z}$	$26\mathbb{Z}_2 \oplus 56\mathbb{Z}$
#7	$11\mathbb{Z}_2 \oplus 38\mathbb{Z}$	$18\mathbb{Z}_2 \oplus 8\mathbb{Z}_4 \oplus 56\mathbb{Z}$
#8	$5\mathbb{Z}_3 \oplus 4\mathbb{Z}_{12} \oplus 38\mathbb{Z}$	$44\mathbb{Z}_2 \oplus 200\mathbb{Z}$
#9	$35\mathbb{Z}$	$15\mathbb{Z}_2 \oplus 31\mathbb{Z}$
#10	$37\mathbb{Z}$	$48\mathbb{Z}_2 \oplus 88\mathbb{Z}$
#11	$6\mathbb{Z}_3 \oplus 35\mathbb{Z}$	$39\mathbb{Z}_2 \oplus 46\mathbb{Z}$
#12	$8\mathbb{Z}_4 \oplus 33\mathbb{Z}$	$9\mathbb{Z}_2 \oplus 55\mathbb{Z}$
#13	$12\mathbb{Z}_2 \oplus 37\mathbb{Z}$	$9\mathbb{Z}_4 \oplus 55\mathbb{Z}$
#14	$6\mathbb{Z}_8 \oplus 4\mathbb{Z}_{48} \oplus 27\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 3\mathbb{Z}_{36} \oplus 31\mathbb{Z}$
#15	$4\mathbb{Z}_3 \oplus 8\mathbb{Z}_{12} \oplus 33\mathbb{Z}$	$45\mathbb{Z}_2 \oplus 199\mathbb{Z}$
#16	$6\mathbb{Z}_2 \oplus 2\mathbb{Z}_4 \oplus 2\mathbb{Z}_{24} \oplus 31\mathbb{Z}$	$45\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 3\mathbb{Z}_{36} \oplus 85\mathbb{Z}$
#17	$6\mathbb{Z}_2 \oplus 2\mathbb{Z}_4 \oplus 2\mathbb{Z}_8 \oplus 31\mathbb{Z}$	$9\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 3\mathbb{Z}_{36} \oplus 31\mathbb{Z}$
#18	$4\mathbb{Z}_3 \oplus 5\mathbb{Z}$	$18\mathbb{Z}_2 \oplus 46\mathbb{Z}$
#19	$41\mathbb{Z}$	$27\mathbb{Z}_2 \oplus 55\mathbb{Z}$
#20	$2\mathbb{Z}_3 \oplus 2\mathbb{Z}_6 \oplus 41\mathbb{Z}$	$24\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 55\mathbb{Z}$
#21	$37\mathbb{Z}$	$12\mathbb{Z}_2 \oplus 3\mathbb{Z}_4 \oplus 31\mathbb{Z}$

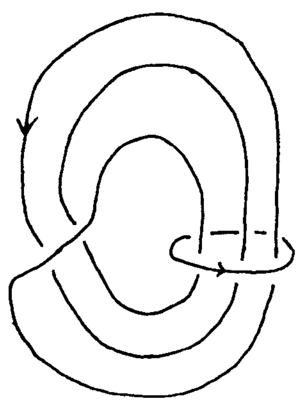
Notes: If we compare the results from Homeob with those of LS and LD we find the following coincidences:

#1 and #2 These are π_1 and $\pi_1 * \pi_1$ of the link exterior hence are just invariants of the link complement.

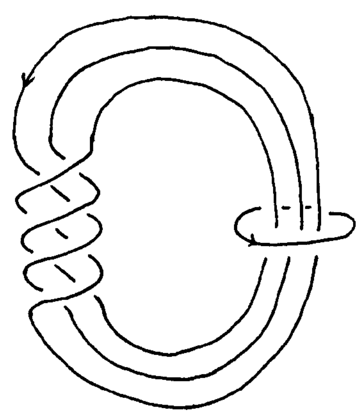
#9 and #19. For these examples we can produce an isomorphism between G_R(Homeob) and G_R(LD). However for the links L3 and L4 below we have

	$AG^{(2)}(G_R(L3))$	$AG^{(2)}(G_R(L4))$
#9	$6\mathbb{Z}_3 \oplus 11\mathbb{Z}$	$2\mathbb{Z}_3 \oplus 11\mathbb{Z}$
#19	$8\mathbb{Z}_3 \oplus 17\mathbb{Z}$	$17\mathbb{Z}$

#7, #10, #21 For these examples we have been unable to decide as to whether or not G_R(Homeob) and G_R(LD) are isomorphic, and whether G_R(L3) and G_R(L4) are isomorphic, so the proof or otherwise as to whether these invariants are only invariants of the link exterior will have to be left open.



L3



L4

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