Effective geometry of curve graphs

by

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Declarations

I declare that the work in this thesis is my own except where it is indicated otherwise. I confirm that this thesis has not been submitted for a degree at another university.

Material in Chapters 5, 6 and 7 has been submitted for peer-reviewed publication.

Chapter 3 represents joint work with Sebastian Hensel and Piotr Przytycki who have given kind permission to include a short account of [30] in this thesis, and this separate article has been submitted for peer-reviewed publication.

Abstract

We show that curve graphs and arc graphs are uniformly hyperbolic i.e. all of these graphs are hyperbolic with some fixed hyperbolicity constant. This is joint work with Sebastian Hensel and Piotr Przytycki. Note that curve graphs were shown to be hyperbolic by Masur and Minsky, arc graphs were shown to be hyperbolic by Masur and Schleimer, and curve graphs were shown to be uniformly hyperbolic by Aougab, Bowditch and Clay–Rafi–Schleimer independently.

We show that if every vertex of a fixed geodesic in the curve graph cuts some fixed subsurface then the image of that geodesic under the corresponding subsurface projection has its diameter bounded from above by 62 (or 50) if the subsurface is an annulus (or otherwise). This is an effective version of the bounded geodesic image theorem of Masur and Minsky, indeed, there is a universal constant that can be taken as a bound.

In terms of the complexity of a surface we provide an exponential upper bound on the so-called slices of tight geodesics. This is an effective version of a theorem of Bowditch. Using this we provide algorithms to compute tight geodesic axes of pseudo-Anosovs in the curve graph.

All of our proofs are combinatorial in nature. In particular, we do not use any geometric limiting arguments.

Chapter 1

Introduction

The curve complex of a surface was introduced by Bill Harvey [27] who was inspired by the work of Borel and Serre on arithmetic groups [8]. Each vertex of the curve complex is an isotopy class of loop on a surface and a simplex spans a subset of vertices if the loops can be realized disjoint. The curve complex is homotopy equivalent to a wedge of spheres [26] and this is particularly useful for showing that the mapping class group is a virtual duality group and for computing the virtual cohomological dimension of the mapping class group [26].

The perception of the curve complex dramatically changed after Masur and Minsky showed that the curve complex is hyperbolic [39]. One motivation for showing that the curve complex is hyperbolic is to associate the Gromov boundary of the curve complex with the ending lamination space endowed with the coarse Hausdorff topology. This is known as Klarreich's theorem [31] (see [53] for a topological proof that does not use Teichmüller theory).

These theorems form the backbone of the proof of the classification of Kleinian surface groups for which Yair Minsky was the main pioneer, see Minsky [40] and Brock–Canary–Minsky [19] (see also [9]). This is an important step in the Ending Lamination Theorem, which was conjectured by Thurston.

The curve graph is the 1-skeleton of the curve complex and the natural inclusion is a quasi-isometry. The curve graph is a graph that is locally infinite at every vertex, and so projecting vertices to the link of a fixed vertex can be interesting from a coarse perspective. Indeed, vertices of the curve graph correspond to isotopy classes of curves of the surface S, and so the link of a vertex "corresponds" to curve graphs of subsurfaces of S. This leads us to the notion of *subsurface projection.* Masur and Minsky [36] showed that, apart from the obvious counterexamples, the image of a geodesic in the curve graph under subsurface projection has its diameter bounded from above only in terms of the surface S. This is referred to as the Bounded Geodesic Image Theorem, and it is an interesting aspect of the hyperbolicity of the curve graph.

Between a typical pair of vertices of the curve graph there are infinitely many geodesics. Masur and Minsky [36] introduced the notion of a *tight geodesic* between two vertices of the curve graph. A tight geodesic is a special kind of geodesic, where each vertex satisfies a local topological condition. Masur and Minsky introduced *hierarchies of tight geodesics*: a hierarchy is a collection of tight geodesics in curve graphs of subsurfaces of S—at most one tight geodesic per subsurface and exactly one tight geodesic for the whole surface S which is called the "main" geodesic. These are especially important for constructing the model manifold in [40], indeed, in this case the "main" geodesic connects a pair of ending laminations in the Gromov boundary of the curve graph of S. An interesting consequence of hierarchies is that there are only finitely many tight geodesics between a fixed pair of vertices in the curve graph [36]—this can be thought of as a remedy to address the non-locally compact property that the curve graphs have.

Bowditch [14] improved on Masur and Minsky's work by showing that tight geodesics behave like geodesics in a hyperbolic group, for example, the set of tight geodesics between a fixed pair of vertices have uniformally bounded "slices". He then used this property to show that the action of the mapping class group on the curve graph is *acylindrical* and that pseudo-Anosovs have rational stable lengths on the curve graph $\mathcal{C}(S)$ with denominators that are uniformally bounded in terms of S. Bowditch also used tight geodesics to show that there are only finitely many atoroidal surface by surface bundles with a given base and fibre [15].

Another motivation for introducing hierarchies of tight geodesics is to provide quasi-geodesics for the mapping class group and a distance estimate between two mapping classes purely in terms of subsurface projections and distances in curve graphs [36]. This has led to breakthroughs for the geometry of the mapping class group. For instance, a quick consequence of the Masur–Minsky theory is that every infinite cyclic subgroup of the mapping class group is quasi-isometrically embedded (this wasn't even known for Dehn twists, see Problem 2.16 of Kirby's list of problems).

There are many consequences of the Masur–Minsky theory for mapping class groups. For a linear conjugator bound for pseudo-Anosov elements see [36]. Later, Jing Tao proved that there is a linear conjugator bound for elements in general [48]. The mapping class group is quasi-isometrically rigid [2], [23], [10]. The rank of the mapping class group has been computed, see [3], [23]. The mapping class group has finite asymptotic dimension [4]. The list of applications continues to go on.

The framework for applying the Masur–Minsky theory to mapping class groups has inspired new directions for studying the outer automorphism group of a free group $Out(F_n)$. However there is no clear analogue of the curve graph in the $Out(F_n)$ setting. Bestvina and Feighn showed that the free factor graph is hyperbolic [6] and later Handel and Mosher showed that the free splitting graph is hyperbolic [25]. Using this, Brian Mann showed that the cyclic splitting graph is hyperbolic [35]. As for subsurface projections, again there is also no unique analogue, see for example [5], [45], [49]. Furthermore there is yet to be an analogue of subsurface projection that detects "twisting" in $Out(F_n)$ similar to how Dehn twisting on a surface is detected by annular subsurface projection [20]. Finally, there is yet to be an $Out(F_n)$ version for tight geodesics and hierarchies.

The coarse geometry of the curve graph also has applications to Teichmüller space, see for example [16], [17], [37], [42], [43]. Summarizing, there are a wide range of applications in hyperbolic 3–manifolds, mapping class groups and Teichmüller theory.

Here we shall discuss the hyperbolicity of the curve graphs, the bounded geodesic image theorem and Bowditch's results on tight geodesics. These theorems are stated without reference to any geometry, however the original proofs of these theorems relied on Teichmüller theory and hyperbolic 3–manifolds. In this thesis we give new proofs that are elementary and constructive where applicable, thus removing any reliance on geometry. In particular we are able to state how these theorems depend on the underlying surface S: the curve graphs are uniformally hyperbolic, there is a universal constant for the bounded geodesic image theorem, and tight geodesics have slices that are bounded exponentially in terms of $\chi(S)$. At the end of this chapter we discuss some examples of how this may be applied to provide effective versions of theorems in hyperbolic geometry and mapping class groups.

Overview and main results

Throughout the thesis we assume that $S = S_{g,n}$, the connected, compact and orientable surface with genus g and n boundary components, and that $\xi(S_{g,n}) \coloneqq 3g + n - 3 \geq 2$.

In Chapter 2 we provide the reader with the definitions of the curve graph $\mathcal{C}(S)$, the arc graph $\mathcal{A}(S)$ and subsurface projection κ_Y, as well as the notion of hyperbolicity.

In Chapter 3 we show the following two theorems.

Theorem 3.1.5. *There exists a uniform constant δ such that the graph $\mathcal{A}(S)$ is δ-hyperbolic.*

We remark that the arc graphs (and many other graphs e.g. the disc graph of a handlebody) were originally shown to be hyperbolic by Masur and Schleimer [38]. Uniform hyperbolicity of the arc graphs was not known. The proof that we give in Chapter 3 will be the simplest proof that the arc graphs are hyperbolic, and this proof is a short version of joint work with Sebastian Hensel and Piotr Przytycki [30]. In [30], it is shown that for each geodesic triangle in the arc graph there exists a vertex that is contained within the 7-neighbourhood of any of the three sides.

Theorem 3.2.1. *There exists a uniform constant δ such that the graph $\mathcal{C}(S)$ is δ-hyperbolic whenever $S = S_{g,n}$ and $\xi(S) \geq 2$.*

The curve graphs were first shown to be hyperbolic by Masur and Minsky [39]. Bowditch [13] showed that we may take $\delta = \delta(S)$ to be a function that is bounded from above by a logarithm in $\xi(S)$. Hamenstädt also gave another proof of the hyperbolicity of the curve graphs [24].

It was independently shown by Aougab [1], Bowditch [11] and Clay–Rafi–Schleimer [21] that the curve graphs are uniformly hyperbolic. The proof that we give in Chapter 3 is the simplest proof that the curve graphs are (uniformly) hyperbolic, and this is a short version of joint work with Sebastian Hensel and Piotr Przytycki [30]. In [30], it is shown that for each geodesic triangle in the curve graph there exists a vertex that is contained within the 17-neighbourhood of any of the three sides. The hyperbolicity constants provided elsewhere are at least 1000.

In Chapter 4 we prove the bounded geodesic image theorem [36]. The theorem, originally due to Masur and Minsky, states that if a fixed geodesic in $\mathcal{C}(S)$ is such that each of its vertices cut a fixed subsurface then the image of the geodesic under the corresponding subsurface projection has diameter bounded from above only in terms of S. Here, we give explicit constants that are independent of S by using the unicorn arcs defined in Chapter 3. This improves on previous work of the author [52]. We remark that an unpublished proof of Chris Leininger combined with recent work of Bowditch [11] can also provide a uniform bound—this proof uses the technology in [13] that involves singular euclidean structures.

First we show the following.

Theorem 4.1.7. *Let $S = S_{g,n}$ with $n > 0$. Let Y be a subsurface of S and let $Q = (c_j)_{j\in I}$ be a geodesic in $\mathcal{C}(S)$. The following statements hold.*

1. *If Y is annular and for each $j \in I$ we have that $d_{\mathcal{AC}(S)}(c_j, \partial Y) \geq 13$ then* $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 7$.

2. *If Y is non-annular and for each $j \in I$ we have that $d_{\mathcal{AC}(S)}(c_j, \partial Y) \geq 12$ then* $\operatorname{diam}_{\mathcal{AC}(S)}(\kappa_Y(Q)) \leq 3$.

It is interesting that there is such a small bound on the diameter of the image in the case provided above. We use this to show the general case, which includes closed surfaces.

Theorem 4.2.1. *Let $S = S_{g,n}$ with $\xi(S) \geq 2$. Let Y be a subsurface of S. Suppose that $Q = (c_j)_{j \in I}$ is a geodesic in $\mathcal{C}(S)$ such that c_j cuts Y for all $j \in I$. Then the following statements hold.*

1. *If Y is annular then* $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 62$.

2. *If Y is non-annular then* $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 50$.

In Chapter 5 we introduce the notion of a *tight filling multipath*, and from there onwards throughout the thesis we discuss tight geodesics. The main theorem below is an effective version of theorems originally due to Bowditch [14, Theorems 1.1 and 1.2]. The notation of this theorem is explained at the start of Chapter 7, and the definitions of tight multigeodesic and so on are given in Chapter 5.

Theorem 7.1.2. *Fix $\delta \geq 3$ such that $\mathcal{C}(S)$ is δ-hyperbolic for all surfaces S with $\xi(S) \geq 2$. The following statements hold, where K is a uniform constant.*

1. *For any $a, b \in \mathcal{C}_0(S)$ and for any curve $c \in Q \in \mathcal{L}(a, b)$ we have that $|G(a, b) \cap N_{\mathcal{C}(S)}(c; \delta)| \leq K^{\xi(S)}$.*

2. *For any $r \geq 0$ and $a, b \in \mathcal{C}_0(S)$ such that $d_{\mathcal{C}(S)}(a, b) \geq 2r + 2k + 1$ (where $k = 10\delta + 1$) for any curve $c \in Q \in \mathcal{L}(a, b)$ such that $c \notin N_{\mathcal{C}(S)}(a; r + k) \cup N_{\mathcal{C}(S)}(b; r + k)$ we have that $|G(a, b; r) \cap N_{\mathcal{C}(S)}(c; 2\delta)| \leq K^{\xi(S)}$.*

Our proof strategy for Theorem 7.1.2 is as follows. Fix vertices c_a and c_b that are elements of Q and are close, but not too close, to c. Given any tight multigeodesic Q_T connecting a to b, by hyperbolicity, Q_T passes through the δ-neighbourhood of c' whenever $c' \in \{c_a, c, c_b\}$. In Chapter 6 we describe how to construct a tight filling multipath P connecting c_a to c_b such that the length of P is uniformly bounded and P coincides with Q_T in the δ-neighbourhood of c (Theorem 6.2.3). In Chapter 5 we show that the number of curves that "appear" in some tight filling multipath of length at most L between a fixed pair of vertices is bounded from above only in terms of S and L (Theorem 5.2.7). This means that the curves of Q_T in the δ-neighbourhood of c is a subset of a finite set determined only by c_a and c_b, which were fixed only in terms of c and Q, this completes the proof.

This strategy is partly inspired by work of Shackleton [46]. He introduced the notion of a 3-embedded multipath, which is similar but not equivalent to the notion of a filling multipath which we introduce in Chapter 5. Comparisons between our work are explained in Chapters 5 and 6. We also note that Yohsuke Watanabe [50] has another combinatorial proof of [14, Theorems 1.1 and 1.2] that uses the notion of subsurface projection. His proof is somewhat simpler yet his bounds are doubly exponential in $\xi(S)$ whereas ours are exponential.

Using the results in Chapters 5 and 6, we are also able to give effective constants for the acylindrical action of the mapping class group of S on the curve graph of S (Theorem 7.2.3). Furthermore we give algorithms that accept a pseudo-Anosov mapping class and return the invariant tight multigeodesics, and, the stable length of the pseudo-Anosov on the curve graph (Theorems 7.3.2 and 7.3.1 respectively).

Future directions

By Theorem 3.2.1, Theorem 4.2.1 and Theorem 7.2.3, we are able to make effective the following constant due to Mangahas [34]: There exists a constant $L = L(S)$ such that for any finite generating set for the mapping class group of S, there exists a pseudo-Anosov with word length less than L. In fact, if we insist that the finite generating set contains an element of infinite order then we can obtain an exponential upper bound on L in terms of $\xi(S)$. By considering the Humphries' generators [22, Chapter 4], we see that L is at least some linear function of $\xi(S)$. It would be interesting to know if there is a polynomial upper bound on L in terms of $\xi(S)$.

Finally, it would be interesting to make Masur and Minsky's distance estimate [36] for the mapping class group effective. This would provide a good step towards making effective versions of theorems of Brock [17, 18]. For example, let ϕ be a pseudo-Anosov mapping class of S, it would be interesting to know how good an estimate, in terms of S, there is for the volume of the mapping torus of ϕ in terms of the translation distance of ϕ on the pants graph. We believe that Theorems 3.2.1 and 4.2.1 would be of great help in this direction.

Chapter 2

Preliminaries

The purpose of this chapter is to give almost all the necessary definitions required to understand this thesis. We give proofs, or sketches of proofs, of lemmas which are used throughout the thesis. This chapter contains no original work of the author.

2.1 Sequences, paths, geodesics, and hyperbolicity

2.1.1 Sequences

We say that $I \subset \mathbb{Z}$ is *convex* if whenever $i_1, i_2 \in I$, $k \in \mathbb{Z}$ and $i_1 \leq k \leq i_2$ then $k \in I$.

Let $\mathcal{S}$ be a set. A *sequence* of elements of $\mathcal{S}$ is a function $f\colon I \to \mathcal{S}$ where $I \subset \mathbb{Z}$ is convex. Throughout we use the notation $(v_j)_{j\in I}$ for a sequence, where $v_j = f(j)$. We call I the *set of indices*. An index $i \in I$ is *initial* if $i-1 \notin I$ and an index $i \in I$ is *terminal* if $i+1 \notin I$.

Remark 2.1.1. Unless it is explicitly stated otherwise, we shall assume throughout that I is non-empty.

Remark 2.1.2. Let P be a sequence and write $P = (v_j)_{j\in I}$. Throughout, we often treat sequences as sets, where we write $v \in P$ if $v = v_j$ for some $j \in I$.

Let $(v_j)_{j\in I}$ be a sequence and let $k \in \mathbb{Z}$. Set $I' = \{i + k : i \in I\}$. We say that the sequence $(v_{j-k})_{j\in I'}$ is a *reparametrization* of the sequence $(v_j)_{j\in I}$.

Throughout, if two sequences P and P' are such that P' is a reparametrization of P then for simplicity we write $P = P'$. This enables us to drop the set of indices in our notation and write (v, v', v'') for example.

Suppose that $P = (a_j)_{j\in I_1}$ and $P' = (b_j)_{j\in I_2}$ are sequences such that P has a terminal index $i_1 \in I_1$, P' has an initial index $i_2 \in I_2$ and we have that $a_{i_1} = b_{i_2}$.

Then after a reparametrization we may write $P = (v_j)_{j \in I_1'}$ and $P' = (v_j)_{j \in I_2'}$ such that $i \in I_1'$ is terminal and $i \in I_2'$ is initial. Set $I_3 = I_1' \cup I_2'$. The sequence $(v_j)_{j \in I_3}$ is the *concatenation* of P and P'.

Let $P = (v_j)_{j \in I}$ be a sequence. We define the *length* of P, written $\mathrm{length}(P)$, to be the cardinality of the set I less one. If I is an infinite set then $\mathrm{length}(P) = \infty$ and we write $\infty \geq k$ for all $k \in \mathbb{Z}$. This is relevant when we write $\mathrm{length}(P) \geq 3$ for example.

2.1.2 Graphs, paths and spanned subgraphs

For our purposes, a *graph* $\mathcal{G}$ is a set $V = V(\mathcal{G})$ together with a set $E = E(\mathcal{G})$ which is a subset of the *pairs* of $V(\mathcal{G})$, that is $E(\mathcal{G}) \subset \{\{x, y\} \subset V : x \neq y\}$. The set V is called the *vertex set* and the set E is called the *edge set*. We say that x and y *span an edge* if $\{x, y\} \in E$.

Remark 2.1.3. Throughout we think of $\mathcal{G}$ as a discrete space.

A *path* in a graph $\mathcal{G}$ is a sequence of vertices $P = (v_j)_{j \in I}$ such that whenever $i, i+1 \in I$ we have that $v_i = v_{i+1}$ or v_i and v_{i+1} span an edge in $\mathcal{G}$. If $i \in I$ is initial and $k \in I$ is terminal then we say that P *connects* v_i *to* v_k.

A graph $\mathcal{G}$ is *connected* if for any two vertices $v, v' \in V(\mathcal{G})$ there exists a path P in $\mathcal{G}$ that connects v to v'.

Let A be a subset of vertices of a graph $\mathcal{G}$. The *subgraph spanned by* A is the graph with vertex set A such that a pair of vertices a and b of A span an edge if and only if a and b span an edge in $\mathcal{G}$. This notion is used in the proofs of Theorem 3.1.5 and Theorem 3.1.9.

Suppose that a graph $\mathcal{G}$ is connected. We define $d_{\mathcal{G}} \colon V(\mathcal{G}) \times V(\mathcal{G}) \to \mathbb{Z}$ by setting $d_{\mathcal{G}}(x, y)$ to be the largest integer k such that whenever a path P connects x to y then $k \leq \mathrm{length}(P)$. Observe that $(\mathcal{G}, d_{\mathcal{G}})$ is a metric space.

We say that $P = (v_j)_{j \in I}$ is a *geodesic* if whenever $i, j \in I$ we have that $d_{\mathcal{G}}(v_i, v_j) = |i - j|$. In particular, a geodesic is a path.

Let A be a subset of $V(\mathcal{G})$. The *diameter* of A, written $\mathrm{diam}_{\mathcal{G}}(A)$, is the smallest integer k such that whenever $x, y \in A$ then $k \geq d_{\mathcal{G}}(x, y)$. If such an integer does not exist then we set $\mathrm{diam}_{\mathcal{G}}(A) = \infty$.

We say that a one-to-many function $f \colon \mathcal{G} \to \mathcal{G}'$ is k-Lipschitz if whenever (v, v') is a path in $\mathcal{G}$ then $\mathrm{diam}_{\mathcal{G}'}(f(v) \cup f(v')) \leq k$. For A a subset of $V(\mathcal{G})$, we adopt the usual convention that $f(A) = \cup_{a \in A} f(a)$.

For non-empty subsets A and B of $V(\mathcal{G})$ we define

$$d_{\mathcal{G}}(A, B) \coloneqq \inf\{d_{\mathcal{G}}(x, y) : (x, y) \in A \times B\}.$$

Similarly, we define $d_{\mathcal{G}}$ on sequences by treating them as sets. This is a convenient abuse of notation and no confusion will arise. For $v \in V(\mathcal{G})$ and A a subset of $V(\mathcal{G})$ we often write $d_{\mathcal{G}}(v, A)$ for $d_{\mathcal{G}}(\{v\}, A)$.

Let A be a subset of $V(\mathcal{G})$ and let $k \geq 0$. We define the set

$$N_{\mathcal{G}}(A; k) \coloneqq \{x \in V(\mathcal{G}) : d_{\mathcal{G}}(x, A) \leq k\}$$

and we write $N_{\mathcal{G}}(v; k) = N_{\mathcal{G}}(\{v\}; k)$ whenever $v \in V(\mathcal{G})$. We call $N_{\mathcal{G}}(A; k)$ the *k-neighbourhood of A* in $\mathcal{G}$.

2.1.3 Hyperbolicity

Let P_1, P_2 and P_3 be geodesics in $\mathcal{G}$ that connect v_1 to v_2, v_2 to v_3, and v_3 to v_1 respectively. We call this a *geodesic triangle formed by* P_1, P_2 and P_3, or just simply a *geodesic triangle.*

Similarly we define *geodesic bigon formed by* P_1 and P_2, and, *geodesic rectangle formed by* P_1, P_2, P_3 and P_4.

Let $\delta \geq 0$. We say that a geodesic triangle is *δ-slim* if any one of the three geodesics that form the triangle is contained in the δ-neighbourhood of the other two geodesics.

We say that a graph $\mathcal{G}$ is *δ-hyperbolic* if every geodesic triangle in $\mathcal{G}$ is δ-slim. We say that $\mathcal{G}$ is *hyperbolic* if there exists $\delta \geq 0$ such that $\mathcal{G}$ is δ-hyperbolic.

2.2 Ambient isotopy, the curve graph and its relatives

2.2.1 Surfaces we consider and ambient isotopy classes

We write $S_{g,n}$ for the connected, compact and orientable surface with genus g and n boundary components. We define $\xi(S) \coloneqq 3g+n-3$ and $\xi(S)$ is called the *complexity* of S.

In general, for a manifold M with or without boundary, we write ∂M for the boundary of M and we write $\mathrm{int}M = M - \partial M$. This notation will be used for surfaces and closed intervals.

An *ambient isotopy* of S is a continuous map $f \colon S \times [0, 1] \to S$ such that for all $t \in [0, 1]$, the map $f_t \colon S \to S$ is a homeomorphism of S where $f_t(x) \coloneqq f(x, t)$, and f_0 is the identity function. We use the notation f_t for an ambient isotopy of S.

We say that a subset A of S is *ambiently isotopic to* a subset B of S if there exists an ambient isotopy f_t of S such that $f_1(A) = B$. We write $[A]_S$ for the equivalence class of subsets of S which are ambiently isotopic to A. We say that A and B are *non-ambiently isotopic* if A is not ambiently isotopic to B.

We say that $[A]_S$ *misses* $[B]_S$ if there exist representatives A' and B' of the classes $[A]_S$ and $[B]_S$ respectively such that $A' \cap B' = \emptyset$. If $[A]_S$ does not miss $[B]_S$ then we say that the classes *cut*.

2.2.2 Curves and the curve graph

An *embedded loop* α is a subset of S such that there exists an open subset $n(\alpha)$ of S and a homeomorphism $f\colon S^1 \times (-1,1) \to n(\alpha)$ where $f(S^1 \times \{0\}) = \alpha$.

An embedded loop $\alpha \subset S$ is *inessential* if there is a simply connected component D of $S - \alpha$. Write $\overline{D}$ for the closure of D in S. By the classification theorem of compact and orientable surfaces [22, Theorem 1.1] we have that $\overline{D}$ is an embedded closed 2-disc in S.

A *peripheral annulus* is a closed subset A of S with a homeomorphism $f\colon S^1 \times [0,1] \to A$ such that $f(S^1 \times \{0\})$ is an embedded loop in S and $f(S^1 \times \{1\})$ is a connected component of ∂S. An embedded loop α is *peripheral* if there exists a peripheral annulus A and a homeomorphism $f\colon S^1 \times [0,1] \to A$ such that $f(S^1 \times \{0\}) = \alpha$.

We say that an embedded loop $\alpha \subset S$ is *essential* if it is not inessential. We say that α is *non-peripheral* if it is not peripheral.

Let α be an essential and non-peripheral embedded loop in S. We call $[\alpha]_S$ a *curve* of S. We write $\mathcal{C}_0(S)$ for the set of curves of S.

Suppose that $\xi(S) \geq 2$. The *curve graph* $\mathcal{C}(S)$ is the graph with vertex set $\mathcal{C}_0(S)$ such that an edge spans c and c' if and only if c misses c'.

Suppose that $\xi(S) = 1$, i.e. $S = S_{1,1}$ (or $S = S_{0,4}$) then $\mathcal{C}(S)$ is the graph with vertex set $\mathcal{C}_0(S)$ such that an edge spans c and c' if and only if $i(c,c') = 1$ (or $i(c,c') = 2$). See Section 2.3.1 for the definition of $i(c,c')$. The graph $\mathcal{C}(S)$ is isomorphic to the Farey graph. We do not discuss this case for our main theorems. It is known that the Farey graph is 1-hyperbolic, see for example [41, Section 3], [33, Proposition 2.2.6].

Lemma 2.2.1. *Suppose that $S = S_{g,n}$ with $\xi(S) \geq 2$ then the curve graph $\mathcal{C}(S)$ is connected.*

Proof. See [22, Theorem 4.3]. $\square$

2.2.3 Arcs and the arc graph

A *properly embedded interval* α is a subset of S such that there exists an open subset $n(\alpha)$ of S and a homeomorphism $f\colon [0,1]\times(-1,1) \to n(\alpha)$ where $f([0,1]\times\{0\}) = \alpha$. Note that $f(\{0,1\} \times (-1,1)) \subset \partial S$.

A properly embedded interval $\alpha \subset S$ is *inessential* if there is a simply connected component D of $S-\alpha$. Write $\overline{D}$ for the closure of D in S. By the classification theorem of compact and orientable surfaces [22, Theorem 1.1] we have that $\overline{D}$ is an embedded closed 2-disc in S.

We say that a properly embedded interval $\alpha \subset S$ is *essential* if it is not inessential.

Let α be an essential properly embedded interval in S. We call $[\alpha]_S$ an *arc* of S. We remind the reader that ambient isotopy of S need not fix ∂S, and so our arcs do not have fixed endpoints. We write $\mathcal{A}_0(S)$ for the set of arcs of S.

The *arc graph* $\mathcal{A}(S)$ is the graph with vertex set $\mathcal{A}_0(S)$ such that an edge spans $\boldsymbol{a}$ and $\boldsymbol{a}'$ if and only if $\boldsymbol{a}$ misses $\boldsymbol{a}'$.

2.2.4 The arc and curve graph

Set $\mathcal{AC}_0(S) \coloneqq \mathcal{C}_0(S) \cup \mathcal{A}_0(S)$.

The *arc and curve graph* $\mathcal{AC}(S)$ is the graph with vertex set $\mathcal{AC}_0(S)$ such that an edge spans x and y if and only if x misses y.

Lemma 2.2.2. *Let c and c' be curves. Suppose that $d_{\mathcal{AC}(S)}(c,c') \leq L$ where $L \geq 2$. Then $d_{\mathcal{C}(S)}(c,c') \leq 2L-2$.*

Sketch of proof. See [36, Lemma 2.2] and [30, Remark 5.1]. There is a retraction $r\colon \mathcal{AC}_0(S) \to \mathcal{C}_0(S)$ such that r restricted to $\mathcal{C}_0(S)$ is the identity. We have that r is 2-Lipschitz. Moreover if $v \in \mathcal{AC}(S)$ and $d_{\mathcal{AC}(S)}(c,v) = 1$ then $d_{\mathcal{C}(S)}(c, r(v)) \leq 1$.

There exists a geodesic $(v_j)_{j\in I}$ in $\mathcal{AC}(S)$ that connects c to c' and we may assume that $I = \{0,...,L\}$. We have that $d_{\mathcal{C}(S)}(v_0, r(v_1)) \leq 1$ and $d_{\mathcal{C}(S)}(r(v_{L-1}), v_L) \leq 1$. Furthermore for $i \in \{1,...,L-2\}$ we have that $d_{\mathcal{C}(S)}(r(v_i), r(v_{i+1})) \leq 2$. We deduce that $d_{\mathcal{C}(S)}(v_0, v_L) \leq 1 + 2(L-2) + 1$ as required. □

2.2.5 Multicurves and the multicurve graph

Let α be a non-empty union of pairwise disjoint, pairwise non-ambiently isotopic, essential and non-peripheral embedded loops in S. We call $[\alpha]_S$ a *multicurve* of S. We write $\mathcal{MC}_0(S)$ for the set of multicurves of S.

The *multicurve graph* $\mathcal{MC}(S)$ is the graph with vertex set $\mathcal{MC}_0(S)$ such that an edge spans m and m' if and only if m misses m'.

Remark 2.2.3. A curve of S is a multicurve of S.

Lemma 2.2.4. *There is a one-to-one correspondence between multicurves of S and non-empty subsets of $\mathcal{C}_0(S)$ of which each pair miss.*

Sketch of proof. Let $\alpha \subset S$ represent m. Then for each connected component α' of α, the class $[\alpha']_S$ is a curve. We may take the union of all such curves and this is the required subset of $\mathcal{C}_0(S)$. It is straightforward that this is well-defined.

Conversely suppose that $\mathcal{S}$ is a set of curves of S with the assumed properties. Enumerate $\mathcal{S}$ and use Lemma 2.3.4 to construct a non-empty union α of pairwise disjoint, pairwise non-ambiently isotopic, essential and non-peripheral embedded loops of S. Then $[\alpha]_S$ is a multicurve of S.

One needs to check that $[\alpha]_S$ is well-defined. Suppose that α and α' are two such constructions as above. Then by Lemma 2.3.4 we may assume that α and α' are disjoint. Then by Lemma 2.3.6, each component of α cobounds an annulus with some component of α' and furthermore these annuli are disjoint. Therefore α and α' are ambiently isotopic in S. $\square$

With regard to Lemma 2.2.4 we may think of multicurves as subsets of $\mathcal{C}_0(S)$. Let c be a curve of S and let m be a multicurve of S. We abuse notation by saying that c is a curve of m, or $c \in m$, if c is an element of the subset of $\mathcal{C}_0(S)$ corresponding to m. We say that m and m' have a *common curve* if there is some curve c such that c is a curve of m and c is a curve of m'.

2.2.6 Subsurfaces

Let $\mathcal{Y}$ be a non-empty, proper and closed subset of S such that $\mathcal{Y}$ is homeomorphic to a connected surface and every connected component of $\partial\mathcal{Y} - \partial S$ is an essential and non-peripheral embedded loop in S. We call $[\mathcal{Y}]_S$ a *subsurface* of S.

Let $Y = [\mathcal{Y}]_S$ be a subsurface of S. If there is a homeomorphism $f\colon S^1 \times [-1,1] \to \mathcal{Y}$ then we say that Y is *annular* and we call $[f(S^1 \times \{0\})]_S$ the *core curve of* Y. We write ∂Y for the core curve of Y.

We say that Y is *non-annular* if Y is not annular. Now $\partial\mathcal{Y} - \partial S$ is a non-empty union of pairwise disjoint, essential and non-peripheral embedded loops in S. (Note that they are not necessarily pairwise non-ambiently isotopic.) Therefore each connected component represents a curve of S. We write ∂Y for the union of these curves of S thus $\partial Y \subset \mathcal{C}_0(S)$. We have that ∂Y is a multicurve of S.

We write $Y = S_{g,n}$ if a representative $\mathcal{Y}$ of Y is homeomorphic to $S_{g,n}$.

2.3 Geometric intersection number, minimal position and uniqueness

2.3.1 Geometric intersection number, bigons and half-bigons

Let A and B be subsets of S. We write $|A \cap B|$ for the number of connected components of $A \cap B$.

Let α be a non-empty union of pairwise disjoint embedded loops and properly embedded intervals in S. Likewise, let β be a such a union. We say that α and β *intersect transversely* if for every $p \in \alpha \cap \beta$ there exists an open subset $n(p)$ of S and a homeomorphism $f\colon (-1,1) \times (-1,1) \to n(p)$ such that $f((0,0)) = p$, $f((-1,1) \times \{0\}) = n(p) \cap \alpha$ and $f(\{0\} \times (-1,1)) = n(p) \cap \beta$. By definition if $\alpha \cap \beta = \emptyset$ then α and β intersect transversely. Note that there exists β' such that α and β' intersect transversely and β' is ambiently isotopic to β.

A subset α' of S is a *closed interval* if it is homeomorphic to $[0,1]$.

Let α and β intersect transversely. We say that α and β *share a bigon* if there exists an embedded closed 2-disc D in S such that $\partial D = \alpha' \cup \beta'$ where $\alpha' \subset \alpha$, $\beta' \subset \beta$, and α' and β' are closed intervals. We say that α and β *share a half-bigon* if there exists an embedded closed 2-disc D in S such that $\partial D = \alpha' \cup \beta' \cup \gamma'$ where $\alpha' \subset \alpha$, $\beta' \subset \beta$, $\gamma' \subset \partial S$, and α', β' and γ' are closed intervals. A bigon D is *innermost* if there does not exist a bigon D' such that $D' \subset D$. A half-bigon D is *innermost* if there does not exist a bigon or half-bigon D' such that $D' \subset D$. We define $i([\alpha]_S, [\beta]_S)$ to be the largest integer k such that whenever α' and β' intersect transversely, and α' and β' are ambiently isotopic to α and β respectively then $k \leq |\alpha' \cap \beta'|$.

Lemma 2.3.1. *Let α and β be embedded loops in S such that α and β intersect transversely. Then $|\alpha \cap \beta| = i([\alpha]_S, [\beta]_S)$ if and only if α and β do not share a bigon.*

Sketch of proof. We refer to [22, Proposition 1.7]. There it is proved that if a bigon is shared by α and β then there exists β' such that α and β' intersect transversely, $|\alpha \cap \beta'| \leq |\alpha \cap \beta| - 2$, and β' is ambiently isotopic to β. Furthermore, the other direction is proved for essential and non-peripheral embedded loops in [22]. Now we sketch how to prove the lemma when α is either inessential or peripheral.

Suppose that α is inessential then $\alpha = \partial D$ for some embedded closed 2-disc D in S. If $\alpha \cap \beta \neq \emptyset$ there exists a bigon D' shared by α and β such that $D' \subset D$.

Similarly if α is peripheral then $\alpha \subset \partial A$ for a peripheral annulus A in S. If $\alpha \cap \beta \neq \emptyset$ then there exists a bigon D shared by α and β such that $D \subset A$. $\square$

Lemma 2.3.2. *Let α be a non-empty union of pairwise disjoint embedded loops in S. Likewise, let β be such a union. Suppose that α and β intersect transversely. Then $|\alpha \cap \beta| = i([\alpha]_S, [\beta]_S)$ if and only if α and β do not share a bigon.*

Sketch of proof. This follows directly from Lemma 2.3.1. □

Suppose that $\partial S \neq \emptyset$. Let $i_1 \colon S_1 \to S$ and $i_2 \colon S_2 \to S$ be homeomorphisms. Consider the disjoint union $S_1 \sqcup S_2$ and identify the points $p_1 \in \partial S_1$ and $p_2 \in \partial S_2$ if $i_1(p_1) = i_2(p_2)$. The resulting space is called the *double* of S and will be written DS. For a subset A of S there are subsets $A_1 \subset S_1$ and $A_2 \subset S_2$ such that $i_1(A_1) = A$ and $i_2(A_2) = A$. The *double* of A is the image of $A_1 \cup A_2$ in the quotient space DS and is written DA.

Lemma 2.3.3 (Bigon/half-bigon criterion)**.** *Let α be a non-empty union of pairwise disjoint embedded loops and properly embedded intervals in S. Likewise, let β be such a union. Suppose that α and β intersect transversely. Then $|\alpha \cap \beta| = i([\alpha]_S, [\beta]_S)$ if and only if α and β do not share a bigon or a half-bigon.*

Sketch of proof. Suppose that $|\alpha \cap \beta| > i([\alpha]_S, [\beta]_S)$. There exists β' such that α and β' intersect transversely, $|\alpha \cap \beta'| = i([\alpha]_S, [\beta]_S)$ and β' is ambiently isotopic to β. Now $D\beta$ is ambiently isotopic to $D\beta'$ in DS. Therefore $D\alpha$ and $D\beta$ share a bigon by Lemma 2.3.2. Therefore α and β share a bigon or a half-bigon.

The other direction is easy. □

2.3.2 Minimal position and its generic uniqueness

Following on from Lemma 2.3.3 we say that α and β are in *minimal position* if α and β intersect transversely and α and β do not share a bigon or a half-bigon.

Lemma 2.3.4 (Minimal position exists)**.** *Suppose that $\gamma_1, ..., \gamma_{k-1}$ are embedded loops or properly embedded intervals in S such that each pair is in minimal position. Given γ_k an embedded loop or properly embedded interval in S, there exists γ_k' such that γ_k' is in minimal position with γ_i whenever $i \in \{1, ..., k-1\}$, and γ_k' is ambiently isotopic to γ_k in S.*

Sketch of proof. After an ambient isotopy we may assume that γ_k and γ_i intersect transversely for each $i \in \{0, ..., k-1\}$.

Suppose that some pair is not in minimal position. By Lemma 2.3.3 there exists an innermost bigon or an innermost half-bigon D shared by γ_k and γ_i for some $i \in \{1, ..., k-1\}$. Now we may ambiently isotope γ_k past D to reduce $|\gamma_k \cap \gamma_i|$ while $|\gamma_k \cap \gamma_{i'}|$ remains constant for each $i' \neq i$. Repeat this process. □

Lemma 2.3.5. *Let $a \in \mathcal{A}_0(S)$ and $c \in \mathcal{C}_0(S)$. If $i(a, c) \leq 2$ then $d_{\mathcal{AC}(S)}(a, c) \leq 2$.*

Sketch of proof. By Lemma 2.3.4 there exist representatives α and γ of a and c respectively such that the representatives are in minimal position.

The lemma is clear if $\alpha \cap \gamma = \emptyset$. If $|\alpha \cap \gamma| = 1$ then we may take a closed regular neighbourhood $N = N(\alpha \cup \gamma)$ of $\alpha \cup \gamma$. Then consider $N' = \overline{\partial N - \partial S}$. There are two connected components of N' and they are both properly embedded intervals in S. If both connected components of N' are inessential then this implies that S is an annulus, a contradiction. So at least one connected component of N' is essential and disjoint from both γ and α as required.

If $|\alpha \cap \gamma| = 2$ then again we consider N' as constructed above. There are two cases.

Case 1. We have that N is planar. Therefore N' is a disjoint union of two embedded loops and two properly embedded arcs. If each of these is either inessential or peripheral in S then S is homeomorphic to a 2-disc, annulus, or a pair of pants, a contradiction.

Case 2. We have that N is not planar. Therefore N' is a disjoint union of two properly embedded intervals in S. If each interval is inessential then we have that $S = S_{1,1}$, a contradiction. □

Let α and α' be embedded loops in S. Suppose that there exists an embedded closed annulus $A \subset S$ such that $\partial A = \alpha \cup \alpha'$. Then we say that α and α' *cobound an annulus A*.

Let α and α' be properly embedded intervals in S. Suppose that there is an embedded closed 2-disc D in S and a homeomorphism $f \colon [0, 1] \times [-1, 1] \to D$ such that $f(\{0, 1\} \times [-1, 1]) \subset \partial S$ and $f([0, 1] \times \{-1, 1\}) = \alpha \cup \alpha'$. Then we say that α and α' *cobound a square D*.

Lemma 2.3.6. *Suppose that α and α' are ambiently isotopic. If α is an essential embedded loop in S then α and α' cobound an annulus. If α is an essential properly embedded interval in S then α and α' cobound a square.*

Sketch of proof. The last statement follows from the first by considering the double of S.

For the first statement we can use hyperbolic geometry and refer to [22, Proof of Proposition 1.10]. Alternatively a proof by contradiction can be given: by considering $S - (\alpha \cup \alpha')$ then constructing an explicit arc or curve x of S such that $i([\alpha]_S, x) \neq 0$ and $i([\alpha']_S, x) = 0$. □

In informal terms, the next lemma states that in most cases minimal position is unique.

Lemma 2.3.7 (Uniqueness of minimal position)**.** *Let α be a non-empty union of pairwise disjoint, essential embedded loops and essential properly embedded intervals in S. Let β be a non-empty union of pairwise disjoint, pairwise non-ambiently isotopic, essential embedded loops and essential properly embedded intervals in S. Suppose that no connected component of α is ambiently isotopic to a connected component of β, and that α and β are in minimal position. If α' and β' are ambiently isotopic to α and β respectively, and α' and β' are in minimal position, then the subsets $\alpha \cup \beta$ and $\alpha' \cup \beta'$ are ambiently isotopic in S.*

Sketch of proof. We may assume that $\alpha = \alpha'$ and that β and β' intersect transversely. Now one constructs an ambient isotopy f_t of S such that $f_t(\alpha) = \alpha$ for all $t \in [0, 1]$ and $f_1(\beta) = \beta'$. For the construction of this ambient isotopy we refer to [22, Lemma 2.9]. $\square$

2.4 In minimal position with a subsurface

Let $\mathcal{Y}$ be a representative of a subsurface Y of S. Let m be a multicurve. By Lemma 2.3.4 there is a representative α of m such that α and $\partial\mathcal{Y}$ are in minimal position. If a connected component α' of α is a peripheral embedded loop in $\mathcal{Y}$ then since α' is non-peripheral in S there exists an ambient isotopy of S that moves α' disjoint from $\mathcal{Y}$. Thus we make the following definition.

We say that $\mathcal{Y}$ and α are in *minimal position* if α and $\partial\mathcal{Y}$ are in minimal position and whenever α' is a connected component of α then α' is not a peripheral embedded loop in $\mathcal{Y}$.

Similarly when α is an essential properly embedded interval in S, we say that $\mathcal{Y}$ and α are in *minimal position* if α and $\partial\mathcal{Y}$ are in minimal position.

2.4.1 Non-annular subsurface projection

Let Y be a non-annular subsurface of S and let c be a curve.

Suppose that c cuts Y. Then c is not a curve of ∂Y. By Lemma 2.3.4 there exist representatives $\mathcal{Y}$ and γ of Y and c respectively such that $\mathcal{Y}$ and γ are in minimal position. Now $\gamma \cap \mathcal{Y}$ is a non-empty union of pairwise disjoint and essential properly embedded intervals in $\mathcal{Y}$ or an essential and non-peripheral embedded loop in $\mathcal{Y}$, therefore there is a corresponding subset of $\mathcal{AC}_0(\mathcal{Y})$ which we write $\kappa_{\mathcal{Y}}(\gamma)$. By Lemma 2.3.7, if γ' is ambiently isotopic to γ, and $\mathcal{Y}$ and γ' are in minimal position then there exists an ambient isotopy of S that preserves $\mathcal{Y}$ but moves γ to γ'. Therefore we deduce that $\kappa_{\mathcal{Y}}(c)$ is a well-defined subset of $\mathcal{AC}_0(\mathcal{Y})$.

If c misses Y then we set $\kappa_{\mathcal{Y}}(c) = \emptyset$.

We say that a pair (A, B) of subsets of S is ambiently isotopic to another pair (A', B') of subsets of S if there exists an ambient isotopy f_t of S such that $f_1(A) = A'$ and $f_1(B) = B'$. This is an equivalence relation. Write $[(A, B)]_S$ for the equivalence class of (A, B).

We write $\mathcal{AC}_0(Y)$ for the set of equivalence classes $[(\mathcal{Y}, \alpha)]_S$ where $\mathcal{Y}$ is a representative of Y and α is a representative of an element $x \in \mathcal{AC}_0(\mathcal{Y})$. There is a natural map $f_{\mathcal{Y}} \colon \mathcal{AC}_0(\mathcal{Y}) \to \mathcal{AC}_0(Y)$ defined by $f_{\mathcal{Y}}([\alpha]_{\mathcal{Y}}) = [(\mathcal{Y}, \alpha)]_S$. Since any ambient isotopy of $\mathcal{Y}$ can be extended to an ambient isotopy of S, we observe that the map $f_{\mathcal{Y}}$ is well-defined.

We set $\kappa_Y(c) = f_{\mathcal{Y}}(\kappa_{\mathcal{Y}}(c))$. This is well-defined, for if f_t is an ambient isotopy of S, and $\mathcal{Y}$ and γ are in minimal position then $f_{\mathcal{Y}}(\kappa_{\mathcal{Y}}(\gamma)) = f_{f_1\mathcal{Y}}(\kappa_{f_1(\mathcal{Y})}(f_1(\gamma)))$.

We define $\kappa_Y(a)$ similarly where a is an arc of S.

For a multicurve m of S we set $\kappa_Y(m)$ equal to the union of $\kappa_Y(c)$ over all curves c of m.

Lemma 2.4.1. *The map $f_{\mathcal{Y}}$ is injective and surjective.*

Sketch of proof. Clearly it is surjective.

For injectivity, it suffices to check that if f_t is an ambient isotopy of S such that $f_1(\mathcal{Y}) = \mathcal{Y}$ then there exists an ambient isotopy h_t of $\mathcal{Y}$ such that $h_1 = f_1|_{\mathcal{Y}}$. This is a consequence of [22, Theorem 3.18]. □

Now $\mathcal{AC}(\mathcal{Y})$ is a graph, and by using the bijection $f_{\mathcal{Y}}$ we construct a graph $\mathcal{AC}(Y)$ with vertex set $\mathcal{AC}_0(Y)$ where x and y span an edge if and only if $f_{\mathcal{Y}}^{-1}(x)$ and $f_{\mathcal{Y}}^{-1}(y)$ span an edge in $\mathcal{AC}(\mathcal{Y})$.

The map κ_Y is called the *subsurface projection.* It is not hard to see that κ_Y is 1-Lipschitz.

2.4.2 Annular subsurface projection

Our main reference for this subsection is [36, Section 2.4]. Let Y be an annular subsurface of S and let c be a curve of S.

We endow int$S$ with an arbitrary finite area, complete hyperbolic metric. We write $\tilde{S}$ for the universal cover of int$S$, which inherits a complete hyperbolic metric from int$S$. By the Cartan–Hadamard theorem we have that $\tilde{S}$ is isometric to the hyperbolic plane $\mathbb{H}^2$. There is a compactification $\overline{\mathbb{H}^2}$ of the hyperbolic plane, see [22, Subsection 1.1.2].

Let $\mathcal{Y}$ be a representative of Y. Let $p \in \mathcal{Y}$. There is a natural injective homomorphism $\pi_1(\mathcal{Y}, p) \to \pi_1(\text{int}S, p)$. This subgroup corresponds to a subgroup of deck transformations of the universal cover $\tilde{S} \to \text{int}S$, which is generated by a hyperbolic isometry $f \colon \tilde{S} \to \tilde{S}$. The map f extends continuously to $\overline{\mathbb{H}^2}$ and write $q_-, q_+ \in \overline{\mathbb{H}^2}$ for the fixed points. We write $\hat{Y}$ for the quotient of $\overline{\mathbb{H}^2} - \{p_-, p_+\}$ by the subgroup $\pi_1(\mathcal{Y}, p)$. We have that $\hat{Y}$ is homeomorphic to a closed annulus.

We write $\mathcal{AC}_0(Y)$ for the set of essential properly embedded intervals in $\hat{Y}$, modulo homotopies that fix the endpoints of the intervals. We call these classes *arcs*. The graph $\mathcal{AC}(Y)$ has vertex set $\mathcal{AC}_0(Y)$ where a pair of vertices span an edge if they have representatives with disjoint interiors.

For $a, b \in \mathcal{AC}_0(Y)$ we define $|a \cap b|$ to be the largest integer k such that whenever α and β represent a and b, and $\text{int}\alpha$ and $\text{int}\beta$ intersect transversely then $k \leq |\text{int}\alpha \cap \text{int}\beta|$.

Lemma 2.4.2 ([36])**.** *For $a \neq b$ we have that $d_{\mathcal{AC}(Y)}(a, b) \leq |a \cap b| + 1$.* □

Suppose that c misses Y then we set $\kappa_Y(c) = \emptyset$.

Suppose that c cuts Y. Let $\mathcal{Y}$ and γ be representatives of Y and c respectively. We set $\kappa_Y(c)$ to be the subset of $\mathcal{AC}_0(Y)$ corresponding to the union of properly embedded intervals in the lift of c to $\hat{Y}$. This is well-defined, indeed, any ambient isotopy of S induces an ambient isotopy of $\hat{Y}$.

Chapter 3

Uniform hyperbolicity of arc graphs and curve graphs

The goal of this chapter is to show that arc graphs and curve graphs are uniformly hyperbolic. In Subsection 3.1.1 we describe *unicorn paths* between two arcs in the arc graph. Similar to how geodesic triangles are slim in hyperbolic graphs, we shall show that triangles of unicorn paths are uniformly slim in the arc graph (Lemma 3.1.4). Using this and an astounding hyperbolicity criterion of Masur and Schleimer [38, Theorem 3.15], [11, Proposition 3.1] we shall deduce that arc graphs are uniformly hyperbolic.

In Subsection 3.1.3 we show that arc and curve graphs are uniformly hyperbolic for surfaces with boundary. The proof makes use of unicorn paths and the Masur and Schleimer criterion. It follows that curve graphs are uniformly hyperbolic for surfaces with boundary since they are uniformly quasi-isometric to arc and curve graphs. Finally we prove directly that if $\mathcal{C}(S_{g,1})$ is δ-hyperbolic, where $g \geq 2$, then $\mathcal{C}(S_{g,0})$ is δ-hyperbolic.

The material in Subsection 3.1.1 is introductory and not original work of the author. The rest of this chapter is a short account of joint work of Sebastian Hensel, Piotr Przytycki and the author [30].

3.1 Surfaces with boundary

Throughout this section we assume that $S = S_{g,n}$ where $n > 0$ and $\xi(S) \geq 2$.

3.1.1 Unicorn paths

Let $a \neq b \in \mathcal{A}_0(S)$. By Lemma 2.3.4 there exist representatives α and β of a and b respectively such that α and β are in minimal position. Fix $p_\alpha \in \alpha \cap \partial S$ and $p_\beta \in \beta \cap \partial S$.

There exists a homeomorphism $i\colon [0,1] \to \alpha$ such that $i(1) = p_\alpha$. For $t \in [0,1]$ we write $\alpha(t) = i([t,1])$.

For $t \in [0,1]$ if $\alpha(t) \cap \beta \neq \emptyset$ then there is a unique element $p = p(t) \in \alpha(t) \cap \beta$ which is nearest to p_β in the closed interval β. We write α_p for the subinterval of α with boundary equal to $\{p, p_\alpha\}$. We write β_p for the subinterval of β with boundary equal to $\{p, p_\beta\}$. Because p is nearest to p_β we have that $\alpha_p \cup \beta_p$ is a properly embedded interval in S.

Lemma 3.1.1. *We have that $\alpha_p \cup \beta_p$ is essential.*

Proof. We assume that $S - \alpha_p \cup \beta_p$ has a simply connected component D. We shall show that α and β are not in minimal position. Indeed, write $\bar{D} \subset S$ for the closure of D in S. We have that $\bar{D}$ is homeomorphic to a 2-disc. Furthermore we have that $\alpha_p \cup \beta_p \subset \partial \bar{D}$. We have that $\bar{D}$ is a half-bigon shared by α and β as required. $\square$

We write $v(p) = [\alpha_p \cup \beta_p]_S$. For $t \in [0,1]$ if $p(t)$ is defined then set $\mathcal{S}(t) = \{p(t)\}$ and if not then set $\mathcal{S}(t) = \emptyset$. Set $\mathcal{S} = \cup_{t \in [0,1]} \mathcal{S}(t)$. In fact there is an order on $\mathcal{S}$ which is induced by the natural order on $[0,1]$. So we may write $\mathcal{S} = \{p_1, ..., p_{k-1}\}$. If $\mathcal{S} = \emptyset$ then we set $k = 1$. By considering $t = 0$ and $\alpha(t) = \alpha$ we see that $\mathcal{S} = \emptyset$ is equivalent to $\alpha \cap \beta = \emptyset$. We remark that whenever $i < j$ we have that p_i is nearer to p_β than p_j in the interval β. Also whenever $i < j$ we have that p_j is nearer to p_α than p_i in the interval α. There is a sequence $(v_1, ..., v_{k-1})$ such that $v_i = v(p_i)$, this sequence is empty if $\mathcal{S} = \emptyset$. We set $v_0 = a$ and $v_k = b$.

Remark 3.1.2. If we exchange the pairs (α, p_α) and (β, p_β) then this induces the opposite order on $\mathcal{S}$ and the opposite order on the sequence $(v_0, ..., v_k)$.

Lemma 3.1.3. *The sequence $(v_0, ..., v_k)$ is a path in $\mathcal{A}(S)$.*

Proof. If $\mathcal{S} = \emptyset$ then $\alpha \cap \beta = \emptyset$ and we have that $a = v_0$ misses $v_1 = b$. So suppose that $\mathcal{S} \neq \emptyset$.

Write $N(\alpha)$ and $N(\beta)$ for closed regular neighbourhoods of α and β respectively. There is a homeomorphism from $[0,1] \times [-1,1]$ to $N(\alpha)$. We say that a subset is *vertical* in $N(\alpha)$ if it is of the form $\{t\} \times [-1,1]$ and *horizontal* in $N(\alpha)$ if it is of the form $[0,1] \times \{t\}$.

We may assume that α is horizontal in $N(\alpha)$ and that $\beta \cap N(\alpha)$ is a union of vertical sets in $N(\alpha)$.

Let $p_i, p_{i+1} \in \mathcal{S}$. We may assume that $\alpha_{p_i} = [t_i, 1] \times \{0\}$ and that $\alpha_{p_{i+1}} = [t_{i+1}, 1] \times \{0\}$ in $N(\alpha)$. We have that $t_i < t_{i+1}$. We may assume that each connected component of $\beta_{p_{i+1}} \cap N(\alpha)$ is either $\{t'\} \times [-1, 1]$ where $t' < t_i$, $\{t_i\} \times [-1, 1]$ or $\{t_{i+1}\} \times [0, 1]$ in $N(\alpha)$.

Fix ϵ such that $0 < \epsilon < 1$. Write $\alpha'_{p_{i+1}}$ for the interval $[t_{i+1}, 1] \times \{\epsilon\}$. Write $\beta'_{p_{i+1}}$ for the unique subinterval of $\beta_{p_{i+1}}$ such that $\alpha'_{p_{i+1}} \cup \beta'_{p_{i+1}}$ is a properly embedded interval. Write $\gamma_{i+1} = \alpha'_{p_{i+1}} \cup \beta'_{p_{i+1}}$. We have that γ_{i+1} is ambiently isotopic to $\alpha_{p_{i+1}} \cup \beta_{p_{i+1}}$. We have that $\gamma_{i+1} \cap (\alpha_{p_i} \cup \beta_{p_i}) = \beta_{p_i}$ and therefore there exists a small perturbation of γ_{i+1} which is disjoint from $\alpha_{p_i} \cup \beta_{p_i}$. This shows that v_{i+1} misses v_i.

Proving that v_0 misses v_1 is similar but easier. By Remark 3.1.2 we have that v_{k-1} misses v_k. $\square$

The sequence $(v_0, ..., v_k)$ is called a *unicorn path* in [30]. These paths originally appear in Hatcher's short proof that the arc complex is contractible [28]. They were also used by Hensel, Osadja and Przytycki [29] to show that a finite subgroup of the mapping class group fixes a clique in the arc graph $\mathcal{A}(S)$. The path depends on the choice of p_α and p_β and so there are at most four possibilities. We write $P(a, b)$ for the union of the unicorn paths. We have that $P(a, b)$ is well-defined by Lemma 2.3.7 (Uniqueness of minimal position).

If $a = b$ where $a, b \in \mathcal{A}_0(S)$ then we set $P(a, b) = \{a\}$.

3.1.2 Uniform hyperbolicity of arc graphs

Lemma 3.1.4. *Let $a, b, c \in \mathcal{A}_0(S)$. The following statements hold.*

1. *The subgraph in $\mathcal{A}(S)$ spanned by $P(a, b)$ is connected.*
2. *If a misses b then $P(a, b) = \{a\} \cup \{b\}$.*
3. *We have that $P(a, c) \subset N_{\mathcal{A}(S)}(P(a, b) \cup P(b, c); 1)$.*

Proof of 1. The set $P(a, b)$ consists of vertices in $\mathcal{A}(S)$ that form unicorn paths that connect a to b so the subgraph spanned by these vertices is connected.

Proof of 2. Using the notation from earlier discussion, we have that $\mathcal{S} = \emptyset$ hence $\{v_0\} \cup \{v_1\} = P(a, b)$.

Proof of 3. There exist representatives α, β and γ of a, b and c respectively such that each pair is in minimal position and $\alpha \cap \beta \cap \gamma = \emptyset$ by Lemma 2.3.4. We use the notation from our earlier construction of unicorn paths. We are given $v \in P(a, c)$ so there exist $p_\alpha \in \alpha \cap \partial S$, $p_\gamma \in \gamma \cap \partial S$, and a representative $\alpha_p \cup \gamma_p$ of v such that $\alpha_p \subset \alpha$ and $\gamma_p \subset \gamma$. Fix $p_\beta \in \beta \cap \partial S$.

If $\beta \cap (\alpha_p \cup \gamma_p) = \emptyset$ then we are done.

If $\beta \cap (\alpha_p \cup \gamma_p) \neq \emptyset$ then there exists q in this non-empty set which is nearest to p_β in the interval β. Since $\alpha \cap \beta \cap \gamma = \emptyset$ either $q \in \alpha_p$ or $q \in \gamma_p$ but not both. Without loss of generality $q \in \alpha_p$. There exists $\beta_q \subset \beta$ and $\alpha_q \subset \alpha_p \subset \alpha$ such that $\alpha_q \cup \beta_q$ is a properly embedded interval in S. Note that $(\alpha_q \cup \beta_q) \cap \gamma_p = \emptyset$. By definition $[\alpha_q \cup \beta_q]_S \in P(a, b)$. Furthermore we have that $(\alpha_q \cup \beta_q) \cap (\alpha_p \cup \gamma_p) = \alpha_q$ so there exists a small perturbation which makes these properly embedded intervals disjoint. We conclude that v misses $[\alpha_q \cup \beta_q]_S$ as required. □

Theorem 3.1.5. *There exists a uniform constant δ such that the graph $\mathcal{A}(S)$ is δ-hyperbolic.*

Proof. We set $\mathcal{L}(a, b)$ to be the subgraph of $\mathcal{A}(S)$ spanned by $P(a, b)$ and use a hyperbolicity criterion of Masur and Schleimer see [11, Proposition 3.1]. This hyperbolicity criterion originally appears in [38, Theorem 3.15]. By Lemma 3.1.4 the criteria required for $\mathcal{L}(a, b)$ are satisfied. □

3.1.3 Uniform hyperbolicity of arc and curve graphs

Let $x \in \mathcal{AC}_0(S)$. We define

$$\mathscr{A}(x) := \begin{cases} \{a \in \mathcal{A}_0(S) : d_{\mathcal{AC}(S)}(a, x) = 1\} & \text{if } x \in \mathcal{C}_0(S) \\ \{x\} & \text{if } x \in \mathcal{A}_0(S). \end{cases}$$

Lemma 3.1.6. *Let $a, a' \in \mathcal{A}_0(S)$ and $c \in \mathcal{C}_0(S)$. Suppose that*

$$d_{\mathcal{AC}(S)}(a, c), d_{\mathcal{AC}(S)}(a', c) = 1$$

then $P(a, a') \subset N_{\mathcal{AC}(S)}(c; 1)$

Proof. There exist representatives α, α' and γ of a, a' and c respectively such that each pair is in minimal position by Lemma 2.3.4. By definition for any $v \in P(a, a')$ there is a representative of v which is a subset of $\alpha \cup \alpha'$ but we have that $\gamma \cap (\alpha \cup \alpha') = \emptyset$ therefore $v \in N_{\mathcal{AC}(S)}(c; 1)$ as required. □

Lemma 3.1.7. *Let $x, y \in \mathcal{C}_0(S)$ and suppose that x misses y then there exists $a \in \mathscr{A}(x) \cap \mathscr{A}(y)$. In general, for any multicurve m there exists $a \in \mathcal{A}_0(S)$ such that a misses m.*

Proof. The last statement implies the first. So let α be a representative of a multicurve m.

There exists an embedded closed interval γ in S such that one endpoint of γ is in ∂S, one endpoint is in α, and the interior of γ is disjoint from α. Let $\alpha' \subset \alpha$ be the unique connected component that intersects γ. We have that α' is essential and non-peripheral.

Take a closed regular neighbourhood $N = N(\alpha' \cup \gamma)$ of $\alpha' \cup \gamma$. Then $\overline{\partial N - \partial S}$ consists of a properly embedded arc γ' and an embedded loop in S. If γ' is inessential then $S - \gamma'$ has a simply connected component D. Clearly this component cannot contain α' otherwise α' is inessential. Therefore $D \cap N = \emptyset$. However $N \cup D$ is a peripheral annulus that contains α', a contradiction. Therefore γ' is essential and we set $\mathscr{a} = [\gamma']_S$ as required. $\square$

Let $x, y \in \mathcal{AC}_0(S)$. We define

$$\mathcal{P}(x, y) := \{x\} \cup \{y\} \cup \left(\bigcup_{\mathscr{a} \in \mathscr{A}(x), \mathscr{b} \in \mathscr{A}(y)} P(\mathscr{a}, \mathscr{b}) \right).$$

Lemma 3.1.8. *Let $x, y, z \in \mathcal{AC}_0(S)$. The following statements hold.*

1. *The subgraph in $\mathcal{AC}(S)$ spanned by $\mathcal{P}(x, y)$ is connected.*
2. *Whenever $d_{\mathcal{AC}(S)}(x, y) = 1$ we have that* $\operatorname{diam}_{\mathcal{AC}(S)}(\mathcal{P}(x, y)) \leq 5$.
3. *We have that $\mathcal{P}(x, z) \subset N_{\mathcal{AC}(S)}(\mathcal{P}(x, y) \cup \mathcal{P}(y, z); 1)$.*

Proof of 1. This is clear by statement *1.* of Lemma 3.1.4.
Proof of 2. If $x, y \in \mathcal{A}_0(S)$ then by Lemma 3.1.4 the diameter is less than or equal to 1.

If $x \in \mathcal{A}_0(S)$ and $y \in \mathcal{C}_0(S)$ then for $v \in \mathcal{P}(x, y)$ such that $v \notin \{x, y\}$ we have that $v \in P(\mathscr{a}, \mathscr{b})$ for some $\mathscr{a} \in \mathscr{A}(x)$ and for some $\mathscr{b} \in \mathscr{A}(y)$. By Lemma 3.1.6 we have that v and y are adjacent vertices of $\mathcal{AC}(S)$. Since there is a path that connects v to y and its length is less than or equal to 1 we conclude that the diameter is less than or equal to 2.

If $x, y \in \mathcal{C}_0(S)$ then for $v \in \mathcal{P}(x, y)$ such that $v \notin \{x\} \cup \{y\}$ we have that $v \in P(\mathscr{a}, \mathscr{b})$ for some $\mathscr{a} \in \mathscr{A}(x)$ and for some $\mathscr{b} \in \mathscr{A}(y)$. By Lemma 3.1.7 there exists an arc $\mathscr{c} \in \mathscr{A}(x) \cap \mathscr{A}(y)$. By statement *3.* of Lemma 3.1.4 there exists an arc $v' \in P(\mathscr{a}, \mathscr{c}) \cup P(\mathscr{c}, \mathscr{b})$ such that v' misses v. By Lemma 3.1.6 we have that v' misses x or y. Therefore there is some path that connects v to x or connects v to y and its length is less than or equal to 2. Since x and y are adjacent we conclude that the diameter is less than or equal to 5.

Proof of 3. Given $v \in \mathcal{P}(x,z)$ if $v \in \{x\} \cup \{z\}$ then $v \in \mathcal{P}(x,y) \cup \mathcal{P}(y,z)$. If $v \notin \{x\} \cup \{z\}$ then by definition $v \in P(\mathfrak{a}, \mathfrak{c})$ for some $\mathfrak{a} \in \mathcal{A}(x)$ and for some $\mathfrak{c} \in \mathcal{A}(z)$. Let $\mathfrak{b} \in \mathcal{A}(y)$. Then by Lemma 3.1.4 we have that $v \in N_{\mathcal{AC}(S)}(P(\mathfrak{a}, \mathfrak{b}) \cup P(\mathfrak{b}, \mathfrak{c}); 1)$. Since $P(\mathfrak{a}, \mathfrak{b}) \subset \mathcal{P}(x,y)$ and $P(\mathfrak{b}, \mathfrak{c}) \subset \mathcal{P}(y,z)$ we are done. □

Theorem 3.1.9. *There exists a uniform constant δ such that the graph $\mathcal{AC}(S)$ is δ-hyperbolic.*

Proof. We set $\mathcal{L}(x,y)$ to be the subgraph in $\mathcal{AC}(S)$ spanned by $\mathcal{P}(x,y)$. By Lemma 3.1.8 we have that $\mathcal{L}(x,y)$ satisfies the criteria of [11, Proposition 3.1]. □

Theorem 3.1.10. *Let $S = S_{g,n}$ where $n > 0$. Suppose that $\xi(S) \geq 2$. There exists a uniform constant δ such that the graph $\mathcal{C}(S)$ is δ-hyperbolic.*

Sketch of proof. The graph $\mathcal{C}(S)$ is 1-dense in $\mathcal{AC}(S)$ and the natural inclusion map satisfies $d_{\mathcal{C}(S)}(c,c') \leq 2d_{\mathcal{AC}(S)}(c,c')$ see [36, Lemma 2.2] or Lemma 2.2.2. Therefore $\mathcal{C}(S)$ and $\mathcal{AC}(S)$ are quasi-isometric with uniform constants.

Since this is the only place where we use the terms quasi-isometric and k-dense, we do not define them. See [12] for these definitions.

We have that $\mathcal{AC}(S)$ is uniformly hyperbolic by Theorem 3.1.9. By [12, Theorem 6.19] we have a hyperbolicity constant for $\mathcal{C}(S)$ that depends on the hyperbolicity constant of $\mathcal{AC}(S)$ and the quasi-isometry constants. Therefore $\mathcal{C}(S)$ is uniformly hyperbolic. □

Remark 3.1.11. In [30] it is shown that for each geodesic triangle in $\mathcal{C}(S)$ there is a curve which is contained in the 17-neighbourhood of any geodesic of the triangle i.e. the geodesic triangle is 17-*thin*.

3.2 Surfaces without boundary

Let $g \geq 2$ be an integer. Set $S = S_{g,0}$. Let $\overline{D}$ be an embedded closed 2-disc in S and write D for its interior. Write $S' = S - D$. We have that S' is homeomorphic to $S_{g,1}$. There is a natural inclusion

$$\iota : S' \to S$$

and this induces a map $\iota_* : \mathcal{C}_0(S') \to \mathcal{C}_0(S)$ where

$$\iota_*([\gamma]_{S'}) := [\iota(\gamma)]_S.$$

Any ambient isotopy of S' extends to an ambient isotopy of S so ι_* is well-defined. Furthermore ι_* is 1-Lipschitz.

Theorem 3.2.1. *There exists a uniform constant δ such that the graph $\mathcal{C}(S)$ is δ-hyperbolic whenever $S = S_{g,n}$ and $\xi(S) \geq 2$.*

Proof. By Theorem 3.1.10 there exists δ such that whenever $n > 0$ we have that $\mathcal{C}(S_{g,n})$ is δ-hyperbolic. We shall show that $\mathcal{C}(S_{g,0})$ is δ-hyperbolic where $g \geq 2$.

Let P_1, P_2 and P_3 form a geodesic triangle T in $\mathcal{C}(S)$. By Lemma 2.3.4 there exist representatives γ_v for each $v \in T$ such that whenever $v, v' \in T$ and $v \neq v'$ we have that γ_v and $\gamma_{v'}$ are in minimal position. There exists an embedded closed 2-disc $\overline{D}$ such that $\overline{D}$ is disjoint from γ_v for every $v \in T$. Write $D = \mathrm{int}\overline{D}$ and $S' = S - D$. For every $v \in T$ we have that $\gamma_v \subset S'$ and γ_v is essential in S'. From earlier discussion we have that $\iota_*([\gamma_v]_{S'}) = [\gamma_v]_S$.

Write $P_1 = (v_0, ..., v_n)$. Since the map ι_* is 1-Lipschitz, we have that $P_1' = ([\gamma_{v_0}]_{S'}, ..., [\gamma_{v_n}]_{S'})$ is a geodesic in $\mathcal{C}(S')$. In a similar fashion we define P_2' and P_3'. Then P_1', P_2' and P_3' form a geodesic triangle T' in $\mathcal{C}(S')$. But $\mathcal{C}(S')$ is δ-hyperbolic so the triangle T' is δ-slim. Since ι_* is 1-Lipschitz we have that T is δ-slim. The geodesic triangle T was arbitrary hence $\mathcal{C}(S)$ is δ-hyperbolic. $\square$

Chapter 4

A uniform bound for the bounded geodesic image theorem

Masur and Minsky [36] originally proved that if a fixed geodesic Q in $\mathcal{C}(S)$ is such that each of its vertices cut a fixed subsurface Y, then the diameter of the image of Q under the subsurface projection κ_Y is bounded from above by K, where K only depends on S. In this chapter we give an elementary proof that we can take K to be 62, or better still, 50 if Y is non-annular (Theorem 4.2.1).

The statement of this theorem should be compared to the following phenomena in hyperbolic spaces. The diameter of the image of a (suitably far away) geodesic under projection to a horosphere in $\mathbb{H}^n$ is uniformly bounded from above. The total image under nearest point projection of a geodesic far from a quasi-convex set in a hyperbolic geodesic metric space has its diameter uniformly bounded from above. The bounded geodesic image theorem is an aspect of the hyperbolicity of the curve graph, and the proof given here follows this philosophy.

We use many of the concepts from Chapter 3.

The case where Y is non-annular is much simpler. When Y is non-annular and S is closed, we may deduce the theorem from the case where Y is non-annular and S has boundary. Indeed, we will use the strategy of the proof of Theorem 3.2.1 where we removed an open disc of S.

So assume that S has boundary. We pick an arc $\mathscr{a}$ that misses ∂Y. Given any two curves c and c' of Q, we pick arcs $\mathscr{b}$ and $\mathscr{b}'$ of S that miss c and c' respectively. The unicorn paths between $\mathscr{b}$ and $\mathscr{b}'$ are contained in a small neighbourhood of the subsequence of Q between c and c' (Lemma 4.1.1). By Lemma 4.1.2 there is a unicorn arc $\mathscr{v}''$ of $\mathscr{b}$ and $\mathscr{b}'$ that is adjacent to a unicorn arc $\mathscr{v}$ of $\mathscr{b}$ and $\mathscr{a}$, and a unicorn arc $\mathscr{v}'$ of $\mathscr{b}'$ and $\mathscr{a}$. Now if the subsequence of Q between c and c' is sufficiently far away from ∂Y in $\mathcal{C}(S)$ then we can show that $\kappa_Y(\mathscr{v})$ and $\kappa_Y(\mathscr{b})$ have an arc in common (Lemma 4.1.6), and similarly for $\kappa_Y(\mathscr{v}')$ and $\kappa_Y(\mathscr{b}')$. These arguments show that the diameter of $\kappa_Y(c) \cup \kappa_Y(c')$ is bounded above by 3 when the geodesic between these curves is sufficiently far from ∂Y (Theorem 4.1.7). It is surprising that such a small bound can be established. In general, Q can be close to ∂Y in $\mathcal{C}(S)$ but this only happens on a subinterval of Q of universally bounded length - the complement of which we can show has bounded image under κ_Y. This is why we take a bound no less than 50.

The case where Y is annular is harder and the strategy is as follows. When S is closed, we pick a subsurface Z such that $\xi(Z) = 1$ and the core curve of Y is a curve of Z. We construct a new surface S' by removing a small open disc of S disjoint from fixed representatives of Y, Z and the curves of the geodesic Q. There are subsurfaces Y' and Z' of S' that Y and Z naturally correspond to, and similarly there is a geodesic Q' of $\mathcal{C}(S')$ corresponding to Q.

We construct an arc $\mathscr{a}$ of S' that misses Y' and $\partial Z'$. For any curves c and c' of the geodesic Q' we pick arbitrary arcs $\mathscr{b}$ and $\mathscr{b}'$ of S' that miss c and c' respectively. We consider the unicorn paths between the three vertices $\mathscr{a}$, $\mathscr{b}$ and $\mathscr{b}'$. By Lemma 4.1.2 there is a unicorn arc $\mathscr{v}''$ of $\mathscr{b}$ and $\mathscr{b}'$ that is adjacent to a unicorn arc $\mathscr{v}$ of $\mathscr{b}$ and $\mathscr{a}$, and a unicorn arc $\mathscr{v}'$ of $\mathscr{b}'$ and $\mathscr{a}$. If the subsequence of Q' between c and c' is sufficiently far from $\partial Y'$ then we may find a common arc of $\kappa_{Z'}(\mathscr{v})$ and $\kappa_{Z'}(\mathscr{b})$ that cuts $\partial Y'$ in Z' (Lemma 4.1.5), and similarly for $\kappa_{Z'}(\mathscr{v}')$ and $\kappa_{Z'}(\mathscr{b}')$. The condition that this arc cuts Y' is necessary in order to apply Lemma 4.1.3, which is the key lemma for bounding the diameter of $\kappa_{Y'}(Q')$. The conclusion is that the diameter of $\kappa_{Y'}(c) \cup \kappa_{Y'}(c')$ is bounded from above by 7 if Q' is sufficiently far from $\partial Y'$ (Theorem 4.1.7). In general Q' may get close to $\partial Y'$, but only on an interval of bounded length, hence our bound is no less than 62.

In summary, we use $\kappa_{Z'}(Q')$ to learn about $\kappa_{Y'}(Q')$ by using Lemma 4.1.3. The images $\kappa_Z(Q)$ and $\kappa_{Z'}(Q')$ are in natural correspondence because Z and Z' are, and so Lemma 4.1.3 is used again to bound the diameter of $\kappa_Y(Q)$, finishing the proof.

It is important to note that it cannot immediately be said that $\kappa_Y(Q)$ and $\kappa_{Y'}(Q')$ are comparable because the definition of κ_Y relies on taking a cover of S. For this reason the details in this chapter concentrate on the case where Y is annular.

4.1 Surfaces with boundary revisited

Lemma 4.1.1. *Let c and c' be curves and let $\mathfrak{a} \in \mathcal{A}(c)$ and $\mathfrak{b} \in \mathcal{A}(c')$. Suppose that $Q = (c_j)_{j \in I}$ is a geodesic in $\mathcal{C}(S)$ that connects c to c'. Then for every $v \in P(\mathfrak{a}, \mathfrak{b})$ there exists $c_k \in Q$ such that $d_{\mathcal{AC}(S)}(v, c_k) \leq 8$.*

Proof. We refer to the second paragraph of the proof of Theorem 1.1 in [30]. $\square$

Lemma 4.1.2. *Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathcal{A}_0(S)$. There exist $v \in P(\mathfrak{a}, \mathfrak{b})$, $v' \in P(\mathfrak{b}, \mathfrak{c})$ and $v'' \in P(\mathfrak{c}, \mathfrak{a})$ such that any pair of the arcs v, v' and v'' miss.*

Proof. We refer to [30, Lemma 3.4]. $\square$

The following lemma is important for Theorem 4.2.1 in the case that the subsurface is annular.

Lemma 4.1.3. *Let Y and Z be subsurfaces and suppose that Y is annular and $\partial Y \in \mathcal{C}_0(Z)$. Let $c, c' \in \mathcal{C}_0(S)$. Suppose that there exists a path $(a_0, ..., a_L)$ in $\mathcal{AC}_0(Z)$ such that for each a_i we have that a_i cuts ∂Y, $a_0 \in \kappa_Z(c)$ and $a_L \in \kappa_Z(c')$. Then* $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(c) \cup \kappa_Y(c')) \leq L + 4$.

Proof. Let $\mathcal{Y}$ be a representative of Y. There exists a representative $\mathcal{Z}$ of Z such that $\partial\mathcal{Y}$ is a union of two non-peripheral embedded loops in $\mathcal{Z}$.

By Lemma 2.3.4 there exist representatives γ and γ' of c and c' respectively such that each pair of γ, γ', $\mathcal{Y}$ and $\mathcal{Z}$ are in minimal position where applicable.

We may assume that $\gamma \cap \mathcal{Z}$ and $\gamma' \cap \mathcal{Z}$ are in minimal position in $\mathcal{Z}$, for if they were not then by Lemma 2.3.3 there exists an innermost half-bigon shared by $\gamma \cap \mathcal{Z}$ and $\gamma' \cap \mathcal{Z}$ in $\mathcal{Z}$. We can isotope γ' across this *triangle region* (an embedded closed 2-disc D such that $\partial D = \epsilon_1 \cup \epsilon_2 \cup \epsilon_3$ where ϵ_1, ϵ_2 and ϵ_3 are closed intervals and subsets of γ, γ' and $\partial\mathcal{Z}$) cobounded by γ, γ' and $\partial\mathcal{Z}$ in order to remove the half-bigon inside $\mathcal{Z}$. Each time a half-bigon is removed the intersection between $\gamma \cap \mathcal{Z}$ and $\gamma' \cap \mathcal{Z}$ decreases by 1 so repeating this procedure eventually constructs the required representatives.

We may assume that any triangle region cobounded by $\partial\mathcal{Y}$, γ and γ' is contained in $\mathcal{Y}$.

Suppose that $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(c) \cup \kappa_Y(c')) \geq L+5$. By Lemma 2.4.2 there exist arcs $[\delta^*] \in \kappa_Y(c)$ and $[\epsilon^*] \in \kappa_Y(c')$ such that $|[\delta^*] \cap [\epsilon^*]| \geq L + 4$. Following a claim from [38, Section 10] we have that $|\delta^* \cap \epsilon^* \cap \mathcal{Y}'| \geq L+2$ where $\mathcal{Y}'$ is a homeomorphic lift of $\mathcal{Y}$ in $\hat{Y}$, δ^* is a connected component of the lift of γ and ϵ^* is a connected component of the lift of γ'.

There exists a connected component α_0 of $\gamma \cap \mathcal{Z}$ that represents a_0 and there exists a connected component α_L of $\gamma' \cap \mathcal{Z}$ that represents a_L.

Fix a connected component α'_0 of $\alpha_0 \cap \mathcal{Y}$ and fix a connected component α'_L of $\alpha_L \cap \mathcal{Y}$.

Now $\mathcal{Y}'$ is the homeomorphic lift of $\mathcal{Y}$ so write δ' for the subset of $\mathcal{Y}'$ that corresponds to $\alpha'_0 \subset \mathcal{Y}$. Similarly we write ϵ' for the subset of $\mathcal{Y}'$ that corresponds to $\alpha'_L \subset \mathcal{Y}$.

We have that δ' is disjoint from δ^*, or $\delta' = \delta^* \cap \mathcal{Y}'$, and that $\mathcal{Y}'$ is an annulus therefore $|\delta' \cap \epsilon^*| \geq L + 1$. Similarly we have that $|\delta' \cap \epsilon'| \geq L$. Therefore $|\alpha'_0 \cap \alpha'_L| \geq L$.

By Lemma 2.3.4 there exist representatives $\alpha_1, ..., \alpha_{L-1}$ in $\mathcal{Z}$ of $a_1, ..., a_{L-1}$ respectively such that each pair of $\alpha_0, ..., \alpha_L$ is in minimal position and that α_i and $\mathcal{Y}$ are in minimal position for each i. Hence for each $i \in \{0, ..., L-1\}$ we have that $\alpha_i \cap \alpha_{i+1} = \emptyset$.

Now a_i cuts ∂Y so $\alpha_i \cap \mathcal{Y} \neq \emptyset$. For each $i \in \{1, ..., L-1\}$ fix a connected component α'_i of $\alpha_i \cap \mathcal{Y}$.

Recall that $|\alpha'_0 \cap \alpha'_L| \geq L$ and therefore we have that $|\alpha'_0 \cap \alpha'_{L-1}| \geq L-1$. We can argue inductively and deduce that $|\alpha'_0 \cap \alpha'_1| \geq L-(L-1) = 1$. This contradicts the earlier statement that $\alpha_0 \cap \alpha_1 = \emptyset$. □

We require the following lemmas (Lemma 4.1.5 and Lemma 4.1.6) to control the subsurface projection of unicorn paths. These lemmas state that if a vertex of a unicorn path is sufficiently far away from ∂Y in $\mathcal{AC}(S)$ then we can control its image under subsurface projection. This is the crucial step in our proof of Theorem 4.1.8. However we require Lemma 4.1.4 to prove Lemma 4.1.5.

Lemma 4.1.4. *Let Y and Z be subsurfaces. Suppose that Y is annular, $Z = S_{0,4}$ and $\partial Y \in \mathcal{C}_0(Z)$. Let $a_1, a_2 \in \mathcal{AC}_0(Z)$. If $i(a_1, \partial Y) \geq 3$ and a_1 misses a_2 then a_2 cuts ∂Y.*

Proof. Suppose for a contradiction that a_2 misses ∂Y.

There exist representatives γ and $\mathcal{Z}$ of ∂Y and Z respectively such that $\gamma \subset \mathcal{Z}$. Following the proof of Lemma 2.3.4 there exist representatives α_1 and α_2 in $\mathcal{Z}$ of a_1 and a_2 respectively such that $\alpha_1 \cap \alpha_2 = \emptyset$, α_1 and γ are in minimal position and $\alpha_2 \cap \gamma = \emptyset$.

Now $\mathcal{Z} - \gamma$ has two components. Therefore α_2 is a subset of one such component $\mathcal{P}$. We have that $\overline{\mathcal{P}}$ is a pair of pants. Now $\alpha_1 \cap \gamma$ is a non-empty set and its cardinality is greater than or equal to 3 therefore there exists a subinterval ϵ of

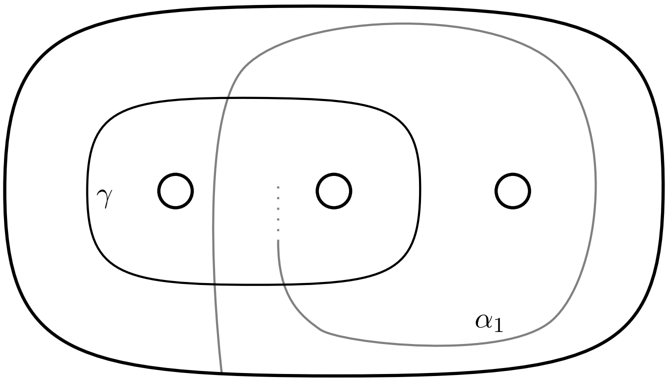

Figure 4.1: The proof of Lemma 4.1.4.

α_1 such that ϵ is an essential properly embedded interval of $\overline{\mathcal{P}}$ and both endpoints of ϵ are in γ. We have that ϵ must intersect α_2, a contradiction. $\square$

Lemma 4.1.5. *Suppose that Y is annular, $\partial Y \in \mathcal{C}_0(Z)$ and Z is a subsurface of S, where $\xi(Z) = 1$, $S = S_{g,n}$ and $n > 0$. Let a_Z be a curve of ∂Z. Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}_0(S)$. Suppose that $\mathfrak{a}$ misses Y and $\mathfrak{a}$ misses ∂Z. Then for any $v \in P(\mathfrak{a}, \mathfrak{b})$ at least one of the following statements holds.*

1. *There exists $\mathfrak{a}' \in \kappa_Z(v) \cap \kappa_Z(\mathfrak{b})$ such that $\mathfrak{a}'$ cuts ∂Y.*
2. *We have that $d_{\mathcal{AC}(S)}(v, \partial Y) \leq 3$.*

Proof. Write $\mathcal{Y}$ for a representative of the core curve of Y.

There exists a representative $\mathcal{Z}$ of Z such that $\mathcal{Y} \subset \mathcal{Z}$ and $\mathcal{Y}$ is non-peripheral in $\mathcal{Z}$. Following the proof of Lemma 2.3.4 there exist representatives α and β of $\mathfrak{a}$ and $\mathfrak{b}$ respectively such that α and β are in minimal position, β and $\partial\mathcal{Z}$ are in minimal position, β and $\mathcal{Y}$ are in minimal position, $\alpha \cap \mathcal{Y} = \emptyset$ and $\alpha \cap (\partial\mathcal{Z} - \partial S) = \emptyset$.

We have that $v \in P(\mathfrak{a}, \mathfrak{b})$ so by definition there exist subintervals $\alpha_p \subset \alpha$ and $\beta_p \subset \beta$ such that $\alpha_p \cup \beta_p$ is a representative of v. Write $\gamma = \alpha_p \cup \beta_p$.

There do not exist any bigons between β and $\partial\mathcal{Z}$ and α is disjoint from $\partial\mathcal{Z}$. We deduce that γ and $\partial\mathcal{Z}$ do not share a bigon so by Lemma 2.3.3 we have that γ and $\partial\mathcal{Z}$ are in minimal position. Similarly we have that γ and $\mathcal{Y}$ are in minimal position. Now we can understand $\kappa_Z(v)$ by considering $\gamma \cap \mathcal{Z}$.

Write α_Z for a representative of a_Z where $\alpha_Z \subset \partial\mathcal{Z}$.

If $|\gamma \cap \mathcal{Y}| \leq 2$ then by Lemma 2.3.5 we have that $d_{\mathcal{AC}(S)}(v, \partial Y) \leq 2$ as required. If $|\gamma \cap \alpha_Z| \leq 2$ then similarly we are done.

The remaining case is that $|\gamma \cap \mathcal{Y}| \geq 3$ and $|\gamma \cap \alpha_Z| \geq 3$.

Suppose that $Z = S_{0,4}$. Suppose that there is no connected component γ' of $\gamma \cap \mathcal{Z}$ such that $\gamma' \subset \beta_p$ and $\gamma' \cap \mathcal{Y} \neq \emptyset$. So then there exists a unique connected component $\gamma_1' \subset \gamma \cap \mathcal{Z}$ such that $|\gamma_1' \cap \mathcal{Y}| \geq 3$ and we have that $\alpha_p \subset \gamma_1'$. However $|\gamma \cap \alpha_Z| \geq 3$ so there exists a connected component $\gamma_2' \subset \gamma \cap \mathcal{Z}$ such that $\gamma_2' \subset \beta_p$. We have that γ_1' and γ_2' are disjoint and essential properly embedded intervals in $\mathcal{Z}$. There are no bigons shared by γ_1' and $\mathcal{Y}$ in $\mathcal{Z}$ so we deduce that $i([\gamma_1']_{\mathcal{Z}}, [\mathcal{Y}]_{\mathcal{Z}}) \geq 3$. Since $\gamma_2' \cap \mathcal{Y} = \emptyset$ we contradict Lemma 4.1.4. Therefore there exists a connected component γ' of $\gamma \cap \mathcal{Z}$ such that $\gamma' \subset \beta_p \subset \beta$ and $\gamma' \cap \mathcal{Y} \neq \emptyset$. We set $a' = [\gamma']_{\mathcal{Z}} \in \mathcal{AC}_0(Z)$.

Suppose that $Z = S_{1,1}$. Since $\alpha_p \cap (\partial\mathcal{Z} - \partial S) = \emptyset$ we have that $\alpha_p \cap \mathcal{Z} = \emptyset$. Pick an arbitrary connected component γ' of $\gamma \cap \mathcal{Z}$ such that $\gamma' \cap \mathcal{Y} \neq \emptyset$ and set $a' = [\gamma']_{\mathcal{Z}} \in \mathcal{AC}_0(Z)$.

We have that $a' \in \kappa_Z(v) \cap \kappa_Z(\mathfrak{b})$ and a' cuts ∂Y. □

Lemma 4.1.6. *Suppose that Z is a non-annular subsurface of S, where $S = S_{g,n}$ and $n > 0$. Let a_Z be a curve of ∂Z. Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{A}_0(S)$. Suppose that $\mathfrak{a}$ misses ∂Z. Then for any $v \in P(\mathfrak{a}, \mathfrak{b})$ at least one of the following statements holds.*

1. *There exists $a' \in \kappa_Z(v) \cap \kappa_Z(\mathfrak{b})$.*

2. *We have that $d_{\mathcal{AC}(S)}(v, a_Z) \leq 2$.*

Proof. The proof is analogous to the proof of Lemma 4.1.5 and it is easier.

Fix a representative $\mathcal{Z}$ of Z. Write α_Z for a representative of a_Z such that $\alpha_Z \subset \partial\mathcal{Z} - \partial S$. By Lemma 2.3.4 there exist representatives α and β for $\mathfrak{a}$ and $\mathfrak{b}$ respectively such that each pair of α, β and $\mathcal{Z}$ are in minimal position. We have that $\alpha \cap \partial\mathcal{Z} = \emptyset$.

By definition there exists a representative γ of v such that $\gamma = \alpha_p \cup \beta_p$ where $\alpha_p \subset \alpha$ and $\beta_p \subset \beta$.

If $|\gamma \cap \alpha_Z| \leq 2$ then by Lemma 2.3.5 we have that $d_{\mathcal{AC}(S)}(v, a_Z) \leq 2$ as required.

So instead assume that $|\gamma \cap \alpha_Z| \geq 3$. Since $\alpha_p \cap \partial\mathcal{Z} = \emptyset$ we have that $|\beta_p \cap \alpha_Z| \geq 3$. So there exists a connected component $\gamma' \subset \gamma \cap \mathcal{Z}$ such that $\gamma' \subset \beta_p \subset \beta$, see Figure 4.2. Since $\mathcal{Z}$ and β are in minimal position we have that γ' is essential in $\mathcal{Z}$ and we set $a' = [\gamma']_{\mathcal{Z}}$. We have that $a' \in \kappa_Z(v)$ and since $\gamma' \subset \beta$ we have that $a' \in \kappa_Z(\mathfrak{b})$ as required. □

Theorem 4.1.7. *Let $S = S_{g,n}$ with $n > 0$. Let Y be a subsurface of S and let $Q = (c_j)_{j \in I}$ be a geodesic in $\mathcal{C}(S)$. The following statements hold.*

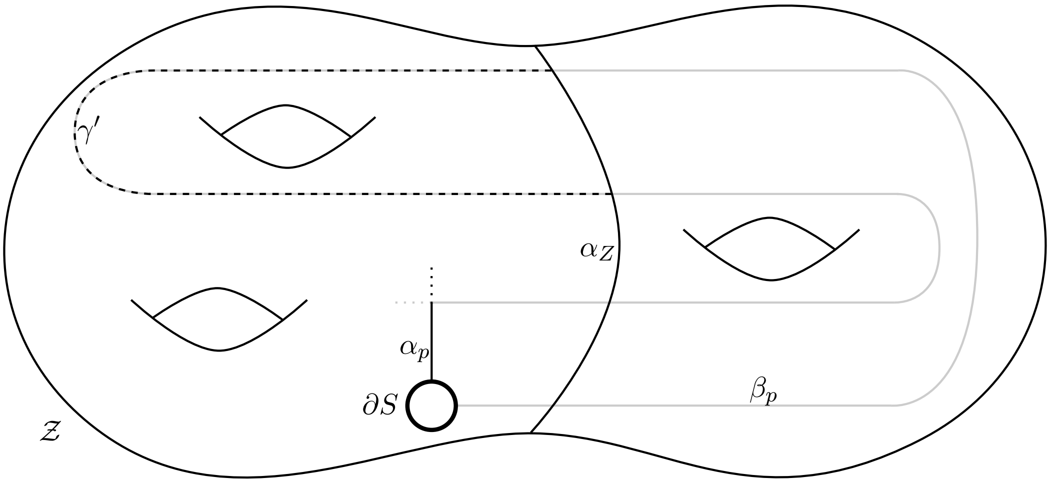

Figure 4.2: The proof of Lemma 4.1.6. Here $S = S_{3,1}$ and $Z = S_{2,2}$. The representative $\gamma = \alpha_p \cup \beta_p$ of v is illustrated. A connected component γ' of $\gamma \cap \mathcal{Z}$ such that $\gamma' \subset \beta_p \subset \beta$ is dotted and $[\gamma']_{\mathcal{Z}}$ is the required arc.

1. *If Y is annular and for each $j \in I$ we have that $d_{\mathcal{AC}(S)}(c_j, \partial Y) \geq 13$ then* $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 7$.

2. *If Y is non-annular and for each $j \in I$ we have that $d_{\mathcal{AC}(S)}(c_j, a_Y) \geq 12$, for some curve a_Y of ∂Y, then* $\operatorname{diam}_{\mathcal{AC}(S)}(\kappa_Y(Q)) \leq 3$.

Proof of 1. It suffices to show that for each $i, i' \in I$ we have that $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(c_i) \cup \kappa_Y(c_{i'})) \leq 7$. After taking a subpath, we may assume that i is initial and i' is terminal in Q.

There exists a subsurface Z of S such that $\partial Y \in \mathcal{AC}_0(Z)$ and $\xi(Z) = 1$. By Lemma 3.1.7 there exists an arc $\mathfrak{a} \in \mathcal{A}_0(S)$ such that $\mathfrak{a}$ misses ∂Y and ∂Z.

Let $\mathfrak{b} \in \mathscr{A}(c_i)$ and $\mathfrak{b}' \in \mathscr{A}(c_{i'})$. By Lemma 4.1.2 there exist $v'' \in P(\mathfrak{b}, \mathfrak{b}')$, $v \in P(\mathfrak{a}, \mathfrak{b})$ and $v' \in P(\mathfrak{a}, \mathfrak{b}')$ such that any pair of the arcs v, v' and v'' miss. By Lemma 4.1.1 we have that $d_{\mathcal{AC}(S)}(v'', Q) \leq 8$. Therefore $d_{\mathcal{AC}(S)}(v, Q), d_{\mathcal{AC}(S)}(v', Q) \leq 9$. Therefore we have that $d_{\mathcal{AC}(S)}(v', \partial Y), d_{\mathcal{AC}(S)}(v, \partial Y) \geq 4$. By Lemma 4.1.5 there exists an arc $\mathfrak{a}_1 \in \kappa_Z(\mathfrak{b}) \cap \kappa_Z(v)$ such that $\mathfrak{a}_1$ cuts ∂Y. Similarly there exists an arc $\mathfrak{a}_2 \in \kappa_Z(\mathfrak{b}') \cap \kappa_Z(v')$ such that $\mathfrak{a}_2$ cuts ∂Y.

Let $\mathfrak{a}_0 \in \kappa_Z(c_i)$ be an arc such that $\mathfrak{a}_0$ cuts ∂Y, and similarly let $\mathfrak{a}_3 \in \kappa_Z(c_{i'})$ be an arc such that $\mathfrak{a}_3$ cuts ∂Y. We have that the sequence $(\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3)$ is a path in $\mathcal{AC}(Z)$ such that each term cuts ∂Y. By Lemma 4.1.3 we have that $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(c_i) \cup \kappa_Y(c_{i'})) \leq 3 + 4 = 7$ as required.

Proof of 2. This proof is similar to the previous. We set $Z = Y$, set $a_Z = a_Y$ and find $\mathfrak{a}$ such that $\mathfrak{a}$ misses ∂Z, let $\mathfrak{b}$ and $\mathfrak{b}'$ be as before and replace Lemma

4.1.5 with Lemma 4.1.6. By Lemma 4.1.6 we construct $a_1 \in \kappa_Y(v) \cap \kappa_Y(b)$ and $a_2 \in \kappa_Y(v') \cap \kappa_Y(b')$. For any $a_0 \in \kappa_Y(c_i)$ and $a_3 \in \kappa_Y(c_{i'})$ we have that (a_0, a_1, a_2, a_3) is a path in $\mathcal{AC}(Y)$ hence there is an upper bound of 3 on the diameter. $\square$

Theorem 4.1.8. *Let $S = S_{g,n}$ with $n > 0$. Let Y be a subsurface of S. Suppose that $Q = (c_j)_{j \in I}$ is a geodesic in $\mathcal{C}(S)$ such that c_j cuts Y for all $j \in I$. Then the following statements hold.*

1. *If Y is annular then* $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 62$.

2. *If Y is non-annular then* $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 50$.

Proof of 1. If there does not exist $i \in I$ such that $d_{\mathcal{AC}(Y)}(c_i, \partial Y) \leq 12$ then we are done by Theorem 4.1.7.

If $i, i' \in I$ are such that $d_{\mathcal{AC}(S)}(c_i, \partial Y) \leq 12$ and $d_{\mathcal{AC}(S)}(c_{i'}, \partial Y) \leq 12$ then $d_{\mathcal{AC}(S)}(c_i, c_{i'}) \leq 24$ so $|i - i'| \leq 46$ by Lemma 2.2.2. So pick i smallest in I and i' largest in I with these upper bounds on distance to ∂Y. Write $I' = \{i, ..., i'\} \subset I$, and set $Q' = (c_j)_{j \in I'}$. Since $|i - i'| \leq 46$ and κ_Y is 1-Lipschitz we have that $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q')) \leq 46$.

Now write $I_1 = \{j \in I : j < i\}$ and $I_2 = \{j \in I : j > i'\}$. By Theorem 4.1.7 we have that $\operatorname{diam}_{\mathcal{AC}(Y)}((c_j)_{j \in I_1}) \leq 7$ and $\operatorname{diam}_{\mathcal{AC}(Y)}((c_j)_{j \in I_2}) \leq 7$. Since κ_Y is 1-Lipschitz and Q is a path, we have that $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 7+1+46+1+7 = 62$.
Proof of 2. We write $Z = Y$ and fix a curve $a_Y = a_Z$ of ∂Y. Then we argue similarly to the proof above, replacing ∂Y with a_Y. The appropriate constants in the argument in order are 11, 22, 42 and $3 + 1 + 42 + 1 + 3 = 50$. $\square$

4.2 Surfaces without boundary revisited

Now we finish the proof of the bounded geodesic image theorem for the case when S is without boundary. When Y is non-annular we use the same strategy as the proof for Theorem 3.2.1. However if Y is annular then a more careful proof must be constructed since the definition of κ_Y relies on taking a cover $\hat{Y}$ of S.

Theorem 4.2.1. *Let $S = S_{g,n}$ with $\xi(S) \geq 2$. Let Y be a subsurface of S. Suppose that $Q = (c_j)_{j \in I}$ is a geodesic in $\mathcal{C}(S)$ such that c_j cuts Y for all $j \in I$. Then the following statements hold.*

1. *If Y is annular then* $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 62$.

2. *If Y is non-annular then* $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 50$.

Proof. By Theorem 4.1.8 we may assume that $n = 0$. We have that $g \geq 2$.

Suppose that Y is non-annular. By Lemma 2.3.4 there exists a representative $\gamma(c_j)$ of c_j for each $j \in J$, and there exists a representative of $\mathcal{Y}$, such that each pair of representatives is in minimal position.

Now there exists an embedded closed 2-disc $\overline{D}$ which is disjoint from $\mathcal{Y}$ and $\gamma(c_j)$ for each $j \in I$. Write D for the interior of $\overline{D}$.

Write $S' = S - D$. We have that $Q' = ([\gamma(c_j)]_{S'})_{j \in I}$ is a geodesic in $\mathcal{C}(S')$. Write $Y' = [\mathcal{Y}]_{S'}$. Furthermore by Lemma 2.3.3 each pair of the representatives $\gamma(c_j)$ (for each $j \in I$) and $\mathcal{Y}$ are in minimal position in S'. Therefore each vertex of Q' cuts Y'. Therefore we may use Theorem 4.1.8 to deduce that $\operatorname{diam}_{\mathcal{AC}(Y')}(\kappa_{Y'}(Q')) \leq 50$.

However we may pick representatives $\gamma'(c_j) \subset \gamma(c_j) \cap \mathcal{Y}$ for $\kappa_Y(c_j)$. But also we have that $\gamma'(c_j)$ is a representative of $\kappa_{Y'}([\gamma(c_j)]_{S'})$. The identity map on $\mathcal{Y}$ induces an isometry $f \colon \mathcal{AC}(Y) \to \mathcal{AC}(Y')$. Furthermore $f(\kappa_Y(c_j)) = \kappa_{Y'}([\gamma'(c_j)]_{S'})$. Therefore $\operatorname{diam}_{\mathcal{AC}(Y)}(\kappa_Y(Q)) \leq 50$ as required.

Suppose that Y is annular. There exists a subsurface Z of S such that $\partial Y \in \mathcal{AC}_0(Z)$ and $\xi(Z) = 1$. Now just as before, construct S', Y', Z' and Q'. As in the proof of Theorem 4.1.7, we find subpaths of Q' which are far from a curve $a_{Z'}$ of $\partial Z'$, then we constructed arcs in $\mathcal{AC}(Z')$ to bound the diameter of the projection of these subpaths of Q' in $\mathcal{AC}(Y')$. Descending these arcs via the one-to-one correspondence between $\mathcal{AC}(Z')$ and $\mathcal{AC}(Z)$ gives us a bound of 7 for the image under κ_Y, by using Lemma 4.1.3. Following along the lines of Theorem 4.1.8, we finish the proof. $\square$

Chapter 5

Control on tight filling multipaths

This chapter consists of two sections. In Section 5.1 we shall define *multipath* and *multigeodesic*, and what it means for a multipath to be *tight*. We introduce the notion of a *filling* multipath. This is similar, but not equivalent, to the notion of a 3-*embedded* multipath introduced by Shackleton [46] and we shall discuss the difference before Lemma 5.1.3. Then we describe what it means to *tighten* a multipath and we show that a tightened filling multipath is a filling multipath (Lemma 5.1.3). Furthermore, we generalize Masur–Minsky's tightening procedure for geodesics [36] to the wider context of filling multipaths. The tightening procedure is important for Theorem 6.2.3, which in turn is used in our proofs of Theorems 7.1.2 and 7.3.2. At the end of Section 5.1 we discuss the different notions of "tight geodesic" in the literature and explain why Theorem 7.1.2 applies to all notions.

In Section 5.2 we consider all the tight filling multipaths of length at most L that connect a fixed pair of curves a and b. We write $C(a, b; L)$ for the set of curves that "appear" in any multicurve of such a tight filling multipath. We show that there is a bound $B = B(S, L)$ that only depends on S and L such that the cardinality of the set $C(a, b; L)$ is bounded from above by B. In fact, fixing L, the bound B is bounded from above by an exponential in $\xi(S)$ (Theorem 5.2.7). Theorem 5.2.7 is a generalization of a theorem of Masur and Minsky [36] that the number of tight geodesics between a pair of curves is finite. In fact, our proof is constructive and therefore gives a new algorithm to compute the distance between two curves in the curve graph. Older algorithms were discovered by Leasure [33] and Shackleton [47]. At the start of Section 5.2 we give a brief overview of the strategy of our proof.

We remark that Theorem 5.2.7 gives the first effective bound on the number of tight multigeodesics between a and b, only in terms of S and $d_{\mathcal{C}(S)}(a,b)$. We remark that prior to the work here, Shackleton provided a bound that only depends on S and the geometric intersection number of a and b [47]. Yohsuke Watanabe independently discovered the idea for the proof of Theorem 5.2.7 [51].

5.1 Tightness

Let m and m' be multicurves with no common curve. There exist representatives α and α' of m and m' respectively such that α and α' are in minimal position. Write $n(\alpha \cup \alpha')$ for an open regular neighbourhood of $\alpha \cup \alpha'$. The *subsurface filled by* α and α', written $\mathcal{F}(\alpha, \alpha')$, is the closure of the union of $n(\alpha \cup \alpha')$ with its complementary components homeomorphic to 2-discs or peripheral annuli. We write $F(\alpha, \alpha')$ for the ambient isotopy class of $\mathcal{F}(\alpha, \alpha')$. In fact $F(\alpha, \alpha')$ is determined by m and m' by Lemma 2.3.4 so instead we may write $F(m, m')$ and we call this the *subsurface filled by* m and m'.

If $\mathcal{F}(\alpha, \alpha') \neq S$ then by construction $\partial\mathcal{F}(\alpha, \alpha') - \partial S$ is a non-empty subset that consists of essential and non-peripheral embedded loops in S that are disjoint from α and α'. We write $\partial F(m, m') \subset \mathcal{C}_0(S)$ for the multicurve which is a union of curves c such that some connected component of $\partial\mathcal{F}(\alpha, \alpha')$ represents c.

We say that m and m' *fill* S if $d_{\mathcal{MC}(S)}(m, m') \geq 3$, or equivalently, if $F(m, m') = S$.

A *multipath* is a sequence of multicurves $(m_j)_{j \in I}$ such that whenever $i, i+1 \in I$ we have that m_i misses m_{i+1}. A *multigeodesic* is a multipath where whenever $i_1, i_2 \in I$ we have that $d_{\mathcal{C}(S)}(m_{i_1}, m_{i_2}) = |i_1 - i_2|$.

Definition 5.1.1. A multipath $P = (m_j)_{j \in I}$ is *filling* if $\mathrm{length}(P) \geq 3$ and whenever $i_1, i_2 \in I$ and $|i_1 - i_2| \geq 3$ we have that m_{i_1} and m_{i_2} fill S.

Note that for a filling multipath $(m_j)_{j \in I}$ whenever $i, i+2 \in I$ we have that m_i and m_{i+2} have no common curve. This is important since we will never consider the subsurface filled by m and m' when the multicurves have a common curve.

Lemma 5.1.2. *Let $(m_j)_{j \in I}$ be a filling multipath with $i-1, i, i+1 \in I$ and let c be a curve. If c misses $\partial F(m_{i-1}, m_{i+1})$ and c cuts m_i then c misses m_{i-1} and m_{i+1}.*

Proof. By Lemma 2.3.4 there exist representatives α_{i-1}, α_i and α_{i+1} of m_{i-1}, m_i and m_{i+1} respectively that are pairwise in minimal position. Construct $\mathcal{F}(\alpha_{i-1}, \alpha_{i+1})$ and write $\mathcal{F}$ for this subset of S. By definition of $\mathcal{F}$ we have that $\alpha_i \cap \mathcal{F} = \emptyset$.

Following the proof of Lemma 2.3.4 there exists a representative γ of c such that $\gamma \cap \partial\mathcal{F} = \emptyset$. Since γ is connected we either have $\gamma \cap \mathcal{F} = \emptyset$ or $\gamma \subset \mathcal{F}$. Since

c cuts m_i we have that $\gamma \cap \alpha_i \neq \emptyset$. Since $\alpha_i \cap \mathcal{F} = \emptyset$ we deduce that $\gamma \cap \mathcal{F} = \emptyset$ therefore we have that $\gamma \cap (\alpha_{i-1} \cup \alpha_{i+1}) = \emptyset$ and we are done. $\square$

A filling multipath $(m_j)_{j\in I}$ is *tight at index* i if $m_i = \partial F(m_{i-1}, m_{i+1})$. If the filling multipath is not tight at index i (and i is neither initial nor terminal in I) then it makes sense to *tighten* the multipath at index i by replacing m_i with $\partial F(m_{i-1}, m_{i+1})$.

Shackleton [46] introduced 3-*embedded* multipaths. A multipath $(m_j)_{j\in I}$ is 3-embedded if whenever $i_1, i_2 \in I$ and $|i_1 - i_2| \geq 3$ we have that $d_{\mathcal{C}(S)}(m_{i_1}, m_{i_2}) \geq 3$. It is immediate that such a pair m_{i_1} and m_{i_2} fill S and so a 3-embedded multipath is a filling multipath. However the converse is not true. For example, there exist multicurves m and m' such that m and m' fill S but $d_{\mathcal{C}(S)}(m, m') = 2$. Moreover, it is remarked by Shackleton [46, Section 2.3] that tightening a 3-embedded multipath may not necessarily result with another 3-embedded multipath. However, tightening a filling multipath does result with a filling multipath.

Lemma 5.1.3. *Suppose that we tighten a filling multipath $(m_j)_{j\in I}$ at index i. Then the resulting multipath is filling.*

Proof. Suppose that there exists a curve c such that c misses $\partial F(m_{i-1}, m_{i+1})$ and c misses $m_{i'}$ where $|i - i'| \geq 3$. Without loss of generality we assume that $i' \geq i$. We have that m_i and $m_{i'}$ fill S so c cuts m_i. By Lemma 5.1.2 we have that c misses m_{i-1}. But this contradicts the assumption that m_{i-1} and $m_{i'}$ fill S. $\square$

Lemma 5.1.4. *Let m' be a multicurve and let $(m_j)_{j\in I}$ be a filling multipath with $i-2, i-1, i \in I$. Suppose that m' misses $\partial F(m_{i-2}, m_i)$ but that each curve of m' cuts m_{i-2}. Then there exist representatives α', α_{i-2} and α_i of m', m_{i-2} and m_i respectively such that each pair is in minimal position and $\alpha' \subset \mathcal{F}(\alpha_{i-2}, \alpha_i)$.*

Proof. By Lemma 2.3.4 there exist representatives α_{i-2} and α_i of m_{i-2} and m_i such that α_{i-2} and α_i are in minimal position. Write $\mathcal{F} = \mathcal{F}(\alpha_{i-2}, \alpha_i)$. Following the proof of Lemma 2.3.4 there exists a representative α'' of m' such that $\alpha'' \cap \partial\mathcal{F} = \emptyset$. By assumption for each connected component $\alpha \subset \alpha''$ we have that $\alpha \cap \alpha_{i-2} \neq \emptyset$. We deduce that $\alpha'' \subset \mathcal{F}$. Again by following the proof of Lemma 2.3.4 there exists α' such that α' and α'' are ambiently isotopic, $\alpha' \subset \mathcal{F}$ and each pair of α', α_{i-2} and α_i is in minimal position. $\square$

The following lemma and its proof is a generalization of Masur–Minsky [36, Lemma 4.5]. They proved the analogous statement for geodesics.

Lemma 5.1.5. *Let $(m_j)_{j\in I}$ be a filling multipath with $i-2, i-1, i, i+1 \in I$. Suppose that $(m_j)_{j\in I}$ is tight at index $i-1$. After tightening at index i, the resulting filling multipath remains tight at index $i-1$.*

Proof. Write $m_i' = \partial F(m_{i-1}, m_{i+1})$. We shall show that $F(m_{i-2}, m_i) = F(m_{i-2}, m_i')$. We have that m_i' misses m_{i+1} so each curve of m_i' cuts m_{i-2}. By Lemma 5.1.4 there exist representatives α_i', α_i and α_{i-2} of m_i', m_i and m_{i-2} respectively such that each pair is in minimal position and $\alpha_i' \subset \mathcal{F}(\alpha_{i-2}, \alpha_i)$. We may assume that an open regular neighbourhood $n(\alpha_{i-2} \cup \alpha_i')$ of $\alpha_{i-2} \cup \alpha_i'$ satisfies $n(\alpha_{i-2} \cup \alpha_i') \subset \mathcal{F}(\alpha_{i-2}, \alpha_i)$. No component of $\overline{S - \mathcal{F}(\alpha_{i-2}, \alpha_i)}$ is either a 2-disc or a peripheral annulus therefore by definition we have that $\mathcal{F}(\alpha_{i-2}, \alpha_i') \subset \mathcal{F}(\alpha_{i-2}, \alpha_i)$.

Each component of α_i intersects α_{i-2} hence each connected component of $\mathcal{F}(\alpha_{i-2}, \alpha_i)$ intersects $\mathcal{F}(\alpha_{i-2}, \alpha_i')$. Therefore in order to show that $\mathcal{F}(\alpha_{i-2}, \alpha_i')$ is ambiently isotopic to $\mathcal{F}(\alpha_{i-2}, \alpha_i)$ it suffices to show that for each connected component $\gamma \subset \partial\mathcal{F}(\alpha_{i-2}, \alpha_i') - \partial S$ we have that γ is peripheral in $\mathcal{F}(\alpha_{i-2}, \alpha_i)$.

So suppose instead that γ is non-peripheral in $\mathcal{F}(\alpha_{i-2}, \alpha_i)$. However γ is essential and non-peripheral in S. Write $c = [\gamma]_S$. By following the proof of Lemma 2.3.4 there exists a representative γ' of c such that $\gamma' \subset \mathcal{F}(\alpha_{i-2}, \alpha_i)$, $\gamma' \cap \alpha_{i-2} = \emptyset$ and γ' and α_i are in minimal position. But γ' is not peripheral in $\mathcal{F}(\alpha_{i-2}, \alpha_i)$ therefore we have that $\gamma \cap \alpha_i \neq \emptyset$ and so c cuts m_i. By definition of c we have that c misses $m_i' = \partial F(m_{i-1}, m_{i+1})$. By Lemma 5.1.2 we have that c misses m_{i+1}. By definition of c we have that c misses m_{i-2}. This contradicts the assumption that m_{i-2} and m_{i+1} fill S. □

Let $P = (m_i)_{i\in I}$ be a multigeodesic or a filling multipath. We say P is *tight* if for each index i (that is neither initial nor terminal in I) we have $m_i = \partial F(m_{i-1}, m_{i+1})$.

Given any filling multipath $P = (m_i)_{i\in I}$ we may enumerate the non-initial and non-terminal indices at which P is not tight. Then one by one we can tighten at these indices. By Lemma 5.1.3 after each tightening the resulting multipath is filling. By Lemma 5.1.5 we have that the resulting multipath retains the indices at which P was tight. We refer to this as the *tightening procedure* see [36, Lemma 4.5]. We use the tightening procedure to prove Theorem 6.2.3 and Theorem 7.3.2. Theorem 6.2.3 is a crucial step in proving the effective bounds in Theorem 7.1.2 which is one of our main theorems.

Before we move on we remark that there are several notions of "tight geodesic" in the literature. The definition of tight multigeodesic given here agrees with Masur and Minsky's notion of a *tight sequence* (see [36, Definition 4.1]) - this is the original definition. Bowditch in [14] defines a tight multigeodesic to be a multigeodesic $(m_j)_{j\in I}$ where at each non-initial and non-terminal index i we have that m_i is a subset of $\partial F(m_{i-1}, m_{i+1})$ - note that this is a weaker notion than the one given here - let us call this a *Bowditch tight multigeodesic*. Bowditch defines a tight geodesic to be a sequence $(c_j)_{j\in I}$ of curves such that there exists a Bowditch multigeodesic $(m_j)_{j\in I}$ with $c_i \in m_i$ for each index $i \in I$ - let us call this a *Bowditch tight geodesic*.

Now we make an observation. Suppose that $(c_0, ..., c_n)$ is a Bowditch tight geodesic. By definition there exists a Bowditch tight multigeodesic $(m_0, ..., m_n)$ with $c_i \in m_i$ for each i. Observe that for each c_i there is a Masur–Minsky tight sequence Q_T connecting m_0 to m_n such that c_i is a curve of some multicurve in Q_T. This follows from the tightening procedure when applied to the Bowditch multigeodesic $(m_0, ..., m_n)$ starting with the index i. We conclude that the finiteness results such as Theorem 7.1.2 and Theorem 5.2.7 apply to Masur–Minsky tight sequences and Bowditch tight geodesics.

5.2 Filling multiarcs

The goal of this section is to provide a bound on the number of tight filling multipaths connecting two vertices purely in terms of S and the lengths of the multipaths (Theorem 5.2.7).

The strategy of the proof of Theorem 5.2.7 is to show that the possibilities for m_1, in terms of m_0 and m_k, is finite, where $(m_0, m_1, ..., m_k)$ is a tight filling multipath. In fact, the bound on the number of possibilities will only depend on S and k. By induction, the theorem follows. To bound the possibilities for m_1, we consider the arcs $m_i - m_0$ in the surface $S - m_0$. In Lemma 5.2.4, we observe that the multicurve m_1 is determined only by the ambient isotopy classes of arcs of $m_2 - m_0$ in $S - m_0$ and we write $\kappa_{m_0}(m_2)$ for these classes. Lemma 5.2.4 only uses the fact that the multipath is tight at m_1. It is worth noting that m_1 is independent of the numbers of parallel copies of arcs of $m_2 - m_0$ in $S - m_0$, and this is why $\kappa_{m_0}(m_2)$ is just the collection of ambient isotopy classes of such arcs.

Finally, we observe that there are only finitely many possibilities for $\kappa_{m_0}(m_i)$ in terms of $\kappa_{m_0}(m_{i+1})$ whenever $i+1 \geq 3$ (Lemma 5.2.1). This is because $\kappa_{m_0}(m_{i+1})$ cuts $S - m_0$ into discs and peripheral annuli, and therefore the arcs that are disjoint from $\kappa_{m_0}(m_{i+1})$ fall into a finite list of possibilities. We use the Catalan numbers to find an exponential bound on the possibilities of $\kappa_{m_0}(m_i)$ in terms of $\xi(S)$.

Summarizing, by an inductive argument we have that $\kappa_{m_0}(m_2)$ has only finitely many possibilities in terms of $\kappa_{m_0}(m_k)$, and, $\kappa_{m_0}(m_2)$ determines m_1 and we are done. We now provide the details.

Given a multicurve $m \in \mathcal{MC}_0(S)$, let α be a representative of m. Write $\mathcal{Z} = S - n(\alpha)$, where $n(\alpha)$ is an open neighbourhood of α. We define $\mathcal{MA}_0(S, m)$ to be the set of ambient isotopy classes of non-empty unions of pairwise disjoint, pairwise non-ambiently isotopic and essential properly embedded intervals in $\mathcal{Z}$, such that each properly embedded interval has its endpoints in $\partial\mathcal{Z} - \partial S$. We call such an element a *multiarc* of (S, m). One should compare this to the definition of $\mathcal{AC}(Y)$. As usual, $\mathcal{MA}(S, m)$ is the graph with vertex set $\mathcal{MA}_0(S, m)$ and two vertices span an edge if they miss.

Write $\mathcal{FMA}_0(S, m)$ for the set of multiarcs $\boldsymbol{m} \in \mathcal{MA}_0(S, m)$ such that there exist representatives α of m and γ of $\boldsymbol{m}$ where $\gamma \subset \mathcal{Z} = S - n(\alpha)$ and $\mathcal{Z} - n(\gamma)$ is a union of discs and peripheral annuli in S, where $n(\gamma)$ is an open regular neighbourhood of γ in $\mathcal{Z}$. Such a multiarc $\boldsymbol{m}$ is called a *filling multiarc of* (S, m).

Define $\mathcal{FMA}(S, m)$ to be the graph with vertex set $\mathcal{FMA}_0(S, m)$, and edges span a pair of vertices $\boldsymbol{m}$ and $\boldsymbol{m}'$ if and only if $\boldsymbol{m}$ misses $\boldsymbol{m}'$.

Lemma 5.2.1. *Let $m \in \mathcal{MC}_0(S)$ and $\boldsymbol{m} \in \mathcal{FMA}_0(S, m)$. Suppose that $\boldsymbol{m}' \in \mathcal{MA}_0(S, m)$ misses $\boldsymbol{m}$. Then there are at most d_0 possibilities for $\boldsymbol{m}'$ where $d_0 = 2^{24n+72g-72} = 2^{24\xi(S)}$.*

Proof. Fix a representative α of m. Write $n(\alpha)$ for an open regular neighbourhood of α. Write $\mathcal{Z} = S - n(\alpha)$.

There exist representatives γ and γ' in $\mathcal{Z}$ for $\boldsymbol{m}$ and $\boldsymbol{m}'$ respectively such that $\gamma \cap \gamma' = \emptyset$ by Lemma 2.3.4.

Now let D be a connected component of $\mathcal{Z} - \gamma$. We define $s(D)$, the number of *sides* of D, to be equal to twice the number of connected components of the boundary of D which are subsets of $\partial\mathcal{Z}$.

Either the interior of D is an open 2-disc or the interior of D is an annulus, whose core loop is peripheral in S.

For each D we shall bound the number of possibilities for $\gamma' \cap D$ up to ambient isotopy of $\mathcal{Z}$ that preserves γ. The bound will be an exponential in $s(D)$.

We remind the reader of the so-called Catalan numbers for which we write C_k. We define $C_0 = 1$ and the number of full triangulations of a convex polygon with $k + 2$ sides is equal to C_k. We shall not prove this but it is known that the numbers C_k satisfy $C_{k+1} = 2(2k+1)C_k/(k+2)$, see [32, p108]. From this we deduce that $C_k \leq 4^k$.

Now when D is simply connected with $s = s(D)$ sides, we can consider a maximal collection of properly embedded intervals in D that are essential in $\mathcal{Z}$, pairwise disjoint and pairwise non-isotopic. There are only C_k such maximal collections in D (where $k = s/2 - 2$) up to ambient isotopy of $\mathcal{Z}$ that preserves γ. Furthermore such a maximal collection in D contains exactly $s/2 - 1 + s/2 - 2 = s - 3$ properly embedded intervals. Now $\gamma' \cap D$ is a subset of such a maximal collection of intervals in D therefore we have that there are at most $2^{s-3}.C_k$ possibilities for $\gamma' \cap D$ up to ambient isotopy of $\mathcal{Z}$ that preserves γ, where $k = s/2 - 2$. We deduce that there are at most $2^{s-3}.2^{s-4} = 2^{2s-7}$ possibilities for $\gamma' \cap D$ up to ambient isotopy of $\mathcal{Z}$ that preserves γ.

When the interior of D is an annulus and D has $s = s(D)$ sides, again we consider a maximal collection of properly embedded intervals in D that are essential in $\mathcal{Z}$, pairwise disjoint, pairwise non-isotopic and have endpoints on $\partial\mathcal{Z} - \partial S$. Now $\gamma' \cap D$ is a subset of such a maximal collection. Since the core loop is peripheral in S, any maximal collection has a properly embedded arc β such that there exists a connected component D' of $D - \beta$ such that the interior of D' is an annulus and $s(D') = 2$. The other connected component D'' of $D - \beta$ is simply connected and we have that $s(D'') = s + 2$. Therefore there are at most $s.C_k$ possibilities for a maximal collection up to ambient isotopy of $\mathcal{Z}$ preserving γ, where $k = s/2 - 1$. Furthermore there are $s - 1$ properly embedded intervals in a maximal collection. We deduce that there are at most $s.2^{s-2}.2^{s-1}$ possibilities for $\gamma' \cap D$ up to ambient isotopy of $\mathcal{Z}$ preserving γ. This is at most 2^{3s-3}.

Now we multiply the number of possibilities in each D to get a bound for the possibilities of m'. It suffices to bound $\Sigma s(D)$ where the sum is taken over all connected components D of $\mathcal{Z} - \gamma$. We use the Euler characteristic χ to do this.

Write N for the number of connected components of $\mathcal{Z} - \gamma$. We attach closed 2-discs to ∂S along their boundaries to obtain a surface X homeomorphic to $S_{g,0}$. Now $\chi(X - n(\alpha)) = \chi(X)$. The space $X - n(\alpha)$ is a collection of N 2-discs glued along $\Sigma s(D)/4$ disjoint intervals. Therefore $\chi(X) = N - \Sigma s(D)/4$ and therefore $\Sigma s(D) = 4N - 4\chi(S_{g,0})$.

So it suffices to bound N. Assume that N is maximal, thus each connected component D of $\mathcal{Z} - \gamma$ is either simply connected and $s(D) = 6$ (let's say there are h such connected components), or, the interior of D is an annulus and $s(D) = 2$ (let's say there are h' such connected components). We observe that $\chi(X - n(\alpha)) = (h' + h) - (\frac{h'}{2} + \frac{3h}{2})$ thus $h = h' - 2\chi(S_{g,0})$ thus $N \leq 2h' - 2\chi(S_{g,0})$ and so $3\Sigma s(D) \leq 3(8h' - 12\chi(S_g)) \leq 24n - 36(2 - 2g) = 24n + 72g - 72$.

We conclude that there are at most $2^{24n+72g-72}$ possibilities for m'. $\square$

Let $m \in \mathcal{MC}_0(S)$. Write $\mathcal{MC}(S; m, 3) = \{m' \in \mathcal{MC}_0(S) : d_{\mathcal{MC}(S)}(m, m') \geq 3\}$. Define $\mathcal{MC}(S; m, 2) = N_{\mathcal{MC}(S)}(\mathcal{MC}(S; m, 3); 1)$. This set is a subset of the multicurves of distance at least 2 from m, but generally it is not an equality, because one may have distance 2 multicurves (in $\mathcal{MC}(S)$) that have a common curve. But any curve of $m' \in \mathcal{MC}(S;m,2)$ must cut m. Define $\kappa_m : \mathcal{MC}(S;m,2) \to \mathcal{MA}_0(S,m)$ similarly to that of κ_Y for a subsurface Y. Note that the restriction of κ_m to $\mathcal{MC}(S;m,3)$ maps into $\mathcal{FMA}_0(S,m)$.

Define a relation $p_m : \mathcal{FMA}(S,m) \to \mathcal{MA}(S,m)$ where $m \mapsto m'$ if and only if m' is a submultiarc of m.

Lemma 5.2.2. *Given $m \in \mathcal{MC}_0(S)$ and $(m_0, m_1, ..., m_l)$ a multipath with $m_i \in \mathcal{MC}(S;m,3)$ for each i, then $\kappa_m(m_0) \in N_{\mathcal{FMA}(S,m)}(\kappa_m(m_l);l) \subset \mathcal{FMA}_0(S,m)$ and the set $N_{\mathcal{FMA}(S,m)}(\kappa_m(m_l);l)$ has an effective bound on its cardinality in terms of l and $\xi(S)$.*

Proof. It suffices to show that $\kappa_m(m_i)$ misses $\kappa_m(m_{i+1})$, and this is clear. Then apply Lemma 5.2.1 to deduce the cardinality is bounded by d_0^{l+1}. $\square$

Lemma 5.2.3. *Let $m, m', m'' \in \mathcal{MC}_0(S)$, suppose that m' misses m'' and that m and m'' fill S. Then $\kappa_m(m') \in p_m(N_{\mathcal{FMA}(S,m)}(\kappa_m(m'');1))$ and the latter set contains at most d_0 multiarcs.*

Proof. Note that $\kappa_m(m') \cup \kappa_m(m'')$ is a filling multiarc of (S,m) that misses $\kappa_m(m'')$, hence $\kappa_m(m')$ is a submultiarc of $\kappa_m(m') \cup \kappa_m(m'') \in N_{\mathcal{FMA}(S,m)}(\kappa_m(m'');1)$. The number of possibilities, following the proof of Lemma 5.2.1, is at most d_0. $\square$

Let $m \in \mathcal{MC}_0(S)$ and $\mathfrak{m} \in \mathcal{MA}_0(S,m)$. Take representatives α, $\mathcal{Z} = S - n(\alpha)$ and $\gamma \subset \mathcal{Z}$. Write $n(\gamma)$ for an open regular neighbourhood of γ in $\mathcal{Z}$. Take $n(\alpha) \cup n(\gamma)$ union with its complementary connected components that are homeomorphic to discs or peripheral annuli. We set $F(m;\mathfrak{m})$ to be the resulting subsurface.

Define a relation $T_m : \mathcal{MA}(S,m) \to \mathcal{MC}_0(S)$ by $\mathfrak{m} \mapsto \partial F(m;\mathfrak{m})$. Here, a filling multiarc would relate to the empty set.

Lemma 5.2.4. *Suppose that $m \in \mathcal{MC}_0(S)$ and $m' \in \mathcal{MC}(S;m,2)$ then we have that $F(m;\kappa_m(m')) = F(m,m')$.*

Proof. Let γ and γ' be representatives of m and m' that are in minimal position. Let $n(\gamma')$ be an open regular neighbourhood of γ. Write $\mathcal{Z} = S - n(\gamma)$. Let $A_1, ..., A_k$ be the partition of connected components of $\gamma' \cap \mathcal{Z}$ into their ambient isotopy classes in $\mathcal{Z}$. Write $I = \{1, ..., k\}$. Therefore a connected component in A_i is ambiently isotopic in $\mathcal{Z}$ to a connected component in A_j if and only if $i \neq j$.

For each i pick an arbitrary choice α_i of connected component in A_i. Thus, $\gamma'' = \cup_{i \in I} \alpha_i$ is a representative of $\kappa_m(m')$ in $\mathcal{Z}$.

Now let $n(\gamma \cup \gamma')$ be an open regular neighbourhood of $\gamma \cup \gamma'$. Since ambiently isotopic and disjoint properly embedded intervals cobound squares (Lemma 2.3.6), if A_i consists of n properly embedded intervals in $\mathcal{Z}$ then there are $n-1$ squares in $\mathcal{Z}$ inbetween the collection of properly embedded intervals A_i. Thus after adjoining these squares to $n(\gamma \cup \gamma')$, the resulting set is $n(\gamma) \cup n(\gamma'')$, for some open regular neighbourhood $n(\gamma'')$ of γ'' in $\mathcal{Z}$.

By the definition of $\mathcal{F}(\gamma, \gamma')$ we are done. □

Corollary 5.2.5. *Suppose that (m_0, m_1, m_2, m_3) is a filling multipath that is tight at m_1. Then $m_1 = T_{m_0}(\kappa_{m_0}(m_2))$.* □

Lemma 5.2.6. *Suppose that $(m_0, m_1, ..., m_l)$ is a tight filling multipath such that $a = m_0$ and $b = m_l$. If i is an integer such that $0 \leq i \leq l-3$ then*

$$m_{i+1} \in T_{m_i}(p_{m_i}(N_{\mathcal{FMA}(S,m_i)}(\kappa_{m_i}(b); l-i-2))).$$

The cardinality of $T_{m_i}(p_{m_i}(N_{\mathcal{FMA}(S,m_i)}(\kappa_{m_i}(b); l-i-2)))$ as a set of multicurves is bounded by d_0^{l-i-1}.

Proof. For each j such that $i+3 \leq j \leq l$ we have that $\kappa_{m_i}(m_j) \in \mathcal{FMA}_0(S, m_i)$. By Lemma 5.2.2 we have that $\kappa_{m_i}(m_{i+3}) \in N_{\mathcal{FMA}(S,m_i)}(\kappa_{m_i}(b); l-i-3)$. By Lemma 5.2.3 we have that

$$\kappa_{m_i}(m_{i+2}) \in p_{m_i}(N_{\mathcal{FMA}(S,m_i)}(\kappa_{m_i}(m_{i+3}); 1)) \subset p_{m_i}(N_{\mathcal{FMA}(S,m_i)}(\kappa_{m_i}(b); l-i-2))$$

and Corollary 5.2.5 finishes the proof. □

Given a subset $M \subset \mathcal{MC}_0(S)$, we define

$$I_j(M, b) \coloneqq \bigcup \{T_m(p_m(N_{\mathcal{FMA}(S,m)}(\kappa_m(b); j))) : m \in M \cap \mathcal{MC}(S; b, 3)\}.$$

One imagines the elements of this set as 'neighbours' of M in the direction of b. We define

$$T(M, b) \coloneqq \bigcup \{\partial F(m; b) : m \in M \cap \mathcal{MC}(S; b, 2)\}.$$

This set consists of the tightenings of multicurves of M with b. We can rephrase the first claim of Lemma 5.2.6 as $m_{i+1} \in I_{l-i-2}(m_i, b)$.

We write $C(a, b; L) \subset \mathcal{C}_0(S)$ for the set of curves c such that there exists a tight filling multipath P connecting a to b with length$(P) \leq L$ and $c \in m \in P$ for some $m \in P$. Note that when $L \geq d_{\mathcal{C}(S)}(a, b) \geq 3$, the tight multigeodesics from

a to b lie in this set. The following theorem, then, is a generalization of Masur–Minsky: that there are only finitely many tight geodesics between two given curves. Furthermore, this is the first effective bound in terms of the distance of the curves and the surface. This improves earlier work of Shackleton [47].

Theorem 5.2.7. *Given S and an integer $L \geq 0$ there exists $B = B(S, L)$ such that given any $a, b \in \mathcal{C}_0(S)$ we have that $|C(a, b; L)| \leq B$. We may take $B = 2\xi(S)Ld_0^{L^2}$ where d_0 is as in Lemma 5.2.1.*

Proof. The theorem is vacuous if a and b do not fill S. So suppose that a and b fill S. Set $C_0 = \{a\}$ and set $C_i = C_{i-1} \cup T(C_{i-1}, b) \cup I_{L-i-1}(C_{i-1}, b)$. We set $C = C_{L-1} \cup \{b\}$. Note that $C_i \subset C_{i+1}$. We claim that $C(a, b; L) \subset C$.

Given any tight filling multipath $(m_0, m_1, ..., m_k)$ that connects a to b with $k \leq L$, by Lemma 5.2.6 and induction, for all integers i with $0 \leq i \leq k-3$ we have that $m_{i+1} \in C_{i+1}$. Also, we have that $m_{k-1} = \partial F(m_{k-2}, m_k) \in T(C_{k-2}, b) \subset C_{k-1}$. By definition of C we have $b \in C$.

Now we bound $|C|$. First we bound the multicurves from the $I_{L-i-1}(C_{i-1}, b)$, forgetting the $T(C_{i-1}, b)$ for the moment. By Lemma 5.2.6 inductively, this is at most $1 + d_0^{L-1} + d_0^{L-1+L-2} + ... + d_0^{L-1+L-2+...+2} \leq Ld_0^{L^2}$. Now we take care of the $T(C_{i-1}, b)$. Note that this completes the tight filling multipath connecting a to b. Since there are at most $Ld_0^{L^2}$ multicurves m that are elements of some $I_{L-i-1}(C_{i-1}, b)$, the possibilities of multicurve m' adjacent to b in such a filling multipath is bounded from above by $Ld_0^{L^2}$, because m' is determined by m and b. We deduce that there are at most $2Ld_0^{L^2}$ possible multicurves that are elements of such a filling multipath. Finally, given any multicurve, there are at most $\xi(S)$ components, therefore there are at most $B = 2\xi(S)Ld_0^{L^2}$ curves. $\square$

Remark 5.2.8. Fixing L, we see that $B(S, L)$ is bounded above by an exponential function in $\xi(S)$.

Remark 5.2.9. When $L = d_{\mathcal{C}(S)}(a, b)$ there is a sharper bound. This is the content of Theorem 7.1.2.

Remark 5.2.10. The proof of Theorem 5.2.7 is constructive. Therefore we can compute $C(a, b; L)$ and compute all tight multigeodesics that connect a and b. This algorithm was discovered independently by Yohsuke Watanabe [51]. It would be interesting to understand how efficient this algorithm is. This algorithm is new but the result is not, see Leasure [33] and Shackleton [47].

Chapter 6

Constructions of tight filling multipaths

The concatenation of two multigeodesics is not necessarily a filling multipath: indeed there may be a 'shortcut' nearby where the multigeodesics meet at their ends. Under some mild assumptions, after replacing pieces of the concatenation with shortcuts, the result is a filling multipath that covers almost all of the concatenation of the original multigeodesics (Proposition 6.1.4). Proposition 6.1.4 will be proved in Section 6.1.

In particular, Proposition 6.1.4 allows us to construct a filling multipath that coincides with a long submultipath of a tight multigeodesic Q_T. We may use the tightening procedure on the filling multipath to produce a tight filling multipath, which also coincides with a long submultipath of Q_T. Our useful setting for this technique is described in Theorem 6.2.3, see Figure 6.2. Theorem 6.2.3 is a key step in our proofs of Theorems 7.1.2 and 7.3.2, and will be proved in Section 6.2.

The arguments in this chapter are elementary. Some of the work in this chapter was inspired by Shackleton [46] who introduced the notion of 3-*embedded* multipath. The difference between Shackleton's strategy in [46] and the strategy employed here is that Shackleton performs a sequence of shortcuts and tightenings (these may alternate) that eventually terminates with a tight, 3-embedded multipath such that most of the original multipaths remain intact. On the other hand, we perform shortcuts first to construct a filling multipath and then we perform tightenings to construct a tight filling multipath.

6.1 Constructing filling multipaths

Let $P = (m_i)_{i \in I}$ be a multipath such that length$(P) \geq 3$. We say that m_i and m_j *fail to fill in* $(m_i)_{i \in I}$ if $i, j \in I$, $j - i \geq 3$ and m_i and m_j do not fill S.

We start with an easy lemma.

Lemma 6.1.1. *If* $(m_0, ..., m_n)$ *is a multipath and* $d_{\mathcal{C}(S)}(m_0, m_n) = n$ *then* $(m_0, ..., m_n)$ *is a multigeodesic.* □

Lemma 6.1.2. *If* $(m_0, ..., m_n)$ *and* $(m_{n-2}, m_{n-1}, m_n, ...)$ *are multigeodesics and* $d_{\mathcal{C}(S)}(m_0, m_j) \geq n$ *whenever* $j \geq n$ *then* $(m_0, m_1, ...)$ *is a filling multipath.*

Proof. Suppose for contradiction that m_i and m_j fail to fill in $(m_i)_{i \in I}$. By the multigeodesic criterion we have $i \leq n - 3$ and $j \geq n + 1$. We have that $d_{\mathcal{C}(S)}(m_0, m_j) \leq d_{\mathcal{C}(S)}(m_0, m_i) + 2 = i + 2 \leq n - 1$ whereas $j \geq n + 1$ and so $d_{\mathcal{C}(S)}(m_0, m_j) \geq n$, a contradiction. □

Sometimes we can pinpoint where a multipath fails to fill with the following lemma.

Lemma 6.1.3. *Suppose that* $(m_0, ..., m_n)$ *and* $(m_n, m_{n+1}, m_{n+2}, m_{n+3}, ...)$ *are multigeodesics and* $d_{\mathcal{C}(S)}(m_0, m_i) \geq n + 1$ *whenever* $i \geq n + 1$.
Write $P = (m_0, ..., m_{n-1}, m_n, m_{n+1}, ...)$. *If* P *is not a filling multipath then only* m_{n-1} *and* m_{n+2}*, or,* m_{n-1} *and* m_{n+3} *can fail to fill in* P.

Proof. If m_i and m_j fail to fill in P then by the multigeodesic criterion we have $i \leq n - 1$ and $j \geq n + 1$. Suppose for contradiction that $i \leq n - 2$ then

$$d_{\mathcal{C}(S)}(m_0, m_j) \leq d_{\mathcal{C}(S)}(m_0, m_i) + d_{\mathcal{C}(S)}(m_i, m_j) \leq i + 2 \leq n$$

whereas $d_{\mathcal{C}(S)}(m_0, m_j) \geq n + 1$. Therefore $i = n - 1$. Now, there exists c such that (m_{n-1}, c, m_j) is a multipath but (m_n, m_{n-1}, c, m_j) is a multipath whose length is less than or equal to 3 hence $j \leq n + 3$. So $j = n + 2$ or $n + 3$ because $j - i \geq 3$. □

The following is the most technical in this section but is very important.

Proposition 6.1.4. *Suppose that* $(m_0, ..., m_r)$ *and* $(m_r, m_{r+1}, ..., m_{r+5}, ...)$ *are multigeodesics and that* $d_{\mathcal{C}(S)}(m_0, m_i) \geq r + 1$ *whenever* $i \geq r + 1$. *Then there is a filling multipath* P *starting at* m_0*, with submultipath* $(m_{r+5}, ...)$*, and the length of the submultipath from* m_0 *to* m_{r+5} *in* P *is at most* $r + 5$.

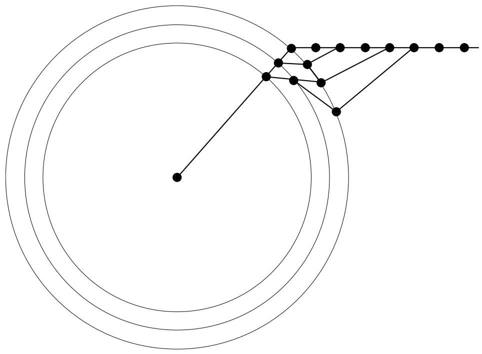

Figure 6.1: The argument for Proposition 6.1.4 going through the worst case scenario: $j = r+2, j' = r+4, j'' = r+5$.

Proof. The proof is by exhaustion. See Figure 6.1.

Write $P_0 = (m_0, ..., m_r, m_{r+1}, ...)$. By Lemma 6.1.3, with $n = r$, if P_0 is not a filling multipath then m_{r-1} and m_j fail to fill in P_0 where $j = r+2$ or $r+3$. We take j to be maximal. So there exists c_1 such that (m_{r-1}, c_1, m_j) is a multipath. It follows that $(m_0, ..., m_{r-1}, c_1, m_j)$ is a multigeodesic by Lemma 6.1.1 and its length is equal to $r+1$. Write $P_1 = (m_0, ..., m_{r-1}, c_1, m_j, m_{j+1}...)$. If $j = r+3$ then $(m_r, m_{r-1}, c_1, m_j, m_{j+1}, ...)$ is a multigeodesic. Hence by Lemma 6.1.2, with $n = r+1$, P_1 is a filling multipath.

The other case is when $j = r+2$. If P_1 is not a filling multipath then c_1 and $m_{j'}$ fail to fill in P_1 where $j' \geq r+4$. We took j to be maximal earlier, so it cannot be the case that m_{r-1} and some other multicurve fail to fill in P_1.

So there exists c_2 such that $(c_1, c_2, m_{j'})$ is a multipath. But we observe that $(m_r, m_{r-1}, c_1, c_2, m_{j'})$ is a multipath and its length is equal to 4 hence $j' \leq r+4$ and so $j' = r+4$. It follows that $(m_{r-1}, c_1, c_2, m_{r+4}, m_{r+5}, ...)$ is a multigeodesic. Write $P_2 = (m_0, ..., m_{r-1}, c_1, c_2, m_{r+4}, m_{r+5}, ...)$.

If $d_{\mathcal{C}(S)}(m_0, c_2) = r+1$ then by Lemma 6.1.1 we have that $(m_0, ..., m_{r-1}, c_1, c_2)$ is a multigeodesic and by Lemma 6.1.2 with $n = r+1$ we have that P_2 is filling.

The other case is if $d_{\mathcal{C}(S)}(m_0, c_2) = r$. By Lemma 6.1.3 with $n = r-1$, if P_2 is not a filling multipath then only m_{r-2} and c_2, or, m_{r-2} and m_{r+4} can fail to fill in P_2. But $d_{\mathcal{C}(S)}(m_0, m_{r+4}) \geq r+1$ and $d_{\mathcal{C}(S)}(m_0, m_{r-2}) = r-2$ so m_{r-2} and m_{r+4} fill S. So there exists c_3 such that (m_{r-2}, c_3, c_2) is a multipath. Write $P_3 = (m_0, ..., m_{r-2}, c_3, c_2, m_{r+4}, m_{r+5}, ...)$. By Lemma 6.1.1 we have that $(m_0, ..., m_{r-2}, c_3, c_2, m_{r+4})$ is a multigeodesic and its length is equal to $r+1$.

If P_3 is not a filling multipath then by Lemma 6.1.3, with $n = r$, we have that c_3 and $m_{j''}$ fail to fill in P_3 where $j'' = r+5$ or $r+6$. So there exists c_4 such that $(c_3, c_4, m_{j''})$ is a multipath. But $(m_r, m_{r-1}, m_{r-2}, c_3, c_4, m_{j''})$ is a multipath and its length is equal to 5 so $j'' \le r+5$ and so $j'' = r+5$. It follows that $(m_{r-2}, c_3, c_4, m_{r+5}, ...)$ is a multigeodesic. Also, $(m_0, ..., m_{r-2}, c_3, c_4)$ is a multigeodesic by Lemma 6.1.1 and its length is equal to r. By Lemma 6.1.2, with $n = r$, $P_4 = (m_0, ..., m_{r-2}, c_3, c_2, c_4, m_{r+5}, ...)$ is a filling multipath with the required properties. $\square$

6.2 Constructing tight filling multipaths

Now we aim to use Proposition 6.1.4 and the following two straightforward lemmas to give a procedure for constructing tight filling multipaths (Theorem 6.2.3). This is crucial for our proofs in Chapter 7.

Lemma 6.2.1. *Suppose that $Q_T = (m_i)_{i \in I}$ is a multigeodesic, $r \ge 0$, and there are vertices $c_a, c, c_b \in \mathcal{C}_0(S)$ such that $d_{\mathcal{C}(S)}(c', Q_T) \le r$ whenever $c' \in \{c_a, c, c_b\}$, $d_{\mathcal{C}(S)}(c_a, c) + d_{\mathcal{C}(S)}(c, c_b) = d_{\mathcal{C}(S)}(c_a, c_b)$ and $d_{\mathcal{C}(S)}(c_a, c) = d_{\mathcal{C}(S)}(c, c_b) = 4r+1$. Then we may reorder the indices of Q_T such that whenever*

$$d_{\mathcal{C}(S)}(c_a, m_i), d_{\mathcal{C}(S)}(c, m_j), d_{\mathcal{C}(S)}(c_b, m_k) \le r \tag{6.1}$$

then $i < j < k$.

Proof. First suppose that Equation (6.1) holds for some $i, j, k \in I$. We have that

$$|j - i| = d_{\mathcal{C}(S)}(m_i, m_j) \le d_{\mathcal{C}(S)}(m_i, c_a) + d_{\mathcal{C}(S)}(c_a, c) + d_{\mathcal{C}(S)}(c, m_j) \le 6r + 1$$

and similarly $|k - j| \le 6r + 1$, and $|k - i| \le 10r + 2$. Also, $4r + 1 = d_{\mathcal{C}(S)}(c_a, c) \le r + |j - i| + r$ so $2r + 1 \le |j - i|$. Similarly, $2r + 1 \le |k - j|$ and $8r + 2 \le r + |k - i| + r$ so $6r + 2 \le |k - i|$. It is now clear that i, j and k are distinct.

Suppose that $i < k < j$. Then $6r + 2 \le k - i \le j - i \le 6r + 1$, a contradiction. This argument also rules out the cases $j < k < i$, $j < i < k$ and $k < i < j$.

If $i > j > k$ then we reorder Q_T and continue. So instead suppose that $i < j < k$ and suppose for contradiction that there exist $i' > j' > k'$ such that Equation (6.1) holds with i', j' and k' in place of i, j and k respectively. Now Q_T is a multigeodesic so we have $|i' - i| \le 2r$, and as above, $|i' - j| \ge 2r + 1$, and so $i' < j$ and $j - i' \ge 2r + 1$. Similarly, we must have $|j - j'| \le 2r$, but $j - j' = (j - i') + (i' - j') \ge 2r + 1 + 1$, a contradiction. We must have $i' < j' < k'$. $\square$

Lemma 6.2.2. *Suppose that $Q_T = (m_i)_{i \in I}$ is a multigeodesic, $r \ge 3$, and there are vertices $c_a, c, c_b \in \mathcal{C}_0(S)$ such that $d_{\mathcal{C}(S)}(c', Q_T) \le r$ whenever $c' \in \{c_a, c, c_b\}$, $d_{\mathcal{C}(S)}(c_a, c) + d_{\mathcal{C}(S)}(c, c_b) = d_{\mathcal{C}(S)}(c_a, c_b)$ and $d_{\mathcal{C}(S)}(c_a, c) = d_{\mathcal{C}(S)}(c, c_b) = 4r + 1$.*

Assume further that Q_T is ordered with respect to Lemma 6.2.1. Then $Q_T \cap N_r(c) \subset m_{i+6} \cup ... \cup m_{k-6} \subset \mathcal{C}_0(S)$ where $i \in I$ is largest such that $d_{\mathcal{C}(S)}(m_i, c_a) \leq r$ and $k \in I$ is smallest such that $d_{\mathcal{C}(S)}(m_k, c_b) \leq r$.

Proof. Suppose that $v \in Q_T \cap N_r(c)$ then $v \in m_j$ for some j. By Lemma 6.2.1 we have $i < j < k$. Also, $j - i \geq 2r + 1 \geq 6$ and similarly $k - j \geq 6$. Thus $j \in \{i + 6, ..., k - 6\}$. $\square$

Theorem 6.2.3. *Let $Q_T = (m_i)_{i \in I}$ be a tight multigeodesic and $r \geq 3$. Then the following statements hold.*

1. *If there exist $c_a, c, c_b \in \mathcal{C}_0(S)$ such that $d_{\mathcal{C}(S)}(c', Q_T) \leq r$ whenever $c' \in \{c_a, c, c_b\}$, $d_{\mathcal{C}(S)}(c_a, c) + d_{\mathcal{C}(S)}(c, c_b) = d_{\mathcal{C}(S)}(c_a, c_b)$ and furthermore we have that $d_{\mathcal{C}(S)}(c_a, c) = d_{\mathcal{C}(S)}(c, c_b) = 4r + 1$, then there exists a tight filling multipath P that connects c_a to c_b such that* $\text{length}(P) \leq 12r + 2$ *and $Q_T \cap N_r(c) \subset P \cap N_r(c)$.*

2. *Suppose b is an endpoint of Q_T and that there exist $c_a, c \in \mathcal{C}_0(S)$ such that $d_{\mathcal{C}(S)}(c', Q_T) \leq r$ whenever $c' \in \{c_a, c\}$, $d_{\mathcal{C}(S)}(c_a, c) + d_{\mathcal{C}(S)}(c, b) = d_{\mathcal{C}(S)}(c_a, b)$, $d_{\mathcal{C}(S)}(c_a, c) = 4r + 1$ and $d_{\mathcal{C}(S)}(c, b) \leq 4r + 1$. Then there exists a tight filling multipath P that connects c_a to b such that* $\text{length}(P) \leq 12r + 2$ *and $Q_T \cap N_r(c) \subset P \cap N_r(c)$.*

Proof. For brevity we shall only write out the proof of *1.* since the proof of *2.* is analogous.

Use Lemma 6.2.1 to order Q_T and define $i \in I$ to be largest such that $d_{\mathcal{C}(S)}(m_i, c_a) \leq r$ and $k \in I$ to be smallest such that $d_{\mathcal{C}(S)}(m_k, c_b) \leq r$. We have $6r + 2 \leq k - i \leq 10r + 2$.

By Lemma 6.2.2 we have $Q_T \cap N_r(c) \subset m_{i+6} \cup ... \cup m_{k-6}$. Therefore if P contains these multicurves it would follow that $Q_T \cap N_r(c) \subset P \cap N_r(c)$.

Let $c'_a \in m_i$ and $c'_b \in m_k$ be curves. There exists a geodesic Q_a (Q_b) connecting c_a to c'_a (c'_b to c_b). Let $Q' = (m'_i)_{i \in I'}$ be the multigeodesic such that $m'_i = c'_a$, $m'_k = c'_b$, and whenever $i < j < k$ we have $m'_j = m_j$, and $I' = \{i, ..., k\}$. Using Proposition 6.1.4 with the concatenation of Q_a and Q' we obtain a filling multipath P_1 connecting c_a to m_k with submultipath $m_{i+5}, ..., m_k$ and the length of the submultipath from c_a to m_{i+5} in P_1 is at most $r + 5$. Similarly, using Proposition 6.1.4 with Q' and Q_b, we obtain a filling multipath P_2 connecting m_i to c_b with submultipath $m_i, ..., m_{k-5}$ and the length of the submultipath from m_{k-5} to c_b in P_2 is at most $r + 5$.

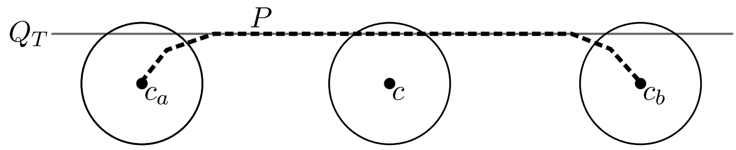

Figure 6.2: A cartoon of Theorem 6.2.3. The required tight filling multipath P is dotted and it coincides with Q_T inside $N_r(c)$.

Now we construct a multipath P_3 connecting c_a to c_b such that length$(P_3) \leq 12r+2$. We take P_3 to be the concatenation of P_1 from c_a to m_{k-5} and P_2 from m_{k-5} to c_b. This multipath has submultipath $\tilde{Q} = m_{i+5}, ..., m_{k-5}$. Now P_3 is filling: suppose for contradiction that $(P_3)_x$ and $(P_3)_y$ fail to fill in P_3. Then since P_3 as a set is a subset of the union of P_1 and P_2, we must have that $(P_3)_x \notin P_2$ and $(P_3)_y \notin P_1$, and therefore since $\tilde{Q} \subset P_1 \cap P_2$ we have that $(P_3)_x, (P_3)_y \notin \tilde{Q}$. Therefore, $d_{\mathcal{C}(S)}((P_3)_x, c_a) \leq r+4$ and $d_{\mathcal{C}(S)}((P_3)_y, c_b) \leq r+4$, but $(P_3)_x$ and $(P_3)_y$ do not fill S hence $d_{\mathcal{C}(S)}(c_a, c_b) \leq 2r+10$. This contradicts $d_{\mathcal{C}(S)}(c_a, c_b) = 8r+2$.

Now P_3 has submultipath $(m_{i+5}, ..., m_{k-5})$ and this is a tight multigeodesic. Now we use Lemmas 5.1.3 and 5.1.5 to tighten P_3 at every index (which is neither initial nor terminal) to construct a tight filling multipath P_4 that connects c_a to c_b such that length$(P_4) \leq 12r+2$. Furthermore, P_4 has submultipath $m_{i+6}, ..., m_{k-6}$ and hence $Q_T \cap N_r(c) \subset P_4$. We set $P = P_4$ and the theorem is proved. $\square$

Chapter 7

Bounds on slices

In this chapter we use the results of Chapters 5 and 6 to give an exponential upper bound on the 'slices' of tight multigeodesics (Theorem 7.1.2) in Section 7.1, to give effective bounds for the acylindrical action of the mapping class group on the curve graph (Theorem 7.2.3) in Section 7.2, and to give a finite time algorithm to compute invariant tight multigeodesics and stable lengths of pseudo-Anosovs on the curve graph (Theorems 7.3.1 and 7.3.2) in Section 7.3.

7.1 Upper bounds on slices

We use notation from [14]. We write $\mathcal{L}(a,b)$ for the set of geodesics in $\mathcal{C}(S)$ that connect a to b. We write $\mathcal{L}(A,B)$ for the union of $\mathcal{L}(a,b)$ over $a \in A$ and $b \in B$. We set $\mathcal{L}(a,b;r) = \mathcal{L}(N_{\mathcal{C}(S)}(a;r), N_{\mathcal{C}(S)}(b;r))$. We write $\mathcal{L}_T(a,b)$ for the tight multigeodesics that connect a to b, and similarly define $\mathcal{L}_T(a,b;r)$. We write $G(a,b)$ for the set of curves c such that $c \in m \in Q \in \mathcal{L}_T(a,b)$ for some multicurve m, and similarly write $G(a,b;r)$ for the set of curves c such that $c \in m \in Q \in \mathcal{L}_T(a,b;r)$ for some multicurve m.

Lemma 7.1.1. *Let $r \geq 0$ and $a,b \in \mathcal{C}_0(S)$. Let $c \in Q \in \mathcal{L}(a,b)$ and suppose that $d_{\mathcal{C}(S)}(c,\{a,b\}) \geq r+2\delta+1$ then for any $Q_T \in \mathcal{L}(a,b;r)$ we have $Q_T \cap N_{\mathcal{C}(S)}(c;2\delta) \neq \emptyset$.*

Proof. Given a geodesic Q_T that connects a' to b' such that $d_{\mathcal{C}(S)}(a,a'), d_{\mathcal{C}(S)}(b,b') \leq r$, there exists a geodesic Q_a connecting a to a', a geodesic Q_b connecting b to b' and a geodesic $\tilde{Q}$ connecting a to b'.

By Theorem 3.2.1, the geodesic triangle formed by Q, Q_b and $\tilde{Q}$ is δ-slim so c is an element of the δ-neighbourhood of $Q_b \cup \tilde{Q}$. Let $v \in Q_b$ then by hypothesis $r + 2\delta + 1 \leq d_{\mathcal{C}(S)}(b,c) \leq d_{\mathcal{C}(S)}(b,v) + d_{\mathcal{C}(S)}(v,c) \leq r + d_{\mathcal{C}(S)}(v,c)$ implying that

$2\delta + 1 \le d_{\mathcal{C}(S)}(v, c)$. But $v \in Q_b$ was arbitrary hence $2\delta + 1 \le d_{\mathcal{C}(S)}(Q_b, c)$. Thus, there exists $\tilde{c} \in \tilde{Q}$ such that $d_{\mathcal{C}(S)}(c, \tilde{c}) \le \delta$. We have that $d_{\mathcal{C}(S)}(\{a, b\}, \tilde{c}) \ge r + \delta + 1$.

A similar argument applied to the geodesic triangle formed by Q_T, Q_a and $\tilde{Q}$ shows that there exists $c' \in Q_T$ such that $d_{\mathcal{C}(S)}(\tilde{c}, c') \le \delta$ and $d_{\mathcal{C}(S)}(c', \{a, b\}) \ge r + 1$. In particular we have that $c' \in Q_T \cap N_{\mathcal{C}(S)}(c; 2\delta)$. □

The following theorem is a statement of [14, Theorems 1.1 and 1.2] with the addition of effective bounds. We refer to the subsets $G(a, b) \cap N_{\mathcal{C}(S)}(c; \delta)$ and $G(a, b; r) \cap N_{\mathcal{C}(S)}(c; 2\delta)$ as the 'slices'.

Theorem 7.1.2. *Fix $\delta \ge 3$ such that $\mathcal{C}(S)$ is δ-hyperbolic for all surfaces S with $\xi(S) \ge 2$. The following statements hold, where K is a uniform constant.*

1. *For any $a, b \in \mathcal{C}_0(S)$ and for any curve $c \in Q \in \mathcal{L}(a, b)$ we have that $|G(a, b) \cap N_{\mathcal{C}(S)}(c; \delta)| \le K^{\xi(S)}$.*

2. *For any $r \ge 0$ and $a, b \in \mathcal{C}_0(S)$ such that $d_{\mathcal{C}(S)}(a, b) \ge 2r + 2k + 1$ (where $k = 10\delta + 1$) for any curve $c \in Q \in \mathcal{L}(a, b)$ such that $c \notin N_{\mathcal{C}(S)}(a; r + k) \cup N_{\mathcal{C}(S)}(b; r + k)$ we have that $|G(a, b; r) \cap N_{\mathcal{C}(S)}(c; 2\delta)| \le K^{\xi(S)}$.*

Proof of 1. Recall the notation of Theorem 5.2.7: there is an effective upper bound $B(S, L)$ (that only depends on S and L) on the cardinality of the set $C(v_1, v_2; L)$ of curves c such that there exists a tight filling multipath P connecting v_1 to v_2 with length$(P) \le L$ and $c \in m \in P$ for some multicurve m.

Given $c \in Q \in \mathcal{L}(a, b)$, first suppose that $d_{\mathcal{C}(S)}(c, \{a, b\}) \ge 4\delta + 1$. Let $c_a, c_b \in Q$ be curves such that $d_{\mathcal{C}(S)}(c_a, c) = d_{\mathcal{C}(S)}(c, c_b) = 4\delta + 1$, $d_{\mathcal{C}(S)}(c_a, c) + d_{\mathcal{C}(S)}(c, c_b) = d_{\mathcal{C}(S)}(c_a, c_b)$ and $d_{\mathcal{C}(S)}(a, c_a) < d_{\mathcal{C}(S)}(a, c_b)$. We shall show that $G(a, b) \cap N_{\mathcal{C}(S)}(c; \delta) \subset C(c_a, c_b; 12\delta + 2)$ to deduce that $|G(a, b) \cap N_{\mathcal{C}(S)}(c; \delta)| \le B(S, 12\delta + 2)$.

Given any $v \in G(a, b) \cap N_{\mathcal{C}(S)}(c; \delta)$ we have $v \in Q_T$, for some $Q_T \in \mathcal{L}_T(a, b)$. The geodesic bigon formed by Q and Q_T is δ-slim by Theorem 3.2.1. Hence $d_{\mathcal{C}(S)}(c', Q_T) \le \delta$ whenever $c' \in \{c_a, c, c_b\}$. By Theorem 6.2.3, with $r = \delta$, there exists a tight filling multipath P that connects c_a to c_b such that length$(P) \le 12\delta + 2$ and $v \in Q_T \cap N_{\mathcal{C}(S)}(c; \delta) \subset P \cap N_{\mathcal{C}(S)}(c; \delta)$. Thus $v \in C(c_a, c_b; 12\delta + 2)$ and the theorem is proved.

If $d_{\mathcal{C}(S)}(c, b) < 4\delta + 1$ and $d_{\mathcal{C}(S)}(a, c) \ge 4\delta + 1$ then we make an analogous argument with c_a, c and b and use Theorem 6.2.3.

If $d_{\mathcal{C}(S)}(a, c), d_{\mathcal{C}(S)}(c, b) < 4\delta + 1$ then every tight multigeodesic that connects a and b is a tight filling multipath and its length is less than or equal to $8\delta + 2$. Therefore its length is less than or equal to $12\delta + 2$ thus $|G(a, b)| \le B(S, 12\delta + 2)$.

By Theorem 3.2.1 and Theorem 5.2.7 we are done.

Proof of 2. Given $c \in Q \in \mathcal{L}(a,b)$ such that $d_{\mathcal{C}(S)}(c,\{a,b\}) \geq r+10\delta+2$, we fix curves $c_a, c_b \in Q$ such that $d_{\mathcal{C}(S)}(c_a,c) = d_{\mathcal{C}(S)}(c,c_b) = 8\delta+1$, $d_{\mathcal{C}(S)}(c_a,c) + d_{\mathcal{C}(S)}(c,c_b) = d_{\mathcal{C}(S)}(c_a,c_b)$ and $d_{\mathcal{C}(S)}(a,c_a) < d_{\mathcal{C}(S)}(a,c_b)$. Then $d_{\mathcal{C}(S)}(\{c_a,c_b\},\{a,b\}) \geq r+2\delta+1$.

Given any $v \in G(a,b;r) \cap N_{\mathcal{C}(S)}(c;2\delta)$, we have $v \in Q_T$, for some $Q_T \in \mathcal{L}_T(a,b;r)$. By Lemma 7.1.1 we have that $d_{\mathcal{C}(S)}(c',Q_T) \leq 2\delta$ whenever $c' \in \{c_a,c,c_b\}$. So by Theorem 6.2.3, with $r = 2\delta$, there exists a tight filling multipath P that connects c_a to c_b such that $v \in Q_T \cap N_{\mathcal{C}(S)}(c;2\delta) \subset P \cap N_{\mathcal{C}(S)}(c;2\delta)$ and $\mathrm{length}(P) \leq 24\delta+2$. We conclude that $G(a,b;r) \cap N_{\mathcal{C}(S)}(c;2\delta) \subset C(c_a,c_b;24\delta+2)$ and thus $|G(a,b;r) \cap N_{\mathcal{C}(S)}(c;2\delta)| \leq B(S,24\delta+2)$. By Theorem 3.2.1 and Theorem 5.2.7 we are done. □

We write $\mathcal{MCG}(S)$ for the *mapping class group*, which is the group of homeomorphisms of S modulo ambient isotopy of S. We shall not define pseudo-Anosov here, see [22] for a definition.

The mapping class group of S naturally acts on the curve graph of S. We say that $\phi \in \mathcal{MCG}(S)$ *preserves* $(m_i)_{i\in I}$ if there exists $k \in \mathbb{Z}$ such that $\phi(m_i) = m_{i+k}$ for all i. As a corollary of [14, Section 3] using our bounds from Theorem 7.1.2 we have the following.

Theorem 7.1.3. *There exists an effective $m = m(\xi(S))$ such that whenever $\phi \in \mathcal{MCG}(S)$ is pseudo-Anosov we have that the mapping class ϕ^m preserves a geodesic in $\mathcal{C}(S)$. Furthermore m is bounded by $\exp(\exp(K'\xi(S)))$ where K' is a uniform constant.*

Proof. For the reader's convenience we use the notation from [14]: we write P for the bounds in Theorem 7.1.2 i.e. we take $P = K^{\xi(S)}$. By [14, Lemma 3.4], for any pseudo-Anosov ϕ there exists $m' \leq P^2$ such that $\phi^{m'}$ preserves a geodesic in $\mathcal{C}(S)$. Write $m = \mathrm{lcm}(1,...,P^2)$. Since m' divides m we have that ϕ^m preserves a geodesic in $\mathcal{C}(S)$. We can bound m by $P^{2\pi(P^2)}$ where π is the prime-counting function. By [44] we have that $\pi(P^2) \leq \frac{2P^2}{\log(P^2)}$. Now P is bounded by $K^{\xi(S)}$ by Theorem 7.1.2 and we are done. □

7.2 Acylindrical action constants

Definition 7.2.1. A group G acts on a metric space (X,d_X) *acylindrically* if for all $r \geq 0$, there exists R, N such that whenever elements $a,b \in X$ satisfy $d_X(a,b) \geq R$ then there are at most N elements $g \in G$ such that $d_X(a,ga), d_X(b,gb) \leq r$.

The only setting in which we discuss acylindricity is where $G = \mathcal{MCG}(S)$ and $(X, d_X) = (\mathcal{C}(S), d_{\mathcal{C}(S)})$. Suppose $r = 0$. Then we have to find R_0 and N_0 such that the cardinality of $\mathrm{stab}(a) \cap \mathrm{stab}(b)$ is at most N_0 whenever $d_{\mathcal{C}(S)}(a, b) \geq R_0$. We may take $R_0 = 3$ and $N_0 = 8\xi(S)$ by the following lemma.

Lemma 7.2.2. *Suppose m and m' fill $S = S_{g,n}$. Then* $\mathrm{stab}(m) \cap \mathrm{stab}(m') \subset \mathcal{MCG}(S)$ *is a finite subgroup, bounded by* $24g + 4n - 24 \leq 8\xi(S) = N_0$.

Proof. Suppose $[g] \in \mathrm{stab}(m) \cap \mathrm{stab}(m')$. Then we may take a representative f of $[g]$, due to Lemma 2.3.4, such that f preserves the subset $\gamma \cup \gamma' \subset S$, where γ and γ' are in minimal position and represent m and m' respectively.

Each complementary region D of $\gamma \cup \gamma'$ has a number of *sides*—this can be defined to be twice the number of connected components of $\partial N(\gamma) \cap D$ where $N(\gamma)$ is a closed regular neighbourhood of γ. A *square* region is a simply connected complementary component of $\gamma \cup \gamma'$ with exactly four sides.

By considering the complementary regions of $\gamma \cup \gamma'$, we see that $[f]$ is determined by f restricted to one non-square complementary region, by extension and Alexander's trick.

A complementary region of $\gamma \cup \gamma'$ is *holed* if it is not simply connected. A *bigon* is a complementary region with two sides. A bigon is necessarily holed because γ and γ' are in minimal position. There are at most n holed bigons, and $[f]$ is determined by the restriction of f to one such holed bigon, whence there are at most $2n$ possibilities for $[f]$.

So suppose that there are no holed bigons. This means that we may glue closed 2-discs to ∂S to obtain a new closed surface S', and, γ and γ' will be in minimal position in S' by Lemma 2.3.3 (Bigon criterion). Since there are no holed bigons to start with, we cannot have that S is planar because $\chi(S') \leq 0$.

In the case where $g = 1$ i.e. $S = S_{1,n}$, all regions have exactly four sides by an Euler characteristic argument. Thus we can bound the possibilities of $[f]$ from above by $4n$ by considering the restriction of f to a holed, four-sided region.

The remaining case is where $g \geq 2$, with no holed bigons. Glue closed 2-discs to ∂S (if possible) to obtain a new surface $S_{g,0}$. We have that γ and γ' are in minimal position in $S_{g,0}$ so it suffices to find an upper bound for $S_{g,0}$.

Pick an arbitrary non-square complementary region and write s for its number of sides. There are at most

$$\frac{8g - 8}{s - 4}$$

complementary regions with s sides. Therefore there are at most $(8g - 8)s/(s - 4)$ possibilities for $[f]$, and since $s \geq 6$ this quantity is at most $24g - 24$. □

The following theorem was originally proved by Bowditch [14] but here we provide effective constants. The main point is to describe the behaviour of R and N in terms of r and $\xi(S)$. This may be of interest.

Theorem 7.2.3. *The group $\mathcal{MCG}(S)$ acts on the metric space $\mathcal{C}(S)$ acylindrically. Fix δ and K as in Theorem 7.1.2. We may take $R = 4r + 24\delta + 7$ and $N = N_0(2r + 4\delta + 1)(8\delta + 7)K^{2\xi(S)}$, where N_0 is as in Lemma 7.2.2. Fixing r we have that R is constant and N grows exponentially with respect to $\xi(S)$. Fixing $\xi(S)$ we have that R and N grow linearly with respect to r.*

Proof. Set $R = 4r + 24\delta + 7$. Suppose that $d_{\mathcal{C}(S)}(a, b) \geq R$. We pick an arbitrary tight multigeodesic $Q_T \in \mathcal{L}_T(a, b)$. There are curves x and y such that $x \in m_x \in Q_T$ and $y \in m_y \in Q_T$ for some multicurves m_x and m_y. Furthermore there exist such x and y with $d_{\mathcal{C}(S)}(x, y) = 3$ and $d_{\mathcal{C}(S)}(\{x, y\}, \{a, b\}) \geq r + (10\delta + 1) + (2\delta + r) + 1$—we remind the reader that $d_{\mathcal{C}(S)}$ of a pair of non-empty sets is defined to be the infimum of the distances between any two of their respective elements.

Suppose that we have $\phi \in \mathcal{MCG}(S)$ such that $d_{\mathcal{C}(S)}(a, \phi a), d_{\mathcal{C}(S)}(b, \phi b) \leq r$. Write x' and y' for nearest point projections of ϕx and ϕy onto Q_T respectively.

We claim that $d_{\mathcal{C}(S)}(x, x') \leq r + 2\delta$, and similarly for y and y'. Without loss of generality we have that $d_{\mathcal{C}(S)}(x', b) \leq d_{\mathcal{C}(S)}(x, b)$. By Lemma 7.1.1 we have that $d_{\mathcal{C}(S)}(x', \phi x) \leq 2\delta$ and so $d_{\mathcal{C}(S)}(\phi x, \phi b) \leq 2\delta + d_{\mathcal{C}(S)}(x', b) + d_{\mathcal{C}(S)}(b, \phi b)$. Now $d_{\mathcal{C}(S)}(x', b) = d_{\mathcal{C}(S)}(x, b) - d_{\mathcal{C}(S)}(x, x') = d_{\mathcal{C}(S)}(\phi x, \phi b) - d_{\mathcal{C}(S)}(x, x')$. Thus $d_{\mathcal{C}(S)}(x, x') \leq 2\delta + d_{\mathcal{C}(S)}(b, \phi b)$ and the claim follows.

Now write m'_x for the multicurve such that $x' \in m'_x \in Q_T$. By the claim above, there are at most $(2r + 4\delta + 1)$ possibilities for m'_x, given x. We have that $d_{\mathcal{C}(S)}(\{x', y'\}, \{a, b\}) \geq r + 10\delta + 2$ and so the proof of Theorem 7.1.2 shows that there are at most $(2r + 4\delta + 1)K^{\xi(S)}$ possibilities for $\phi(x)$, given x.

Similarly write m'_y for the multicurve such that $y' \in m'_y \in Q_T$. Now $d_{\mathcal{C}(S)}(x', y') \leq d_{\mathcal{C}(S)}(x', \phi x) + d_{\mathcal{C}(S)}(\phi x, \phi y) + d_{\mathcal{C}(S)}(\phi y, y') \leq 4\delta + 3$. So there are at most $(8\delta + 7)$ possibilities for m'_y, given m'_x. Therefore there are $(8\delta + 7)K^{\xi(S)}$ possibilities for $\phi(y)$, given m'_x.

In conclusion there are at most $(2r + 4\delta + 1)(8\delta + 7)K^{2\xi(S)}$ possibilities for the pair $(\phi x, \phi y)$, given x. Therefore there are at most $N = N_0(2r + 4\delta + 1)(8\delta + 7)K^{2\xi(S)}$ possibilities for ϕ, by Lemma 7.2.2. By Theorem 7.1.2 we obtain the last statement. □

Remark 7.2.4. In [14] it is remarked that a refinement of the argument in [7] shows that if $d_{\mathcal{C}(S)}(a, b) \geq 2r + 3$ then the cardinality of the set

$$\{\phi \in \mathcal{MCG}(S) : d_{\mathcal{C}(S)}(a, \phi a) \leq r, d_{\mathcal{C}(S)}(b, \phi b) \leq r\}$$

is finite. However it is not known if the cardinality of this set is bounded from above independently of a and b. In other words, in the definition of acylindricity, it is not known if we can take $R = 2r + 3$. Yohsuke Watanabe [50] has another combinatorial proof of Theorem 7.1.2 which is different to ours here but his bounds are doubly exponential in $\xi(S)$ rather than exponential. However, there is a chance that his techniques might give a better bound for R.

7.3 Computing stable lengths of pseudo-Anosovs and invariant tight geodesic axes

The *stable length* $||\phi||$ of an element $\phi \in \mathcal{MCG}(S)$ (acting on $\mathcal{C}(S)$) is the quantity

$$\lim_{j \to \infty} \frac{d_{\mathcal{C}(S)}(c, \phi^j c)}{j}$$

where $c \in \mathcal{C}_0(S)$ is arbitrary. Note that the limit exists because the sequence is decreasing and non-negative. Furthermore the limit does not depend on the choice of c.

Now we outline a procedure which accepts a surface S and a pseudo-Anosov mapping class $\phi \in \mathcal{MCG}(S)$ and returns the stable length $||\phi||$, answering a question asked by Brian Bowditch at the 'Aspects of hyperbolicity in geometry, topology, and dynamics' workshop in July 2011.

We won't make an effort here to optimize running times.

Theorem 7.3.1. *There exists a finite time algorithm that accepts a surface S and a pseudo-Anosov mapping class $\phi \in \mathcal{MCG}(S)$ and returns the stable length of ϕ on the curve graph of S.*

Proof. In Theorem 7.1.3 we constructed an effective constant $m = m(\xi(S))$ such that ϕ^m preserves some geodesic axis. Now apply [47, Proposition 7.1]. □

Note that ϕ^{2m} preserves a geodesic with even translation length therefore by the tightening procedure (Lemmas 5.1.3 and 5.1.5) a tight multigeodesic is also preserved by ϕ^{2m}.

Theorem 7.3.2. *There exists a finite time algorithm that accepts a surface S and a pseudo-Anosov mapping class $\phi \in \mathcal{MCG}(S)$ and returns all invariant tight multigeodesics of ϕ. The collection of multigeodesics are returned as a collection of finite sets of curves such that the orbit under ϕ on each set is a tight multigeodesic.*

Proof. We start by computing all invariant tight multigeodesics of ϕ^{2m} by considering $\psi = \phi^{2m(32\delta+32)}$ where m is as in Theorem 7.1.3. Of course these include the tight multigeodesics preserved by ϕ. Then we check whether or not these tight multigeodesics are preserved by ϕ and remove those that are not. We have that $||\psi|| \geq 4||\phi^{2m}|| + 32\delta + 28$.

Fix $v \in \mathcal{C}_0(S)$ and compute a geodesic Q from v to $\psi(v)$ using the techniques of the proof of Theorem 5.2.7. Write $C'(a,b)$ for the set of tight filling multipaths P that connect a to b such that $\text{length}(P) \leq d_{\mathcal{C}(S)}(a,b) + 8\delta$. We compute $C'(a,b)$ for each $a, b \in Q$ by using the constructive proof of Theorem 5.2.7.

Now we show that for any tight multigeodesic Q_T preserved by ϕ^{2m} there exists a submultipath Q'_T of Q_T (whose orbit under ϕ^{2m} is Q_T) such that Q'_T is a submultipath of some $P \in C'(a,b)$ with $a, b \in Q$. Write $v' \in Q_T$ for a nearest point projection of v to Q_T. Write $\tilde{Q}_T = (m_i)_{i\in I}$ for the multigeodesic that connects v' to $\psi(v')$ such that for each index $i \in I$ which is neither initial nor terminal we have that $m_i \in Q_T$. Furthermore we choose I such that v' is initial and $\psi(v')$ is terminal in $\tilde{Q}_T$. There exists a geodesic Q_v connecting v to v'. Consider the geodesic rectangle formed by Q_v, $\tilde{Q}_T$, $\psi(Q_v)$ and Q.

Now $\text{length}(\tilde{Q}_T) = ||\psi|| \geq 4||\phi^{2m}|| + 32\delta + 28$ so there exists $c' \in \tilde{Q}_T$ such that $d_{\mathcal{C}(S)}(c', \{v', \psi(v')\}) \geq 2||\phi^{2m}|| + 16\delta + 14$. Since v' is a nearest point projection we have that $d_{\mathcal{C}(S)}(c', Q_v) \geq ||\phi^{2m}|| + 8\delta + 7$ (in fact one can obtain a better bound here but for brevity we choose not to). Similarly we have that $d_{\mathcal{C}(S)}(c', \psi(Q_v)) \geq ||\phi^{2m}|| + 8\delta + 7$. Now following the proof of Lemma 7.1.1 there exists $c \in Q$ such that $d_{\mathcal{C}(S)}(c, c') \leq 2\delta$. We have that $d_{\mathcal{C}(S)}(c, Q_v \cup \psi(Q_v)) \geq ||\phi^{2m}|| + 6\delta + 7$. Therefore there exist $a, b \in Q$ such that $d_{\mathcal{C}(S)}(a, c) = d_{\mathcal{C}(S)}(c, b) = ||\phi^{2m}|| + 4\delta + 6$ with $d_{\mathcal{C}(S)}(v, a) < d_{\mathcal{C}(S)}(v, b)$. We have that $d_{\mathcal{C}(S)}(\{a, b\}, Q_v \cup \psi(Q_v)) \geq 2\delta + 1$ so we may apply the proof of Lemma 7.1.1 again to deduce that there exist $a', b' \in \tilde{Q}_T$ such that $d_{\mathcal{C}(S)}(a, a') \leq 2\delta$ and $d_{\mathcal{C}(S)}(b, b') \leq 2\delta$. With respect to the order of the indices of $\tilde{Q}_T$, we may assume $a' \in m_i$ to be last possible, in the sense that $d_{\mathcal{C}(S)}(a, m_{i'}) \geq 2\delta + 1$ for $i' > i$. Similarly we may assume b' to be first possible. By applying Proposition 6.1.4 and Lemmas 5.1.3 and 5.1.5, with very much the same technique as that used to prove Theorem 6.2.3, there exists a tight filling multipath P that connects a to b which has a submultipath $Q'_T \subset Q_T$ such that $\text{length}(Q'_T) \geq d_{\mathcal{C}(S)}(a,b) - 8\delta - 12 \geq 2||\phi^{2m}||$ and furthermore $\text{length}(P) \leq d_{\mathcal{C}(S)}(a,b) + 8\delta$. This is the required submultipath Q'_T and the required vertices are a and b.

Now we have a finite list of multipaths P such that P is a submultipath of some element of $C'(a,b)$ for some $a,b \in Q$. The rest of the algorithm discards those multipaths P whose orbit under ϕ is not a tight multigeodesic. First we check whether or not the concatenation of P and $\phi^{2m}(P)$ is a tight multigeodesic, and if not then discard P. Now discard the remaining multipaths that are not minimal in length. The length of any remaining multipath is equal to $||\phi^{2m}||$. It follows that for any remaining multipath P we have that the orbit under ϕ^{2m} of P is a tight multigeodesic. Now for each remaining multipath P, check whether or not $\phi(P)$ is a submultipath of the concatenation of P and $\phi^{2m}(P)$ and if not then discard P. If the condition on $\phi(P)$ does hold then the orbit of P under ϕ^{2m} is preserved by ϕ, in other words the orbit of P under ϕ is a tight multigeodesic preserved by ϕ. For those P that remain, take any submultipath P' of P such that $\text{length}(P') = \frac{1}{2m}\text{length}(P)$. These are the required sets of curves. $\square$

Bibliography

[1] T. Aougab, *Uniform hyperbolicity of the graphs of curves.* `http://arxiv.org/abs/1212.3160` To appear in *Geometry and Topology.*

[2] Jason Behrstock, Bruce Kleiner, Yair Minsky, and Lee Mosher, *Geometry and rigidity of mapping class groups*, Geom. Topol. **16** (2012), no. 2, 781–888. MR2928983

[3] Jason A. Behrstock and Yair N. Minsky, *Dimension and rank for mapping class groups*, Ann. of Math. (2) **167** (2008), no. 3, 1055–1077. MR2415393 (2009d:57031)

[4] M. Bestvina, K. Bromberg, and K. Fujiwara, *Constructing group actions on quasi-trees and applications to mapping class groups.* `http://arxiv.org/abs/1006.1939v3`.

[5] M. Bestvina and M. Feighn, *Subfactor projections.* `http://arxiv.org/abs/1211.1730`.

[6] Mladen Bestvina and Mark Feighn, *Hyperbolicity of the complex of free factors*, Adv. Math. **256** (2014), 104–155. MR3177291

[7] Mladen Bestvina and Koji Fujiwara, *Bounded cohomology of subgroups of mapping class groups*, Geom. Topol. **6** (2002), 69–89 (electronic). MR1914565 (2003f:57003)

[8] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491. Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault. MR0387495 (52 #8337)

[9] B. H. Bowditch, *The ending lamination theorem.* `http://homepages.warwick.ac.uk/~masgak/papers/elt.pdf`.

[10] ———, *Rank and rigidity properties of spaces associated to a surface.* `http://www.warwick.ac.uk/~masgak/papers/rigidity.pdf`.

[11] B.H. Bowditch, *Uniform hyperbolicity of the curve graphs.* `http://homepages.warwick.ac.uk/~masgak/papers/uniformhyp.pdf` To appear in *Pacific J. Math.*

[12] Brian H. Bowditch, *A course on geometric group theory*, MSJ Memoirs, vol. 16, Mathematical Society of Japan, Tokyo, 2006. MR2243589 (2007e:20085)

[13] ———, *Intersection numbers and the hyperbolicity of the curve complex*, J. Reine Angew. Math. **598** (2006), 105–129. MR2270568 (2009b:57034)

[14] ———, *Tight geodesics in the curve complex*, Invent. Math. **171** (2008), no. 2, 281–300. MR2367021 (2008m:57040)

[15] ———, *Atoroidal surface bundles over surfaces*, Geom. Funct. Anal. **19** (2009), no. 4, 943–988. MR2570310 (2011d:57045)

[16] Jeffrey Brock, Howard Masur, and Yair Minsky, *Asymptotics of Weil-Petersson geodesics II: bounded geometry and unbounded entropy*, Geom. Funct. Anal. **21** (2011), no. 4, 820–850. MR2827011 (2012j:32008)

[17] Jeffrey F. Brock, *The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores*, J. Amer. Math. Soc. **16** (2003), no. 3, 495–535 (electronic). MR1969203 (2004c:32027)

[18] ———, *Weil-Petersson translation distance and volumes of mapping tori*, Comm. Anal. Geom. **11** (2003), no. 5, 987–999. MR2032506 (2004k:32018)

[19] Jeffrey F. Brock, Richard D. Canary, and Yair N. Minsky, *The classification of Kleinian surface groups, II: The ending lamination conjecture*, Ann. of Math. (2) **176** (2012), no. 1, 1–149. MR2925381

[20] Matt Clay and Alexandra Pettet, *Relative twisting in outer space*, J. Topol. Anal. **4** (2012), no. 2, 173–201. MR2949239

[21] M.T. Clay, K. Rafi, and S. Schleimer, *Uniform hyperbolicity of the curve graph via surgery sequences*. `http://arxiv.org/abs/1302.5519`.

[22] Benson Farb and Dan Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, vol. 49, Princeton University Press, Princeton, NJ, 2012. MR2850125 (2012h:57032)

[23] U. Hamenstädt, *Geometry of the mapping class groups III: Quasi-isometric rigidity*. `http://arxiv.org/pdf/math/0512429v2`.

[24] Ursula Hamenstädt, *Geometry of the complex of curves and of Teichmüller space*, Handbook of Teichmüller theory. Vol. I, 2007, pp. 447–467. MR2349677 (2009b:32018)

[25] Michael Handel and Lee Mosher, *The free splitting complex of a free group, I: hyperbolicity*, Geom. Topol. **17** (2013), no. 3, 1581–1672. MR3073931

[26] John L. Harer, *The virtual cohomological dimension of the mapping class group of an orientable surface*, Invent. Math. **84** (1986), no. 1, 157–176. MR830043 (87c:32030)

[27] W. J. Harvey, *Boundary structure of the modular group*, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978), 1981, pp. 245–251. MR624817 (83d:32022)

[28] Allen Hatcher, *On triangulations of surfaces*, Topology Appl. **40** (1991), no. 2, 189–194. MR1123262 (92f:57020)

[29] S. Hensel, D. Osadja, and P. Przytycki, *Realisation and dismantlability*. `http://arxiv.org/abs/1205.0513`.

[30] S. Hensel, P. Przytycki, and R. C. H. Webb, *Slim unicorns and uniform hyperbolicity for arc graphs and curve graphs*. `http://arxiv.org/abs/1301.5577`.

[31] E. Klarreich, *The boundary at infinity of the curve complex and the relative teichmller space*. `http://www.nasw.org/users/klarreich/research.htm`.

[32] Thomas Koshy, *Catalan numbers with applications*, Oxford University Press, Oxford, 2009. MR2526440 (2010g:05008)

[33] Jason Paige Leasure, *Geodesics in the complex of curves of a surface*, ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)–The University of Texas at Austin. MR2705485

[34] J. Mangahas, *A recipe for short-word pseudo-Anosovs*, Amer. J. Math. **135** (2013), no. 4, 1087–1116.

[35] B. Mann, *Hyperbolicity of the Cyclic Splitting Graph.* http://arxiv.org/pdf/1212.2986v2.

[36] H. A. Masur and Y. N. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, Geom. Funct. Anal. **10** (2000), no. 4, 902–974. MR1791145 (2001k:57020)

[37] Howard Masur and Saul Schleimer, *The pants complex has only one end*, Spaces of Kleinian groups, 2006, pp. 209–218. MR2258750 (2007g:57035)

[38] ______, *The geometry of the disk complex*, J. Amer. Math. Soc. **26** (2013), no. 1, 1–62. MR2983005

[39] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. **138** (1999), no. 1, 103–149. MR1714338 (2000i:57027)

[40] Yair Minsky, *The classification of Kleinian surface groups. I. Models and bounds*, Ann. of Math. (2) **171** (2010), no. 1, 1–107. MR2630036 (2011d:30110)

[41] Yair N. Minsky, *A geometric approach to the complex of curves on a surface*, Topology and Teichmüller spaces (Katinkulta, 1995), 1996, pp. 149–158. MR1659683 (2000g:32016)

[42] K. Rafi, *Hyperbolicity in Teichmüller space.* http://www.math.ou.edu/~rafi/research/Fellow.pdf.

[43] Kasra Rafi, *A combinatorial model for the Teichmüller metric*, Geom. Funct. Anal. **17** (2007), no. 3, 936–959. MR2346280 (2008j:30063)

[44] J. Barkley Rosser and Lowell Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64–94. MR0137689 (25 #1139)

[45] L. Sabalka and D. Savchuk, *Submanifold projection.* http://front.math.ucdavis.edu/1211.3111.

[46] Kenneth J. Shackleton, *An acylindricity theorem for the mapping class group*, New York J. Math. **16** (2010), 563–573. MR2740591 (2012a:57025)

[47] ______, *Tightness and computing distances in the curve complex*, Geom. Dedicata **160** (2012), 243–259. MR2970053

[48] Jing Tao, *Linearly bounded conjugator property for mapping class groups*, Geom. Funct. Anal. **23** (2013), no. 1, 415–466. MR3037904

[49] S. Taylor, *Right-angled Artin groups and* $out(f_n)$ *I: quasi-isometric embeddings.* http://arxiv.org/abs/1303.6889.

[50] Y. Watanabe, *Local finiteness of the curve graph via subsurface projections and a uniform bound of tight geodesics.* http://arxiv.org/abs/1312.5040.

[51] ______, *A polygon decomposition of a surface by filling curves and tight geodesics.* http://www.math.utah.edu/~ywatanab/Yohsuke_Watanabe_files/distancebound.pdf.

[52] R. C. H. Webb, *Uniform bounds for bounded geodesic image theorems.* Submitted.

[53] Alexander S. Wickens, *Ending laminations and the curve complex.* University of Warwick MPhil thesis.