Improved Approximation Algorithms for Degree-bounded Network Design Problems with Node Connectivity Requirements

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ABSTRACT

We consider *degree bounded* network design problems with element and vertex connectivity requirements. In the degree bounded Survivable Network Design (SNDP) problem, the input is an undirected graph $G = (V, E)$ with weights $w(e)$ on the edges and degree bounds $b(v)$ on the vertices, and connectivity requirements $r(uv)$ for each pair uv of vertices. The goal is to select a minimum-weight subgraph H of G that meets the connectivity requirements and it satisfies the degree bounds on the vertices: for each pair uv of vertices, H has $r(uv)$ disjoint paths between u and v; additionally, each vertex v is incident to at most $b(v)$ edges in H. We give the first $(O(1), O(1)\cdot b(v))$ bicriteria approximation algorithms for the degree-bounded SNDP problem with *element connectivity* requirements and for several degree-bounded SNDP problems with *vertex connectivity* requirements. Our algorithms construct a subgraph H whose weight is at most $O(1)$ times the optimal such that each vertex v is incident to at most $O(1)\cdot b(v)$ edges in H. We can also extend our approach to network design problems in directed graphs with out-degree constraints to obtain $(O(1), O(1)\cdot b^+(v))$ bicriteria approximation.

1. INTRODUCTION

Network design is an important area in combinatorial optimization and approximation algorithms with several practical applications. Many problems that arise in real-world networks can be modeled as follows: we are given a weighted graph and the goal is to choose a minimum weight subgraph that meets certain connectivity requirements between pairs of vertices. This framework captures well-studied problems such as the Minimum Spanning Tree, the Minimum Steiner Tree and Forest, and their generalizations to higher connectivity. A central problem in this area is the Survivable Network Design (SNDP) problem in which the input is an undirected edge-weighted graph $G = (V, E)$ together with integer connectivity requirements $r(uv)$ for each pair uv of vertices. The objective is to select a minimum weight subgraph H of G that satisfies the connectivity requirements for all pairs of vertices. In the *edge connectivity* SNDP (EC-SNDP) problem, the subgraph H satisfies the connectivity requirements if, for each pair uv of vertices, H contains $r(uv)$ *edge disjoint* paths between u and v. In the *vertex connectivity* SNDP (VC-SNDP) problem, the paths are required to be *internally vertex disjoint*. In addition to edge and vertex connectivity, another notion of connectivity that plays an important role in this area is the notion of *element connectivity*. In this setting, the vertices of the graph are partitioned into two sets, the set R of reliable vertices and the set W of unreliable vertices. The vertices of W and the edges are called *elements*. The element connectivity of a pair uv of vertices is the maximum number of element-disjoint paths from u to v; these paths do not share any unreliable nodes or edges, but they may share reliable nodes. In the *element connectivity* SNDP (Elem-SNDP) problem, the paths are required to be *element disjoint*.

Survivable network design problems have received considerable attention over the years and their study has led to the development of powerful algorithmic tools based on primal dual and iterated rounding. In a celebrated paper [11], Jain gave a 2-approximation for EC-SNDP based on iterated rounding, which was later extended to element connectivity by Fleischer et al. [4]. Vertex connectivity network design problems such as the Rooted k-Connectivity problem, k-Connected Subgraph, and VC-SNDP have been studied extensively over the past decade, leading to several breakthroughs in terms of both algorithms and hardness.

In this paper, we consider the *degree bounded* variants of these problems in which we are additionally given a degree bound $b(v)$ for each vertex v and the goal is to select a minimum weight subgraph H that satisfies the connectivity requirements and moreover the degree in H of each vertex v is at most its degree bound $b(v)$. A classical problem in this setting is the degree bounded MST problem. A simple reduction from the Hamiltonian Path problem shows that it is **NP**-complete to decide whether there is a tree that satisfies

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the degree constraints and thus the problem is inapproximable. A long line of work has focused on designing *bicriteria* approximation algorithms that construct a low weight spanning tree that violates the degree bounds by a small amount (see [9, 12, 13, 10, 19] and references thereof). This sequence of results culminated with the result of Singh and Lau [19] that settled the approximability of the problem; the algorithm of [19] constructs a spanning tree with weight at most the optimal that violates each degree constraint by at most *one*, and it is based on iterated rounding.

In a subsequent breakthrough, Lau et al. [14] and Lau and Singh [16] extended the iterated rounding framework to degree bounded SNDP problems. An important contribution of the work of [14, 16] is the development of iterated rounding techniques that can simultaneously handle *covering* constraints corresponding to the SNDP problem and *packing* constraints corresponding to the degree bounds. These techniques have led to several powerful results for degree bounded network design problems; we refer the reader to the recent book of Lau, Ravi, and Singh [15] for an overview of these results. We remark that these results focus almost entirely on *edge connectivity* problems, and the iterated rounding techniques of [14, 16] are not sufficient for the element and vertex connectivity settings. It was only recently that Nutov gave the first approximation algorithms for several degree bounded network design problems with element and vertex connectivity requirements [18]. We remark that these algorithms lead to degree violations that are exponential in the maximum requirement k of any pair. Shortly after, Fukunaga et al. [8, 7] significantly improved the best guarantees for these problems in which the degree violations are only polynomial in k.

1.1 Our techniques and contributions

The main question left open by the recent work on degree bounded network design problems with element and vertex connectivity requirements is whether we can simultaneously achieve a constant factor approximation in both the weight of the solution and the violation on the degrees. In this paper, we answer this question affirmatively for the degree bounded Elem-SNDP problem and several vertex connectivity problems. Our work closes the gap between several edge connectivity problems and their element or vertex counterparts by providing approximation algorithms that nearly match the best guarantees for the edge connectivity setting. As it is common in this area, we use iterated rounding techniques but our analyses are a departure from the approach of Fukunaga and Ravi [8] and Fukunaga, Nutov, and Ravi [7]. Our main technical insight is inspired by the work of Nutov [18] and it takes advantage of the special structure of the SNDP problem. More precisely, our algorithms consider the problem of covering biset functions that belong to a *proper sub-class* of skew bisupermodular functions; this sub-class is general enough to allow us to use the iterated rounding framework but it has additional structure that we exploit in the analysis. This is a departure from most of the previous work — including the work on *edge connectivity* problems — that solves the more general problem of covering an arbitrary skew (bi)supermodular function; we defer the definitions related to biset functions to Section 2. We now give the precise statements of our main results.

In the following, we say that an algorithm achieves an $(\alpha, \beta b(v)+\gamma)$ bicriteria approximation if it always constructs a solution H whose weight is at most α times the weight of the optimal solution such that the degree in H of each vertex v is at most $\beta b(v) + \gamma$.

Our first result is an $(O(1), O(1)\cdot b(v))$ bicriteria approximation for the degree bounded Elem-SNDP problem.

THEOREM 1.1. *There is a polynomial time approximation algorithm for the degree bounded* Elem-SNDP *problem in undirected graphs that achieves an* $(O(1), O(1)\cdot b(v))$ *bicriteria approximation.*

By combining the result above with the reduction of Chuzhoy and Khanna [3], we obtain the following result for the degree bounded VC-SNDP.

COROLLARY 1.2. *There is a polynomial time approximation algorithm for the degree bounded* VC-SNDP *problem in undirected graphs that achieves an* $(O(k^3 \log n), O(k^3 \log n)\cdot b(v))$ *bicriteria approximation.*

Our next result is for the k-Connected Subgraph problem. An undirected graph G is k-vertex-connected (or k-connected for short) if it has at least $k+1$ vertices and the removal of any set of $k-1$ vertices leaves a connected graph. In the k-Connected Subgraph problem, we are given an edge-weighted undirected graph $G = (V, E)$ and the goal is to select a minimum-weight spanning subgraph $H = (V, E')$ of G such that H is k-connected.

The k-Connected Subgraph problem has been studied actively over the last decade; we refer the reader to the work of Cheriyan of Vegh [2] for an overview of the approximation results that are known for the problem. In a recent breakthrough result, Cheriyan and Vegh [2] gave the first constant factor approximation for instances of the k-Connected Subgraph problem in which the number of vertices n is not too small compared to k. In this paper, we extend the result of [2] to the degree-bounded setting.

THEOREM 1.3. *There is a polynomial time approximation algorithm for the degree bounded* k-Connected Subgraph *problem in undirected graphs that achieves an* $(O(1), O(1)\cdot b(v))$ *bicriteria approximation when* $|V| \geq (k-1)^3 - k$.

The Rooted k-Connectivity problem is a well-studied problem that has several applications in combinatorial optimization. The input consists of an edge-weighted undirected graph $G = (V, E)$, an integer k, and a vertex r called the root. The goal is to select a minimum-weight spanning subgraph $H = (V, E')$ of G such that H contains k internally vertex disjoint paths from r to v for each vertex $v \in V - \{r\}$. Frank and Tardos [6] gave an exact polynomial time algorithm for the directed version of the problem, which implies a 2-approximation for the undirected problem. In this paper, we give the first $(O(1), O(1)\cdot b(v))$ approximation for the degree bounded Rooted k-Connectivity problem.

THEOREM 1.4. *There is a polynomial time approximation algorithm for the degree bounded* Rooted k-Connectivity *problem in undirected graphs that achieves an* $(O(1), O(1)\cdot b(v))$ *bicriteria approximation.*

We note that our approach can be extend to degree-bounded network design problems in directed graphs, such as a directed counterpart of the Rooted k-Connectivity problem. We defer these results to a longer version of this paper.

THEOREM 1.5. *There is a polynomial time approximation algorithm for the degree bounded* Rooted k-Outconnectivity *problem in directed graphs that achieves an* $(O(1), O(1) \cdot b^+(v))$ *bicriteria approximation.*

Table 1 summarizes our main results and the best approximations for the problems we consider that were given in previous work. We refer the reader to [8] for additional results and references on degree bounded network design problems with element and vertex connectivity requirements.

1.2 Related work

Survivable network design problems have been studied extensively, and the edge connectivity problems have received the most attention. In a seminal paper [11], Jain gave a 2-approximation for EC-SNDP based on iterated rounding. The EC-SNDP problem can be viewed as a *cut covering* problem. By Menger's theorem, a subgraph H is a feasible solution for the EC-SNDP instance if and only if, for any cut $(S, V-S)$, there are $f(S) = \max_{u \in S,\ v \in V-S} r(uv)$ edges of H that are crossing the cut. Thus the problem is equivalent to finding a minimum-weight subgraph H that *covers* this set function f, that is, $|\delta_H(S)| \geq f(S)$ for each set S. Jain's algorithm is based on a natural LP relaxation for the problem of covering such a set function f using a graph. He showed that, if the function f satisfies certain *uncrossing* properties, any basic solution to the LP for covering f has an edge variable whose fractional value is at least $1/2$ (or there is an edge variable whose fractional value is zero, in which case we can simply remove the edge from the graph). The algorithm iteratively rounds up variables with fractional value at least $1/2$ and resolves the LP for the residual problem until the resulting integral solution becomes feasible.

Fleischer et al. [4] generalized Jain's result to the Elem-SNDP problem. The element connectivity problem requires us to work with a function f that is defined on pairs of sets (or bisets) instead of sets; we defer the precise definitions to Section 2. Fleischer et al. proved an analogous result on the structure of basic solutions to the LP relaxation for covering a function f defined on pairs of sets. This result also crucially relies on certain uncrossing properties of the function f and it gives a 2-approximation for Elem-SNDP via iterated rounding.

Vertex connectivity problems such as the k-Connected Subgraph problem and VC-SNDP are considerably more challenging than their edge and element counterparts. One of the reasons for this discrepancy is that our current iterated rounding techniques are unsuitable for this setting, since the uncrossing properties that play a key role for edge and element connectivity do not hold for vertex connectivity. Aazami et al. [1] showed that there exist basic solutions for the natural LP relaxation for the k-Connected Subgraph problem in which each variable is at most $\Theta(1/\sqrt{k})$. Most of the algorithmic results for vertex connectivity problems are obtained via reductions to element connectivity (or more generally, to the problem of covering a skew bisupermodular function). Chuzhoy and Khanna [3] showed that the VC-SNDP problem can be reduced to solving $O(k^3 \log n)$ instances of the Elem-SNDP problem, where n is the number of vertices in the graph; this reduction gives an $O(k^3 \log n)$ approximation for the vertex connectivity problem. An interesting and challenging open question in this area is whether there is an algorithm for VC-SNDP that achieves an approximation guarantee that depends only on k; the $\log n$ factor is unavoidable using the techniques of Chuzhoy and Khanna and new insights are needed to resolve this question. In particular, it seems fruitful to understand whether we can extend some of the iterated rounding techniques to the vertex connectivity setting, and we hope that the work in this paper will provide some insights in this direction.

The study of *degree bounded* network design problems has led to the development of new and powerful iterated rounding techniques. Lau et al. [14] and Lau and Singh [16] extended the iterated rounding approach of Jain to the setting in which we have mixed *covering* and *packing* constraints; the former are the cut covering constraints capturing the EC-SNDP problem and the latter are degree bound constraints on the vertices. The breakthrough results of Lau et al. [14] and Lau and Singh [16] give a $(2, 2b(v) + 3)$ approximation and a $(2, b(v) + O(k))$ approximation for the degree bounded EC-SNDP problem. The former result was improved to $(2, 2b(v) + 2)$ by Louis and Vishnoi [17]. The techniques developed in these papers have been extended to a variety of problems with edge connectivity requirements; we refer the reader to [15] for more details. The element and vertex connectivity problems have proved to be more challenging with these techniques. The results of [14] and [16] can be extended to the element connectivity setting provided that the degree bounds are only on the *terminals*[1] and removing this restriction required additional insights.

Recently, Nutov [18] considered several degree bounded network design problems with element and vertex connectivity requirements. Nutov showed that one can combine the augmentation framework of Williamson et al. [20] with iterated rounding techniques in order to construct a solution that violates the degree bounds by a multiplicative factor that is exponential in k. Some of Nutov's results include an $(O(\log k), O(2^k)b(v))$ approximation for Elem-SNDP and Rooted k-Connectivity, and an $(O(k), O(2^k)b(v))$ approximation for k-Connected Subgraph; the latter result holds for all values of k. Shortly after, Fukunaga and Ravi [8] significantly improved the approximation guarantees for these problems; their algorithms are based on iterated rounding and their analyses use a novel technical insight. The results of [8] include an $(O(k), O(k)b(v))$ approximation for Elem-SNDP, a $(4, 2b(v)+O(k))$ approximation for Rooted k-Connectivity, a $(2, 2b^+(v) + O(k))$ approximation for Rooted k-Outconnectivity, and an $(O(k), b(v) + O(k^2))$ approximation for k-Connected Subgraph; the latter result holds for all values of k. In joint work with Nutov [7], the authors of [8] strengthened their analysis and obtained an $(O(1), O(1)b(v) + O(k))$ approximation for Elem-SNDP. By combining the approach of Cheriyan and Vegh [2] for the k-Connected Subgraph problem without degree bounds with the algorithm of [8] for the degree bounded Rooted k-Connectivity problem and the algorithm of [7] for the degree bounded Elem-SNDP problem, one obtains an $(O(1), O(1)b(v) + O(k))$ approximation for the degree bounded k-Connected Subgraph problem when the number of nodes is at least $(k-1)^3 - k$.

2. PRELIMINARIES

2.1 Bisets and biset functions

[1] A vertex is a terminal if it participates in a pair with nonzero connectivity requirement.

Problem	Previous work	This paper
Elem-SNDP	$(O(1), O(1)b(v) + O(k))$ [7]	$(O(1), O(1)b(v))$
k-Connected Subgraph ($n = \Omega(k^3)$)	$(O(1), O(1)b(v) + O(k))$ [8, 7, 2]	$(O(1), O(1)b(v))$
Rooted k-Connectivity	$(O(1), O(1)b(v) + O(k))$ [8]	$(O(1), O(1)b(v))$
VC-SNDP	$(O(k^3 \log n), O(k^3 \log n)(b(v) + O(k))$ [7]	$(O(k^3 \log n), O(k^3 \log n)b(v))$
Rooted k-Outconnectivity	$(O(1), O(1)b^+(v) + O(k))$ [8, 7]	$(O(1), O(1)b^+(v))$

Figure 1: Summary of results.

In this section, we introduce some definitions and notation; we follow the conventions used in previous work such as [18]. We start with some definitions related to bisets and biset functions.

Let V be a ground set. A *biset* $\mathbb{A} = (A, A')$ is a pair of sets such that $A \subseteq A' \subseteq V$. We refer to A and A' as the inner and outer parts of $\mathbb{A}$, respectively. We refer to the set $A' - A$ as the boundary of $\mathbb{A}$ and we use $\mathrm{bd}(\mathbb{A})$ to denote the set $A' - A$. We define the intersection, union, and difference of two bisets $\mathbb{A} = (A, A')$ and $\mathbb{B} = (B, B')$ as follows: $\mathbb{A} \cap \mathbb{B} = (A \cap B, A' \cap B')$, $\mathbb{A} \cup \mathbb{B} = (A \cup B, A' \cup B')$, $\mathbb{A} - \mathbb{B} = (A - B', A' - B)$.

We define a partial order $\subseteq$ on bisets as follows. If $\mathbb{A} = (A, A')$ and $\mathbb{B} = (B, B')$ are two bisets, we have $\mathbb{A} \subseteq \mathbb{B}$ if and only if $A \subseteq B$ and $A' \subseteq B'$. If $\mathbb{A} \subseteq \mathbb{B}$, we say that $\mathbb{A}$ is contained in $\mathbb{B}$. Note that $\mathbb{A} \cap \mathbb{B} \subseteq \mathbb{A}$, $\mathbb{A} \cap \mathbb{B} \subseteq \mathbb{B}$, $\mathbb{A} - \mathbb{B} \subseteq \mathbb{A}$, and $\mathbb{B} - \mathbb{A} \subseteq \mathbb{B}$.

Two bisets $\mathbb{A} = (A, A')$ and $\mathbb{B} = (B, B')$ are *disjoint* if $A \cap B$ is empty; if $\mathbb{A}$ and $\mathbb{B}$ are not disjoint, they are *intersecting*. They are *strongly disjoint* if $A' \cap B$ and $A \cap B'$ are both empty; if $\mathbb{A}$ and $\mathbb{B}$ are not strongly disjoint, they are *overlapping*. A family of bisets is *bilaminar* if, for any two bisets $\mathbb{A}$ and $\mathbb{B}$ in the family, one of the following holds: $\mathbb{A} \subseteq \mathbb{B}$, $\mathbb{B} \subseteq \mathbb{A}$, or $\mathbb{A}$ and $\mathbb{B}$ are disjoint. A family of bisets is *strongly bilaminar* if, for any two bisets $\mathbb{A}$ and $\mathbb{B}$ in the family, one of the following holds: $\mathbb{A} \subseteq \mathbb{B}$, $\mathbb{B} \subseteq \mathbb{A}$, or $\mathbb{A}$ and $\mathbb{B}$ are strongly disjoint.

Let $f : 2^V \times 2^V \to \mathbb{Z}$ be a function on bisets. The function f is *bisupermodular* if, for any two bisets $\mathbb{A}$ and $\mathbb{B}$, we have

$$f(\mathbb{A}) + f(\mathbb{B}) \leq f(\mathbb{A} \cap \mathbb{B}) + f(\mathbb{A} \cup \mathbb{B}).$$

The function is *intersecting* bisupermodular if the inequality above holds for any two bisets $\mathbb{A}$ and $\mathbb{B}$ that intersect. The function is *positively* bisupermodular if the inequality above holds for any two bisets $\mathbb{A}$ and $\mathbb{B}$ such that $f(\mathbb{A}) > 0$ and $f(\mathbb{B}) > 0$. The function is *positively intersecting* bisupermodular if the inequality above holds for any two bisets $\mathbb{A}$ and $\mathbb{B}$ such that $\mathbb{A}$ and $\mathbb{B}$ intersect, $f(\mathbb{A}) > 0$, and $f(\mathbb{B}) > 0$. The function f is *bisubmodular* if $-f$ bisupermodular.

The function f is *binegamodular* if, for any two bisets $\mathbb{A}$ and $\mathbb{B}$, we have

$$f(\mathbb{A}) + f(\mathbb{B}) \leq f(\mathbb{A} - \mathbb{B}) + f(\mathbb{B} - \mathbb{A}).$$

The function f is *biposimodular* if $-f$ is binegamodular.

The function f is *skew bisupermodular* if, for any two bisets $\mathbb{A}$ and $\mathbb{B}$, we have

$$f(\mathbb{A}) + f(\mathbb{B}) \leq \max\{f(\mathbb{A} \cap \mathbb{B}) + f(\mathbb{A} \cup \mathbb{B}), f(\mathbb{A} - \mathbb{B}) + f(\mathbb{B} - \mathbb{A})\}.$$

The function is *positively* skew bisupermodular if the inequality holds for any two bisets $\mathbb{A}$ and $\mathbb{B}$ such that $f(\mathbb{A}) > 0$ and $f(\mathbb{B}) > 0$.

If F is a set of undirected edges, we let $\delta_F(\mathbb{A})$ be the set of all edges $e \in F$ such that e has one endpoint in A and the other endpoint in $V - A'$. We let $\chi(\delta_F(\mathbb{A}))$ be the characteristic vector of $\delta_F(\mathbb{A})$, i.e., the $|F|$-dimensional vector that has an entry for each edge $e \in F$ that is equal to 1 if $e \in \delta_F(\mathbb{A})$ and 0 otherwise.

The following lemmas are well-known (see e.g., [15]).

LEMMA 2.1. *For any set F of edges and any positive weight function w on F, the function $f(\mathbb{A}) = w(\delta_F(\mathbb{A}))$ is both bisubmodular and biposimodular. That is, for any two bisets $\mathbb{A}$ and $\mathbb{B}$, we have*

- $w(\delta_F(\mathbb{A})) + w(\delta_F(\mathbb{B})) \geq w(\delta_F(\mathbb{A} \cap \mathbb{B})) + w(\delta_F(\mathbb{A} \cup \mathbb{B}))$, *and*
- $w(\delta_F(\mathbb{A})) + w(\delta_F(\mathbb{B})) \geq w(\delta_F(\mathbb{A} - \mathbb{B})) + w(\delta_F(\mathbb{B} - \mathbb{A}))$.

The proof of the following lemma is a straightforward case analysis.

LEMMA 2.2. *For any two bisets $\mathbb{A}$ and $\mathbb{B}$, we have*

- $|\mathrm{bd}(\mathbb{A})| + |\mathrm{bd}(\mathbb{B})| = |\mathrm{bd}(\mathbb{A} \cap \mathbb{B})| + |\mathrm{bd}(\mathbb{A} \cup \mathbb{B})|$, *and*
- $|\mathrm{bd}(\mathbb{A})| + |\mathrm{bd}(\mathbb{B})| = |\mathrm{bd}(\mathbb{A} - \mathbb{B})| + |\mathrm{bd}(\mathbb{B} - \mathbb{A})| + 2|\mathrm{bd}(\mathbb{A}) \cap \mathrm{bd}(\mathbb{B})|$.

2.2 Problem definitions

In this section, we formally define the problems we consider in this paper. Using Menger's theorem, we formulate the problems as covering certain biset functions. Let $G = (V, E)$ be an undirected graph and let $f : 2^V \times 2^V \to \mathbb{R}$ be a biset function. We say that G *covers* f if we have $|\delta_G(\mathbb{A})| \geq f(\mathbb{A})$ for each biset $\mathbb{A}$. When the vertex set V is clear from the context, we simply say that a set F of edges covers f, i.e., we have $|\delta_F(\mathbb{A})| \geq f(\mathbb{A})$ for each biset $\mathbb{A}$.

In the following, if H is an undirected graph and v is a vertex in H, we let $\delta_H(v)$ denote the set of edges of H that are incident to v. If H is a directed graph and v is a vertex of H, we let $\delta_H^-(v)$ and $\delta_H^+(v)$ denote the set of all arcs of H that are entering and leaving v, respectively.

Degree-bounded Elem-SNDP. Let $G = (V, E)$ be an undirected graph with weights $w(e)$ on the edges and degree bounds $b(v)$ on the nodes. The vertices of V are partitioned into two sets, the set R of reliable vertices and the set W of unreliable vertices. The vertices of W and the edges are called *elements*. The element-connectivity of a pair uv of nodes is the maximum number of element-disjoint paths from u to v; these paths do not share any unreliable nodes or edges, but they may share reliable nodes. Two vertices u and v are ℓ-element-connected if their element connectivity is at least ℓ. In the Degree-bounded Elem-SNDP problem, in addition to the graph G, we are given integer requirements $r(uv)$ on pairs of vertices such that $r(uv) > 0$ only if $\{u, v\} \subseteq R$. The goal is to select a minimum-weight subgraph H of G such that each pair $uv \in V \times V$ of vertices is

$r(uv)$-element-connected in H and $|\delta_H(v)| \leq b(v)$ for each vertex v.

Let $\mathrm{r}_{\mathrm{elt}} : 2^V \times 2^V \to \mathbb{Z}_+$ be a biset function such that $\mathrm{r}_{\mathrm{elt}}(\mathbb{A}) = \max_{u \in A, v \in V - A'} r(uv)$ if $\mathrm{bd}(\mathbb{A}) \subseteq W$ and $\mathrm{r}_{\mathrm{elt}}(\mathbb{A}) = 0$ otherwise. Let $f_{\mathrm{elt}} : 2^V \times 2^V \to \mathbb{Z}$ be the biset function such that $f_{\mathrm{elt}}(\mathbb{A}) = \mathrm{r}_{\mathrm{elt}}(\mathbb{A}) - |\mathrm{bd}(\mathbb{A})|$. By Menger's theorem, a subgraph H satisfies the element connectivity requirements if and only if $|\delta_H(\mathbb{A})| \geq f_{\mathrm{elt}}(\mathbb{A})$ for each biset $\mathbb{A}$.

LEMMA 2.3 (FLEISCHER ET AL. [4]). *The functions* $\mathrm{r}_{\mathrm{elt}}$ *and* f_{elt} *are positively skew bisupermodular.*

Degree-bounded VC-SNDP. Let $G = (V, E)$ be an undirected graph with weights $w(e)$ on the edges and degree bounds $b(v)$ on the nodes. The vertex-connectivity of a pair uv of vertices is the maximum number of paths between u and v that are internally vertex disjoint. Two vertices u and v are ℓ-vertex-connected if their vertex connectivity is at least ℓ. In the Degree-bounded VC-SNDP problem, in addition to the graph G, we are given integer requirements $r(uv)$ for each pair uv of vertices. The goal is to select a minimum-weight subgraph H of G such that each pair $uv \in V \times V$ of vertices is $r(uv)$-vertex-connected in H and $|\delta_H(v)| \leq b(v)$ for each vertex v.

Degree-bounded k-Connected Subgraph. Let $G = (V, E)$ be an undirected graph with weights $w(e)$ on the edges and degree bounds $b(v)$ on the vertices. A graph H is k-vertex-connected if each pair uv of vertices is k vertex connected in H. In the Degree-bounded k-Connected Subgraph problem, the goal is to select a minimum-weight spanning subgraph $H = (V, E')$ of G such that H is k-vertex-connected and $|\delta_H(v)| \leq b(v)$ for each vertex v.

Let $\mathrm{r}_{\mathrm{sg}} : 2^V \times 2^V \to \mathbb{Z}_+$ be the following biset function: $\mathrm{r}_{\mathrm{sg}}(\mathbb{A}) = k$ if $A \neq \emptyset$ and $A' \neq V$, and $\mathrm{r}_{\mathrm{sg}}(\mathbb{A}) = 0$ otherwise. Let $f_{\mathrm{sg}} : 2^V \times 2^V \to \mathbb{Z}$ be the biset function such that $f_{\mathrm{sg}}(\mathbb{A}) = \mathrm{r}_{\mathrm{sg}}(\mathbb{A}) - |\mathrm{bd}(\mathbb{A})|$ for each biset $\mathbb{A}$. By Menger's theorem, a subgraph H is k-vertex-connected if and only if $|\delta_H(\mathbb{A})| \geq f_{\mathrm{sg}}(\mathbb{A})$ for each biset $\mathbb{A}$.

Degree-bounded Rooted k-Connectivity. Let $G = (V, E)$ be an undirected graph with weights $w(e)$ on the edges and degree bounds $b(v)$ on the vertices. A vertex r is k-vertex-connected in H to each other vertex if, for any vertex $v \in V - \{r\}$, the pair rv is k-vertex-connected in H. In the Degree-bounded Rooted k-Connectivity problem, we are given a root vertex r and the goal is to select a minimum-weight subgraph H of G such that the root r is k-vertex-connected in H to every other vertex and $|\delta_H(v)| \leq b(v)$ for each vertex v.

Let $\mathrm{r}_{\mathrm{rc}} : 2^V \times 2^V \to \mathbb{Z}_+$ be a biset function such that $\mathrm{r}_{\mathrm{rc}}(\mathbb{A}) = k$ if $A \neq \emptyset$ and $r \in V - A'$, and $\mathrm{r}_{\mathrm{rc}}(\mathbb{A}) = 0$ otherwise. Let $f_{\mathrm{rc}} : 2^V \times 2^V \to \mathbb{Z}$ be the biset function such that $f_{\mathrm{rc}}(\mathbb{A}) = \mathrm{r}_{\mathrm{rc}}(\mathbb{A}) - |\mathrm{bd}(\mathbb{A})|$. By Menger's theorem, the root is k-vertex-connected in H to every other vertex if and only if $|\delta_H(\mathbb{A})| \geq f_{\mathrm{rc}}(\mathbb{A})$ for each biset $\mathbb{A}$. The following lemma is well-known, see for instance [5, 18].

LEMMA 2.4 ([5]). *The functions* r_{rc} *and* f_{rc} *are positively intersecting bisupermodular.*

Degree-bounded Rooted k-Outconnectivity. Let $G = (V, E)$ be a directed graph with weights $w(e)$ on the edges and out-degree bounds $b^+(v)$ on the vertices. In the Degree-bounded Rooted k-Outconnectivity problem, we are given a root vertex r and the goal is to select a minimum-weight subgraph H of G such that, for each vertex $v \in V - \{r\}$, there are k internally vertex disjoint paths in H from r to v and, for each vertex $v \in V$, $|\delta_H^+(v)| \leq b^+(v)$.

3. MAIN TECHNICAL RESULTS AND APPLICATIONS

In this section, we state our main technical results and their applications to the degree-bounded network design problems that we consider in this paper.

Our algorithms for network design problems in undirected graphs use the following abstract problem as a subroutine.

DEFINITION 3.1 (DEGREE-BOUNDED RESIDUAL COVER). *Let $G = (V, E)$ be an undirected graph with weights $w(e)$ on the edges and degree bounds $b(v)$ on the vertices. In the* Degree-bounded Residual Cover *problem, we are given a function $f : 2^V \times 2^V \to \mathbb{Z}$ satisfying $f(\mathbb{A}) = \mathrm{r}(\mathbb{A}) - |\mathrm{bd}(\mathbb{A})| - |\delta_F(\mathbb{A})|$ for each biset $\mathbb{A}$, where r is a biset function and $F \subseteq E$ is a set of edges, and the goal is to select a minimum weight set $F' \subseteq E - F$ of edges such that $|\delta_{F'}(\mathbb{A})| \geq f(\mathbb{A})$ for each biset $\mathbb{A}$ and $|\delta_{F'}(v)| \leq b(v)$ for each vertex v.*

One of our main contributions is an iterated rounding algorithm for the Degree-bounded Residual Cover problem that achieves an $(O(1), O(1)b(v))$ approximation provided that the requirement function r satisfies a certain technical condition. The following theorem states our main result for undirected graphs. We prove the theorem in Section 4.

THEOREM 3.2. *Consider an instance of the* Degree-bounded Residual Cover *problem in which the function f satisfies $f(\mathbb{A}) = \mathrm{r}(\mathbb{A}) - |\mathrm{bd}(\mathbb{A})| - |\delta_F(\mathbb{A})|$, where r is an integer-valued biset function and $F \subseteq E$ is a set of edges. Let* OPT *be the weight of an optimal solution for the instance. Suppose that r and f satisfy the following conditions:*

- *For each biset (A, A') and each vertex $v \in A' - A$, we have $\mathrm{r}((A, A')) \leq \mathrm{r}((A, A' - v))$.*
- *The function f is positively skew bisupermodular.*

Then there is a polynomial time iterated rounding algorithm that selects a set $F' \subseteq E - F$ of edges such that $w(F') \leq 3\mathrm{OPT}$ and $|\delta_{F'}(v)| \leq |\delta_F(v)| + 6b(v) + 5$ for each vertex v.

We remark that the requirement function arising from instances of SNDP satisfies the technical conditions imposed by Theorem 3.2, since the requirement of a biset is the maximum requirement of a pair separated by the biset and thus removing a vertex from the boundary of a biset cannot decrease the requirement. We use this observation to show that Theorem 3.2 immediately implies an $(O(1), O(1)b(v))$ approximation for Elem-SNDP.

In order to get a $(O(1), O(1)b(v))$ approximation for the Elem-SNDP problem, we apply Theorem 3.2 with $F = \emptyset$, $f = f_{\mathrm{elt}}$, and $\mathrm{r} = \mathrm{r}_{\mathrm{elt}}$, where f_{elt} and $\mathrm{r}_{\mathrm{elt}}$ are the functions defined in Section 2.2. It is straightforward to verify that $\mathrm{r}_{\mathrm{elt}}$ satisfies the first condition of the theorem. Additionally, f_{elt} satisfies the second condition by Lemma 2.3. Therefore we have the following result.

THEOREM 3.3. *There is a polynomial time $(3, 6b(v) + 5)$ approximation for the* Degree-bounded Elem-SNDP *problem in undirected graphs.*

We can also prove an analogue of Theorem 3.2 for directed graphs; we defer this result to a longer version of this paper. As a corollary, we obtain the following result.

THEOREM 3.4. *There is a polynomial time* $(3, 6b^+(v) + 3)$ *approximation for the* Degree-bounded Rooted k-Outconnectivity *problem in directed graphs.*

We can reduce the degree-bounded Rooted k-Connectivity problem in undirected graphs to the degree-bounded Rooted k-Outconnectivity problem in directed graphs as follows. We first make the graph directed by bidirecting each edge; more precisely, we replace each undirected edge uv by two directed edges, $\vec{uv}$ and $\vec{vu}$, and we set $w(\vec{uv}) = w(\vec{vu}) = w(uv)$. For each vertex v, we set $b^+(v) = b(v)$. Note that the resulting instance of the degree-bounded Rooted k-Outconnectivity problem has a solution of weight at most 2OPT that satisfies the out-degree constraints, where OPT is the weight of the optimal solution for the degree-bounded Rooted k-Connectivity instance. Thus, using Theorem 3.4, we can find in polynomial time a set F of directed edges such that $w(F) \leq 6\text{OPT}$ and $|\delta_F^+(v)| \leq 6b^+(v) + 3$ for each vertex v. Let F' be the set of undirected edges corresponding to F. We can show that F' is a $(6, 7b(v) + 3)$-approximate solution to the initial Rooted k-Connectivity instance as follows. The set F' has weight at most 6OPT and it k-connects each vertex to the root. Thus it suffices to show that each vertex v has degree at most $7b(v) + 3$ in F'. We have $|\delta_{F'}(v)| \leq |\delta_F^-(v)| + |\delta_F^+(v)| \leq |\delta_F^-(v)| + 6b(v) + 3$. It is well-known that, in an inclusion-minimal solution to the Rooted k-Outconnectivity problem, all in-degrees are at most k (each vertex other than the root has in-degree exactly k). Thus we have $|\delta_F^-(v)| \leq k \leq b(v)$ and hence $|\delta_{F'}(v)| \leq 7b(v) + 3$.

THEOREM 3.5. *There is a polynomial time* $(6, 7b(v) + 3)$ *approximation for the* Degree-bounded Rooted k-Connectivity *problem in undirected graphs.*

Now we turn our attention to the k-Connected Subgraph problem in undirected graphs. In the following, we show how to extend the algorithm of Cheriyan and Vegh [2] for the k-Connected Subgraph problem to the degree-bounded setting. We start with a high-level overview of the algorithm of [2].

Recall from Section 2.2 that the k-Connected Subgraph problem is equivalent to covering a biset function f_{sg} satisfying $f_{\text{sg}}(\mathbb{A}) = \text{r}_{\text{sg}}(\mathbb{A}) - |\text{bd}(\mathbb{A})|$ for each biset $\mathbb{A}$. The requirement $\text{r}_{\text{sg}}(\mathbb{A})$ of a biset $\mathbb{A}$ is equal to k if $\mathbb{A}$ is a non-trivial biset ($A \neq \emptyset$ and $A' \neq V$) and it is equal to zero otherwise. If the function f_{sg} was positively skew bisupermodular, we could use the algorithm of Fleischer et al. [4] to construct a 2-approximate cover of the function. Unfortunately, the function f_{sg} is not positively skew bisupermodular (since r_{sg} is not positively skew bisupermodular). The key insight in [2] is that, if the number of nodes is large[2] compared to k, we can partially cover some of the bisets so that the residual function becomes positively skew bisupermodular. More precisely, Cheriyan and Vegh show that we can find a set $F \subseteq E$ of edges such that the function $g(\mathbb{A}) = \text{r}_{\text{sg}}(\mathbb{A}) - |\text{bd}(\mathbb{A})| - |\delta_F(\mathbb{A})|$ is positively skew bisupermodular. Once we have F, we can use the algorithm of Fleischer et al. [4] to select a set $F' \subseteq E - F$ of edges that cover the residual function g. The set $F \cup F'$ is a feasible cover of f_{sg}. The preprocessing phase that constructs the set F of edges proceeds in two steps which we now sketch.

In the first step, we pick an arbitrary subset R_1 consisting of k vertices. Once we have R_1, we construct an instance of the Rooted k-Outconnectivity problem as follows. We start with our (undirected) graph G and we make it directed by bidirecting each of its edges; each undirected edge is replaced by two directed edges, one in each direction, each of which has the same weight as the original edge. Then we introduce a root vertex r_1 and we add a directed edge from r_1 to each vertex in R_1; these edges receive weight zero. This gives us an instance of the Rooted k-Outconnectivity problem with root r_1, and we use an algorithm for the problem — such as the algorithm of Frank and Tardos [6] — to find a set F_1 of directed edges that provide k internally vertex disjoint paths from r_1 to each vertex in V; the set F_1 corresponds to a set $F_1' \subseteq E$ in the undirected graph G.

Once we have the set F_1', we proceed to the second step. Let R_2 be an appropriately chosen set consisting of k vertices; we will describe how to choose R_2 later. Now we construct a second instance of the Rooted k-Outconnectivity problem as follows. We start with the graph G. We set the weight of each edge in F_1' to zero (the weights of the edges in $E - F_1'$ remain unchanged). Finally, we make the graph directed by bidirecting each of the edges. We add a root r_2 and a zero-weight edge from r_2 to each vertex in R_2. We solve the resulting instance of the Rooted k-Outconnectivity problem and find a set F_2 of directed edges that provide k internally vertex disjoint paths from r_2 to each vertex in V; the set F_2 corresponds to a set $F_2' \subseteq E - F_1'$ of undirected edges.

If the number of nodes is large, for any set R_1 of k vertices and any solution F_1 for the first Rooted k-Outconnectivity instance, we can find in polynomial time a set R_2 of k vertices satisfying the following key property. Consider the second Rooted k-Outconnectivity instance that we construct based on F_1, and let F_2 be a solution for this instance. Let F_1' and F_2' be the undirected edges corresponding to F_1 and F_2, respectively. Let g be the function such that $g(\mathbb{A}) = \text{r}_{\text{sg}}(\mathbb{A}) - |\text{bd}(\mathbb{A})| - |\delta_{F_1' \cup F_2'}(\mathbb{A})|$. Then the function g is positively skew bisupermodular. The second preprocessing step uses such a set R_2. This completes our overview of the preprocessing phase and thus of the algorithm of Cheriyan and Vegh.

Now we turn our attention to the *degree bounded* k-Connected Subgraph problem. Consider an instance of the degree bounded problem in which the number of nodes in the graph is at least $(k-1)^3 - k$. Let F^* be an optimal solution for the instance and let OPT be its weight. The algorithm follows the high level outline described above. We first perform the following preprocessing phase. We proceed in two steps.

In the first step, we pick an arbitrary set R_1 consisting of k vertices and we define the first instance of the Rooted k-Outconnectivity problem as before. Additionally, we assign out-degree bounds to the vertices as follows: $b^+(v) = b(v)$ for each vertex v. Using the algorithm in Theorem 3.4, we construct a solution F_1 for the resulting instance of the degree-bounded Rooted k-Outconnectivity problem. Let F_1' be the set of undirected edges corresponding to F_1. We have $w(F_1') \leq w(F_1) \leq 6\text{OPT}$ and $|\delta_{F_1'}(v)| \leq 7b(v) + 3$ for each

[2]Cheriyan and Vegh [2] showed that $n \geq k^3(k-1) + k$ suffices. Fukunaga et al. gave a simple proof that shows that $n \geq (k-1)^3 - k$ suffices [7].

Undir-LP $\langle\langle$*Input:* $(G=(V,E), f, X, b)\rangle\rangle$

$$\begin{array}{lll} \min & \sum_{e\in E} w(e)x(e) & \\ \text{s.t.} & x(\delta_E(\mathbb{A})) \geq f(\mathbb{A}) & \forall \mathbb{A} : f(\mathbb{A}) > 0 \\ & x(\delta_E(v)) \leq b(v) & \forall v \in X \\ & 0 \leq x(e) \leq 1 & \forall e \in E \end{array}$$

Figure 2: LP relaxation for the Degree-bounded Residual Cover problem in undirected graphs.

vertex v.

In the second step, we pick a set R_2 consisting of k vertices; we construct R_2 using the approach of Cheriyan and Vegh [2] mentioned above. We define the second instance of the Rooted k-Outconnectivity problem as before. Additionally, we assign out-degree bounds to the vertices as follows: $b^+(v) = b(v)$ for each vertex v. Using the algorithm in Theorem 3.4, we construct a solution F_2 for the resulting instance of the degree-bounded Rooted k-Outconnectivity problem. Let F_2' be the set of undirected edges corresponding to F_2. We have $w(F_2') \leq w(F_2) \leq 6\mathrm{OPT}$ and $|\delta_{F_2'}(v)| \leq 7b(v)+3$ for each vertex v. This completes the preprocessing phase.

Let g be the biset function such that $g(\mathbb{A}) = \mathrm{r}_{\mathrm{sg}}(\mathbb{A}) - |\mathrm{bd}(\mathbb{A})| - |\delta_{F_1'\cup F_2'}(\mathbb{A})|$ for each biset $\mathbb{A}$. Note that $F^* - (F_1' \cup F_2')$ covers g. As mentioned above, we choose R_2 in such a way that the function g is positively skew bisupermodular. Additionally, it is straightforward to verify that r_{sg} satisfies the first condition of Theorem 3.2. Thus we can apply Theorem 3.2 with $F = F_1' \cup F_2'$, $f = g$, $\mathrm{r} = \mathrm{r}_{\mathrm{sg}}$, and degree bounds given by b in order to get a cover F_3' of g. We have $w(F_3') \leq 3\mathrm{OPT}$ and $|\delta_{F_3'}(v)| \leq |\delta_{F_1'\cup F_2'}(v)| + 6b(v) + 5 \leq 2(7b(v)+3) + 6b(v) + 5$ for each vertex v. Our final solution is $F = F_1' \cup F_2' \cup F_3'$.

It follows from the discussion above that F is a feasible cover of f_{sg} of weight at most $15\mathrm{OPT}$. Moreover, we have $|\delta_F(v)| \leq 4(7b(v)+3) + 6b(v) + 5$ for each vertex v.

THEOREM 3.6. *There is a polynomial time* $(15, 34b(v) + 17)$ *approximation for the* Degree-bounded k-Connected Subgraph *problem in undirected graphs with at least* $(k-1)^3 - k$ *nodes.*

4. ITERATED ROUNDING ALGORITHM FOR UNDIRECTED GRAPHS

In this section, we prove Theorem 3.2. Our algorithm for the Degree-bounded Residual Cover problem is based on a standard iterated rounding approach; we give the algorithm in Figure 3. Our main contribution is the following key theorem that shows that the algorithm terminates with a feasible solution.

THEOREM 4.1. *Consider an iteration of* Undir-algo. *Let* $G' = (V, E')$ *be the residual subgraph at the beginning of this iteration. Let* F' *be the set of edges selected in the previous iterations. Let* X' *be the set of vertices that have degree bounds, and let* $b' : X' \to \mathbb{R}$ *be the degree bounds on* X' *at the beginning of this iteration. Let* $f' : 2^V \times 2^V \to \mathbb{Z}$ *be the function satisfying* $f'(\mathbb{A}) = \mathrm{r}(\mathbb{A}) - |A' - A| - |\delta_{F\cup F'}(\mathbb{A})|$ *for each biset* $\mathbb{A}$. *If* x *is a basic solution to* Undir-LP *for the input* (G', f', X', b'), *one of the following holds:*

- *There is an edge* $e \in E'$ *such that* $x(e) = 0$.
- *There is an edge* $e \in E'$ *such that* $x(e) \geq 1/3$.
- *There is a vertex* $v \in X'$ *such that* $|\delta_{E'}(v)| \leq |\delta_F(v)| + 3b(v) + 5$.

Undir-Algo $\langle\langle$*Input:* $\langle G=(V,E), \mathrm{r}, F, X, b\rangle\rangle$

Let $E' \leftarrow E - F$, $F' \leftarrow \emptyset$, $X' \leftarrow X$
Let $b'(v) \leftarrow b(v)$ for each $v \in X$

While E' is non-empty
  Let $f' : 2^V \times 2^V \to \mathbb{Z}$ such that
  $f'(\mathbb{A}) = \mathrm{r}(\mathbb{A}) - |\mathrm{bd}(\mathbb{A})| - |\delta_{F\cup F'}(\mathbb{A})|$ for each biset $\mathbb{A}$

  Compute an optimal basic solution to Undir-LP
  for the input $(G' = (V, E'), f', X', b')$

  If there is an edge $e \in E'$ such that $x(e) = 0$
    $E' \leftarrow E' - \{e\}$ $\langle\langle$*Remove e from the graph*$\rangle\rangle$
  If there is an edge $e = uv \in E'$ such that $x(e) \geq 1/3$
    $F' \leftarrow F' \cup \{e\}$ $\langle\langle$*Add e to the solution*$\rangle\rangle$
    $E' \leftarrow E' - \{e\}$ $\langle\langle$*Remove e from the graph*$\rangle\rangle$
    If $u \in X'$
      $b'(u) \leftarrow b'(u) - x(e)$
    If $v \in X'$
      $b'(v) \leftarrow b'(v) - x(e)$
  Else
    Let $v \in X'$ be a vertex such that
    $|\delta_{E'}(v)| \leq |\delta_F(v)| + 3b(v) + 5$

    $X' \leftarrow X' - \{v\}$ $\langle\langle$*Drop degree bound on v*$\rangle\rangle$
Return F'

Figure 3: Iterated rounding algorithm for the Degree-bounded Residual Cover problem in undirected graphs.

Our proof of Theorem 4.1 is inspired by the approach of Nutov [18]. The proof of the theorem is quite technical and we omit it from this extended abstract; at the end of this section, we give a simpler proof that shows a weaker version of the theorem with larger constants. We remark that the weaker theorem suffices to show $(O(1), O(1)b(v))$ approximations for all the problems we consider.

Using Theorem 4.1, we can analyze the iterated rounding algorithm as follows. It follows that the algorithm terminates and it outputs a feasible cover of f. Additionally, since each edge added to the solution satisfies $x(e) \geq 1/3$, it is straightforward to verify that the weight of the solution is at most $3\mathrm{OPT}$. Thus it suffices to upper bound the degree of each vertex in the final solution. The proof of the following lemma is straightforward and it is based on the fact that each edge that is added to the solution satisfies $x(e) \geq 1/3$.

LEMMA 4.2. *Consider an iteration of* Undir-algo. *Let* F' *be the set of edges selected in the previous iterations, let* X' *be the set of vertices that have degree bounds, and let* $b' : X' \to \mathbb{R}$ *be the degree bounds on* X' *at the beginning of*

the iteration. For each vertex $v \in X'$, we have $|\delta_{F'}(v)| \leq 3(b(v) - b'(v))$, where $b(v)$ is the initial degree bound on v.

Using Theorem 4.1 and a straightforward inductive argument, we get the following theorem.

THEOREM 4.3. *Let F' be the solution constructed by* Undir-algo. *The set F' satisfies the following:*

- $|\delta_{F'}(\mathbb{A})| \geq f(\mathbb{A})$ *for each biset* $\mathbb{A}$.
- *The total weight of F' is at most* 3OPT.
- *For each vertex v, $|\delta_{F'}(v)| \leq |\delta_F(v)| + 6b(v) + 5$.*

Theorem 3.2 is an immediate corollary of Theorem 4.3. Thus it only remains to prove Theorem 4.1. In the following subsection, we prove a weaker version of the theorem that highlights the main ideas behind the proof.

4.1 A weaker version of Theorem 4.1

In this section, we prove the following weaker version of Theorem 4.1. We consider the variant of Undir-algo that removes the degree bound constraint of a vertex v if $|\delta_{E'}(v)| \leq 4|\delta_F(v)| + 12b(v) + 12$. We refer to the modified algorithm as Undir-algo-weaker. The two algorithms only differ in the body of the Else statement.

THEOREM 4.4. *Consider an iteration of* Undir-algo-weaker. *Let $G' = (V, E')$ be the residual subgraph at the beginning of this iteration. Let F' be the set of edges selected in the previous iterations. Let X' be the set of vertices that have degree bounds, and let $b' : X' \to \mathbb{R}$ be the degree bounds on X' at the beginning of this iteration. Let $f' : 2^V \times 2^V \to \mathbb{Z}$ be the function satisfying $f'(\mathbb{A}) = \mathrm{r}(\mathbb{A}) - |A' - A| - |\delta_{F \cup F'}(\mathbb{A})|$ for each biset $\mathbb{A}$. If x is a basic solution to* Undir-LP *for the input (G', f', X', b'), one of the following holds:*

- *There is an edge $e \in E'$ such that $x(e) = 0$.*
- *There is an edge $e \in E'$ such that $x(e) \geq 1/3$.*
- *There is a vertex $v \in X'$ such that $|\delta_{E'}(v)| \leq 4|\delta_F(v)| + 12b(v) + 12$.*

We devote the rest of this section to the proof of Theorem 4.4. We start with the following theorem whose proof is based on a standard uncrossing argument [4].

THEOREM 4.5 (COROLLARY 4.6 IN [4]). *Let x be a basic solution to* Undir-LP *for an input $(G = (V, E), f, X, b)$. If f is positively skew bisupermodular, there is a collection $\mathcal{L}$ of bisets and a set $C \subseteq X$ of vertices with the following properties:*

- *For each biset $\mathbb{A} \in \mathcal{L}$, $x(\delta_E(\mathbb{A})) = f(\mathbb{A}) > 0$.*
- *For each vertex $v \in C$, $x(\delta_E(v)) = b(v)$.*
- *The family $\mathcal{L}$ is strongly bilaminar.*
- $|\mathcal{L}| + |C| = |E|$.
- *The vectors $\{\chi(\delta_E(\mathbb{A})) \mid \mathbb{A} \in \mathcal{L}\} \cup \{\chi(\delta_E(v)) \mid v \in C\}$ are linearly independent.*

Note that the function f' is positively skew bisupermodular. To see this, recall that $f'(\mathbb{A}) = f(\mathbb{A}) - |\delta_{F'}(\mathbb{A})|$ for each biset $\mathbb{A}$, where f is the input function for the Degree-bounded Residual Cover instance. As stated in Theorem 3.2, the function f is positively skew bisupermodular. By Lemma 2.1, the function $|\delta_{F'}(\cdot)|$ is bisubmodular and biposimodular.

Thus we can apply Theorem 4.5 to x and $(G' = (V, E'), f', X', b')$ in order to get a collection $\mathcal{L}$ of tight bisets and a set C of tight vertices. (A biset $\mathbb{A}$ is tight if $x(\delta_{E'}(\mathbb{A})) = f'(\mathbb{A})$, and a vertex $v \in X'$ is tight if $x(\delta_{E'}(v)) = b'(v)$.)

We assume for contradiction that we have $0 < x(e) < 1/3$ for each edge $e \in E'$ and $|\delta_{E'}(v)| > 4|\delta_F(v)| + 12b(v) + 12$ for each vertex $v \in X'$. We use $\mathcal{L}$ and C to derive a contradiction as follows. Note that we may assume that $\mathcal{L}$ is non-empty, since otherwise we already have a contradiction. We have two tokens for each edge in E'; the total number of tokens is therefore $2|E'|$. Our goal is to show that we can rearrange the tokens so that each biset in $\mathcal{L}$ gets at least 2 tokens, each vertex in C gets at least 2 tokens, and each maximal biset in $\mathcal{L}$ gets at least 4 tokens. If we can reassign the tokens so that these conditions are satisfied, it would then follow that the number of tokens is greater than $2(|\mathcal{L}| + |C|) = 2|E'|$, which is a contradiction.

We view $\mathcal{L}$ as a rooted forest. A biset $\mathbb{B}$ is a child of a biset $\mathbb{A} \neq \mathbb{B}$ if $\mathbb{B} \subseteq \mathbb{A}$ and there does not exist a biset $\mathbb{C} \in \mathcal{L} - \{\mathbb{A}, \mathbb{B}\}$ such that $\mathbb{B} \subseteq \mathbb{C} \subseteq \mathbb{A}$. A biset $\mathbb{B}$ is a descendant of a biset $\mathbb{A}$ if $\mathbb{B} \subseteq \mathbb{A}$ (note that a biset is a descendant of itself); $\mathbb{B}$ is a proper descendant of $\mathbb{A}$ if $\mathbb{B}$ is a descendant of $\mathbb{A}$ and $\mathbb{B} \neq \mathbb{A}$. A biset is a leaf if it does not have any proper descendants and it is a root if it is a maximal biset of $\mathcal{L}$. (A biset $\mathbb{A}$ is a maximal biset of $\mathcal{L}$ if there does not exist a biset $\mathbb{B} \in \mathcal{L} - \{\mathbb{A}\}$ such that $\mathbb{A} \subseteq \mathbb{B}$.)

We will need the following standard lemma, which follows from the fact that the vectors $\{\chi(\delta_{E'}(\mathbb{S})) \mid \mathbb{S} \in \mathcal{L}\}$ are linearly independent, f' is integer valued, and $0 < x(e) < 1/3$ for each edge $e \in E'$.

LEMMA 4.6. *Let $\mathbb{A}$ be a biset that has a unique child $\mathbb{B}$ in $\mathcal{L}$. Then one of the following holds:*

- *$\delta_{E'}(\mathbb{A}) - \delta_{E'}(\mathbb{B})$ or $\delta_{E'}(\mathbb{B}) - \delta_{E'}(\mathbb{A})$ has at least 4 edges.*
- *$\delta_{E'}(\mathbb{A}) - \delta_{E'}(\mathbb{B})$ and $\delta_{E'}(\mathbb{B}) - \delta_{E'}(\mathbb{A})$ are both non-empty.*

Now we turn our attention to the token argument. We start by assigning the tokens to $\mathcal{L} \cup C$ as follows. For each biset $\mathbb{A} \in \mathcal{L}$, let $\mathrm{in}(\mathbb{A})$ be the set of all edges e with one endpoint in A and the other in A'. Since $\mathcal{L}$ is strongly bilaminar, for each edge $e \in E'$ there is at most one *minimal* biset $\mathbb{A} \in \mathcal{L}$ such that $e \in \mathrm{in}(\mathbb{A})$. Additionally, for each vertex v, there is at most one *minimal* biset of $\mathcal{L}$ that contains v in its inner part.

Initial token assignment. Each edge $e = uv \in E'$ has two tokens $t_{e,u}$ and $t_{e,v}$, one for each endpoint. The edge e distributes its tokens to $\mathcal{L} \cup C$ as follows.

(Rule A) If $u \in C$, the token $t_{e,u}$ is assigned to u (similarly for the case $v \in C$).

(Rule B) If $u \notin C$, the token $t_{e,u}$ is assigned to the minimal biset $\mathbb{A} = (A, A') \in \mathcal{L}$ such that $e \in \delta_{E'}(\mathbb{A})$ and $u \in A$. If there does not exist such a biset, the token $t_{e,u}$ is assigned to the minimal biset $\mathbb{B}$ such that $e \in \mathrm{in}(\mathbb{B})$ (similarly for the case $v \notin C$).

It follows from the discussion above that each token is assigned to at most one member of $\mathcal{L} \cup C$. We will reassign some of the tokens in two stages. We will need the following key definition.

DEFINITION 4.7. *A biset $\mathbb{A} \in \mathcal{L}$ is* ***relevant*** *for a vertex $v \in V$ if all of the following conditions hold:*

- *$\mathbb{A}$ is not a leaf of $\mathcal{L}$ and it has only one child in $\mathcal{L}$.*

- *The vertex v is on the boundary of $\mathbb{A}$ but not on the boundary of the child of $\mathbb{A}$.*

LEMMA 4.8. *Let v be a vertex in V. If $\mathcal{L}$ is a bilaminar family, the relevant bisets of v are pairwise disjoint.*

Proof: Let $\mathbb{A}_1$ and $\mathbb{A}_2$ be two bisets in $\mathcal{L}$ that are relevant bisets for v. Suppose for contradiction that $\mathbb{A}_1$ and $\mathbb{A}_2$ are not disjoint. Since $\mathcal{L}$ is bilaminar, we have $\mathbb{A}_1 \subseteq \mathbb{A}_2$ or $\mathbb{A}_2 \subseteq \mathbb{A}_1$. Without loss of generality, assume $\mathbb{A}_2 \subseteq \mathbb{A}_1$. Let $\mathbb{B}_1$ be the unique child of $\mathbb{A}_1$ in $\mathcal{L}$. Suppose for contradiction that $\mathbb{A}_2 \subseteq \mathbb{B}_1$. Since $\mathbb{A}_2 \subseteq \mathbb{B}_1 \subseteq \mathbb{A}_1$ and v is on the boundary of both $\mathbb{A}_1$ and $\mathbb{A}_2$, it follows that v is on the boundary of $\mathbb{B}_1$. But this contradicts the fact that $\mathbb{A}_1$ is a relevant biset for v. Therefore $\mathbb{A}_2 \not\subseteq \mathbb{B}_1$. Since $\mathcal{L}$ is bilaminar, either $\mathbb{A}_2$ and $\mathbb{B}_1$ are disjoint or $\mathbb{B}_1 \subseteq \mathbb{A}_2$. If $\mathbb{B}_1$ and $\mathbb{A}_2$ are disjoint, $\mathbb{A}_1$ has at least two children, which contradicts the fact that $\mathbb{A}_1$ is relevant for v (each relevant biset has only one child). Therefore we must have $\mathbb{B}_1 \subseteq \mathbb{A}_2 \subseteq \mathbb{A}_1$. Since $\mathbb{B}_1$ is a child of $\mathbb{A}_1$ and $\mathbb{A}_1 \neq \mathbb{A}_2$, we must have $\mathbb{B}_1 = \mathbb{A}_2$. But this cannot happen, since v is on the boundary of $\mathbb{A}_2$ but v is not on the boundary of $\mathbb{B}_1$. □

Our token argument relies on the following key theorem that we prove at the end of the section.

THEOREM 4.9. *Let v be a vertex in C. Let $\rho(v)$ be the number of bisets in $\mathcal{L}$ that are relevant for v. We have $2\rho(v) + 6 \leq |\delta_{E'}(v)|$.*

Once we have Theorem 4.9, we can rearrange the tokens as follows.

First token reassignment. After the initial token assignment, each vertex in C receives $|\delta_{E'}(v)|$ tokens. For each vertex $v \in C$, we reassign some of the tokens of v as follows. For each biset $\mathbb{A} \in \mathcal{L}$ that is relevant for v, the vertex v gives two tokens to $\mathbb{A}$. Additionally, v gives 4 tokens to the minimal biset $\mathbb{A} = (A, A') \in \mathcal{L}$ such that $v \in A$.

Note that each vertex v gives $2\rho(v)+4$ tokens, where $\rho(v)$ is the number of bisets in $\mathcal{L}$ that are relevant for v. By Theorem 4.9, each vertex $v \in C$ has at least 2 tokens left over after the first token reassignment.

Second token reassignment. Now we consider the bisets in $\mathcal{L}$. Each biset in $\mathcal{L}$ has some tokens that were assigned by the initial token assignment and the first token reassignment. We show that we can rearrange these tokens so that each biset in $\mathcal{L}$ has at least two tokens and each maximal biset in $\mathcal{L}$ has at least 4 tokens.

THEOREM 4.10. *We can rearrange the tokens of the bisets in $\mathcal{L}$ such that each biset gets at least 2 tokens and each maximal biset in $\mathcal{L}$ gets at least 4 tokens.*

Proof Sketch: Recall that we view $\mathcal{L}$ as a rooted forest. We consider each tree of $\mathcal{L}$ separately. Let $\mathcal{T}$ be a tree of $\mathcal{L}$. We prove the theorem by induction on the height of the tree $\mathcal{T}$.

Let $\mathbb{A} = (A, A')$ be a leaf biset of $\mathcal{T}$. Since $x(\delta_{E'}(\mathbb{A})) = f'(\mathbb{A}) \geq 1$ and $x(e) < 1/3$ for each edge, we have $|\delta_{E'}(\mathbb{A})| \geq 4$. Consider an edge $e \in \delta_{E'}(\mathbb{A})$ and let v be the endpoint of e that is in A. If $v \in C$, $\mathbb{A}$ receives 4 tokens from v in the first token reassignment. Otherwise, $\mathbb{A}$ receives the token $t_{e,v}$ in the initial token assignment (by (Rule B)). It follows that $\mathbb{A}$ has at least 4 tokens.

Let $\mathbb{A} = (A, A')$ be a non-leaf biset of $\mathcal{T}$. For each child $\mathbb{B}$ of $\mathbb{A}$, it follows by induction that we can rearrange the tokens in the subtree rooted at $\mathbb{B}$ in such a way that each biset in the subtree gets at least 2 tokens and $\mathbb{B}$ gets at least 4 tokens. A child $\mathbb{B}$ of $\mathbb{A}$ has 4 tokens and it gives 2 of them to $\mathbb{A}$. If $\mathbb{A}$ has at least two children, it receives at least 4 tokens from its children and we are done.

Therefore we may assume that $\mathbb{A}$ has only one child $\mathbb{B}$. The biset $\mathbb{A}$ receives 2 tokens from $\mathbb{B}$ and therefore it suffices to show that $\mathbb{A}$ received 2 other tokens. In the following, we show that $\mathbb{A}$ received 2 additional tokens in the initial token assignment and the first token reassignment.

Let $E_1 = \delta_{E'}(\mathbb{A}) - \delta_{E'}(\mathbb{B})$ and $E_2 = \delta_{E'}(\mathbb{B}) - \delta_{E'}(\mathbb{A})$. By Lemma 4.6, one of the following holds: $|E_1| \geq 4$, $|E_2| \geq 4$, or E_1 and E_2 are both non-empty. We consider each of these cases in turn.

Suppose that $|E_1| \geq 4$. Consider an edge $e \in E_1$ and let u be the endpoint of e that is in A. If $u \in C$, $\mathbb{A}$ receives 4 tokens from u in the first token reassignment, since $\mathbb{A}$ is the minimal biset of $\mathcal{L}$ that contains u in its inner part. If $u \notin C$, $\mathbb{A}$ receives the token $t_{e,u}$ in the initial token assignment (by (Rule B)). Thus $\mathbb{A}$ receives at least 4 additional tokens.

Suppose that $|E_2| \geq 4$. Consider an edge $e \in E_2$ and let v be the endpoint of e that is in $A' - B'$. Note that $\mathbb{A}$ is the minimal biset such that $e \in \text{in}(\mathbb{A})$. Since $\mathcal{L}$ is strongly bilaminar and $\mathbb{B}$ is the unique child of $\mathbb{A}$, one can verify that there does not exist a biset $\mathbb{S} = (S, S') \in \mathcal{L}$ such that $e \in \delta_{E'}(\mathbb{S})$ and $v \in S$. Thus, if $v \notin C$, $\mathbb{A}$ receives the token $t_{e,v}$ in the initial token assignment (by (Rule B)). If $v \in C$, $\mathbb{A}$ receives at least 2 tokens from v in the first token reassignment: if v is on the boundary of $\mathbb{A}$, $\mathbb{A}$ is a relevant biset for v; otherwise, $\mathbb{A}$ is the minimal biset such that $v \in A$. Therefore $\mathbb{A}$ receives at least 2 additional tokens.

Finally, suppose that E_1 and E_2 are both non-empty. Let e be an edge in E_1 and let u be the endpoint of e that is in A. If $u \in C$, $\mathbb{A}$ receives 4 tokens from u in the first token reassignment, since $\mathbb{A}$ is the minimal biset of $\mathcal{L}$ that contains u in its inner part. If $u \notin C$, $\mathbb{A}$ receives the token $t_{e,u}$ in the initial token assignment (by (Rule B)). Let e' be an edge in E_2 and let v be the endpoint of e' that is in $A' - B'$. If $v \in C$, $\mathbb{A}$ receives at least 2 tokens from v in the first token reassignment: if v is on the boundary of $\mathbb{A}$, $\mathbb{A}$ is a relevant biset for v; otherwise, $\mathbb{A}$ is the minimal biset such that $v \in A$. If $v \notin C$, $\mathbb{A}$ receives the token $t_{e',v}$ in the initial token assignment (by (Rule B)); this follows from the fact that there does not exist a biset $\mathbb{S} = (S, S')$ such that $e' \in \delta_{E'}(\mathbb{S})$ and $v \in S$. Thus $\mathbb{A}$ receives at least 2 additional tokens. □

Thus, after reassigning the tokens, each vertex in C has at least 2 tokens, each biset in $\mathcal{L}$ has at least 2 tokens, and each maximal biset in $\mathcal{L}$ has at least 4 tokens. Since $\mathcal{L}$ is non-empty, the total number of tokens is strictly greater than $2(|\mathcal{L}| + |C|) = 2|E'|$, which is a contradiction. Thus it only remains to prove Theorem 4.9; we give the proof in the remainder of the section.

Proof of Theorem 4.9. Consider a vertex $v \in C$. We partition the bisets that are relevant for v into two categories: (i) bisets $\mathbb{A}$ such that $|\text{in}(\mathbb{A}) \cap \delta_{E'}(v)| \geq 4$, and (ii) bisets $\mathbb{A}$ such that $|\text{in}(\mathbb{A}) \cap \delta_{E'}(v)| < 4$. Let $\rho_1(v)$ and $\rho_2(v)$ denote the number of bisets in the first and second category, respectively. We upper bound $\rho_1(v)$ and $\rho_2(v)$ separately.

PROPOSITION 4.11. *Let v be a vertex in C. Let $\rho_1(v)$ be the number of bisets $\mathbb{A}$ such that $\mathbb{A}$ is relevant for v and $|\mathrm{in}(\mathbb{A}) \cap \delta_{E'}(v)| \geq 4$. Then $\rho_1(v) \leq |\delta_{E'}(v)|/4$.*

Proof: By Lemma 4.8, any two bisets that are relevant for v are disjoint. Therefore the edge sets $\{\mathrm{in}(\mathbb{A}) \cap \delta_{E'}(v) \mid \mathbb{A} \text{ is relevant for } v\}$ are pairwise disjoint. Thus we have $\rho_1(v) \leq |\delta_{E'}(v)|/4$. □

LEMMA 4.12. *Let v be a vertex in C. Let $\rho_2(v)$ be the number of bisets $\mathbb{A}$ such that $\mathbb{A}$ is relevant for v and $|\mathrm{in}(\mathbb{A}) \cap \delta_{E'}(v)| < 4$. Then $\rho_2(v) \leq |\delta_{F \cup F'}(v)|$ and thus $\rho_2(v) \leq (|\delta_{E'}(v)| - 12)/4$.*

Note that Theorem 4.9 follows from Proposition 4.11 and Lemma 4.12. We devote the rest of this section to the proof of Lemma 4.12.

A key insight is that, if $\mathbb{A}$ is a biset in the second category, then $\mathrm{in}(\mathbb{A})$ contains an edge of $\delta_{F \cup F'}(v)$. More precisely, we have the following lemma. We remark that this is the only place where we use the fact that the requirement function r satisfies $\mathrm{r}((A, A')) \leq \mathrm{r}(A, A' - v)$ for each biset (A, A') and each vertex $v \in A' - A$.

LEMMA 4.13. *Let v be a vertex and let $\mathbb{A} = (A, A')$ be a tight biset that contains v on its boundary. If $\delta_{F \cup F'}(v) \cap \mathrm{in}(\mathbb{A})$ is empty, $|\delta_{E'}(v) \cap \mathrm{in}(\mathbb{A})| \geq 4$.*

Proof Sketch: Let $\mathbb{B} = (A, A' - \{v\})$. We claim that $x(\delta_{E'}(\mathbb{B})) - x(\delta_{E'}(\mathbb{A})) \geq 1$. To see this, note that we have $x(\delta_{E'}(\mathbb{B})) - x(\delta_{E'}(\mathbb{A})) \geq f'(\mathbb{B}) - f'(\mathbb{A})$, since x is feasible and $\mathbb{A}$ is tight. One can verify that $f'(\mathbb{B}) - f'(\mathbb{A}) \geq 1$ as follows. Recall that $f'(\mathbb{S}) = r(\mathbb{S}) - |\mathrm{bd}(\mathbb{S})| - |\delta_{F \cup F'}(\mathbb{S})|$ for each biset $\mathbb{S}$. We have $r(\mathbb{B}) \geq r(\mathbb{A})$ (see the statement of Theorem 3.2). Additionally, $|\mathrm{bd}(\mathbb{B})| = |\mathrm{bd}(\mathbb{A})| - 1$. Therefore

$$f'(\mathbb{B}) - f'(\mathbb{A}) \geq 1 + |\delta_{F \cup F'}(\mathbb{A})| - |\delta_{F \cup F'}(\mathbb{B})|.$$

Since $\delta_{F \cup F'}(v) \cap \mathrm{in}(\mathbb{A})$ is empty, we have $\delta_{F \cup F'}(\mathbb{A}) = \delta_{F \cup F'}(\mathbb{B})$ and thus $f'(\mathbb{B}) - f'(\mathbb{A}) \geq 1$.

Since $x(\delta_{E'}(\mathbb{B})) - x(\delta_{E'}(\mathbb{A})) \geq 1$ and $x(e) < 1/3$ for each edge $e \in E'$, one can verify that $|\delta_{E'}(\mathbb{B}) - \delta_{E'}(\mathbb{A})| \geq 4$. Since $\delta_{E'}(\mathbb{B}) - \delta_{E'}(\mathbb{A}) \subseteq \delta_{E'}(v) \cap \mathrm{in}(\mathbb{A})$, the lemma follows. □

Proof of Lemma 4.12: By Lemma 4.13, for each biset $\mathbb{A}$ in the second category, $\mathrm{in}(\mathbb{A}) \cap \delta_{F \cup F'}(v)$ is non-empty. Since the bisets in the second category are disjoint (by Lemma 4.8), we have $\rho_2(v) \leq |\delta_{F \cup F'}(v)|$.

To complete the proof, we note that $|\delta_{F \cup F'}(v)| \leq |\delta_F(v)| + 3b(v) \leq (|\delta_{E'}(v)| - 12)/4$: since Undir-algo-weaker only selects edges whose fractional value is at least 1/3, we have $|\delta_{F'}(v)| \leq 3b(v)$; additionally, by our assumption on the vertices of C, we have $|\delta_{E'}(v)| > 4|\delta_F(v)| + 12b(v) + 12$. □

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