The All-or-Nothing Flow Problem in Directed Graphs with Symmetric Demand Pairs*

Chandra Chekuri† Alina Ene‡

April 30, 2014

Abstract

We study the approximability of the All-or-Nothing multicommodity flow problem in directed graphs with symmetric demand pairs (SymANF). The input consists of a directed graph $G = (V, E)$ and a collection of (unordered) pairs of nodes $\mathcal{M} = \{s_1t_1, s_2t_2, \ldots, s_kt_k\}$. A subset $\mathcal{M}'$ of the pairs is *routable* if there is a feasible multicommodity flow in G such that, for each pair $s_it_i \in \mathcal{M}'$, the amount of flow from s_i to t_i is at least one *and* the amount of flow from t_i to s_i is at least one. The goal is to find a maximum cardinality subset of the given pairs that can be routed. Our main result is a poly-logarithmic approximation with constant congestion for SymANF. We obtain this result by extending the well-linked decomposition framework of [11] to the directed graph setting with symmetric demand pairs. We point out the importance of studying routing problems in this setting and the relevance of our result to future work.

1 Introduction

We consider some fundamental maximum throughput routing problems in *directed* graphs. In this setting, we are given a capacitated directed graph $G = (V, E)$ with n nodes and m edges. We are also given source-destination pairs of nodes $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$. The goal is to select a largest subset of the pairs that are simultaneously *routable* subject to the capacities; a set of pairs is routable if there is a multicommodity flow for the pairs satisfying certain constraints that vary from problem to problem (e.g., integrality, unsplittability, edge or node capacities). Two well-studied optimization problems in this context are the Maximum Edge Disjoint Paths (MEDP) and the All-or-Nothing Flow (ANF) problem. In MEDP, a set of pairs is routable if the pairs can be connected using edge-disjoint paths. In ANF, a set of pairs is routable if there is a feasible multicommodity flow that fractionally routes one unit of flow from s_i to t_i for each routed pair (s_i, t_i). ANF, introduced in [14, 10], can be seen as a relaxed version of MEDP where the flow for the routed pairs is not required to be integral.

MEDP and ANF are both **NP**-hard and their approximability has attracted substantial attention. Over the last decade, several non-trivial results on both upper bounds and lower bounds have led to a much better understanding of these problems. At a high level, one can summarize this progress as follows. MEDP and ANF admit poly-logarithmic approximation in *undirected* graphs if one allows constant congestion[1]; in fact, a congestion of 2 is sufficient for MEDP [18] and for ANF no extra

*An extended abstract of this paper is to appear in *Proc. of IPCO*, 2014.

†Supported in part by NSF grants CCF-1016684 and CCF-1319376. Part of this work was done while the author was suppoted by TTI Chicago on a sabbatical visit in Fall 2012.

‡Supported in part by NSF grants CCF-1016684 and CCF-0844872. Part of this work was done while the author was an intern at TTI Chicago.

[1]A routing has congestion c if it violates the capacities by a factor of at most c.

congestion is needed [10]. Moreover, both problems are hard to to approximate to within a factor of $\Omega(\log^{\frac{1-\varepsilon}{c+1}} n)$ for any constant congestion $c \geq 1$ [3]; the hardness is under the assumption that $\mathbf{NP} \not\subseteq \mathbf{ZPTIME}(n^{\text{polylog}(n)})$. In sharp contrast, in *directed* graphs both problems are hard to approximate to within a *polynomial* factor for any constant congestion $c \geq 1$; the hardness factor is $n^{\Omega(1/c)}$ [16]. The upper bounds and lower bounds on the approximability are closely related to corresponding integrality gap bounds on a multicommodity flow relaxation for these problems.

In this paper, with several interrelated motivations in mind that we discuss in detail subsequently, we initiate the study of maximum throughput routing problems in directed graphs in the setting where the demand pairs are *symmetric*. Informally, in a symmetric demand pair instance, the input pairs are *unordered* and a pair s_it_i is routed only if both the ordered pairs (s_i, t_i) and (t_i, s_i) are routed. In particular, we focus our attention on the SymANF problem. The input consists of a directed graph $G = (V, E)$ and a collection of (unordered) pairs of nodes $\mathcal{M} = \{s_1t_1, s_2t_2, \ldots, s_kt_k\}$. A subset $\mathcal{M}'$ of the pairs is *routable* if there is a feasible multicommodity flow in G such that, for each pair $s_it_i \in \mathcal{M}'$, the amount of flow from s_i to t_i is one unit *and* the amount of flow from t_i to s_i is one unit[2]. The goal is to find a maximum cardinality subset of the given pairs that can be routed.

One issue is whether we assume capacities on edges or on nodes or on both. In undirected graphs the node capacitated case is more general, however, in directed graphs this is not the case. In this paper we will assume that G has only node-capacities, in particular that each node has capacity one. The main reason for our choice is to relate routability in the symmetric setting to the notion of directed treewidth.

Our main result is the following theorem that gives a poly-logarithmic approximation with constant congestion for SymANF.

Theorem 1.1. *There is a polynomial time algorithm that, given any instance of the* SymANF *problem in directed graphs, it routes* $\Omega(\text{OPT}/\log^2 k)$ *pairs with constant node congestion, where* OPT *is the value of an optimal fractional solution for the instance.*

The congestion that we guarantee is 64. We believe that the congestion can be improved, but we have not attempted to optimize the constant. Our algorithm uses a natural LP relaxation for the problem as a starting point and we also show a poly-logarithmic upper bound on the integrality gap of the relaxation. Some simple and natural extensions such as handling capacitated graphs and pairs with demand values can be handled via known techniques and we do not address them in this version.

We observe that, via existing results on the hardness of ANF in undirected graphs with congestion [3], one can conclude that SymANF with congestion c is hard to approximate to within a factor of $(\log n)^{\Omega(1/c)}$ for any fixed c unless $\mathbf{NP} \subseteq \mathbf{ZPTIME}\left(n^{\text{polylog}(n)}\right)$.

2 Motivation and connection to related problems

The study of routing problems is motivated by several real-world applications but also by the fundamental role that flows and cuts play in algorithms, combinatorial optimization, and graph theory. For a single pair (s, t) it is well-known that the value of a maximum s-t flow in a directed graph is equal to the value of a minimum s-t cut; moreover, when the capacities are integral, the maximum fractional

[2]There are alternative ways to define routability that captures symmetry. One option is to require a flow of 1/2 unit in each direction which is compatible with a total of one unit of flow entering and leaving each terminal. Another option is to require that for any orientation of the demand pairs, there is a feasible multicommodity for the pairs with one unit for each pair in the direction given by the orientation; however, deciding the routability according to this definition is not easy. For simplicity we require one unit of flow in each direction which results in a factor of 2 loss in the congestion when compared to other models.

s-t flow is equal to the maximum integral s-t flow. These nice structural properties do not hold in the multicommodity setting in undirected or directed graphs even when the number of commodities is three.

The study of *approximate* flow-cut gap results, starting with the seminal work of Leighton and Rao [32], has been extremely fruitful and we now have an optimal upper bound of $\Theta(\log k)$ on multicommodity flow-cut gaps in undirected graphs in a variety of settings [23, 33, 21, 9]. In contrast, it was only fairly recently that polynomial-factor lower bounds were established for flow-cut gaps in directed graphs [17]. On the other hand, poly-logarithmic upper bounds on the flow-cut gap are also known in directed graphs with symmetric demand pairs [31], although the lower and upper bounds are not known to be tight and several open problems remain.

Maximum throughput routing problems, such as MEDP and the related problem of congestion minimization, aim to construct integer flows. These problems are typically tackled via relaxations based on multicommodity flows. Rounding these relaxations is equivalent to upper bounding the gap between fractional and integral flows. A technical issue that arises is the following. For MEDP, even in undirected graphs, the integrality gap of the flow relaxation is known to be $\Omega(\sqrt{n})$ because of a simple topological obstruction in the plane [23]. For this reason, the main focus has been on understanding the following question: what is the gap between the maximum fractional flow and the maximum integral flow with constant congestion? Understanding this question has been a very challenging open problem. Culminating a series of papers that addressed special cases and developed various tools, in a recent breakthrough, Chuzhoy [15] showed a poly-logarithmic upper bound with constant congestion in general undirected graphs. Subsequently, the congestion has been brought down to the optimal bound of 2 in [18]. Building on Chuzhoy's work, the authors of this paper obtained a poly-logarithmic upper bound with constant congestion for the maximum *node*-disjoint paths problem in undirected graphs; note that in undirected graphs the distinction between node-disjoint routing and edge-disjoint routing is important, with the former being more general. Again, in contrast to the undirected graph case, as we already mentioned, in directed graphs there is a polynomial-factor lower bound on the integrality gap of the flow relaxation for *any* constant congestion.

A natural meta-question is the following. Why are routing problems "easy" in undirected graphs and "hard" in directed graphs? In the same vein, why are flow-cut gaps and fractional flow-integral flow gaps "small" (poly-logarithmic) in undirected graphs and "large" (polynomial) in directed graphs? What is the most general setting in which one can obtain good results? It is difficult to give a clean answer to these questions but one can observe that routing problems in directed graphs with symmetric demand pairs straddle the boundary between routing in undirected and directed graphs. Moreover, we already know that the flow-cut gap is poly-logarithmic in directed graphs if the demand pairs are symmetric. Our primary motivation is to understand whether the gap between fractional flows and integral flows is also small in directed graphs with symmetric demand pairs. A direct benefit is a generalization of existing results that show poly-logarithmic gaps for edge *and* node disjoint paths in undirected graphs (with constant congestion). This research agenda will involve the development and understanding of several technical ingredients that have auxiliary benefits. We briefly elaborate on this point below.

Structure of graphs with large (directed) treewidth: Recent progress on routing problems has been accomplished via the following scheme. The well-linked decomposition framework of [11] showed that one can use flow-cut gap results to reduce the problem (to within poly-logarithmic factors) to a graph theoretic question: if the graph G has a "well-linked" set of size k, does it have a routing structure (called a *crossbar*) of size[3] $\tilde{\Omega}(k)$? For node-capacitated routing problems in undirected graphs, the

[3]The $\tilde{\Omega}$ notation hides poly-logarithmic factors.

question can be phrased in terms of the well-understood notion of *treewidth*: If G has treewidth k, does G have a crossbar of size $\tilde{\Omega}(k)$? The question was answered affirmatively (in [8], following Chuzhoy's framework for MEDP) by embedding, in a technical sense, in G an expander of size $\tilde{\Omega}(k)$ with constant congestion; the expander is the desired crossbar. This high-level structural result required several technical ingredients. A key tool developed in this process was a graph splitting procedure that has already found several powerful applications in fixed parameter tractability and graph theory [6]. These results are also closely related to the well-known grid minor theorem of Robertson and Seymour that shows that G contains a grid minor of size k as long as the treewidth of G is at least $f(k)$ for some function f [35]. Very recently the ideas from routing led to a proof that $f(k)$ can chosen to be a polynomial [7], improving upon a previous exponential bound [36].

Johnson, Robertson, Seymour, and Thomas [25] introduced the notion of *directed* treewidth. In undirected graphs treewidth is defined via tree decompositions and is reasonably intuitive. Directed treewidth is defined via arboreal decompositions [25] and is less easy to grasp — we provide the formal definition and some related concepts in Section A. We refer the reader to [25, 34, 1] for some subtle issues that differentiate directed treewidth from treewidth. See [5, 24] for related but different definitions of width parameters for directed graphs. It is believed that understanding directed treewidth better would yield significant dividends in graph theory and algorithms. Fortunately, directed treewidth, like treewdith, can be approximately understood via well-linked sets [34]. There are slightly different notions of linkedness that are closely related. In this paper we say that $X \subseteq V$ is cut-well-linked in a directed graph G if for any two disjoint subsets $A, B \subset X$ of equal size, there are $|A| = |B|$ node-disjoint paths from A to B in G. It is known that (directed) treewidth of a graph G is within a constant factor of the largest well-linked set in G [34]. Moreover, one can approximate (directed) treewidth via algorithms for (directed) sparsest cut and the current best approximation ratio is $O(\sqrt{\log k})$ where k is the (directed) treewidth of G [21, 29].

Our second motivation for studying routing problems in directed graphs with symmetric demand pairs stems from their connections to directed treewidth. In this paper, we extend the well-linked decomposition framework of [11] to this setting and this leads to the following question: If a directed graph G has directed treewidth k (equivalently, has a well-linked set of size $\Omega(k)$), does it have a "routing structure" of size $\tilde{\Omega}(k)$? Answering this question affirmatively would lead to algorithms for disjoint path routing for symmetric pair instances. This question seems to pose several non-trivial technical challenges despite the substantial recent progress made in the undirected graph case. We note that [25] formulated a conjecture that can be viewed as a directed counterpart of the grid minor theorem of Robertson and Seymour. The conjecture posits that there is a function $f(k)$ such that, if G is a directed graph with directed treewidth at least k, then G has a cylinder of size k as a minor[4]; a cylinder is a directed analogue of a grid and it is a crossbar. The cylinder minor conjecture is still open for general graphs. Johnson *et al.* [26] showed that the cylinder conjecture is true in planar graphs for some function $f(k)$; very recently the conjecture was shown for a more general class of graphs [28]. Proving it for general graphs seems very challenging. Moreover, to obtain good approximations for routing, one needs a very strong quantitative bound on f; to obtain an approximation ratio of $\alpha(k)$, we need $f(k)$ to be $\Omega(k/\alpha(k))$. On the other hand, there is flexibility in routing applications in that the crossbar structure that we need to exhibit need not be a cylinder minor, and moreover we can allow constant congestion. In fact such flexibility is needed to obtain a crossbar whose size is a near-linear function of k.

Flow-cut gap in planar graphs: Another interesting direction for future work, and a motivation for us, is to show the following: if G is a planar directed graph with directed treewidth k, then it has a crossbar

[4]There are several ways to define a minor in directed graphs. The minor notion used in the cylinder conjecture is the notion of a butterfly minor. We refer the reader to [25] for the definition of a butterfly minor and a cylinder.

of size $\Omega(k)$. A linear relationship between treewidth and grid-minor size is known in undirected planar graphs [36] (and also in a more general class of graphs [19]) but the known quantitative relationship between directed treewidth and the cylinder minor size in planar graphs is so weak that it is not explicitly specified [26]. A linear relationship would have applications to routing on disjoint paths, but would also give an improved upper bound on the flow-cut gap for symmetric product multicommodity flows in planar directed graphs. Currently, the best upper bound on the flow-cut gap is $O(\log n)$ for product multicommodity flows in both general and planar graphs [32]. The existence of a crossbar of size $\Omega(k)$ will imply that the flow-cut gap is $O(1)$ in planar directed graphs[5], which in turn it will give constant factor approximation guarantees for problems such as the Uniform Sparsest Cut problem in planar directed graphs; such results are known for planar undirected graphs [30]. Currently, the best approximation for the Uniform Sparsest Cut problem is $O(\sqrt{\log n})$ [2] in both planar and general directed graphs. We remark that the crossbar we need for a flow-cut gap result can be much weaker than what one needs for disjoint path problems; it only needs to support fractional routing instead of integral routing. Exploring weaker notions of crossbars, we believe, may help make progress on the difficult questions while also yielding results that are independently interesting.

In this paper, we study the SymANF problem as a first step towards understanding maximum throughput routing problems in directed graphs. We now give a high-level description of our algorithm and we describe in more detail some specific technical contributions that enable us to prove Theorem 1.1.

2.1 Overview of the algorithm and technical contributions

Let $(G, \mathcal{M})$ be an instance of SymANF. Let $\mathcal{T}$ be the set of all nodes that participate in the pairs of $\mathcal{M}$; we refer to the nodes in $\mathcal{T}$ as the *terminals*. Our algorithm for SymANF in directed graphs follows the framework of Chekuri, Khanna, and Shepherd [10, 11] for the ANF problem in undirected graphs. In a nutshell, the framework decomposes an arbitrary instance of ANF into several instances that are *flow-well-linked*. The set of terminals $\mathcal{T} = \{s_1, t_1, \ldots, s_k, t_k\}$ is *flow-well-linked* if *any* matching on the terminals is routable. This is essentially equivalent (modulo a factor of 2 in congestion) to saying that the G admits a symmetric product multicommodity flow where the weight on each terminal is 1 and is 0 on every non-terminal. If the terminals are flow-well-linked, we can route all the input pairs. Thus the heart of the matter is showing that an arbitrary instance can be decomposed into well-linked instances without losing too much flow.

The decomposition has two main components. The first step is a weaker decomposition in which we take a fractional solution to a natural multicommodity flow based LP (described in Section 3.3) and use it to decompose the instance into instances that are only *fractionally* flow-well-linked. More precisely, there is a weight function $\pi : \mathcal{T} \to [0, 1]$ and the terminals are flow-well-linked with respect to these weights; if all terminals have weight 1 then they are flow-well-linked. The second step is a clustering step in which we take a fractionally flow-well-linked instance and we identify a large subset of the pairs such that their endpoints are flow-well-linked. In this paper, we show how to implement these two steps for the SymANF problem in directed graphs. In the first step, we extend the approach of [11] to our setting; we refer the reader to Section 4 for the details of the decomposition. We note that the approximation factor that we lose in the decomposition is proportional to the flow-cut gap; for symmetric instances, the flow-cut gap is only polylog(k). The second step poses several technical difficulties in directed graphs and it is our main technical contribution. We briefly highlight some of the difficulties involved in the clustering step, and we refer the reader to Section 5 for the details. Chekuri, Khanna, and Shepherd [11] gave a simple clustering technique for *edge-capacitated*

[5]The implications of crossbar results for product multicommodity flow-cut gaps is pointed out in [11].

undirected graphs. Roughly speaking, the approach is to take a spanning tree and to partition it into edge-disjoint subtrees where each subtree gathers roughly a unit weight from π. These subtrees are then used to find the desired flow-well-linked subset of pairs/terminals; one terminal is picked from each subtree. The clustering step is more involved in *node-capacitated* undirected graphs. The spanning tree approach, combined with some preprocessing to reduce the degree, gives a clustering for node-capacitated graphs with slightly weaker parameters [11]. In [12], the authors gave a stronger clustering for the node-capacitated setting; this approach is more involved than the spanning tree clustering and it exploits a connection between well-linked sets and treewidth; recent work [7] obtains a stronger result but requires more involved ideas. In directed graphs, there is no simple clustering process akin to using a spanning tree (or even an arborescence). Instead, our approach exploits the connection between well-linked sets and directed treewidth. However, the main challenge is to make this algorithmic. We also mention that, in addition to finding a large flow-well-linked set Y from a fractionally flow-well-linked set X, we also need to ensure that Y contains a large enough matching from the original set of pairs. For this purpose, we rely on a flow augmentation tool developed in [13]. These difficulties are also the reason why we are only able to obtain a constant congestion for SymANF while ANF admits a poly-logarithmic ratio with congestion 1 in edge-capacitated graphs [10] and with congestion $(1+\varepsilon)$ in node-capacitated graphs [11].

2.2 Discussion of related work

The ANF problem, the MEDP problem and its node-capacitated counterpart, the Maximum Node Disjoint Paths (MNDP) problem have been studied extensively in both undirected and directed graphs. We first discuss the decision versions of these problems where we are given G and the pairs, and the goal is to decide if all of them can be routed. It is easy to see that the decision version of ANF is polynomial-time solvable via linear programming — one needs to check whether there is a multicommodity flow that routes one unit of flow for each input pair. On the other hand, the decision versions of MEDP and MNDP, denoted by EDP and NDP respectively, are **NP**-complete if k is part of the input [27, 20]. If k is fixed, Robertson and Seymour, building on their seminal work on graph minors, gave a polynomial-time algorithm for NDP (and hence also for EDP) in undirected graphs. Interestingly, EDP is already **NP**-complete for $k = 2$ in directed graphs [22]. It is useful to note that the undirected graph algorithm of Robertson and Seymour relies heavily on treewidth and the structure of graphs with large treewidth.

ANF, MEDP and MNDP are optimization problems. Although the decision version of ANF is poly-time solvable, ANF is **NP**-hard, and **APX**-hard to approximate, even in capacitated trees [23]; routing is trivial in trees, selecting the pairs to route is not. The best approximation guarantees that are known for the ANF problem in undirected graphs are an $O(\log^2 k)$ approximation in edge-capacitated graphs [11] and an $O(\log^4 k \log n)$ approximation with congestion $(1+\epsilon)$ in node-capacitated graphs [11]; these ratios improve by a logarithmic factor for planar graphs. For node-capacitated graphs, an unpublished manuscript [12] gives an $O(\log^2 k)$-approximation if constant congestion is allowed. The MEDP problem with congestion $c = o(\log n / \log\log n)$ is $\Omega\left((\log n)^{O(1/c)}\right)$-hard to approximate in undirected graphs [4] and $\Omega(n^{O(1/c)})$-hard to approximate in directed graphs [16], unless $\mathbf{NP} \subseteq \mathbf{ZPTIME}\left(n^{\mathrm{polylog}(n)}\right)$. It is useful to note that these hardness results also hold for the ANF problem, which suggests that the current techniques do not distinguish between the difficulty of ANF and MEDP. For MEDP in undirected graphs there is an $O(\sqrt{n})$-approximation with congestion 1 [13] and as we already mentioned, recent work obtains a poly-logarithmic approximation with congestion 2 [18]. In directed graphs, MEDP has an $n^{O(1/c)}$-approximation with congestion c [37], and this approximation carries over to ANF as well. These approximation results use the natural multicommodity flow relaxations as a starting point and they also establish the same upper bound on the

integrality gap of the relaxations.

Organization: Section 3 introduces the main definitions and technical tools that we use, and it describes the approximation algorithm for SymANF. Section 4 and Section 5 describe the well-linked decomposition and clustering technique for directed graphs with symmetric demand pairs.

3 Approximation Algorithm for SymANF

In the following, we work with an instance $(G, \mathcal{M})$ of the SymANF problem, where $G = (V, E)$ is a directed graph with unit node capacities and $\mathcal{M} = \{s_1t_1, \ldots, s_kt_k\}$ is a collection of node pairs. We refer to the nodes participating in the pairs of $\mathcal{M}$ as terminals, and we use $\mathcal{T}$ to denote the set of all terminals. We assume that the pairs $\mathcal{M}$ form a perfect matching on $\mathcal{T}$ and each terminal is a leaf in G, i.e., each terminal is connected to a single neighbor using an edge in each direction. One can reduce an arbitrary instance to an instance that satisfies these assumptions as follows. If a node v participates in several pairs, we make a copy of v for each of the pairs it participates in, and attach the copy v' to v using an edge in each direction; finally we replace v by v' in the pair. Similarly, if a terminal is not a leaf, we make a copy of the terminal, we attach the copy to the original node as a leaf, and we replace the terminal by its copy in the pairs that contain it. Note that, if a set of pairs was routable in the original instance, then it is routable in the new instance with congestion at most 2.

In the following subsection we describe some basic ingredients about (symmetric) multicommodity flows and node separators. Subsection 3.2 formally defines well-linked sets and states the results on well-linked decompositions and clustering that are used in the algorithm for SymANF.

3.1 Multicommodity flows and sparse node separators

Let $G = (V, E, \text{cap})$ be a directed node-capacitated graph with node capacities given by cap. In this paper, we work with path-based flows f that assign a non-negative real value $f(p)$ to each path in G. A flow f is *feasible* if it satisfies the capacity constraints; more precisely, for each node v, $\sum_{p:v\in p} f(p) \leq \text{cap}(v)$. For any ordered pair (u, v) of nodes, the total flow from u to v is $\sum_{p\in\mathcal{P}(u,v)} f(p)$, where $\mathcal{P}(u, v)$ is the set of all paths of G from u to v.

A *multicommodity flow* instance in G is a demand vector $\mathbf{d}$ that assigns a non-negative real value $d(u, v)$ to each ordered pair (u, v) of nodes of G; we refer to $d(u, v)$ as the demand of the pair (u, v). A multicommodity flow instance is *symmetric* if $d(u, v) = d(v, u)$ for all ordered pairs (u, v). A multicommodity flow instance $\mathbf{d}$ is a *product* multicommodity flow instance if $d(u, v) = w(u)w(v)$, where $w : V \to \mathbb{R}_+$ is a weight function on the nodes of G. Note that a product multicommodity flow instance is symmetric. In the following, we only consider symmetric multicommodity flow instances. A multicommodity flow instance $\mathbf{d}$ is *routable* if there is a feasible multicommodity flow in which, for each ordered pair (u, v), the total flow on the paths from u to v is at least $d(u, v)$. We work with the following quantities associated with a *symmetric* multicommodity flow instance, the maximum concurrent flow and the sparsest node separator. The *maximum concurrent flow* is the maximum value $\lambda \geq 0$ such that $\lambda\mathbf{d}$ is routable. A *node separator* is a set $C \subseteq V$ of nodes. The removal of a node separator gives us one or more strongly connected components; we say that a pair uv is separated by C if u and v are *not* in the same strongly connected component of $G - C$. The demand separated by C, denoted by $\text{dem}_{\mathbf{d}}(C)$, is the total demand of all of the unordered pairs separated by C; more precisely, $\text{dem}_{\mathbf{d}}(C) = \sum_{uv \text{ separated by } C} d(u, v)$. The *sparsity* of a node separator C is $\text{cap}(C)/\text{dem}_{\mathbf{d}}(C)$. A *sparsest node separator* is a separator with minimum sparsity. It is straightforward to verify that,

for a symmetric multicommodity flow, the minimum sparsity of a node separator is an upper bound on the maximum concurrent flow. The *flow-cut gap* in G is the maximum value — over all symmetric multicommodity flow instances $\mathbf{d}$ in G — of the ratio between the minimum sparsity of a node separator and the maximum concurrent flow. The flow-cut gap in any graph is $O(\log^2 k)$, where k is the number of commodities (each pair (u, v) with non-zero demand is a commodity) [31]. For product multicommodity flows, the flow-cut gap is $O(\log k)$ [32]. Moreover, it was shown in [32] that there is a polynomial time algorithm that, given a product multicommodity flow instance $\mathbf{d}$ in G, it constructs a node separator C whose sparsity is at most $O(\log k)\lambda$, where λ is the maximum concurrent flow for $\mathbf{d}$; we use such an algorithm in a black box fashion in the well-linked decomposition step that we describe in more detail below.

A *node separation* in G is a partition (A, B, C) of the nodes of G such that there is no edge of G from A to B (note that there can be an edge of G from B to A). The following proposition shows that, given a weight function π on the nodes, and a node separator C, one can choose a node separation (A, B, C) that is "balanced" with respect to π.

Proposition 3.1. *Let $G = (V, E)$ be a directed graph and let $\pi : V \to \mathbb{R}_+$ be a weight function. Let $\mathbf{d}$ be the following product multicommodity flow: $d(u, v) = \pi(u)\pi(v)/\pi(V)$ for each pair (u, v) of nodes. Let C be a node separator in G. There is a node separation (A, B, C) such that $\mathrm{dem}_{\mathbf{d}}(C) \leq 2\min\{\pi(A), \pi(B)\}$. Moreover, given C, we can compute such a node separation in polynomial time.*

Proof: Let C be a node separator. Let $K_1, K_2, \ldots, K_\ell$ be a topological ordering of the strongly connected components of $G - C$ in which each edge of $G - C$ connecting different strongly connected components is oriented from right to left.

Suppose that $\pi(K_i) \leq \pi(V-C)/2$ for each i. Let p be the smallest index such that $\pi(K_1 \cup \cdots \cup K_p) \geq \pi(V-C)/4$. Let $A = V(K_1) \cup \cdots \cup V(K_p)$ and $B = V(K_{p+1}) \cup \cdots \cup V(K_\ell)$. Since $\pi(K_1 \cup \cdots \cup K_{p-1}) < \pi(V-C)/4$ and $\pi(K_p) \leq \pi(V-C)/2$, we have $\pi(A) \leq 3\pi(V-C)/4$ and therefore $\pi(B) \geq \pi(V-C)/4$. Thus (A, B, C) is a node separation satisfying $\min\{\pi(A), \pi(B)\} \geq \pi(V - C)/4$. Note that the total demand of the pairs $(u, v) \in (V - C) \times (V - C)$ is $\pi(V - C) \cdot \pi(V - C)/\pi(V) \leq \pi(V - C)$. Therefore $\mathrm{dem}_{\mathbf{d}}(C) \leq \pi(V - C)/2 \leq 2\min\{\pi(A), \pi(B)\}$.

Therefore we may assume that $\max_i \pi(K_i) > \pi(V - C)/2$. Let K_q be the strongly connected component with maximum π-weight; more precisely, $q = \arg\max_i \pi(K_i)$. We define a partition (A, B) of $V - C$ as follows. If $\pi(K_1 \cup \cdots \cup K_{q-1}) \geq \pi(K_{q+1} \cup \cdots \cup K_\ell)$, we let $A = V(K_1) \cup \cdots \cup V(K_{q-1})$ and $B = V(K_q) \cup \cdots \cup V(K_\ell)$. Otherwise, we let $A = V(K_1) \cup \cdots \cup V(K_q)$ and $B = V(K_{q+1}) \cup \cdots \cup V(K_\ell)$. The partition (A, B, C) is a node separation satisfying $\min\{\pi(A), \pi(B)\} \geq \pi(V - (C \cup K_q))/2$. Note that the total demand of the pairs $(u, v) \in (V - (C \cup K_q)) \times (V - (C \cup K_q))$ is $\pi(V - (C \cup K_q)) \cdot \pi(V - (C \cup K_q))/\pi(V)$. Additionally, the total demand of the pairs $(u, v) \in K_q \times (V - (C \cup K_q))$ is $\pi(K_q)\pi(V - (C \cup K_q))/\pi(V)$. Therefore we have

$$\begin{aligned}
\mathrm{dem}_{\mathbf{d}}(C) &\leq \frac{\pi(K_q)\pi(V - (C \cup K_q))}{\pi(V)} + \frac{\pi(V - (C \cup K_q))\pi(V - (C \cup K_q))}{2\pi(V)} \\
&= \frac{\pi(V - (C \cup K_q))(\pi(K_q) + \pi(V - C))}{2\pi(V)} \\
&\leq \pi(V - (C \cup K_q))
\end{aligned}$$

Therefore $\mathrm{dem}_{\mathbf{d}}(C) \leq 2\min\{\pi(A), \pi(B)\}$. □

3.2 Well-linked sets, decomposition, and clustering

There are two notions of well-linkedness that have been used for routing problems in undirected graphs [11]; one is based on a flow requirement and the other is based on a cut requirement. In the following,

we define directed versions of these two notions and we show some basic properties of these notions.

Flow-well-linked sets: Let G be a directed graph with unit capacities on the nodes. We define a fractional version of flow-well-linkedness as follows. Let $\pi : X \to [0,1]$ be a weight function on X. Let $\mathbf{d}$ be the following demand vector: $d(u,v) = \pi(u)\pi(v)/\pi(X)$ for each ordered pair (u,v) of nodes in X. The set X is **π-flow-well-linked** in G iff $\mathbf{d}$ is routable in G. For a scalar $c \in [0,1]$, we say that X is c-flow-well-linked if X is π-flow-well-linked, where $\pi(v) = c$ for each vertex $v \in X$.

Cut-well-linked sets: A set $X \subseteq V$ is **cut-well-linked** in G iff, for any two disjoint subsets Y and Z of X of equal size, there are $|Y|$ node-disjoint paths from Y to Z in G. Recall that a node is a leaf in G if it is connected to a single neighbor using an edge in each direction. If the nodes of X are leaves in G, an equivalent definition is the following. The set X is cut-well-linked iff, for any node separation (A,B,C) satisfying $X \cap C = \emptyset$, we have $|C| \geq \min\{|X \cap A|, |X \cap B|\}$. We define a fractional version of cut-well-linkedness as follows. Let X be a set of nodes of G and let $\pi : X \to [0,1]$ be a weight function on X. Suppose that all the nodes in X are leaves of G. The set X is **π-cut-well-linked** in G if, for any node separation (A,B,C), we have $|C| \geq \min\{\pi(A), \pi(B)\}$. Note that, since the nodes in X are leaves, it suffices to check this condition for separations (A,B,C) for which $\pi(C) = 0$. Now consider a set X that contains nodes that are not leaves. For each node $x \in X$, we add a new node x' and connect x' to x using two edges, one in each direction. Let X' be the set of new nodes, let G' be the resulting graph, and let $\pi' : X' \to [0,1]$ be the weight function $\pi'(x') = \pi(x)$ for each node $x \in X$. The set X is π-cut-well-linked in G iff X' is π'-cut-well-linked in G'.

The following proposition relates the two notions of well-linkedness.

Proposition 3.2. *Let $G = (V,E)$ be a directed graph. Let X be a set of nodes and let $\pi : X \to [0,1]$ be a weight function on X. Let $\alpha = \alpha(G) \geq 1$ be an upper bound on the worst case flow-cut gap for product multicommodity flows in G. If X is π-flow-well-linked in G then X is $(\pi/2)$-cut-well-linked in G. If X is π-cut-well-linked in G then X is $(\pi/(2\alpha))$-flow-well-linked in G.*

Proof: Let $\mathbf{d}$ be the following product multicommodity flow: $d(u,v) = \pi(u)\pi(v)/\pi(X)$ for each pair (u,v) of nodes in X, and $d(\cdot)$ is zero for all other pairs.

Suppose that X is π-flow-well-linked. Recall that, in order to show that X is $(\pi/2)$-cut-well-linked, it suffices to verify that, for each node separation (A,B,C) such that $\pi(C) = 0$, we have $|C| \geq \min\{\pi(A), \pi(B)\}/2$. Consider a node separation (A,B,C) such that $\pi(C) = 0$. Since X is π-flow-well-linked, $\mathbf{d}$ is routable and therefore $|C| \geq \mathrm{dem}_{\mathbf{d}}(C) = \pi(A)\pi(B)/\pi(X)$. Since $\pi(X) = \pi(A) + \pi(B)$, we have $\pi(A)\pi(B)/\pi(X) \geq \min\{\pi(A), \pi(B)\}/2$, as desired.

Conversely, suppose that X is π-cut-well-linked. By definition, X is $(\pi/(2\alpha))$-flow-well-linked if $\mathbf{d}/(2\alpha)$ is routable. Thus, in order to show that X is $(\pi/(2\alpha))$-flow-well-linked, it suffices to verify that each node separator has sparsity at least $1/2$.

Let C be a sparsest node separator. By Proposition 3.1, there is a node separation (A,B,C) such that $\mathrm{dem}_{\mathbf{d}}(C) \leq 2\min\{\pi(A), \pi(B)\}$. Since X is π-cut-well-linked, we have $|C| \geq \min\{\pi(A), \pi(B)\}$. Therefore the sparsity of C is at least $1/2$. □

Well-linked decomposition: The following theorem is an extension to directed graphs of the well-linked decomposition technique introduced by [11] for routing problems in undirected graphs. The proof follows the outline of the approach in [11] and it can be found in Section 4.

Theorem 3.3. *Let* OPT *be the value of a solution to the* symANF-LP *relaxation for a given instance $(G, \mathcal{M})$ of* SymANF. *Let $\alpha = \alpha(G) \geq 1$ be an upper bound on the worst case flow-cut gap for product multicommodity flows in G. There is a partition of G into node-disjoint induced subgraphs*

$G_1, G_2, \ldots, G_\ell$ and weight functions $\pi_i : V(G_i) \to \mathbb{R}_+$ with the following properties. Let $\mathcal{M}_i$ be the induced pairs of $\mathcal{M}$ in G_i and let X_i be the endpoints of the pairs in $\mathcal{M}_i$. We have

(a) $\pi_i(u) = \pi_i(v)$ for each pair $uv \in \mathcal{M}_i$.

(b) X_i is π_i-flow-well-linked in G_i.

(c) $\sum_{i=1}^{\ell} \pi_i(X_i) = \Omega(\mathrm{OPT}/(\alpha \log \mathrm{OPT})) = \Omega(\mathrm{OPT}/\log^2 k)$.

Moreover, such a partition is computable in polynomial time if there is a polynomial time algorithm for computing a node separator with sparsity at most $\alpha(G)$ times the maximum concurrent flow.

From fractional well-linked sets to well-linked sets: The following theorem describes an algorithm that obtains a well-linked set from a fractionally well-linked set. The proof is given in Section 5.

Theorem 3.4. *Let X be a π-flow-well-linked set in G and let $\mathcal{M}$ be a perfect matching on X such that $\pi(u) = \pi(v)$ for each pair $uv \in \mathcal{M}$. There is a matching $\mathcal{M}' \subseteq \mathcal{M}$ on a set $X' \subseteq X$ such that X' is $1/32$-flow-well-linked in G and $|\mathcal{M}'| = 2|X'| = \Omega(\pi(X))$. Moreover, given X and $\mathcal{M}$, we can construct X' and $\mathcal{M}'$ in polynomial time.*

Routing a flow-well-linked instance: Finally, we observe that, if an instance of SymANF is c-flow-well-linked for some $c \leq 1$, then we can route all of the pairs with congestion at most $2/c$.

Proposition 3.5. *Let $(G, \mathcal{M})$ be an instance of SymANF and let X be the set of all vertices that participate in the pairs of $\mathcal{M}$. If X is c-flow-well-linked for some $c \leq 1$, then we can route all of the pairs of $\mathcal{M}$ with congestion at most $2/c$.*

Proof: Note that it suffices to show that we can route c units of flow for each pair using congestion at most 2; once we have this flow, we can simply scale it by $1/c$ to get a flow that routes one unit of flow for each pair.

Let X_1 be a set consisting of exactly one node from each pair of $\mathcal{M}$, and let $X_2 = X - X_1$ be the set of all partners of the nodes in X_1. Let $\mathbf{d}$ be the following demand vector: $d(u, v) = c/|X|$ for each pair (u, v) of nodes in X, and $d(\cdot)$ is zero for all other pairs. Since X is c-flow-well-linked, there is a feasible flow f that routes $\mathbf{d}$. Note that f gives us a feasible flow in which each node in X_1 sends c units of flow to its partner: consider a node $u \in X_1$ and let v be its partner; we combine the flow paths of f connecting u to X and the flow paths of f connecting X to v in order to get flow paths from u to v carrying at least c units of flow. Similarly, f also gives us a feasible flow in which each node in X_2 sends c units of flow to its partner. The sum of the two flows gives us a congestion two flow that routes c units of flow for each pair of $\mathcal{M}$. □

3.3 The approximation algorithm for SymANF

We now describe our algorithm for SymANF. Let $(G, \mathcal{M})$ be an instance of SymANF. We consider a natural multicommodity flow relaxation for the problem. For each ordered pair (u, v) of nodes of G, let $\mathcal{P}(u, v)$ be the set of all paths in G from u to v. Since $\mathcal{M}$ forms a matching on $\mathcal{T}$, for all $i \neq j$, the sets $\mathcal{P}(s_i, t_i)$, $\mathcal{P}(t_i, s_i)$, $\mathcal{P}(s_j, t_j)$, and $\mathcal{P}(t_j, s_j)$ are pairwise disjoint. Let $\mathcal{P} = \bigcup_{i=1}^{k} (\mathcal{P}(s_i, t_i) \cup \mathcal{P}(t_i, s_i))$. For each path $p \in \mathcal{P}$, we have a variable $f(p)$ that is equal to the amount of flow on p. For each unordered pair $s_i t_i \in \mathcal{M}$ we have a variable x_i to indicate whether to route the pair or not. The LP relaxation ensures the symmetry constraint: there is a flow from s_i to t_i of value x_i *and* a flow from t_i to s_i of value x_i. Recall that we will be working with the node-capacitated problem and each node has unit capacity.

(symANF-LP)

$$\max \quad \sum_{i=1}^{k} x_i$$

$$\text{s.t.} \quad \sum_{p \in \mathcal{P}(s_i,t_i)} f(p) \geq x_i \qquad 1 \leq i \leq k$$

$$\sum_{p \in \mathcal{P}(t_i,s_i)} f(p) \geq x_i \qquad 1 \leq i \leq k$$

$$\sum_{p:\, v \in p} f(p) \leq 1 \qquad v \in V(G)$$

$$x_i \leq 1 \qquad 1 \leq i \leq k$$

$$f(p) \geq 0 \qquad p \in \mathcal{P}$$

The dual of the symANF-LP relaxation has polynomially many variables and exponentially many constraints. The separation oracle for the dual is the shortest path problem. Thus we can solve the relaxation in polynomial time. Alternatively, we can write an equivalent LP relaxation that is polynomial sized.

The algorithm is described below.

(1) Solve the relaxation symANF-LP to get an optimal fractional solution (x, f) for the instance $(G, \mathcal{M})$.

(2) Use the well-linked decomposition (Theorem 3.3) to get a collection $(G_1, \mathcal{M}_1, \pi_1), \ldots, (G_\ell, \mathcal{M}_\ell, \pi_\ell)$ of disjoint instances and weight functions.

(3) For each instance $(G_i, \mathcal{M}_i, \pi_i)$ in the decomposition, use the clustering technique (Theorem 3.4) to get an instance $(G_i, \mathcal{M}'_i)$.

(4) For each instance $(G_i, \mathcal{M}'_i)$, route all of the pairs of $\mathcal{M}'_i$ in G_i (Proposition 3.5). Output the union of these routings.

Let OPT be the value of the symANF-LP for the given instance, which lower bounds the number of pairs routed in an optimum solution. Combining Theorems 3.3 and 3.4 and Proposition 3.5, the number of pairs routed by the algorithm is $\sum_{i=1}^{\ell} |\mathcal{M}'_i| = \sum_{i=1}^{\ell} \Omega(\pi(V(\mathcal{M}_i))) = \Omega(\mathrm{OPT}/\log^2 k)$. Since each instance $(G_i, \mathcal{M}'_i)$ is 1/32-flow-well-linked, the routing in G_i has congestion at most 64. The graphs $G_1, \ldots, G_\ell$ are node disjoint and hence the pairs routed in these graphs do not interfere with each other. This completes the proof of Theorem 1.1.

4 Well-linked decomposition

In this section, we prove Theorem 3.3. We follow the notation and the approach introduced in [11] for edge and node-capacitated multicommodity flow problems in undirected graphs.

Let (x, f) be a solution to the symANF-LP with value $\mathrm{OPT} = \sum_{i=1}^{k} x_i$. The flow f is a symmetric multicommodity flow; as before, we view f as a path-based flow. Let H be a node-induced subgraph of G. For each ordered pair (u, v) of nodes in H, let $\gamma(u, v; H)$ be the total amount of f-flow on paths p from u to v that are completely contained in H. For each unordered pair uv of nodes in

Decomposition Algorithm

Input: Strongly connected subgraph H.
Output: Node-disjoint subgraphs $H_1, H_2, \ldots, H_\ell$ with associated weight functions $\pi_1, \pi_2, \ldots, \pi_\ell$, where each H_i is a node-induced subgraph of H.

(1) Suppose that $0 < \mathbf{w}(H) \leq \alpha \log \mathrm{OPT}$. Let $\pi(u) = w(u; H)/(8\alpha \log \mathrm{OPT})$ for each node $u \in V(H)$. Stop and output H and π.

(2) Suppose that $\mathbf{w}(H) > \alpha \log \mathrm{OPT}$. Let $\mathbf{d}$ be the following demand vector: $d(u, v) = w(u; H)w(v; H)/\mathbf{w}(H)$ for each ordered pair (u, v) of nodes in H. Let λ be the maximum concurrent flow for $\mathbf{d}$.

   (*a*) If $\lambda \geq 1/(8\alpha \log \mathrm{OPT})$, stop the recursive procedure. Let $\pi(u) = w(u; H)/(8\alpha \log \mathrm{OPT})$ for each node $u \in V(H)$. Output H and π.

   (*b*) Otherwise find a node separation (A, B, C) such that $|C| \leq \min\left\{\sum_{a \in A} w(a; H), \sum_{b \in B} w(b; H)\right\}/(4 \log \mathrm{OPT})$. Recursively decompose each strongly connected component of $H - C$. Output the decompositions of the strongly connected components.

H, let $\gamma'(u, v; H) = \gamma'(v, u; H) = \min\{\gamma(u, v; H), \gamma(v, u; H)\}$. For each node u in H, let $w(u; H) = \sum_{v \in V(H)} \gamma'(u, v; H)$. Let $\mathbf{w}(H) = \sum_{u \in V(H)} w(u; H)$.

We will need the following observation. Recall that a node separation in G is a partition (A, B, C) of the nodes of G such that there is no edge of G from A to B.

Proposition 4.1. *Let $G = (V, E)$ be a directed graph. Let $\pi : V \to [0, 1]$ be a weight function. Let $\alpha = \alpha(G) \geq 1$ be an upper bound on the worst case flow-cut gap for product multicommodity flows in G. Suppose that V is not π-flow-well-linked in G. There is a node separation (A, B, C) such that $|C| \leq 2\alpha \min\{\pi(A), \pi(B)\}$. Moreover, we can construct such a separation in polynomial time if there is a polynomial time algorithm for computing a node separator with sparsity at most α times the maximum concurrent flow.*

Proof: Note that it follows from Proposition 3.2 that, for any weight function $\pi : V \to [0, 1]$, either V is π-flow-well-linked or there is a node separation (A, B, C) such that $|C| \leq 2\alpha \min\{\pi(A), \pi(B)\}$. Additionally, we can construct such a separation in polynomial time as follows.

Let $\mathbf{d}$ be the following demand vector: $d(u, v) = \pi(u)\pi(v)/\pi(V)$ for each ordered pair (u, v) of nodes. Since $\mathbf{d}$ is not routable, we can compute in polynomial time a node separator C such that $|C| \leq \alpha \, \mathrm{dem}_{\mathbf{d}}(C)$. By Proposition 3.1, once we have C, we can compute in polynomial time a node separation (A, B, C) such that $\mathrm{dem}_{\mathbf{d}}(C) \leq 2 \min\{\pi(A), \pi(B)\}$. The resulting separation (A, B, C) satisfies $|C| \leq 2\alpha \min\{\pi(A), \pi(B)\}$. □

Note that, in the step (2*b*) of the algorithm, we used Proposition 4.1: let $\pi(u) = w(u; H)/(8\alpha \log \mathrm{OPT})$ for each node u in H; since $\lambda < 1/(8\alpha \log \mathrm{OPT})$, $V(H)$ is not π-flow-well-linked and therefore there is a node separation (A, B, C) such that

$$|C| \leq 2\alpha \min\{\pi(A), \pi(B)\} = \min\left\{\sum_{a \in A} w(a; H), \sum_{b \in B} w(b; H)\right\}/(4 \log \mathrm{OPT}).$$

We apply the decomposition algorithm to each strongly connected component of G in order to get a decomposition of G into node-induced disjoint subgraphs $G_1, G_2, \ldots, G_\ell$ with associated weight functions $\pi_1, \pi_2, \ldots, \pi_\ell$. In the following, we show that this decomposition has the properties required by Theorem 3.3. It is straightforward to verify that the decomposition has the first two properties and thus we focus on the third property.

From the terminating conditions, it follows that $\pi_i(G_i) \geq \mathbf{w}(G_i)/(8\alpha \log \mathrm{OPT})$ for each i. Therefore it suffices to show that $\sum_{i=1}^{k} \mathbf{w}(G_i) \geq \mathbf{w}(G)/2 = \mathrm{OPT}/2$. Equivalently, the total flow lost is at most $\mathbf{w}(G)/2$, where the flow lost is $\mathbf{w}(G) - \sum_{i=1}^{k} \mathbf{w}(G_i)$.

We upper bound the total flow lost as follows. We say that a node of G was cut in the decomposition if the node belongs to a node separator C found in the step $(2b)$. We first note that the total flow lost is at most twice the number of nodes that were cut by the decomposition. We can show this as follows. Let Z be the set of all nodes that are cut by the decomposition. Recall that, for each pair uv of nodes, the amount of f-flow from u to v is equal to the amount of f-flow from v to u; we think of the flow from u to v and the flow from v to u as partner flows. Now consider the f-flow that does not contribute to $\sum_{i=1}^{k} \mathbf{w}(G_i)$: this flow can be partitioned into flows, each of which is on paths that intersect Z or it is the partner of a flow whose paths intersect Z. Since each node has unit capacity, the total f-flow on paths that intersect Z is at most $|Z|$ and thus the total flow lost is at most $2|Z|$. Thus it suffices to show that $|Z|$ is at most $\mathbf{w}(G)/4$.

Lemma 4.2. *The number of nodes cut by the well-linked decomposition is at most* $\mathbf{w}(G)/4$.

Proof: We charge the cut nodes as follows. Consider an iteration of the decomposition algorithm that cuts a node. Let H denote the graph considered in the current iteration and let (A, B, C) be the node separation found in Step $(2b)$. Recall that we have

$$|C| \leq \frac{1}{4 \log \mathrm{OPT}} \min \left\{ \sum_{a \in A} w(a; H), \sum_{b \in B} w(b; H) \right\}.$$

We charge the nodes in C as follows. Let $D = A$ if $\sum_{a \in A} w(a; H) \leq \sum_{b \in B} w(b; H)$ and $D = B$ otherwise. We refer to D as the smaller side of the separation (A, B, C). For each node $u \in D$, we charge $w(u; H)/(4 \log \mathrm{OPT})$ to u. Note that the total charge to the nodes of D is $\sum_{u \in D} w(u; H)/(4 \log \mathrm{OPT}) \geq |C|$.

The total amount charged by the charging scheme is at least the number of nodes that are cut by the decomposition and thus it suffices to upper bound the total amount charged. We can show that the total amount charged is at most $\mathbf{w}(G)/4$ as follows. For each node u, we claim that u is charged at most $w(u; G)/4$. A node u is charged only if it is on the smaller side of the separation found in Step $(2b)$ and therefore it is charged at most $\log(\mathbf{w}(G)) = \log \mathrm{OPT}$ times. Additionally, each charge to u is at most $w(u; G)/(4 \log \mathrm{OPT})$. □

5 From fractional well-linked sets to well-linked sets

In this section, we prove Theorem 3.4. We prove the theorem in two steps. In the first step, we show that there exists a set Y of cardinality $\Omega(\pi(X))$ such that Y is $\Omega(1)$-flow-well-linked. Additionally, the set Y can send flow to X and receive flow from X. In the second step, we use Y to select a matching $\mathcal{M}' \subseteq \mathcal{M}$ of size $\Omega(|Y|)$.

Before we give the details of this procedure, we first give an intuitive (and non-constructive) argument that motivates the approach. The argument is partly inspired by the work in [12] and it

differs from the low-degree spanning tree clustering that has been the main approach in the undirected case. The reader can skip the following paragraph and go straight to the technical proof.

Intuitive argument: In the following, we give an informal argument that illustrates the clustering for fractionally cut-well-linked sets. Suppose that G has a set X that is π-cut-well-linked. Recall that the directed treewidth of G is within a constant factor of the largest cut-well-linked set in G; this approximate duality relation is shown via the notion of *havens* [25] (see Section A for the definition). Using a similar argument, one can show that, if X is π-cut-well-linked in G, the directed treewidth of G is $\Omega(\pi(X))$. By applying the approximate duality again, we get that there is a cut-well-linked Z in G of size $|Z| = \Omega(\pi(X))$. Since the existence of Z was shown via the π-cut-well-linkedness of X, it is intuitive that there is such a set Z that is reachable from X in the following sense: there is a (single commodity) flow from X to Z where each node in Z receives one unit of flow and each node v in X sends $\pi(v)$ units of flow; similarly there is a flow from Z to X. The existence of these flows together with the fact that X is π-cut-well-linked imply that Z is $\Omega(1)$-cut-well-linked. We then have to identify a subset $X' \subset X$ that is $\Omega(1)$-cut-well-linked. Moreover, for the SymANF problem, we need to ensure that for the initial matching $\mathcal{M}$ on X there is a sufficiently large sub-matching of $\mathcal{M}$ induced on X'. These latter arguments require an incremental flow-augmentation technique from [13]. The main technical challenge is to *efficiently* find a Z reachable from X as described above. Surprisingly, we are able to show that a simple greedy iterative approach based on the intuition of the existence argument, with a careful argument, works to give the desired set Z modulo constant congestion. We believe that this is a useful technical building block for further work in this area. Now we give the formal argument.

First step: Finding a large well-linked set. In the first step, we find a set Y with the following properties:

Theorem 5.1. *Let G be a directed graph with unit node capacities. Let X be a set of nodes of G and let $\pi : X \to (0, 1]$ be a weight function on X. Suppose that X is π-flow-well-linked in G. There is a polynomial time algorithm that constructs a set $Y \subseteq V(G)$ with the following properties.*

(P_1) $|Y| = \lfloor \pi(X)/8 \rfloor$.

(P_2) *Y is $1/4$-flow-well-linked in G.*

Additionally, for any subset $X' \subseteq X$ such that $\pi(X') \leq \pi(X)/15$, we have

(Q_1) *There is a single commodity flow in G from X' to Y such that each node $x \in X'$ sends $\pi(x)/64$ units of flow and each node in Y receives at most one unit of flow.*

(Q_2) *There is a single commodity flow in G from Y to X' such that each node $x \in X'$ receives $\pi(x)/64$ units of flow and each node in Y sends at most one unit of flow.*

The main ingredient in the proof of Theorem 5.1 is the following lemma. The lemma shows that, if we have a set X that is π-flow-well-linked, then there exists a set Y of size $\Omega(\pi(X))$ such that Y is $\Omega(1)$-flow-well-linked. The main idea behind the lemma is the following. If X is π-cut-well-linked and Z is a node separator of size less than $\pi(X)/4$, there is a unique strongly connected component $\beta(Z)$ of $G - Z$ whose π-weight is more than half the weight of X. The main insight is that, if we consider the set Y of size $\lfloor \pi(X)/4 \rfloor$ for which $|Y \cup \beta(Y)|$ is minimum, this gives us the desired set. This gives us a non-constructive proof of the existence of such a set Y. Using a simple iterative procedure, we can find such a set Y in polynomial time.

Lemma 5.2. *Let G be a directed graph with unit node capacities. Let X be a set of nodes of G and let $\pi : X \to (0, 1]$ be a weight function on X. Suppose that X is π-cut-well-linked in G. There is a polynomial time algorithm that constructs a set $Y \subseteq V(G)$ with the following properties.*

(R_1) $|Y| = \lfloor \pi(X)/4 \rfloor$.

(R_2) *There is a single commodity flow in G from X to Y such that each node $x \in X$ sends at most $\pi(x)$ units of flow and each node in Y receives one unit of flow.*

(R_3) *There is a single commodity flow in G from Y to X such that each node in Y sends one unit of flow and each node $x \in X$ receives at most $\pi(x)$ units of flow.*

We will need the following simple observation.

Proposition 5.3. *Let G be a directed graph. Let X be a set of nodes of G and let $\pi : X \to [0, 1]$ be a weight function on X. Suppose that X is π-cut-well-linked in G. Then for any set Z such that $|Z| < \pi(X)/4$, there is a unique strongly connected component $\beta(Z)$ of $G - Z$ such that $\pi(\beta(Z)) > \pi(X)/2$.*

Proof: Suppose for contradiction that there is a set Z such that $|Z| < \pi(X)/4$ and, for each strongly component H of $G - Z$, we have $\pi(H) \leq \pi(X)/2$. Let $H_1, H_2, \ldots, H_\ell$ be a topological ordering of the strongly connected components of $G - Z$ in which each edge of $G - Z$ that connects different strongly connected components is oriented from right to left. Let p be the smallest index such that $\pi(H_1 \cup \cdots \cup H_p) \geq \pi(X)/4$. Note that, since $\pi(H_1 \cup \cdots \cup H_{p-1}) < \pi(X)/4$ and $\pi(H_p) \leq \pi(X)/2$, we have $\pi(H_1 \cup \cdots \cup H_p) < 3\pi(X)/4$. Thus we have $\pi(H_{p+1} \cup \cdots \cup H_\ell) > \pi(X)/4$. Let A be the set of all vertices in $H_1 \cup \cdots \cup H_p$ and let B be the set of all vertices in $H_{p+1} \cup \cdots \cup H_\ell$. Note that (A, B, Z) is a node separation in G. Since X is π-cut-well-linked, it follows that $|Z| \geq \min\{\pi(A), \pi(B)\} \geq \pi(X)/4$, which is a contradiction. □

Proof of Lemma 5.2: We start by introducing some notation. If X is a π-cut-well-linked set in G, it follows from Proposition 5.3 that, for each set $Z \subseteq V(G)$ such that $|Z| < \pi(X)/4$, there is a unique strongly connected component $\beta(Z)$ of $G - Z$ such that $\pi(\beta(Z)) > \pi(X)/2$.

Let H_1 be the following network. We start with $H_1 = G$; recall that each node in G has a capacity of one. For each node $x \in X$, we add a node x' to H_1 and an edge from x' to x; the node x' receives a capacity of $\pi(x)$. We add a source node s and a directed edge from s to each node x'. We add a sink node t and an edge from each node in Y to t.

Let H_2 be the following network. We start with $H_2 = G$. We add a source node s and a directed edge from s to each node in Y. For each node $x \in X$, we add a node x' to H_2 and an edge from x to x'; the node x' receives a capacity of $\pi(x)$. We add a sink node t and a directed edge from each node x' to t.

We will maintain a set Y satisfying the first condition. If Y does not satisfy the second or the third condition, we show that we can find a set Y' satisfying the first condition such that $|Y' \cup \beta(Y')| < |Y \cup \beta(Y)|$. Initially, Y is an arbitrary subset of size $\lfloor \pi(X)/4 \rfloor$.

Suppose that Y does not satisfy the second condition. Consider the network H_1 and let X' be the set of all copies of the nodes in X. A triple (A, B, C) is an s-t separation in H_1 if the sets A, B, C partition $V(H_1)$, $s \in A$, $t \in B$, and there is no edge of H_1 from A to B. The capacity of a separation (A, B, C) is the capacity of the nodes in C. Let (A, B, C) be an s-t separation in H_1 with minimum capacity. Since Y does not satisfy the second condition, the capacity of C is smaller than $|Y|$. Let $A' = A - (X' \cup \{s\})$, $B' = B - (X' \cup \{t\})$, and $C' = C - X'$. Since t is in B and there is no edge of H_1 from A to B, we have $Y \subseteq B' \cup C'$.

In the following, we show that $\beta(C') \subseteq A' \cap \beta(Y)$. Since there is no edge of H_1 from A to B, for each node $x \in X \cap B$, we have $x' \in C$: if x' is in A, the edge from x' to x is connecting A to B; if x' is in B, the edge from s to x' is connecting A to B. Therefore $\mathrm{cap}(C) \geq \pi(B) = \pi(B')$ and thus $\pi(B') \leq \pi(X)/4$. Since $\beta(C')$ is a strongly connected component of $G - C'$ and there is no edge of G from A' to B', we have that $\beta(C')$ is completely contained in one of A' and B'. Since $\pi(\beta(C')) > \pi(X)/2$ and $\pi(B') \leq \pi(X)/2$, we have $\beta(C') \subseteq A'$. Since $\beta(C')$ is contained in A', $\beta(C')$ is a strongly connected subgraph of $G - (B' \cup C')$. Since $Y \subseteq B' \cup C'$, there is a unique strongly connected component K of $G - Y$ that contains $\beta(C')$. Since $\beta(C')$ and $\beta(Y)$ overlap at a vertex of X, we have $K = \beta(Y)$. Therefore $\beta(C') \subseteq \beta(Y)$ and thus $\beta(C') \subseteq A' \cap \beta(Y)$, as claimed.

Since $|C'| < |Y|$, we have $|C' \cup \beta(C')| = |C'| + |\beta(C')| < |Y| + |\beta(Y)| = |Y \cup \beta(Y)|$. We let Y' be the set consisting of C' together with an arbitrary subset of $\beta(C')$ of size $\lfloor \pi(X)/4 \rfloor - |C'|$. Then Y' is the desired set.

Therefore we may assume that Y does not satisfy the third condition. The argument is very similar to the previous case, and we include it for completeness. Consider the network H_2 and let X' be the set of all copies of the nodes in X. A triple (A, B, C) is an s-t separation in H_2 if the sets A, B, C partition $V(H_2)$, $s \in A$, $t \in B$, and there is no edge of H_2 from A to B. The capacity of a separation (A, B, C) is the capacity of the nodes in C. Let (A, B, C) be an s-t separation in H_2 with minimum capacity. Since Y does not satisfy the third condition, the capacity of C is smaller than $|Y|$. Let $A' = A - (X' \cup \{s\})$, $B' = B - (X' \cup \{t\})$, and $C' = C - X'$. Since s is in A there is no edge of H_2 from A to B, we have $Y \subseteq A' \cup C'$.

In the following, we show that $\beta(C') \subseteq B' \cap \beta(Y)$. Since there is no edge of H_2 from A to B, for each node $x \in X \cap A$, we have $x' \in C$: if x' is in A, the edge from x' to t is connecting A to B; if x' is in B, the edge from x to x' is connecting A to B. Therefore $\mathrm{cap}(C) \geq \pi(A) = \pi(A')$ and thus $\pi(A') \leq \pi(X)/4$. Since $\beta(C')$ is a strongly connected component of $G - C'$ and there is no edge of G from A' to B', we have that $\beta(C')$ is completely contained in one of A' and B'. Since $\pi(\beta(C')) > \pi(X)/2$ and $\pi(A') \leq \pi(X)/2$, we have $\beta(C') \subseteq B'$. Since $\beta(C')$ is contained in B', $\beta(C')$ is a strongly connected subgraph of $G - (A' \cup C')$. Since $Y \subseteq A' \cup C'$, there is a unique strongly connected component K of $G - Y$ that contains $\beta(C')$. Since $\beta(C')$ and $\beta(Y)$ overlap at a vertex in X, we have $K = \beta(Y)$. Therefore $\beta(C') \subseteq \beta(Y)$ and thus $\beta(C') \subseteq B' \cap \beta(Y)$. Since $|C'| < |Y|$, we have $|C' \cup \beta(C')| = |C'| + |\beta(C')| < |Y| + |\beta(Y)| = |Y \cup \beta(Y)|$. We let Y' be the set consisting of C' together with an arbitrary subset of $\beta(C')$ of size $\lfloor \pi(X)/4 \rfloor - |C'|$. Then Y' is the desired set. □

Lemma 5.4. *Let $G = (V, E)$ be a node-capacitated directed network. Let A and B be two sets of nodes in G. Let $\pi : A \to \mathbb{R}_+$ and $\pi' : B \to \mathbb{R}_+$ be two weight functions. Suppose that A and B satisfy the following conditions:*

- *A is π-flow-well-linked.*
- *There is a feasible single-commodity flow f_1 in G from B to A such that each node $b \in B$ sends $\pi'(b)$ units of flow to A and each node $a \in A$ receives at most $\pi(a)$ units of flow.*
- *There is a feasible single-commodity flow f_2 in G from A to B such that each node $a \in A$ sends at most $\pi(a)$ units of flow and each node $b \in B$ receives $\pi'(b)$ units of flow.*

Then B is $(\pi'/4)$-flow-well-linked in G.

Proof: Let $\mathbf{d}_1$ be the following multicommodity flow instance: $d_1(b, a) = \pi(a)\pi'(b)/\pi(A)$ for each pair $(b, a) \in B \times A$, and $\mathbf{d}_1(\cdot)$ is zero for all other pairs. We claim that we can route $\mathbf{d}_1$ using congestion at most two. In order to prove the claim, we combine the flow f_1 and the flow f that routes the following product multicommodity flow instance $\mathbf{d}$: $d(a, a') = \pi(a)\pi(a')/\pi(A)$ for all pairs of nodes

$(a, a') \in A \times A$. Let $F_1(b, a)$ be the amount of flow sent by f_1 from b to a. We split the flow of f_1 from b to a among the nodes of A as follows: for each node $a' \in A$, the amount of f_1-flow from b to a that we allocate to a' is $F_1(b, a)\pi(a')/\pi(A)$. We split the flow of f from a to a' among the nodes of B as follows: for each node $b \in B$, the amount of f-flow from a to a' that we allocate to b is $F_1(b, a)\pi(a')/\pi(A)$; since $\sum_b F_1(b, a) = \pi(a)\pi'(B)/\pi(A) \leq \pi(a)$, there is enough f-flow from a to a' to allocate to B. Finally, we concatenate the allocated flow paths as follows. Consider a node $b \in B$ and two nodes $a, a' \in A$. We allocated $F_1(b, a)\pi(a')/\pi(A)$ units of f_1-flow to a'; we can represent the allocated flow as a collection $\{(P_i, \epsilon_i)\}$, where P_i is a path from b to a and ϵ_i is the amount of f_1-flow on P_i that we allocated. We allocated $F_1(b, a)\pi(a')/\pi(A)$ units of f-flow to a'; we can represent the allocated flow as a collection $\{(Q_j, \delta_j)\}$, where Q_j is a path from a to a' and δ_j is the amount of f-flow on Q_j that we allocated. By making multiple copies of each path, we may assume that $\epsilon_i = \delta_j = \epsilon$ for all i and j; that is, all flow paths have the same amount ϵ of flow. For each i, we send ϵ units of flow on the path obtained by concatenating P_i and Q_i; more precisely, we replace the flow paths $\{(P_i, \epsilon)\}$ and $\{(Q_i, \epsilon)\}$ by the flow paths $\{(P_iQ_i, \epsilon)\}$. By concatenating all of the allocated flow paths, we get a flow with congestion at most two. For each pair $(b, a') \in B \times A$, the amount of flow from b to a' is $\sum_{a \in A} F_1(b, a)\pi(a')/\pi(A) = \pi'(b)\pi(a')/\pi(A) = d_1(b, a')$.

Let $\mathbf{d}_2$ be the following multicommodity flow instance: $d_2(a, b) = \pi(a)\pi'(b)/\pi(A)$ for each pair $(a, b) \in A \times B$, and $d_2(\cdot)$ is zero for all other pairs. By combining the flows f_2 and f, we can show that $\mathbf{d}_2$ is routable with congestion at most two; the argument is very similar to the previous argument and we omit it.

Let g_1 and g_2 be the congestion two flows that route $\mathbf{d}_1$ and $\mathbf{d}_2$, respectively. In the following, we show how to combine g_1 and g_2 to get a congestion four flow that routes the following product multicommodity flow instance $\mathbf{d}'$: $d'(b, b') = \pi'(b)\pi'(b')/\pi'(B)$ for each pair of nodes $(b, b') \in B \times B$. Consider a node $a \in A$ and two nodes $b_1, b_2 \in B$. The amount of g_1-flow from b_1 to a is $\pi(a)\pi'(b_1)/\pi(A)$; we allocate $\pi(a)\pi'(b_1)\pi'(b_2)/(\pi(A)\pi'(B))$ of this flow to b_2. The amount of g_2-flow from a to b_2 is $\pi(a)\pi'(b_2)/\pi(A)$; we allocate $\pi(a)\pi'(b_1)\pi'(b_2)/(\pi(A)\pi'(B))$ of this flow to b_1. By concatenating the allocated flow paths, we can send $\pi(a)\pi'(b_1)\pi'(b_2)(\pi(A)\pi'(B))$ units of flow from b_1 to b_2 through a; summing over all nodes $a \in A$, the total flow from b_1 to b_2 is $\pi'(b_1)\pi'(b_2)/\pi'(B)$. Therefore the $\mathbf{d}'$ is routable with congestion at most four. Thus B is $(\pi'/4)$-flow-well-linked. □

Now we are ready to prove Theorem 5.1.

Proof of Theorem 5.1: Since X is π-flow-well-linked in G, it follows from Proposition 3.2 that X is $(\pi/2)$-cut-well-linked in G. By Lemma 5.2, there is a set Y with the following properties.

- $|Y| = \lfloor \pi(X)/8 \rfloor$.
- There is a single commodity flow f_1 in G from X to Y such that each node $x \in X$ sends at most $\pi(x)/2$ units of flow and each node in Y receives one unit of flow.
- There is a single commodity flow f_2 in G from Y to X such that each node in Y sends one unit of flow and each node $x \in X$ receives at most $\pi(x)/2$ units of flow.

By Lemma 5.4, Y is 1/4-flow-well-linked; here we applied the lemma with $A = X$, $B = Y$, $\pi(x) = \pi(x)$ for each $x \in X$, and $\pi'(y) = 1$ for each $y \in Y$.

Let $X_1 \subseteq X$ be the set of all nodes $x \in X$ such that x sends at least $\pi(x)/32$ units of flow in f_1. We can show that $\pi(X_1) \geq \pi(X)/15$ as follows. For a set $A \subseteq X$, let $F_1(A)$ be the total amount of f_1-flow sent by the nodes in A. We have $F_1(X) = |Y| \geq \pi(X)/16$. Additionally, since each node $x \in X - X_1$ sends at most $\pi(x)/32$ units of flow in f_1, we have $F_1(X - X_1) \leq (\pi(X) - \pi(X_1))/32$.

It follows that $F_1(X_1) \geq (\pi(X) + \pi(X_1))/32$ and therefore $\pi(X_1)/2 \geq F_1(X_1) \geq (\pi(X) + \pi(X_1))/32$. Thus $\pi(X_1) \geq \pi(X)/15$.

Let $X_2 \subseteq X$ be the set of all nodes $x \in X$ such that x receives at least $\pi(x)/32$ units of flow in f_2. As before, we have $\pi(X_2) \geq \pi(X)/15$.

Now consider a subset $X' \subseteq X$ such that $\pi(X') \leq \pi(X)/15$. Note that $\pi(X') \leq \pi(X_1)$ and $\pi(X') \leq \pi(X_2)$. Consider the following multicommodity flow instance $\mathbf{d}$: $d(x', x) = \pi(x')\pi(x)/(32\pi(X_1))$ for each pair $(x', x) \in X' \times X_1$, $d(x, x') = \pi(x)\pi(x')/(32\pi(X_2))$ for each pair $(x, x') \in X_2 \times X'$, and $d(\cdot)$ is zero for all other pairs. Since $d(a, b) \leq \pi(a)\pi(b)/\pi(X)$ for all pairs of nodes (a, b), there is a feasible flow g that routes $\mathbf{d}$. The flow g satisfies the following properties:

- Each node $x \in X_1$ receives $\pi(x)\pi(X')/(32\pi(X_1)) \leq \pi(x)/32$ units of flow.
- Each node $x' \in X'$ sends $\pi(x')/32$ units of flow.
- Each node $x \in X_2$ sends $\pi(x)\pi(X')/(32\pi(X_2)) \leq \pi(x)/32$ units of flow.
- Each node $x' \in X'$ receives $\pi(x')/32$ units of flow.

By combining the flows f_1 and g, we get a congestion two flow from X' to Y in which each node in Y receives at most one unit of flow and each node $x' \in X'$ sends $\pi(x')/32$ units of flow. Similarly, by combining the flows f_2 and g, we get a congestion two flow from Y to X' in which each node in Y sends at most one unit of flow and each node $x' \in X'$ receives $\pi(x')/32$ units of flow. We scale down these flows by a factor of two to get feasible flows. □

Second step: Finding a matching. In the second step, we use the set Y guaranteed by Theorem 5.1 in order to select a matching $\mathcal{M}' \subseteq \mathcal{M}$.

We will need the following theorem, which is a slight variant of Theorem 2.1 in [13]. Let G be a directed graph with integer arc capacities given by c. Let $s_1, s_2, \ldots, s_k$ be distinct source nodes and let t be a sink node. A non-negative vector $\mathbf{b} = (b_1, b_2, \ldots, b_k)$ is a *feasible* flow vector if there is a feasible flow in G in which each source s_i sends b_i units of flow to t and t receives $\sum_{i=1}^{k} b_i$ units of flow. Let $\mathcal{B}$ be the set of all feasible flow vectors. For a vector $\mathbf{b} \in \mathcal{B}$, let $F(\mathbf{b}) = \sum_{i=1}^{k} b_i$ denote the total flow and let $I(\mathbf{b})$ be the set of all indices i such that b_i is an integer.

Theorem 5.5. *Given $\mathbf{b} \in \mathcal{B}$ and $j \notin I(\mathbf{b})$ with $b_j > 0$, we can compute $\mathbf{b}' \in \mathcal{B}$ in polynomial time with $b'_j = \lceil b_j \rceil$ and $F(\mathbf{b}') \geq F(\mathbf{b})$ such that*

- *$b'_i \leq b_i$ for each $i \in [k] - \{j\}$, and*
- *$b'_i = b_i$ for each $i \in I(\mathbf{b})$.*

The difference between Theorem 5.5 and Theorem 2.1 in [13] is that the former theorem requires that $b'_i \leq b_i$ for each terminal $i \neq j$, whereas the latter theorem requires that $b'_i \leq \lceil b_i \rceil$. One can prove the theorem above using essentially the same argument as in [13]; we include a proof in Section B for completeness.

Note that the flow augmentation theorem (Theorem 5.5) also applies to single-source networks and flows, since we can simply reverse the directions of all of the arcs. It also applies to node-capacitated routing using a standard reduction from node-capacitated directed networks to edge-capacitated directed networks.

Now we are ready to complete the proof of Theorem 3.4. Note that we may assume that $\pi(X) \geq c$, for some large enough constant c, since otherwise we can let $\mathcal{M}'$ be a single pair (u, v) of $\mathcal{M}$ that is

routable in G. In the following, we assume that $\pi(X) \geq 2048$. Additionally, we may assume that $\pi(x) > 0$ for each node $x \in X$, since we can discard from X all the nodes x such that $\pi(x) = 0$.

Using Theorem 5.5, we can identify a large matching whose terminals can send one unit of flow to Y and receive one unit of flow from Y.

Lemma 5.6. *There is a matching $\mathcal{M}' \subseteq \mathcal{M}$ with the following properties. Let X'_1 be a set of nodes containing exactly one node from each pair in $\mathcal{M}'$, and let $X'_2 = V(\mathcal{M}') - X'_1$ be the partners of the nodes in X'_1. We have*

(C_1) $|\mathcal{M}'| = \Omega(|Y|)$.

(C_2) *There is a feasible single-commodity flow in G in which each node in X'_1 sends one unit of flow to Y.*

(C_3) *There is a feasible single-commodity flow in G in which each node in X'_1 receives one unit of flow from Y.*

(C_4) *There is a feasible single-commodity flow in G in which each node in X'_2 sends one unit of flow to Y.*

(C_5) *There is a feasible single-commodity flow in G in which each node in X'_2 receives one unit of flow from Y.*

Proof: Let $\mathcal{M}''$ be any subset of $\mathcal{M}$ such that $\pi(X)/16 \leq \pi(X'') \leq \pi(X)/15$, where X'' is the set of nodes participating in the pairs of $\mathcal{M}''$. Note that such a set $\mathcal{M}''$ exists, since $\pi(X) \geq 240$ and $\pi(x) \leq 1$ for each node $x \in X$. Let X''_1 be a set of nodes containing exactly one node from each pair in $\mathcal{M}''$, and let $X''_2 = V(\mathcal{M}'') - X''_1$ be the partners of the nodes in X''_1.

In the following, we use the flow augmentation theorem (Theorem 5.5) to select a matching $\mathcal{M}' \subseteq \mathcal{M}''$ with the desired properties.

We make four copies of G; let G_1, G_2, G_3, and G_4 denote the four copies of G. For each $i \in \{1, 3\}$, we construct a node-capacitated single-sink network H_i from G_i as follows. We start with $H_i = G_i$ and we assign a capacity of one to each node. We add to H_i a sink node t_i and a directed edge from each node in Y to t_i. For each $i \in \{2, 4\}$, we construct a node-capacitated single-source network H_i from G_i as follows. We start with $H_i = G_i$ and we assign a capacity of one to each node. We add to H_i a source node s_i and a directed edge from s_i to each node in Y.

For each $i \in \{1, 2, 3, 4\}$, we maintain a feasible flow vector $\mathbf{b}_i$ in H_i. If $i \in \{1, 2\}$, $\mathbf{b}_i$ has an entry $\mathbf{b}_i(x)$ for each node $x \in X''_1$. If $i \in \{3, 4\}$, $\mathbf{b}_i$ has an entry $\mathbf{b}_i(x)$ for each node $x \in X''_2$.

We initialize the flow vectors $\mathbf{b}_i$ as follows. Let f_1 be the flow from X'' to Y guaranteed by property (Q_1) (see the statement of Theorem 5.1). Let f_2 be the flow from Y to X'' guaranteed by property (Q_2). Note that, for each $i \in \{1, 3\}$, f_1 translates to a flow in H_i from X'' to t_i; we let $\mathbf{b}_i(x)$ denote the amount of flow from x to t_i. Similarly, for each $i \in \{2, 4\}$, f_2 translates to a flow in H_i from s_i to X''; we let $\mathbf{b}_i(x)$ denote the amount of flow from s_i to x.

Our goal is to use the flow augmentation theorem (Theorem 5.5) in order to select a matching $\mathcal{M}' \subseteq \mathcal{M}''$. The main idea behind the approach is the following. If we have a pair $(u, v) \in \mathcal{M}''$ whose flow is fractional (that is, $\mathbf{b}_1(u) = \mathbf{b}_2(u) = \mathbf{b}_3(v) = \mathbf{b}_4(v) \in (0, 1)$), we use Theorem 5.5 in each copy G_i to increase the flow of u and v to 1. We repeatedly apply this procedure until the flow of each pair is either 0 or 1. The pairs with unit flow will give us the desired matching. We now describe this approach more formally.

If $\mathbf{b}$ is a flow vector on $Z \subseteq X''$, we let $F(\mathbf{b}) = \sum_{x \in Z} \mathbf{b}(x)$ be the total flow and $I(\mathbf{b})$ denote the set of all nodes $x \in Z$ such that $\mathbf{b}(x) = 1$. We will maintain flow vectors $\mathbf{b}_i$ that satisfy the following invariants:

(I_1) For each $u \in X''_1$, we have $\mathbf{b}_1(u) = \mathbf{b}_2(u) = \mathbf{b}_3(v) = \mathbf{b}_4(v)$, where v is the partner of u.

(I_2) For each $i \in \{1, 2, 3, 4\}$, we have $F(\mathbf{b}_i) \geq (|Y|/256) - 4|I(\mathbf{b}_1)|$.

We can verify that the initial flow vectors satisfy the invariants as follows. For each pair $uv \in \mathcal{M}$, we have $\mathbf{b}_1(u) = \mathbf{b}_2(u) = \pi(u)/64$ and $\mathbf{b}_3(v) = \mathbf{b}_4(v) = \pi(v)/64$. Since $\pi(a) = \pi(b)$ for each pair $ab \in \mathcal{M}$, the flow vectors satisfy the first invariant. Note that $I(\mathbf{b}_i)$ is empty, since $\mathbf{b}_i(x) = \pi(x)/64 < 1$ for each $x \in X''$. Moreover, we have $F(\mathbf{b}_i) = \pi(X''_1)/64 = \pi(X'')/128$. Since $\pi(X'') \geq \pi(X)/16$ and $|Y| = \lfloor \pi(X)/8 \rfloor$, we have $F(\mathbf{b}_i) = \pi(X'')/128 \geq \pi(X)/2048 \geq |Y|/256$. Thus the flow vectors satisfy the second invariant.

Now consider flow vectors $\mathbf{b}_i$ that satisfy the invariants (I_1) and (I_2) above. Suppose that, for each $u \in X''_1$, we have $\mathbf{b}_1(u) \in \{0, 1\}$. Let $X'_1 = I(\mathbf{b}_1)$ and $X'_2 = I(\mathbf{b}_3)$; by (I_1), X'_2 is the set of all partners of the nodes of X'_1. Let $\mathcal{M}'$ be the set of all pairs $uv \in \mathcal{M}''$ such that $u \in X'_1$ and $v \in X'_2$. We can verify that $\mathcal{M}'$ is the desired matching as follows. We have $|X'_1| = F(\mathbf{b}_1) \geq (|Y|/256) - 4|X'_1|$ and thus $|X'_1| \geq |Y|/1280$. Thus X'_1 and X'_2 satisfy the conditions (C_1)-(C_5) in the theorem statement and we are done.

Therefore we may assume that there is a node $u \in X''_1$ such that $\mathbf{b}_1(u) \in (0, 1)$. Let v be the partner of u. Recall that we have $\mathbf{b}_1(u) = \mathbf{b}_2(u) = \mathbf{b}_3(v) = \mathbf{b}_4(v)$. For each $i \in \{1, 2\}$, we apply Theorem 5.5 with $G = H_i$, $\mathbf{b} = \mathbf{b}_i$, and $b_j = u$ in order to get a feasible flow vector $\mathbf{b}'_i$ such that $I(\mathbf{b}'_i) \supseteq I(\mathbf{b}_i) \cup \{u\}$. For each $i \in \{3, 4\}$, we apply Theorem 5.5 with $G = H_i$, $\mathbf{b} = \mathbf{b}_i$, and $b_j = v$ in order to get a feasible flow vector $\mathbf{b}'_i$ such that $I(\mathbf{b}'_i) \supseteq I(\mathbf{b}_i) \cup \{v\}$. We construct flow vectors $\mathbf{b}''_i$ as follows. For each pair $zw \in \mathcal{M}''$, we let $\mathbf{b}''_1(z) = \mathbf{b}''_2(z) = \mathbf{b}''_3(w) = \mathbf{b}''_4(w) = \min\{\mathbf{b}'_1(z), \mathbf{b}'_2(z), \mathbf{b}'_3(w), \mathbf{b}'_4(w)\}$.

In the following, we show that the flows $\mathbf{b}''_i$ satisfy the second invariant. This follows from the properties guaranteed by Theorem 5.5. When we augment the flow of u to 1 in G_1 (or G_2), we decrease the total flow of all other pairs by at most 1. Similarly, when we augment the flow of v to 1 in G_3 (or G_4), we decrease the total flow of all other pairs by at most 1. Thus the total flow decrease in G_1, G_2, G_3, and G_4 is at most 4, and we charge this flow to the pair uv.

More formally, we claim that $F(\mathbf{b}''_i) \geq F(\mathbf{b}_i) - 4$ for each $i \in \{1, 2, 3, 4\}$. Consider an index $i \in \{1, 2\}$. Note that it suffices to show that $\sum_{z \in X''_1 - \{u\}} \Big(\mathbf{b}_i(z) - \mathbf{b}''_i(z)\Big) \leq 4$. We have

$$\begin{aligned}
&\sum_{z \in X''_1 - \{u\}} \Big(\mathbf{b}_i(z) - \mathbf{b}''_i(z)\Big) = \sum_{zw \in \mathcal{M}'' - \{uv\}} \Big(\mathbf{b}_i(z) - \min\big\{\mathbf{b}'_1(z), \mathbf{b}'_2(z), \mathbf{b}'_3(w), \mathbf{b}'_4(w)\big\}\Big) \\
&= \sum_{zw \in \mathcal{M}'' - \{uv\}} \max\big\{\mathbf{b}_1(z) - \mathbf{b}'_1(z), \mathbf{b}_2(z) - \mathbf{b}'_2(z), \mathbf{b}_3(w) - \mathbf{b}'_3(w), \mathbf{b}_4(w) - \mathbf{b}'_4(w)\big\} \\
&\qquad \Big(\text{Since } \mathbf{b}_1(z) = \mathbf{b}_2(z) = \mathbf{b}_3(w) = \mathbf{b}_4(w)\Big) \\
&\leq \sum_{zw \in \mathcal{M}'' - \{uv\}} (\mathbf{b}_1(z) - \mathbf{b}'_1(z) + \mathbf{b}_2(z) - \mathbf{b}'_2(z) + \mathbf{b}_3(w) - \mathbf{b}'_3(w) + \mathbf{b}_4(w) - \mathbf{b}'_4(w)) \\
&\qquad \Big(\text{The terms in the max are non-negative by the first bullet in Theorem 5.5}\Big) \\
&= \sum_{i=1}^{4} \Big(F(\mathbf{b}_i) - F(\mathbf{b}'_i)\Big) + \sum_{i=1}^{4} \Big(\mathbf{b}_i(u) - \mathbf{b}'_i(u)\Big) + \sum_{i=1}^{4} \Big(\mathbf{b}_i(v) - \mathbf{b}'_i(v)\Big) \\
&\leq 4
\end{aligned}$$

The last inequality follows from the fact that $F(\mathbf{b}_i) \leq F(\mathbf{b}'_i)$ for all $i \in \{1,2,3,4\}$, $\mathbf{b}_i(u) - \mathbf{b}'_i(u)$ is at most 0 if $i \in \{1,2\}$ and at most 1 if $i \in \{3,4\}$, $\mathbf{b}_i(v) - \mathbf{b}'_i(v)$ is at most 1 if $i \in \{1,2\}$ and at most 0 if $i \in \{3,4\}$.

A very similar argument shows that, for each $i \in \{3,4\}$, we have $F(\mathbf{b}''_i) \geq F(\mathbf{b}_i) - 4$. Thus we have $F(\mathbf{b}''_i) \geq F(\mathbf{b}_i) - 4 \geq (|Y|/256) - 4(|I(\mathbf{b}_1)| + 1)$ for each $i \in \{1,2,3,4\}$. Since $|I(\mathbf{b}''_1)| \geq |I(\mathbf{b}_1)| + 1$, we have $F(\mathbf{b}''_i) \geq (|Y|/256) - 4|I(\mathbf{b}''_1)|$ and thus the second invariant is also satisfied.

We repeatedly apply the flow augmentation procedure until the flow of each pair is either zero or one. The pairs with unit flow will give us the desired matching $\mathcal{M}'$. □

Let $\mathcal{M}'$ be the set of pairs guaranteed by Lemma 5.6 and let X' be the set of terminals participating in the pairs of $\mathcal{M}'$. We can show that X' is 1/32-flow-well-linked as follows. Note that the properties $(C_2) - (C_5)$ gives us the following flows: a congestion two flow from X' to Y in which each node in X' sends one unit of flow and each node in Y receives at most two units of flow, and a congestion two flow from Y to X' in which each node in Y sends at most two units of flow and each node in X' receives one unit of flow. We scale these flows by a factor of 8 to ensure that each node in Y sends and receives at most 1/4 units of flow. Since Y is 1/4-flow-well-linked, it follows from Lemma 5.4 that X' is 1/32-flow-well-linked; here we applied the lemma with $A = Y$, $\pi(y) = 1/4$ for each $y \in Y$, $B = X'$, $\pi'(x') = 1/8$ for each $x' \in X'$. This completes the proof of Theorem 3.4.

References

[1] I. Adler. Directed tree-width examples. *Journal of Combinatorial Theory, Series B*, 97(5):718 – 725, 2007.

[2] A. Agarwal, M. Charikar, K. Makarychev, and Y. Makarychev. $O(\sqrt{\log n})$ approximation algorithms for min UnCut, min 2CNF deletion, and directed cut problems. In *Proc. of ACM STOC*, pages 573–581, 2005.

[3] M. Andrews, J. Chuzhoy, V. Guruswami, S. Khanna, K. Talwar, and L. Zhang. Inapproximability of edge-disjoint paths and low congestion routing on undirected graphs. *Combinatorica*, 30(5):485–520, 2010.

[4] M. Andrews, J. Chuzhoy, S. Khanna, and L. Zhang. Hardness of the undirected edge-disjoint paths problem with congestion. In *Proc. of IEEE FOCS*, pages 226–241, 2005.

[5] D. Berwanger, A. Dawar, P. Hunte, S. Kreutzer, and J. Obdržálek. The dag-width of directed graphs. *Journal of Combinatorial Theory, Series B*, 102(4):900 – 923, 2012.

[6] C. Chekuri and J. Chuzhoy. Large-treewidth graph decompositions and applications. In *Proc. of ACM STOC*, 2013.

[7] C. Chekuri and J. Chuzhoy. Polynomial bounds for the grid-minor theorem. In *Proc. of ACM STOC*, 2014.

[8] C. Chekuri and A. Ene. Poly-logarithmic approximation for maximum node disjoint paths with constant congestion. In *Proc. of ACM-SIAM SODA*, 2013.

[9] C. Chekuri, S. Kannan, A. Raja, and P. Viswanath. Multicommodity flows and cuts in polymatroidal networks. In *Proc. of ITCS*, pages 399–408, 2012.

[10] C. Chekuri, S. Khanna, and F. B. Shepherd. The all-or-nothing multicommodity flow problem. *SIAM Journal on Computing*, 42(4):1467–1493, 2013.

[11] C. Chekuri, S. Khanna, and F.B. Shepherd. Multicommodity flow, well-linked terminals, and routing problems. In *Proc. of ACM STOC*, pages 183–192, 2005.

[12] C. Chekuri, S. Khanna, and F.B. Shepherd. Well-linked terminals for node-capacitated routing problems. Manuscript, 2005.

[13] C. Chekuri, S. Khanna, and F.B. Shepherd. An $O(\sqrt{n})$ approximation and integrality gap for disjoint paths and unsplittable flow. *Theory of Computing*, 2(7):137–146, 2006.

[14] C. Chekuri, M. Mydlarz, and F.B. Shepherd. Multicommodity demand flow in a tree and packing integer programs. *ACM Transactions on Algorithms*, 3(3):27, 2007.

[15] J. Chuzhoy. Routing in undirected graphs with constant congestion. *ArXiv preprint ArXiv:1107.2554*, 2011. Extended abstract in STOC 2012.

[16] J. Chuzhoy, V. Guruswami, S. Khanna, and K. Talwar. Hardness of routing with congestion in directed graphs. In *Proc. of ACM STOC*, pages 165–178, 2007.

[17] J. Chuzhoy and S. Khanna. Polynomial flow-cut gaps and hardness of directed cut problems. *Journal of the ACM*, 56(2):6, 2009.

[18] J. Chuzhoy and S. Li. A polylogarithimic approximation algorithm for edge-disjoint paths with congestion 2. In *Proc. of IEEE FOCS*, 2012.

[19] E. Demaine and M. T. Hajiaghayi. Linearity of grid minors in treewidth with applications through bidimensionality. *Combinatorica*, 28:19–36, 2008.

[20] S. Even, A. Itai, and A. Shamir. On the complexity of timetable and multicommodity flow problems. *SIAM Journal on Computing*, 5(4):691–703, 1976.

[21] U. Feige, M.T. Hajiaghayi, and J.R. Lee. Improved approximation algorithms for minimum weight vertex separators. *SIAM Journal on Computing*, 38:629–657, 2008.

[22] S. Fortune, J. Hopcroft, and J. Wyllie. The directed subgraph homeomorphism problem. *Theoretical Computer Science*, 10(2):111–121, 1980.

[23] N. Garg, V.V. Vazirani, and M. Yannakakis. Primal-dual approximation algorithms for integral flow and multicut in trees. *Algorithmica*, 18(1):3–20, 1997.

[24] Paul Hunter and Stephan Kreutzer. Digraph measures: Kelly decompositions, games, and orderings. *Theoretical Computer Science*, 399(3):206 – 219, 2008.

[25] T. Johnson, N. Robertson, P. D. Seymour, and R. Thomas. Directed tree-width. *Journal of Combinatorial Theory, Series B*, 82(1):138–154, 2001.

[26] T. Johnson, N. Robertson, P. D. Seymour, and R. Thomas. Excluding a grid minor in digraphs. Manuscript, 2001.

[27] R. M. Karp. Reducibility among combinatorial problems. *Complexity of Computer Computations*, pages 85–103, 1972.

[28] K. Kawarabayashi and S. Kreutzer. An excluded grid theorem for digraphs with forbidden minors. In *Proc. of ACM-SIAM SODA*, 2014.

[29] S. Kintali, N. Kothari, and A. Kumar. Approximation algorithms for directed width parameters. *ArXiv preprint ArXiv:1107.4824*, 2011.

[30] P. N. Klein, S. A. Plotkin, and S. Rao. Excluded minors, network decomposition, and multicommodity flow. In *Proc. of ACM STOC*, pages 682–690, 1993.

[31] P. N. Klein, S. A. Plotkin, S. Rao, and E. Tardos. Approximation algorithms for Steiner and directed multicuts. *Journal of Algorithms*, 22(2):241–269, 1997.

[32] T. Leighton and S. Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *Journal of the ACM*, 46(6):787–832, 1999.

[33] N. Linial, E. London, and Y. Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, 1995.

[34] B. Reed. Introducing directed tree width. *Electronic Notes in Discrete Mathematics*, 3:222–229, 1999.

[35] N. Robertson and P. D. Seymour. Graph minors V. Excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 41(1):92–114, 1986.

[36] N. Robertson, P. D. Seymour, and R. Thomas. Quickly excluding a planar graph. *Journal of Combinatorial Theory, Series B*, 62(2):323–348, 1994.

[37] A. Srinivasan. Improved approximations for edge-disjoint paths, unsplittable flow, and related routing problems. In *Proc. of IEEE FOCS*, pages 416–425, 1997.

A Background on directed treewidth

We define the notions of directed treewidth and havens that were introduced in [25].

Directed treewidth. We start by defining *arboreal* decompositions, which are directed analogues of *tree* decompositions.

An *arborescence* is a directed graph R with a vertex r_0 called the root such that, for each vertex $r \in V(R)$, there is a unique directed walk from r_0 to r in R. If r and r' are two nodes in $V(R)$, we write $r > r'$ if $r \neq r'$ and there is a directed walk in R from r to r'. If $e \in E(R)$, we write $r' > e$ if either $r' = r$ or $r' > r$, where r is the head of e. If e is incident to r, we write $e \sim r$.

Let D be a directed graph and let $Z \subseteq V(D)$. Let $D - Z$ denote the graph obtained from D by removing the vertices in Z. We say that a set $S \subseteq V(D) - Z$ is *Z-normal* if there is no directed walk in $D - Z$ with first and last vertex in S that uses a vertex of $D - (S \cup Z)$. One can show that a set S is Z-normal if and only if the vertex sets of the strongly connected components of $D - Z$ can be numbered $S_1, S_2, \ldots, S_d$ such that

(a) if $1 \leq i < j \leq d$, then no edge of D has its head in S_i and its tail in S_j, and

(b) either $S = \emptyset$ or $S = S_i \cup S_{i+1} \cup \cdots \cup S_j$ for some indices i and j such that $1 \leq i \leq j \leq d$.

An *arboreal decomposition* of a directed graph D is a triple (R, X, W), where R is an arborescence, and $X = (X_e \colon e \in E(R))$ and $W = (W_r \colon r \in V(R))$ are sets of vertices of D that satisfy

(D_1) $(W_r \colon r \in V(R))$ is a partition of $V(D)$ into non-empty sets, and

(D_2) if $e \in E(R)$, then $\bigcup \{W_r \colon r \in V(R), r > e\}$ is X_e-normal.

The *width* of (R, X, W) is the least integer w such that for all $r \in V(R)$, $|W_r \cup \bigcup_{e \sim r} X_e| \leq w+1$. The directed treewidth of D is the least integer w such that D has an arboreal decomposition of width w.

Havens. Let D be a directed graph and let w be an integer. A *haven* of order w in D is a function β assigning to each set $Z \subseteq V(D)$ with $|Z| < w$ the vertex set of a strongly connected component of $D - Z$ in such a way that if $Z' \subseteq Z \subseteq V(D)$ then $\beta(Z) \subseteq \beta(Z')$. The following lemma relates directed treewidth to havens and provides an approximate duality relationship.

Lemma A.1 ([25])**.** *Let D be a directed graph and let w be an integer. If D has a haven of order w then its directed treewidth is at least $w - 1$. Additionally, either D has directed treewidth at most $3w - 2$ or it has a haven of order w.*

One can relate well-linked sets to havens (see Proposition 5.3) which can be used to derive a relationship between directed treewdith and well-linked sets.

B Proof of Theorem 5.5

As we have already mentioned in Section 5, Theorem 5.5 is a slight variant of Theorem 2.1 in [13]. The difference between Theorem 5.5 and Theorem 2.1 in [13] is that the former theorem requires that $b'_i \leq b_i$ for each terminal $i \neq j$, whereas the latter theorem requires that $b'_i \leq \lceil b_i \rceil$. One can prove the theorem above using essentially the same argument as in [13]; we include a proof below for completeness.

Proof of Theorem 5.5: Recall that $G = (V, A)$ is an edge-capacitated network with arc capacities given by c. In the following, we assume for simplicity that, for each pair uv of nodes, at most one of the arcs $u \to v$, $v \to u$ is in A; if this is not the case, we simply subdivide one of the arcs.

Since $\mathbf{b}$ is feasible, there is a flow f that routes $\mathbf{b}$. We construct a residual graph G_f from G and f in a standard way. More precisely, we define a residual capacity function $c_f : V \times V \to \mathbb{R}_+$ as follows. For each pair (u, v), we have

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in A \\ f(v \to u) & \text{if } v \to u \in A \\ 0 & \text{otherwise} \end{cases}$$

The residual graph G_f has the same vertex set as G and the following arc set. For each pair (u, v) such that $c_f(u \to v) > 0$, we add the arc $u \to v$ to G_f and we assign capacity $c_f(u \to v)$ to it.

Suppose that G_f has a directed path p that starts at s_j and it ends at either t or a terminal s_h with $h \notin I(\mathbf{b}) \cup \{j\}$. Consider the flow f' in G obtained from f by augmenting along p. More precisely, f' is the following flow in G. Let $\delta = \min_{u \to v \in p} c_f(u \to v)$ be the minimum residual capacity on p. For each pair (u, v), we have:

$$f'(u \to v) = \begin{cases} f(u \to v) + \delta & \text{if } u \to v \in p \\ f(u \to v) - \delta & \text{if } v \to u \in p \\ f(u \to v) & \text{otherwise} \end{cases}$$

Note that f' is a feasible flow in G. For each vertex u, let $\Delta_f(u) = \sum_{u \to v \in A} f(u \to v) - \sum_{v \to u \in A} f(v \to u)$ be the net f-flow from u to t. We define $\Delta_{f'}(u)$ analogously. We claim that, for each vertex u, we have

$$\Delta_{f'}(u) = \begin{cases} \Delta_f(u) + \delta & \text{if } u \text{ is the start node of } p \\ \Delta_f(u) - \delta & \text{if } u \text{ is the end node of } p \\ \Delta_f(u) & \text{otherwise} \end{cases}$$

Suppose that u is the start node of p. Then p contains only one arc $u \to z$ that is incident to u. For each arc $u \to v \in A$, we have

$$f'(u \to v) = \begin{cases} f(u \to v) + \delta & \text{if } v = z \\ f(u \to v) & \text{otherwise} \end{cases}$$

For each arc $v \to u \in A$, we have

$$f'(v \to u) = \begin{cases} f(v \to u) - \delta & \text{if } v = z \\ f(v \to u) & \text{otherwise} \end{cases}$$

Thus we have

$$\begin{aligned} \Delta_{f'}(u) &= \sum_{u \to v \in A} f'(u \to v) - \sum_{v \to u \in A} f'(v \to u) \\ &= \Delta_f(u) + \delta \cdot (|\{u \to z, z \to u\} \cap A|) \\ &= \Delta_f(u) + \delta \end{aligned}$$

In the last line we have used the fact that, since $c_f(u \to z) > 0$, one of the arcs $u \to z$ or $z \to u$ is present in G. Additionally, it follows from our assumption on G that only one of $u \to z$ and $z \to u$ is present in G.

Suppose that u is the end vertex of p. Then p contains only one arc $w \to u$ that is incident to u. For each arc $u \to v \in A$, we have

$$f'(u \to v) = \begin{cases} f(u \to v) - \delta & \text{if } v = w \\ f(u \to v) & \text{otherwise} \end{cases}$$

For each arc $v \to u \in A$, we have

$$f'(v \to u) = \begin{cases} f(v \to u) + \delta & \text{if } v = w \\ f(v \to u) & \text{otherwise} \end{cases}$$

Thus we have

$$\begin{aligned} \Delta_{f'}(u) &= \sum_{u \to v \in A} f'(u \to v) - \sum_{v \to u \in A} f'(v \to u) \\ &= \Delta_f(u) - \delta \cdot (|\{u \to w, w \to u\} \cap A|) \\ &= \Delta_f(u) - \delta \end{aligned}$$

As before, since $c_f(w \to u) > 0$, one of the arcs $u \to w$ and $w \to u$ is present in G. Additionally, by our assumption on G, only one of the arcs $u \to w$ and $w \to u$ is present in G.

Suppose that u is an intermediate node on p. Then p contains two arcs $u \to z$ and $w \to u$ that are incident to u. For each arc $u \to v \in A$, we have

$$f'(u \to v) = \begin{cases} f(u \to v) + \delta & \text{if } v = z \\ f(u \to v) - \delta & \text{if } v = w \\ f(u \to v) & \text{otherwise} \end{cases}$$

For each arc $v \to u \in A$, we have

$$f'(v \to u) = \begin{cases} f(v \to u) + \delta & \text{if } v = w \\ f(v \to u) - \delta & \text{if } v = z \\ f(v \to u) & \text{otherwise} \end{cases}$$

Thus we have

$$\begin{aligned} \Delta_{f'}(u) &= \sum_{u \to v \in A} f'(u \to v) - \sum_{v \to u \in A} f'(v \to u) \\ &= \Delta_f(u) + \delta \cdot (|\{u \to z, z \to u\} \cap A|) - \delta \cdot (|\{u \to w, w \to u\} \cap A|) \\ &= \Delta_f(u) \end{aligned}$$

As before, since $c_f(u \to z) > 0$, one of the arcs $u \to z$ and $z \to u$ is present in G. Additionally, by our assumption on G, only one of the arcs $u \to z$ and $z \to u$ is present in G. Similarly, exactly one of the arcs $u \to w$ and $w \to u$ is present in G. Finally, if u does not appear on p, we have $\Delta_{f'}(u) = \Delta_f(u)$.

For each terminal s_ℓ, we have $b_\ell = \Delta_f(s_\ell)$. Let $b'_\ell = \Delta_{f'}(s_\ell)$. Since p starts at s_j, we have $b'_j = b_j + \delta > b_j$ and $b'_\ell \leq b_\ell$ for all $\ell \neq j$. Since the end vertex on p is not in $I(\mathbf{b}) \cup t$, $b'_\ell = b_\ell$ for each $\ell \in I(\mathbf{b})$. Finally, $F(\mathbf{b}') \geq F(\mathbf{b})$.

Thus, while there exists an augmenting path from s_j to either t or a terminal s_h with $h \notin I(\mathbf{b}) \cup \{j\}$, we can augment the flow along p. The augmentation will increase the flow of s_j while maintaining the properties in the theorem statement. Therefore it suffices to show that, as long as b_j is fractional, there is always such an augmenting path p.

Suppose for contradiction that b_j is fractional but there does not exist an augmenting path from s_j to either t or a terminal s_h with $h \notin I(\mathbf{b}) \cup \{j\}$. Let S be the set of all nodes reachable from s_j in the residual graph G_f. It follows that $t \notin S$ and, for each terminal s_h with $h \notin I(\mathbf{b}) \cup \{j\}$, $s_h \notin S$. Since no arcs leave S in G_f, for each arc $a \in \delta_G^-(S)$ of G that enters S, we have $f(a) = 0$. Additionally, for each arc $a \in \delta_G^+(S)$ of G that leaves S, we have $f(a) = c(a)$. It follows that $\sum_{a \in \delta_G^+(S)} c(a) = \sum_{i \in S} b_i$. Note that the only terminals in S are s_j and some terminals from $I(\mathbf{b})$. Therefore $\sum_{i \in S} b_i$ cannot be integral, since s_j is fractional and all other terms in the sum are integral. But this contradicts the fact that the sum $\sum_{a \in \delta_G^+(S)} c(a)$ is integral.

Hence there always exists an augmenting path with the desired properties and the theorem follows. □