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MONOTONICITY OF THE VALUE FUNCTION FOR A TWO-DIMENSIONAL OPTIMAL STOPPING PROBLEM

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We consider a pair (X, Y) of stochastic processes satisfying the equation $dX = a(X)Y\,dB$ driven by a Brownian motion and study the monotonicity and continuity in y of the value function $v(x, y) = \sup_\tau E_{x,y}[e^{-q\tau} g(X_\tau)]$, where the supremum is taken over stopping times with respect to the filtration generated by (X, Y). Our results can successfully be applied to pricing American options where X is the discounted price of an asset while Y is given by a stochastic volatility model such as those proposed by Heston or Hull and White. The main method of proof is based on time-change and coupling.

1. Introduction. Consider a two-dimensional strong Markov process $(X, Y) = (X_t, Y_t, t \geq 0)$ with state space $\mathbb{R} \times \mathcal{S}$, $\mathcal{S} \subseteq (0, \infty)$, given on a family of probability spaces $(\Omega, \mathcal{F}, P_{x,y}, (x, y) \in \mathbb{R} \times \mathcal{S})$ which satisfies the stochastic differential equation

$$dX = a(X)Y\,dB, \tag{1.1}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, and $a : \mathbb{R} \to \mathbb{R}$ is a measurable function.

Processes of this type are common in mathematical finance, and in this context, X would be the discounted price of an asset while Y is a process giving the so-called stochastic volatility.

We shall refer to this application in the examples, as it was our motivation in the beginning. However, the methods used are of a broader nature and can be applied in a wider context.

This paper mainly deals with the regularity of the value function

$$v(x, y) = \sup_{0\leq\tau\leq T} E_{x,y}\big[e^{-q\tau} g(X_\tau)\big], \qquad (x, y) \in \mathbb{R} \times \mathcal{S}, \tag{1.2}$$

with respect to the optimal stopping problem given by (X, Y), a discount rate $q > 0$, a time horizon $T \in [0, \infty]$ and a measurable gain function $g : \mathbb{R} \to \mathbb{R}$. But for financial applications (see Section 5), a slightly modified value function of type

$$v(x, y) = \sup_{0\leq\tau\leq T} E_{x,y}\big[e^{-r\tau} g\big(e^{r\tau} X_\tau\big)\big] \tag{1.2$'$}$$

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is also considered where r stands for the instantaneous interest rate.

The supremum in (1.2) and (1.2′) is taken over all *finite* stopping times with respect to the filtration generated by the pair of processes (X, Y).

To ensure the well-posedness of this problem, we assume the integrability condition (recall that T may be infinite)

$$E_{x,y}\Big[\sup_{0\le t\le T} e^{-qt}|g(X_t)|I(t<\infty)\Big]<\infty \qquad \text{for all } (x,y)\in\mathbb{R}\times\mathcal{S}, \tag{1.3}$$

which is a common assumption in the context of optimal stopping problems.

Note that this condition is satisfied if g is bounded. For more general functions, verifying this condition can be fairly difficult, and its validity may depend on the particular choice of the dynamics for (X, Y).

Our main focus is on proving the monotonicity of $v(x, y)$ with respect to $y \in \mathcal{S}$, and we are able to verify this property in the case of the following two classes of strong Markov processes under not too restrictive conditions (see Theorems 2.5 and 3.5):

- *Regime-switching*: Y is a skip-free continuous-time Markov chain (see page 1561) which is independent of the Brownian motion B driving equation (1.1).
- *Diffusion*: Y solves a stochastic differential equation of the type

$$dY = \eta(Y)\,dB^Y + \theta(Y)\,dt, \tag{1.4}$$

  where $B^Y = (B_t^Y)_{t\ge 0}$ is a standard Brownian motion such that the quadratic covariation satisfies $\langle B, B^Y\rangle_t = \delta t, t \ge 0$, for some real parameter $\delta \in [-1, 1]$ and $\eta, \theta : \mathbb{R} \to \mathbb{R}$ are measurable functions.

Note that, in the second class, the joint distribution of X and Y is uniquely determined if the system of equations (1.1), (1.4) admits a weakly unique solution, and the process Y does not have to be independent of the driving Brownian motion B, whereas, in the case of the first class, the process Y is not given by an equation, and the assumed independence of Y and B is a natural way of linking X and Y if there is too little information about the structure of the pair (X, Y).

Our technique is based on time-change and coupling. Equation (1.1) goes back to a volatility model used by Hobson in [6] who also applies time-change and coupling but for comparing prices of European options. As far as we know, our paper is the first paper dealing with the extra difficulty of applying this technique in the context of optimal stopping. It should be mentioned that Ekström [3], Theorem 4.2, can compare prices of American options if $Y \equiv 1$ in equation (1.1) and a also depends on time. Nevertheless, it seems to be that his method cannot be applied in the case of nontrivial processes Y.

We provide some examples to illustrate the results. In the case of regime-switching, we look at the pricing of perpetual American put options which, for $a(x) = x$, was studied by Guo and Zhang [4] for a two-state Markov chain and by Jobert and Rogers [9] for a finite-state Markov chain. While the former, since the situation is much easier, gave a closed-form expression for the price, the latter could only provide a numerical algorithm to approximate the value function which gives the price of the contract. It turns out that the algorithm in the case of a chain with many states can be very time-intensive if the unknown *thresholds* which characterize the optimal stopping rule are not known to be in a specific order when labeled by the different volatility states *before* the algorithm starts. However, based on our result that the value function $v(x, y)$ is monotone in y, we are now able to give conditions under which these *thresholds* must be in a monotone order.

Ultimately, in the case where Y is a diffusion, we verify the continuity and monotonicity of the value function $v(x, y)$ with respect to $y \in \mathcal{S} = (0, \infty)$ for two important volatility models, the Heston [5] and the Hull and White [7] model. Note that, using entirely different methods, differentiability and monotonicity in the volatility parameter of European option prices under the Hull and White model were studied in [1, 12]. The authors of [12] also showed a connection between the monotonicity in the volatility parameter and the ability of an option to complete the market. Another motivation to study the monotonicity of the value function in the volatility parameter y is that the numerical solution of the corresponding free-boundary problem becomes a lot easier if we know that the continuation region is monotonic in y and if we know that the corresponding free-boundary is continuous. Moreover, we will show, in a sequel, under the assumption of continuity, how to solve a game-theoretic version of the American put problem corresponding to model uncertainty for the stochastic volatility.

The structure of this paper is as follows. In Section 2 the monotonicity of the value function $v(x, y)$ with respect to $y \in \mathcal{S} = \{y_i : i = 1, 2, \dots, m\} \subseteq (0, \infty)$ is shown in the case of regime-switching, and the main method is established. In Section 3 the main method is adapted to the case of a system of stochastic differential equations (1.1), (1.4) which is the diffusion case, while in Section 4 we use monotonicity to show the continuity of the value function $v(x, y)$ with respect to $y \in \mathcal{S} = (0, \infty)$ in the diffusion case. In Section 5 we reformulate our results in the context of option pricing. Then all our examples are discussed in detail in Section 6 and, in the Appendix, we prove auxiliary results and some of the corollaries.

Finally, it should be mentioned that all our results and proofs would not change in principle if the state space of (X, Y) is $\mathbb{R} \times \mathcal{S}$ with $\mathcal{S} \subseteq (-\infty, 0)$ instead of $\mathcal{S} \subseteq (0, \infty)$. The only change in this case [see Corollary 2.7(ii)] would be to *order*: *increasing* becomes *decreasing*. However, as pointed out in the proof of Corollary 2.7(ii), our method cannot be applied to show the monotonicity of $v(x, y)$ in $y \in \mathcal{S}$ if $\mathcal{S}$ contains a neighborhood of zero. We do not know either how to generalize our method to the nonmartingale case.

2. The regime-switching case. Suppose $(X, Y) = (X_t, Y_t, t \geq 0)$ is a strong Markov process given on a family of probability spaces $(\Omega, \mathcal{F}, P_{x,y}, (x, y) \in \mathbb{R} \times \mathcal{S})$ which satisfies the following conditions:

(C1) The process (X, Y) is adapted with respect to a filtration $\mathcal{F}_t$, $t \geq 0$, of sub-σ-algebras of $\mathcal{F}$ and, for every $(x, y) \in \mathbb{R} \times \mathcal{S}$, there is an $\mathcal{F}_t$ Brownian motion B on $(\Omega, \mathcal{F}, P_{x,y})$ independent of Y such that

$$X_t = X_0 + \int_0^t a(X_s) Y_s \, dB_s, \qquad t \geq 0, \, P_{x,y}\text{-a.s.};$$

(C2) The process Y is a continuous-time Markov chain on the finite state space $\mathcal{S} = \{y_i : i = 1, 2, \ldots, m\} \subset (0, \infty)$ with Q-matrix $(q[y_i, y_j])$.

REMARK 2.1. (i) Because of the condition $\min\{y_1, \ldots, y_m\} > 0$ we have that $P_{x,y}(\lim_{t\uparrow\infty} \int_0^t Y_s^2 \, ds = \infty) = 1$ for all $(x, y) \in \mathbb{R} \times \mathcal{S}$.

(ii) From the above assumptions it immediately follows that, for every initial condition $x \in \mathbb{R}$, there exists a weak solution to the stochastic differential equation $dG = a(G)\, dW$ driven by a Brownian motion W. To see this fix $(x, y) \in \mathbb{R} \times \mathcal{S}$, and write

$$X_t = x + \int_0^t a(X_s) \, dM_s, \qquad t \geq 0, \, P_{x,y}\text{-a.s.},$$

where $M_s = \int_0^s Y_u \, dB_u$ is well defined since $\int_0^s Y_u^2 \, du < \infty$, $P_{x,y}$-a.s., for all $s \geq 0$. But time-changing X by the inverse of $\langle M \rangle$, which exists by (i) above, yields

$$G_t = x + \int_0^t a(G_s) \, dW_s, \qquad t \geq 0, \, P_{x,y}\text{-a.s.},$$

where $G = X \circ \langle M \rangle^{-1}$ is $\mathcal{F}_{\langle M \rangle_t^{-1}}$-adapted, and $W = M \circ \langle M \rangle^{-1}$ is an $\mathcal{F}_{\langle M \rangle_t^{-1}}$ Brownian motion by the Dambis–Dubins–Schwarz theorem; see [11], Theorem V.1.6. The equation does indeed hold for all $t \geq 0$ since $P_{x,y}(\lim_{t\uparrow\infty} \int_0^t Y_s^2 \, ds = \infty) = 1$.

(iii) Because $\langle M \rangle^{-1} = \int_0^\cdot Y_{\langle M \rangle_s^{-1}}^{-2} \, ds$, an easy calculation shows that the process $Y \circ \langle M \rangle^{-1}$ is a continuous-time Markov chain with Q-matrix $(y_i^{-2} q[y_i, y_j])$, $y_i, y_j \in \mathcal{S}$.

We can now formulate the condition on the coefficient a needed for our method.

(C3) Let $a : \mathbb{R} \to \mathbb{R}$ be measurable functions such that the stochastic differential equation $dG = a(G)\, dW$ driven by a Brownian motion W has a weakly unique strong Markov solution with state space $\mathbb{R}$.

The law of the strong Markov process given by (C3) is entirely determined by its semigroup of transition kernels. Multiplying these transition kernels and the transition kernels of a continuous-time Markov chain on $\mathcal{S} \times \mathcal{S}$ both marginals of which are determined by the Q-matrix $(y_i^{-2} q[y_i, y_j])$, $y_i, y_j \in \mathcal{S}$, results in a semigroup of transition kernels of a strong Markov process (G, Z, Z') with G being independent of (Z, Z'). Now choose a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$

such that (G, Z, Z') starts from fixed $(x, y, y') \in \mathbb{R} \times \mathcal{S} \times \mathcal{S}$. Let $\mathcal{F}_t^{G,Z,Z'}$ denote the augmentation of the filtration $\sigma(\{G_s, Z_s, Z'_s : s \le t\}), t \ge 0$, and assume that $(G, Z), (G, Z')$ are strong Markov processes with respect to $\mathcal{F}_t^{G,Z,Z'}$—an example will be given in the proof of Theorem 2.5.

Moreover, by the martingale problem associated with the strong Markov process G, $G_t - x$ is a continuous local $\mathcal{F}_t^{G,Z,Z'}$-martingale with quadratic variation $\int_0^t a(G_s)^2\, ds, t \ge 0$. Thus, by a well-known result going back to Doob (see [8], Theorem II 7.1′, e.g.), there is a Brownian motion W such that

$$G_t - x = \int_0^t a(G_s)\, dW_s, \qquad t \ge 0, \tilde{P}\text{-a.s.} \tag{2.1}$$

The construction of W on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ (or on a canonical enlargement of it[2]) as given in the proof of Theorem II 7.1′ in [8] shows that the pair (G, W) is also independent of (Z, Z'). But note that W might only be a Brownian motion with respect to a filtration $\tilde{\mathcal{F}}_t$ larger than $\mathcal{F}_t^{G,Z,Z'}$, $t \ge 0$, so that the stochastic integral in (2.1) can only be understood with respect to the larger filtration.

COROLLARY 2.2. *For given $(x, y, y') \in \mathbb{R} \times \mathcal{S} \times \mathcal{S}$, there is a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ equipped with two filtrations $\mathcal{F}_t^{G,Z,Z'} \subseteq \tilde{\mathcal{F}}_t, t \ge 0$, which is big enough to carry four basic processes G, W, Z, Z' such that: (G, W) is a weak $\tilde{\mathcal{F}}_t$-adapted solution of $dG = a(G)\, dW$ starting from x independent of (Z, Z'), the processes Z and Z' are Markov chains with Q-matrices $(y_i^{-2} q[y_i, y_j]), y_i, y_j \in \mathcal{S}$, starting from y and y', respectively, and $(G, Z), (G, Z')$ are strong Markov processes with respect to $\mathcal{F}_t^{G,Z,Z'}, t \ge 0$.*

The goal of this section is to show that, under some not too restrictive conditions, for fixed $x \in \mathbb{R}$ and $y, y' \in \mathcal{S}$,

$$\text{if } y \le y' \text{ then } v(x, y) \le v(x, y'), \tag{2.2}$$

where the value function v is given by (1.2).

Choosing x and $y \le y'$, we will construct two processes $(\tilde{X}, \tilde{Y})$ and $(\tilde{X}', \tilde{Y}')$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that $(\tilde{X}, \tilde{Y})$ has the same law as (X, Y) under $P_{x,y}$, and $(\tilde{X}', \tilde{Y}')$ has the same law as (X, Y) under $P_{x,y'}$. As a consequence we obtain that

$$\begin{aligned} v(x, y) &= \sup_{0 \le \tilde{\tau} \le T} \tilde{E}[e^{-q\tilde{\tau}} g(\tilde{X}_{\tilde{\tau}})], \\ v(x, y') &= \sup_{0 \le \tilde{\tau}' \le T} \tilde{E}[e^{-q\tilde{\tau}'} g(\tilde{X}'_{\tilde{\tau}'})], \end{aligned} \tag{2.3}$$

where $\tilde{\tau}$ and $\tilde{\tau}'$ are finite stopping times with respect to the filtrations generated by $(\tilde{X}, \tilde{Y})$ and $(\tilde{X}', \tilde{Y}')$, respectively.

[2] Our convention is to use $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ for the enlarged space, too.

To see this note that each stopping time τ with respect to the filtration generated by (X, Y) can easily be associated with a stopping time $\tilde{\tau}$ with respect to the filtration generated by $(\tilde{X}, \tilde{Y})$ such that

$$E_{x,y}[e^{-q\tau} g(X_\tau)] = \tilde{E}[e^{-q\tilde{\tau}} g(\tilde{X}_{\tilde{\tau}})]$$

and vice versa proving the first equality in (2.3). The second equality follows of course by the same argument.

Hence, we can now work on only ONE probability space. This is an important part of our method for proving (2.2) which is based on time-change and coupling and which is demonstrated below.

Let G, W, Z, Z' be given on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ as described in Corollary 2.2, and define

$$\Gamma_t = \int_0^t Z_s^{-2}\, ds, \qquad t \geq 0.$$

This process $\Gamma = (\Gamma_t)_{t\geq 0}$ is of course continuous but also strictly increasing since Z only takes nonzero values. Moreover, condition (C2) on page 1556 implies that

$$\Gamma_t < \infty, t \geq 0, \text{a.s.} \quad \text{and} \quad \lim_{t\uparrow\infty} \Gamma_t = \infty \text{ a.s.}, \tag{2.4}$$

since, by Remark 2.1(iii), Z has the same law as $Y \circ \langle M\rangle^{-1}$ under $P_{x,y}$ with $\int_0^\cdot Y^{-2}_{\langle M\rangle_s^{-1}}\, ds$ being the inverse of $\int_0^\cdot Y_s^2\, ds$. Thus $A = \Gamma^{-1}$ is also a continuous and strictly increasing process satisfying

$$A_t < \infty, t \geq 0, \text{ a.s.} \quad \text{and} \quad \lim_{t\uparrow\infty} A_t = \infty \text{ a.s.} \tag{2.5}$$

As a consequence, the two technical properties:

(P1) $\Gamma_{A_t} = A_{\Gamma_t} = t$ for all $t \geq 0$ a.s. and
(P2) $s < \Gamma_t$ if and only if $A_s < t$ for all $0 \leq s, t < \infty$ a.s.

must hold.

Of course, Γ is adapted to both filtrations $\mathcal{F}_t^{G,Z,Z'}$ and $\tilde{\mathcal{F}}_t$, $t \geq 0$. However, $A = \Gamma^{-1}$ is considered an $\mathcal{F}_t^{G,Z,Z'}$ time change in the following lemma. We denote by $\mathcal{M}$ and $\mathcal{T}$ the families of stopping times with respect to the filtrations $(\mathcal{F}_t^{G,Z,Z'})_{t\geq 0}$ and $(\mathcal{F}_{A_t}^{G,Z,Z'})_{t\geq 0}$, respectively.

LEMMA 2.3. *If $\rho \in \mathcal{M}$ then $\Gamma_\rho \in \mathcal{T}$, and if $\tau \in \mathcal{T}$ then $A_\tau \in \mathcal{M}$.*

A similar lemma can be found in [13]. Since the above lemma is going to be used to reformulate the original optimal stopping problem (1.2) in both the case where Y is a Markov chain and the case where Y is a diffusion, its proof is given in the Appendix for completeness.

The reformulation of (1.2) is based on the existence of a suitable solution to (1.1) which is constructed next.

Since Z is $\tilde{\mathcal{F}}_t$-adapted, one can rewrite (2.1) to get

$$G_t = x + \int_0^t a(G_s) Z_s \, d\tilde{M}_s, t \geq 0, \text{a.s.} \quad \text{where} \quad \tilde{M}_s = \int_0^s \frac{dW_u}{Z_u}, s \geq 0.$$

Observe that the stochastic integral defining $\tilde{M}$ exists by (2.4). Time changing the above equation by A yields

$$\tilde{X}_t = x + \int_0^t a(\tilde{X}_s)\tilde{Y}_s \, d\tilde{B}_s, \qquad t \geq 0, \text{a.s.}$$

for $\tilde{X} = G \circ A$, $\tilde{Y} = Z \circ A$, $\tilde{B} = \tilde{M} \circ A$. Of course, $(\tilde{X}, \tilde{Y})$ is $\tilde{\mathcal{F}}_{A_t}$-adapted, and $\tilde{B}$ is an $\tilde{\mathcal{F}}_{A_t}$ Brownian motion by Dambis–Dubins–Schwarz' theorem [11], Theorem V.1.6. Thus $(\tilde{X}, \tilde{Y})$ gives a weak solution to (1.1) starting from (x, y). Moreover, $\tilde{B}$ and $\tilde{Y}$ are independent since W and Z are independent. The proof of this is contained in the Appendix; see Lemma A.1 on page 1581.

PROPOSITION 2.4. *Let $G, \tilde{X}, \tilde{Y}$ be the processes on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ introduced above and starting from $G_0 = \tilde{X}_0 = x$ and $\tilde{Y}_0 = y$. If the stochastic differential equation*

$$dX = a(X)Y \, dB, \qquad (X, Y) \textit{ unknown},$$

driven by a Brownian motion B, where Y is required to be a continuous-time Markov chain independent of B with Q-matrix $(q[y_i, y_j])$, $y_i, y_j \in \mathcal{S}$, admits a weakly unique solution then, for any $T \in [0, \infty]$,

$$v(x, y) = \sup_{\tau \in \mathcal{T}_T} \tilde{E}\big[e^{-q\tau} g(\tilde{X}_\tau)\big] = \sup_{\rho \in \mathcal{M}_T} \tilde{E}\big[e^{-q\Gamma_\rho} g(G_\rho)\big],$$

where

$$\mathcal{T}_T = \{\tau \in \mathcal{T} : 0 \leq \tau \leq T\} \quad \textit{and} \quad \mathcal{M}_T = \{\rho \in \mathcal{M} : 0 \leq \rho \leq A_T\}.$$

Here, $\mathcal{T}$ and $\mathcal{M}$ denote the families of finite *stopping times with respect to the filtrations $(\mathcal{F}^{G,Z,Z'}_{A_t})_{t \geq 0}$ and $(\mathcal{F}^{G,Z,Z'}_t)_{t \geq 0}$, respectively.*

PROOF. First note that Γ is a continuous, strictly increasing, perfect additive functional of (G, Z) which satisfies (2.4) and recall that (G, Z) is a strong Markov process with respect to $\mathcal{F}^{G,Z,Z'}_t$, $t \geq 0$, by Corollary 2.2. So $(G \circ A, Z \circ A) = (\tilde{X}, \tilde{Y})$ must possess the strong Markov property with respect to $\mathcal{F}^{G,Z,Z'}_{A_t}$, $t \geq 0$, by [13], Theorem 65.9. But $A = \Gamma^{-1} = \int_0^\cdot \tilde{Y}_s^2 \, ds$ by time-changing the integral defining Γ. So $\tilde{Y}$ is a continuous-time Markov chain with Q-matrix $(q[y_i, y_j])$, $y_i, y_j \in \mathcal{S}$. Combining these statements, $(\tilde{X}, \tilde{Y})$ has the same law as (X, Y) under $P_{x,y}$, since both pairs satisfy the equation $dX = a(X)Y \, dB$ in the sense explained in the proposition and this equation admits a weakly unique solution. As a consequence it follows from (2.3) that

$$v(x, y) = \sup_{0 \le \tau \le T} \tilde{E}\big[e^{-q\tau} g(\tilde{X}_\tau)\big], \tag{2.6}$$

where the finite stopping times τ are with respect to the filtration $\mathcal{F}_{A_t}^{G,Z,Z'}$, $t \ge 0$. Here one should mention that the stopping times used in (2.3) are with respect to the filtration generated by $(\tilde{X}, \tilde{Y})$ which might be smaller than $\mathcal{F}_{A_t}^{G,Z,Z'}$, $t \ge 0$. However, it is well known that the corresponding suprema are the same if the underlying process, in this case $(\tilde{X}, \tilde{Y})$, is also strong Markov with respect to the bigger filtration. For completeness we sketch the proof of (2.6) in the Appendix on page 1579.

It remains to show that

$$\sup_{\tau \in \mathcal{T}_T} \tilde{E}\big[e^{-q\tau} g(\tilde{X}_\tau)\big] = \sup_{\rho \in \mathcal{M}_T} \tilde{E}\big[e^{-q\Gamma_\rho} g(G_\rho)\big]. \tag{2.7}$$

Fix $\tau \in \mathcal{T}_T$, and observe that

$$\tilde{E}\big[e^{-q\tau} g(\tilde{X}_\tau)\big] = \tilde{E}\big[e^{-q\Gamma_{A_\tau}} g(G_{A_\tau})\big]$$

by property (P1) and the construction of $\tilde{X}$. Also A_τ is an $\mathcal{F}_t^{G,Z,Z'}$ stopping time by Lemma 2.3. The right-hand side above does not change if a finite version of A_τ is chosen which still is an $\mathcal{F}_t^{G,Z,Z'}$ stopping time, since the filtration satisfies the usual conditions. Thus $A_\tau \in \mathcal{M}_T$, and it follows that

$$\tilde{E}\big[e^{-q\tau} g(\tilde{X}_\tau)\big] \le \sup_{\rho \in \mathcal{M}_T} \tilde{E}\big[e^{-q\Gamma_\rho} g(G_\rho)\big].$$

Similarly, for fixed $\rho \in \mathcal{M}_T$, the equality $\tilde{E}[e^{-q\Gamma_\rho} g(G_\rho)] = \tilde{E}[e^{-q\Gamma_\rho} g(\tilde{X}_{\Gamma_\rho})]$ leads to

$$\tilde{E}\big[e^{-q\Gamma_\rho} g(G_\rho)\big] \le \sup_{\tau \in \mathcal{T}_T} \tilde{E}\big[e^{-q\tau} g(\tilde{X}_\tau)\big]$$

finally proving (2.7). □

Of course, the conclusion of Proposition 2.4 remains valid for $v(x, y')$, $\tilde{X}'$, $\tilde{Y}'$, $\mathcal{T}_T'$, $\mathcal{M}_T'$, A' and Γ' if these objects are constructed by using Z' instead of Z. Notice that the solution G is the same.

We are now in the position to formulate and prove the main result of this section about the validity of (2.2). The following notion of a skip-free Markov chain is needed: a continuous-time Markov chain with Q-matrix $(q[y_i, y_j])$ taking the states $y_1 < \cdots < y_m$ is called *skip-free* if the matrix Q is tridiagonal.

THEOREM 2.5. *Let (X, Y) be a strong Markov process given on a family of probability spaces $(\Omega, \mathcal{F}, P_{x,y}, (x, y) \in \mathbb{R} \times \mathcal{S})$ and let $g : \mathbb{R} \to \mathbb{R}$ be a measurable gain function such that $\{g \geq 0\} \neq \varnothing$. Assume (1.3), that (X, Y) satisfies conditions* (C1), (C2) *on page 1556 and condition* (C3) *on page 1557 and that all pairs of processes satisfying conditions* (C1), (C2) *have the same law. Further suppose that Y is skip-free. Define $\mathcal{K}_T^{g+}$ to be the collection of all finite stopping times $\tau \leq T$ with respect to the filtration generated by (X, Y) such that $g(X_\tau) \geq 0$. Fix $(x, y) \in \mathbb{R} \times \mathcal{S}$ and assume that $v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}[e^{-q\tau} g(X_\tau)]$. Then*

$$v(x, y) \leq v(x, y') \qquad \text{for all } y' \in \mathcal{S} \text{ such that } y \leq y',$$

so that $v(x, y)$ is a lower bound for $v(x, \cdot)$ on $[y, \infty) \cap \mathcal{S}$.

REMARK 2.6. (i) The condition $v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}[e^{-q\tau} g(X_\tau)]$ is a technical condition which states that the optimum $v(x, y)$ as defined by (1.2) can be achieved by stopping at nonnegative values of g only. It is of course trivially satisfied for all $(x, y) \in \mathbb{R} \times \mathcal{S}$ if the gain function is nonnegative and in this case the theorem means that $v(x, \cdot)$ is increasing.

(ii) In the case of an infinite time horizon $T = \infty$, it easily follows from the section theorem [11], Theorem IV.5.5, that

$$P_{x,y}\bigl(\inf\{t \geq 0 : g(X_t) \geq 0\} < \infty\bigr) = 1 \qquad \text{for all } (x, y) \in \mathbb{R} \times \mathcal{S}$$

is sufficient for $v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}[e^{-q\tau} g(X_\tau)]$ to be true for all $(x, y) \in \mathbb{R} \times \mathcal{S}$ since (X, Y) is strong Markov. Indeed, if a process always hits the set $\{g \geq 0\}$ with probability one, then it is quite natural that maximal gain is obtained while avoiding stopping at negative values of g. One can easily construct processes satisfying this sufficient condition where the gain function g takes both positive *and* negative values.

(iii) In the case where $T < \infty$, the only reasonable sufficient condition the authors can find is the trivial condition $g(x) \geq 0$ for all $x \in \mathbb{R}$. This is because, in general a process is not almost surely guaranteed to hit a subset of the state space in finite time.

(iv) The monotonicity result of this theorem supports the intuition that the larger the diffusion coefficient (volatility) of a diffusion without drift, the faster this diffusion moves and hence the sooner it reaches the points where the gain function g is large. As the killing term of the cost functional defining $v(x, y)$ punishes the elapsed time, $v(x, y')$ should indeed be larger than $v(x, y)$, for $y' > y$, if the volatility process starting from y' stays above the volatility process starting from y, and this is ensured by the skip-free property of the Markov chain.

PROOF OF THEOREM 2.5. Fix $x \in \mathbb{R}$ and $y, y' \in \mathcal{S}$ such that $y \leq y'$, and let G, W, Z, Z' be given on a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ as described in

Corollary 2.2. While in Corollary 2.2 the coupling of the two chains Z and Z' was not specified any further we now choose a particular coupling associated with a Q-matrix $\mathfrak{Q}$ which allows us to compare Z and Z' directly. Denoting the Q-matrix corresponding to the independence coupling by $\mathfrak{Q}^{\perp}$, we set

$$\mathfrak{Q}\left[\begin{array}{c|c} y_i & y_j \\ y_k & y_l \end{array}\right] = \begin{cases} \mathfrak{Q}^{\perp}\left[\begin{array}{c|c} y_i & y_j \\ y_k & y_l \end{array}\right], & i \neq k, \\ y_i^{-2} q[y_i, y_j], & i = k, j = l, \\ 0, & i = k, j \neq k \end{cases}$$

for $y_i, y_j, y_k, y_l \in \mathcal{S}$; that is, Z and Z' move independently until they hit each other for the first time and then they move together. It follows from the skip-free-assumption that Z cannot overtake Z' before they hit each other for the first time. Hence

$$Z_0 = y \leq y' = Z'_0 \qquad \text{implies } Z_t \leq Z'_t, t \geq 0, \text{a.s.},$$

which results in the inequality

$$\Gamma_t = \int_0^t Z_s^{-2}\, ds \geq \int_0^t (Z'_s)^{-2}\, ds = \Gamma'_t, \qquad t \geq 0, \text{a.s.} \tag{2.8}$$

Note that then the inverse increasing processes $A = \Gamma^{-1}$ and $A' = (\Gamma')^{-1}$ must satisfy the relation $A_t \leq A'_t, t \geq 0$, a.s.

Now recall the definition of $\mathcal{M}_T$ in Proposition 2.4, and note that the above comparison allows us to conclude that

$$\tilde{E}[e^{-q\Gamma_\rho} g(G_\rho)] \leq \tilde{E}[e^{-q\Gamma'_\rho} g(G_\rho)] \qquad \text{for every } \rho \in \mathcal{M}_T^+, \tag{2.9}$$

where $\mathcal{M}_T^+ = \{\rho \in \mathcal{M}_T : g(G_\rho) \geq 0 \text{ a.s.}\}$. Thus

$$\tilde{E}[e^{-q\Gamma_\rho} g(G_\rho)] \leq \sup_{\rho' \in \mathcal{M}'_T} \tilde{E}[e^{-q\Gamma'_{\rho'}} g(G_{\rho'})] \qquad \text{for every } \rho \in \mathcal{M}_T^+$$

since $A_T \leq A'_T$ a.s. implies that every stopping time in $\mathcal{M}_T^+$ has a version which is in $\mathcal{M}'_T$. Putting these results together, we obtain

$$\sup_{\rho \in \mathcal{M}_T^+} \tilde{E}[e^{-q\Gamma_\rho} g(G_\rho)] \leq \sup_{\rho' \in \mathcal{M}'_T} \tilde{E}[e^{-q\Gamma'_{\rho'}} g(G_{\rho'})].$$

But, if $\mathcal{T}_T^+$ denotes $\{\tau \in \mathcal{T}_T : g(\tilde{X}_\tau) \geq 0 \text{ a.s.}\}$, then the equality

$$\sup_{\tau \in \mathcal{T}_T^+} \tilde{E}[e^{-q\tau} g(\tilde{X}_\tau)] = \sup_{\rho \in \mathcal{M}_T^+} \tilde{E}[e^{-q\Gamma_\rho} g(G_\rho)]$$

can be shown in the same way that (2.7) was shown in the proof of Proposition 2.4 (note that in this proof we may choose versions of certain stopping times and this is the reason the qualification "a.s." appears in the definitions of $\mathcal{M}_T^+$ and $\mathcal{T}_T^+$).

Furthermore,

$$\text{if } v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}\big[e^{-q\tau} g(X_\tau)\big] \text{ then } v(x, y) \le \sup_{\tau \in \mathcal{T}_T^+} \tilde{E}\big[e^{-q\tau} g(\tilde{X}_\tau)\big]$$

since the law of $(\tilde{X}, \tilde{Y})$ is equal to the law of (X, Y) under $P_{x,y}$ and the filtration $\mathcal{F}_{A_t}^{G,Z,Z'}$, $t \ge 0$, is at least as big as the filtration generated by $(\tilde{X}, \tilde{Y})$. So, under the condition $v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}[e^{-q\tau} g(X_\tau)]$, we can finally deduce that

$$\begin{aligned} v(x, y) &\le \sup_{\tau \in \mathcal{T}_T^+} \tilde{E}\big[e^{-q\tau} g(\tilde{X}_\tau)\big] \\ &= \sup_{\rho \in \mathcal{M}_T^+} \tilde{E}\big[e^{-q\Gamma_\rho} g(G_\rho)\big] \le \sup_{\rho' \in \mathcal{M}'_T} \tilde{E}\big[e^{-q\Gamma'_{\rho'}} g(G_{\rho'})\big] = v\big(x, y'\big), \end{aligned}$$

where the last equality is due to Proposition 2.4 applied to $(\tilde{X}', \tilde{Y}')$. $\square$

COROLLARY 2.7. (i) *If $\{g \ge 0\} = \varnothing$, but all other assumptions of Theorem 2.5 are satisfied, then in the infinite time horizon case where $T = \infty$,*

$$v(x, y) \ge v\big(x, y'\big) \qquad \textit{for all } x \in \mathbb{R} \textit{ and } y, y' \in \mathcal{S} \textit{ such that } y \le y',$$

so that $v(x, \cdot)$ is decreasing.

(ii) *Let the assumptions of Theorem 2.5 be based on $\mathcal{S} \subseteq (-\infty, 0)$, fix $(x, y) \in \mathbb{R} \times \mathcal{S}$ and assume that $v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}[e^{-q\tau} g(X_\tau)]$. Then*

$$v(x, y) \ge v\big(x, y'\big) \qquad \textit{for all } y' \in \mathcal{S} \textit{ such that } y \le y',$$

so that $v(x, y)$ is an upper bound for $v(x, \cdot)$ on $[y, \infty) \cap \mathcal{S}$.

PROOF. If $\{g \ge 0\} = \varnothing$ then, instead of (2.9), we obtain

$$\tilde{E}\big[e^{-q\Gamma_\rho} g(G_\rho)\big] \ge \tilde{E}\big[e^{-q\Gamma'_\rho} g(G_\rho)\big] \qquad \text{for every } \rho \in \mathcal{M}_T$$

and $\mathcal{M}_T$ is (up to versions) equal to $\mathcal{M}'_T$ since $T = \infty$. Hence (i) can be deduced directly from Proposition 2.4. Note that the above inequality cannot be used in the case where $T < \infty$ since there can be stopping times in $\mathcal{M}'_T$ which are not in $\mathcal{M}_T$.

If $\mathcal{S} \subseteq (-\infty, 0)$ then $Z_t \le Z'_t$, $t \ge 0$, a.s., does not imply (2.8) but instead

$$\Gamma_t = \int_0^t Z_s^{-2}\, ds \le \int_0^t \big(Z'_s\big)^{-2}\, ds = \Gamma'_t, \qquad t \ge 0, \text{a.s.},$$

hence, interchanging the roles of y and y', (ii) can be proved like Theorem 2.5. Note that $Z_t \le Z'_t$, $t \ge 0$, a.s., would not lead to any comparison between Γ and Γ' if $y < 0 < y'$. Hence our method cannot be applied to show the monotonicity of $v(x, y)$ in $y \in \mathcal{S}$ if $\mathcal{S}$ contains a neighbourhood of zero. $\square$

3. The diffusion case. Fix $\delta \in [-1, 1]$, and suppose that (X, Y) is a strong Markov process given on a family of probability spaces $(\Omega, \mathcal{F}, P_{x,y}, (x, y) \in \mathbb{R} \times \mathcal{S})$ which satisfies the following conditions:

(C1′) the process (X, Y) is adapted with respect to a filtration $\mathcal{F}_t, t \geq 0$, of sub-σ-algebras of $\mathcal{F}$ and, for every $(x, y) \in \mathbb{R} \times \mathcal{S}$, there is a pair (B, B^Y) of $\mathcal{F}_t$ Brownian motions on $(\Omega, \mathcal{F}, P_{x,y})$ with covariation $\langle B, B^Y\rangle_t = \delta t, t \geq 0$, such that

$$X_t = X_0 + \int_0^t a(X_s) Y_s \, dB_s \quad \text{and} \quad Y_t = Y_0 + \int_0^t \eta(Y_s) \, dB_s^Y + \int_0^t \theta(Y_s) \, ds$$

for all $t \geq 0$, $P_{x,y}$-a.s.;

(C2′) the process Y takes values in $\mathcal{S} \subseteq (0, \infty)$ and

$$P_{x,y}\left(\lim_{t\uparrow\infty} \int_0^t Y_s^2 \, ds = \infty\right) = 1 \qquad \text{for all } (x, y) \in \mathbb{R} \times \mathcal{S}.$$

REMARK 3.1. Under the assumptions above, for every $(x, y) \in \mathbb{R} \times \mathcal{S}$, there exists a weak solution to the system of stochastic differential equations

$$\begin{cases} dG = a(G) \, dW, \\ d\xi = \eta(\xi)\xi^{-1} \, dW^\xi + \theta(\xi)\xi^{-2} \, dt, \\ \xi_t \in \mathcal{S}, \qquad t \geq 0, \end{cases} \tag{3.1}$$

driven by a pair of Brownian motions with covariation $\langle W, W^\xi\rangle_t = \delta t, t \geq 0$. Such a solution can be given by $(X \circ \langle M\rangle^{-1}, Y \circ \langle M\rangle^{-1})$ where M denotes the continuous local martingale $M_s = \int_0^s Y_u \, dB_u, s \geq 0$, as in Remark 2.1(ii). Here $W = M \circ \langle M\rangle^{-1}$ and $W^\xi = \int_0^{\langle M\rangle_\cdot^{-1}} Y_s \, dB_s^Y$ are $\mathcal{F}_{\langle M\rangle_t^{-1}}$ Brownian motions by Dambis–Dubins–Schwarz' theorem (see [11], Theorem V.1.6) with covariation

$$\begin{aligned} \langle W, W^\xi\rangle_t &= \left\langle \int_0^\cdot Y_s \, dB_s, \int_0^\cdot Y_s \, dB_s^Y \right\rangle_{\langle M\rangle_t^{-1}} \\ &= \delta \int_0^{\langle M\rangle_t^{-1}} Y_s^2 \, ds = \delta \langle M\rangle_{\langle M\rangle_t^{-1}} = \delta t, \qquad t \geq 0, P_{x,y}\text{-a.s.,} \end{aligned}$$

where the last equality is ensured by condition (C2′).

We want to show (2.2) using a method similar to the method applied in Section 2. The main difference to the case discussed in Section 2 is that the pair (X, Y) is now determined by a system of stochastic differential equations. So, instead of constructing $\tilde{X}$ by time-changing a solution of the single equation $dG = a(G) \, dW$ as in Section 2, we now construct $\tilde{X}$ by time-changing a solution of a system of stochastic differential equations. Furthermore, in Section 2 we constructed the coupling of Z and Z' in the proof of Theorem 2.5 from a given generator. In this section we will couple ξ and ξ'—both satisfying the second equation in (3.1) but starting from $y \le y'$, respectively—we will do so directly from the stochastic differential equation. As a consequence, the next condition appears to be slightly stronger than the corresponding condition (C3) of the last section. However, in Theorem 2.5 we needed (C3), a skip-free Markov chain and weak uniqueness of (1.1) while below, in the corresponding Theorem 3.5, we will only need:

(C3$'$) Let a, η, θ be measurable functions such that the system of stochastic differential equations (3.1) has, for all initial conditions $(G_0, \xi_0) \in \mathbb{R} \times \mathcal{S}$, a unique nonexploding strong solution taking values in $\mathbb{R} \times \mathcal{S}$.

Now choose a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ big enough to carry a pair of Brownian motions (W, W^ξ) with covariation $\langle W, W^\xi\rangle_t = \delta t, t \ge 0$, and denote by $\tilde{\mathcal{F}}_t, t \ge 0$, the usual augmentation of the filtration generated by (W, W^ξ). Let (G, ξ) be the unique solution of the system (3.1) starting from $G_0 = x \in \mathbb{R}$ and $\xi_0 = y \in \mathcal{S}$ given on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ by (W, W^ξ).

Define $\Gamma = (\Gamma_t)_{t\ge 0}$ by

$$\Gamma_t = \int_0^t \xi_u^{-2}\, du, \qquad t \ge 0,$$

and remark that Γ satisfies (2.4). Indeed, by Remark 3.1, $Y \circ \langle M\rangle^{-1}$ solves the second equation of (3.1), and hence condition (C3$'$) implies that ξ has the same law as $Y \circ \langle M\rangle^{-1}$ under $P_{x,y}$. Property (2.4) therefore follows from (C2$'$) since $\int_0^\cdot Y^{-2}_{\langle M\rangle_s^{-1}}\, ds$ is the inverse of $\int_0^\cdot Y_s^2\, ds$.

Of course, we may deduce from (2.4) together with the fact that ξ never vanishes, that Γ is a continuous and strictly increasing process. Thus, $A = \Gamma^{-1}$ is also a continuous and strictly increasing process satisfying (2.5). As a consequence, the two technical properties (P1) and (P2) on page 1559 must again be valid.

As ξ is $\tilde{\mathcal{F}}_t$-adapted, we see that

$$G_t = x + \int_0^t a(G_s)\xi_s\, d\tilde{M}_s, \qquad t \ge 0, \text{a.s.}, \tag{3.2}$$

$$\xi_t = y + \int_0^t \eta(\xi_s)\, d\tilde{M}_s^\xi + \int_0^t \theta(\xi_s)\, d\Gamma_s > 0, \qquad t \ge 0, \text{a.s.}, \tag{3.3}$$

where (2.4) implies that the continuous local martingales $\tilde{M}$ and $\tilde{M}^\xi$ given by the stochastic integrals

$$\tilde{M}_s = \int_0^s \xi_u^{-1}\, dW_u \quad \text{and} \quad \tilde{M}_s^\xi = \int_0^s \xi_u^{-1}\, dW_u^\xi$$

exist for each $s \ge 0$. Now it immediately follows from (3.2), (3.3) that the $\tilde{\mathcal{F}}_{A_t}$-adapted processes $\tilde{X} = G \circ A$ and $\tilde{Y} = \xi \circ A$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ constitute a nonexploding weak solution of the system (1.1), (1.4) with $\tilde{Y}_t \in \mathcal{S}, t \ge 0$, since $\tilde{B} = \tilde{M} \circ A$

and $\tilde{B}^Y = \tilde{M}^\xi \circ A$ are $\tilde{\mathcal{F}}_{A_t}$ Brownian motions by Dambis–Dubins–Schwarz' theorem [11], Theorem V.1.6 and

$$\langle \tilde{B}, \tilde{B}^Y \rangle_t = \langle \tilde{M}, \tilde{M}^\xi \rangle_{A_t} = \delta \int_0^{A_t} \xi_u^{-2}\, du = \delta \Gamma_{A_t} = \delta t, \quad t \geq 0, \text{a.s.},$$

by property (P1).

REMARK 3.2. (i) Combining Remark 3.1 and condition (C3′), it follows from the construction above that $(\tilde{X}, \tilde{Y})$ must have the same distribution as (X, Y) under $P_{x,y}$.

(ii) The filtration $\tilde{\mathcal{F}}_{A_t}, t \geq 0$, might be bigger than the filtration generated by $(\tilde{X}, \tilde{Y})$. However, it is straightforward to show the strong Markov property of $(\tilde{X}, \tilde{Y})$ with respect to $\tilde{\mathcal{F}}_{A_t}, t \geq 0$, since $(\tilde{X}, \tilde{Y})$ was obtained by time-changing a unique strong solution of a system of stochastic differential equation driven by Brownian motions.

This remark makes clear that the following proposition can be proved by applying the ideas used in the proof of Proposition 2.4 in Section 2 (so we omit its proof).

PROPOSITION 3.3. *Let $G, \tilde{X}, \tilde{Y}$ be the processes on the filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, t \geq 0, \tilde{P})$ introduced above and starting from $G_0 = \tilde{X}_0 = x \in \mathbb{R}$ and $\tilde{Y}_0 = y \in \mathcal{S}$. Then, for any $T \in [0, \infty]$, it follows that*

$$v(x, y) = \sup_{\tau \in \mathcal{T}_T} \tilde{E}[e^{-q\tau} g(\tilde{X}_\tau)] = \sup_{\rho \in \mathcal{M}_T} \tilde{E}[e^{-q\Gamma_\rho} g(G_\rho)],$$

where

$$\mathcal{T}_T = \{\tau \in \mathcal{T} : 0 \leq \tau \leq T\} \quad \text{and} \quad \mathcal{M}_T = \{\rho \in \mathcal{M} : 0 \leq \rho \leq A_T\}.$$

Here, $\mathcal{T}$ and $\mathcal{M}$ denote the families of finite stopping times with respect to the filtrations $(\tilde{\mathcal{F}}_{A_t})_{t \geq 0}$ and $(\tilde{\mathcal{F}}_t)_{t \geq 0}$, respectively.

REMARK 3.4. The above representation of $v(x, y)$, $(x, y) \in \mathbb{R} \times \mathcal{S}$, could be extended to cases where $\mathcal{S}$ is bigger than $(0, \infty)$. However, in such cases, the equation for ξ in (3.1) must admit solutions starting from $\xi_0 = 0$ which is an additional constraint, since ξ is in the denominator on the right-hand side of this equation. Furthermore, in addition to the assumption that $P_{x,y}(\lim_{t \uparrow \infty} \int_0^t Y_s^2\, ds = \infty) = 1$ one would need to assume that $\int_0^{\cdot} Y_s^2\, ds$ is strictly increasing $P_{x,y}$-a.s. as, in principle, the process Y could now spend time at zero.

Recall that, in contrast to the case of regime-switching, the process $\tilde{Y}$ above was constructed by time-change from a solution of a stochastic differential equation and this results in some small variations from the proof of Theorem 2.5. Note that the conclusion of Proposition 3.3 remains valid for $v(x, y')$, $\tilde{X}'$, $\mathcal{T}'_T$, $\mathcal{M}'_T$, A' and Γ' if these objects are constructed using a different starting point $y' \in \mathcal{S}$.

THEOREM 3.5. *Let (X, Y) be a strong Markov process given on a family of probability spaces $(\Omega, \mathcal{F}, P_{x,y}, (x, y) \in \mathbb{R} \times \mathcal{S})$, and let $g : \mathbb{R} \to \mathbb{R}$ be a measurable gain function such that $\{g \geq 0\} \neq \varnothing$. Assume* (1.3), *that (X, Y) satisfies conditions* (C1′) *and* (C2′) *on page* 1565 *and that condition* (C3′) *on page* 1566 *holds true for system* (3.1). *Define $\mathcal{K}_T^{g+}$ to be the collection of all finite stopping times $\tau \leq T$ with respect to the filtration generated by (X, Y) such that $g(X_\tau) \geq 0$. Fix $(x, y) \in \mathbb{R} \times \mathcal{S}$ and assume that $v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}[e^{-q\tau} g(X_\tau)]$. Then*

$$v(x, y) \leq v(x, y') \qquad \textit{for all } y' \in \mathcal{S} \textit{ such that } y \leq y',$$

so that $v(x, y)$ is a lower bound for $v(x, \cdot)$ on $[y, \infty) \cap \mathcal{S}$.

PROOF. Fix $x \in \mathbb{R}$ and $y, y' \in \mathcal{S}$ with $y \leq y'$ and choose a complete probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ large enough to carry a pair of Brownian motions (W, W^ξ) with covariation $\langle W, W^\xi \rangle_t = \delta t, t \geq 0$. Let (G, ξ) and (G, ξ') be the solutions of (3.1) starting from (x, y) and from (x, y'), respectively, which are both given by (W, W^ξ) on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Remark that G is indeed the same for both pairs since (3.1) is a system of decoupled equations.

Define $C = \inf\{t \geq 0 : \xi_t > \xi'_t\}$ and set $\bar{\xi}_t = \xi_t I(t < C) + \xi'_t I(t \geq C)$ so that $\bar{\xi}_t \leq \xi'_t$ for all $t \geq 0$. Obviously, $(G, \bar{\xi})$ solves system (3.1) starting from $G_0 = x$ and $\bar{\xi}_0 = y$ hence $\xi_t = \bar{\xi}_t \leq \xi'_t, t \geq 0$, a.s., by strong uniqueness.

Construct $\tilde{X} = G \circ A$ and $\tilde{X}' = G \circ A'$ using the above (G, ξ) and (G, ξ'), and observe that $\Gamma_t \geq \Gamma'_t$ follows immediately from $0 < \xi_t \leq \xi'_t$ for all $t \geq 0$ a.s. Thus, simply using Proposition 3.3 instead of Proposition 2.4, the rest of the proof can be copied from the corresponding part of the proof of Theorem 2.5. □

REMARK 3.6. For a discussion of the technical condition $v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}[e^{-q\tau} g(X_\tau)]$ we refer the reader to Remark 2.6. Corollary 2.7 remains true if it is reformulated in the terms of the theorem above instead of Theorem 2.5.

4. Continuity in the diffusion-case.

Let $\mathcal{S}$ be an open subset of $(0, \infty)$, fix $x \in \mathbb{R}$ and suppose that all the assumptions of Theorem 3.5 are satisfied. Furthermore, suppose that

$$v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}\bigl[e^{-q\tau} g(X_\tau)\bigr] \qquad \text{for all } y \in \mathcal{S}. \tag{4.1}$$

For a sequence $(y_n)_{n=1}^\infty \subseteq \mathcal{S}$ converging to $y_0 \in \mathcal{S}$ as $n \to \infty$, denote by (G, ξ^n) the solution of (3.1), starting from $G_0 = x$ and $\xi_0^n = y_n, n = 0, 1, 2, \ldots,$ given by a pair (W, W^ξ) of Brownian motions with covariation $\langle W, W^\xi \rangle_t = \delta t, t \geq 0$, on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$. Using (G, ξ^n) construct $\Gamma^n, A^n, n = 0, 1, 2, \ldots,$ like Γ, A in Section 3.

LEMMA 4.1. *Suppose that the* ξ*-component corresponding to the unique strong solution to* (3.1) *on page* 1565 *is a Feller process with state space* $\mathcal{S}$. *If the sequence* $(y_n)_{n=1}^{\infty}$ *is monotone, that is, either* $y_n \downarrow y_0$ *or* $y_n \uparrow y_0$ *when* $n \to \infty$, *then*

$$\Gamma_t^n \to \Gamma_t^0 \quad \textit{and} \quad A_t^n \to A_t^0 \qquad \textit{as } n \to \infty \textit{ for all } t \geq 0 \textit{ a.s.} \tag{4.2}$$

PROOF. Here, we will use without further comment the elementary fact that if U and V are two random variables with the same law and $U \geq V$ a.s. then, in fact, $U = V$ a.s.

Suppose that $y_n \downarrow y_0$ as $n \to \infty$. By the coupling argument in the proof of Theorem 3.5, without loss of generality one may chose ξ^n such that

$$\xi_t^1 \geq \xi_t^2 \geq \cdots \geq \xi_t^n \geq \cdots \geq \xi_t^0 > 0, \qquad t \geq 0, n = 0, 1, 2, \ldots; \tag{4.3}$$

hence the pathwise limit $\lim_n \xi_t^n$, $t \geq 0$, exists. It follows from the Feller property that the two processes ξ^0 and $(\lim_n \xi_t^n)_{t \geq 0}$ must have the same law by comparing their finite-dimensional distributions. As (4.3) also yields the inequalities $\lim_n \xi_t^n \geq \xi_t^0 > 0$, $t \geq 0$, we see that

$$\Gamma_t^0 \geq \int_0^t \left(\lim_n \xi_u^n\right)^{-2} du = \lim_n \Gamma_t^n, \qquad t \geq 0,$$

by monotone convergence. But, if ξ^0 and $(\lim_n \xi_t^n)_{t \geq 0}$ have the same law, then the same must hold true for Γ^0 and $\int_0^t (\lim_n \xi_u^n)^{-2} du$, $t \geq 0$. Thus $\Gamma_t^0 \geq \int_0^t (\lim_n \xi_u^n)^{-2} du$ implies $\Gamma_t^0 = \int_0^t (\lim_n \xi_u^n)^{-2} du$ a.s. for each $t \geq 0$. The desired result, $\Gamma_t^0 = \int_0^t (\lim_n \xi_u^n)^{-2} du$, $t \geq 0$, a.s., now follows since both processes have continuous paths.

Thus $\Gamma_t^n \uparrow \Gamma_t^0$, $t \geq 0$, a.s. Since A^n, A^0 are the right-inverses of the continuous increasing processes Γ^n and Γ^0, respectively, we have $A_t^n \downarrow A_t^0$, $t \geq 0$, a.s., completing the proof in the case where the (y_n) are decreasing.

In the case where $y_n \uparrow y_0$ as $n \to \infty$, we see that

$$0 < \xi_t^1 \leq \xi_t^2 \leq \cdots \leq \xi_t^n \leq \cdots \leq \xi_t^0$$

and

$$\Gamma_t^0 \leq \int_0^t \left(\lim_n \xi_u^n\right)^{-2} du = \lim_n \Gamma_t^n, \qquad t \geq 0,$$

by Lebesgue's dominated convergence theorem. This ensures that $\Gamma_t^n \downarrow \Gamma_t^0$, $t \geq 0$, a.s., and $A_t^n \uparrow A_t^0$, $t \geq 0$, a.s. $\square$

In what follows, in addition to the assumptions of Theorem 3.5, we impose the assumption of Lemma 4.1 and the following condition (C4′) is used to summarize these conditions, that is:

(C4′) – the gain function g satisfies $\{g \geq 0\} \neq \varnothing$;

– the process (X, Y) satisfies conditions (1.3), (C1′), (C2′) and the value function v satisfies (4.1) for the chosen value of x;
– condition (C3′) holds true for the system (3.1) and the second equation in (3.1) has a Feller solution.

Note that, in many cases, the conditions one imposes on the coefficients η and θ to ensure condition (C3′) also imply that the whole solution of (3.1) is a Feller process.

We now discuss the continuity of the value function $v(x, \cdot)$ which we subdivide into left-continuity and right-continuity.

PROPOSITION 4.2. *Assume condition* (C4′). *Then*, *when* $T = \infty$, $v(x, \cdot)$ *is left-continuous*.

PROOF. First observe that Theorem 3.5 implies that

$$\limsup_{n\to\infty} v(x, y_n) \le v(x, y_0),$$

whenever $y_n \uparrow y_0$ in $\mathcal{S}$, so it remains to show that

$$v(x, y_0) \le \liminf_{n\to\infty} v(x, y_n).$$

Recall the definition of $\mathcal{M}$ from Proposition 3.3, and choose $\rho \in \mathcal{M}$. Then

$$e^{-q\Gamma_\rho^n} g(G_\rho) \ge -e^{-q\Gamma_\rho^0} \big|g(G_\rho)\big| = -e^{-q\Gamma_\rho^0} \big|g\big(\tilde{X}^0_{\Gamma_\rho^0}\big)\big| \tag{4.4}$$

for all $n = 1, 2, \dots,$ since $\Gamma_\rho^n \ge \Gamma_\rho^0$. But the right-hand side of (4.4) is integrable by (1.3). Thus the inequality

$$\tilde{E} e^{-q\Gamma_\rho^0} g(G_\rho) \le \liminf_{n\to\infty} \tilde{E} e^{-q\Gamma_\rho^n} g(G_\rho) \tag{4.5}$$

follows from Fatou's lemma and Lemma 4.1.

Now $\tilde{E} e^{-q\Gamma_\rho^n} g(G_\rho) \le \sup_{\rho'\in\mathcal{M}} \tilde{E} e^{-q\Gamma_{\rho'}^n} g(G_{\rho'})$, and so Proposition 3.3 gives

$$\tilde{E} e^{-q\Gamma_\rho^0} g(G_\rho) \le \liminf_{n\to\infty} v(x, y_n) \tag{4.6}$$

since $\mathcal{M}_T$ can be replaced by $\mathcal{M}$ in the case where $T = \infty$. So, taking the supremum over $\rho \in \mathcal{M}$ in the left-hand side of (4.6) completes the proof. □

REMARK 4.3. (i) The fact that $v(x, y_0) \le \liminf_{n\to\infty} v(x, y_n)$ when $y_n \downarrow y_0$ in $\mathcal{S}$ is an immediate consequence of Theorem 3.5. As $v(x, y_0) \le \liminf_{n\to\infty} v(x, y_n)$, $y_n \uparrow y_0$, was shown in the proof above, $v(x, \cdot)$ is, under condition (C4′), lower semicontinuous on $\mathcal{S}$ when $T=\infty$ without any continuity-assumption on the gain function g.

(ii) From (i) above it follows that, to establish right-continuity in the case where $T=\infty$, it remains to show that $\limsup_{n\to\infty} v(x,y_n)\le v(x,y_0)$ when $y_n\downarrow y_0$ in $\mathcal{S}$. We are only able to prove this using the extra integrability condition of Proposition 4.4 below. Note that the combination of Propositions 4.2 and 4.4 gives continuity of $v(x,\cdot)$ for fixed x in the case where $T=\infty$ without the requirement that the gain function g is continuous.

(iii) If $T<\infty$, then the proof of Proposition 4.2 fails. Indeed, in this case, ρ cannot be chosen from $\mathcal{M}$ as it belongs to a different class $\mathcal{M}_T^n$ for each $n=0,1,2,\dots$ We are able to show left- and right-continuity in the case where $T<\infty$ under the additional assumption that the gain function g is continuous; see Proposition 4.5.

PROPOSITION 4.4. *Assume, in addition to condition* (C4′), *that for each $y\in\mathcal{S}$ there exists $\bar{y}>y$ such that $(y,\bar{y})\subseteq\mathcal{S}$ and*

$$\sup_{y\le y'<\bar{y}} E_{x,y'}\Big[\sup_{t\ge N} e^{-qt}\big|g(X_t)\big|\Big]\to 0 \qquad \textit{as } N\uparrow\infty.$$

Then, when $T=\infty$, $v(x,\cdot)$ is right-continuous.

PROOF. Choose $y\in\mathcal{S}$ and $y'\in(y,\bar{y})$. Applying Proposition 3.3 with respect to x and y' yields $v(x,y')=\sup_{\rho\in\mathcal{M}}\tilde{E}[e^{-q\Gamma'_\rho}g(G_\rho)]$ since $T=\infty$. Fix an arbitrary $\varepsilon>0$, and choose an ε-optimal stopping time $\rho'_\varepsilon\in\mathcal{M}$ for $v(x,y')$ so that

$$0\le v(x,y')-v(x,y)\le\varepsilon+\tilde{E}\big[e^{-q\Gamma'_{\rho'_\varepsilon}}g(G_{\rho'_\varepsilon})-e^{-q\Gamma_{\rho'_\varepsilon}}g(G_{\rho'_\varepsilon})\big]. \tag{4.7}$$

Because $\Gamma_{\rho'_\varepsilon}\ge\Gamma'_{\rho'_\varepsilon}$, the right-hand side of (4.7) can be dominated by

$$\begin{aligned}
&\varepsilon+\tilde{E}\big(1-e^{-q(\Gamma_{\rho'_\varepsilon}-\Gamma'_{\rho'_\varepsilon})}\big)e^{-q\Gamma'_{\rho'_\varepsilon}}\big|g(G_{\rho'_\varepsilon})\big|I\big(\rho'_\varepsilon\le A'_N\big)\\
&\quad+\tilde{E}e^{-q\Gamma'_{\rho'_\varepsilon}}\big|g(G_{\rho'_\varepsilon})\big|I\big(\rho'_\varepsilon> A'_N\big)\\
&\qquad\le\varepsilon+\tilde{E}\big(1-e^{-q(\Gamma_{\rho'_\varepsilon}-\Gamma'_{\rho'_\varepsilon})}\big)e^{-q\Gamma'_{\rho'_\varepsilon}}\big|g(G_{\rho'_\varepsilon})\big|I\big(\rho'_\varepsilon\le A'_N\big)\\
&\qquad\quad+\tilde{E}\Big[\sup_{t\ge N}e^{-qt}\big|g(\tilde{X}'_t)\big|\Big],
\end{aligned}$$

where

$$\tilde{E}\Big[\sup_{t\ge N}e^{-qt}\big|g(\tilde{X}'_t)\big|\Big]=E_{x,y'}\Big[\sup_{t\ge N}e^{-qt}\big|g(X_t)\big|\Big]$$

and both

$$\begin{aligned} e^{-q\Gamma'_{\rho'_\varepsilon}}\left|g(G_{\rho'_\varepsilon})\right| &\le \sup_{t\le A'_N} e^{-q\Gamma'_t}\left|g(G_t)\right| \\ &\le \sup_{t\le A^{\bar y}_N} e^{-q\Gamma^{\bar y}_t}\left|g(G_t)\right| \qquad (4.8)\\ &= \sup_{t\le A^{\bar y}_N} e^{-q\Gamma^{\bar y}_t}\left|g\big(\tilde X^{\bar y}_{\Gamma^{\bar y}_t}\big)\right| \le \sup_{t\le N} e^{-qt}\left|g\big(\tilde X^{\bar y}_t\big)\right| \end{aligned}$$

and

$$\begin{aligned} \Gamma_{\rho'_\varepsilon} - \Gamma'_{\rho'_\varepsilon} &= \int_0^{\rho'_\varepsilon} \big(\xi_u^{-2} - (\xi'_u)^{-2}\big)\,du \le \int_0^{A'_N} \big(\xi_u^{-2} - (\xi'_u)^{-2}\big)\,du \qquad (4.9)\\ &= \Gamma_{A'_N} - N \end{aligned}$$

on $\{\rho'_\varepsilon \le A'_N\}$. Hence choosing N large enough that

$$\sup_{y\le y'<\bar y} E_{x,y'}\Big[\sup_{t\ge N} e^{-qt}\left|g(X_t)\right|\Big] \le \varepsilon,$$

we obtain from (4.7)

$$\left|v\big(x,y'\big) - v(x,y)\right| \le 2\varepsilon + \tilde E\big(1 - e^{-q(\Gamma_{A'_N} - N)}\big) \sup_{t\le N} e^{-qt}\left|g\big(\tilde X^{\bar y}_t\big)\right| \qquad (4.10)$$

for some N depending on y but NOT on y'.

Now, in inequality (4.10), replace y and y' by y_0 and y_n, respectively, with the (y_n) bounded above by $\bar y$ and decreasing to y_0. Since $\sup_{t\le N} e^{-qt}|g(\tilde X^{\bar y_0}_t)|$ is integrable, it follows by dominated convergence that

$$\lim_{n\to\infty}\left|v(x,y_n) - v(x,y_0)\right| \le 2\varepsilon + \tilde E\big(1 - e^{-q(\lim_n \Gamma^0_{A^n_N} - N)}\big) \sup_{t\le N} e^{-qt}\left|g\big(\tilde X^{\bar y_0}_t\big)\right|,$$

and so

$$\lim_{n\to\infty}\left|v(x,y_n) - v(x,y_0)\right| \le 2\varepsilon$$

by Lemma 4.1. Since ε is arbitrary we conclude with the desired result. □

PROPOSITION 4.5. *Assume, in addition to condition* (C4′), *that the gain function g is continuous. Then, when $T < \infty$, $v(x,\cdot)$ is continuous.*

PROOF. Following the proof of the previous proposition choose $y, y' \in \mathcal S$ with $y < y'$, fix an arbitrary $\varepsilon > 0$ and choose an ε-optimal stopping time $\rho'_\varepsilon \in \mathcal M'_T$ so that

$$\begin{aligned} 0 &\le v\big(x,y'\big) - v(x,y) \qquad (4.11)\\ &\le \varepsilon + \tilde E\big[e^{-q\Gamma'_{\rho'_\varepsilon}} g(G_{\rho'_\varepsilon}) - e^{-q\Gamma_{\rho'_\varepsilon\wedge A_T}} g(G_{\rho'_\varepsilon\wedge A_T})\big]. \end{aligned}$$

Note that $\rho'_\varepsilon \le A'_T$ and that $\rho'_\varepsilon \wedge A_T$ is used since one cannot conclude that $v(x,y) \ge \tilde{E}e^{-q\Gamma_\rho} g(G_\rho)$ for stopping times ρ which may exceed A_T with positive probability. Therefore, in contrast to the case where $T=\infty$, dominating the right-hand side of (4.11) leads to an upper bound of

$$\varepsilon + \tilde{E}\bigl(1-e^{-q(\Gamma_{\rho'_\varepsilon}-\Gamma'_{\rho'_\varepsilon})}\bigr)e^{-q\Gamma'_{\rho'_\varepsilon}}\bigl|g(G_{\rho'_\varepsilon})\bigr| \tag{4.12}$$

$$+\,\tilde{E}\bigl(1-e^{-q(T-\Gamma'_{\rho'_\varepsilon})}\bigr)e^{-q\Gamma'_{\rho'_\varepsilon}}\bigl|g(G_{\rho'_\varepsilon})\bigr|I\bigl(A_T<\rho'_\varepsilon\le A'_T\bigr) \tag{4.13}$$

$$+\,\tilde{E}e^{-qT}\bigl|g(G_{\rho'_\varepsilon})-g(G_{A_T})\bigr|I\bigl(A_T<\rho'_\varepsilon\le A'_T\bigr) \tag{4.14}$$

by adding $-e^{-qT}g(G_{\rho'_\varepsilon})+e^{-qT}g(G_{\rho'_\varepsilon})$ in the case where $A_T<\rho'_\varepsilon\le A'_T$.

Now replace y and y' by y_n and y_0, respectively, with $y_n \uparrow y_0$ in $\mathcal{S}$. Suppose for now that Lebesgue's dominated convergence theorem can be applied to interchange limit and expectation in (4.12), (4.13), (4.14). Then it can be shown that

$$\lim_{n\to\infty}\bigl|v(x,y_n)-v(x,y_0)\bigr|\le\varepsilon$$

proving left-continuity since ε was arbitrary. To see this first dominate

$$\Gamma^n_{\rho^0_\varepsilon}-\Gamma^0_{\rho^0_\varepsilon}\quad\text{by}\quad\Gamma^n_{A^0_T}-T$$

performing a calculation similar to (4.9), but using T instead of N. Then (4.12) tends to ε as $n\to\infty$ by Lemma 4.1. Second, since $\{A^n_T<\rho^0_\varepsilon\le A^0_T\}=\{\Gamma^0_{A^n_T}<\Gamma^0_{\rho^0_\varepsilon}\le T\}$, both (4.13) and (4.14) converge to zero as $n\to\infty$ by Lemma 4.1 and the continuity of g.

Finally it remains to justify the application of the dominated convergence theorem. Observe that

$$e^{-qT}\bigl|g(G_{\rho^0_\varepsilon})\bigr|\le e^{-q\Gamma^0_{\rho^0_\varepsilon}}\bigl|g(G_{\rho^0_\varepsilon})\bigr|=e^{-q\Gamma^0_{\rho^0_\varepsilon}}\bigl|g\bigl(\tilde{X}^0_{\Gamma^0_{\rho^0_\varepsilon}}\bigr)\bigr|\le\sup_{0\le t\le T}e^{-qt}\bigl|g\bigl(\tilde{X}^0_t\bigr)\bigr|$$

since $\rho^0_\varepsilon\le A^0_T$ and

$$e^{-qT}\bigl|g(G_{A^n_T})\bigr|=e^{-q\Gamma^n_{A^n_T}}\bigl|g(G_{A^n_T})\bigr|\le\sup_{t\le A^0_T}e^{-q\Gamma^0_t}\bigl|g(G_t)\bigr|\le\sup_{0\le t\le T}e^{-qt}\bigl|g\bigl(\tilde{X}^0_t\bigr)\bigr|$$

since $A^n_T\le A^0_T$ for all $n\ge1$ which, by (1.3), gives an integrable bound with respect to all three terms (4.12), (4.13), (4.14).

For the right-continuity, replace y and y' by y_0 and y_n, respectively, assuming $y_n\downarrow y_0$ in $\mathcal{S}$. Note that

$$e^{-qT}\bigl|g(G_{\rho^n_\varepsilon})\bigr|\le e^{-q\Gamma^n_{\rho^n_\varepsilon}}\bigl|g(G_{\rho^n_\varepsilon})\bigr|\le\sup_{t\le T}e^{-qt}\bigl|g\bigl(\tilde{X}^1_t\bigr)\bigr|,$$

where the second inequality is obtained following the line of inequalities in (4.8) but using T and y_1 instead of N and $\bar{y}$, respectively. As $e^{-qT}|g(G_{A^0_T})|\le$

$\sup_{0\le t\le T} e^{-qt}|g(\tilde{X}_t^0)|$ too, dominated convergence can be applied again by (1.3) with respect to all three terms (4.12), (4.13), (4.14). Then (4.12) tends to ε as $n\to\infty$ by Lemma 4.1 since $\Gamma^0_{\rho^n_\varepsilon}-\Gamma^n_{\rho^n_\varepsilon}$ can be estimated by $\Gamma^0_{A^n_T}-T$. Furthermore, (4.13) and (4.14) converge to zero as $n\to\infty$ by Lemma 4.1 and the continuity of g since $T-\Gamma^n_{\rho^n_\varepsilon}\le T-\Gamma^n_{A^0_T}$ on $\{A^0_T<\rho^n_\varepsilon\le A^n_T\}$. So, making ε arbitrarily small completes the proof. $\square$

5. Application to option pricing. Assume that the dynamics of X are given by

$$dX = XY\,dB, \tag{1.1$'$}$$

which is the special case $a(x)=x$ of equation (1.1). In mathematical finance (1.1′) describes a simple model for the discounted price of an asset with stochastic volatility Y.

If exercised at a stopping time τ, the American options we have in mind would pay off $g(e^{r\tau}X_\tau)$ where $r>0$ stands for the instantaneous interest rate which is assumed to be constant. So, for notational convenience, the discount rate q is replaced by r throughout this section.

In this setup, assuming the measure $P_{x,y}$ is used for pricing when $X_0=x$ and $Y_0=y$, the price of such an option with maturity $T\in[0,\infty]$ is

$$v(x,y)=\sup_{0\le\tau\le T} E_{x,y}\bigl[e^{-r\tau}g\bigl(e^{r\tau}X_\tau\bigr)\bigr], \tag{1.2$'$}$$

where the supremum is taken over all finite stopping times with respect to the filtration generated by (X,Y). This value function differs from the value function given by (1.2) since g is not applied to X_τ but to $e^{r\tau}X_\tau$ and, as a consequence, some of the conditions for our results have to be adjusted slightly.

First, the condition

$$E_{x,y}\Bigl[\sup_{0\le t\le T} e^{-rt}\bigl|g\bigl(e^{rt}X_t\bigr)\bigr|I(t<\infty)\Bigr]<\infty \qquad \text{for all } (x,y)\in\mathbb{R}\times\mathcal{S} \tag{1.3$'$}$$

is now assumed throughout. Then

$$v(x,y)=\sup_{\tau\in\mathcal{T}_T}\tilde{E}\bigl[e^{-r\tau}g\bigl(e^{r\tau}\tilde{X}_\tau\bigr)\bigr]=\sup_{\rho\in\mathcal{M}_T}\tilde{E}\bigl[e^{-r\Gamma_\rho}g\bigl(e^{r\Gamma_\rho}G_\rho\bigr)\bigr]$$

is the analogue to what was obtained in Propositions 2.4 and 3.3 for the value function given by (1.2). However, in order to conclude the results of Theorems 2.5 and 3.5 for the new value function, a new condition has to be imposed on g.

COROLLARY 5.1. *Let v be the value function given by* (1.2′). *In addition to the assumptions made in either Theorem* 2.5 *or* 3.5 *assume that g is a decreasing function. Define $\mathcal{K}_T^{g+}$ to be the collection of all finite stopping times $\tau\le T$ with* *respect to the filtration generated by* (X, Y) *such that* $g(e^{r\tau} X_\tau) \geq 0$. *Fix* $(x, y) \in \mathbb{R} \times \mathcal{S}$ *and assume that* $v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}[e^{-r\tau} g(e^{r\tau} X_\tau)]$. *Then*

$$v(x, y) \leq v(x, y') \qquad \textit{for all } y' \in \mathcal{S} \textit{ such that } y \leq y',$$

so that $v(x, y)$ *is a lower bound for* $v(x, \cdot)$ *on* $[y, \infty) \cap \mathcal{S}$.

REMARK 5.2. (i) The proofs of this and the next corollary are contained in the Appendix.

(ii) If g is a monotone function, then it has a left and a right-continuous version. Note that the proof of Corollary 5.1 does not depend on choosing a specific version for g. But, when applying the corollary to show continuity properties of the value function, we will choose the right-continuous version in what follows.

(iii) Of course, Corollary 5.1 does not depend on the specific choice of the diffusion coefficient a in this section as long as (1.3′) and all other assumptions of Theorems 2.5 or 3.5 are satisfied.

(iv) If $a(x) = x$, then conditions (C2) or (C2′) assumed in Corollary 5.1 ensures that the discounted price X is a positive exponential local martingale of the form

$$X_t = x \exp\left\{\int_0^t Y_s \, dB_s - \frac{1}{2}\int_0^t Y_s^2 \, ds\right\}, \qquad t \geq 0, P_{x,y}\text{-a.s.},$$

since the stochastic integrals $\int_0^t Y_s \, dB_s, t \geq 0$, are all well defined. Furthermore, because $\lim_{t\uparrow\infty} \int_0^t Y_s^2 \, ds = \infty$ $P_{x,y}$-a.s., X_t tends to zero for large t as in the Black–Scholes model.

(v) From (iv) above it follows immediately that, in the case $a(x) = x$, all processes satisfying conditions (C1) and (C2) on page 1556 have the same law.

(vi) Note that, in this section, the equation for G in (3.1) on page 1565 coincides with the linear equation $dG = G\, dW$ which has a unique nonexploding strong solution for all $G_0 \in \mathbb{R}$. Hence condition (C3′) on page 1566 becomes a condition only on the coefficients η, θ of the equation for ξ in (3.1).

We now consider the diffusion case and discuss the results of Section 4 for the value function given by (1.2′). So, let $\mathcal{S}$ be an open subset of $(0, \infty)$, fix $x \in \mathbb{R}$ and replace condition (C4′) on page 1569 by:

(C4″): – the gain function g is decreasing and satisfies $\{g \geq 0\} \neq \emptyset$;
– the process (X, Y) satisfies conditions (1.3′), (C1′), (C2′), and the value function v satisfies

$$v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}\big[e^{-r\tau} g\big(e^{r\tau} X_\tau\big)\big] \qquad \text{for all } y \in \mathcal{S} \tag{4.1′}$$

for the chosen x (using the definition of $\mathcal{K}_T^{g+}$ given in Corollary 5.1);

– condition (C3′) holds true for system (3.1), and the second equation in (3.1) has a Feller solution.

COROLLARY 5.3. *Let v be the value function given by* (1.2′). *Assume condition* (C4′′).

(i) *If g is bounded from below then, when $T = \infty$, $v(x, \cdot)$ is left-continuous and lower semicontinuous.*

(ii) *If g is continuous and if for each $y \in \mathcal{S}$ there exists $\bar{y} > y$ such that $(y, \bar{y}) \subseteq \mathcal{S}$ and*

$$\sup_{y \le y' < \bar{y}} E_{x,y'}\Big[\sup_{t \ge N} e^{-rt}\big|g\big(e^{rt}X_t\big)\big|\Big] \to 0 \qquad \text{as } N \uparrow \infty,$$

then, when $T = \infty$, $v(x, \cdot)$ is right-continuous.

(iii) *If g is bounded from below and continuous then, when $T < \infty$, $v(x, \cdot)$ is continuous.*

6. Examples. We now discuss three models used in option pricing and explain the impact of our results.

Pricing of American Puts via Jobert and Rogers [9] using the Markov modulated model

$$dX = XY\,dB, \qquad Y \text{ finite state Markov chain.}$$

Notice that the value function in [9] is more general than ours as the authors allow for an interest rate which depends on Y. So in what follows we always mean a constant interest rate when applying our results to the value function[3] in [9].

Obviously, the gain function $g(x) = \max\{0, K - x\}$ where K is the strike price is decreasing and satisfies both condition (1.3′) and

$$v(x, y) = \sup_{\tau \in \mathcal{K}_T^{g+}} E_{x,y}\big[e^{-r\tau} g\big(e^{r\tau}X_\tau\big)\big], \qquad (x, y) \in \mathbb{R} \times \mathcal{S}.$$

So, recalling Remark 5.2(iv) + (v), Corollary 5.1 implies that, for fixed $x \in \mathbb{R}$, the value function $v(x, y)$ in [9] is monotonously increasing in $y \in \mathcal{S} = \{y_1, \dots, y_m\}$, provided Y is skip-free.

Knowing this monotonicity property of the value function massively reduces the computational complexity of PROBLEM 1 on page 2066 in [9]. The authors verified that the value function is uniquely attained at a stopping time of the form[4]

$$\tau^\star = \inf\{t \ge 0 : X_t < b[Y_t]\},$$

[3]Note that the notation of the value function in [9] is different because our Markov chain Y is, in their terms, a function σ applied to the Markov chain playing the role of their volatility process.

[4]We again adapted the author's notation to ours in the definition of $\tau^\star$.

where the vector $b[y_i], i = 1, \ldots, m$, is indexed by the states of the Markov chain Y and their PROBLEM 1 consists in finding the so-called *thresholds* $b[y_i]$ which are assumed to be in the order $b[y_1] \geq \cdots \geq b[y_m]$. It is then stated in a footnote on the same page, 2066, that "When it comes in practice to identifying the thresholds, no assumption is made on the ordering, and all possible orderings are considered." Of course, this approach has exponential complexity. Our result on the monotonicity of the value function would reduce this complexity to choosing one ordering $b[y_1] > \cdots > b[y_m]$ if $y_1 < \cdots < y_m$ and Y is skip-free. Indeed, since $\tau^\star$ is the unique optimal stopping time for this problem, by general theory, it must coincide with the first time the process (X, Y) enters the stopping region $\{(x, y) : v(x, y) = g(x)\}$. Thus, as it is not optimal to stop when g is zero, we obtain that

$$v(x, y_i) = g(x) \text{ for } x \leq b[y_i] \quad \text{while} \quad v(x, y_i) > g(x) \text{ for } x > b[y_i]$$

for each $i = 1, \ldots, m$ which gives the unique ordering of the thresholds since g is strictly decreasing on $\{g > 0\}$.

The Hull and White model [7]:

$$dX = X\sqrt{V}\, dB \quad \text{and} \quad dV = 2\eta V\, dB^Y + \kappa V\, dt,$$

where $\eta, \kappa > 0$ and B, B^Y are independent Brownian motions.[5] Setting $Y = \sqrt{V}$ transforms the above system into

$$dX = XY\, dB \quad \text{and} \quad dY = \eta Y\, dB^Y + \theta Y\, dt,$$

where $\theta = (\kappa - \eta^2)/2$. Assuming a positive initial condition, this equation has a pathwise unique positive solution for every $\eta, \theta \in \mathbb{R}$. Calculating the equation for ξ in (3.1) on page 1565 gives a constant diffusion coefficient η, and if Z denotes ξ/η, then

$$dZ = dW^\xi + \frac{\theta}{\eta^2} Z^{-1}\, dt,$$

which formally is an equation for a Bessel process of dimension $\phi = 1 + 2\theta/\eta^2$. This equation, and so the equation for ξ, only has a unique nonexploding strong solution if $\phi \geq 2$, and this solution stays positive when started from a positive initial condition. As made clear in Section 3, the fact that Y satisfies condition (C2′) on page 1565 can be derived from condition (2.4) with respect to

$$\Gamma_t = \int_0^t \frac{1}{\xi_u^2}\, du = \eta^2 \int_0^t \frac{1}{Z_u^2}\, du, \qquad t \geq 0.$$

Now, by applying Proposition A.1(ii)–(iii) in [6] with respect to the second time integral above, we see that Γ satisfies condition (2.4) if $\phi \geq 2$. So, assuming $\phi \geq 2$,

[5]Remark that $\langle B, B^Y \rangle \neq 0$ is possible but we follow Hull and White's original setup.

Remark 5.2(iv) + (vi) ensures that there is a unique strong Markov process (X, Y) which satisfies conditions (C1′) and (C2′) on page 1565 and that the system (3.1) satisfies condition (C3′) on page 1566 in this example. Since Bessel processes are Feller processes (see [11], page 446), the second equation of (3.1) has a Feller solution.

Therefore if $\phi \geq 2$ (i.e., $\kappa \geq 2\eta^2$), then the conclusions of Corollaries 5.1 and 5.3 apply to perpetual American options whenever the corresponding pay-off function g satisfies the conditions stated.

The Heston model [5]:

$$dX = X\sqrt{V}\,dB \quad \text{and} \quad dV = 2\eta\sqrt{V}\,dB^Y + \kappa(\lambda - V)\,dt,$$

where $\eta, \kappa, \lambda > 0$ are constants, and B, B^Y are Brownian motions, this time with covariation $\delta \in [-1, 1]$. The equation for V describes the so-called Cox–Ingersoll–Ross process, and it is well known (see [2], page 391) that, with a positive initial condition, this equation has a pathwise unique positive solution if $\kappa\lambda \geq 2\eta^2$. Setting $Y = \sqrt{V}$ transforms the system into

$$dX = XY\,dB \quad \text{and} \quad dY = \eta\,dB^Y + \left(\frac{\theta_1}{Y} - \theta_2 Y\right)dt$$

with $\theta_1 = (\kappa\lambda - \eta^2)/2$ and $\theta_2 = \kappa/2$. It is clear that the pathwise uniqueness of the equation for V ensures the pathwise uniqueness of positive solutions of the equation for Y. Calculating the equation for ξ in (3.1) on page 1565 yields

$$d\xi = \frac{\eta}{\xi}\,dW^\xi + \left(\frac{\theta_1}{\xi^3} - \frac{\theta_2}{\xi}\right)dt,$$

and hence $Z = \xi^2/(2\eta)$ satisfies

$$dZ = dW^\xi + \left(\frac{\phi - 1}{2Z} - \frac{\theta_2}{\eta}\right)dt$$

with $\phi = \theta_1/\eta^2 + 3/2$. By changing to an equivalent probability measure, this equation for Z is transformed into an equation for a Bessel process of dimension ϕ which only has a unique nonexploding strong solution if $\phi \geq 2$, and this unique strong solution stays positive when started from a positive initial condition. All these properties and the Feller property of Bessel processes carry over to the solutions of the equation for ξ. Finally, the process

$$\Gamma_t = \int_0^t \frac{1}{\xi_u^2}\,du = \frac{1}{2\eta}\int_0^t \frac{1}{Z_u}\,du, \qquad t \geq 0,$$

satisfies (2.4) if $\phi \geq 2$ (apply Proposition A.1(ii)–(iii) in [6] to the second integral) which implies condition (C2′) on page 1565 following the arguments given in Section 3. So, as in the previous example, all conditions imposed on X, Y, ξ in the Corollaries 5.1 and 5.3 are satisfied if $\phi \geq 2$ or equivalently $\kappa\lambda \geq 2\eta^2$.

APPENDIX

PROOF OF LEMMA 2.3. Fix $\rho \in \mathcal{M}$ and $r \geq 0$, and set

$$\Omega_0 = \{\omega \in \Omega : s < \Gamma_t(\omega) \text{ if and only if } A_s(\omega) < t \text{ for all } 0 \leq s, t < \infty\}.$$

Then

$$\{\Gamma_\rho \leq r\} \cap \Omega_0 \cap \{A_r < \infty\} = \{\rho \leq A_r\} \cap \Omega_0 \cap \{A_r < \infty\}$$

implies

$$\{\Gamma_\rho \leq r\} \in \mathcal{F}_{A_r}^{G,Z,Z'}$$

since both $P(\Omega_0 \cap \{A_r < \infty\}) = 1$ by property (P2), (2.5) and $\{\rho \leq A_r\} \in \mathcal{F}_{A_r}^{G,Z,Z'}$. Note that $\Omega_0 \cap \{A_r < \infty\} \in \mathcal{F}_{A_r}^{G,Z,Z'}$ as $\mathcal{F}_0^{G,Z,Z'}$ already contains all $\tilde{P}$-null sets.

Similarly, if $\tau \in \mathcal{T}$, then $\{A_\tau \leq r\} \in \mathcal{F}_{A_{\Gamma_r}}^{G,Z,Z'}$ where $A_{\Gamma_r} = r$ a.s. by property (P1). Thus the inclusion $\mathcal{F}_{A_{\Gamma_r}}^{G,Z,Z'} \subseteq \mathcal{F}_r^{G,Z,Z'}$ must be true. $\square$

PROOF OF (2.6). By (2.3), we only have to show that

$$\sup_{0 \leq \tilde{\tau} \leq T} \tilde{E}[e^{-q\tilde{\tau}} g(\tilde{X}_{\tilde{\tau}})] = \sup_{0 \leq \tau \leq T} \tilde{E}[e^{-q\tau} g(\tilde{X}_\tau)], \tag{A.1}$$

where $\tilde{\tau}$ on the above left-hand side corresponds to finite stopping times with respect to the filtration $\mathcal{F}_t^{\tilde{X},\tilde{Y}}, t \geq 0$, generated by the pair of processes $(\tilde{X}, \tilde{Y})$ while τ on the above right-hand side corresponds to finite stopping times with respect to the possibly bigger filtration $\mathcal{F}_{A_t}^{G,Z,Z'}, t \geq 0$. In what follows we assume that $\mathcal{F}_t^{\tilde{X},\tilde{Y}}, t \geq 0$, was augmented. Without loss of generality, we also assume that there exist a family $\{\theta_t, t \geq 0\}$ of shift operators on our chosen probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$.

We are going to show that

$$\sup_{\tilde{\tau} \in \mathcal{O}_0^T} \tilde{E}[e^{-q\tilde{\tau}} g(\tilde{X}_{\tilde{\tau}})] = \sup_{0 \leq \tau \leq T} \tilde{E}[e^{-q\tau} g(\tilde{X}_\tau)],$$

where $\mathcal{O}_s^T$ stands for the family of all finite $\mathcal{F}_t^{\tilde{X},\tilde{Y}}$-stopping times $\tilde{\tau}$ satisfying $s \leq \tilde{\tau} \leq T$ and $\tilde{\tau} - s \overset{\text{a.s.}}{=} \gamma \circ \theta_s$ for some $\mathcal{F}_\infty^{\tilde{X},\tilde{Y}}$-measurable random variable γ. This obviously proves (A.1) because the above left-hand side is less than or equal to the left-hand side of (A.1).

First observe that

$$\mathcal{F}_0^{\tilde{X},\tilde{Y}} = \mathcal{F}_0^{G,Z,Z'} = \mathcal{F}_{A_0}^{G,Z,Z'} = \sigma \qquad (\tilde{P}\text{-null sets})$$

hence

$$\sup_{\tilde\tau\in\mathcal{O}_0^T} \tilde E\big[e^{-q\tilde\tau} g(\tilde X_{\tilde\tau})\big] \overset{\text{a.s.}}{=} \tilde V_0$$

$$\text{where } \tilde V_t = \operatorname*{ess\,sup}_{\tilde\tau\in\mathcal{O}_t^T} \tilde E\big[e^{-q\tilde\tau} g(\tilde X_{\tilde\tau})|\mathcal{F}_t^{\tilde X,\tilde Y}\big]$$

and

$$\sup_{0\le\tau\le T} \tilde E\big[e^{-q\tau} g(\tilde X_\tau)\big] \overset{\text{a.s.}}{=} V_0$$

$$\text{where } V_t = \operatorname*{ess\,sup}_{t\le\tau\le T} \tilde E\big[e^{-q\tau} g(\tilde X_\tau)|\mathcal{F}_{A_t}^{G,Z,Z'}\big].$$

Note that $t\in\mathcal{O}_t^T$ gives $\tilde V_t \ge e^{-qt} g(\tilde X_t)$ almost surely for each $t\ge 0$.

Second, since $(\tilde X,\tilde Y)$ has the same law as (X,Y) under $P_{x,y}$, the process $(\tilde X,\tilde Y)$ is strong Markov with respect to $\mathcal{F}_t^{\tilde X,\tilde Y}$, $t\ge 0$. Therefore

$$\begin{aligned}\tilde E\big[e^{-q\tilde\tau} g(\tilde X_{\tilde\tau})|\mathcal{F}_t^{\tilde X,\tilde Y}\big] &\overset{\text{a.s.}}{=} \tilde E\big[e^{-q\tilde\tau} g(\tilde X_{\tilde\tau})|\sigma(\tilde X_t,\tilde Y_t)\big] \\ &\overset{\text{a.s.}}{=} \tilde E\big[e^{-q\tilde\tau} g(\tilde X_{\tilde\tau})|\mathcal{F}_{A_t}^{G,Z,Z'}\big]\end{aligned} \tag{A.2}$$

for all $\tilde\tau\in\mathcal{O}_t^T$ because $(\tilde X,\tilde Y)$ is strong Markov with respect to $\mathcal{F}_{A_t}^{G,Z,Z'}$, $t\ge 0$, too. Note that we have only used the Markov property to get (A.2).

Third, (A.2) implies that $\tilde V_t$, $t\ge 0$, is an $\mathcal{F}_{A_t}^{G,Z,Z'}$-supermartingale. Using assumption (1.3), the proof of this fact is almost identical to part 1^o of the proof of Theorem 2.2 in [10]. The only difference is concerned with $\mathcal{F}_{A_t}^{G,Z,Z'}$-stopping times of type

$$\tau = \tilde\tau_1 I_\Gamma + \tilde\tau_2 I_{\tilde\Omega\setminus\Gamma}$$

given by

$$\Gamma = \big\{\tilde E\big[e^{-q\tilde\tau_1} g(\tilde X_{\tilde\tau_1})|\mathcal{F}_{A_t}^{G,Z,Z'}\big] \ge \tilde E\big[e^{-q\tilde\tau_2} g(\tilde X_{\tilde\tau_2})|\mathcal{F}_{A_t}^{G,Z,Z'}\big]\big\}$$

and $\tilde\tau_1,\tilde\tau_2\in\mathcal{O}_t^T$ where $t\ge 0$ is fixed. We need to show that $\tau\in\mathcal{O}_t^T$. But, if $\tilde\tau_1 - t \overset{\text{a.s.}}{=} \gamma_1\circ\theta_t$ and $\tilde\tau_2 - t \overset{\text{a.s.}}{=} \gamma_2\circ\theta_t$, then, by (A.2),

$$\tau(\omega) - t = \gamma_1(\theta_t\omega) I_{\Gamma_0}(\theta_t\omega) + \gamma_1(\theta_t\omega) I_{\tilde\Omega\setminus\Gamma_0}(\theta_t\omega) \tag{A.3}$$

for almost every $\omega\in\tilde\Omega$. Here Γ_0 stands for a set of type $\{\phi_1(\tilde X_0,\tilde Y_0)\ge\phi_2(\tilde X_0,\tilde Y_0)\}$ where $\phi_1,\phi_2:\mathbb{R}^2\to\mathbb{R}$ are Borel-measurable functions satisfying

$$\phi_1(\tilde X_t,\tilde Y_t) \overset{\text{a.s.}}{=} \tilde E\big[e^{-q\tilde\tau_1} g(\tilde X_{\tilde\tau_1})|\sigma(\tilde X_t,\tilde Y_t)\big]$$

and

$$\phi_2(\tilde X_t,\tilde Y_t) \overset{\text{a.s.}}{=} \tilde E\big[e^{-q\tilde\tau_2} g(\tilde X_{\tilde\tau_2})|\sigma(\tilde X_t,\tilde Y_t)\big],$$

and hence (A.3) justifies $\tau \in \mathcal{O}_t^T$.

Now, by Theorem 2.2 in [10], the Snell envelope $V_t, t \geq 0$, is the smallest $\mathcal{F}_{A_t}^{G,Z,Z'}$-supermartingale dominating the gain process and hence $\tilde{V}_t \geq V_t$ almost surely for each $t \geq 0$ proving

$$\sup_{\tilde{\tau} \in \mathcal{O}_0^T} \tilde{E}[e^{-q\tilde{\tau}} g(\tilde{X}_{\tilde{\tau}})] \geq \sup_{0 \leq \tau \leq T} \tilde{E}[e^{-q\tau} g(\tilde{X}_\tau)].$$

The reverse inequality is obvious. $\square$

LEMMA A.1. *Let $W, Z, A, \tilde{M}$ be given on the filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_t, t \geq 0, \tilde{P})$ as introduced in Section 2. Then the time-changed processes $\tilde{B} = \tilde{M} \circ A$ and $\tilde{Y} = Z \circ A$ are independent.*

PROOF. Let $\mathcal{F}_t^W, t \geq 0$, denote the augmentation of the filtration generated by W and define the so-called big filtration by

$$\mathcal{F}_t^{\text{big}} = \mathcal{F}_t^W \vee \sigma(\{Z_s : s \geq 0\}), \qquad t \geq 0.$$

Note that W is an $\mathcal{F}_t^{\text{big}}$ Brownian motion since W and Z are independent, and hence the stochastic integral $\tilde{M}$ is a continuous $\mathcal{F}_t^{\text{big}}$ local martingale. Since A is a functional of Z, it must be an $\mathcal{F}_t^{\text{big}}$ time-change by the definition of the big filtration. As A satisfies (2.5), it follows from Dambis–Dubins–Schwarz' theorem [11], Theorem V.1.6, that $\tilde{B} = \tilde{M} \circ A$ is an $\mathcal{F}_{A_t}^{\text{big}}$ Brownian motion. But $\tilde{Y} = Z \circ A$ is a functional of Z, so it must be independent of $\tilde{B}$ since $\sigma(\{Z_s : s \geq 0\}) \subseteq \mathcal{F}_0^{\text{big}} = \tilde{\mathcal{F}}_{A_0}^{\text{big}}$, and $\tilde{B}$ is independent of $\mathcal{F}_{A_0}^{\text{big}}$. $\square$

PROOF OF COROLLARY 5.1. The only part of the proof where the additional condition on g is needed is the verification of (2.9). But, for (1.2′), the modification of (2.9) reads

$$\tilde{E}[e^{-r\Gamma_\rho} g(e^{r\Gamma_\rho} G_\rho) \leq \tilde{E}[e^{-r\Gamma'_\rho} g(e^{r\Gamma'_\rho} G_\rho)] \qquad \text{for every } \rho \in \mathcal{M}_T^+,$$

and the above inequality is indeed true because $\Gamma_t \geq \Gamma'_t, t \geq 0$, a.s., and g is decreasing. Note that the above set of stopping times $\mathcal{M}_T^+$ now denotes the set $\{\rho \in \mathcal{M}_T : g(e^{r\Gamma_\rho} G_\rho) \geq 0 \text{ a.s.}\}$. $\square$

PROOF OF COROLLARY 5.3. First observe that Lemma 4.1 follows by simply applying Corollary 5.1 instead of Theorem 3.5 and can therefore be used in the proof below.

Now, as the left-hand side of the estimate (4.4) is trivially bounded from below since g is bounded from below, we obtain

$$\tilde{E} e^{-r\Gamma_\rho^0} g(e^{r\Gamma_\rho^0} G_\rho) \leq \liminf_{n \to \infty} \tilde{E} e^{-r\Gamma_\rho^n} g(e^{r\Gamma_\rho^n} G_\rho)$$

using Fatou's lemma, Lemma 4.1 and Remark 5.2(ii). The remaining arguments below (4.5) used to show Proposition 4.2 also apply in the case where (1.2′) holds proving the left-continuity claimed in part (i). And finally, the lower semicontinuity follows by the argument for lower semicontinuity given in Remark 4.3(i).

The proof of part (ii) is along the lines of the proof of Proposition 4.4 with some small changes emphasized below.

First, using the value function defined in (1.2′), the right-hand side of (4.7) is dominated by

$$\begin{aligned}\varepsilon &+ \tilde{E}\big(1-e^{-r(\Gamma_{\rho'_\varepsilon}-\Gamma'_{\rho'_\varepsilon})}\big)e^{-r\Gamma'_{\rho'_\varepsilon}}\big|g\big(e^{r\Gamma'_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)\big|I(\rho'_\varepsilon\le A'_N)\\ &+\tilde{E}e^{-r\Gamma_{\rho'_\varepsilon}}\big|g\big(e^{r\Gamma'_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)-g\big(e^{r\Gamma_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)\big|I(\rho'_\varepsilon\le A'_N)\\ &+\tilde{E}\Big[\sup_{t\ge N}e^{-rt}\big|g(\tilde{X}'_t)\big|\Big]+\tilde{E}\Big[\sup_{t\ge N}e^{-rt}\big|g(\tilde{X}_t)\big|\Big],\end{aligned}$$

where the middle term

$$\tilde{E}e^{-r\Gamma_{\rho'_\varepsilon}}\big|g\big(e^{r\Gamma'_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)-g\big(e^{r\Gamma_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)\big|I(\rho'_\varepsilon\le A'_N)\tag{A.4}$$

is new. Note that the ε-optimal stopping time ρ'_ε can be chosen from the set $(\mathcal{M}')^+=\{\rho\in\mathcal{M}:g(e^{r\Gamma'_\rho}G_\rho)\ge 0\}$ and so

$$\begin{aligned}e^{-r\Gamma'_{\rho'_\varepsilon}}\big|g\big(e^{r\Gamma'_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)\big| &= e^{-r\Gamma'_{\rho'_\varepsilon}}g\big(e^{r\Gamma'_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)\\ &\le e^{-r\Gamma^{\bar{y}}_{\rho'_\varepsilon}}g\big(e^{r\Gamma^{\bar{y}}_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)\\ &\le \sup_{t\ge 0}e^{-r\Gamma^{\bar{y}}_t}\big|g\big(e^{r\Gamma^{\bar{y}}_t}G_t\big)\big|\le\sup_{t\ge 0}e^{-rt}\big|g\big(e^{rt}\tilde{X}^{\bar{y}}_t\big)\big|.\end{aligned}$$

Using this in place of the upper bound on the right-hand side of (4.8), we obtain that

$$\begin{aligned}\big|v(x,y')-v(x,y)\big| &\le 3\varepsilon+\tilde{E}\big(1-e^{-r(\Gamma_{A'_N}-N)}\big)\sup_{t\ge 0}e^{-rt}\big|g\big(e^{rt}\tilde{X}^{\bar{y}}_t\big)\big|\\ &\quad+\tilde{E}e^{-r\Gamma_{\rho'_\varepsilon}}\big|g\big(e^{r\Gamma'_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)-g\big(e^{r\Gamma_{\rho'_\varepsilon}}G_{\rho'_\varepsilon}\big)\big|I(\rho'_\varepsilon\le A'_N).\end{aligned}$$

So, after y and y' were replaced by y_0 and y_n respectively, it only remains to show that

$$\lim_{n\to\infty}\tilde{E}e^{-r\Gamma^0_{\rho^n_\varepsilon}}\big|g\big(e^{r\Gamma^n_{\rho^n_\varepsilon}}G_{\rho^n_\varepsilon}\big)-g\big(e^{r\Gamma^0_{\rho^n_\varepsilon}}G_{\rho^n_\varepsilon}\big)\big|I(\rho^n_\varepsilon\le A^n_N)=0.\tag{A.4′}$$

This limit refers to the new term in (A.4) which was not considered in the proof of Proposition 4.4. But, by dominated convergence, (A.4′) would follow if, for almost every $\omega\in\Omega$, the equality

$$\begin{aligned}\lim_{n\to\infty}&\big|g\big(e^{r\Gamma^n_{\rho^n_\varepsilon(\omega)}(\omega)}G_{\rho^n_\varepsilon(\omega)}(\omega)\big)-g\big(e^{r\Gamma^0_{\rho^n_\varepsilon(\omega)}(\omega)}G_{\rho^n_\varepsilon(\omega)}(\omega)\big)\big|\\ &\times I\big(\rho^n_\varepsilon(\omega)\le A^n_N(\omega)\big)=0\end{aligned}\tag{A.5}$$

holds, and this is true. Indeed, choose $\omega \in \Omega$ such that both $\Gamma^0_{A^n_N(\omega)}(\omega) \to N$ as $n \to \infty$ and $t \mapsto G_t(\omega)$ is continuous. Define

$$c_1 = \sup_{t \le A^{\bar{y}_0}_N(\omega)} |G_t(\omega)|, \qquad c_2 = \Gamma^0_{A^{\bar{y}_0}_N(\omega)}(\omega),$$

and observe that

$$0 \le \rho^n_\varepsilon(\omega) I(\rho^n_\varepsilon(\omega) \le A^n_N(\omega)) \le A^{\bar{y}_0}_N(\omega) I(\rho^n_\varepsilon(\omega) \le A^n_N(\omega)), \qquad n = 1, 2, \dots,$$

since $y_n \downarrow y_0$ and $y_1 < \bar{y}_0$ by assumption. The functions g and $t \mapsto e^{rt}$ are uniformly continuous on $[-e^{rc_2}c_1, e^{rc_2}c_1]$ and $[0, c_2]$, respectively. Hence, for the chosen ω, equality (A.5) follows from

$$0 \le (\Gamma^0_{\rho^n_\varepsilon(\omega)}(\omega) - \Gamma^n_{\rho^n_\varepsilon(\omega)}(\omega)) I(\rho^n_\varepsilon(\omega) \le A^n_N(\omega)) \le (\Gamma^0_{A^n_N(\omega)}(\omega) - N) \to 0$$

as $n \to \infty$ and almost all ω are indeed of this type since the map $t \mapsto G_t$ is almost surely continuous and $\lim_{n\to\infty} \Gamma^0_{A^n_N}$ is almost surely equal to N by Lemma 4.1.

Part (iii) can be shown by combining the ideas of the proof of part (ii) and the proof of Proposition 4.5. In addition to (4.12), (4.13), (4.14) there will be an extra term like (A.4). We only need to justify why Lebesgue's dominated convergence theorem can be applied with respect to this extra term after substituting the sequence $y_n, n = 1, 2, \dots,$ and here, but only in the case of $y_n \uparrow y_0$, one needs g to be bounded from below. □

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