A Topological proof of Bloch's conjecture

The British postgraduate student is a lonely, forlorn soul, uncertain of what he is doing or whom he is trying to please

(D. Lodge)

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Introduction

In this thesis I give a topological proof of a conjecture of Spencer Bloch [6] concerning the rationality of Chern Cheeger Simons invariants of flat bundles. More precisely I prove that the degree three Chern-Cheeeger-Simons class of any flat bundle associated to a representation of the fundamental group of a smooth projective complex algebraic variety is always rational.

After some time I had started studying this problem, I was convinced that Bloch's conjecture could be proved using Lefschetz's theorems on the homology of smooth projective algebraic varieties to construct oriented 3-dimensional manifolds M_i with a 'nice' fundamental group, and maps f_i from M_i into the projective algebraic variety such that the images of the fundamental classes of the M_i would give a generating set for the three dimensional homology of the variety. Having done this it is straightforward to compute the values of the Chern Cheeger Simons invariants using naturality properties. Shortly after I had started working in this direction Retznikov gave a proof of the rationality of the degree three Chern-Cheeger-Simons classes in the case of representations in $SL(2,\mathbb{C})$ in [39]. He later generalized this result and proved that all Chern Cheeger Simons classes are rational in [40]. However his methods are quite different. In particular he uses some deep results of Donaldson and Corlette about the existence of harmonic sections of flat bundles, and of Siu and Sampson on rigidity properties of harmonic mappings (see [39], [40]). So I thought that in any case it would be interesting to see if I could succede in proving Bloch's conjecture by elementary topological methods using the strategy outlined above.

Abstract

The following is an outline of the structure of the thesis.

- In the first chapter, after reviewing the Chern Weil construction of characteristic classes of bundles I introduce the constructions of secondary invariants given by Chern and Simons [13] and by Cheeger and Simons [14].

- In the second chapter I introduce Bloch's conjecture and give some of its motivations. In particular we define Deligne cohomology and its smooth analogue, and prove that the latter is isomorphic to Cheeger and Simons ring of differential characters.

- In the third chapter I review Lefschetz Theorems and related results about the topology of smooth projective algebraic varieties, including an analysis of the local structure near a singular hyperplane section.

- The fourth chapter is the heart the thesis. I prove the two main results Theorem 4.0.1 and Theorem 4.0.2, asserting that the rational three dimensional homology of a smooth projective algebraic variety is generated by the images of the fundamental classes of $S^2 \times S^1$ and of connected sums of $S^1 \times S^2$ under some appropriate maps. The rationality of the Chern-Cheeger-Simons class follows directly from this result. I also discuss briefly the difficulties which I encountered when trying to generalize the construction to give inductively generators for all the odd homology groups of smooth projective algebraic manifolds.

Aknowledgments

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Declaration

The work in this thesis is, to the best of my knowledge, original, except where attributed to others. No part of this work has been submitted for other degrees.

Contents

Chapter 1

What are secondary invariants?

A characteristic class is an invariant of a vector bundle (or a principal bundle) with values in the cohomology of the base space. The classical examples are Stiefel-Whitney, Chern and Pontrjagin classes. These are invariants of the topological isomorphism class of the bundle, so in the study of 'finer' geometrical structures (e.g. metrics, holomorphic structures, foliations ...) it is desirable to define other invariants when the primary ones vanish, hence the name *secondary* invariants. In [32], Karoubi gives the following as a first example. Holomorphic line bundles over a compact Kähler manifold are classified by the cohomology group $H^1(X;\mathcal{O}^*)$. Here $\mathcal{O}$ is the sheaf of holomorphic functions on X, and $\mathcal{O}^*$ is the sheaf of nonvanishing holomorphic functions. There is an exact sequence

$$0 \to T^h \to H^1(X;\mathcal{O}^*) \xrightarrow{c_1} H^2(X;\mathbb{Z}) \to \dots$$

where h is the dimension of $H^1(X;\mathbb{R})$, T^h is a torus of dimension h, and c_1 is the first Chern class. Topologically line bundles are classified by their first Chern class, so when two line bundles L_1 and L_2 are topologically isomorphic their 'difference' as holomorphic bundles is given by an element of T^h. This element can be considered as a secondary invariant. The generalization of this example is one of the main themes in Bloch's conjecture.

The first construction of secondary characteristic classes was given by Chern and Simons using Chern-Weil theory. Given a Lie group G and a principal G-bundle with a connection, Chern-Weil theory associates to an invariant polynomial on the Lie algebra of G a characteristic class by 'evaluating' the polynomial on the curvature form of the connection. In this

way a closed differential form is obtained, and thus a de Rham cohomology class. It turns out that these characteristic classes do not depend on the specific connection chosen, but are topological invariants of the bundle. In this framework Chern and Simons in [13] described a construction to produce secondary characteristic classes whenever an invariant polynomial evaluated on the curvature form of a connection is zero. One of the reasons why these invariants are interesting is that in many cases they depend on the connection, so they can detect richer structures on the bundles. This construction was generalized by Cheeger and Simons in [14] who defined an appropriate functor from manifolds to graded rings in which these invariants take values.

A complete survey of all the different definitions, generalizations, and applications of secondary invariants would be far out of the scope of this work. To mention but a few, there is the theory of secondary invariants of foliations [8], [30], the Atiyah-Patodi Singer index theorem, [3] applications to symplectic geometry (Maslov class) [43], and in physics in the theory of anomalies see [4] and references therein. A unified treatment of the different situations in which secondary invariants arise is given in [32]. In the rest of this chapter we introduce the constructions of Chern-Simons and Cheeger-Simons invariants. This is the plan of the chapter.

- In section 1.1 we review the Chern-Weil construction.
- In section 1.2 we introduce the Chern-Simons classes.
- In section 1.3 we define the ring of differential characters and the Cheeger-Simons invariants.
- In section 1.4 we review a few properties of secondary invariants of flat bundles.

1.1 The Chern-Weil homomorphism

The main reference for this section is [12]. The Chern-Weil construction can be carried out using either real or complex coefficients. For simplicity we work with real coefficients, but everything we say carries over, with some obvious modification, to the case of complex coefficients.

Let G be a Lie group and E be a principal G bundle over a differential manifold M. We are going to use the definition of a connection on E as a global $\mathfrak{g}$ valued differential 1-form θ defined on the total space of E satisfying the two following properties.

- For all $x \in E$, let $i_x : \mathfrak{g} \to T_xE$ be the isomorphism between the subspace of vertical vectors (i.e. the vectors tangent to the fibre) in T_xE. The map i_x is the differential at the identity of the map sending an element g of G to $xg \in E$. Then θ must satisfy

$$\theta_x \circ i_x = \mathrm{Id}_{\mathfrak{g}}$$

- Given $g \in G$, let $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ be the adjoint action and let $R_g : E \to E$ be right translation by g given by the action of G on E. Then θ must satisfy

$$R_g^*\theta = \mathrm{Ad}(g^{-1}) \circ \theta$$

The curvature of the connection is the two form on E with values in $\mathfrak{g}$ defined by

$$\Theta = \mathrm{d}\theta - \frac{1}{2}[\theta, \theta].$$

From the properties of θ it follows that Θ is horizontal (i.e. vanishes whenever any of its two arguments is a vertical vector) and equivariant.

Let h be an integer, and consider h -linear functions

$$F : \mathfrak{g} \times \ldots \times \mathfrak{g} \to \mathbb{R}$$

satisfying the two following properties.

- F is invariant under permutations of the variables.
- F is invariant under the adjoint action of G on $\mathfrak{g}$.

Multilinear maps of h variables with the two properties above are called called *invariant polynomials* of degree h. In fact for any vector space V, pick a basis $\{e_1, \ldots, e_n\}$ and set $v = \sum_i x_i e_i$. Then we get an isomorphism of $S^h(V^*)$ with $\mathbb{R}[x_1, \ldots, x_n]^h$, the homogeneous polynomials of degree h, sending the map F to

$$\sum_{i_1, \ldots, i_n} x_{i_1} \cdots x_{i_n} F(e_{i_1}, \ldots, e_{i_n})$$

Hence the invariant polynomials of degree h are the elements of $S^h(\mathfrak{g}^*)$ invariant under the action of G. The set of all invariant polynomials forms a graded ring, denoted by $I(G)$.

Given a connection on E and an invariant polynomial F of degree h, we can evaluate F with the curvature form Θ to get a $\mathbb{R}$-valued $2h$ differential form $F(\Theta)$ defined over P in the following way: Θ^h is a $2h$-form with values in $\bigotimes^h \mathfrak{g}$, the Lie algebra of G tensored with itself h times. Then the composition

$$\bigwedge^{2h} \mathrm{T}^*M \xrightarrow{\Theta^h} \bigotimes^h \mathfrak{g} \xrightarrow{F} \mathbb{R}$$

gives the $\mathbb{R}$-valued $2h$-form $F(\Theta)$.

For example, when $\mathfrak{g}$ is an algebra of matrices, the curvature can be represented by a matrix of forms, and we can take as F the trace or the determinant of an appropriate polynomial in the matrix.

From the invariance property of F and the fact that Θ is horizontal and equivariant it follows that $F(\Theta)$ descends to a globally defined $2h$-form on M; i.e. there is a form F_Θ on M such that if $\pi : E \to M$ is the projection of the bundle we have

$$\pi^*(F_\Theta) = F(\Theta).$$

There is the well known theorem (see e.g. [12])

Theorem 1.1.1 *The form F_Θ is closed. Hence it determines a (de Rham) cohomology class*

$$[F_\Theta] \in H^{2h}(M; \mathbb{R}).$$

Moreover this cohomology class does not depend on the connection, and it is a characteristic class of the bundle (i.e. it is natural).

Another way of stating this theorem is to say that given a bundle with a connection over the manifold M, there is a well defined map

$$w : I(G) \to H^*(M; \mathbb{R})$$

sending $F \in I(G)$ to $[F_\Theta]$. The map w does not depend on the connection and it can be proved that w is a ring homomorphism. Naturality means that

if $f : N \to M$ is a map of smooth manifolds, then the pullback connection $f^*(\theta)$ on f^*E has curvature $f^*\Theta$ and we have

$$[F_{f^*(\Theta)}] = f^*[F_\Theta].$$

Narasimhan and Ramanan [37] proved the existence of n classifying objects for bundles with a connection. A principal bundle $\pi_n : E_n \to A_n$ with a connection θ_n is n-classifying if the two following conditions are satisfied:

- for any G-bundle $E \to M$ with connection θ and dimension of M less than n there is a map $f : M \to A_n$ such that $E = f^*E_n$ and $\theta = f^*\theta_n$. The map f is called a classifying map for the bundle E and the connection θ.
- Any two classifying maps for the bundle E and the connection θ are homotopic.

Then the naturality property implies that w factors as

$$\begin{array}{ccc} I(G) & \xrightarrow{w} & H^*(M;\mathbb{R}) \\ {\scriptstyle w_n}\searrow & & \nearrow{\scriptstyle f^*} \\ & H^*(A_n;\mathbb{R}) & \end{array} \tag{1.1}$$

For every Lie group G there also exists a universal classifying (i.e. n-classifying for every n) space for G bundles, noted BG and a principal G bundle

$$EG \to BG$$

such that isomorphism classes of G bundles over M are in one to one correspondence with homotopy classes of maps $f : M \to BG$. The space BG is defined only up to homotopy and is not a manifold. However, it can be constructed as the realisation of a simplicial set, so using simplicial techniques (see e.g. [16]) it is possible to generalize the Chern-Weil construction to give a homomorphism

$$w_G : I(G) \to H^*(BG, \mathbb{R})$$

with the following universal property: if E is a principal bundle over M, and $f : M \to BG$ is a classifying map for E, the Weil homomorphism $w : I(G) \to H^*(M;\mathbb{R})$ factors as above as

$$f^* \circ w_G$$

If G is compact $w_G : I(G) \to H^*(BG, \mathbb{R})$ is an isomorphism identifying the invariant polynomials with the characteristic classes for G-bundles.

1.2 Chern Simons classes

The main references for this section are [12] and [13]. We keep the notation of the previous section. Theorem 1.1.1 states that characteristic classes obtained by the Chern-Weil construction do not depend on the connection. This follows from the following construction. Let Q be a principal bundle over a manifold X, and let θ_t be a path of connections on Q. The family θ_t defines a connection on the bundle $Q \times I \to X \times I$. Let $p_1 : X \times I \to X$ be the projection; we have $p_1^* Q \cong Q \times I$. Apply the Chern-Weil construction to θ to get a $2h$ form F_Θ. Integration along the fibre gives a $2h-1$ form on X, $p_{1*} F_\Theta$. Define $T(F, \theta_t) = p_{1*} F_\Theta$. we have

$$\mathrm{d}T(F, \theta_t) = F_{\Theta_1} - F_{\Theta_0}.$$

As the space of connections on a bundle is an affine space, it is path connected, so this construction proves that the characteristic classes obtained by the Chern-Weil construction do not depend on the connection chosen.

Let E be a principal bundle over M and let π be the projection. The pullback $\pi^*(E)$ of E over its total space has a canonical trivialisation, so we can identify connections on $\pi^*(E)$ with differential forms on E. In particular the form θ corresponds to a connection given by the differential form $\pi^*(\theta)$ on the total space of $\pi^*(E)$. The curvature of $\pi^*(\theta)$ is $\pi^*(\Theta)$. Let F be an invariant polynomial. Using the notation of the previous section, the Chern-Weil construction gives a form $F_{\pi^*(\Theta)}$ defined on the base E of the bundle $\pi^* E$. Evaluating F on the curvature of the connection θ we get another form $F(\Theta)$ defined on E. Then we have

$$F_{\pi^*(\Theta)} = F(\Theta).$$

By the triviality of $\pi^*(E)$ all its characteristic classes vanish, hence the form $F_{\pi^*(\Theta)}$ is exact. Applying the previous construction to the bundle $\pi^* E$ and the straight line of connections joining $\pi^*\theta$ to the trivial connection, we get a $2h-1$ form $TF(\theta)$ on E with the following property.

$$\mathrm{d}TF(\theta) = F(\Theta). \tag{1.2}$$

The forms $TF(\theta)$ depend in general on the connection. The operator T is called the *transgression operator.* It is natural and so gives a canonical way of writing $F(\Theta)$ as a coboundary. Calculating the form $TF(\theta)$ explicitly gives

$$TF(\theta) = h\int_0^1 F(\theta, \Theta_t, \ldots, \Theta_t)\mathrm{d}t$$

where $\Theta_t = t\mathrm{d}\theta - \frac{1}{2}t^2[\theta,\theta] = t\Theta + \frac{1}{2}(t-t^2)[\theta,\theta]$, and $F(\theta, \Theta_t, \ldots, \Theta_t)$ is the form obtained from the composition

$$\bigwedge^{2h-1} \mathrm{T}^*M \xrightarrow{\theta\wedge\Theta^{h-1}} \bigotimes^{h} \mathfrak{g} \xrightarrow{F} \mathbb{R}$$

as in the previous section. Another formula for $TF(\theta)$ is given by

$$TF(\theta) = \sum_{i=0}^{h-1}(-1)^i \frac{h!(h-1)!}{2^i(h+i)!(h-1-i)!} F(\theta, \overbrace{[\theta,\theta], \ldots, [\theta,\theta]}^{i}, \overbrace{\Theta, \ldots, \Theta}^{h-1-i}).$$

If $F(\Theta) = 0$ it follows from (1.2) that the form $TF(\theta)$ is closed, hence determines a cohomology class

$$[TF(\theta)] \in H^{2h-1}(E, \mathbb{R}).$$

This class is a secondary characteristic class of Chern-Simons type.

Let $m \in M$ be a point, E_m be the fibre of E over m. Then the restriction of the form $TF(\theta)$ to E_m is closed. This restriction pulls back to a bi-invariant closed $2h-1$ form TF that does not depend on the connection. By construction, its cohomology class $[TF] \in H^{2h-1}(G; \mathbb{R})$ has a representative which is the restriction of a cochain on the total space of the bundle, and whose coboundary is the lift of a cocycle on the base. Classes with this property are called transgressive (see Borel [7]).

Chern-Simons classes take values in the cohomology of the total space of the bundle E. If E is trivial, it is possible to pull back the classes $[TF(\theta)]$ to the base M of E via a section s to get classes

$$s^*[TF(\theta)] \in H^*(M; \mathbb{R}).$$

These classes depend on the section chosen. However, if the polynomial F corresponds via the Chern-Weil construction to a cohomology class with

coefficients in a proper subring Λ of $\mathbb{R}$, $s^*[TF(\theta)]$ determines a well defined class in $H^*(M;\mathbb{R}/\Lambda)$.

The transgression operator T and hence the secondary classes can be defined in an analogous way in the complex case.

1.3 Cheeger-Simons differential characters

In this section we introduce the ring of differential characters. This is a graded ring, functorial in the base space, in which secondary invariants of a bundle take values. These invariants are a generalization of Chern-Simons classes. The main reference for Cheeger-Simons differential characters is [14], from which the definition of differential characters and their properties given below are taken.

Let M be a C^∞ manifold, $\Omega^*(M;\mathbb{R})$ the de Rham complex of real valued differential forms on M. We denote by

$$C_k(M;G) \supset Z_k(M;G) \supset B_k(M;G)$$

the groups of smooth singular chains, cycles, boundaries on M with coefficients in G. Let Λ be a discrete subring of the real numbers. Furthermore, let $\Omega_0^k(M)$ be the *closed* differential k-forms whose cohomology classes (identifying de Rham and singular cohomology) are in the image of the inclusion $H^k(M;\Lambda) \to H^k(M;\mathbb{R})$.

Any differential k-form ω defines a singular cochain with values in $\mathbb{R}$ by integration on the elements of $C_k(M;\mathbb{R})$. We denote by $\widetilde{\ }$ the reduction modulo Λ. Also, with a slight abuse of notation, we write $\tilde{\omega}$ for the $\mathbb{R}/\Lambda$ cochain obtained reducing the values of ω modulo Λ.

Remark A nonzero differential form ω never takes values only in a discrete subring of $\mathbb{R}$. It follows that if we denote with $C^k(M;\mathbb{R}/\Lambda)$ the singular cochains with values in $\mathbb{R}/\Lambda$), the map

$$\omega \mapsto \tilde{\omega} \in C^k(M;\mathbb{R}/\Lambda) \tag{1.3}$$

is an injection. Hence we can identify $\Omega^k(M;\mathbb{R})$ with its image in $C^k(M;\mathbb{R}/\Lambda)$.

Definition 1.3.1 *Let $\partial : C_{k+1}(M;\mathbb{R}/\Lambda) \to C_k(M;\mathbb{R}/\Lambda)$ be the boundary operator. The group of k-differential characters is defined as follows.*

$$\hat{H}^k(M;\mathbb{R}/\Lambda) := \{f \in Hom(Z_k(M),\mathbb{R}/\Lambda) \mid f \circ \partial \in \Omega^{k+1}(M;\mathbb{R})\}$$

(with the above identification).

By this definition the groups of differential characters are subgroups of $\mathrm{Hom}(Z_k(M),\mathbb{R}/\Lambda)$, so obvious functoriality properties are satisfied.

We now define homomorphisms

$$\delta_1 : \hat{H}^k(M;\mathbb{R}/\Lambda) \to \Omega_0^{k+1}(M) \text{ and } \delta_2 : \hat{H}^k(M;\mathbb{R}/\Lambda) \to H^{k+1}(M;\Lambda). \quad (1.4)$$

Let f be an element of $\hat{H}^k(M;\mathbb{R}/\Lambda)$. There exists a cochain $a \in C^k(M;\mathbb{R})$ such that on $Z_k(M;\mathbb{R}/\Lambda)$ we have $\tilde{a} = f$. Then

$$\delta\tilde{a} = f \circ \partial = \omega \in \Omega^{k+1}(M;\mathbb{R})$$

by the definition of differential character. Hence there is a $c \in C^{k+1}(M;\Lambda)$ such that $\omega - \delta a = c$. Applying the coboundary we have

$$\mathrm{d}\omega = \delta c.$$

By the remark above this is possible if and only if $\mathrm{d}\omega = \delta c = 0$. Let $u \in H^{k+1}(M;\Lambda)$ be the cohomology class of c. If

$$r : H^{k+1}(M;\Lambda) \to H^{k+1}(M;\mathbb{R})$$

is the coefficient homomorphism we have $r(u) = [\omega]$. Neither u nor ω depend on the choice of the lift a of f, hence we can define the homomorphisms (1.4) setting $\delta_1(f) = \omega$ and $\delta_2(f) = u$.

From the discussion above, it follows that

$$\hat{H}^k(M;\mathbb{R}/\Lambda) \cong \{(f,\omega) \in \mathrm{Hom}(Z_k(M),\mathbb{R}/\Lambda) \oplus \Omega^{k+1}(M) \mid f \circ \partial = \omega\}.$$

Moreover, let $\phi_1, \phi_2 \in C^k(M;\mathbb{R}/\Lambda)$ be extensions of f. Then $\phi_1 - \phi_2$ is a coboundary , so we have (see [18])

$$\hat{H}^k(M;\mathbb{R}/\Lambda) \cong \frac{\{(\phi,\omega) \in C^k(M,\mathbb{R}/\Lambda) \oplus \Omega^{k+1}(M) \mid \mathrm{d}\omega = 0, \tilde{\omega} = \delta\phi\}}{\delta C^{k-1}(M,\mathbb{R}/\Lambda)} \quad (1.5)$$

Definition 1.3.2 *Let $m : B^* \to C^*$ be a map between two cochain complexes, the cone on m, denoted by $cone(B^* \xrightarrow{m} C^*)$, is the cochain complex having $B^{k+1} \oplus C^k$ in degree k and whose differential is given by*

$$d(b,c) = (-d_B b, m(b) + d_C c).$$

For a complex K, let $K[p]$ be the complex with $K^i[p] = K^{i-p}$. We also define $K^{\geq k}$, the truncation of K in degree less then k, to be the complex whose terms of degree less then k are zero, and whose degree i term , for $i \geq k$ is K^i. The truncation of K in degree greater than k, $K^{<k}$, is defined in an analogous way.

Having established this notation, it follows from (1.5) that there is an isomorphism

$$\hat{H}^k(M;\mathbb{R}/\Lambda) \cong H^k(\text{cone}(\Omega^{\geq k+1}(M) \to C^*(M;\mathbb{R}/\Lambda))). \tag{1.6}$$

Definition 1.3.3 *Let*

$$R^k(M;\Lambda) = \{(\omega, u) \in \Omega_0^{k+1}(M) \times H^k(M;\Lambda) \mid [\omega] = r(u)\}$$

and let $R^*(M,\Lambda) = \oplus_k R^k(M;\Lambda)$.

There is an obvious product between elements of $R^*(M;\Lambda)$ giving it the structure of graded ring.

Theorem 1.3.1 *(See [14]) There are short exact sequences*

$$0 \to H^k(M;\mathbb{R}/\Lambda) \to \hat{H}^k(M;\mathbb{R}/\Lambda) \xrightarrow{\delta_1} \Omega_0^{k+1}(M) \to 0 \tag{1.7}$$

$$0 \to \frac{\Omega^k(M;\mathbb{R})}{\Omega_0^k(M)} \to \hat{H}^k(M;\mathbb{R}/\Lambda) \xrightarrow{\delta_2} H^{k+1}(M;\Lambda) \to 0 \tag{1.8}$$

$$0 \to \frac{H^k(M;\mathbb{R})}{r(H^k(M;\Lambda))} \to \hat{H}^k(M;\mathbb{R}/\Lambda) \xrightarrow{(\delta_1,\delta_2)} R^{k+1}(M;\Lambda) \to 0 \tag{1.9}$$

Proof: Sequence (1.7) follows from (1.6). Let $\imath$ be the map $\Omega^{\geq k+1} \to C^*(M;\mathbb{R}/\Lambda)$ given by (1.3) and $C(\imath)$ be the cone on $\imath$. There is a short exact sequence of complexes

$$0 \to C^*(M;\mathbb{R}/\Lambda) \to C(\imath) \to \Omega^{\geq k+1}(M)[1] \to 0$$

giving a long exact sequence

$$\to H^{k-1}(\Omega^{\geq k+1}(M)[1]) \to H^k(C^*(M;\mathbb{R}/\Lambda)) \to H^k(C(\iota)) \to H^k(\Omega^{\geq k+1}(M)[1]) \to$$

We have $H^{k-1}(\Omega^{\geq k+1}(M)[1]) = 0$ and $H^k(\Omega^{\geq k+1}(M)[1])$ is the group of closed $k+1$ forms. Now if a closed $k+1$ form ω is in the image of the map from the group of degree k differential characters, then there must be a cochain $\phi \in C^k(M;\mathbb{R}/\Lambda)$ such that for every $k+1$ cycle z we have

$$\widetilde{\int_z} \omega = \langle \delta\phi, z\rangle = \langle \phi, \partial z\rangle = 0.$$

So sequence (1.7) is established. We will prove (1.8) in the next chapter, and for the proof of (1.9) we refer to [14].

Let B be the Bockstein homomorphism in the exact sequence

$$\ldots \to H^k(M;\mathbb{R}) \to H^k(M;\mathbb{R}/\Lambda) \xrightarrow{B} H^{k+1}(M;\Lambda) \to \ldots$$

given by the short exact sequence of the coefficients.

Proposition 1.3.2 *(See [14])*

- $\delta_2 = -B$ *on the image of* $H^k(M;\mathbb{R}/\Lambda)$ *in* $\hat{H}^k(M;\mathbb{R}/\Lambda)$.
- $\delta_1 = d$ *on the image of* $\Omega^k(M;\mathbb{R})/\Omega_0^k(M)$ *in* $\hat{H}^k(M;\mathbb{R}/\Lambda)$.

Let G be a group with finitely many components, and define a category $\epsilon(G)$ whose objects are triples

$$P = (E, M, \theta)$$

where E is a principal G-bundle over M, and θ is a connection on E. The morphisms of $\epsilon(G)$ are connection preserving bundle maps.

Define

$$K^k(G;\Lambda) = \{(F,u) \in I(G) \times H^{2k}(BG;\Lambda) \mid w_G(F) = r(u)\}$$

(where $I(G)$ is the ring of invariant polynomials and w_G is the Chern-Weil homomorphism as in Section 1.1). The following Theorem asserts the existence of secondary invariants in $\hat{H}^*(M;\mathbb{R}/\Lambda)$ lifting the classes obtained by the Chern-Weil construction.

Theorem 1.3.3 (Cheeger-Simons [14]) *Let $(F,u) \in K^{2k}(G;\Lambda)$. For any $P \in \epsilon(G)$ there is a unique class $S_{F,u}(P) \in \hat{H}^{2k-1}(M;\mathbb{R}/\Lambda)$ such that*

- *If Θ denotes the curvature of θ, $\delta_1(S_{F,u}(P)) = F_\Theta$.*
- *If $f : M \to BG$ is a classifying map for P, $\delta_2(S_{F,u}(P)) = f^*(u)$.*
- *If $P' \in \epsilon(G)$ and $\eta : P \to P'$ is a morphism , $\eta_M^*(S_{F,u}(P')) = S_{F,u}(P)$ (where η_M is the corresponding map on the bases).*

The proof of the theorem rests on the existence of n-classifying objects in $\epsilon(G)$ (see Section 1.1 above). Let $P_n = (E_n, A_n, \theta_n) \in \epsilon(G)$ be n-classifying. As the odd cohomology with real coefficients of the universal classifying space for G is trivial, we have that for n big enough also $H^{2k-1}(A_n;\mathbb{R}) = 0$. It follows from the exact sequence (1.9) in Theorem 1.3.1 that

$$(\delta_1, \delta_2) : \hat{H}^{2k-1}(A_n;\mathbb{R}/\Lambda) \to R^{2k}(M;\Lambda)$$

is an isomorphism. Setting

$$S_{F,u}(P_n) = (\delta_1, \delta_2)^{-1}((F_{\Theta_n}, u))$$

(where F_{Θ_n} is the form obtained by the Chern Weil construction as in Section 1.1) defines the invariants in the case of n classifying objects. If $\eta : P \to P_n$ is a morphism to a n-classifying object by naturality one must have

$$S_{F,u}(P) = \eta_M^*(S_{F,u}(P_n)).$$

To finish the proof of the theorem it is only left to check that the definition of the differential characters $S_{F,u}(P)$ does not depend on the particular classifying object chosen. We refer to [14] for the proof this fact.

From the above theorem it follows that if $F_\Theta = 0$ and f is a classifying map for P

$$S_{F,u}(P) \in H^{2k-1}(M;\mathbb{R}/\Lambda) \quad \text{and} \quad B(S_{F,u}(P)) = -f^*(u).$$

Let $\pi : E \to M$ be the projection of P. Then the relation between $S_{F,u}(P)$ and the form $TF(\theta)$ defined in the previous section is given by

$$\pi^*(S_{F,u}(P)) = \widetilde{TF(\theta)}_{|Z_{2k-1}(E)}.$$

If P admits a global cross section then we have

$$S_{F,u}(P) = s^*(\widetilde{[TF(\theta)]})$$

as elements of $H^{2k-1}(M;\mathbb{R}/\Lambda)$.

1.4 Flat bundles

In this thesis we are going to be concerned with secondary invariants of flat bundles, or equivalently of representations of the fundamental group of a manifold M. Let G be a Lie group with finitely many components, and

$$\rho : \pi_1(M) \to G$$

be a representation. An object $P_\rho = (E_\rho, M, \theta_\rho)$ of $\epsilon(G)$, is determined byρ in the following way. If $\tilde{M}$ denotes the universal cover of M, then E_ρ is $\tilde{M} \times_\rho G$, the quotient of $\tilde{M} \times G$ by the action of $\pi_1(M)$. Let θ_G be the one form defined by $\theta_G(g) = \mathrm{d}L_{g^{-1}}(g)$ (as maps from the tangent space of G at g and $\mathfrak{g} \cong T_{1_G}G$). The canonical flat connection (or Maurer Cartan connection) on $\tilde{M} \times G$ is $p_2^*\theta_G$, where p_2 is the projection of $\tilde{M} \times G$ to the second factor. The canonical flat connection on $\tilde{M} \times G$ induces a connection on $\tilde{M} \times_\rho G$ since the form $p_2^*\theta_G$ is invariant under the action of ρ. The connection θ_ρ is this induced connection, whose holonomy representation is the representation ρ.

For any $(F,u) \in K^{2k}(G,\Lambda)$ we define $u(\rho) = u(E_\rho)$ and $S_{F,u}(\rho) = S_{F,u}(P_\rho)$. For all $u \in H^{2k}(BG;\Lambda)$, $u(\rho)$ is torsion, hence it follows from Theorem 1.3.1 that

- $S_{F,u}(\rho) \in H^{2k-1}(M;\mathbb{R}/\Lambda)$.
- $B(S_{F,u}(\rho)) = -u(\rho)$ where B is the Bockstein operator.

Definition 1.4.1 *Let C_k be the k-th Chern polynomial and c_k the k-th universal Chern class, we will write*

$$cs_{2k-1}(\rho) \in H^{2k-1}(M;\mathbb{R}/\Lambda)$$

for the $\mathbb{R}/\Lambda$ cohomology class determined by $S_{C_k,c_k}(\rho)$.

We are going to use essentially two properties of secondary invariants of representations. The first one is a restatement of the naturality of differential characters in the language of representations: let M and N be differentiable manifolds and $f : M \to N$ be a smooth map. A representation $\rho : \pi_1(N) \to G$ can be pulled back to give a representation

$$f^*(\rho) = \rho \circ f_* : \pi_1(M) \to G.$$

Then we have

$$S_{F,u}(f^*(\rho)) = f^*(S_{F,u}(\rho)).$$

The second important property is rigidity under deformations.

Theorem 1.4.1 (Cheeger Simons [14]) *Let $(F, u) \in K^{2k}(G, \Lambda)$, $k \geq 2$ and*

$$\rho_t : [0,1] \times \pi_1(M) \to G$$

a smooth family of representations. Then $S_{F,u}(\rho_0) = S_{F,u}(\rho_1)$.

Example 1 The condition $k \geq 2$ in the Theorem is necessary. Representations $\rho : \pi_1(S^1) \to U_1 \cong S^1$ are in one to one correspondence with elements of $H^1(S^1; U_1)$ as U_1 is abelian, and $H^1(S^1; U_1) \cong H^1(S^1; \mathbb{R}/\mathbb{Z})$. From Theorem 1.3.1 we also have isomorphisms

$$H^1(S^1; \mathbb{R}/\mathbb{Z}) \cong \hat{H}^1(S^1; \mathbb{R}/\Lambda) \cong \frac{\Omega^1(M)}{\Omega^1_0(M)} \cong \frac{H^1(S^1; \mathbb{R})}{H^1(S^1; \mathbb{Z})}$$

Let $(C_1, c_1) \in K^2(U_1, \mathbb{Z})$ be the first Chern polynomial and the universal first Chern class. Using the identification of $\mathfrak{u}_1$ with $i\mathbb{R}$, we can represent a connection with a real valued 1-form ω. Moreover for all ρ the associated flat bundle P_ρ is trivial, so we can identify the space of flat U_1 connections with $\Omega^1(M)$. Two such connections have the same holonomy representation if and only if they are gauge equivalent, and from the isomorphisms above it follows that this happens if and only if their forms differ by a closed form with integer periods. Hence if we identify $\hat{H}^1(S^1; \mathbb{R}/\Lambda)$ with $\Omega^1(M)/\Omega^1_0(M)$, $cs_1(\rho) = S_{C_1,c_1}(\rho)$ is just the gauge equivalence class of the connection, i.e. the projection of a connection form A to the equivalence class of $(2\pi i)^{-1}A$. If we instead consider $cs_1(\rho)$ as an element of $H^1(S^1, \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}(\mathbb{Z}, R/\mathbb{Z}) \cong$

U_1, and we identify the representation with the image $e^{2\pi i\alpha}$ of a generator of $\pi_1(S^1)$, we have

$$cs_1(e^{2\pi i\alpha}) = \alpha = \log(e^{2\pi i\alpha}) \mod \mathbb{Z}$$

As the one dimensional homology of any manifold is spanned by the classes determined by simple closed curves in the manifold, by the naturality of the Chern Cheeger Simons class we can conclude that $cs_1(\rho)$ is not rigid for representations ρ of the fundamental group of any manifold.

Remark Connection forms of $GL(n,\mathbb{C})$-bundles are complex valued, and the Chern Polynomials are complex valued. However the Chern classes are always real. Thus the differential characters $S_{\mathcal{R}eC_n,c_n}$ ($\mathcal{R}e$ is the real part) lift the real part of the Chern form, which also represents the Chern class in de Rahm cohomology, so they also lift the Chern class itself (i.e. $\delta_1(S_{\mathcal{R}eC_n,c_n})$ is the real part of the Chern form and $\delta_2(S_{\mathcal{R}eC_n,c_n}) = c_n$). The pullback of $S_{\mathcal{R}eC_n,c_n}$ to the total space of the bundle is (integration of) the real part of the form $TC_n(\theta)$ (here T is transgression of a complex form) restricted to the real cycles of the total space of the bundle (cfr. [25]).

It is also possible to define complex valued differential characters (see [18],[25]) by setting for Λ a subring of $\mathbb{C}$

$$\hat{H}^k(M;\mathbb{C}/\Lambda) := \{f \in \mathrm{Hom}(Z_k(M),\mathbb{C}/\Lambda) \mid f\circ\partial \in \Omega^{k+1}(M;\mathbb{C})\}.$$

All the properties seen for the real valued differential characters are valid (with obvious modifications) also for the complex valued ones (see e.g. [25]). In particular if $\jmath : \Omega^*(M;\mathbb{C}) \to C^*(M;\mathbb{C}/\Lambda))$ is the map given by integration and reduction modulo Λ as in (1.3), we have that $\jmath$ is an injection if Λ is totally disconnected. In this case, if we write $\Omega^{\geq k+1}(M)$ for the truncation of $\Omega^*(M;\mathbb{C})$, we have as in the real case,

$$\hat{H}^k(M;\mathbb{C}/\Lambda) \cong H^k(\mathrm{cone}(\Omega^{\geq k+1}(M) \to C^*(M;\mathbb{C}/\Lambda)).$$

Chapter 2

Secondary invariants and algebraic geometry

In this chapter we discuss the Bloch conjecture concerning the rationality of the Chern Cheeger Simons classes (see [6]). We mention some of the motivations coming from algebraic geometry, algebraic K-theory and algebraic number theory. However we will make no attempt to write a detailed, exhaustive review on these subjects.

In Section 2.1 we discuss the case of line bundles which is well understood. Then in Section 2.3 we will move to the case of higher dimensional bundles, in order to compare it with the case of line bundles. In particular we will see that in both cases the Chern-Cheeger-Simons invariants of a flat bundle E over a smooth projective complex algebraic variety X determine points on the intermediate Jacobians of X. These points are the images of the corresponding Chern classes with values in the Chow groups of X via the Abel-Jacobi map. Hence we can say loosely that the Chern-Cheeger-Simons classes are compatible with the Chern classes with values in the Chow groups. However, while the values of the first Chern-Cheeger-Simons class determine all the points of the Jacobian torus of X, the higher dimensional secondary invariants determine only discrete subsets of the intermediate Jacobians because of their rigidity property.

2.1 Back to line bundles

At the beginning of last Chapter we gave as an example of secondary invariants of holomorphic bundles over a compact complex manifold X elements in a torus T^h fitting in an exact sequence

$$0 \to T^h \to H^1(X;\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X;\mathbb{Z}) \to \dots. \tag{2.1}$$

This sequence comes from the exponential sequence of sheaves

$$0 \to \mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

where $\mathcal{O}_X$ and $\mathcal{O}_X^*$ are the sheaves of holomorphic function and nowhere vanishing holomorphic functions on X respectively. The exponential sequence gives rise to a long exact cohomology sequence

$$\dots \to H^1(X;\mathbb{Z}) \to H^1(X;\mathcal{O}_X) \to H^1(X;\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X;\mathbb{Z}) \to \dots$$

that gives the sequence 2.1 in which T^h is the Jacobian torus

$$J(X) = \frac{H^1(X;\mathcal{O}_X)}{H^1(X;\mathbb{Z})}.$$

Let $Hg^k(X)$ denote the integral classes of type k,k , i.e the elements of $H^{2k}(X;\mathbb{Z})$ whose image under the map $H^{2k}(X;\mathbb{Z}) \to H^{2k}(X;\mathbb{C})$ is of type k,k. Chern classes of holomorphic bundles are of type $1,1$, hence we can rewrite (2.1) as a short exact sequence

$$0 \to J(X) \to H^1(X;\mathcal{O}_X^*) \to Hg^1(X) \to 0. \tag{2.2}$$

This exact sequence has a well known geometrical interpretation: the classes in $Hg^1(X)$ are in one to one correspondence with the smooth isomorphism classes of line bundles over X admitting a holomorphic structure; the group $H^1(X;\mathcal{O}_X^*)$ is isomorphic to the group of holomorphic isomorphism classes of line bundles, and $J(X)$ parametrizes holomorphic line bundles that are topologically trivial.

Now let X be a smooth projective algebraic variety. The k-th Chow group of X, $CH^k(X)$, is the group of complex codimension k subvarieties of X modulo rational equivalence. There is the cycle map

$$cl : CH^k(X) \to H^{2k}(X;\mathbb{Z})$$

sending the class of a complex codimension k subvariety V to the singular cohomology class determined by V (which is a submanifold of real codimension $2k$). We denote by $CH_0^k(X)$ the kernel of the cycle map, (i.e. the subgroup of $CH^k(X)$ consisting of the classes of homologically trivial subvarieties) and by $H_{alg}^{2k}(X;\mathbb{Z})$ its image. There is a theory of Chern classes c_k^{alg} with values in the Chow groups (see [23]). These classes are compatible with the topological Chern classes c_k^{top} in the sense that the cycle map sends c_k^{alg} to c_k^{top}.

The group $CH^1(X)$ is isomorphic to $H^1(X;\mathcal{O}_X^*)$, and $H_{alg}^2(X;\mathbb{Z}) = Hg^1(X)$ (see [28] p.161-163). Furthermore the Abel-Jacobi map $CH_0^1(X) \to J(X)$ is an isomorphism, hence we can rewrite the short exact sequence 2.2 as

$$0 \to CH_0^1(X) \to CH^1(X) \to Hg^1(X) \to 0.$$

Let L_ρ be the flat bundle associated to a representation $\rho : \pi_1(X) \to \mathbb{C}^*$. The bundle L_ρ is holomorphic, so now we will compare the class $c_1^{alg}(L_\rho)$ with the Chern-Cheeger-Simons class

$$cs_1(\rho) \in H^1(X;\mathbb{C}/\mathbb{Z}).$$

We know from the last chapter that $cs_1(\rho)$ is sent to $c_1^{top}(L_\rho)$ by the Bockstein $b : H^1(X;\mathbb{C}/\mathbb{Z}) \to H^1(X;\mathbb{C})$. Since L_ρ is flat, its first Chern class is a torsion class. Hence there exist a natural number N such that $Ncs_1(\rho)$ is the image of a (unique) element of $H^1(X;\mathbb{C})/H^1(X;\mathbb{Z})$ which we still indicate with $Ncs_1(\rho)$. Similarly, $Nc_1^{alg}(L)$ is the image of a unique element $Nc_1^{alg}(L)$ of $J(X)$. There is a map

$$p : \frac{H^1(X;\mathbb{C})}{H^1(X;\mathbb{Z})} \to J(X)$$

induced by $H^1(X;\mathbb{C}) \to H^1(X;\mathcal{O})$.

Proposition 2.1.1

$$p(Ncs_1(\rho)) = Nc_1^{alg}(L_\rho).$$

Proof: Let $\mathcal{C}^*$ be the sheaf of nowhere vanishing C^∞ functions on X. It is known (see e.g.[28] p.139-140) that $H^1(X,\mathcal{C}^*)$ is isomorphic to the group of topological isomorphism classes of line bundles over X. There is an isomorphism $H^1(X,\mathcal{C}^*) \cong H^2(X;\mathbb{Z})$ sending (the class of) a line bundle to its first Chern class. The composition

$$H^1(X;\mathbb{C}/\mathbb{Z}) \cong H^1(X;\mathbb{C}^*) \to H^1(X,\mathcal{C}^*)$$

fits in the following commutative diagram

$$\begin{array}{ccc} H^1(X,\mathcal{C}^*) & \xrightarrow{\cong} & H^2(X;\mathbb{Z}) \\ \nwarrow & & \nearrow \\ & H^1(X;\mathbb{C}/\mathbb{Z}) & \end{array}$$

Since the image of $cs_1(\rho)$ in $H^2(X;\mathbb{Z})$ is the first Chern class of the flat bundle L_ρ, it follows from the diagram that $cs_1(\rho)$ is mapped to the element determined by L_ρ in $H^1(X,\mathcal{C}^*)$. Now, $CH^1(X) \cong H^1(X;\mathcal{O}^*)$, and the class $c_1^{alg}(L_\rho)$ is sent to the element determined by L_ρ in $H^1(X;\mathcal{O}^*)$. The commutative diagram

$$\begin{array}{ccc} H^1(X,\mathcal{C}^*) & \longleftarrow & H^1(X;\mathcal{O}_X^*) \\ \nwarrow & & \nearrow \\ & H^1(X;\mathbb{C}/\mathbb{Z}) & \end{array}$$

shows that $c_1^{alg}(L_\rho)$ and $cs_1(\rho)$ have the same image in $H^1(X;\mathcal{O}^*)$, namely the class determined by the Čech cocycle given by the locally constant transition functions of L_ρ, and the proposition follows.

2.2 Deligne cohomology

The main tool in giving a generalization of the constructions of the preceding section for the case of higher dimensional bundles is Deligne cohomology.

Let Λ be a subring of $\mathbb{C}$. Let $\Lambda(k)$ be the subgroup $(2\pi i)^k\Lambda \subset \mathbb{C}$. Let Ω^p_{hol} be the sheaf of holomorphic complex valued p-forms on X.

Definition 2.2.1 *The holomorphic Deligne complex $\mathfrak{D}$ is the following complex of sheaves:*

$$\Lambda(k) \to \Omega^0_{hol} \to \ldots \to \Omega^{k-1}_{hol}$$

where $\Lambda(k)$ is in degree zero and Ω^k_{hol} is in degree $k+1$. The holomorphic Deligne cohomology $H^*_{\mathfrak{D}}(X;\Lambda(k))$ is the hypercohomology of the holomorphic Deligne complex.

The smooth Deligne complex $\mathfrak{D}_\infty$ and smooth Deligne cohomology $H^*_{\mathfrak{D}_\infty}(X;\Lambda(k))$ are defined in the same way using the sheaves of C^∞ forms Ω^p_∞. Our main references for Deligne cohomology are [22] and [10]. Recall that if K is a complex , we donote by $K[p]$ the complex with $K^i[p] = K^{i-p}$.

Lemma 2.2.1 *The holomorphic and smooth Deligne complexes are quasi isomorphic to the complexes*

$$\mathcal{D} = cone(\Omega_{hol}^{\geq k} \oplus \Lambda(k) \to \Omega_{hol}^{*})[-1]$$

$$\mathcal{D}_{\infty} = cone(\Omega_{\infty}^{\geq k} \oplus \Lambda(k) \to \Omega_{\infty}^{*})[-1]$$

respectively.

Proof: In fact this is Beilinson's definition of the Deligne complex, see [5]. We drop the subscrips *hol* and ∞ as the proof is identical in the two cases. As in the previous chapter, let $\Omega^{\geq k}$ be the complex which is zero in degree less than k and whose degree i term is Ω^i for $i \geq k$. There is a short exact sequence of complexes

$$0 \to \text{cone}(\Omega^{\geq k} \to \Omega^{\geq k}) \to \mathcal{D} \to \mathfrak{D} \to 0$$

and the cone over the identity is acyclic.

The next ingredient we need to generalize our discussion of the case of line bundles to higher dimensional bundles is a generalization of the Jacobian. Let $H^{p,q}(X)$ the subspace of classes of type p,q in $H^{p+q}(X;\mathbb{C})$. Griffith's intermediate Jacobian is defined as

$$J^k(X) = \frac{H^{2k-1}(X;\mathbb{C})}{H^{2k-1}(X;\mathbb{Z}) \oplus H^{2k-1,0}(X) \oplus \ldots \oplus H^{k,k-1}(X)}. \tag{2.3}$$

For $k = 1$ this definition agrees with the one given in the previous section because of the isomorphism

$$\frac{H^1(X;\mathbb{C})}{H^1(X;\mathbb{Z}) \oplus H^{1,0}(X)} \cong \frac{H^1(X;\mathcal{O}_X)}{H^1(X;\mathbb{Z})}.$$

Theorem 2.2.2 *For $\Lambda = \mathbb{Z}$ holomorphic Deligne cohomology groups fit in the following short exact sequence*

$$0 \to J^k(X) \to H_{\mathcal{D}}^{2k}(X;\mathbb{Z}(k)) \to Hg^k(X) \to 0 \tag{2.4}$$

Proof: We have the short exact sequence of complexes

$$0 \to \Omega_{hol}^{*}[-1] \to \mathcal{D} \to \Omega_{hol}^{\geq k} \oplus \mathbb{Z}(k) \to 0$$

giving the long exact sequence of hypercohomology groups

$$\ldots \to H^{2k-1}(\Omega^{\geq k}_{hol} \oplus \mathbb{Z}(k)) \to H^{2k-1}(\Omega^{*}_{hol}) \to H^{2k}(\mathcal{D}) \to H^{2k}(\Omega^{\geq k}_{hol} \oplus \mathbb{Z}(k)) \to \ldots$$

The $2k-1$ hypercohomology of $\Omega^{\geq k}_{hol}$ is the k-th term in the Hodge filtration of $H^{2k-1}(X;\mathbb{C})$, i.e.

$$H^{2k-1,0}(X) \oplus H^{2k-2,1}(X) \oplus \ldots \oplus H^{k,k-1}(X)$$

so the injection of the intermediate Jacobian in Deligne cohomology follows from the above exact sequence. Similarly $H^{2k}(\Omega^{\geq k}_{hol})$ is the k-th term of the Hodge filtration of $H^{2k}(X;\mathbb{C})$, and the image of the map from Deligne cohomology to $H^{2k}(\Omega^{\geq k}_{hol} \oplus \mathbb{Z}(k))$ is $Hg^k(X)$ (see [22]).

Hence holomorphic Deligne cohomology allows to generalize the exponential sequence to higher dimensional bundles in the following sense: there is a generalization of cycle map

$$cl_{\mathcal{D}} : CH^k(X) \to H^{2k}_{\mathcal{D}}(X;\mathbb{Z}(k))$$

taking values in Deligne cohomology, such that the short exact sequence 2.4 is compatible with the short exact sequence for the cycle homomorphism; i.e. there is a commutative diagram (see [21])

$$\begin{array}{ccccccccc}
0 & \to & CH^k(X)_0 & \to & CH^k(X) & \to & H^{2k}_{alg}(X;\mathbb{Z}) & \to & 0 \\
 & & \downarrow \Phi & & \downarrow & & \downarrow & & \\
0 & \to & J^k(X) & \to & H^{2k}_{\mathcal{D}}(X;\mathbb{Z}(k)) & \to & Hg^k(X) & \to & 0
\end{array}$$

When $k = 1$ there is an isomorphism $H^1_{\mathcal{D}}(X;\mathbb{Z}(1)) \cong H^1(X;\mathcal{O}^*)$, so we find again the exponential sequence. Geometrical interpretations of 2.4 generalizing the interpretation of the short exact sequence 2.2 are given in [10] and [24], who show that Deligne cohomology groups can be interpreted as isomorphism classes of certain geometric objects (gerbes in [10] and bundles with fibre the iterated classifying space of $\mathbb{C}^*$ in [24]).

We now turn to examine more closely smooth Deligne cohomology. Let $\Omega^{k-1}(X;\mathbb{C})$ and $\Omega^{k-1}_0(X)$ denote the smooth $\mathbb{C}$-valued $k-1$-differential forms and the closed $k-1$ forms with periods in $\Lambda(k)$ respectively.

Theorem 2.2.3 *Smooth Deligne cohomology fits in the following exact sequence*

$$0 \to \frac{\Omega^{k-1}(X;\mathbb{C})}{\Omega_0^{k-1}(X)} \to H^k_{\mathcal{D}_\infty}(X;\Lambda(k)) \to H^k(X;\Lambda(k)) \to 0 \qquad (2.5)$$

Proof: As in the previous chapter, let $\Omega_\infty^{<k}$ the truncation above degree $k-1$ of the complex of sheaves of smooth differential forms. Consider the short exact sequence of complexes

$$0 \to \Omega_\infty^{<k}[-1] \to [\Lambda(k) \to \Omega^0_{\infty,X} \to \ldots \to \Omega^{k-1}_{\infty,X}] \to \Lambda(k) \to 0.$$

Then the sequence (2.5) follows from the long exact sequence of the hypercohomology by the following remarks.

- The k-th hypercohomology group of the complex consisting only of the constant sheaf $\Lambda(k)$ in degree 0 is $H^k(X;\Lambda(k))$, the ordinary cohomolgy with coefficients in $\Lambda(k)$.
- The hypercohomology of $\Omega_\infty^{<k}$ is isomorphic to the cohomology of the complex of the global sections. Hence $H^{k+1}(\Omega_\infty^{<k}[-1]) = 0$ and

  $$H^{k+1}(\Omega_\infty^{<k}[-1]) = \frac{\Omega^{k-1}(X;\mathbb{C})}{d\Omega^{k-2}(X;\mathbb{C})}.$$

  Moreover if an element of $\Omega^{k-1}(X;\mathbb{C})/d\Omega^{k-2}(X;\mathbb{C})$ is in the image of the map from $H^{k-1}(X;\Lambda(k))$, it is obviously (the equivalence class of) a closed form with periods in $\Lambda(k)$.

Sequence (2.5) for smooth Deligne cohomolgy is the complex version of sequence (1.8) of Theorem 1.3.1 for real valued Cheeger Simons differential characters. The reason for this is in the following theorem, which states that Deligne cohomology groups are isomorphic to the groups of complex valued differential characters. This also gives the proof, in the complex case, of the existence of the exact sequence (1.8) that we promised in the previous Chapter.

Theorem 2.2.4 *There is an isomorphism*

$$\hat{H}^{k-1}(X;\mathbb{C}/\Lambda(k)) \cong H^k_{\mathcal{D}_\infty}(X;\Lambda(k))$$

(for different proofs see [11],[18]).

Proof: Let $\Omega^*(X)$ denote the complex valued differential forms on X, i.e. the global sections of Ω^*_∞ and $\Omega^{\geq k}(X)$ be the truncation as usual. Let $C^*_X(\mathbb{C}/\Lambda(k))$ be the complex of sheaves of singular $\mathbb{C}/\Lambda(k)$ cochains, $\mathcal{C}(\imath)$ be the complex

$$\mathrm{cone}(\Omega^{\geq k}_\infty \xrightarrow{\imath} C^*_X(\mathbb{C}/\Lambda(k))$$

and $C(\imath)$ the complex of the global sections of $\mathcal{C}(\imath)$. We have seen in the previous chapter that $\hat{H}^{k-1}(X;\mathbb{C}/\Lambda(k)) \cong H^{k-1}(C(\imath))$. This cohomology group is isomorphic to the hypercohomology $H^{k-1}(\mathcal{C}(\imath))$. The complex $C^*_X(\mathbb{C}/\Lambda(k))$ is quasi isomorphic to the complex $\mathrm{cone}(\Lambda(k) \to \Omega^*_\infty)$, which we denote by $\mathcal{K}$. In fact we have the short exact sequence of complexes

$$0 \to \mathcal{K} \to \mathrm{cone}(\mathbb{C} \to \Omega^*_\infty) \to \mathbb{C}/\Lambda(k)[1] \to 0$$

in which $\mathrm{cone}(\mathbb{C} \to \Omega^*_\infty)$ is acyclic. It follows that $\mathcal{K}$ is quasi isomorphic to the complex consisting only of the constant sheaf $\mathbb{C}/\Lambda(k)$ in degree 0, which is quasi isomorphic to $C^*_X(\mathbb{C}/\Lambda(k))$. It then follows that the the complex $\mathcal{C}(\imath)$ is quasi isomorphic to the complexes

$$\mathrm{cone}(\Omega^{\geq k}_\infty \to \mathcal{K}) \cong \mathrm{cone}(\Omega^{\geq k}_\infty \oplus \Lambda(k) \to \Omega^*_\infty)$$

which are quasi isomorphic because of a general fact about cones that states that if $m_1 : A \to C$ and $m_2 : B \to C$ are maps of complexes, then $\mathrm{cone}(A \xrightarrow{m_1} \mathrm{cone}(B \xrightarrow{-m_2} C))$ is quasi isomorphic to $\mathrm{cone}(A \oplus B \xrightarrow{m_1 - m_2} C)$ (see [22]). Summing up we have

$$\hat{H}^{k-1}(X;\mathbb{C}/\Lambda(k)) \cong H^{k-1}(\mathrm{cone}(\Omega^{\geq k}_\infty \oplus \Lambda(k) \to \Omega^*_\infty)) \cong H^k(\mathcal{D}_\infty).$$

2.3 Higher dimensional bundles

Consider the exact sequence

$$0 \to \frac{H^{2k-1}(X;\mathbb{C})}{H^{2k-1}(X;\mathbb{Z})} \to H^{2k-1}(X;\mathbb{C}/\mathbb{Z}) \to H^{2k}(X;\mathbb{Z}) \to \dots \qquad (2.6)$$

Let E_ρ be the flat bundle associated to a representation

$$\rho : \pi_1(X) \to GL_n(\mathbb{C})$$

and consider the algebraic Chern class $c_k^{alg}(E_\rho) \in CH^k(X)$. This class is sent to the topological class $c_k^{top}(E_\rho)$ under the cycle map

$$cl : CH^k(X) \to H^{2k}(X;\mathbb{Z}).$$

As the bundle E_ρ is flat, as in section 2.1 there is a natural number N such that

$$Nc_k^{alg}(E) \in CH^k(X)_0.$$

The intermediate Jacobian $J^k(X)$ is isomorphic to the torus

$$\frac{H^{2k-1}(X;\mathbb{R})}{H^{2k-1}(X;\mathbb{Z})}.$$

As in section 2.1, denote by $Ncs_{2k-1}(\rho) \in H^{2k-1}(X;\mathbb{C})/H^{2k-1}(X;\mathbb{Z})$ the point mapped to N times the Chern-Cheeger-Simons class in $H^{2k-1}(X;\mathbb{C}/\mathbb{Z})$. If ρ is unitary, $cs_{2k-1}(\rho)$ is real, hence $Ncs_{2k-1}(\rho) \in J^k(X)$. In [6] Bloch proves the following generalization of Proposition 2.1.1.

Theorem 2.3.1 *Assume that ρ is a unitary representation. Then*

$$\Phi(Nc_k^{alg}(E_\rho)) = Ncs_{2k-1}(\rho).$$

The proof of this theorem, and of its generalization below uses Beilinson's theory of Chern classes $c_k^b \in H_{\mathcal{D}}^{2k}(X;\mathbb{Z}(k))$ with values in the Deligne cohomology (see [5]). It can be shown that these classes are compatible with the topological ones and with the ones with values in the Chow groups (see [5], [22]); i.e. $cl_{\mathcal{D}}(c_k^{alg}) = c_k^b$ by , and the map

$$H_{\mathcal{D}}^{2k}(X;\mathbb{Z}(k))) \to Hg^k(X) \subset H^{2k}(X;\mathbb{Z})$$

takes c_k^b to the topological Chern class.

We now compare Beilinson's Chern classes c_k^b with Cheeger-Chern-Simons invariants. There is a natural map

$$H^{2k-1}(X;\mathbb{C}/\mathbb{Z}(k)) \to H_{\mathcal{D}}^{2k}(X;\mathbb{Z}(k)).$$

Composing this map with the twist of the coefficients $H^{2k-1}(X;\mathbb{C}/\mathbb{Z}) \xrightarrow{(2\pi i)^k} H^{2k-1}(X;\mathbb{C}/\mathbb{Z}(k))$ we get a map

$$\epsilon : H^{2k-1}(X;\mathbb{C}/\mathbb{Z}) \to H_{\mathcal{D}}^{2k}(X;\mathbb{Z}(k)).$$

Let $\rho : \pi_1(X) \to GL_n(\mathbb{C})$ be a representation of the fundamental group of X. The following theorem has been proved in various versions in [18], [25] and [41].

Theorem 2.3.2 *Let E_ρ be the flat bundle associated to ρ. Then we have*

$$\epsilon(cs_{2k-1}(\rho)) = c_k^b(E_\rho).$$

This theorem is essentialy a consequence of Theorem 2.2.4. The images of the classes c_k^b via the composition

$$H_{\mathcal{D}}^{2k}(X;\mathbb{Z}(k)) \to H_{\mathcal{D},\infty}^{2k}(X;\mathbb{Z}(k)) \cong \hat{H}^{2k-1}(X;\mathbb{C}/\mathbb{Z}(k))$$

are sent to the classes c_k^{top} by the map $\hat{H}^{2k-1}(X;\mathbb{C}/\mathbb{Z}(k)) \to H^{2k}(X;\mathbb{Z}(k))$. Moreover these images have the same naturality property of Cheeger Simons differential characters, as the c_k^b are natural too. So the Theorem follows by the unicity of the differential characters S_{C_k,c_k}.

As a corollary we get the following generalization of Theorem 2.3.1 to representations in $GL_n(\mathbb{C})$. Let N be an integer such that $Nc_k^{top}(E_\rho) = 0$. Then as before $Nc_k^{alg}(E)$ is in $CH^k(X)_0$ and $Ncs_{2k-1}(\rho)$ determines a unique element $Ncs_{2k-1}(\rho) \in H^{2k-1}(X;\mathbb{C})/H^{2k-1}(X;\mathbb{Z})$.

Theorem 2.3.3 *Let $\Phi : CH^k(X)_0 \to J^k(X)$ be the Abel Jacobi map. Then $\Phi(Nc_k^{alg}(E))$ is equal to the image of $Ncs_{2k-1}(\rho)$ under the projection*

$$\frac{H^{2k-1}(X;\mathbb{C})}{H^{2k-1}(X;\mathbb{Z})} \to J^k(X).$$

Bloch's conjecture concerning the rationality of secondary invariants of flat bundles over algebraic varieties can be seen as a particular case of a more general conjecture asserting the rationality of Chern-Cheeger-Simons invariants of flat bundles over arbitrary smooth manifolds. But secondary invariants of algebraic bundles deserve interest in their own merit. We have seen that they can be used to compute the Beilinson classes c_k^b in Deligne cohomology. Using these classes Beilinson in [5] defined regulator maps from algebraic K-theory to Deligne cohomology generalizing the Borel regulator.

There are many interesting conjectures linking these regulators maps to polylogarithms and the values of L-functions at integer points (see the book [38]). For example we have seen that the first Cheeger-Chern-Simons class can be expressed as a logarithm. Bloch in [6], and Dupont [17] proved that also the degree three Cheeger-Chern-Simons class can be computed using a suitable modification of the dilogarithm function. Bloch also introduced in [6] a group, called the Bloch group, which fits in a complex whose cohomology in the appropriate degree is isomorphic (after tensoring with the rationals) to the indecomposable part of $K_3(\mathbb{C})$. Conjecturally it should be possible to find formulae expressing all Beilinson's regulators (i.e. Beilinson's or Chern-Cheeger-Simons classes) in terms of appropriate modifications of polylogarithms. It is also conjectured that it should be possible to define higher dimensional Bloch's groups , fitting in complexes whose cohomology should be isomorphic (after tensoring with $\mathbb{Q}$) with the terms of a filtration of algebraic K-theory. Results in this direction have been obtained recently by Goncharov [26], [27].

Chapter 3

The topology of smooth projective algebraic varieties

In this chapter we recall following Lamotke [35] some known results about the topology of smooth projective algebraic varieties. The idea of using iterated hyperplane sections to study the topology of a projective variety goes back to Lefschetz [33], [34]. Lefschetz's main results on this subject are the so called *Weak Lefschetz theorem* and *Hard Lefschetz theorem.* Both these results have been proved later by several authors. For the former Andreotti and Fraenkel [1, 2], Bott [9], Kas [31], Wallace [44], and Lamotke [35]. For the latter see Hodge ([29]) and Deligne et al. [42]

3.1 Pencils of hyperplanes and the modification of a variety

Roughly speaking, we study a smooth projective algebraic variety $X \subset \mathbb{CP}^m$ of complex dimension n by intersecting it with a hyperplane of $\mathbb{CP}^m$. Then we let the hyperplane vary in a family to see how the intersection changes. A natural notion of family of hyperplanes is that of a pencil.

Definition 3.1.1 *Let $A \subset \mathbb{CP}^m$ be an $(m-2)$-dimensional subspace. Then the pencil of hyperplanes with axis A is the family of all hyperplanes in $\mathbb{CP}^m$ containing A.*

Pencils of hyperplanes in $\mathbb{CP}^m$ correspond to projective lines in the dual projective space, i.e. the space whose points parametrize hyperplanes in the original projective space. So we denote by $\{H_t\}_{t\in\mathbb{CP}^1}$ pencils of hyperplanes. Define $X_t := H_t \cap X$ for each $t \in \mathbb{CP}^1$. We call $\{X_t\}_{t\in\mathbb{CP}^1}$ a *pencil of hyperplane sections of* X .

Given a pencil of hyperplane sections, we would like to employ fibration techniques to investigate the topology of X. However we cannot define a map $X \to \mathbb{CP}^1$ by the prescription that each $x \in X$ is mapped to the $t \in \mathbb{CP}^1$ such that $x \in X_t$. The reason for this is that if A is the axis of the pencil and $X' = X \cap A$, then every $x \in X'$ is in X_t for every t. The complement of X', $X \setminus X'$ can be regarded as a fibration over $\mathbb{CP}^1$ with fibres $X_t \setminus X'$. For our purposes however, it is more convenient to blow up the axis of the pencil and define the modification of X (with respect to the pencil H_t)

$$Y = \{(x,t) \in X \times \mathbb{CP}^1 \mid x \in H_t\}$$

There are two projections $p : Y \to X$, and $f : Y \to \mathbb{CP}^1$. The former restricts to a relative homeomorphism between $Y \setminus p^{-1}(X')$ and $X \setminus X'$. We call the inverse images of points by f the fibres of f, even though f is not a fibration. For each t, the fibre $f^{-1}(t)$ is $X_t \times \{t\}$, thus we can identify the two spaces.

If a hyperplane H_t of the pencil is tangent to X, the corresponding section X_t is singular. In general this cannot be avoided, but we can consider pencils with the following properties (see [35]):

1. The axis of the pencil intersects X transversally, so X' is nonsingular and has complex dimension $n-2$.

2. The modification Y of X along X' is a projective variety. It is irreducible and nonsingular.

3. The projection $f : Y \to \mathbb{CP}^1$ has a finite number r of critical values, and there are exactly r critical points. So in each fibre of f there is at most one critical point.

4. Every critical point is nondegenerate, i.e. with respect to local coordinates the Hessian of f has maximal rank at the critical point.

Pencils with these properties are called *Lefschetz pencils.* In [35] and [42] it is proved that Lefschetz pencils exist and are generic in the following sense. The set of points in the dual m-projective space corresponding to hyperplanes tangent to X forms a variety $\check{X}$ whose dimension is at most $m-1$. However the dimension of $\check{X}$ can be $m-1$ even if n, the dimension of X, is less than $m-1$. If p is a point in the dual projective space not lying on $\check{X}$ then the corresponding hyperplane H_p is transversal to X. The set of all pencils containing H_p corresponds to the set L of all projective lines through p. This is a $m-1$ dimensional projective space. There is a nonempty open subset of L such that pencils corresponding to points of this subset satisfy the conditions above.

Let $\{H_t\}_{t\in\mathbb{CP}^1}$ be a Lefschetz pencil and $\{X_t\}_{t\in\mathbb{CP}^1}$ the corresponding pencil of hyperplane sections. Let Y be the modification of X with respect to this pencil, and $t_1,\ldots,t_r \in \mathbb{CP}^1$ be the critical values of the projection $f: Y \to \mathbb{CP}^1$. These are exactly the points corresponding to hyperplanes of the pencil which are tangent to X. Hence the hyperplane sections X_{t_i} are singular. We call the fibres $f^{-1}(t_i)$ the singular fibres of f. Let

$$Y^* := Y \setminus \bigcup_{i=1}^{r} Y_{t_i} \quad \text{and} \quad S^* := \mathbb{CP}^1 \setminus \{t_1,\ldots,t_r\}.$$

Theorem 3.1.1 (Ehresmann Fibration Theorem, see [35], [48]) *Let $f: E \to B$ be a proper differentiable mapping between differentiable manifolds without boundary. Suppose that the rank of f is equal to the dimension of B for all the points in E. Then f is a locally trivial fibration. If E has a boundary ∂E and the rank of the restriction of f to ∂E is equal to the dimension of B everywhere, then f fibres the pair $(E,\partial E)$ locally trivially. An analogous statement holds for any pair (E,E_0), where $E_0 \subset E$ is a closed submanifold with the property that the restriction of f to it has everywhere rank equal to the dimension of B.*

The restriction of f to Y^* is proper and has no critical points . Hence by the Ehresmann fibration theorem

$$f_{|Y^*} : Y^* \to S^*$$

is a locally trivial fibration. Let b be a point of S^*. Then the fibre $f^{-1}(b) = X_b \times \{b\}$ is a smooth projective variety of complex dimension $n-1$, the generic hyperplane section of X.

Recall that we denoted p the projection from Y to X. Since X' is the intersection of X with the axis of the pencil, it follows that $p^{-1}(X') = X' \times \mathbb{CP}^1$. Hence the inclusion $p^{-1}(X') \to Y$ induces a homomorphism

$$H_*(X' \times \mathbb{CP}^1) \to H_*(Y).$$

Using the Kunneth formula, we can define the following composition

$$H_{*-2}(X') \to H_*(X' \times \mathbb{CP}^1) \to H_*(Y)$$

which we denote by k. The following proposition, proved in [35], compares the homology of X with the homology of Y.

Proposition 3.1.2 *For every q the natural homomorphism $k : H_{q-2}(X') \to H_q(Y)$ fits in the following split short exact sequence.*

$$0 \to H_{q-2}(X') \xrightarrow{k} H_q(Y) \xrightarrow{p_*} H_q(X) \to 0.$$

In particular p induces isomorphisms

$$H_{2n-1}(Y) \cong H_{2n-1}(X) \text{ and } H_1(Y) \cong H_1(X).$$

3.2 The local structure of the singular fibres

Here we summarize some results from [35] leading to the Weak Lefschetz Theorem (Theorem 3.2.3) and we state some equivalent formulations of the Hard Lefschetz Theorem (Theorem 3.2.5). For proofs and more details we refer to [35].

Let X be an algebraic variety and Y be its modification with respect to a Lefschetz pencil. From now on we will identify $\mathbb{CP}^1$ with S^2. So consider the projection $f : Y \to S^2$ and let $\{x_1, \ldots, x_r\}$ be the critical points of f. Denote by D_+ and D_- the closed upper and lower hemispheres of S^2 respectively. We can assume that all the critical values $t_1 = f(x_1), \ldots, t_r = f(x_r)$ are interior points of D_+. Pick a regular value $b \in D_+ \cap D_-$ of f as a basepoint and let

$$Y_\pm := f^{-1}(D_\pm)$$

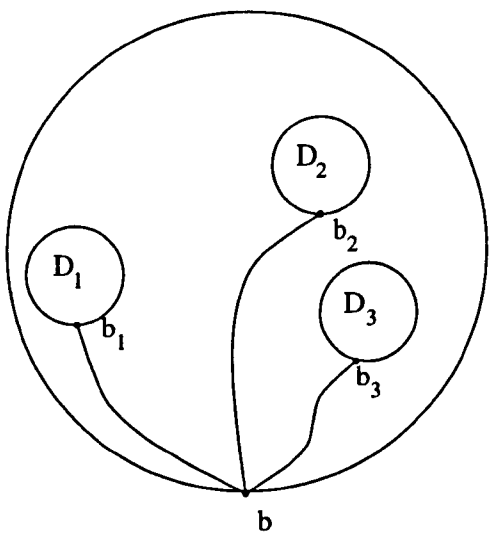
and

$$Y_b := f^{-1}(b).$$

Notation: We establish the convention that whenever A is a subset of S^2, Y_A is the subset $f^{-1}(A) \subset Y$. However when $x \in S^2$ is a point, we will denote the fibre over x by Y_x (rather than $Y_{\{x\}}$).

As in the previous section denote by S^* the complement of the critical values of f in S^2 and let $Y^* = f^{-1}(S^*)$. With the choices that we made, D_- is contained in S^*. Since $f_{|Y^*}$ is a locally trivial fibration and D_- is contractible, it follows that Y_- is diffeomorphic to the product $Y_b \times D_-$ via a fibre preserving diffeomorphism.

We are now going to see how the singular fibres determine the homology of Y_+. Since the pencil is a Lefschetz pencil, all critical points of f are nondegenerate, and no two lie in the same fibre of f. Choose coordinates to identify D_+ with the unit disk in $\mathbb{C}$ so that b corresponds to 1. Choose also small, mutually disjoint disks $D_i \subset D_+$ of radius η centered at t_i. Pick regular values $b_i \in \partial D_i$ as additional basepoints, and embedded C^∞ paths ℓ_i from b to b_i meeting only in b, so that $\ell := \bigcup_{i=1}^r \ell_i$ retracts to b and $k := \ell \cup \bigcup_{i=1}^r D_i$ is a deformation retract of D_+ (see figure).



We now describe the local structure of a neighbourhood of a singular fibre Y_{x_i}. Since the singular point x_i of f is nondegenerate, we can choose a neighbourhood of x_i in Y in which f can be written with respect to local complex coordinates $(z_1, \ldots, z_n)$

$$f(z) = t_i + z_1^2 + \cdots + z_n^2$$

(we can suppose that x_i is the origin in this local system). Choose $\epsilon > 0$ small enough so that the closed ball in $\mathbb{C}$ of radius ϵ centered at the origin is contained in the range of the local coordinates. Let B be the corresponding subset of Y. We can choose η small enough (namely $\eta < \epsilon^2$) so that $D_i \subset f(B)$. Let

$$T := Y_{D_i} \cap B, \quad \text{and} \quad F := Y_{b_i} \cap B$$

then in local coordinates

$$T = \{z \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 \leq \epsilon^2, \quad \text{and} \quad |z_1^2 + \cdots + z_n^2| \leq \eta\}$$

$$F = \{z \in T : z_1^2 + \cdots + z_n^2 = \eta\}$$

(see [35] and references therein). From this local coordinates description, it follows that T is contractible, and F is diffeomorphic to the unit disk bundle of the tangent bundle of the sphere S^{n-1}.

The space T can be seen as a 'singular fiber space' with generic fibre the disk bundle DTS^{n-1}, and with just one singular fibre over the origin consisting of DTS^{n-1} with the zero section shrunk to a point. Let N_i be the closure of the complement of T in Y_{D_i}, i.e. $N_i = Y_{D_i} \setminus \overset{\circ}{B}$. The next lemma describes the structure of N_i.

Lemma 3.2.1 *There is a fibre preserving diffeomorphism from N_i to the trivial bundle $\overline{(Y_{b_i} \setminus F)} \times D_i$.*

Proof: This is a consequence of the Ehresmann fibration theorem for manifolds with boundary. The restriction of f to N_i has maximal rank everywhere, so no critical point. Hence $f : N_i \to D_i$ is a locally trivial fibration. As D_i retracts to b_i we can lift this contraction to a contraction of N_i to $\overline{(Y_{b_i} \setminus F)}$.

From this local analysis one can prove the following important result.

Proposition 3.2.2 *For any ring of coefficients R*

$$H_q(Y_+, Y_b) = \begin{cases} R^r & \text{if } q = n \\ 0 & \text{otherwise} \end{cases}$$

The proof consists of the following steps (see [35]):

1. $H_q(T,F) = 0$ for $q \neq n$, and $H_n(T,F)$ is free of rank 1. A generator of $H_n(T,F)$ is given by an orientation of the real n-disk

$$\{z \in T : z_1, \ldots, z_n \text{ are real}\}.$$

The map $\partial : H_n(T,F) \to H_{n-1}(F) \cong H_{n-1}(S^{n-1})$ is an isomorphism.

2. The inclusion $(T,F) \hookrightarrow (Y_{D_i}, Y_{b_i})$ induces isomorphisms in homology.

3. The fibre Y_b is a deformation retract of Y_ℓ, as ℓ retracts to b and Y_ℓ contains no critical points of f, hence is diffeomorphic to the trivial fibration. Thus $H_q(Y_+, Y_\ell) \cong H_q(Y_+, Y_b)$ for all q.

4. Let $L = (\bigcup_{i=1}^r D_i) \cup \ell$. Then Y_+ retracts onto Y_L, so we have isomorphisms $H_q(Y_+, Y_\ell) \cong H_q(Y_L, Y_\ell)$. The inclusion

$$(\bigcup_{i=1}^r Y_{D_i}, \bigcup_{i=1}^r Y_{b_i}) \hookrightarrow (Y_L, Y_\ell)$$

is an excision. Since the union is disjoint, for all q we have the chain of isomorphisms

$$\bigoplus_{i=1}^r H_q(Y_{D_i}, Y_{b_i}) \xrightarrow{\cong} H_q(Y_+, Y_\ell) \xleftarrow{\cong} H_q(Y_+, Y_b) \tag{3.1}$$

that proves the proposition.

Theorem 3.2.3 (Weak Lefschetz Theorem) *The inclusion of the hyperplane section $X_b \hookrightarrow X$ induces isomorphisms in homology up to dimension $n-2$ and a surjection in dimension $n-1$, i.e. for all $q \leq n-1$ we have*

$$H_q(X, X_b) = 0.$$

The Weak Lefschetz Theorem follows as a corollary of the previous proposition by induction on the dimension of X. In fact X' is an hyperplane section of the smooth projective $n-1$-variety X_b, and there is a long exact sequence (see 3.6.5 in [35])

$$\to H_{q+2}(Y_+, Y_b) \to H_{q+2}(X, X_b) \to H_q(X_b, X') \to H_{q+1}(Y_+, Y_b) \to \tag{3.2}$$

that gives isomorphisms $H_{q+2}(X, X_b) \cong H_q(X_b, X')$ for $q \neq n-2, n-1$.

Remark: A fact that we are going to use quite often follows from the exact sequence (3.2): the map

$$H_n(Y_+, Y_b) \to H_n(X, X_b)$$

induced by the projection $p : Y \to X$ restricted to Y_+ is surjective, as the next term in the sequence is zero by the Weak Lefschetz Theorem.

Definition 3.2.1 *The module of the vanishing cycles $V \subseteq H_{n-1}(Y_b)$ is the image of the connecting homomorphism $\partial : H_n(Y_+, Y_b) \to H_{n-1}(Y_b)$.*

Hence we have

$$V = \mathrm{Im}[\partial : H_n(Y_+, Y_b) \to H_{n-1}(Y_b)] = \mathrm{Ker}[H_{n-1}(Y_b) \to H_{n-1}(Y_+)].$$

There is an isomorphism $H_{n-1}(X) \cong H_{n-1}(Y_+)$ (to see this compare the exact sequences of the pairs (Y_+, Y_b) and (X, X_b) and use the previous remark), and as we identify the hyperplane section X_b with the fibre of f, Y_b, we have that V is isomorphic to the kernel of the map

$$H_{n-1}(X_b) \to H_{n-1}(X).$$

Remark Consider the commutative diagram

$$\begin{array}{ccccccc}
\cdots \to & H_n(Y, Y_b) & \to & H_{n-1}(Y_b) & \to & H_{n-1}(Y) & \to \cdots \\
& \downarrow & & \downarrow\cong & & \downarrow & \\
\cdots \to & H_n(X, X_b) & \to & H_{n-1}(X_b) & \to & H_{n-1}(X) & \to \cdots
\end{array}$$

The vertical morphism at the right is a surjection by Proposition 3.1.2. The middle one is an isomorphism and the left one is surjective since the map $H_n(Y_+, Y_b) \to H_n(X, X_b)$ is a surjection and is equal to the composition

$$H_n(Y_+, Y_b) \to H_n(Y, Y_b) \to H_n(X, X_b).$$

Hence, identifying the two groups $H_{n-1}(Y_b) \cong H_{n-1}(X_b)$, the map $H_{n-1}(Y_b) \to H_{n-1}(Y)$ has the same kernel as $H_{n-1}(X_b) \to H_{n-1}(X)$ by a standard diagram chase argument.

Consider the locally trivial fibration with fibre Y_b

$$f_{|Y^*} : Y^* \to S^*$$

There is an action of $\pi_1(S^*, b)$ on the homology of Y_b, the monodromy of the fibration f. A set of generators for $\pi_1(S^*, b)$ is given by closed paths ω_i going round the singular value t_i and separating it from the other singular values. For example, one could choose $\omega_i = \ell_i^{-1}\partial D_i \ell_i$, where the ℓ_i and the D_i are paths and disks we chose earlier in this section when describing the structure of the singular fibres. Moreover if the t_i's are suitably ordered, then one can choose the ω_i so that the only relation in $\pi_1(S^*, b)$ is

$$[\omega_1] \cdot [\omega_2] \cdots [\omega_r] = 1.$$

Using the isomorphisms (3.1) we can define a composition

$$H_n(Y_{D_i}, Y_{b_i}) \to H_n(Y_+, Y_b) \xrightarrow{\partial} H_{n-1}(Y_b)$$

Let $\delta_i \in H_{n-1}(Y_b)$ be the image of the generator of $H_n(Y_{D_i}, Y_{b_i})$ via this composition, and denote by $\cdot$ the intersection pairing of cycles. Choosing the generators ω_i as above, the action of $\pi_1(S^*, b)$ on the homology of Y_b is given by the following formulae (see [35] and references therein).

Theorem 3.2.4 (Picard-Lefschetz Formulae) *1. The action of $\pi_1(S^*, b)$ on $H_q(Y_b)$ is trivial for all $q \neq n-1$.*

2. Let $x \in H_{n-1}(Y_b)$. Then $[\omega_i]$ acts on x by

$$[\omega_i](x) = x + (-1)^{\frac{n(n+1)}{2}}(x \cdot \delta_i)\delta_i \tag{3.3}$$

So a necessary and sufficient condition for an element $x \in H_{n-1}(Y_b)$ to be invariant under the action of the fundamental group of S^* is to have zero intersection with all the vanishing cycles. This motivates the following definition.

Definition 3.2.2

$$I := \{x \in H_{n-1}(Y_b) \mid (x \cdot \delta_i) = 0 \text{ for } i = 1, \ldots, r\}$$

is called the module of invariant cycles.

We can now state the Hard Lefschetz Theorem:

Theorem 3.2.5 (Hard Lefschetz Theorem) *If coefficients in a field of characteristic zero are taken then*

$$H_{n-1}(Y_b) \cong H_{n-1}(X_b) \cong I \oplus V.$$

In [35] it is proved that this is equivalent to any of the following statements:

1. $H_{n-1}(X_b) \to H_{n-1}(X)$ *maps* I *isomorphically onto* $H_{n-1}(X)$.
2. *Let* $u \in H^2(X)$ *be the Poincaré dual of the image of the fundamental class of* X_b *in* $H_{2n-2}(X)$, *then* $u^q \cap : H_{n+q}(X) \to H_{n-q}(X)$ *is an isomorphism for* $q = 1, \ldots, n$.
3. *The intersection form on* $H_{n-1}(Y_b) \cong H_{n-1}(X_b)$ *splits into the direct sum of its restrictions to* I *and* V; *the two restrictions remain non-degenerate.*

Chapter 4

Homology classes in algebraic varieties and the Chern Cheeger Simons invariant

In this Chapter we prove that the third Chern Cheeger Simons invariant of any representation of the fundamental group of a smooth projective algebraic variety is always rational. We now describe briefly the method we use. If M is a manifold, and $\alpha \in H_p(M)$ is a homology class, we say for brevity that α is represented by a p-dimensional oriented manifold N if there is a map $g : N \to M$ such that $g_*([N]) = \alpha$, where $[N]$ is the fundamental class of N. First we consider the case of an algebraic surface and prove the following theorem.

Theorem 4.0.1 *Let X be a smooth projective algebraic surface, then there is a generating set for the three dimensional homology of X (with rational coefficients) whose classes are all represented by products $S^2 \times S^1$.*

By the Weak Lefschetz theorem, in order to prove an analogous statement for a projective algebraic variety of any dimension, we have only to consider the case of a threefold, where 'new' (in the sense that they are not carried by the inclusion of an hyperplane section) three dimensional homology classes appear. We call these classes 'extra' classes. We prove that all extra classes can be represented either by 3-spheres or connected sums of $S^1 \times S^2$. Hence we have the following theorem:

Theorem 4.0.2 *Let X be a smooth complex projective variety. Then there exist a generating set for the rational three dimensional homology of X whose classes can be represented by*

- $S^2 \times S^1$
- *3-spheres*
- *connected sums of $S^2 \times S^1$*

Finally, we prove that the Chern Simons invariant of representations of the fundamental group of the 3-manifolds listed in the above theorem in a Lie group G with finitely many components is always rational.

4.1 Construction of homology classes in an algebraic surface

In this section we work in complex dimension 2, so using the notations of the previous chapter, X will be a projective algebraic surface and Y its modification with respect to a Lefschetz pencil. Again we take a regular value b of the projection $f : Y \to S^2$ as a reference fibre. We identify the fibre over b, Y_b with the hyperplane section X_b and we use it as reference fibre. As in the previous chapter, we choose small disks D_i around the singular values x_i and additional reference points b_i on ∂D_i so that Y_{b_i} is a regular fibre.

Remark: To prove Theorem 4.0.1 we need only to consider the case in which there are critical values of the projection f. This happens when the dimension of the dual variety $\check{X}$ in the dual projective m-dimensional space is $m-1$. In fact the hyperplane section X_t is singular (i.e. Y_t contains a critical point for f) if the hyperplane H_t is tangent to X, that is if $t \in \check{X}$ in the dual projective space. If the dimension of $\check{X}$ is less than $m-1$, the generic projective line in the dual space will not meet $\check{X}$, so the generic pencil of hyperplane sections will not have singular hyperplane sections (cf. section 3.1). In this case $f : Y \to S^2$ is a genuine fibre bundle of Riemann surfaces over S^2. The structure group of bundles with fibre a Riemann surface of genus $g \geq 2$ is the discrete group M_g, the mapping class group of the surface. Then the classifying space for Riemann surface bundles has the homotopy

type of a $K(1, M_g)$ (see [19]), hence any such bundle over a sphere is trivial, and Theorem 4.0.1 is true.

So, let $\{x_1, \ldots, x_r\}$ be the critical points of f. Choose small disks D_i with centre $f(x_i)$, as in Section 3.2, so that Lemma 3.2.1 holds. In complex dimension two, Y_b is a Riemann surface, so the space that we denoted by F in section 3.2 is diffeomorphic to a cylinder (the unit disk bundle of the tangent bundle of S^1). Then Lemma 3.2.1 states that the submanifold $N_i \subset Y_{D_i}$ is diffeomorphic to a product

$$(Y_{b_i} \setminus \tilde{C}_i) \times D_i$$

where $\tilde{C}_i$ is a cylinder embedded in Y_{b_i}.

Then we can choose diffeomorphisms

$$\psi_i : N_i \to (Y_{b_i} \setminus \tilde{C}_i) \times D_i$$

Choose embedded paths ℓ_i from b to b_i as in 3.2. These give diffeomorphisms

$$\phi_i : Y_{\ell_i} \to Y_b \times I$$

(we consider the ϕ_i's and the ψ_i's fixed once and for all).

Remark on notation and conventions:
We parametrize ℓ_i so that the fibre above the point $\ell_i(t)$ is $\phi_i^{-1}(\{Y_b\}, t)$, $\ell_i(0) = b$ and $\ell_i(1) = b_i$ for every i. Then $\phi_i^{-1}(\cdot, t)$, t fixed, is a diffeomorphism between Y_b and $Y_{\ell_i(t)}$. We can assume $\phi_i^{-1}(\cdot, 0)$ is the identity on Y_b. We adopt similar conventions for the diffeomorphism ψ_i: we identify D_i with the complex unit disk, so that b_i corresponds to the point $z = 1$, and $\psi_i^{-1}(\cdot, 1)$ is the identity of $Y_{b_i} \setminus \tilde{C}_i$.

Denote by $p_1 : Y_b \times I \to Y_b$ be the projection onto the first factor, and let

$$C_i = p_1(\phi_i(\tilde{C}_i)) \subset Y_b.$$

By definition C_i is the image of the cylinder $\tilde{C}_i$ under a diffeomorphism, hence topologically it is a cylinder embedded in Y_b.

Proposition 4.1.1 *Let γ be a simple closed curve in $Y_b \setminus \bigcup_{i=1}^r C_i$. Then there is a circle bundle $\Gamma \to S^2$ embedded in Y whose fibre at b is γ.*

Proof: Let $\ell = \bigcup_{i=1}^{r} \ell_i$. Given a simple closed curve $\gamma \subset Y_b \setminus \bigcup_{i=1}^{r} C_i$, using the trivializations given by the ϕ_i's, we can define a subbundle of Y_ℓ whose fibre over $\ell_i(t)$ is $\phi_i^{-1}(\{\gamma\}, t)$. Now let $L = \ell \cup \bigcup_{i=1}^{r} D_i$. For every i we have

$$\phi_i^{-1}(\{\gamma\}, 1) \subset Y_{b_i} \setminus \tilde{C}_i$$

so we can extend the subbundle defined over ℓ to a subbundle defined over L using the trivializations ψ_i. The fibre over the point $f(\psi^{-1}(\{Y_{b_i} \setminus \tilde{C}_i\}, z))$ is

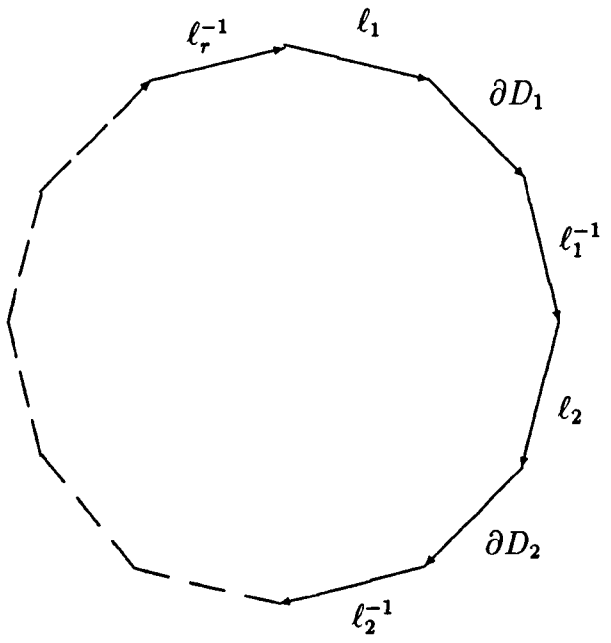
$$\psi^{-1}((\phi_i^{-1}(\{\gamma\}, 1)), z).$$

In this way we have defined a circle bundle Γ_L embedded in Y_L. To extend Γ_L over the whole of S^2 take a disk D and consider the attaching map

$$\alpha : \partial D \to \bigcup_{i=1}^{r} \ell_i \cup \bigcup_{i=1}^{r} \partial D_i$$

defined as follows: divide $S^1 \cong \partial D$ in $3r$ intervals (i.e. triangulate it with $3r$ vertices), and

1. glue the $3(i-1)+1^{th}$ to ℓ_i
2. glue the $3(i-1)+2^{nd}$ to ∂D_i
3. the $3i^{th}$ to ℓ_i^{-1} (i.e. ℓ_i with the opposite orientation)



It is clear that the map is well defined on the vertices. Now take $\gamma \times D$ and glue it to Γ_L with the map $\gamma \times \partial D \to \Gamma_L$ which identifies

$$\gamma \times \{z\} \subset \gamma \times \partial D \text{ with } \phi_i^{-1}(\gamma, \alpha(z)) \quad \text{if} \quad \alpha(z) \in \ell_i$$

$$\gamma \times \{z\} \subset \gamma \times \partial D \text{ with } \psi^{-1}(\{\phi_i^{-1}(\{\gamma\}, 1)\}, \alpha(z)) \quad \text{if} \quad \alpha(z) \in \partial D_i.$$

In this way we obtain the bundle Γ as required.

Let u be the Poincaré dual of $[Y_b]$ in $H^2(Y;\mathbb{Q})$. To prove that the bundles Γ constructed above generate $H_3(Y;\mathbb{Q})$ (and hence their images by the projection p generate $H_3(X;\mathbb{Q})$), we will show that the elements of the form $u \cap [\Gamma]$ span $H_1(Y;\mathbb{Q})$. We have that $H_1(Y_b) \to H_1(Y)$ is a surjection for any ring of coefficients. The map $u\cap : H_3(Y) \to H_1(Y)$ factors as

$$H_3(Y) \xrightarrow{i_!} H_1(Y_b) \to H_1(Y).$$

From the previous proposition we know that if $\gamma \subset Y_b \setminus \bigcup_{i=1}^r C_i$ is a simple closed curve, we can construct a circle bundle $\Gamma \subset Y$ over S^2, such that the fibre over b is γ. If $a \in H^1(Y;\mathbb{Q})$ is the Poincaré dual of $[\Gamma]$, then the Poincaré dual of $u \cup a$ is represented by the transverse intersection of Γ and Y_b (see e.g. [15]). Then by construction up to a sign we have

$$u \cap [\Gamma] = (u \cup a) \cap [Y] = [\gamma] \in H_1(Y)$$

(with an abuse of notation we will denote by $[\gamma]$ the class of γ in $H_1(Y)$, in $H_1(Y_b)$ and in $H_1(Y_b \setminus \bigcup_{i=1}^r C_i)$).

The closure of $Y_b \setminus \bigcup_{i=1}^r C_i$ is a compact surface with boundary (with more than one component). Choosing diffeomorphisms from every component to the standard models of compact surfaces with boundary one can see that its one dimensional homolgy (any coefficients) is generated by embedded simple closed curves. Thus we need only to prove:

Proposition 4.1.2 *The composite*

$$H_1(Y_b \setminus \bigcup_{i=1}^r C_i;\mathbb{Q}) \to H_1(Y_b;\mathbb{Q}) \to H_1(Y;\mathbb{Q})$$

is a surjection.

Proof: For the rest of this section homology will be with rational coefficients unless otherwise stated. Let $i_1 : Y_b \setminus \bigcup_{i=1}^r C_i \to Y_b$ and $i_2 : \bigcup_{i=1}^r C_i \to Y_b$ be the inclusions. The commutative diagram

$$\begin{array}{ccccc} \bigoplus_{i=1}^r H_2(T,F) & \cong & \bigoplus_{i=1}^r H_2(Y_{D_i}, Y_{b_i}) & \cong & H_2(Y_+, Y_b) \\ \cong \downarrow & & & & \downarrow \\ \bigoplus_{i=1}^r H_1(F) \cong \bigoplus_{i=1}^r H_1(\tilde{C}_i) & \xrightarrow{\oplus \phi_{i*}} & H_1(\bigcup_{i=1}^r C_i) & \xrightarrow{i_{2*}} & H_1(Y_b) \end{array}$$

shows that any vanishing cycle $v \in V := \mathrm{Im}[H_1(Y_+, Y_b) \to H_1(Y_b)]$ is in the image of i_{2*}. We want to prove that $\mathrm{Im}[i_{1*} : H_1(Y_b \setminus \bigcup_{i=1}^r C_i) \to H_1(Y_b)] = I$, the module of invariant cycles. Let $v \in V$, $\alpha \in H_1(Y_b)$ and let α^* be the Poincaré dual of α. Let $c \in H_1(\bigcup_{i=1}^r C_i)$ be such that $i_{2*}(c) = v$ and denote by $\langle , \rangle$ the pairing between homology and cohomology. Then

$$\alpha \cdot v = \langle \alpha^*, v \rangle = \langle \alpha^*, i_{2*}(c) \rangle = \langle i_2^*(\alpha^*), v \rangle.$$

We have a sign commutative diagram

$$\begin{array}{ccc} H^1(Y_b) & \xrightarrow{i_2^*} & H^1(\bigcup_{i=1}^r C_i) \\ \text{p.d.} \downarrow & & \downarrow \text{a.d.} \\ H_1(Y_b) & \xrightarrow{j} & H_1(Y_b, Y_b \setminus \bigcup_{i=1}^r C_i) \end{array}$$

in which the arrows p.d. and a.d. are isomorphisms given by Poincaré and Alexander duality. It follows that

$$\alpha \in I \Leftrightarrow \forall v \in V \quad \alpha \cdot v = 0 \Leftrightarrow i_2^*(\alpha^*) = 0 \Leftrightarrow j(\alpha) = 0 \Leftrightarrow \alpha \in \mathrm{Im}\, i_{1*}.$$

By the Hard Lefschetz theorem, the proposition is proved.

To get a generating set for $H_3(Y)$ then, we can take a generating set $\{[\gamma_1], \ldots, [\gamma_q]\}$ of $H_1(Y \setminus \bigcup_{i=1}^r C_i)$ and construct the corresponding Γ_i's. Let $\gamma_i : S^1 \to Y_b$ be maps such that $\gamma_{i*}[S^1] = [\gamma_i]$. As the classes $[\gamma_i]$ are not torsion classes, the commutative diagram

$$\begin{array}{ccc} S^1 & \xrightarrow{\gamma_i} & Y_b \\ \downarrow & & \downarrow \\ \Gamma_i & \to & Y \end{array}$$

implies that $\pi_1(\Gamma_i) = \mathbb{Z}$. Then Γ is trivial (see e.g. [45] p.194-195).

The map p_*, induced by the projection $p : Y \to X$, is an isomorphism between $H_3(Y)$ and $H_3(X)$, hence the elements $p_*([\Gamma])$ are set of generators for $H_3(X)$. This ends the proof of 4.0.1.

Remark: It is in this step that we have to use the Hard Lefschetz theorem. In fact in general, it is not true that the inclusion $H_1(\Sigma \setminus \bigcup_{i=1}^r C_i) \to H_1(\Sigma)$ where Σ is a surface is a surjection over $H_1(\Sigma)/\mathrm{Im} H_1(\bigcup_{i=1}^r C_i) \to H_1(\Sigma)$: consider a torus T, $\alpha, \beta \in H_1(T;\mathbb{Z})$ standard generators of $H_1(T;\mathbb{Z})$ and let C be a neighbourhood of a circle representing α; then the image of the inclusion $H_1(T \setminus C) \to H_1(T)$ is generated by α, while β is sent to a nonzero element in $H_1(T, T \setminus C)$ representing a nonzero element of $H^1(C)$. Geometrically this element is the intersection of (a representative of) β with C giving an element of $H_1(C, \partial C)$ that intersects α in exactly one point.

4.2 Extra classes and the kernel of the Hurewicz map

In this section we want to find representatives for the three-dimensional homology classes of a threefold that are not in the image of the inclusion of an algebraic surface as a hyperplane section, thus completing the proof of Theorem 4.0.2. So X will be a complex projective variety of (complex) dimension 3, Y its modification with respect to a Lefschetz pencil, $Y_b \cong X_b$ a nonsingular hyperplane section, and Y_+ will be as in section 3.2.

Consider the diagram

$$\begin{array}{ccccccccc}
\cdots \to & H_3(Y_b) & \to & H_3(Y_+) & \to & H_3(Y_+, Y_b) & \to & H_2(Y_b) & \to \cdots \\
& \updownarrow & & \downarrow & & \downarrow & & \updownarrow & \\
\cdots \to & H_3(X_b) & \to & H_3(X) & \to & H_3(X, X_b) & \to & H_2(X_b) & \to \cdots
\end{array}$$

From the isomorphisms $H_*(Y_b) \cong H_*(X_b)$ given by the identification of the two spaces and the surjectivity of the map $H_3(Y_+, Y_b) \to H_3(X, X_b)$ (remark after Weak Lefschetz theorem in section 3.2), using a diagram chase it is easy to see that $H_3(Y_+) \to H_3(X)$ is a surjection. So we will work with the long exact sequence

$$\ldots \to H_3(Y_b) \to H_3(Y_+) \to H_3(Y_+, Y_b) \to H_2(Y_b) \to \ldots$$

Lemma 4.2.1 *Every relative homology class in $H_3(Y_+, Y_b)$ can be represented by a map*

$$\Delta : (D^3, S^2) \to (Y_+, Y_b).$$

Moreover this map can be chosen so that its geometric boundary is an embedded 2-sphere in Y_b representing the class of a vanishing cycle. (Lefschetz called these maps -or rather their images- 'thimbles').

Remark: This Lemma is an immediate consequence of the relative Hurewicz theorem: in fact the Weak Lefschetz theorem is still valid if we replace relative homology groups with relative homotopy groups (see [35], [36]), and the Lemma follows from the analogous statement for the pair (Y_+, Y_b). The Lemma essentially asserts the surjectivity of the relative Hurewicz map, which holds even though Y_b is not simply connected. The proof we give here however is more constructive, and illustrates the techniques we are going to use in this section.

Proof: We return to the sketch of the proof of Proposition 3.2.2 to construct these maps: from there it follows that the composition

$$H_3(T, F) \cong H_3(Y_{D_i}, Y_{b_i}) \to H_3(Y_+, Y_b)$$

is a monomorphism, sending a generator $\mathcal{D}$ of $H_3(T, F)$, given by a map of pairs

$$(D^3, S^2) \to (T, F)$$

to one of the generators of the free group $H_3(Y_+, Y_b)$, $[\Delta_i]$. So geometrically the generator of $H_3(Y_{D_i}, Y_{b_i})$ is given by a map

$$\Gamma : (D^3, S^2) \to (Y_{D_i}, Y_{b_i}).$$

To construct a map representing $[\Delta_i]$, we note that (Γ can be chosen so that) $\Gamma(S^2) \subset Y_{b_i}$ is an embedded 2-sphere S_i. As Y_{ℓ_i} (we are using the notations of section 3.2) is diffeomorphic to the product $Y_{b_i} \times \ell_i$, we can choose an embedding $j : Y_{b_i} \times \ell_i \to Y_+$ such that

$$j(Y_{b_i} \times \ell_i) = Y_{\ell_i} \qquad j(y, b_i) = y \qquad f \circ j(y, \lambda) = \lambda$$

The space obtained from the three disk and $S^2 \times \ell_i$ identifying the boundary of the three disk with $S^2 \times \{b_i\}$ is diffeomorphic to a three disk (after smoothing the corners), hence we can define the map

$$\Delta_i : (D^3, S^2) \to (Y_+, Y_b)$$

glueing the maps Γ and $j_{|S_i \times \ell_i}$. The map Δ_i represents $[\Delta_i] \in H_3(Y_+, Y_b)$, and the image of its restriction to the boundary S^2 is an embedded 2-sphere in Y_b representing a vanishing cycle.

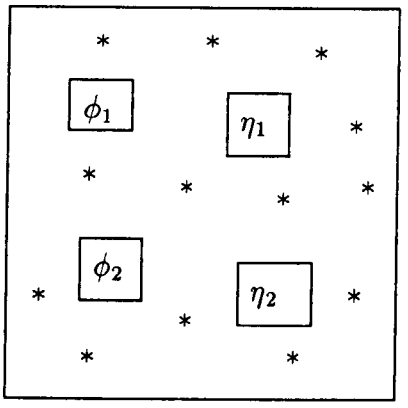
We adopt the convention that for any map $\Delta : (D^3, S^2) \to (Y_+, Y_b)$ we will indicate with $[\Delta]_H$ its homology class in $H_3(Y_+, Y_b)$ and with $[\Delta]_\pi$ the homotopy class in $\pi_3(Y_+, Y_b)$. We will omit the subscript when the context is clear.

Let $a \in H_3(Y_+)$ be a class not in the image of $H_3(Y_b) \to H_3(Y_+)$. Then a is sent to a nonzero class $[\Delta] \in H_3(Y_+, Y_b)$ such that $\partial[\Delta]_H = 0$ in $H_2(Y_b)$. If also $\partial[\Delta]_\pi = 0$ in $\pi_2(Y_b)$, then the class a is represented by a three-sphere, i.e. there is a map $s : S^3 \to Y_+$ such that $s_*[S^3] = a$ in $H_3(Y_+)$. If instead $\partial[\Delta]_\pi \neq 0$ in $\pi_2(Y_b)$ then $\alpha := \partial[\Delta]_\pi$ is in the kernel of the Hurewicz map $h : \pi_2(Y_b) \to H_2(Y_b)$, hence it can be written in the form (see e.g. [45] p.154-169, [46] p.53)

$$\alpha = \sum_{i=1}^{d} (\xi_i \cdot \alpha_i - \alpha_i)$$

where $\xi_i \in \pi_1(Y_b)$, $\alpha_i \in \pi_2(Y_b)$ and $\cdot$ is the action of $\pi_1(Y_b)$ on $\pi_2(Y_b)$.

Let $\phi_i : S^2 \to Y_b$ and $\eta_i : S^2 \to Y_b$ be maps representing the classes α_i and $\xi_i \cdot \alpha_i$ respectively. Let U be the disjoint union of $2d$ small 2-disks in S^2. Then α can be represented by a map $\Psi : S^2 \to Y_b$ which is constant on the complement of U in S^2, agrees with ϕ_i on the $(2i-1)$-th disk and agrees with η_i on the $2i$-th disk (see picture in next page).



$$S^2 = I^2/\partial I^2$$

Let $F : S^2 \times I \to Y_b$ be an homotopy between $\Delta_{|S^2} = F_0$ and $\Psi = F_1$. Identifying $S^2 = \partial D^3$ with $S^2 \times \{0\}$ we form a space

$$B = D^3 \cup_{S^2} S^2 \times I.$$

Since $\Delta_{|S^2} = F_0$ we can glue the maps Δ and F to obtain a map

$$\Delta_1 : (B, \partial B) \to (Y_+, Y_b).$$

The space B is still a disk, and we have $[\Delta_1] = [\Delta]$ both in $H_3(Y_+, Y_b)$ and in $\pi_3(Y_+, Y_b)$.

The boundary ∂B of B is $S^2 \times \{1\} \subset S^2 \times I \subset B$. Let $\sim$ be the relation that identifies all the points of $(S^2 \times \{1\}) \setminus U$ to a single point, and define the pair $(K, \bigvee_1^{2d} S^2) := (B, S^2)/\sim$. From the discussion above it follows that one can define a map m such that Δ_1 factors as

$$\begin{array}{ccc} (B, S^2) & \xrightarrow{\Delta_1} & (Y_+, Y_b) \\ \searrow & & \nearrow m \\ & (K, \bigvee_1^{2d} S^2) & \end{array} \tag{4.1}$$

Lemma 4.2.2 *Let $D_1^3, \ldots, D_{2d}^3$ be disjoint small disks in S^3. There exist a map*

$$p : (S^3 \setminus \bigsqcup_{i=1}^{2d} D_i^3, \partial \bigsqcup_{i=1}^{2d} D_i^3) \to (K, \bigvee_1^{2d} S^2)$$

such that

$$m \circ p : (S^3 \setminus \bigsqcup_{i=1}^{2d} D_i^3, \partial \bigsqcup_{i=1}^{2d} D_i^3) \to (Y_+, Y_b)$$

represents $[\Delta] \in H_3(Y_+, Y_b)$

Proof: Let $N \subset S^3$ be a 3-disk containing $\bigsqcup_{i=1}^{2d} D_i^3$. Write the open 3-disk $S^3 \setminus N$ as $S \cup C$, where S is another 3-disk and C is a cylinder $S^2 \times I$. As B is a 3-disk D^3 with a cylinder attached to the boudary, it is possible to define our map p on $S^3 \setminus N$ identifying $S \cup C$ with $D^3 \cup (S^2 \times (0, \frac{3}{4}) \subset B$ and composing with the projection onto the quotient $B \to K$ so that we have commutative diagrams

$$\begin{tikzcd} & D^3 \subset B \arrow[d] \\ \mathrm{S} \arrow[ur] \arrow[r, "p_{|S}"'] & \mathrm{K} \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & S^2 \times (0, \frac{3}{4}) \subset B \arrow[dr] & \\ C \arrow[ur] \arrow[r, "p_{|C}"'] & & \mathrm{K} \end{tikzcd}$$

To define p on $N \setminus \bigsqcup_{i=1}^{2d} D_i^3$ now, we observe that $K \setminus \mathrm{image}(p_{|(S^3 \setminus N)})$ is an $S^2 \times I$ with a boundary S^2 collapsed to a wedge of spheres. The space $N \setminus \bigsqcup_{i=1}^{2d} D_i^3$ is almost the same thing: if we contract to points paths in N connecting the disjoint 2-spheres, we can identify the two spaces. We define p on N to be the composition of this contraction and this identification.

By construction, since the restriction to S of the composition $m \circ p$ factors as $S \to D^3 \xrightarrow{m} Y_+$, from the commutative diagram (4.1) it follows that the difference of the chains defined by $m \circ p$ (choosing an appropriate triangulation) and Δ_1 is contained in Y_b (at least up to an homotopy). Hence if we denote by G the composition $m \circ p$ we have $[G] = [\Delta]$ both in relative homotopy and relative homology.

Remark The map g has the property that $g_{|S^2_{2i-1}} = \phi_i$ and $g_{|S^2_{2i}} = \eta_i$.

The classes α_i and $\xi_i \cdot \alpha_i$ are freely homotopic in Y_b (see [46]). Hence for each $i = 1, \ldots, d$, there are maps $h_i : S^2 \times I \to Y_b$ such that $h_{i_0} = \phi_i$, and $h_{i_1} = \eta_i$. We can then glue d copies of $S^2 \times I$ to K identifying each sphere of $\bigvee_1^{2d} S^2$ with one component of the boundary of one of the cylinders in such

a way to obtain a space $\hat{K}$ and a map

$$\hat{m} : (\hat{K}, \bigcup_{1}^{2d} S^2 \times I) \to (Y_+, Y_b)$$

defined glueing the maps m and $h_1, \ldots, h_d$. In the same way, we can attach d copies of $S^2 \times I$ to $S^3 \setminus \bigsqcup_{i=1}^{2d} D_i^3$, to obtain (after smoothing the corners) a manifold M diffeomorphic to a connected sum of $S^2 \times S^1$. In fact a three sphere minus two disks with a cylinder attached is an $S^2 \times S^1$. We can see S^3 minus $2d$ disks as a connected sum of d copies S^3 each with two disks removed. Attaching the cylinders we have identification of M with a connected sum of $S^2 \times S^1$. We can then extend p to a map $\hat{p} : M \to \hat{K}$ that identifies each copy of $S^2 \times I \subset M$ with a copy of $S^2 \times I \subset \hat{K}$.

As before it is easy to see that $\hat{m} \circ \hat{p} : (M, \bigcup S^2 \times I) \to (Y_+, Y_b)$ represents the homology class $[\Delta] \in H_3(Y_+, Y_b)$. But M is a closed manifold, so it represent a class $[M] \in H_3(Y_+)$ such that $[M] \mapsto [\Delta]$ via the inclusion $H_3(Y_+) \to H_3(Y_+, Y_b)$. Hence we have proved Theorem 4.0.2 for the case of a smooth projective variety of complex dimension 3. The general case follows from the Weak Lefschetz Theorem.

4.3 Rationality of the Chern Cheeger Simons invariant

In this section we prove our main theorem, namely:

Theorem 4.3.1 *Let X be a smooth projective algebraic variety and $\rho : \pi_1(X) \to G$ be a representation in a Lie group with finitely many components. Then $cs_3(\rho) \in H^3(X; \mathbb{R}/\mathbb{Z})$ is a rational class.*

First we show that it is enough to check that $cs_3(\rho)$ takes rational values on a set of generators of $H_3(X; \mathbb{Z})$. We have

$$H^3(X; \mathbb{R}/\mathbb{Z}) \cong \mathrm{Hom}(H_3(X; \mathbb{Z}), \mathbb{R}/\mathbb{Z}).$$

In fact by the universal coefficient theorem we have an exact sequence

$$0 \to \mathrm{Ext}(H_2(X; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \to H^3(X; \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(H_3(X; \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \to 0$$

We have $\mathrm{Ext}(H_2(X;\mathbb{Z}),\mathbb{R}/\mathbb{Z}) = \mathrm{Ext}(T_2,\mathbb{R}/\mathbb{Z})$ where T_2 is the torsion submodule. As T_2 is finitely generated it is enough to verify that $\mathrm{Ext}(\mathbb{Z}_q,\mathbb{R}/\mathbb{Z}) = 0$ for all q. Let $\mathbb{Z} \xrightarrow{q} \mathbb{Z} \to \mathbb{Z}_q$ be the natural free resolution. By the universal property of Ext we have an exact sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}_q,\mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z},\mathbb{R}/\mathbb{Z}) \xrightarrow{q_*} \mathrm{Hom}(\mathbb{Z}_q,\mathbb{R}/\mathbb{Z}) \to \mathrm{Ext}(\mathbb{Z}_q,\mathbb{R}/\mathbb{Z}) \to 0$$

Thus we must show that $q_* : \mathrm{Hom}(\mathbb{Z},\mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z},\mathbb{R}/\mathbb{Z})$ induced by multiplication by q in $\mathbb{Z}$ is surjective. Every $f \in \mathrm{Hom}(\mathbb{Z},\mathbb{R}/\mathbb{Z})$ is determined by $f(1)$, so given f define g by $g(1) = f(1)/q$. Then $q(g) = f$.

The torsion of $\mathbb{R}/\mathbb{Z}$ is $\mathbb{Q}/\mathbb{Z}$, so every cohomology class in $H^3(X;\mathbb{R}/\mathbb{Z})$ is always rational on T_3, the torsion part of $H_3(X;\mathbb{Z})$. Hence to check that $cs_3(\rho)$ is rational we must check that it assumes rational values on the free part F_3.

If $r_i : R_i \to X$ are singular manifolds generating $H_3(X;\mathbb{Q})$, then there are integers z_i such that the classes

$$\xi_i := \frac{1}{z_i} r_{i*}[R_i]$$

form a generating set of $H_3(X;\mathbb{Z})/T_3$. Then

$$\langle cs_3(\rho), \xi_i \rangle = \frac{1}{z_i} \langle cs_3(\rho), [r_i] \rangle.$$

By naturality

$$\langle cs_3(\rho), r_{i*}[R_i] \rangle = \langle r_i^* cs_3(\rho), [R_i] \rangle = \langle cs_3(r_i^*(\rho)), [R_i] \rangle$$

where $r_i^*(\rho) : \pi_1(R_i) \to G$ is the pullback of the representation ρ via the function r_i.

We know from Theorem 4.0.2 that we can find a set of generators

$$r_i : R_i \to X$$

for the three dimensional homology of a smooth projective algebraic variety, whose non simply connected R_i are either $S^2 \times S^1$ (abelian fundamental group), or connected sums of $S^2 \times S^1$. Hence the proof of 4.3.1 is completed by these two lemmas:

Lemma 4.3.2 *If M is a three dimensional manifold with abelian fundamental group, and $\rho : \pi_1(M) \to G$ is a representation into a group with finitely many components, then $cs_3(\rho) = 0$ in $H^3(M; \mathbb{R}/\mathbb{Q})$.*

Proof: This is true for a manifold of any dimension $n > 1$ and for higher dimensional Chern Simons invariants $cs_k(\rho) \in H^k(M; \mathbb{R}/\mathbb{Z})$ $(k > 1)$. Assume first that G is connected. If $\pi_1(M)$ is abelian, any representation must take values in a maximal torus of G, so it can be path connected to the trivial representation. If G has a finite number of connected components and $g \in G$ is not in the component of the identity, then g and g^2 are not in the same component of G. As G has only finitely many components, it follows that for all $g \in G$ there is a natural number n such that g^n is in the component of the identity. Hence there is a finite cover $\tilde{M}$ of M such that the induced representation of $pi_1(\tilde{M})$ takes values in the connected component of the identity of G. Then the Lemma follows from the multiplicativity of the Chern-Cheeger-Simons classes.

Lemma 4.3.3 *Chern-Cheeger-Simons invariants of representations of the fundamental group of a connected sum of $S^2 \times S^1$ into a group G with finitely many components are always rational.*

Proof: The fundamental group of such manifolds is free, hence any map from a set of free generators of the group to G can be extended to an homomorphism, i.e. to a representation. Hence we can path connect any representation to one that sends every generator to an element of finite order in G. So if we lift such a representation to a finite cover of X, we get the trivial one. By multiplicativity of the Chern-Cheeger-Simons we have the Lemma.

By these two lemmas, given any representation $\rho : \pi_1(X) \to G$, $cs_3(\rho)$ is rational on the free part of $H_3(X; \mathbb{Z})$. This completes the proof of 4.3.1.

4.4 Epilogue

It is natural to ask whether the method used here to prove the rationality of the third Chern-Cheeger-Simons class could be generalized to give a proof of the rationality of all Chern-Cheeger-Simons invariants of representations of the fundamental group of smooth projective algebraic varieties. The inductive nature of Lefschetz Theorems suggests that it should be possible to give

a procedure to construct inductively representatives for all odd dimensional rational homology classes of smooth projective algebraic varieties. In order to be more precise we review some notation. Let X be a smooth projective algebraic variety of complex dimension n, X_b a nonsingular hyperplane section belonging to a Lefschetz pencil of hyperplane sections. Let X' be the intersection of X with the axis of the pencil of hyperplanes. Finally let $u \in H^2(X;\mathbb{Z})$ be the Poincaré dual of the homology class determined by the image of $[X_b]$ in $H_{2n-2}(X;\mathbb{Z})$ as in Theorem 3.2.5.

Definition 4.4.1 *Let α be an element of $H_{n+q}(X;\mathbb{Q})$, where q is an integer between 0 and n. The class α is* primitive *if*

$$u^{q+1} \cap \alpha = 0.$$

Then the Hard Lefschetz Theorem is equivalent to the following primitive decomposition theorem (see [35]).

Theorem 4.4.1 *Every class $\alpha \in H_{n+q}(X;\mathbb{Q})$ can be written in a unique way as*

$$\alpha = \alpha_0 + u \cap \alpha_1 + u^2 \cap \alpha_2 + \dots$$

Every class $\alpha \in H_{n-q}(X;\mathbb{Q})$ can be written as

$$\alpha = u^q \cap \alpha_0 + u^{q+1} \cap \alpha_1 + u^{q+2} \cap \alpha_2 + \dots$$

where the $\alpha_i \in H_{n+q+2i}(X;\mathbb{Q})$ are primitive classes.

For any k cap product with u factors as

$$H_{k+2}(X) \xrightarrow{i_!} H_k(X_b) \to H_k(X).$$

It follows that if a class $\alpha \in H_{n+q}(X;\mathbb{Q})$ can be written as $u \cap \alpha_0$ with $\alpha_0 \in H_{n+q+2}(X;\mathbb{Q})$, then it is in the image of the map

$$H_{n+q}(X_b;\mathbb{Q}) \to H_{n+q}(X;\mathbb{Q})$$

induced by the inclusion of X_b in X. To construct inductively representatives of a generating set for the odd dimensional cohomology of X it is enough to construct representatives of the primitive classes. To do this one would proceed in the following way:

- Assume inductively to have constructed manifolds M_i fibered over the sphere and maps $f_i : M_i \to X$ such that the classes $f_{i*}[M_i]$ span the subspace of $H_{n+q}(X;\mathbb{Q})$ consisting of those classes which can be written as $u \cap \beta$ with $\beta \in H_{n+q+2}(X;\mathbb{Q})$. (For this inductive hypothesis to hold one should also generalize to higher dimension the treatment of extra classes given in Section 4.2).

- Construct bundles E_i over S^2 with fibre M_i and maps

$$g_i : E_i \to X$$

  such that $u \cap g_{i*}[E_i] = f_{i*}[M_i]$. Then using Hard Lefschetz and Primitive Decomposition show that the classes $g_{i*}[E_i]$ span the subspace of primitive elements in $H_{n+q+2}(X;\mathbb{Q})$.

In the case of the three dimensional homology in an algebraic surface, we were able to construct circle bundles over S^2 and maps from these bundles into the algebraic surface using the following implication: If the homological intersection of the class determined by a circle γ embedded in section X_b with a vanishing cycle is zero, then we can assume that γ does not intersect a neighbourhood of the vanishing cycle.

However in higher dimension this implication is no longer immediate. For example, let X a smooth projective threefold, X_b a nonsingular hyperplane section. Let $\alpha \in H_3(X_b)$ be the homology class determined by one of the (trivial) circle bundles over S^2 given by the proof of Theorem 4.0.1, and $\beta \in H_2(X_b)$ be the image of any of the generators of $H_3(X, X_b)$.

Lemma 4.4.2 *Let a and b be the Poincaré duals (in X_b) of α and β respectively, we have*

$$a \cup b = 0.$$

Proof: The class β is represented by an immersed two sphere S (cfr. Section 4.2 and [31]). Hence, if i is the immersion of S in X_b, cap product with b factors as

$$H_3(X_b) \xrightarrow{i_*} H_1(S) \to H_1(X_b).$$

So $b\cap : H_3(X_b) \to H_1(X_b)$ is the zero map, and the lemma follows.

However, already in this case we have not been able to prove that we can choose our circle bundles over S^2 so that they do not intersect a neighbourhood of the vanishing cycle.

The heart of the problem lies in the fact that while there are strong results on the homology of projective algebraic varieties, the results about homotopy are much weaker. For example, Wallace proved in [44] that up to homology vanishing cycles in any smooth projective X of complex dimension n are determined by spheres in X_b obtained by glueing two thimbles in X_b associated to a Lefschetz pencil of hyperplane sections of X_b. That is embeddings

$$(D^{n-1}, S^{n-2}) \to (X_b, X')$$

representing generators of $H_{n-1}(X_b, X')$. If this result could be improved to assert that these spheres are the vanishing cycles up to isotopy, then we could carry on our inductive argument, as it is not too difficult to give an inductive argument to show that the geometric intersection of these spheres and the bundles we construct is empty.

Another way of proceeding would be the following. Let Y be the modification of X with respect to a Lefschetz pencil, c a critical point of the projection $f : Y \to S^2$. Let $D \subset S^2$ a small disk centered in c and not containing any other critical point of f. The manifold $f^{-1}(\partial D)$ is a fibre bundle over the circle. Let $b \in \partial D$ be be a basepoint, and denote by $h : Y_b \to Y_b$ the monodromy of the bundle. The Picard Lefschetz formulae tell us that the action of h on the homology is trivial in all dimensions different from $n-1$ and in dimension $n-1$ is trivial on classes having zero homological intersection with the vanishing cycle. Thus if $g : M \to Y_b$ is a map from a manifold of dimension different from $n-1$, or of a $n-1$ dimensional manifold such that $g_*[M]$ has intersection zero with the vanishing cycle, the cycles determined by $g(M)$ and $hg(M)$ are homologous. If this could be improved to say g and hg are homotopic as maps from M to Y_b, then it would be immediate to construct a map

$$G : S^1 \times M \to f^{-1}(\partial D).$$

The map G could then be extended to a map $F : D \times M \to f^{-1}(D)$ using the following fact (see [42] exposé 13 and [20] appendix C). There exist a retraction $r : Y_b \to f^{-1}(c)$ such that $rh = r$. Using r it is possible to define a map

$$R : f^{-1}(\partial D) \to f^{-1}(c)$$

such that if we denote by M_R the mapping cylinder

$$f^{-1}(\partial D) \times [0,1]/(f^{-1}(\partial D) \times \{0\} \xrightarrow{R} f^{-1}(c))$$

and by M_c the mapping cylinder $S^1 \times [0,1]/(S^1 \times \{0\} \to c)$, then $f^{-1}(\partial D)$ can be identified with M_R, and there is a commutative diagram

$$\begin{array}{ccc} M_R & \xrightarrow{\cong} & f^{-1}(\partial D) \\ \downarrow & & \downarrow \\ M_c & \xrightarrow{\cong} & D \end{array}$$

Hence it would be possible to extend G to a map from a disjoint union of products $D \times M$ into neighbourhoods of the singular fibres of f. It would then be straightforward to extend this map to a map from a bundle over S^2 with fibre M to Y proceeding as in Section 4.1.

Womb ? Weary?

He rests. He has travelled.

With?

Sinbad the Sailor and Tinbad the Tailor and Jinbad the Jailer and Whinbad the Whaler and Ninbad the Nailer and Finbad the Failer and Binbad the Bailer and Pinbad the Pailer and Minbad the Mailer and Hinbad the Hailer and Rinbad the Railer and Dinbad The Kailer and Vinbad the Quailer and Linbad the Yailer and Xinbad the Phthailer.

James Joyce

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