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Invariant tensors for the spin representation of $\mathfrak{so}(7)$

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Abstract

We give a graphical calculus for the invariant tensors of the eight dimensional spin representation of the quantum group $U_q(B_3)$. This leads to a finite confluent presentation of the centraliser algebras of the tensor powers of this representation and a construction of a cellular basis.

1. *Introduction*

In his classic book [**Wey39**] Weyl raised the problem of studying the invariant tensors of an irreducible representation of a complex simple Lie group and the related problem of studying the centraliser algebra of the tensor powers of the representation. The book then treats the examples where the representation is a vector representation of a classical group.

There is a comparatively short list of further examples which have been treated. This problem was treated for any irreducible representation of SL (2) in the 19th century. More recently the fundamental representations of rank two Lie algebras are treated in [**Kup96a**] and the particular case of G_2 is developed further in [**Wes06**]. Also the adjoint representation of SL (n) is treated in [**BD02**]. This paper follows the approach of [**Kup96a**] and is a response to [**Kup94**, question 3·6].

After Weyl's book was published Chevalley gave an integral form of the enveloping algebra of a complex simple Lie algebra. More recently a deformation of these enveloping algebras was found by Drinfeld and Jimbo and Lusztig has given a $\mathbb{Z}[q, q^{-1}]$ form of these Hopf algebras. This raises the problems of studying the invariant tensors of an irreducible representation of the Drinfeld–Jimbo quantised enveloping algebra and the further problem of finding $\mathbb{Z}[q, q^{-1}]$ forms of these vector spaces. A general method for studying these problems is given in [**Lus93**] using the canonical bases. A general method for understanding the combinatorics of these problems is given by the theory of crystal graphs.

In this paper we treat the eight dimensional spin representation of $\mathfrak{so}(7)$. This is the first spin representation which is not included in the above examples. The reason for this is that for $3 \leqslant n \leqslant 6$ there is a classical group which is the double cover of the group SO (n). These groups are:

3	4	5	6
SU (2)	SU (2) × SU (2)	Sp (4)	SU (4)

Furthermore the spin representations are vector representations of these classical groups.

We use diagrams to represent tensors. In the physics literature this approach was pioneered by Penrose and Cvitanovic. In the mathematical literature this is an accepted approach to the Brauer algebras and the Temperley–Lieb algebras. The diagrams in this paper can easily be converted to the index notation for tensors but this would make the paper harder to read. The reason for this is that given a diagram there are many ways of writing it in index notation. These tensors are all equal but this may not be apparent in the index notation. The basic tensor operations are tensor product, contraction and raising and lowering indices. In terms of rectangular diagrams the tensor product corresponds to putting rectangles side-by-side, contraction corresponds to connecting points, raising an index corresponds to moving it from the bottom edge of a rectangle to the top edge and conversely for lowering an index.

We use a single line to denote the spin representation Δ and a double line to denote the vector representation V. The fundamental tensor is the tensor shown in (1). If we read this as an invariant map $V \otimes \Delta \to \Delta$ and take $q \to 1$ then this is the action of vectors on spinors which generates the action of the Clifford algebra. The results of this paper show that every invariant tensor can be built from this tensor using the basic tensor operations and taking linear combinations. We then give some linear relations among invariant tensors and show that these are a complete set of relations. We also give a finite presentation for each centraliser algebra and show that these presentations are confluent. Finally we construct a cellular basis for each centraliser algebra.

2. *Diagrams*

The co-efficients or scalars in this paper are polynomials in an indeterminate δ with integer co-efficients. There are two sequences of Chebychev polynomials that we use. One is the sequence $[n]$ which is defined by the initial conditions $[0] = 0$, $[1] = 1$ and the recurrence relation

$$[n+1] - \delta[n] + [n-1] = 0.$$

The other is the sequence $([2n]/[n])$ which is defined by the initial conditions $([0]/[0]) = 2$, $([2]/[1]) = \delta$, $([4]/[2]) = \delta^2 - 2$ and the same recurrence relation. The polynomials that appear in this paper are given in the following table.

$[1]$	$[2]$	$[3]$	$[4]$
1	δ	$\delta^2 - 1$	$\delta(\delta^2 - 2)$

$\frac{[4]}{[2]}$	$\frac{[6]}{[3]}$	$\frac{[8]}{[4]}$	$\frac{[10]}{[5]}$
$\delta^2 - 2$	$\delta(\delta^2 - 3)$	$\delta^4 - 4\delta^2 + 2$	$\delta(\delta^4 - 5\delta^2 + 5)$

There are three specialisations of the ring $\mathbb{Z}[\delta]$ which are of particular interest. One is the specialisation $\mathbb{Z}[\delta] \to \mathbb{Z}$ determined by $\delta \mapsto 2$. Under this specialisation we have

$$[n] \longmapsto n \qquad \frac{[2n]}{[n]} \longmapsto 2.$$

Another specialisation is the homomorphism $\mathbb{Z}[\delta] \to \mathbb{Z}[q, q^{-1}]$ determined by $\delta \mapsto q + q^{-1}$. Under this specialisation we have

$$[n] \longmapsto \frac{q^n - q^{-n}}{q - q^{-1}} \qquad \frac{[2n]}{[n]} \longmapsto q^n + q^{-n}.$$

The third specialisation is $\mathbb{Z}[\delta] \to \mathbb{R}$ determined by $\delta \mapsto 2\cos(\theta)$. Under this specialisation

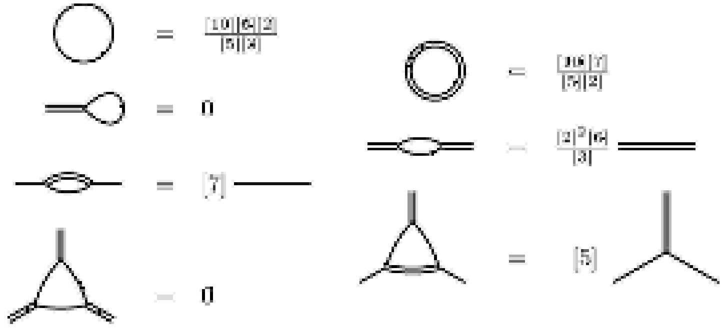


Fig. 1. Basic relations.

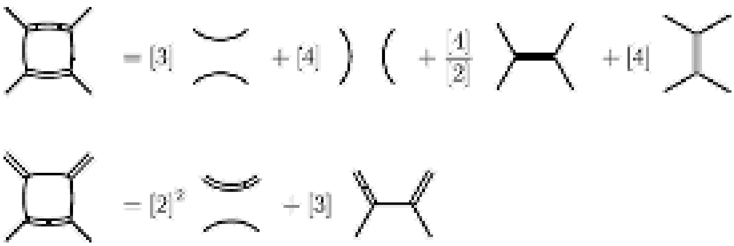
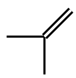


Fig. 2. Square relations.

we have

$$[n] \longmapsto \frac{\sin(n\theta)}{\sin(\theta)} \qquad \frac{[2n]}{[n]} \longmapsto 2\cos(n\theta).$$

First we describe the planar graphs that we will be using. There are two types of edge which we draw as a single line and a double line. There is just one type of vertex, namely the trivalent vertex with two single lines and one double line.



(1)

Define a diagram to be a trivalent graph embedded in a rectangle with boundary points on the top and bottom edges. Each edge is labelled as being either single or double such that there are two single edges and one double edge at each trivalent vertex. Let $\widetilde{\mathcal{D}}$ be the category whose morphisms are isotopy equivalence classes of such diagrams. The composition of morphisms is given by putting one rectangle on top of the other. The tensor product of morphisms is given by putting the rectangles together side by side and the dual diagram is obtained by rotating through half a revolution.

Then we take the free $\mathbb{Z}[\delta]$-linear category on $\widetilde{\mathcal{D}}$ and impose relations. This defines the category $\mathcal{D}'$. The defining relations consist of the basic relations given in Figure 1, the two square relations given in Figure 2, the pentagon relation given in Figure 3 and the hexagon relation given in Figure 4.

The objects of $\widetilde{\mathcal{D}}$ and $\mathcal{D}'$ are finite sequences of single and double edges. The category $\mathcal{D}$ is the full subcategory of $\mathcal{D}'$ whose objects are finite sequences of single edges.

This presentation is not terminal. The reason is that the hexagon relation is not a rewrite rule. If we have a hexagon in a diagram then we can obtain a second diagram by interchanging the single and double edges in the diagram. In order to make the hexagon relation into

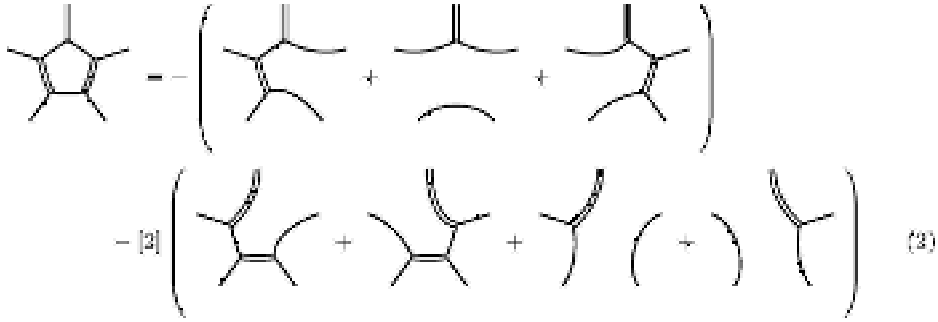


Fig. 3. Pentagon relation.

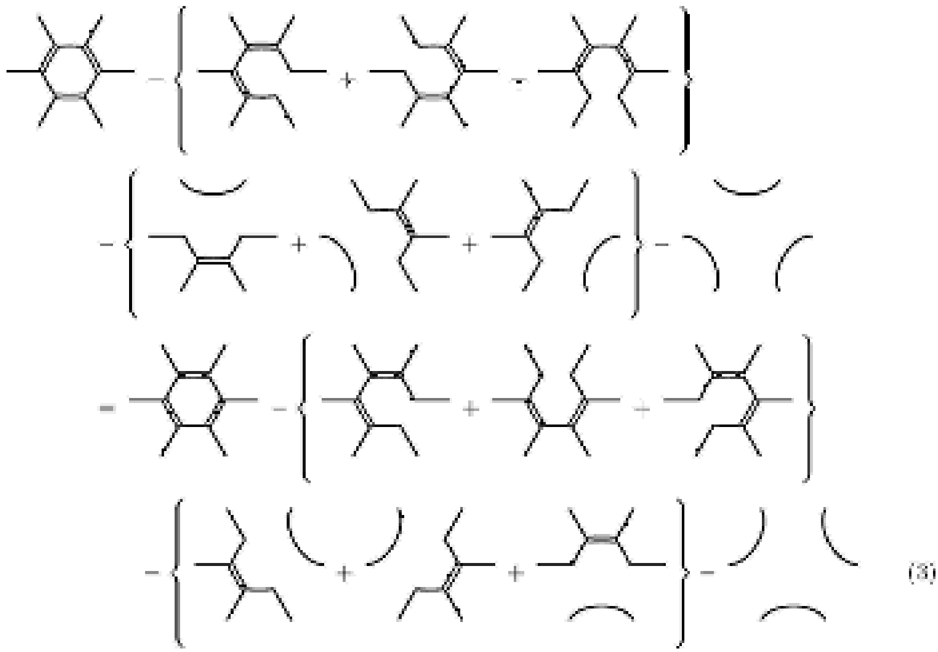


Fig. 4. Hexagon relation.

a rewrite rule we need to know which of these two diagrams is simpler. In order to remedy this we introduce a new type of vertex with six single edges and then we replace the hexagon relation in Figure 4 by the reduction rule in Figure 5.

Then these reduction rules are not locally confluent and so lead to further reduction rules. It would be interesting to know if this leads to a finite confluent presentation. For a discussion of confluence in this context see [**SW**].

3. *Centraliser algebras*

In this section we define a sequence of finitely presented algebras. For $n \geqslant 0$ the algebra $A_P(n)$ is generated by the set

$$\{U_i, K_i, H_i \colon 1 \leqslant i \leqslant n-1\}.$$

The motivation for the relations is that they are satisfied in $\mathcal{D}$. The diagrams for the generators are obtained from (4) by adding $i-1$ vertical lines on the left and $n-i-1$ vertical

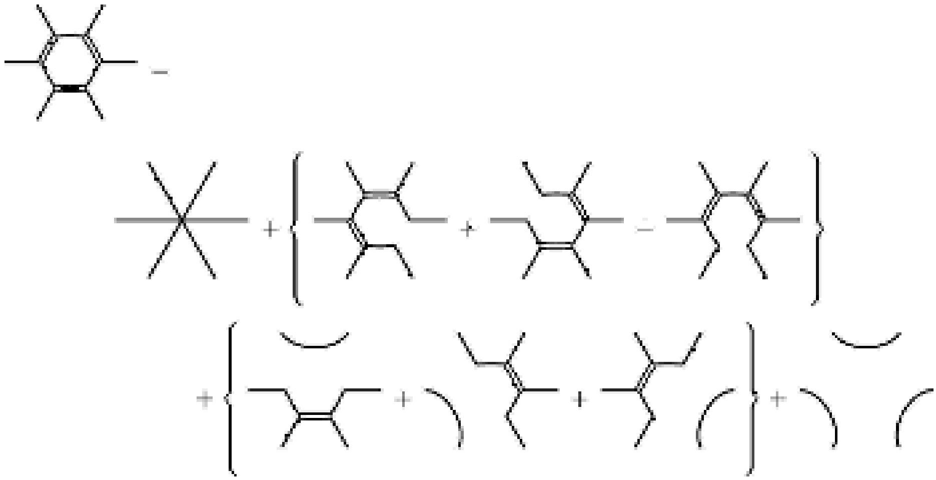
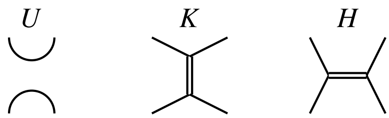


Fig. 5. Hexagonal reduction relation.

lines on the right.



(4)

First we have the commuting relations. For $1 \leqslant i, j \leqslant n-1$ with $|i-j| > 1$, if $a \in \{U_i, K_i, H_i\}$ and $b \in \{U_j, K_j, H_j\}$ then $ab = ba$.

The two string relations are:

$$\begin{aligned}
&U_iU_i = \frac{[10][6][2]}{[5][3]}U_i, U_iH_i = [7]U_i, \quad U_iK_i = 0,\\
&H_iU_i = [7]U_i, \quad H_i^2, = [3] + \frac{[4]}{[2]}H_i + [4]U_i + [4], K_iH_iK_i = -[5]K_i,\\
&K_iU_i = 0, \quad K_iH_i = -[5]K_i, \quad K_iK_i = \frac{[2][2][6]}{[3]}K_i.
\end{aligned}$$

The eigenvalues of H_i are $\{-1, [3], -[5], [7]\}$. The relation for H_i^2 comes from the square relation in Figure 2.

These relations imply that $A(2)$ is a commutative algebra with basis $\{1, U_1, K_1, H_1\}$.

The three string relations are the following. The first relations come from isotopy of diagrams:

$$\begin{aligned}
U_iU_{i\pm1}U_i &= U_i & H_iU_{i\pm1}U_i &= K_{i\pm1}U_i\\
U_iH_{i\pm1}U_i &= 0 & K_iU_{i\pm1}U_i &= H_{i\pm1}U_i\\
U_iU_{i\pm1}H_i &= U_iK_{i\pm1} & K_iU_{i\pm1}K_i &= H_{i\pm1}U_iH_{i\pm1}\\
U_iU_{i\pm1}K_i &= U_iH_{i\pm1}.
\end{aligned}$$

The next set of relations are relations in the ideal generated by U_i which come from the basic relations in Figure 1.

$$
\begin{aligned}
U_iK_{i\pm1}U_i &= [7]U_i & U_iH_{i\pm1}K_i &= \frac{[2]^2[6]}{[3]}U_iH_{i\pm1}\\
U_iK_{i\pm1}K_i &= -[5]U_iH_{i\pm1} & K_iH_{i\pm1}U_i &= \frac{[2]^2[6]}{[3]}H_{i\pm1}U_i\\
K_iK_{i\pm1}U_i &= -[5]H_{i\pm1}U_i & U_iH_{i\pm1}H_i &= -[5]U_iH_{i\pm1}\\
U_iH_{i\pm1}U_i &= 0 & H_iH_{i\pm1}U_i &= -[5]H_{i\pm1}U_i.
\end{aligned}
$$

The next relations are relations in the ideal generated by U_i and come from the square relations in Figure 2.

$$
\begin{aligned}
U_iK_{i\pm1}H_i &= [3]U_i + [4]U_iU_{i\pm1} + \frac{[4]}{[2]}U_iK_{i\pm1} + [4]U_iH_{i\pm1}\\
H_iK_{i\pm1}U_i &= [3]U_i + [4]U_{i\pm1}U_i + \frac{[4]}{[2]}K_{i\pm1}U_i + [4]H_{i\pm1}U_i.
\end{aligned}
$$

The next relation comes from one of the basic relations and the other three come from the square relation in Figure 2.

$$
\begin{aligned}
K_iH_{i\pm1}K_i &= 0\\
K_iK_{i\pm1}K_i &= [2]^2K_i + [3]K_iU_{i\pm1}K_i\\
H_iH_{i\pm1}K_i &= [2]^2U_{i\pm1}K_i + [3]H_{i\pm1}K_i\\
K_iH_{i\pm1}H_i &= [2]^2K_iU_{i\pm1} + [3]K_iH_{i\pm1}.
\end{aligned}
$$

The next relations come from the pentagon relation in Figure 3.

$$
\begin{aligned}
K_iK_{i\pm1}H_i &= -\left(K_iK_{i\pm1} + K_iU_{i\pm1} + K_iU_{i\pm1}H_i\right)\\
&\quad - [2]\left(K_i + K_iH_{i\pm1} + K_iU_{i\pm1}K_i + K_iU_{i\pm1}U_i\right)\\
H_iK_{i\pm1}K_i &= -\left(K_{i\pm1}K_i + U_{i\pm1}K_i + H_iU_{i\pm1}K_i\right)\\
&\quad - [2]\left(K_i + H_{i\pm1}K_i + K_iU_{i\pm1}K_i + U_iU_{i\pm1}K_i\right)\\
H_iH_{i\pm1}H_i &= -\left(H_iH_{i\pm1} + H_{i\pm1} + H_{i\pm1}H_i\right)\\
&\quad - [2]\left(U_iH_{i\pm1} + H_{i\pm1}U_i + K_iH_{i\pm1} + H_{i\pm1}K_i\right).
\end{aligned}
$$

Finally we have the relation which comes from the hexagon relation in Figure 4.

$$
\begin{aligned}
&H_{i+1}\left(1 - [4]U_i - \frac{[4]}{[2]}H_i - [4]K_i\right)H_{i+1}\\
&\quad = H_i\left(1 - [4]U_{i+1} - \frac{[4]}{[2]}H_{i+1} - [4]K_{i+1}\right)H_i.
\end{aligned}
$$

This can be rewritten using previous relations as

$$
\begin{aligned}
&H_{i+1}K_iH_{i+1} + H_{i+1}U_iH_{i+1} - U_{i+1}H_i - H_iU_{i+1} - K_{i+1}H_i - H_iK_{i+1} - U_{i+1} - K_{i+1}\\
&\quad = H_iK_{i+1}H_i + H_iU_{i+1}H_i - U_iH_{i+1} - H_{i+1}U_i - K_iH_{i+1} - H_{i+1}K_i - U_i - K_i.
\end{aligned}
$$

The braid matrices are given by

$$
\begin{aligned}
(q + q^{-1})\sigma_i &= -q^{-10} - q^{-5}U_i + q^{-7}K_i + q^{-8}H_i\\
(q + q^{-1})\sigma_i^{-1} &= -q^{10} - q^5U_i + q^7K_i + q^8H_i.
\end{aligned} \tag{5}
$$

3·1. *Yang–Baxter*

These relations can be described very succinctly using the Yang–Baxter equation (with spectral parameter). There are two solutions of the Yang–Baxter equation.

The first solution is discussed in [**GLWX90**], [**Oka90**] and [**HM91**] and is associated with the affine quantum group $U_q(B_3^{(1)})$. This solution can also be found using the tensor product graph method introduced in [**Mac91**] and [**ZGB91**]. The tensor product graph is

$$[0,0,0] \xrightarrow{-10} [0,1,0] \xrightarrow{-2} [0,0,2] \xrightarrow{6} [1,0,0].$$

The eigenvalues of $-(q-q^{-1})^3 R_i(u)$ are

$$\begin{aligned}
&[0,0,0]\ (uq^5 - u^{-1}q^{-5})(uq - u^{-1}q^{-1})(uq^{-3} - u^{-1}q^3)\\
&[0,1,0]\ (u^{-1}q^5 - uq^{-5})(uq - u^{-1}q^{-1})(uq^{-3} - u^{-1}q^3)\\
&[0,0,2]\ (u^{-1}q^5 - uq^{-5})(u^{-1}q - uq^{-1})(uq^{-3} - u^{-1}q^3)\\
&[1,0,0]\ (u^{-1}q^5 - uq^{-5})(u^{-1}q - uq^{-1})(u^{-1}q^{-3} - uq^3).
\end{aligned}$$

Then this satisfies

$$\begin{aligned}
R_i(1) &= [5][3] & R_i(q^3) &= -[3]K_i\\
R_i(q^5) &= -[5][3]U_i & R_i(q^2) &= [3]H_i.
\end{aligned}$$

Hence $R_i(u)$ can be written as

$$\begin{aligned}
(q-q^{-1})^2(q^2-q^{-2})R_i(u) = &-(uq^{-5}-u^{-1}q^5)(uq^{-3}-u^{-1}q^3)(uq^{-2}-u^{-1}q^2)\\
&-(u-u^{-1})(uq^{-3}-u^{-1}q^3)(uq^{-2}-u^{-1}q^2)U_i\\
&+(u-u^{-1})(uq^{-5}-u^{-1}q^5)(uq^{-2}-u^{-1}q^2)K_i\\
&+n(u-u^{-1})(uq^{-5}-u^{-1}q^5)(uq^{-3}-u^{-1}q^3)H_i.
\end{aligned}$$

Then this satisfies the following relations:

$$\begin{aligned}
R_i(u)R_j(v) &= R_j(v)R_i(u) \text{ if } |i-j|>1\\
R_i(u)R_{i+1}(uv)R_i(v) &= R_{i+1}(v)R_i(uv)R_{i+1}(u)\\
U_iU_{i\pm1}R_i(u) &= -U_iR_{i\pm1}(q^5u^{-1})\\
R_i(u)U_{i\pm1}U_i &= -R_{i\pm1}(q^5u^{-1})U_i\\
R_i(u)U_{i+1}R_i(v) &= -R_{i+1}(q^5u^{-1})U_iR_{i+1}(q^5v^{-1})\\
U_iR_{i\pm1}(u)U_i &= -\frac{(uq^{-10}-u^{-1}q^{10})(uq^{-6}-u^{-1}q^6)(uq^{-2}-u^{-1}q^2)}{(q-q^{-1})^3}U_i.
\end{aligned}$$

There are various algebra relations which can be expressed succintly using this R-matrix

$$\begin{aligned}
U_iR_{i\pm1}(q^{10})U_i &= 0 & U_iR_{i\pm1}(q^8)K_i &= 0\\
K_iR_{i\pm1}(q^8)U_i &= 0 & K_iR_{i\pm1}(q^6)K_i &= 0.
\end{aligned}$$

and the hexagon relation can be written as a particular case of the Yang–Baxter equation

$$H_iR_{i\pm1}(q^4)H_i = H_{i\pm1}R_i(q^4)H_{i\pm1}.$$

There is a second solution of the Yang–Baxter equation. This solution is associated to the affine quantum group $U_q(D_4^{(2)})$. This solution to the Yang–Baxter equation was first found in [**DGZ96**, section 4·3].

The solution is described by the tensor product graph

$$[0,0,0] \xrightarrow{-6} [1,0,0] \xrightarrow{-4} [0,1,0] \xrightarrow{-2} [0,0,2].$$

The eigenvalues of $(q-q^{-1})^2 S_i(u)$ are

$$\begin{aligned}
&[0,0,0]\ (uq^3-u^{-1}q^{-3})(uq^2+u^{-1}q^{-2})(uq-u^{-1}q^{-1})\\
&[1,0,0]\ (u^{-1}q^3-uq^{-3})(uq^2+u^{-1}q^{-2})(uq-u^{-1}q^{-1})\\
&[0,1,0]\ (u^{-1}q^3-uq^{-3})(u^{-1}q^2+uq^{-2})(uq-u^{-1}q^{-1})\\
&[0,0,2]\ (u^{-1}q^3-uq^{-3})(u^{-1}q^2+uq^{-2})(u^{-1}q-uq^{-1}).
\end{aligned}$$

In particular we have

$$S_i(1) = [3]\frac{[4]}{[2]} \qquad S_i(q^3) = \frac{[4]^2}{[2]}U_i.$$

These satisfy relations

$$\begin{aligned}
S_i(u)S_j(v) &= S_j(v)S_i(u) \text{ if } |i-j|>1\\
S_i(u)S_{i+1}(uv)S_i(v) &= S_{i+1}(v)S_i(uv)S_{i+1}(u)\\
U_iS_{i\pm1}(u)U_i &= \frac{(uq^{-4}-u^{-1}q^4)(uq^{-6}-u^{-1}q^6)}{(q-q^{-1})^2}(uq^{-5}+u^{-1}q^5)U_i.
\end{aligned} \tag{6}$$

This solution to the Yang–Baxter equation gives a sequence of rank one central idempotents. These idempotents are defined inductively by $E_0 = 1$ and for $n > 0$,

$$E_{n+1} = \frac{[n+2]}{[n+1][2n+4][n+3]}E_nS_n(q^{-n})E_n. \tag{7}$$

Let $\Delta^{(n)}$ be the irreducible highest weight representation of $U_q(B_3)$ with highest weight $[0,0,n]$ so that $\Delta^{(0)}$ is the trivial representation and $\Delta^{(1)}$ is the spin representation. Then the quantum dimension of $\Delta^{(n)}$ can be found by applying the principal gradation to the Weyl character formula, see [**Kac85**, proposition 10·10]. The result is

$$\dim_q(\Delta^{(n)}) = \frac{[2n+8][2n+6][2n+4][n+5][n+3][n+1]}{[8][6][5][4][3]}. \tag{8}$$

Taking the specialisation $q \to 1$ shows the sequence whose nth term is $\dim(\Delta^{(n)})$ is sequence A040977 in [**Slo06**].

An alternative way of finding this formula is to note that $\dim_q(\Delta^{(n)}) = \tau(E_n)$ where τ is the trace defined in Definition 3·1. Applying (6) to the definition (7) gives

$$\dim_q(\Delta^{(n+1)}) = \frac{[n+2][n+4][n+6][2n+10]}{[n+1][2n+4][n+3][n+5]}\dim_q(\Delta^{(n)})$$

in agreement with (8).

The sequence of algebras $A_P(n)$ can also be defined by the commuting relations, the two string relations and both Yang–Baxter equations.

3·2. *Representations*

Next we give irreducible representations of dimensions 1,2,3 and 4 of $A(3)$. In each case we only give the matrices representing U_1, K_1, H_1 and σ_1. The reason is that, for each of these representations of dimension n, the matrices representing U_2, K_2, H_2 and σ_2 are obtained by applying the involution

$$A_{ij} \longleftrightarrow A_{n-j+1,n-i+1}.$$

Also the matrices for σ_i^{-1} are given by applying the involution $q \leftrightarrow q^{-1}$ to the entries of the matrices for σ_1.

The reason we have included the matrices for σ_1 even though they are determined by (5) is that we found these matrices first using the results in [**TW01**] and then we calculated the matrices representing the generators from these.

The four dimensional representation is given by

$$U_1 = \begin{bmatrix} \frac{[10][6][2]}{[5][3]} & -\frac{[10][6]}{[5][3]} & -\frac{[6][4]}{[3][2]} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 0 & [2]\frac{[6]}{[3]} & -\frac{[6]}{[3]} & -1 \\ 0 & [2]^2\frac{[6]}{[3]} & -[2]\frac{[6]}{[3]} & -[2] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} [7] & -[6] & 0 & 0 \\ 0 & -[5] & [4] & 0 \\ 0 & 0 & [3] & -[2] \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$\sigma_1 = \begin{bmatrix} -q^6 & q - q^3 + q^5 & q^{-2} - 1 + q^2 & -q^{-3} \\ 0 & 1 & q^{-3} - q^{-1} & -q^{-4} \\ 0 & 0 & q^{-4} & -q^{-5} \\ 0 & 0 & 0 & -q^{-6} \end{bmatrix}.$$

For $1 \leqslant k \leqslant 3$ there is a representation of dimension $4 - k$. The matrices representing U_1, K_1, H_1 and $\sigma_1^{\pm 1}$ are obtained by deleting the first k rows and columns from the matrix representing the same element in the four dimensional representation.

This means that there is a spanning set for $A_P(3)$ which consists of the words of length at most two the hexagon $H_1K_2H_1$ and the following four words in the ideal generated by U_2:

$$\begin{matrix} H_1U_2H_1 & H_1U_2K_1 \\ K_1U_2H_1 & K_1U_2K_1. \end{matrix}$$

This gives a total of thirty words in the spanning set. The sum of the squares of the dimensions of the above representations is thirty and these thirty words are linearly independent in this thirty dimensional algebra. This shows that these thirty words are a basis for $A(3)$.

It follows from these relations that the criterion of [**Wes97**] is satisfied. This means that

$$A_P(n+1) = A_P(n) + A_P(n)U_nA_P(n) + A_P(n)K_nA_P(n) + A_P(n)H_nA_P(n). \qquad (9)$$

The first consequence of this is that there is a conditional expectation $\varepsilon_n\colon A_P(n) \to A_P(n-1)$. This satisfies $U_naU_n = \varepsilon(a)U_n$ for all $a \in A_P(n)$.

Definition 3·1. For each n, the conditional expectation $\varepsilon_{n+1}\colon A_P(n+1) \to A_P(n)$ is uniquely determined by the properties

$$\begin{array}{ll} \varepsilon(a) = \dfrac{[10][6][2]}{[5][3]}a & \varepsilon(aK_na') = [7]aa' \\ \varepsilon(aU_na') = aa' & \varepsilon(aH_na') = 0 \end{array}$$

for all $a, a' \in A_P(n-1)$. These conditional expectations then determine a trace map τ_n on $A_P(n)$ for each $n > 0$ by

$$\tau_{n+1}(a) = \frac{[5][3]}{[10][6][2]}\tau_n(\varepsilon_{n+1}(a))$$

for all $a \in A_P(n)$.

The second consequence of (9) is that for any $n \geqslant 0$, there is a finite set of words in the generators which span $A_P(n)$. In fact we can be much more specific.

Definition 3·2. Take an array of integers $\{a(i,j) | i \geqslant 1, j \geqslant 1, i+j \leqslant n+1\}$ such that $0 \leqslant a(i,j) \leqslant 3$, $a(i,j) \leqslant a(i',j)$ if $i < i'$, $a(i,j) \leqslant a(i,j')$ if $j < j'$. Then associated to this array is the following word in non-commuting indeterminates $E_i^{(k)}$ where $1 \leqslant i \leqslant n-1$ and $0 \leqslant k \leqslant 3$

$$\prod_{j=n-1}^{1}\left(\prod_{i=1}^{n-j} E_{n-i-j+1}^{a(i,j)}\right).$$

Then substitute 1 for $E_i^{(0)}$, U_i for $E_i^{(1)}$, K_i for $E_i^{(2)}$, H_i for $E_i^{(3)}$ to get a word in the generators of $A_P(n)$. We will call these the irreducible words.

It is clear that the irreducible words span $A_P(n)$ since if we take any irreducible word and multiply by a generator then by using the relations we can write this word as a linear combination of irreducible words. In fact the irreducible words are a basis and we will give two proofs of this; one in Section 5 using representation theory and one in Section 6 using the diamond lemma.

The third consequence of (9) is:

LEMMA 3·3. *For all $n > 0$, $U_nA_P(n+1)U_n = A_P(n-1)U_n$.*

Proof. Let $a \in A_P(n+1)$. Then by (9), a can be written as a linear combination of terms of the form bX_nb' where $b, b' \in A_P(n)$ and $X_n \in \{1, U_n, K_n, H_n\}$. Then applying (9) a second time, each b can be written as a linear combination of elements of the form $cY_{n-1}c'$ where $c, c' \in A_P(n-1)$ and $Y_{n-1} \in \{1, U_{n-1}, K_{n-1}, H_{n-1}\}$. Hence U_naU_n can be written as a linear combination of terms of the form

$$cU_nY_{n-1}X_n(c'b')U_n$$

where $c \in A_P(n-1)$ and $(c'b') \in A_P(n)$. Each word of the form $U_nY_{n-1}X_n$ can be written as a linear combination of the words

$$\{U_n, U_nU_{n-1}, U_nK_{n-1}, U_nH_{n-1}\}$$

using the defining relations. This shows that U_naU_n is a element of $U_nA_P(n)U_n$ and so is an element of $A_P(n-1)U_n$.

4. *Crystal graphs*

Next we discuss the combinatorics of Littelmann paths for the spin representation. The first observation is that the weights of this representation are the orbit of the highest weight under the action of the Weyl group. This implies that all weight spaces have dimension one and that the Littelmann paths are just the straight lines from the origin to the weight. Then when we concatenate these paths we get a sequence of weights $\omega_1, \omega_2, \dots$ such that for all $i > 2$, $\omega_{i+1} - \omega_i$ is a weight of the representation. Furthermore the sequence of weights corresponds to a dominant path if and only if every weight in the sequence is dominant.

Next instead of working with the basis consisting of the fundamental weights we change basis by

$$(w_1, w_2, w_3) \longmapsto (2w_1 + 2w_2 + w_3, 2w_2 + w_3, w_3)$$
$$(s_1, s_2, s_3) \longmapsto \left(\frac{s_1 - s_2}{2}, \frac{s_2 - s_3}{2}, s_3\right).$$

In the new coordinates the weights of the representation are the eight vectors $(\pm 1, \pm 1, \pm 1)$. Also the dominant weights correspond to vectors (s_1, s_2, s_3) such that $s_1 \geqslant s_2 \geqslant s_3 \geqslant 0$. This shows that Littelmann paths correspond to triples of non-crossing Dyck paths.

The number of triples of non-crossing Dyck paths is calculated in [**dSCV86**] where it is shown to be

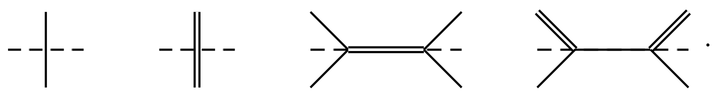
$$\prod_{1 \leqslant i \leqslant j \leqslant n} \frac{i + j + 6}{i + j}.$$

This is sequence A006149 in [**Slo06**].

In this section we use the algorithms introduced in [**Wes06**] to find a set of diagrams which gives a basis for the invariant tensors.

Definition 4·1. Assume we are given a diagram. Let A and B be two boundary points which are not marked points. Then a cut path from A to B is a path from A to B such that each component of the intersection with the embedded graph is either an isolated transverse intersection point or else is an edge of the graph.

The diagrams for these four cases and the associated weights are:



$(0, 0, 1)$ $(1, 0, 0)$ $(0, 1, 0)$ $(0, 1, 0)$

The weight of a cut path is (a, b, c) if it crosses a double edges transversally, c single edges transversally and contains b edges. A cut path is minimal if there is no cut path with the same endpoints and lower weight.

Now suppose we have a diagram with n boundary points which are all single edges. Then we draw the diagram in a triangle AXY with the edge XY horizontal and with all boundary points on the edge XY. Then we choose marked points $X_0 = X, X_1, \dots X_n = Y$ such there is precisely one boundary point between each marked point. Now for $0 \leqslant i \leqslant n$ let ω_i be the minimum weight such there is a cut path from A to X_i of weight ω_i. Then $\omega_0 = (0, 0, 0) = \omega_n$.

The diagram model for the crystal graph of the eight dimensional spin representation is given in Figure 6 and the diagram model for the crystal graph of the seven dimensional vector representation is given in Figure 7. These crystal graphs are given in [**HK02**, section 8·1].

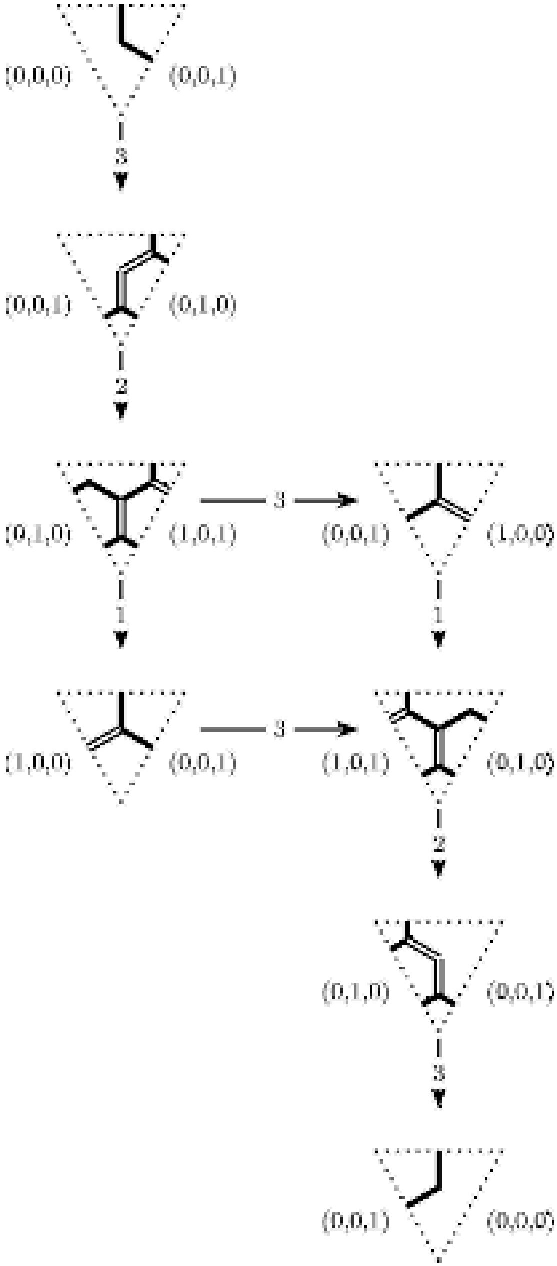


Fig. 6. Crystal graph for the spin representation.

Remark 4·2. There is a different interpretation of these labelled directed graphs. These two representations are miniscule which means that the action of the Weyl group W on the weights is transitive. The stabiliser of a point is a parabolic subgroup W_0. Each coset has a unique representative of minimal length. In these examples W has type B_3; for the vector representation W_0 has type B_2 and for the spin representation W_0 has type A_2. The Weyl group of type B_3 is generated by the reflections in the three simple roots, s_1,s_2,s_3. For each vertex, take a directed path from the highest weight (at the top) to the vertex. Then take the

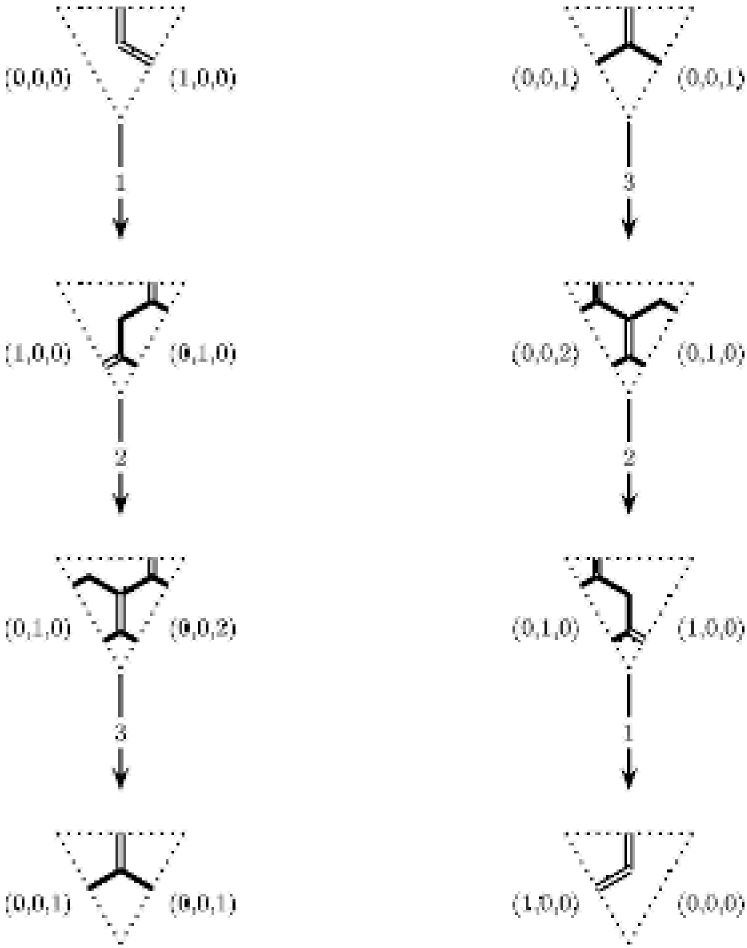
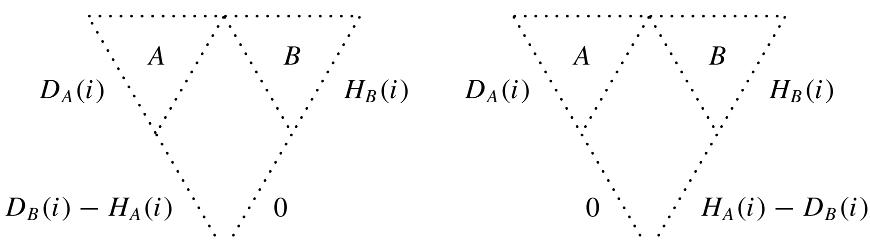


Fig. 7. Crystal graph for the vector representation.

sequence of labels and regard it as a word in the generators. These words are then reduced expressions for the minimal length coset representative.

Next we discuss the tensor product rule. This tensor product rule gives a procedure which associates a triangular diagram to any word in the weights of the spin representation and the vector representation. This tensor product rule is similar to the tensor product rule given in [**Wes06**]. First we draw a triangular grid. The triangles in top row is the sequence of triangles associated to the word. This also gives the weights of the edges of these triangles. Next we label each edge of the triangular grid by a dominant weight using the rule below. This rule is derived from the tensor product of crystals.



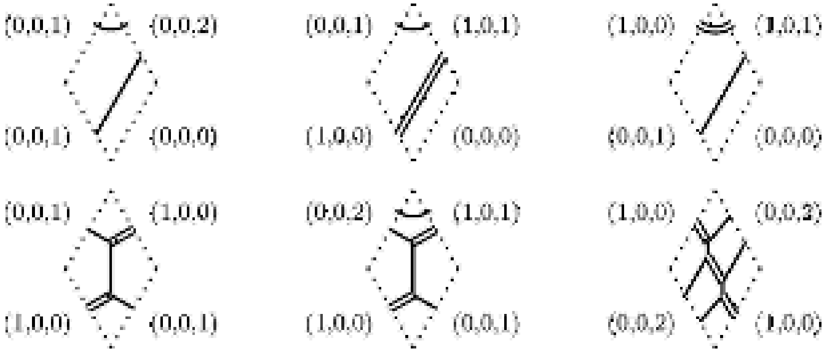


Fig. 8. Diamonds.

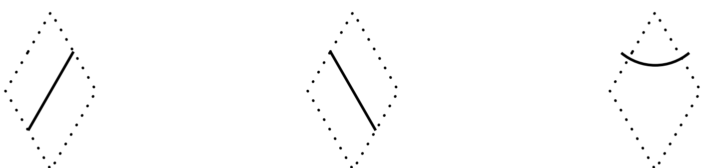
Now we complete the triangular diagram by drawing a graph in each diamond. Note that the weights H_A and D_B are both elements of the set

$$\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(1,0,1),(0,0,2)\}.$$

This gives thirty six different diamonds. Furthermore note that for each of these diamonds the other two weights are also elements of this set. This implies that for any word every diamond will have one of these thirty six labellings. Therefore to complete the diagram it is sufficient to know how to fill in a diamond with each of these labellings.

The general principle for filling in these diagrams is that each edge of the diamond is required to be a minimal cut path whose weight is given by the label and the diagram is required to be irreducible. By construction the two paths from the bottom of the diamond to the top are cut paths of the same weight.

The simplest case is where two of the edges are labelled with the zero weight. These can be filled in using the following diagrams:



In these diagrams any solid line can be removed or replaced by any sequence of parallel lines.

This leaves twenty cases. However these come in pairs since any diamond can be reflected in a vertical line. This leaves ten cases which we describe in Figures 8 and 9.

Next we associate an element of $\otimes^n \Delta$ to each of these triangular diagrams with n points on the top edge. This follows [**FK97**]. First a triangular diagram interpreted as a tensor is an intertwiner

$$V(H) \otimes V(D) \longrightarrow \otimes^n \Delta$$

so applying this map to the tensor product of highest weight vectors gives an element of $\otimes^n \Delta$.

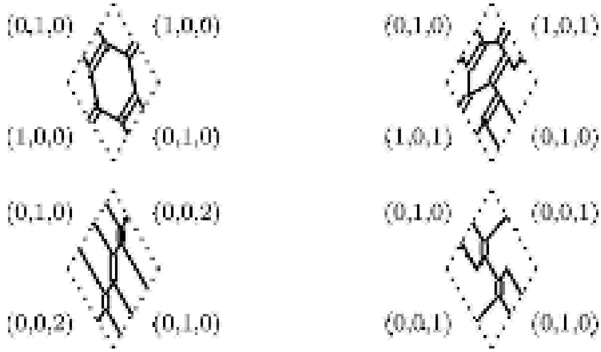


Fig. 9. Diamonds.

CONJECTURE 4·3. *If we write the tensor associated to a triangular diagram in terms of the tensor product basis we get an expression of the form*

$$b_D = \sum_{b' \leqslant b} \alpha_b^{b'} b'_T \tag{10}$$

where $\alpha_b^{b'} \in \mathbb{Z}[q, q^{-1}]$ *and* $\alpha_b^{b'} = 1$ *if* $b' = b$.

This means that the tensors associated to the triangular diagrams are a basis of $\otimes^n \Delta$ and the change of basis matrices are integral and unitriangular. The following is a sketch of a proof of this result.

Consider the diagrams in which only some of the diamonds have been filled in. If we start at the top left corner of one of these diagrams we have a sequence of alternating South-East and North-East boundary edges which take us to the top right corner. Each of these edges is labelled by a dominant weight. Then the diagram can be considered as map from the tensor product of the corresponding sequence of highest weight representations to $\otimes^n \Delta$. Applying this map to the tensor product of the highest weight vectors gives an element of $\otimes^n \Delta$. Then we prove that each of these vectors has the property in (10). The proof is by induction on the number of diamonds that have been filled in. If no diamonds have been filled in then this construction just gives the tensor product of the basis vectors. It is this property that determined the diagrams we drew in the first place. The inductive step follows from the observation that if we add one diamond then we multiply by a triangular matrix.

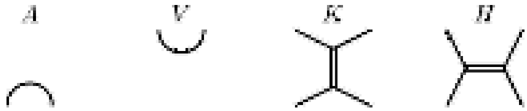
A corollary of this conjecture is that the diagrams with $H = 0$ and $D = 0$ are a basis of the space of invariant tensors. This corollary follows from the results in Section 5 and is part of Proposition 7·2.

5. *Comparison*

The aim of this section is to explain the relationships between the constructions of the previous three sections.

Let $A_D(n)$ be the endomorphism algebra of n in $\mathcal{D}$. There are algebra homomorphisms $A_P(n) \to A_D(n)$ for $n \geqslant 0$. This map is defined on the generators by (4) and the defining relations for $A_P(n)$ are satisfied by construction.

In Section 2 we used trivalent graphs to construct the category $\widetilde{D}$. Here we take a more algebraic approach. The full subcategory of $\widetilde{\mathcal{D}}$ whose objects are finite sequences of single edges is generated as a category by morphisms $A_i : n \to n-2$, $V_i : n-2 \to n$, $K_i : n \to n$ and $H_i : n \to n$. The diagrams for these morphisms are given in Figure 10 and the diagram



(11)

Fig. 10. Generators.

with index i is obtained by putting $i-1$ vertical lines on the left and $n-i-1$ vertical lines on the right.

LEMMA 5·1. *Each morphism can be written as a composable sequence of these generators.*

Proof. Given a diagram in a rectangle we draw it so that each horizontal line intersects the diagram in a finite set of points. Generically each of these intersection points is either a transverse intersection, a maximum, a minimum or a trivalent vertex. Then we can isotope the diagram so that each horizontal line meets the diagram in at most one point which is not a transverse intersection. Then we put in horizontal lines so that between any consecutive pair of these lines we have one horizontal line which has an intersection point which is not transverse. Then between each consecutive pair of horizontal lines we have the diagram of a generator.

Definition 5·2. The width of the morphism to be the maximum value of n.

LEMMA 5·3. *A diagram in $A_D(n)$ of width n can be written as a word in the generators of $A_P(n)$.*

Proof. Since $U_i = V_i A_i$, this means that we can find a composable sequence of these generators such that every V_i is succeeded by A_i and whose diagram is isotopic to the given diagram. First replace each V_{n-k} by

$$(V_{n-k}A_{n-k})(V_{n-k+1}A_{n-k+1})\cdots(V_{n-1}A_{n-1})V_n$$

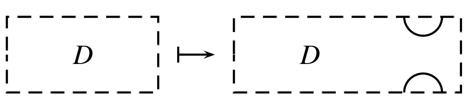
and replace each A_{n-k} by

$$A_n(V_{n-1}A_{n-1})(V_{n-2}A_{n-2})\cdots(V_{n-k}A_{n-k}).$$

Then we draw the diagram in the strip $0 \leqslant x \leqslant n+1$ so that for $1 \leqslant m \leqslant n$ the object m the x-coordinates of the m points are $1, 2, \ldots, m$. Then in this diagram there is a maximum below each minimum. This means that the maxima and minima are paired by vertical lines which do not intersect the diagram. Then for each pair we perform an isotopy which shrinks this vertical line. If each of these vertical lines is sufficiently short then the resulting diagram has the required form.

LEMMA 5·4. *For $n \geqslant 0$, the homomorphism $A_P(n) \to A_D(n)$ is surjective.*

Proof. We show that if $k > 0$ then a diagram $D \in A_D(n)$ of width $n+2k$ can be written as linear combination of diagrams of width $n+2k-2$. Given D add k minima at the top right and k maxima at the bottom right. This means we repeat the following operation until we get a diagram whose width is at most the number of endpoints on the top and bottom edge



.

This gives a diagram in $A_D(n+2k)$ of width $n+2k$. This can then be written as a word in the generators of $A_P(n+2k)$. This word is of the form $U_{n+2k-1}aU_{n+2k-1}$ for $a \in A_P(n+2k)$. This word is an element of $A_P(n+2k-2)U_{n+2k-1}$ by Lemma 3·3 which means that D has been written as a linear combination of diagrams of width $n+2k-2$.

Let $\mathcal{T}_{\mathbb{Q}}$ be the $\mathbb{Q}(q)$-linear category of invariant tensors for the spin representation, Δ. This has objects 0, 1, 2, ... and a morphism $n \to m$ is a linear map $\otimes^n \Delta \to \otimes^m \Delta$ which intertwines the actions of the Drinfeld-Jimbo quantum group $U_q(B_3)$. This quantum group is defined to be a Hopf algebra over the field $\mathbb{Q}(q)$.

Then we construct a functor $\widetilde{\mathcal{D}} \to \mathcal{T}_{\mathbb{Q}}$. This functor is constructed by defining it on the generators in Figure 10. The tensors K_i and H_i are constructed from the basic tensor $\Gamma\colon V \otimes \Delta \to \Delta$. The tensors A_i and V_i correspond to raising and lowering operators. Then given a diagram, by Lemma 5·1 we can read it as a word in these generators and from this we obtain the corresponding tensor. This tensor depends on a choice since there are many ways of reading a diagram as a word in the generators. However if we take any two ways of doing this there is a finite set of local moves that take us from one diagram to the other. These local moves all correspond to identities satisfied by the tensor operations of tensor product, contracting indices and raising and lowering indices. Therefore the tensor we obtain only depends on the diagram. This gives a well-defined map from diagrams to tensors. It is clear that this is a functor. By construction it is compatible with the tensor operations.

PROPOSITION 5·5. *The defining relations for $\mathcal{D}$ are satisfied so this functor factorises through $\mathcal{D}$ to give a functor $\mathcal{D} \to \mathcal{T}_{\mathbb{Q}}$.*

This result is the motivation for the relations.

Proof. Let B_3 be the three string braid group. Then B_3 acts on the invariant tensors in $\otimes^3\Delta$. The dimensions of these representations are known from the decomposition of $\otimes^3\Delta$. Also the eigenvalues of the standard generators are known. Since each representation has dimension at most five this information determines the representations by [**TW01**]. This gives the representations Section 3·2. Given these representations and using the tensor operations one can check that the defining relations for $\mathcal{D}$ are satisfied.

Let $A_T(n)_{\mathbb{Q}}$ be the endomorphism algebra of n in $\mathcal{T}_{\mathbb{Q}}$. Then in the remainder of this section we will use the homomorphisms

$$A_P(n) \longrightarrow A_D(n) \longrightarrow A_T(n)_{\mathbb{Q}} \tag{12}$$

to compare these algebras.

Let $\mathcal{D}_{\mathbb{Q}}$ be the free $\mathbb{Q}(q)$-linear category on $\mathcal{D}$ so $\mathcal{D}_{\mathbb{Q}} = \mathcal{D} \otimes_{\mathbb{Z}[\delta]} \mathbb{Q}(q)$. Then for $n \geqslant 0$ we can define $A_D(n)_{\mathbb{Q}}$ either as the endomorphism algebra of the object n of $\mathcal{D}_{\mathbb{Q}}$ or as $A_D(n) \otimes_{\mathbb{Z}[\delta]} \mathbb{Q}(q)$. Then the functor $\mathcal{D} \to \mathcal{T}_{\mathbb{Q}}$ extends to a functor $\mathcal{D}_{\mathbb{Q}} \to \mathcal{T}_{\mathbb{Q}}$ and induces an algebra homomorphism $A_D(n)_{\mathbb{Q}} \to A_T(n)_{\mathbb{Q}}$ for each $n \geqslant 0$.

For $n > 1$ we have generators $U_1, \ldots, U_{n-1} \in A_P(n)$. We also consider these as elements of the algebra $A_P(n)_{\mathbb{Q}}$ and as elements of $A_T(n)_{\mathbb{Q}}$ using the homomorphisms in (12). Then we define $H_P(n)_{\mathbb{Q}}$ to be the quotient of $A_P(n)_{\mathbb{Q}}$ by the ideal generated by these elements and we define $H_T(n)_{\mathbb{Q}}$ to be the quotient of $A_T(n)_{\mathbb{Q}}$ by the ideal generated by these elements. Then, for $n > 0$ we have an induced homomorphism $H_P(n)_{\mathbb{Q}} \to H_T(n)_{\mathbb{Q}}$.

In [**Wen90**] it is shown that these homomorphisms are isomorphisms. Furthermore these algebras are direct sums of matrix algebras and the Bratteli diagram is the same as the

Bratteli diagram of the centraliser algebras of the tensor powers of the four dimensional fundamental representation of the simple Lie algebra of type B_2. This representation can be taken to be the vector representation of $\mathfrak{sp}(4)$ or the spin representation of $\mathfrak{so}(5)$.

PROPOSITION 5·6. *For $n > 0$, the homomorphism $A_P(n)_{\mathbb{Q}} \to A_T(n)_{\mathbb{Q}}$ is surjective.*

Proof. The proof is by induction on n. We have

$$A_T(n+1)_{\mathbb{Q}} \cong A_T(n)_{\mathbb{Q}} U_n A_T(n)_{\mathbb{Q}} \oplus H_T(n)_{\mathbb{Q}}.$$

The inductive step now follows from the inductive hypothesis and the result that the homomorphism $H_P(n)_{\mathbb{Q}} \to H_T(n)_{\mathbb{Q}}$ is surjective.

COROLLARY 5·7. *For $n > 0$, the homomorphism $A_P(n)_{\mathbb{Q}} \to A_T(n)_{\mathbb{Q}}$ is an isomorphism.*

Proof. The number of irreducible words in Definition 3·2 is dim $A_T(n)_{\mathbb{Q}}$. Hence the observation that the set of irreducible words is a spanning set for dim $A_P(n)_{\mathbb{Q}}$ shows that dim $A_P(n)_{\mathbb{Q}} \leqslant$ dim $A_T(n)_{\mathbb{Q}}$.

COROLLARY 5·8. *For all $n > 0$, the homomorphism $A_P(n) \to A_D(n)$ is an isomorphism.*

Proof. Corollary 5·7 implies that this homomorphism is injective. The result follows from this observation and Lemma 5·4.

COROLLARY 5·9. *The functor $\mathcal{D}_{\mathbb{Q}} \to \mathcal{T}_{\mathbb{Q}}$ is an isomorphism.*

Proof. For $n+m$ even put $n+m = 2p$. Then using the diagrammatic raising and lowering operators we have an isomorphism $\mathrm{Hom}_{\mathcal{D}}(n, m) \to A_D(p)$. Using the tensorial raising and lowering operators we have an isomorphism $\mathrm{Hom}_{\mathcal{T}}(n, m) \to A_T(p)$. The functor is compatible with raising and lowering operators and induces an isomorphism $A_D(p) \to A_T(p)$. Therefore it also induces an isomorphism $\mathrm{Hom}_{\mathcal{D}}(n, m) \to \mathrm{Hom}_{\mathcal{T}}(n, m)$.

If $n + m$ is odd then $\mathrm{Hom}_{\mathcal{D}}(n, m)$ and $\mathrm{Hom}_{\mathcal{T}}(n, m)$ are both the zero module so there is nothing to prove.

This result can be rephrased to say that the category $\mathcal{D}$ is an integral form or order for the category $\mathcal{T}_{\mathbb{Q}}$. Another construction of an integral form for $\mathcal{T}_{\mathbb{Q}}$ is given in [**Lus93**, part IV]. Then Conjecture 4·3 implies that these two integral forms are equivalent. Both of these integral forms are closed under the tensor operations.

6. *Confluence*

In this section we apply the theory of rewrite rules to the presentations of the algebras $A_P(n)$. This theory originates from the diamond lemma in [**New42**] and our account is based on [**Sim94**, chapter 2]. This theory has been applied to the Hecke and Temperley–Lieb algebras in [**KLLO02**]. Our approach differs from these standard approaches in that we take $A_P(n)$ to be a quotient of the free algebra on a commutation monoid (instead of a free monoid) and we only require a reduction order to be a partial order (instead of a linear order).

Let X be a set and $W \subset X \times X$ a relation. We write $x \to y$ if $(x, y) \in W$. Let the relation $\overset{*}{\to}$ be the reflexive transitive closure of the relation W. Let $\simeq$ be the equivalence relation generated by W. The first result is the following.

Definition 6·1. A rewrite system is confluent if either of the following two equivalent conditions is satisfied:

(i) if $u \simeq v$ then there exists an x such that $u \stackrel{*}{\to} x$ and $v \stackrel{*}{\to} x$;

(ii) if $u \stackrel{*}{\to} x$ and $u \stackrel{*}{\to} y$ then there exists a v such that $x \stackrel{*}{\to} v$ and $y \stackrel{*}{\to} v$.

A rewrite system is locally confluent if whenever $u \to x$ and $u \to y$ then there exists a v such that $x \stackrel{*}{\to} v$ and $y \stackrel{*}{\to} v$.

It is clear that a confluent rewrite system is locally confluent but there are examples of locally confluent rewrite systems that are not confluent.

Definition 6·2. A rewrite system is terminal if there is no infinite sequence $x_0, x_1, \dots$ such that $x_{i-1} \to x_i$ for all $i > 0$.

Note that if a rewrite system is terminal then the relation $\stackrel{*}{\to}$ is a partial order.

PROPOSITION 6·3. *A rewrite system that is terminal and locally confluent is confluent.*

Now we apply this to finitely presented algebras. Let K be a commutative ring and M a monoid. Let KM be the monoid algebra of M over K. Then every element $u \in KM$ can be written uniquely as $\sum_{m \in S} r_m m$ where S is a finite subset of M and $r_m \in K \setminus \{0\}$. The subset S is called the support of u and is denoted by $\mathrm{supp}\,(u)$.

Let R be a finite set of ordered pairs (p, P) where $p \in M$, $P \in KM$ and $p \notin \mathrm{supp}\,(P)$. Then let $I \subset KM$ be the ideal generated by the set $\{p - P | (p, P) \in R\}$ and let A be the K-algebra KM/I.

Now define a relation W on the set KM by

$$rxpy + u \to rxPy + u$$

where $r \in K \setminus \{0\}$, $x, y \in M$, $u \in KM$, $(p, P) \in R$ and $p \notin \mathrm{supp}\,(u)$. Then the equivalence relation $u \simeq v$ is the equivalence relation $u - v \in I$.

Definition 6·4. An element $u \in M$ is reducible if it can be written as $u = apb$ where $a, b \in M$ and for some $P \in KM$ $(p, P) \in R$. An element $u \in M$ is irreducible if it is not reducible.

LEMMA 6·5. *The rewrite system W is confluent if and only if A is the free K-module on the set of irreducible elements of M.*

The following Proposition 6·7 gives a criterion which can be checked by a finite calculation and which implies that the rewrite system is locally confluent.

Definition 6·6. An overlap consists of elements $u, v, w \in M$ together with elements $P, Q \in KM$ such that $v \neq 1$, $(uv, P) \in R$ and $(vw, Q) \in R$. This overlap is unambiguous if there is an $x \in KM$ such that $Pw \stackrel{*}{\to} x$ and $uQ \stackrel{*}{\to} x$.

PROPOSITION 6·7. *The rewrite system W is locally confluent if and only if every overlap is unambiguous.*

Definition 6·8. A reduction order on M is a partial order which is invariant under both left and right translations and such that, for each $m \in M$, the set $\{m' \in M | m' < m\}$ is finite.

Then the standard method of showing that a rewrite system is terminal is to construct a reduction order on M such that for each $(p, P) \in R$ we have $a < p$ for each $a \in \mathrm{supp}\,(P)$.

Next we apply this theory to the presentations of the algebras $A_P(n)$.

Definition 6·9. For $n \geqslant 1$ let $C(n)$ be the monoid with the same set of generators as $A_P(n)$ and with defining relations the commuting relations. That is, for $1 \leqslant i, j \leqslant n-1$ with $|i-j| > 1$, if $a \in \{U_i, K_i, H_i\}$ and $b \in \{U_j, K_j, H_j\}$ then $ab = ba$.

Let the type of a word in the generators be the sequence of subscripts. Construct a rewrite system for $A_P(3)$ by taking the set of pairs (p, P) where the words p are the words of type $(1, 1)$ and $(2, 2)$, the words of type $(2, 1, 2)$ and the words of type $(1, 2, 1)$ which are not irreducible words. Then, for each p, we can use the relations to write p uniquely as a linear combination of irreducible words. Then, by construction, this is a confluent rewrite system for $A_P(3)$.

Then, for $n > 3$, we obtain a rewrite system by taking the union over $1 \leqslant i \leqslant n-2$ of the sets obtained by changing each subscript 1 to i and each subscript 2 to $i+1$. This gives a finite set of rewrite rules, $R(n)$. Although we have described these rewrite rules using words, all of these words are well-defined elements of $C(n)$.

Define a partial order on $C(n)$ recursively. This definition is similar to the definition of a wreath product order on a free monoid. On the free monoid generated by $\{U_i, K_i, H_i\}$ take the length plus reverse lexicographic order with the generators ordered by $U_i < K_i < H_i$. Let w be a word in the generators of $C(n+1)$. Then we obtain a word u in the generators $\{U_n, K_n, H_n\}$ by deleting all generators whose subscript is not n and a word v in the generators of $C(n)$ by deleting all generators whose subscript is n. Now note that u and $v \in C(n)$ depend only on $w \in C(n+1)$ and not on the choice of word representing w. Then let $w_1, w_2 \in C(n+1)$ with corresponding words u_1, u_2 and elements $v_1, v_2 \in C(n)$. Then we define $w_1 < w_2$ if $u_1 < u_2$ or $u - 1 = u_2$ and $v_1 < v_2$.

Then this is a reduction order and has the property that if $(p, P) \in R(n)$ and $a \in \mathrm{supp}\,(P)$ then $a < p$. This shows that the rewrite system $W(n)$ associated to $R(n)$ is terminal. Note that the set of irreducible words defined in Definition 3·2 coincides with the set of irreducible elements for the rewrite system $W(n)$ in Definition 6·4. It now follows from the theory of rewrite systems that the following are equivalent, for any $n > 3$:

(1) the rewrite system $W(n)$ for $A_P(n)$ is confluent;
(2) the rewrite system $W(n)$ for $A_P(n)$ is locally confluent;
(3) the $\mathbb{Z}[\delta]$-module underlying $A_P(n)$ is the free module on the set of irreducible words.

The third property has been proved in Proposition 5·6 using the representation theory of the quantum group $U_q(B_3)$ and so we conclude that these rewrite systems are confluent.

However it is also possible to prove that these rewrite systems are locally confluent by directly checking that all overlaps are unambiguous. This depends on the observation that this holds for any $n \geqslant 4$ if and only if it holds for $n = 4$. This holds since any overlap can only involve subscripts of the form $i-1$, i and $i+1$ for some $i > 1$. This case can in principle be checked directly since it is a finite calculation. This case can also be checked indirectly by giving a finite dimensional representation in which the irreducible words are linearly independent. This case can also be checked indirectly using a computer algebra package.

7. *Cellular algebras*

First we extend the definition of a cellular algebra given in [**GL96**] to cellular categories. The basic example is the Temperley–Lieb category discussed in [**Wes95**]. The proof we give here also applies to the categories of diagrams constructed from the rank two simple

Lie algebras in [**Kup96b**]. Further examples are the affine Temperley–Lieb category in [**GL98**], the partition category whose endomorphism algebras are the partition algebras, and the Brauer category whose endomorphism algebras are the Brauer algebras.

Definition 7·1. Let R be a commutative ring with identity. Let $\mathcal{A}$ be a R-linear category with an anti-involution $*$. Then cell datum for $\mathcal{A}$ consists of a partially ordered set Λ, a finite set $M(n, \lambda)$ for each $\lambda \in \Lambda$ and each object n of $\mathcal{A}$, and for $\lambda \in \Lambda$ and n,m any two objects of $\mathcal{A}$ we have an inclusion

$$C: M(n, \lambda) \times M(m, \lambda) \to \mathrm{Hom}_{\mathcal{A}}(n, m)$$
$$C \ : \ (S, T) \mapsto C^{\lambda}_{S,T}.$$

The conditions that this datum is required to satisfy are:

C-1 For all objects n and m, the image of the map

$$C: \coprod_{\lambda\in\Lambda} M(n, \lambda) \times M(m, \lambda) \to \mathrm{Hom}_{\mathcal{A}}(n, m)$$

is a basis for $\mathrm{Hom}_{\mathcal{A}}(n, m)$ as an R-module.

C-2 For all objects n, m, all $\lambda \in \Lambda$ and all $S \in M(n, \lambda)$, $T \in M(m, \lambda)$ we have

$$\left(C^{\lambda}_{S,T}\right)^* = C^{\lambda}_{T,S}.$$

C-3 For all objects p, n, m, all $\lambda \in \Lambda$ and all $a \in \mathrm{Hom}_{\mathcal{A}}(p, n)$, $S \in M(n, \lambda)$, $T \in M(m, \lambda)$ we have

$$aC^{\lambda}(S, T) = \sum_{S'\in M(p,\lambda)} r_a(S', S)C^{\lambda}_{S',T} \quad \mathrm{mod}\ \mathcal{A}(< \lambda)$$

where $r_a(S', S) \in R$ is independent of T and $\mathcal{A}(< \lambda)$ is the R-linear span of

$$\left\{C^{\mu}_{S,T}\middle|\mu < \lambda;\ S \in M(p, \mu), T \in M(m, \mu)\right\}.$$

Another way of formulating C-3 is that for all objects p, n, m, all $\lambda \in \Lambda$ and all $S \in M(p, \lambda)$,$T \in M(n, \lambda)$, $U \in M(n, \lambda)$,$V \in M(m, \lambda)$ we have

$$C^{\lambda}(S, T)C^{\lambda}(U, V) = \langle T, U\rangle C^{\lambda}_{S,V} \quad \mathrm{mod}\ \mathcal{A}(< \lambda)$$

where $\langle T, U\rangle$ is independent of S and V.

A consequence of this definition is that for all $\lambda \in \Lambda$ we have ideals $\mathcal{A}(< \lambda)$ and $\mathcal{A}(\leqslant \lambda)$ where for all objects n, m the subspace $\mathcal{A}(< \lambda)(n, m)$ is the R-linear span of

$$\left\{C^{\mu}_{S,T}\middle|\mu < \lambda;\ S \in M(n, \mu), T \in M(m, \mu)\right\}$$

and the subspace $\mathcal{A}(\leqslant \lambda)(n, m)$ is the R-linear span of

$$\left\{C^{\mu}_{S,T}\middle|\mu \leqslant \lambda;\ S \in M(n, \mu), T \in M(m, \mu)\right\}.$$

This is a generalisation of the definition of a cellular algebra since we can regard any algebra over R as an R-linear category with one object.

More significantly, if $\mathcal{A}$ is a cellular category then $\mathrm{End}(n)$ is a cellular algebra for any object n of $\mathcal{A}$.

A functor $\phi: \mathcal{A} \to \mathcal{A}'$ between cellular categories is cellular if we have a map of partially ordered sets $\phi: \Lambda \to \Lambda'$ and set maps

$$\phi_{\lambda}: M(n, \lambda) \to M'(\phi(n), \phi(\lambda))$$

such that

$$\phi\big(C^{\lambda}_{S,T}\big) = C^{\phi(\lambda)}_{\phi_\lambda(S),\phi_\lambda(T)} \mod \mathcal{A}'(< \phi(\lambda))$$

for all λ and all S, T.

One reason for considering cellular categories instead of just the cellular endomorphism algebras is that for each object n and each $\lambda \in \Lambda$ we have an R-linear functor $\rho(n; \lambda)$ from $\mathcal{A}$ to the category of left $\mathrm{End}_{\mathcal{A}}(n)$-modules which on objects is given by

$$\rho(n; \lambda) : m \longmapsto \mathrm{Hom}_{\mathcal{A}(\lambda)}(n, m).$$

Furthermore if $\mu < \lambda$ then we have a natural transformation from $\rho(n; \mu)$ to $\rho(n; \lambda)$. This is used in [**GL98**].

PROPOSITION 7·2. *The category $\mathcal{D}$ with the $*$-functor given is cellular.*

Proof. The partially ordered set Λ is the set of dominant weights. For n an object of $\mathcal{D}$ and λ a dominant weight we take the set $M(n, \lambda)$ to be a set of diagrams (or morphisms in $\widetilde{\mathcal{D}}$). We require that each diagram in $M(n, \lambda)$ has n boundary points on the top edge and that the bottom edge is a minimal cut path of weight λ (see Definition 4·1). Furthermore we require that the bottom edge is minimal with these properties. This means that each diagram $D \in M(n, \lambda)$ has the property that if $D = D_1 D_2$ is a factorisation in $\widetilde{\mathcal{D}}$ such that the bottom edge of D_1 has weight λ then $D = D_1$.

Next we construct the map C. Let $D' \in M(n, \lambda)$ and $D \in M(m, \lambda)$. Then we define $C(D', D)$ to be the composition in $\widetilde{\mathcal{D}}$, $D' H D^*$ where the diagram H is uniquely determined by the properties that this composition exists and gives an irreducible diagram such that every cut path that traverses the rectangle has weight at least λ.

This gives the data in Definition 7·1. Next we verify that the conditions are satisfied. Each irreducible (n, m)-diagram can be written uniquely as $C(D', D) = D' H D^*$. The weight λ is the minimum weight of a cut path which traverses the rectangle. This shows that the diagram has a factorisation as $D' D^*$ where the bottom edges of D and D' are minimal cut paths of weight λ. Then to obtain the factorisation as $C(D', D) = D' H D^*$ apply the analogue of [**Kup96b**, lemma 6·6] to both the top and bottom edge of the rectangle.

It has been shown in section 5 that the set of irreducible (n, m)-diagrams is a basis of $\mathrm{Hom}_{\mathcal{D}}(n, m)$ and hence property C-1 holds.

The property C-2 holds by construction since

$$C(D, D')^* = (D' H D^*)^* = D H^* (D')^* = C(D', D).$$

The property C-3 holds by inspecting the relations for $\mathcal{D}$.

COROLLARY 7·3. *For $n > 0$ the algebra $A_P(n) \cong A_D(n)$ is cellular over $\mathbb{Z}[\delta]$.*

Let $\Bbbk$ be a field and $\phi : \mathbb{Z}[\delta] \to \Bbbk$ a ring homomorphism. Then by applying the functor $- \otimes_{\mathbb{Z}[\delta]} \Bbbk$ to the algebra $A_P(n) \cong A_D(n)$ we obtain a $\Bbbk$-algebra which we denote by $A(n)_\phi$. In the rest of this section we will write $[r]$ for $\phi([r]) \in \Bbbk$.

The definition of a quasi-hereditary algebra is given in [**CPS88**] and cellular algebras over a field which are quasi-hereditary are characterised in [**KX99**].

PROPOSITION 7·4. *If $\delta(\delta^2 - 2)(\delta^2 - 3)(\delta^4 - 5\delta^2 + 5) \neq 0$ then $A(n)_\phi$ has the following properties*:

(1) *the algebra $A(n)_\phi$ is quasi-hereditary*;

(2) *the algebra* $A(n)_\phi$ *has finite global dimension*;
(3) *the Cartan matrix of* $A(n)_\phi$ *has determinant one.*

Proof. We apply [**KX99**, lemma 2·1(3)] to show that the algebra $A(n)_\phi$ is quasi-hereditary. The result is then an application of [**KX99**, theorem 3·1].

Let $0 = J_0 \subset J_1 \subset \cdots \subset J_N = A(n)_\phi$ be a cell chain of ideals. Then we show that for $1 \leqslant r \leqslant N$ that there is an idempotent e such that $e \in J_r$ and $e \notin J_{r-1}$. It follows that $J_r^2 \nsubseteq J_{r-1}$.

For $1 \leqslant i \leqslant n-1$ we have orthogonal idempotents

$$\frac{1}{[2]^2}U_i,\ \frac{[3]}{[2]^2[6]}K_i,\ \frac{1}{[2]^2} + \frac{1}{[2]^2}H_i + \frac{1}{[2]^3}K_i - \frac{[4][5]}{[2]^3[10]}U_i$$

which we denote by u_i, k_i and e_i. Then for each dominant weight (a, b, c) such that $2a + 2b + 2c \leqslant n$ we have an idempotent

$$(u_1u_3\cdots u_{2a-1})(k_{2a+1}k_{2a+3}\cdots k_{2a+2b-1})(e_{2a+2b+1}e_{2a+2b+3}\cdots e_{2a+2b+2c-1}).$$

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