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CLASSES OF GENERALISED NILPOTENT GROUPS

by

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ABSTRACT

We construct infinite groups by taking a canonical set of generators for the general linear group over the ring $\mathbb{Z}_{p^n}$ and generalising these to a group of linear maps on an R-module, where R is any ring with a 1.

The structure of the groups obtained is related to the structure of the Jacobson radical of the ring, and we show that the groups considered have a wide range of generalised nilpotent properties. We show that our construction can yield infinite simple groups. As far as the author knows these groups have not previously been studied.

We prove that many of the groups are $\overline{SI}$-groups having subgroups which are not $\overline{SI}$-groups. Previously only one example of such a group seems to have been known.

The last chapter of this thesis is concerned with a completely different class of infinite groups. We construct here infinite groups that are generalisations of the Sylow p-subgroup of the symplectic group. The main result obtained is the construction of a group having two ascendant Abelian subgroups whose join is self-normalising. Again only one example of such a group seems to have been known previously.

PREFACE

The following work was carried out at the University of Warwick during the years 1965 - 1968. I would like to express my thanks to Dr. S.E.Stonehewer who was my supervisor for the last two of these three years. I have appreciated greatly the help and encouragement he has given me while I have been carrying out this research. I would like to thank also Mr. R.J.Clarke for several helpful conversations. His interest in the Sylow p-subgroup of the general linear group over the ring $\mathbb{Z}_{p^n}$ gave me the idea to construct the generalisations presented here.

During these three years I have been financed from two sources, an Andrew Bell Scholarship and a St. Andrews University Scholarship. I am most grateful for the financial assistance both these have been to me.

The work presented here is original except for several standard results which I have references to in the text.

CONTENTS

INTRODUCTION

In 1954 D.H.McLain introduced a generalisation of the Sylow p-subgroup of the general linear group over the field GF(p), see [9]. Using this technique he produced his famous characteristically simple group and P.Hall made use of McLain's group to produce many interesting examples of infinite groups.

In the first six chapters of this thesis we use techniques, similar to those McLain developed, to generalise the Sylow p-subgroup of the general linear group over the ring $\mathbb{Z}_{p^n}$, the integers modulo p^n. Not only do we make the dimension of the module infinite but we also work over an arbitrary ring with a 1. This has the advantage of producing groups with a considerably wider range of properties than McLain's groups.

We investigate, in chapter three, how the properties of the groups are related to the structure of the ring. In chapter four we show that some of the groups considered are infinite simple groups which, as far as the author knows, are not isomorphic to any known simple groups.

In 1938 Kurosh and Cernikov posed the question whether every subgroup of an $\overline{SI}$-group, that is a group in which every chief factor is Abelian, is an $\overline{SI}$-group.

This was answered in the negative by P.Hall in 1964, see [6], when he constructed an $\overline{SI}$-group with a subgroup that was free of rank two. We show in chapter five that our construction leads to the discovery of a whole class of $\overline{SI}$-groups having subgroups not in the class $\overline{SI}$.

Chapter six is devoted to a study of the properties of the groups we have considered when a specific ring is chosen. This chapter makes it clear how easy the groups are to handle despite their wide range of properties.

The last chapter of this thesis deals with a completely different class of groups. Again we use McLain's techniques, this time to generalise the Sylow p-subgroup of the symplectic group over GF(p). In particular we construct a group having a self-normalising subgroup that is the join of two ascendant subgroups. As far as the author knows, the Zassenhaus group, which is described in [12], Appendix D, Exercise 23, is the only other known example of a group having these properties.

§1 Basic definitions and notation.

1.1 Closure operations.

We begin by introducing the concept of a closure operation as defined by Hall in [7].

1.1.1 **Definition** We say that $\mathfrak{X}$ is a class of groups if

(i) $\mathfrak{X}$ contains ~~the~~ a trivial group.

(ii) $G \in \mathfrak{X}$ with $G_1 \cong G$ implies $G_1 \in \mathfrak{X}$

1.1.2 **Definition** A map c from the class of group classes to itself is called a closure operation if

$$\mathfrak{X} \leq c\mathfrak{X} = c^2\mathfrak{X} \leq c\mathfrak{D} \quad \text{when } \mathfrak{D} \geq \mathfrak{X}$$

and $c(1) = (1)$ where (1) denotes the class containing only trivial groups.

We now define the closure operations s and q.

1.1.3 **Definition** Let $\mathfrak{X}$ be any class of groups. Then ~~d~~ define $s\mathfrak{X}$ by

$G \in s\mathfrak{X}$ if and only if G is embeddable in a $\mathfrak{X}$-group;

and define $q\mathfrak{X}$ by

$G \in q\mathfrak{X}$ if and only if G is the epimorphic image of a $\mathfrak{X}$-group.

We say that a class $\mathfrak{X}$ is c-closed if $c\mathfrak{X} = \mathfrak{X}$; and clearly for any $\mathfrak{X}$, $c\mathfrak{X}$ is the smallest c-closed class containing $\mathfrak{X}$.

Let G be any group. Then denote the smallest class of groups containing G by (G). Hence (G) consists of all groups isomorphic to G, and all trivial groups.

1.1.4 **Definition** A closure operation c is called

<u>unary</u> if

$$c\mathfrak{B} = \bigcup_{G \in \mathfrak{B}} c(G)$$

for all classes $\mathfrak{B}$.

Not every closure operation is unary, but it is easy to see that S and Q defined above are unary closure operations. For any unary closure operation c and an arbitrary class $\mathfrak{B}$ it is possible to define the class $\mathfrak{B}^c$ to be the largest c-closed class contained in $\mathfrak{B}$, and we shall call it the <u>c-interior</u> of $\mathfrak{B}$.

1.1.5 <u>Definition</u> Let $\mathfrak{B}$ be any class of groups. We define the closure operations N, $\acute{N}$, $\grave{N}$ as follows:-

$G \in N\mathfrak{B}$ if and only if G is the join of its subnormal $\mathfrak{B}$-subgroups.

$G \in \acute{N}\mathfrak{B}$ if and only if G is the join of its ascendant $\mathfrak{B}$-subgroups.

$G \in \grave{N}\mathfrak{B}$ if and only if G is the join of its descendant $\mathfrak{B}$-subgroups.

We refer the reader to section 1.2 for the definitions of ascendant and descendant $\mathfrak{B}$-subgroups.

Let $\mathfrak{A}$ denote the class of Abelian groups. Then $N\mathfrak{A}$ is the class of Baer nilgroups, see [1] ; i.e. the class of all groups generated by their subnormal Abelian subgroups. These groups can also be characterised by the property that every cyclic subgroup is subnormal.

We now define the product of two closure operations.

1.1.6 <u>Definition</u> Let A and B be two closure operations and let $\mathcal{X}$ be any class of groups. Define

$$(AB)\mathcal{X} = A(B\mathcal{X}).$$

Notice that AB need not be a closure operation. For example QS is a closure operation, but SQ is not. We now introduce a partial ordering on closure operations as follows.

1.1.7 <u>Definition</u> Let A and B be two closure operations. Then

$A \leq B$ if and only if $A\mathcal{X} \leq B\mathcal{X}$ for all classes $\mathcal{X}$.

It is clear that $N \leq N'$ and $N \leq \grave{N}$.

Let A and B be two closure operations. Then the product AB is a closure operation if and only if $BA \leq AB$; while if $A \leq B$ then $AB = BA = B$.

We now define two further closure operations.

1.1.8 <u>Definition</u> Let $\mathcal{X}$ be any class of groups. Define closure operations L, R as follows:-

$G \in L\mathcal{X}$ if and only if every finite subset of G is contained in a $\mathcal{X}$-subgroup of G.

$G \in R\mathcal{X}$ if and only if the intersection of all normal subgroups N of G such that $G/N \in \mathcal{X}$ is trivial.

We call the classes $L\mathcal{X}$ and $R\mathcal{X}$ the class of locally-$\mathcal{X}$ groups and the class of residually-$\mathcal{X}$ groups, respectively. Let $\mathfrak{N}$ denote the class of nilpotent groups. Then $L\mathfrak{N}$ is the class of locally nilpotent groups; and denoting the class of finite p-groups by $\mathcal{Y}_p$, we have the following inclusions:-

$$\mathcal{Y}_p < \mathfrak{N} < {}_N\mathfrak{A} < {}_{\acute{N}}\mathfrak{A} < {}_L\mathfrak{N}, \ p \text{ prime}.$$

1.2 Series

We introduce the standard notation for series, see [13].

1.2.1 Definition We say that G has a $\mathfrak{L}$-series of type Ω, for some linearly ordered set Ω, if G has a set of pairs of subgroups

$$(\Lambda_\lambda , V_\lambda : \lambda \in \Omega)$$

satisfying

(i) V_λ is normal in Λ_λ, and $\Lambda_\lambda / V_\lambda \in \mathfrak{L}$, for all $\lambda \in \Omega$;

(ii) $\Lambda_\sigma < V_\rho$ for $\sigma < \rho$; $\sigma, \rho \in \Omega$;

(iii) $G-1 = \bigcup_{\lambda \in \Omega} (\Lambda_\lambda - V_\lambda)$.

1.2.2 Definition We say that G has an invariant $\mathfrak{L}$-series of type Ω, for some linearly ordered set Ω, if G has a set of pairs of normal subgroups

$$(\Lambda_\lambda , V_\lambda : \lambda \in \Omega)$$

satisfying (i), (ii), and (iii) of definition 1.2.1.

Suppose the linearly ordered set Ω is well ordered with order type λ where λ is an ordinal. In this case we call a $\mathfrak{L}$-series of type Ω an ascending $\mathfrak{L}$-series. Then $\Lambda_\sigma = V_{\sigma+1}$, for $\sigma+1 < \lambda$; and defining $V_\lambda = G$ we can write an ascending $\mathfrak{L}$-series in the form

$$(V_\sigma : \sigma < \lambda),$$

where $V_0 = 1$, V_σ is normal in $V_{\sigma+1}$ with $V_{\sigma+1}/V_\sigma \in \mathcal{L}$ for $\sigma < \lambda$, $V_\lambda = G$ and

$$V_\rho = \bigcup_{\sigma<\rho} V_\sigma$$

for limit ordinals $\rho < \lambda$.

If the subgroups V_σ are normal in G, for all $\sigma \in \Omega$, then we say that G has an <u>ascending invariant $\mathcal{L}$-series</u>.

Suppose now that Ω is inversely well ordered. Then we say that a $\mathcal{L}$-series of type Ω is a <u>descending $\mathcal{L}$-series</u>. Let the inverse ordering to Ω have order type λ where λ is an ordinal. We can write a descending $\mathcal{L}$-series in the form

$$(\Lambda_\sigma : \sigma < \lambda)$$

where $\Lambda_0 = G$, $\Lambda_{\sigma+1}$ is normal in Λ_σ and $\Lambda_\sigma/\Lambda_{\sigma+1} \in \mathcal{L}$ for $\sigma < \lambda$, $\Lambda_\lambda = 1$ and

$$\Lambda_\rho = \bigcap_{\sigma<\rho} \Lambda_\sigma$$

for limit ordinals $\rho < \lambda$.

Groups having an $\mathcal{A}$-series will be of special interest in what follows and we define the following classes.

1.2.3 <u>Definition</u> Let $\mathfrak{X}$, $\mathfrak{X}_A$, $\mathfrak{X}_D$ be the classes of groups having an $\mathcal{A}$-series, an ascending $\mathcal{A}$-series, and a descending $\mathcal{A}$-series, respectively. In the Kurosh notation

$$\mathfrak{X} = SN; \quad \mathfrak{X}_A = SN^*; \quad \text{and } \mathfrak{X}_D = SD .$$

1.2.4 <u>Definition</u> Let $\mathfrak{Y}$, $\mathfrak{Y}_A$, be the classes of groups having an invariant $\mathcal{A}$-series, and an ascending invariant

$\mathfrak{A}$-series, respectively. In the Kurosh notation

$$\mathcal{Y} = SI, \quad \text{and} \quad \mathcal{Y}_A = SI^*.$$

We now introduce the concept of a central series. If H and K are subgroups of a group G we denote by

$$[H , K]$$

the subgroup of G generated by all elements of the form

$$[h , k] = h^{-1}k^{-1}hk$$

where $h \in H$ and $k \in K$.

1.2.5 <u>Definition</u> The group G has a <u>central series</u> if G has an $\mathfrak{A}$-series

$$(\Lambda_\lambda , V_\lambda : \lambda \in \Omega)$$

with $[\Lambda_\lambda , G] < V_\lambda$, for all $\lambda \in \Omega$.

Then every central series is an invariant series.

1.2.6 <u>Definition</u> Let $\mathfrak{Z}$, $\mathfrak{Z}_A$, $\mathfrak{Z}_D$ be the classes of groups having a central series, an ascending central series, and a descending central series, respectively.

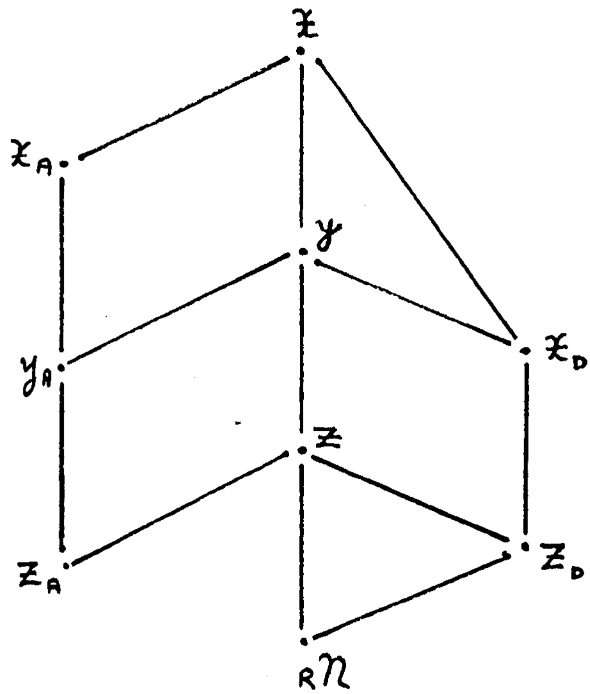
The lower central series of a group G is the descending central series with terms $\gamma_\lambda(G)$ where

$$\gamma_0(G) = G$$
$$\gamma_{\lambda+1}(G) = [\gamma_\lambda(G) , G]$$
$$\gamma_\rho(G) = \bigcap_{\lambda<\rho} \gamma_\lambda(G) \quad \text{for limit ordinals } \rho.$$

In 1.1.8 we defined the closure operation R. We shall be particularly interested in the class $R\mathfrak{N}$ of residually nilpotent groups. This class can be characterised by

$G \in {}_R\mathfrak{N}$ if and only if $\gamma_\omega(G) = 1$.

We can now indicate the inclusion relations between the classes defined above.



We end this section by defining one further closure operation.

1.2.7 <u>Definition</u> Let $\mathfrak{L}$ be any class of groups. Define the closure operation E by

$G \in {}_E\mathfrak{L}$ if and only if G has a finite $\mathfrak{L}$-series.

We note that the class ${}_E\mathfrak{A}$ is the class of soluble groups.

1.3 The McLain groups $T_\Omega(p)$.

We give the definition of the McLain groups $T_\Omega(p)$ (see [9]) and consider some properties of these groups.

Let Ω be a linearly ordered set. Let V be the vector space over the Galois field GF(p) with basis v_λ for $\lambda \in \Omega$. For any $\lambda, \mu \in \Omega$ with $\lambda < \mu$, we define the non-singular linear map $t_{\lambda\mu}: V \longrightarrow V$ by

$$t_{\lambda\mu}: v_\lambda \longrightarrow v_\lambda + v_\mu ,$$

$$t_{\lambda\mu}: v_\nu \longrightarrow v_\nu , \quad \nu \neq \lambda.$$

Let $T_\Omega(p) = \langle t_{\lambda\mu} : \lambda, \mu \in \Omega \text{ with } \lambda < \mu \rangle$. If Ω is a finite set, say $\Omega = \{1, 2, \ldots\ldots n\}$, then $T_\Omega(p)$ has order $p^{n(n-1)/2}$ and is isomorphic to the group of $n \times n$ unitriangular matrices over GF(p). Hence if $\Omega = \{1, \ldots\ldots n\}$, then $T_\Omega(p)$ is isomorphic to the Sylow p-subgroup of the general linear group over GF(p); and $T_\Omega(p)$, for arbitrary Ω, is a generalisation of this Sylow p-subgroup. We quote the following well known properties of McLain groups:-

1.3.1 For any linearly ordered set Ω, $T_\Omega(p) \in L\mathcal{Y}_p$.

1.3.2 For any linearly ordered set Ω, the normal closure of $\langle t_{\lambda\mu} \rangle$ in $T_\Omega(p)$ is an elementary Abelian p-group, and so $T_\Omega(p) \in N\mathcal{a}$.

1.3.3 If Ω is a linearly ordered set isomorphic to the rationals, then $T_\Omega(p)$ is a characteristically simple group.

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1.3.4 For any linearly ordered set Ω , $T_\Omega(p) \in \mathcal{Y}_A$.

1.3.5 The centre of $T_\Omega(p)$ is trivial unless Ω has both a greatest and a least element.

We see that McLain groups have strong generalised nilpotent properties and we consider in the next section a class of groups $\mathfrak{R}$ which could be considered as a generalisation of the McLain groups $T_\Omega(p)$.

§2 Generalisation of the Sylow p-subgroup of $GL(m,p^n)$.

Let p be a prime and let n be an integer with $n \geq 1$. Inthis section we determine the Sylow p-subgroup of the group of non-singular mxm matrices over the ring $\mathbb{Z}_{p^n}$. We then generalise this Sylow p-subgroup to obtain a class of groups $\mathcal{R}$.

2.1 The general linear group over the ring $\mathbb{Z}_{p^n}$.

Consider the group of non-singular mxm matrices over the ring $\mathbb{Z}_{p^n}$. Denote this group by $GL(m,p^n)$. We determine the order of this group, see [4].

2.1.1 Theorem $\quad |GL(m,p^n)| = p^{nm^2 - m(m+1)/2} \prod_{i=1}^{m} (p^i - 1)$.

Proof Let $\quad A = \begin{bmatrix} a_1 & a_2 & \cdots\cdots & a_r \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$

be a matrix with coefficients in the ring $R = \mathbb{Z}_{p^n}$. First, we show that if A is non-singular, then

$$Ra_1 + Ra_2 + \cdots\cdots + Ra_r = R.$$

Suppose not. Then $Ra_1 + Ra_2 + \cdots\cdots + Ra_r < R$. No a_i, $1 \leq i \leq r$, can be prime to p; for, if a_i is prime to p, then a_i^{-1} exists and $Ra_i = R.$

Hence $$a_i = pa_i', \quad 1 < i < r.$$

But then $|A| = p|A'|$, where

$$A' = \begin{bmatrix} a_1' & a_2' & \cdots\cdots & a_r' \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix},$$

a matrix with first row $(a_1', a_2', \ldots\ldots, a_r')$ and all other entries as in A.

Since $|A| = p|A'|$, then $|A|^n = 0$ and A is singular.

Hence $$Ra_1 + Ra_2 + \ldots\ldots + Ra_r = R.$$

Conversely, we show that if

$$Ra_1 + Ra_2 + \ldots\ldots + Ra_r = R,$$

then there exists a non-singular matrix A with $(a_1, a_2, \ldots, a_r)$ as a first row. At least one of the a_i, $1<i<r$, must be prime to p; for if not, then every element of $Ra_1 + \ldots\ldots + Ra_r$ is divisible by p, giving $Ra_1 + \ldots\ldots + Ra_r < R$.

Suppose a_i is prime to p. Let A be the matrix with $(a_1, a_2, \ldots, a_r)$ as the first row and $(a_i, 0, 0, \ldots, 0)$ as the i-th column, and such that the matrix obtained when the first row and the i-th column are deleted is the $(r-1)\times(r-1)$ identity. Then $|A| = \pm a_i$, and so A is non-singular.

Suppose $a_1, a_2, \ldots, a_r \in R$ with $Ra_1 + \ldots + Ra_r = R$. Then let B be a fixed $r \times r$ non-singular matrix with first row $(a_1, a_2, \ldots, a_r)$. Thus every non-singular matrix A with first row $(a_1, a_2, \ldots, a_r)$ may be written uniquely in the form

$$A = CB,$$

where C is a matrix of the form

$$\begin{bmatrix} 1 & 0 \; 0 \; \cdots\cdots \; 0 \\ x_{21} & \\ \vdots & \begin{bmatrix} & c_{ij} & \end{bmatrix} \\ x_{r1} & \end{bmatrix},$$

where (c_{ij}) is a non-singular $(r-1)\times(r-1)$ matrix.

This follows at once, for $BB^{-1} = I$ and AB^{-1} must have the same first row as BB^{-1}, since A has the same first row as B.

Suppose that there are $\varphi_r(p^n)$ choices of $(a_1, \ldots, a_r)$ as the first row of a matrix in $GL(r, p^n)$. We can choose $x_{21}, \ldots, x_{r1}$ in $p^{n(r-1)}$ ways; and since (c_{ij}) is a non-singular $(r-1)\times(r-1)$ matrix, we can choose it in $|GL(r-1, p^n)|$ ways. Thus

$$|GL(m, p^n)| = \varphi_m(p^n) p^{n(m-1)} |GL(m-1, p^n)|.$$

We must calculate $\varphi_r(p^n)$. Let

$$Ra_1 + Ra_2 + \ldots + Ra_r = R. \qquad \text{——} \quad (1)$$

There are p^{rn} ways of choosing $(a_1, a_2, \dots, a_r)$ and $p^{r(n-1)}$ ways of choosing $(a_1, a_2, \dots, a_r)$ such that $a_i = pa_i'$, $1 \leq i \leq r$.

Thus there are $p^{nr} - p^{r(n-1)} = p^{r(n-1)}(p^r - 1)$ ways of choosing $(a_1, \dots, a_r)$ to satisfy (1); and so

$$\varphi_r(p^n) = p^{r(n-1)}(p^r - 1).$$

Then
$$|GL(m,p^n)| = \prod_{i=1}^{m} p^{i(n-1)} p^{n(i-1)}(p^i - 1)$$
$$= p^{m(m+1)(n-1)/2 + nm(m-1)/2} \prod_{i=1}^{m} (p^i - 1)$$
$$= p^{nm^2 - m(m+1)/2} \prod_{i=1}^{m} (p^i - 1).$$

2.1.2 Corollary The order of the Sylow p-subgroup of $GL(m,p^n)$ is $p^{nm^2 - m(m+1)/2}$.

We now find a Sylow p-subgroup of $GL(m,p^n)$. Let I denote the $m \times m$ identity and E_{ij} denote the matrix with 1 in the (i,j)th position and zeros in all other places.

Consider the subgroup

$$\langle I + E_{ij},\ I + pE_{kl} : i < j,\ k \geq l \rangle.$$

The generators of the form $I + E_{ij}$ have order p^n;

while those of the form $I + pE_{ij}$ also have order a power of p. All these generators clearly lie in the subset of all mxm matrices of the form $I + pA + B$ where A is any matrix, and B is a matrix with zeros on and below the main diagonal. We show that the matrices of the form $I + pA + B$ give a Sylow p-subgroup of $GL(m,p^n)$.

$$(I + pA_1 + B_1)(I + pA_2 + B_2)$$

$$= I + p(A_1 + A_2 + pA_1A_2 + A_1B_2 + B_1A_2) + (B_2 + B_1 + B_1B_2).$$

Since every element has finite order, as elements of $GL(m,p^n)$, then matrices of this form give a subgroup of $GL(m,p^n)$. The order of this subgroup is easily seen to be

$$p^{nm(m-1)/2}\ p^{(n-1)(m+1)m/2}$$

$$= p^{nm^2 - m(m+1)/2}$$

By corollary 2.1.2, we see that the set $\{I + pA + B\}$ is a Sylow p-subgroup of $GL(m,p^n)$ and in fact this subgroup is generated by

$$I + E_{ij}\ ,\ i < j,$$

$$I + pE_{ij}\ ,\ i > j.$$

The proof that this is a generating set is straight forward but tedious, and as we shall prove a more general result later (see 2.3.4), we omit the proof here.

We have seen above that the Sylow p-subgroup of $GL(m,p^n)$ is isomorphic to the set of mxm matrices which, modulo $p\mathbb{Z}_{p^n}$ are unitriangular. We shall generalise this by considering matrices, over an arbitrary ring, that are unitriangular modulo the Jacobson radical of the ring.

In the next section we give the basic definitions and results on the Jacobson radical of a ring (see [4]).

2.2 The Jacobson radical of a ring.

Let R be an arbitrary (not necessarily commutative) ring with a 1. Assume in all that follows that $1 \neq 0$. We state the following result, see [4].

2.2.1 Lemma Let R be a ring with a 1. Then the intersection of the maximal right ideals of R is equal to the intersection of the maximal left ideals of R.

We now define the Jacobson radical of a ring with a 1.

2.2.2 Definition Let R be a ring with a 1. We call the intersection of the maximal right ideals the Jacobson radical of R and denote it by $J(R)$.

By lemma 2.2.1, $J(R)$ is also the intersection of the maximal left ideals and hence $J(R)$ is a two-sided ideal of R.

We now introduce the idea of right and left quasi-regular elements.

2.2.3 <u>Definition</u> Let R be an arbitrary ring. Then $x \in R$ is called a <u>right quasi-regular element</u> if there exists a $y \in R$ with

$$x + y + xy = 0.$$

The element y is called ~~the~~ a right <u>quasi-inverse</u> of x.

Similarly $x \in R$ is called a <u>left quasi-regular element</u> if there exists a $y \in R$ with

$$x + y + yx = 0.$$

The element y is called ~~the~~ a left <u>quasi-inverse</u> of x.

Notice that if an element has both a right and left quasi-inverse, these quasi-inverses are equal.

2.2.4 <u>Lemma</u> Let R be a ring with a 1. Then the elements of $J(R)$ are right and left quasi-regular.

We note that, if I is a two-sided ideal of R satisfying

$x \in I$ implies $x^n = 0$ for some integer n,

then $I < J(R)$. In particular $J(R)$ contains all the locally nilpotent two-sided ideals of R.

2.2.5 <u>Definition</u> We call a ring R a <u>radical</u> ring if every element of R is right and left quasi-regular.

We can now show that any radical ring R can be embedded in a ring R^* with a 1 so that $R = J(R^*)$. This result is proved in [4]. We give here a different proof.

2.2.6 <u>Lemma</u> Any radical ring R can be embedded in a ring R^* so that R^* has a 1 and R is the Jacobson radical of R^*.

<u>Proof</u> Let R be a radical ring. Define $R \times \mathbb{Z} = R^*$ to be the ring of ordered pairs (r,n), $r \in R$, $n \in \mathbb{Z}$ with

$$(r_1,n_1) + (r_2,n_2) = (r_1 + r_2, n_1 + n_2)$$

$$(r_1,n_1)(r_2,n_2) = (r_1r_2 + n_1r_2 + n_2r_1, n_1n_2).$$

Now R^* has an identity element $(0,1)$.

We show that $R \leq J(R^*)$. If not, then there exists some element $(r,0) \notin J(R^*)$, where $r \in R$. Then there exists a maximal right ideal M_R say, not containing $(r,0)$. Hence

$$(r,0)R^* + M_R = R^*.$$

So $(r,0)(r_1,n_1) + (0,1) \in M_R$, for some $r_1 \in R$, $n_1 \in \mathbb{Z}$.
Hence $(r',1) \in M_R$, where $r' = rr_1 + n_1r \in R$.
Let s be the right quasi-inverse of r'; that is

$$r' + s + r's = 0.$$

Then $(r,1)(s,1) = (0,1) \in M_R$.
This is a contradiction and so $R \leq J(R^*)$.

Now suppose $R < J(R^*)$. Then there exists $(r,n) \in J(R^*)$ with $n \neq 0$. We show that there is a maximal right ideal not containing (r,n).

Let p be any prime not dividing n. Then $(R,p\mathbb{Z})$ is a maximal right ideal which clearly does not contain (r,n). Hence $R = J(R^*)$ and the proof is complete.

We can now define the class of groups $\mathcal{R}$.

2.3 <u>Definition of the class $\mathcal{R}$.</u>

Let R be a (not necessarily commutative) ring with a 1. Let J be the Jacobson radical of R. Suppose I is any two-sided ideal of R that is contained in J. Let Ω be any linearly ordered set and let V be the free module over R with basis v_λ, for $\lambda \in \Omega$.

Define non-singular linear maps $V \longrightarrow V$ as follows.

For any $\lambda,\mu \in \Omega$, with $\lambda < \mu$, and for any $r \in R$ define

$$t_{\lambda\mu}(r) : v_\lambda \longrightarrow v_\lambda + rv_\mu ,$$

$$t_{\lambda\mu}(r) : v_\nu \longrightarrow v_\nu , \quad \nu \neq \lambda.$$

Since $t_{\lambda\mu}(r)$ is a linear map,

$$t_{\lambda\mu}(r) : sv_\lambda \longrightarrow sv_\lambda + srv_\mu \ , \quad \text{where } s \in R.$$

For any $\alpha, \beta \in \Omega$ with $\alpha > \beta$ and for any $a \in I$ define

$$r_{\alpha\beta}(a) : v_\alpha \longrightarrow v_\alpha + av_\beta \ ,$$

$$r_{\alpha\beta}(a) : v_\nu \longrightarrow v_\nu \ , \quad \nu \neq \alpha.$$

Since $r_{\alpha\beta}(a)$ is a linear map,

$$r_{\alpha\beta}(a) : sv_\alpha \longrightarrow sv_\alpha + sav_\beta \ , \quad \text{where } s \in R.$$

It is easy to see that the map $t_{\lambda\mu}(r)$ has inverse $t_{\lambda\mu}(-r)$, for $\lambda < \mu$ and $r \in R$. The map $r_{\alpha\beta}(a)$ has inverse $r_{\alpha\beta}(-a)$ for $\alpha > \beta$ and $a \in I$.

We now prove

2.3.1 <u>Lemma</u> $r_{\alpha\alpha}(a)$ has an inverse $r_{\alpha\alpha}(b)$, for all $\alpha \in \Omega$, $a \in I$, where $b \in I$.

<u>Proof</u> $r_{\alpha\alpha}(a) : v_\alpha \longrightarrow v_\alpha + av_\alpha = (1 + a)v_\alpha$.

We must find $1+b$ so that

$$(1 + b)(1 + a) = 1 \ ,$$

where $b \in I$.

The element a has a left quasi-inverse b, by 2.2.4, since $a \in J$. Then

$$(1 + b)(1 + a) = 1 + a + b + ba = 1.$$

But $b = -a - ba \in I$, since a is in the two-sided ideal I.

Now a has a right quasi-inverse, by 2.2.4, and this must also be b.

Hence $r_{\alpha\alpha}(b)$ is the inverse of $r_{\alpha\alpha}(a)$.

Now we have shown that the maps

$$\{ t_{\lambda\mu}(r) : \lambda < \mu,\ \lambda,\mu \in \Omega,\ r \in R \}$$

and $$\{ r_{\alpha\beta}(a) : \alpha > \beta\ ,\ \alpha,\beta \in \Omega,\ a \in I \}$$

all have inverses so we can define $\mathcal{R}(\Omega:R:I)$ to be the group of maps $V \longrightarrow V$ generated by

$$\{ t_{\lambda\mu}(r), r_{\alpha\beta}(a) : \lambda,\mu,\alpha,\beta \in \Omega,\ \lambda < \mu,\ \alpha > \beta,\ r \in R,\ a \in I \}.$$

It will sometimes be convenient to think of the elements of $\mathcal{R}(\Omega:R:I)$ as matrices whose rows and columns are indexed by Ω, that is $\Omega\times\Omega$ matrices.

We now define subgroups of $\mathcal{R}(\Omega:R:I)$ as follows. Let I_1 be an ideal of R contained in I. From now on we shall mean "two-sided ideal" when we write "ideal". Define $\mathcal{N}(\Omega:R:I_1)$ to be the group of maps $V \longrightarrow V$ generated by

$$\{ t_{\lambda\mu}(a_1), r_{\alpha\beta}(a_1) : \lambda,\mu,\alpha,\beta \in \Omega,\ \lambda < \mu,\ \alpha > \beta,\ a_1 \in I_1 \}.$$

Clearly $\mathcal{N}(\Omega:R:I_1) < \mathcal{R}(\Omega:R:I)$ and, after proving a useful lemma about 2x2 matrices, we show it is a normal subgroup.

We shall call an element of a ring R with both a left and right inverse, a <u>unit</u> of R.

2.3.2 <u>Lemma</u> Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

be an arbitrary 2x2 matrix over the ring R. Suppose R has a 1 and d and $a - bd^{-1}c$ are units of R. Then we can decompose A in the following way.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & bd^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ c(a-bd^{-1}c)^{-1} & 1 \end{bmatrix} \begin{bmatrix} a-bd^{-1}c & 0 \\ 0 & d \end{bmatrix}$$

<u>Proof</u> The result is easily obtained by multiplying the matrices on the right hand side of the equation.

2.3.3 <u>Lemma</u> Let R be any ring with a 1 and with Jacobson radical J. Let I, I_1 be two ideals such that

$$I_1 \leqslant I \leqslant J.$$

Then $\mathcal{N}(\Omega:R:I_1)$ is normal in $\mathcal{R}(\Omega:R:I)$;

and in particular

$$\mathcal{N}(\Omega:R:I) \text{ is normal in } \mathcal{R}(\Omega:R:I).$$

<u>Proof</u> If I_1 is an ideal of R, then I_1V is a characteristic submodule of V and so the centralizer C of V/I_1V in the automorphism group of V is normal in this automorphism group. By lemma 2.3.4, we see that

$$\mathcal{N}(\Omega:R:I_1) = C \cap \mathcal{R}(\Omega:R:I),$$

hence $\mathcal{N}(\Omega:R:I_1)$ is normal in $\mathcal{R}(\Omega:R:I)$.

2.3.4 Lemma Let R be a ring with a 1 and let I_1 and I_2 be two ideals, I_2 being contained in the Jacobson radical J of the ring R. Let A be an $n{\times}n$ matrix of the form

$$A = \begin{bmatrix} 1+b_{11} & a_{12} & a_{13} & \cdots\cdots & a_{1n} \\ b_{21} & 1+b_{22} & a_{23} & \cdots\cdots & a_{2n} \\ \vdots & \vdots & & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & \cdots\cdots & 1+b_{nn} \end{bmatrix},$$

where $b_{ij} \in I_2$, $a_{ij} \in I_1$.

Then A can be written as a product of a finite number of matrices of the form $I + aE_{ij}$, $i < j$, $a \in I_1$, and $I + bE_{ij}$, $j < i$, $b \in I_2$.
E_{ij} denotes the nxn matrix with 1 in the (i,j)th position and zeros everywhere else, while I is the nxn identity matrix.

<u>Proof</u> We reduce A to an upper triagular matrix by multiplying A on the right by matrices X_i of the form

$$\begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & \bigcirc & \\ \vdots & \vdots & \ddots & & & \\ x_{i1} & x_{i2} & \cdots & x_{i,i-1} & 1 & \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & \cdots & \cdots & \cdots & 1 \end{bmatrix}$$

Define X_n by putting $x_{nj} = -(1+b_{nn})^{-1} b_{nj} \in I_2$, $1 \leq j \leq n-1$.

Then $$AX_n = \begin{bmatrix} 1+b'_{11} & & & a'_{ij} \\ & 1+b'_{22} & & \\ & b'_{ij} & \ddots & \\ 0 & 0 & \cdots\cdots\; 0 & 1+b'_{nn} \end{bmatrix},$$

where $a'_{ij} \in I_1$, $b'_{ij} \in I_2$.

Now use induction on l decreasing. Suppose we can find $X_n, X_{n-1}, \ldots\ldots, X_l$ with the required form so that

$$AX_n \ldots\ldots X_l = \begin{array}{r c} & \begin{bmatrix} 1+b''_{11} & & & \\ & 1+b''_{22} & a''_{ij} & \\ & b''_{ij} & \ddots & \\ & & & \\ & O & & 1+b''_{nn} \end{bmatrix} \end{array},$$

$(l-1)$th row $\rightarrow$
lth row $\rightarrow$

where $a''_{ij} \in I_1$, $b''_{ij} \in I_2$.

Now define X_{l-1} by putting

$$x_{l-1,j} = -(1+b''_{l-1,l-1})^{-1} b_{l-1,j} \in I_2, \text{ for } 1<j<l-2.$$

Then $AX_n \ldots\ldots X_l X_{l-1}$ is a matrix with zeros in the $(l-1)$th row to n-th row up to the main diagonal, otherwise with the same form as the matrix $AX_n \ldots\ldots X_l$.

Let $X = X_n \ldots\ldots X_2$. Then AX is a matrix of the form

$$\begin{bmatrix} 1+\bar{b}_{11} & & \bar{a}_{ij} & \\ & 1+\bar{b}_{22} & & \\ & O & \ddots & \\ & & & 1+\bar{b}_{nn} \end{bmatrix} .$$

Let $\bar{B} =$

$$\begin{bmatrix} (1+\bar{b}_{11})^{-1} & & & O \\ & (1+\bar{b}_{22})^{-1} & & \\ & & \ddots & \\ O & & & (1+\bar{b}_{nn})^{-1} \end{bmatrix} .$$

Then $AX\bar{B} =$

$$\begin{bmatrix} 1 & & \bar{a}_{ij} & \\ & 1 & & \\ & & 1 & \\ O & & & \ddots \ 1 \end{bmatrix} ,$$

where $\bar{a}_{ij}$ are elements of I_1.

We can now define matrices $Y_1, \ldots, Y_{n-1}$ of the form

$$Y_l = \begin{array}{c} \begin{bmatrix} 1 & 0 & & & & O & \\ & 1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & y_{l,l+1} & \cdots & y_{l,n} \\ & & & & \ddots & O & \\ & O & & & & & \\ & & & & & & 1 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow l\text{-th row} \\ \\ \\ \\ \end{array}$$

so that $y_{1j} \in I_1$ and $AX\bar{B}Y_1Y_2\ldots Y_{n-1} = I$.

This can be done in a simpler manner than the definition of the X_i's above.

Clearly X_i is a product of elements of the form $I + bE_{jk}$, $j > k$, while Y_i is a product of elements of the form $I + aE_{jk}$, $j < k$. The matrix $\bar{B}$ is a product of matrices $I + bE_{ii}$ using lemma 2.3.1. This completes the proof of the lemma.

2.3.5 <u>Definition</u> Let R be a ring with a 1 having Jacobson radical J. We define the groups $\mathcal{R}(\Omega:R)$ and $\mathcal{N}(\Omega:R)$ by

$$\mathcal{R}(\Omega:R) = \left\langle t_{\lambda\mu}(r), r_{\alpha\beta}(a) : \lambda<\mu,\ \alpha>\beta,\ r \in R,\ a\in J\right\rangle.$$

$$\mathcal{N}(\Omega:R) = \left\langle t_{\lambda\mu}(a), r_{\alpha\beta}(a) : \lambda<\mu,\ \alpha>\beta,\ a \in J\right\rangle.$$

Clearly we have

$$\mathcal{R}(\Omega:R) = \mathcal{R}(\Omega:R:J),$$

$$\mathcal{N}(\Omega:R) = \mathcal{N}(\Omega:R:J).$$

Now every element of $\mathcal{R}(\Omega:R)$ can be considered as an $\Omega\times\Omega$ matrix differing from the identity matrix in only a finite number of positions while modulo the Jacobson radical of R every matrix is unitriangular. We deduce immediately from lemma 2.3.4 that every $\Omega\times\Omega$ matrix of the above form can be generated by the generators of $\mathcal{R}(\Omega:R)$ and so every $\Omega\times\Omega$ matrix of this form can be considered as an element of $\mathcal{R}(\Omega:R)$.

Let $\Omega_m = \{1,2,\ldots,m\}$ and let $R = \mathbb{Z}_{p^n}$. Now $p\mathbb{Z}_{p^n}$ is the unique maximal ideal of $\mathbb{Z}_{p^n}$, so that the Jacobson radical $J(\mathbb{Z}_{p^n})$ is $p\mathbb{Z}_{p^n}$. Combining the results of section 2.1 with lemma 2.3.4, we have

$$\mathcal{R}(\Omega_m:\mathbb{Z}_{p^n}) \cong \text{Sylow } p\text{-subgroup of } GL(m,p^n).$$

Suppose $R = \mathbb{Z}_p$. Then the Jacobson radical of R is zero and so

$$\mathcal{R}(\Omega:\mathbb{Z}_p) \cong T_\Omega(p),$$

where $T_\Omega(p)$ is the McLain group defined in section 1.3. Hence $\mathcal{R}(\Omega:R)$ can be considered both as a generalisation of the Sylow p-subgroup of $GL(m,p^n)$ and as a generalisation of the McLain group $T_\Omega(p)$.

§3 Generalised nilpotent properties of $\mathcal{R}(\Omega:R)$ and $\mathcal{N}(\Omega:R)$.

We investigate the groups $\mathcal{R}(\Omega:R)$ and $\mathcal{N}(\Omega:R)$ when the ring R has a nilpotent Jacobson radical J.

3.1 The groups $\mathcal{R}(\Omega:R)$ and $\mathcal{N}(\Omega:R)$ when J(R) is nilpotent.

We begin by studying the commutators of generators of $\mathcal{R}(\Omega:R:I)$.

3.1.1 Lemma The generators of $\mathcal{R}(\Omega:R:I)$ satisfy the following commutator relations, where $r_i \in R$, $a_i \in I$, and b_i is the quasi-inverse of a_i. The list covers all possibilities up to inversion.

(i) $[t_{\lambda\mu}(r_1), t_{\mu\rho}(r_2)] = t_{\lambda\rho}(r_1 r_2)$

(ii) $[t_{\lambda\mu}(r_1), t_{\rho\sigma}(r_2)] = 1$, $\mu\neq\rho$, $\lambda\neq\sigma$.

(iii) $[r_{\lambda\mu}(a_1), r_{\mu\rho}(a_2)] = r_{\lambda\rho}(a_1 a_2)$, $\lambda\neq\mu$ or ρ, $\rho\neq\mu$.

(iv) $[r_{\lambda\mu}(a_1), r_{\rho\sigma}(a_2)] = 1$, $\mu\neq\rho$, or λ , $\sigma\neq\rho$ or λ.

(v) $[r_{\lambda\mu}(a_1), r_{\mu\mu}(a_2)] = r_{\lambda\mu}(a_1 a_2)$, $\lambda\neq\mu$

(vi) $[r_{\lambda\mu}(a_1), r_{\lambda\lambda}(a_2)] = r_{\lambda\mu}(b_2 a_1)$, $\lambda\neq\mu$

(vii) $[r_{\lambda\lambda}(a_1), r_{\mu\mu}(a_2)] = 1$, $\lambda\neq\mu$

(viii) $[r_{\lambda\lambda}(a_1), r_{\lambda\lambda}(a_2)]$

$$= r_{\lambda\lambda}(b_2 a_1 - b_1 a_2 + b_2 a_1 a_2 + b_1 b_2 a_1 + b_1 b_2 a_1 a_2)$$

(ix) $[r_{\lambda\mu}(a_1), r_{\mu\lambda}(a_2)]$

$$= r_{\lambda\mu}(a_1a_2a_1(1-a_2a_1)^{-1})r_{\mu\lambda}(-a_2a_1a_2(1-a_1a_2))r_{\lambda\lambda}^{-1}(-a_1a_2)r_{\mu\mu}(-a_2a_1)$$

where $\lambda \neq \mu$.

(x) $[t_{\lambda\mu}(r_1), r_{\mu\sigma}(a_2)] = r_{\lambda\sigma}(r_1a_2)$ if $\lambda>\sigma$, $\sigma\neq\lambda$ or μ.

$t_{\lambda\sigma}(r_1a_2)$ if $\lambda<\sigma$, $\sigma\neq\lambda$ or μ.

(xi) $[t_{\lambda\mu}(r_1), r_{\mu\mu}(a_2)] = t_{\lambda\mu}(r_1a_2)$

(xii) $[t_{\lambda\mu}(r_1), r_{\lambda\lambda}(a_2)] = t_{\lambda\mu}(b_2r_1)$

(xiii) $[t_{\lambda\mu}(r_1), r_{\rho\sigma}(a_2)] = 1$, $\mu \neq \rho$, $\lambda \neq \sigma$.

(xiv) $[t_{\lambda\mu}(r_1), r_{\mu\lambda}(a_2)]$

$$= t_{\lambda\mu}(r_1a_2r_1(1-a_2r_1)^{-1})r_{\mu\lambda}(-a_2r_1a_2(1-r_1a_2))r_{\lambda\lambda}^{-1}(-r_1a_2)r_{\mu\mu}(-a_2r_1).$$

A non trivial commutator can only occur when the inside two subscripts are equal or the two outside subscripts are equal.

Proof All these relations can be checked by applying the maps to the module V. Since the generators are $\Omega\times\Omega$ matrices differing from the identity in only one position, we can compute commutators using only 4x4 matrices.

As an example we prove relation (xiv) where we need only 2x2 matrices.

$$\begin{bmatrix} 1 & -r_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & r_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a_2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+r_1a_2+r_1a_2r_1a_2 & r_1a_2r_1 \\ -a_2r_1a_2 & 1-a_2r_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & r_1a_2r_1(1-a_2r_1)^{-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -a_2r_1a_2(1-r_1a_2) & 1 \end{bmatrix} \begin{bmatrix} (1-r_1a_2)^{-1} & 0 \\ 0 & (1-a_2r_1) \end{bmatrix}$$

Let R be a ring and $M_n(R)$ the ring of nxn matrices over R.

Now $\quad J(M_n(R)) = M_n(J(R))$.

For a proof of this result see [8].

Hence, if $A = (a_{ij})$ is an nxn matrix with $a_{ij} \in J(R)$, $I_n + A$ is invertible, for, $A \in J(M_n(R))$.
We use this fact in the following lemmas where we investigate the central structure of $\mathcal{N}(\Omega:R:I)$.

3.1.2 Lemma Let R be a ring with a 1 and let I be an ideal contained in the Jacobson radical of R. Let A be an nxn matrix (a_{ij}) where $a_{ij} \in I^r$, for some integer r, and let B be an nxn matrix (b_{ij}), where $b_{ij} \in I$. Then

$$[\, I_n + A,\ I_n + B\,] = I_n + H \quad ,$$

where $H = (h_{ij})$ is an nxn matrix with $h_{ij} \in I^{r+1}$.

<u>Proof</u> Let $(I_n+A)^{-1} = I_n+C$ and $(I_n+B)^{-1} = I_n+D$. Then $C = (c_{ij})$, where $c_{ij} \in I^r$, and $D = (d_{ij})$, where $d_{ij} \in I$.

Now $[I_n+A, I_n+B] = (I_n+C)(I_n+D)(I_n+A)(I_n+B)$

$$= (I_n+C+D+CD)(I_n+A+B+AB)$$

But $(I_n+A)(I_n+C) = I_n+A+C+AC = I_n = I_n+A+C+CA$,

and $(I_n+B)(I_n+D) = I_n+B+D+BD = I_n = I_n+B+D+DB$.

Hence $[I_n+A, I_n+B]$

$= I_n+(A+C+CA)+(B+D+DB)+C(B+AB+DB+DA+DAB)+DA+AB+CD+DAB$

$= I_n+CX+DA+DAB+AB+CD$, where $X = B+AB+DA+DB+DAB$.

Now $X = (x_{ij})$, where $x_{ij} \in I$, and so $CX = (y_{ij})$, $y_{ij} \in I^{r+1}$.

Similarly DA, DAB, AB and CD are nxn matrices with coefficients in I^{r+1}; and so writ ing

$$H = CX+DA+DAB+AB+CD,$$

we have the required result.

We make the following definitions.

3.1.3 Definition Let I be any ideal of the ring R and define $\mathcal{E}(I)$ to be the set of all linear maps

$$e : V \longrightarrow V$$

such that there exists a finite set $\lambda_1, \lambda_2, \ldots, \lambda_n \in \Omega$ with

$$e : v_{\lambda_j} \longrightarrow \sum_{i=1}^{n} a_i^{(j)} v_{\lambda_i} \quad , \text{ where } a_i^{(j)} \in I,$$

for $1 \leqslant j \leqslant n$,

$$e : v_\mu \longrightarrow 0 \quad , \quad \text{for } \mu \neq \lambda_i,\ i=1,2,\ldots,n\ ,\ \mu \in \Omega.$$

It is easy to see that the set $\mathcal{E}(I)$ is a ring under the usual addition and composition of linear maps.

Define the set $1+\mathcal{E}(I)$ to be the set of elements of the form

$$1 + x \quad , \qquad x \in \mathcal{E}(I).$$

This becomes a semi-group under the multiplication

$$(1 + x)(1 + y) = 1 + x + y + xy,$$

where $x, y \in \mathcal{E}(I)$.

If $I \leqslant J(R)$, then $1+\mathcal{E}(I)$ is a group, for, if $x \in \mathcal{E}(I)$ we can consider x as an n×n matrix over I, for some n. Then $x \in J(M_n(R))$ and $1+x$ is invertible.

Every map in the group $\mathcal{R}(\Omega{:}R{:}I)$ is of the form

$1+e$, for $e \in \mathcal{E}(R)$, and so

$$\mathcal{R}(\Omega:R:I) < 1 + \mathcal{E}(R).$$

Every map in the group $\mathcal{N}(\Omega:R:I)$ is of the form $1+e$, for $e \in \mathcal{E}(I)$, and so

$$\mathcal{N}(\Omega:R:I) < 1 + \mathcal{E}(I).$$

In the next lemma we show that in fact we have equality in this last relation.

3.1.4 <u>Lemma</u> $\mathcal{N}(\Omega:R:I) = 1 + \mathcal{E}(I)$.

<u>Proof</u> Since $\mathcal{N}(\Omega:R:I) < 1 + \mathcal{E}(I)$, it remains to show that if $x \in \mathcal{E}(I)$, then $1+x \in \mathcal{N}(\Omega:R:I)$.

This is an immediate consequence of lemma 2.3.4.

We are now in a position to study the central structure of $\mathcal{N}(\Omega:R:I)$.

3.1.5 <u>Theorem</u> The group $\mathcal{N}(\Omega:R:I)$ has a descending central series

$$\mathcal{N}(\Omega:R:I) > \mathcal{N}(\Omega:R:I^2) > \ldots\ldots > \mathcal{N}(\Omega:R:I^n) > \ldots\ldots\ldots$$

If I is a nilpotent ideal, then $I^n = 0$ for some positive integer n, where $I^{n-1} \neq 0$. Then $\mathcal{N}(\Omega:R:I)$ is a nilpotent group of class $n-1$, if $|\Omega| \geq 2$.

<u>Proof</u> Let $H_i = \mathcal{N}(\Omega:R:I^i)$; $i = 1,2, \ldots\ldots$

We show that $[H_i, H_1] < H_{i+1}$.

Now $H_i = \langle 1+x : x \in \mathcal{E}(I^i)\rangle$, by lemma 3.1.4.

Consider $[1+x, 1+y]$, where $x \in \mathcal{E}(I^i)$ and $y \in \mathcal{E}(I)$.

Using lemma 3.1.2, we have

$$[1+x, 1+y] = 1+z, \quad \text{where } z \in \mathcal{E}(I^{i+1}).$$

Since $1+z \in H_{i+1}$, we have $[H_i, H_1] < H_{i+1}$.

If $I^n = 0$, then clearly $H_n = 1$.

Suppose $I^{n-1} \neq 0$. Then there is an element $h \in I^{n-1}$ with $h \neq 0$ and

$$h = h_1 h_2 \ldots\ldots h_{n-1}, \quad \text{where } h_i \in I.$$

Thus by 3.1.1 (xi) we have

$$[t_{\lambda\mu}(h_1), r_{\mu\mu}(h_2), r_{\mu\mu}(h_3), \ldots\ldots, r_{\mu\mu}(h_{n-1})] = t_{\lambda\mu}(h).$$

But $t_{\lambda\mu}(h) \neq 1$, since $h \neq 0$.

Hence $\mathcal{N}(\Omega : R : I)$ is nilpotent of class $n-1$.

We now consider rings satisfying the descending chain condition on left ideals, see [8].

3.1.6 <u>Definition</u> A ring R has descending chain condition on left ideals (d.c.c.) if every descending series of left ideals of R,

$$I_1 > I_2 > \ldots\ldots\ldots,$$

becomes stationary after a finite number of steps.

We use the following result which is proved in [8].

3.1.7 <u>Lemma</u> If R is a ring with a 1 satisfying the d.c.c. on left ideals, then the Jacobson radical J is nilpotent.

Combining the results of 3.1.7 and 3.1.5 we have

3.1.8 <u>Corollary</u> Let R be a ring with a 1 satisfying the d.c.c. on left ideals and let I be any ideal contained in the Jacobson radical of R. Then $\mathcal{N}(\Omega:R:I)$ is nilpotent.

We now study homomorphic images of the group $\mathcal{R}(\Omega:R:I)$ induced by homomorphisms of R.

3.1.9 <u>Lemma</u> Let R and S be rings with a 1 and let I be an ideal contained in the Jacobson radical J of R. Let $\varphi : R \longrightarrow S$ be a homomorphism of R onto S taking I onto L. Then

$$\frac{\mathcal{R}(\Omega:R:I)}{\mathcal{R}(\Omega:\ker\varphi:\ker\varphi\cap I)} \cong \mathcal{R}(\Omega:S:L).$$

<u>Proof</u> Define $\bar{\varphi} : \mathcal{R}(\Omega:R:I) \longrightarrow \mathcal{R}(\Omega:S:L)$

by $\bar{\varphi} : a \longrightarrow \bar{a}$

where, if $a : v_{\lambda_j} \longrightarrow v_{\lambda_j} + \sum_{i=1}^{n} a_i^{(j)} v_{\lambda_i}$, $1<j<n$, $a_i^{(j)} \in R$,

$a : v_\mu \longrightarrow v_\mu$, $\mu \neq \lambda_j$, for $1<j<n$,

then

$$\bar{a} : v_{\lambda_j} \longrightarrow v_{\lambda_j} + \sum_{i=1}^{n} \varphi(a_i^{(j)})v_{\lambda_i}, \quad 1 < j < n,$$

$$\bar{a} : v_{\mu} \longrightarrow v_{\mu} \quad , \quad \mu \neq \lambda_j, \text{ for } 1 < j < n.$$

Let $a,b \in \mathcal{R}(\Omega:R:I)$ and suppose that both a and b act trivially on all ba sis elements except $v_{\lambda_1}, \ldots, v_{\lambda_n}$.

Let a be defined as above and let b be defined by

$$b : v_{\lambda_j} \longrightarrow v_{\lambda_j} + \sum_{i=1}^{n} b_i^{(j)}v_{\lambda_i} \ , \ 1<j<n, \quad b_i^{(j)} \in R,$$

$$b : v_{\mu} \longrightarrow v_{\mu} \ , \quad \mu \neq \lambda, \ \text{ for } \ 1 < j < n.$$

We now show that $\bar{\varphi}(ab) = \bar{\varphi}(a)\bar{\varphi}(b)$ using the fact that φ preserves sums and products.

$$\bar{\varphi}(ab) : v_{\lambda_j} \longrightarrow v_{\lambda_j} + \sum_{i=1}^{n}\varphi(b_i^{(j)})v_{\lambda_i} + \sum_{i=1}^{n}\varphi(a_i^{(j)})v_{\lambda_i}$$
$$+ \sum_{i,k=1}^{n}\varphi(a_i^{(j)}b_k^{(i)})v_{\lambda_k}$$

$$\bar{\varphi}(ab) : v_{\mu} \longrightarrow v_{\mu} \quad , \quad \mu \neq \lambda, \text{ for } \ 1 < j < n.$$

Hence $\quad \bar{\varphi}(ab) = \bar{\varphi}(a)\bar{\varphi}(b)$.

Clearly $\ker\bar{\varphi} = \mathcal{R}(\Omega : \ker\varphi : \ker\varphi \cap I)$ and $\mathrm{im}\bar{\varphi} = \mathcal{R}(\Omega:S:L)$.

Hence the result follows.

3.1.10 <u>Corollary</u> The group $\mathcal{R}(\Omega:R:I)/\mathcal{N}(\Omega:R:I)$ is locally nilpotent.

<u>Proof</u> By lemma 3.1.9, we have

$$\mathcal{R}(\Omega:R:I)/\mathcal{N}(\Omega:R:I) \cong \mathcal{R}(\Omega:R/I:0),$$

where 0 is the zero ideal of R/I. But any finite set of elements of $\mathcal{R}(\Omega:R/I:0)$ is contained in some finite dimensional subgroup of upper triangular matrices and so is contained in a nilpotent subgroup.

Hence $\mathcal{R}(\Omega:R/I:0)$ is locally nilpotent and the result follows.

From the above arguments we see that if R is a ring with trivial Jacobson radical, then $\mathcal{R}(\Omega:R) \in {}_L\mathfrak{N}$. If R is a ring satisfying the d.c.c. on left ideals, then $\mathcal{R}(\Omega:R)$ is an extension of the nilpotent group $\mathcal{N}(\Omega:R)$ by the ${}_L\mathfrak{N}$-group $\mathcal{R}(\Omega:R/J)$.

However, we can improve these results considerably and we aim to prove the following theorem.

3.1.11 <u>Theorem</u> Let I be a nilpotent ideal of the ring R. Then $\mathcal{R}(\Omega:R:I) \in {}_N\mathfrak{A}$, the class of Baer nilgroups.

<u>Proof</u> Fix $\alpha, \beta \in \Omega$, $\alpha < \beta$, and fix $r \in R$. Denote $\mathcal{N}(\Omega:R:I)$ by $\mathcal{N}$ and define K by

$$K = \langle t_{\lambda\mu}(r_1) : \lambda<\alpha,\ \mu\geq\beta,\ \alpha,\beta,\lambda,\mu \in \Omega,\ r_1 \in R\rangle.$$

Then clearly

$$\frac{\langle \mathcal{N}, t_{\alpha\beta}(r)\rangle}{\mathcal{N}} \quad \text{is normal in} \quad \frac{\mathcal{N}K}{\mathcal{N}}.$$

We now show that

$$\mathcal{N}K/\mathcal{N} \quad \text{is normal in } \mathcal{R}(\Omega:R:I)/\mathcal{N}.$$

$$\begin{aligned} \text{Now}\quad [t_{\lambda\mu}(r_1), t_{\gamma\delta}(r_2)] &= 1, && \text{if } \mu\neq\gamma \text{ and } \lambda\neq\delta,\\ &= t_{\lambda\delta}(r_1r_2), && \text{if } \mu = \gamma,\\ &= t_{\gamma\mu}(-r_2r_1), && \text{if } \lambda = \delta. \end{aligned}$$

Hence $K^{t_{\gamma\delta}(r_2)} = K$ and so $\mathcal{N}K/\mathcal{N}$ is normal in $\mathcal{R}(\Omega:R:I)/\mathcal{N}$. Thus $\langle \mathcal{N}, t_{\alpha\beta}(r)\rangle$ is subnormal in $\mathcal{R}(\Omega:R:I)$ and to show that $\mathcal{R}(\Omega:R:I) \in N\mathfrak{A}$, it is sufficient to show that $\langle \mathcal{N}, t_{\alpha\beta}(r)\rangle \in N\mathfrak{A}$. For, then $\mathcal{R}(\Omega:R:I)$ is generated by subnormal $N\mathfrak{A}$-subgroups, giving $\mathcal{R}(\Omega:R:I) \in NN\mathfrak{A} = N\mathfrak{A}$.

We require the following definition.

3.1.12 <u>Definition</u> We say that G is an <u>Engel group</u> of class$\leq$n if, given $x,y \in G$, then

$$[x,\underbrace{y,y,y,\ldots,y}_{n}] = 1.$$

We denote the class of Engel groups of class$\leq$n by $\mathcal{E}_n$.

To complete the proof of theorem 3.1.11 we require the following two lemmas - the first of these being a well known result, see [5].

3.1.13 <u>Lemma</u> $\mathcal{E}_n \cap {}_E\mathfrak{A} \leq {}_N\mathfrak{A}$.

3.1.14 <u>Lemma</u> If $I^m = 0$, then $\mathcal{N}(\Omega:R:I)$ is nilpotent of class m-1, and $\langle \mathcal{N}, t_{\alpha\beta}(r)\rangle$ is an Engel group of class 3m-2.

These two lemmas complete the proof of theorem 3.1.11, since $\langle \mathcal{N}, t_{\alpha\beta}(r)\rangle$ is an Engel group of bounded class which is clearly soluble, being an extension of the nilpotent group $\mathcal{N}$ by the cyclic group $\langle t_{\lambda\mu}(r)\rangle$. The theorem then follows using lemma 3.1.13.

<u>Proof of 3.1.14</u> During this proof we use the following commutator relations many times.

$$[a,bc] = [a,c][a,b][a,b,c],$$

$$[ab,c] = [a,c][a,c,b][b,c].$$

These are easily checked directly by expanding the commutators.

Let $\quad H_i = \mathcal{N}(\Omega:R:I^i)\ , \quad i = 1,2,\ldots\ldots$

Suppose $I^m = 0$, then

$$\mathcal{N}(\Omega:R:I) = H_1 \geq H_2 \geq \ldots\ldots \geq H_i \geq \ldots \geq H_m = 1\ ,$$

is a central series for $\mathcal{N}(\Omega:R:I)$, by theorem 3.1.5.

Let $x \in H_r$ and let $y = t_{\alpha\beta}(r)n$, where $n \in \mathcal{N}(\Omega:R:I)$. We shall write $t_{\alpha\beta}(r) = t_{\alpha\beta}$ unless confusion arises. We aim to prove $[x, t_{\alpha\beta}n, t_{\alpha\beta}n, t_{\alpha\beta}n] \in H_{r+1}$.

Now

$$[x, t_{\alpha\beta}n] = [x,n][x,t_{\alpha\beta}][x,t_{\alpha\beta},n],$$

also $[x,n] \in H_{r+1}$ and $[x,t_{\alpha\beta},n] \in H_{r+1}$.

Write (H_{r+1}) for modulo H_{r+1}. Then we have

$$[x, t_{\alpha\beta}n] \equiv [x, t_{\alpha\beta}](H_{r+1}). \qquad (1)$$

Now $x = a_1 a_2 \ldots a_k$, where a_i is a generator of H_r. We shall use decreasing induction on k.

$$[(a_1 \ldots a_{k-1})a_k, t_{\alpha\beta}]$$
$$= [(a_1 \ldots a_{k-1}), t_{\alpha\beta}][(a_1 \ldots a_{k-1}), t_{\alpha\beta}, a_k][a_k, t_{\alpha\beta}].$$

Suppose a_k is not $r_{\beta\alpha}(a)$, for some $a \in J$. Then by the commutator relations of lemma 3.1.1, we see that

$$[a_k, t_{\alpha\beta}] = c(t_{\alpha\beta}),$$

where $c(t_{\alpha\beta})$ is an element of H_r that commutes with $t_{\alpha\beta}$.

Now suppose $a_k = r_{\beta\alpha}(a)$, for some $a \in I^r$. Then

$$[r_{\beta\alpha}(a), t_{\alpha\beta}(r), t_{\alpha\beta}(r)] \equiv t_{\alpha\beta}(-2rar)(H_{r+1}).$$

This relation can be checked working with 2x2 matrices as in 3.1.1, using the fact that modulo H_{r+1} in the group

is equivalent to taking the matrix entries modulo I^{r+1}.
Hence

$$[r_{\beta\alpha}(a), t_{\alpha\beta}, t_{\alpha\beta}] \equiv c(t_{\alpha\beta})(H_{r+1}),$$

where we have $c(t_{\alpha\beta}) \in H_r$ and $c(t_{\alpha\beta})$ commutes with $t_{\alpha\beta}$.

Thus

$$[(a_1 \dots a_{k-1})a_k, t_{\alpha\beta}, t_{\alpha\beta}] \equiv [[(a_1 \dots a_{k-1}), t_{\alpha\beta}][a_k, t_{\alpha\beta}], t_{\alpha\beta}](H_{r+1})$$

$$\equiv [(a_1 \dots a_{k-1}), t_{\alpha\beta}, t_{\alpha\beta}][[a_k, t_{\alpha\beta}], t_{\alpha\beta}](H_{r+1}),$$

since $\quad [(a_1 \dots a_{k-1}), t_{\alpha\beta}, t_{\alpha\beta}, [a_k, t_{\alpha\beta}]] \in H_{r+1}$

Hence

$$[(a_1 \dots a_{k-1})a_k, t_{\alpha\beta}, t_{\alpha\beta}] \equiv [(a_1 \dots a_{k-1}), t_{\alpha\beta}, t_{\alpha\beta}]c(t_{\alpha\beta})(H_{r+1}),$$

where $c(t_{\alpha\beta}) \in H_r$ and commutes with $t_{\alpha\beta}$.

Now using (1) we see that

$$\begin{aligned}[x, t_{\alpha\beta}n, t_{\alpha\beta}n] &\equiv [x, t_{\alpha\beta}, t_{\alpha\beta}](H_{r+1}) \\ &\equiv c(t_{\alpha\beta})(H_{r+1}) \\ &= ch_{r+1}\end{aligned}$$

where c commutes with $t_{\alpha\beta}$, $c \in H_r$, $h_{r+1} \in H_{r+1}$.

Then $[x, t_{\alpha\beta}n, t_{\alpha\beta}n, t_{\alpha\beta}n] = [ch_{r+1}, t_{\alpha\beta}n]$

$$\begin{aligned}&= [ch_{r+1}, n][ch_{r+1}, t_{\alpha\beta}][ch_{r+1}, t_{\alpha\beta}, n] \\ &\equiv [c, n](H_{r+1}) \\ &\equiv 1 \ (H_{r+1}).\end{aligned}$$

Hence $[x, t_{\alpha\beta}n, t_{\alpha\beta}n, t_{\alpha\beta}n] \in H_{r+1}$.

Now suppose that $x = t_{\alpha\beta}y$, where $y \in \mathcal{N}(\Omega{:}R{:}I)$. Notice that this may be taken as a general element of $\langle \mathcal{N}, t_{\alpha\beta} \rangle$, since $t^m_{\alpha\beta}(r_1) = t_{\alpha\beta}(mr_1)$.

Hence $[t_{\alpha\beta}y, t_{\alpha\beta}n] \in \mathcal{N}(\Omega{:}R{:}I)$, since the derived group of $\langle \mathcal{N}, t_{\alpha\beta}(r) \rangle$ must be contained in $\mathcal{N}(\Omega{:}R{:}I)$. So $\langle \mathcal{N}, t_{\alpha\beta}(r) \rangle$ is an Engel group of class $3m-2$, where $H_m = 1$.

From the first part of the proof of theorem 3.1.11 we have,

3.1.15 <u>Corollary</u> $\mathcal{R}(\Omega{:}R)/\mathcal{N}(\Omega{:}R) \in N\mathfrak{A}$ for any linearly ordered set Ω.

Combining theorem 3.1.11 with lemma 3.1.7, we have

3.1.16 <u>Corollary</u> Let R be a ring with a 1 satisfying the d.c.c. on left ideals. Then $\mathcal{R}(\Omega{:}R) \in N\mathfrak{A}$. In particular, if R is a finite ring with a 1, then $\mathcal{R}(\Omega{:}R) \in N\mathfrak{A}$.

3.2 <u>$\mathcal{R}(\Omega{:}R)$ and $\mathcal{N}(\Omega{:}R)$ when J(R) is locally nilpotent.</u>

Every locally nilpotent ring is radical so, given a locally nilpotent ring J, we can embed it in a ring R with a 1 so that $J = J(R)$.

3.2.1 <u>Theorem</u> Let J be a locally nilpotent ring embedded in a ring R with a 1, using the construction of 2.2.6. Then, for any linearly ordered set Ω, $\mathcal{R}(\Omega : R) \in L\mathfrak{N}$.

<u>Proof</u> Let T be a finite set of elements of $\mathcal{R}(\Omega : R)$. Now every generator of $\mathcal{R}(\Omega : R)$ involves only a single element of the ring R. Thus every element of $\mathcal{R}(\Omega : R)$, being expressible as the product of a finite number of generators, can involve only a finite number of elements of R. The finite set T can only involve a finite set of elements of R and we denote this set by S. Then

$$S = \{(x_\lambda, n_\lambda) : x_\lambda \in J,\ n_\lambda \in \mathbb{Z},\ \lambda \in \Lambda\}$$

where Λ is a finite set.

Let X be the subring generated by

$$\{(x_\lambda, 0) : \lambda \in \Lambda\}$$

Then X is nilpotent, being a finitely generated subring of a locally nilpotent ring.

Let I be the subring generated by

$$\{(x_\lambda, 0), (0, 1) : \lambda \in \Lambda\}$$

Since X is a nilpotent two-sided ideal of I, $X < J(\overset{I}{R})$. We show that $X = J(\overset{I}{R})$.

Suppose $(x, n) \in J(I)$, where $n \neq 0$. Choose a prime p which does not divide n.

Define

$$I_p = Rg\{(x_\lambda,0),(o,p) : \lambda \in \Lambda\}$$

Then I_p is a maximal ideal of I which does not contain (x,n). Hence $n = 0$ and $(x,0) \in X$.

Now $T \leq \mathcal{R}(\Omega:I)$. But I has a nilpotent Jacobson radical and so applying theorem 3.1.11 we have

$$\mathcal{R}(\Omega:I) \in {}_N\mathfrak{A}.$$

Hence $\mathcal{R}(\Omega:R) \in {}_{LN}\mathfrak{A}$.

But ${}_N\mathfrak{A} \leq {}_L\mathfrak{N}$ implies ${}_{LN}\mathfrak{A} \leq {}_{LL}\mathfrak{N} = {}_L\mathfrak{N}$.

Hence $\mathcal{R}(\Omega:R) \in {}_L\mathfrak{N}$.

We now study the structure of $\mathcal{N}(\Omega:R)$ more closely and to do this we introduce the following notation.

3.2.2 <u>Definition</u> Let $a \in J$ and $\lambda,\mu \in \Omega$. Define

$$n_{\lambda\mu}(a) = r_{\lambda\mu}(a) \quad \text{if } \lambda > \mu,$$

$$n_{\lambda\mu}(a) = t_{\lambda\mu}(a) \quad \text{if } \lambda < \mu.$$

We see that $\mathcal{N}(\Omega:R) = \langle n_{\lambda\mu}(a) : \lambda,\mu \in \Omega,\ a \in J\rangle$.

Now suppose that J is a commutative locally nilpotent ring embedded in R by the construction of 2.2.6. It follows immediately from 3.2.1 that $\mathcal{N}(\Omega:R) \in {}_L\mathfrak{N}$. The fact that R is now commutative enables us to give a stronger result.

3.2.3 <u>Theorem</u> Let R be a ring with a 1 having a commutative locally nilpotent radical J. Then for any linearly ordered set Ω, $\mathcal{N}(\Omega:R) \in \mathrm{N}\mathfrak{a}$.

<u>Proof</u> We show that every generator of $\mathcal{N}(\Omega:R)$ is subnormal by showing that

$$\mathcal{N}(\Omega:R) = H_0 > H_1 > \;\ldots\ldots > H_n > \ldots\ldots > \langle n_{\lambda\mu}(a)\rangle$$

is a series of subgroups from $\mathcal{N}(\Omega:R)$ to the cyclic group generated by $n_{\lambda\mu}(a)$.

Define

$$H_n = \langle n_{\lambda\mu}(a),\; n_{\rho\sigma}(x_n) : \rho,\sigma \in \Omega,\; x_n \in a^nJ\rangle,$$

for $n = 1,2,3,\ldots$

Now $a \in J$, so $a^m = 0$, for some m, since J is locally nilpotent. Thus $a^mJ = 0$.

Hence $H_m = \langle n_{\lambda\mu}(a)\rangle$, and to complete the proof that $\langle n_{\lambda\mu}(a)\rangle$ is subnormal in $\mathcal{N}(\Omega:R)$, it is sufficient to show that H_n is normal in H_{n-1}.

Let h_n be a generator of H_n and let h_{n-1} be a generator of H_{n-1}. We must show that

$$h_{n-1}^{-1}h_nh_{n-1} \in H_n \;.$$

Suppose $h_n = n_{\rho\sigma}(x_n)$, where $x_n \in a^nJ$.

Then $h_{n-1}^{-1}n_{\rho\sigma}(x_n)h_{n-1} \in H_n$, by lemma 2.3.3.

If h_n and h_{n-1} are both equal to $n_{\lambda\mu}(a)$ they clearly commute.

Suppose $h_n = n_{\lambda\mu}(a)$ and $h_{n-1} = n_{\rho\sigma}(x_{n-1})$, where $x_{n-1} \in a^{n-1}J$.

It follows from lemma 3.1.1, using the fact that J is commutative, that

$$[n_{\lambda\mu}(a), n_{\rho\sigma}(x_{n-1})] \in H_n.$$

Hence every generator of $\mathcal{N}(\Omega:R)$ is subnormal and so

$$\mathcal{N}(\Omega:R) \in {}_N\mathfrak{A}.$$

We recall that the class $\mathcal{Y}_A$ is the class of all groups having an invariant ascending series with Abelian factors. We prove that $\mathcal{N}(\Omega:R) \in \mathcal{Y}_A$ when the Jacobson radical of R is a commutative locally nilpotent ring.

3.2.4 Theorem Let R be a ring with a 1 having a commutative locally nilpotent radical J. Then for any linearly ordered set Ω, $\mathcal{N}(\Omega:R) \in \mathcal{Y}_A$.

Proof We construct an ascending series of ideals

$$\{J_\lambda : \lambda \in \Lambda\},$$

in the following way.

Define $J_0 = 0$.

Suppose J_λ is already defined. Choose, if possible,

$a \in J$ with $a \notin J_\lambda$.

Then $a + J_\lambda$ generates an ideal of J/J_λ which is of the form

$$\{ax + J_\lambda : x \in J\}.$$

This is a nilpotent ideal, since a is nilpotent and J is commutative.

Hence

$$\{ax + J_\lambda : x \in J\}^m = J_\lambda \text{ , for some integer m.}$$

Put $J_{\lambda+1} = \{ax + j_\lambda : x \in J,\ j_\lambda \in J_\lambda\}^{m-1}$ if $m > 1$.

$= J$ if $m = 1$.

For limit ordinals ρ define

$$J_\rho = \bigcup_{\lambda<\rho} J_\lambda.$$

The ascending series of ideals $\{J_\lambda : \lambda \in \Lambda\}$ has the property that $(J_{\lambda+1})^2 \leqslant J_\lambda$, by their construction. Corresponding to this chain of ideals we have a chain of normal subgroups

$$\{\mathcal{N}(\Omega:R:J_\lambda) : \lambda \in \Lambda\}.$$

Now $\mathcal{N}(\Omega:R:J_{\lambda+1})/\mathcal{N}(\Omega:R:J_\lambda) \in \mathfrak{A}$, since $(J_{\lambda+1})^2 \leqslant J_\lambda$, using lemma 3.1.2.

The series

$$\{\mathcal{N}(\Omega:R:J_\lambda) : \lambda \in \Lambda\}$$

is therefore an ascending series with $\mathfrak{A}$-factors, and so

$\mathcal{N}(\Omega:R) \in \mathcal{Y}_A$.

3.3 The group $\mathcal{N}(\Omega:R)$ when $\bigcap_{i<\omega} (J(R))^i = 0$.

We prove a result that is a consequence of 3.1.5.

3.3.1 Theorem Let R be a ring with a 1, and let the Jacobson radical J of R satisfy $\bigcap_{i<\omega} J^i = 0$. Then for any linearly ordered set Ω, $\mathcal{N}(\Omega:R) \in R\mathfrak{N}$, the class of residually nilpotent groups.

Proof By theorem 3.1.5 we have a descending central series for $\mathcal{N}(\Omega:R)$ given by

$$\mathcal{N}(\Omega:R) = H_1 \geqslant H_2 \geqslant \ldots\ldots \geqslant H_n \geqslant \ldots\ldots ,$$

where $H_n = \mathcal{N}(\Omega:R:J^n)$.

To show that $\mathcal{N}(\Omega:R) \in R\mathfrak{N}$ it is sufficient to show that $\bigcap_{i<\omega} H_i = 1$.

Let $h \in \bigcap_{i<\omega} H_i$. Then h can be considered as an $\Omega\times\Omega$ matrix of the form $I + H$, where I is the $\Omega\times\Omega$ identity. But since $h \in H_i$, then the entries of H must be in J^i. The condition $\bigcap_{i<\omega} J^i = 0$ gives $H = 0$.

Thus $h = 1$ and so $\bigcap_{i<\omega} H_i = 1$ as required.

We recall that the class $\acute{N}\mathfrak{A}$ consists of groups generated by their descendant Abelian subgroups.

Let G be a group with a subgroup H, and let Ω be an inversely well ordered set. Let the inverse ordering to Ω be λ, where λ is an ordinal.

We say that

$$(\Lambda_\sigma : \sigma < \lambda)$$

is a descending series of type λ from G to H if

$$\Lambda_0 = G,\ \Lambda_{\sigma+1} \text{ is normal in } \Lambda_\sigma,\ \Lambda_\lambda = H \quad \text{and}$$

$$\Lambda_\rho = \bigcap_{\sigma<\rho} \Lambda_\sigma, \quad \text{for limit ordinals } \rho.$$

We write $G \overset{\lambda}{\trianglerighteq} H$.

We prove the following result.

3.3.2 Theorem Let R be a ring with a 1 having a commutative Jacobson radical J satisfying $\bigcap_{i<\omega} J^i = 0$, and let Ω be any linearly ordered set. Then every generator $n_{\lambda\mu}(\bar{a})$ of $\mathcal{N}(\Omega:R)$ satisfies

$$\mathcal{N}(\Omega:R) \overset{\omega+1}{\trianglerighteq} \langle n_{\lambda\mu}(\bar{a})\rangle,$$

and so $\mathcal{N}(\Omega:R) \in \grave{N}\mathfrak{a}$.

<u>Proof</u> Let λ,μ be fixed elements of Ω and $\bar{a}$ fixed in J. We show that there is a descending series

$$\mathcal{N}(\Omega:R) = H_1 \geq H_2 \geq \ldots \geq H_n \geq \ldots \geq H \geq \langle n_{\lambda\mu}(\bar{a})\rangle,$$

where $H_n = \langle n_{\lambda\mu}(a),\ n_{\alpha\beta}(r)\ :\ \alpha,\beta \in \Omega,\ a \in J,\ r \in J^n\rangle$

for $n = 1, 2, \ldots\ldots$

and $H = \bigcap_{i<\omega} H_i$.

We show firstly that if

$$\bar{H} = \langle n_{\lambda\mu}(a)\ :\ a \in J\rangle,$$

then $H = \bar{H}$.

Clearly $\bar{H} \leq H$, since $\bar{H} \leq H_n$ for all integers n. Let $h \in H$. Then if $h \neq n_{\lambda\mu}(a)$, for some $a \in J$, there must exist $\alpha,\beta \in \Omega$ with

$$h : v_\alpha \longrightarrow v_\alpha + a'v_\beta + \ldots\ldots \text{ a finite number of terms not involving } v_\alpha \text{ or } v_\beta,$$

for some $a' \in J$, $a' \neq 0$, where either $\alpha \neq \lambda$ or $\beta \neq \mu$.

Then $a' \in J^n$, for some greatest n, since $a' \neq 0$, and $\bigcap_{i<\omega} J^i = 0$.

But $a' \notin J^{n+1}$ implies that $h \notin H_{n+1}$, for, suppose $\alpha \neq \lambda$. The only generators in h which act non-trivially on the basis element v_α are those of the form $n_{\alpha\gamma}(s)$

where $s \in J^{n+1}$. Thus $a' \in J^{n+1}$. A similar argument holds if $\beta \neq \mu$. Thus $h = n_{\lambda\mu}(a)$, for some $a \in J$, and so $h \in \overline{H}$.

To show we have a series we must show that H_{i+1} is normal in H_i and $\langle n_{\lambda\mu}(\bar{a})\rangle$ is normal in H.

Let h_{i+1} be a generator of H_{i+1} and h_i a generator of H_i.

Consider $$h_i^{-1}\ h_{i+1}\ h_i\ .$$

If h_i is of the form $n_{\alpha\beta}(r)$ for $r \in J^{i+1}$, then

$$h_i^{-1}\ h_{i+1}\ h_i \in H_{i+1} \qquad \text{by 3.1.2.}$$

Suppose $h_{i+1} = n_{\lambda\mu}(a)$, for some $a \in J$.
Then if $h_i = n_{\lambda\mu}(a')$ we have

$$h_i^{-1}\ h_{i+1}\ h_i = h_{i+1}.$$

This is true even if $\lambda = \mu$, since J is commutative.
Now suppose $h_{i+1} = n_{\lambda\mu}(a)$ for some $a \in J$ and $h_i = n_{\alpha\beta}(r)$ for $r \in J^i$. Then

$$[h_{i+1}, h_i] \in H_{i+1}, \text{ using lemma 3.1.2.}$$

Then in all cases $h_i^{-1}\ h_{i+1}\ h_i \in H_{i+1}$, so H_{i+1} is normal in H_i.

Clearly $\langle n_{\lambda\mu}(\bar{a})\rangle < H$, and since J is commutative, H is Abelian.

Thus $\langle n_{\lambda\mu}(\bar{a})\rangle$ is normal in H and the proof is complete.

A large class of commutative rings R satisfy the condition $\bigcap_{i<\omega} J^i = 0$, e.g. all Noetherian rings.

3.3.3 <u>Definition</u> We say that a commutative ring is <u>Noetherian</u> if R has a 1 and every ideal I of R contains a finite set of elements $a_1, a_2, \ldots, a_n$ so that

$$I = a_1R + a_2R + \ldots + a_nR.$$

We state without proof the Krull intersection theorem on Noetherian rings, see [10].

3.3.4 <u>Theorem</u> [Krull] Let I be an ideal of a Noetherian ring R. Then an element x of R belongs to $\bigcap_{i<\omega} I^i$ if and only if $x = ax$ for at least one $a \in I$.

This gives an immediate corollary to theorem 3.3.2.

3.3.5 <u>Corollary</u> Let R be a Noetherian ring and let Ω be any linearly ordered set. Then $\mathcal{N}(\Omega:R) \in \check{N}\mathcal{A}$.

<u>Proof</u> It is sufficient to show that a Noetherian ring R has a radical J satisfying $\bigcap_{i<\omega} J^i = 0$.

Using 3.3.4 we have

$$x \in \bigcap_{i<\omega} J^i \text{ implies that } x = ax \text{ for some } a \in J.$$

Then $x(1 - a) = 0$. But $1 - a$ is a unit so $x = 0$.

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§4 The classes $\mathcal{R}$, $\mathcal{N}$ and $\mathcal{H}$.

4.1 Some subgroups of the group $\mathcal{R}(\Omega:R)$.

Let R be a ring with a 1 and let J be the Jacobson radical of R. We have defined the group $\mathcal{R}(\Omega:R)$ and its subgroup $\mathcal{N}(\Omega:R)$. we now define further subgroups of the group $\mathcal{R}(\Omega:R)$.

4.1.1 <u>Definition</u> Let $\mathcal{H}(\Omega:R)$, $\mathcal{K}_\lambda(\Omega:R)$, $\mathcal{K}(\Omega:R)$ be the subgroups of $\mathcal{R}(\Omega:R)$ defined by

$$\mathcal{H}(\Omega:R) = \langle n_{\alpha\beta}(a) \ : \ \alpha, \beta \in \Omega, \ \alpha \neq \beta, \ a \in J\rangle.$$

For a fixed $\lambda \in \Omega$,

$$\mathcal{K}_\lambda(\Omega:R) = \langle n_{\lambda\lambda}(a) \ : \ a \in J\rangle.$$

$$\mathcal{K}(\Omega:R) = \langle n_{\alpha\alpha}(a) \ : \ \alpha \in \Omega, \ a \in J\rangle.$$

Suppose J is a radical ring. Then $1 + J$ forms a multiplicative group.

Let G be a group with a set of subgroups K_λ for $\lambda \in \Lambda$. We say that G is the <u>direct product</u> of the subgroups K_λ if

(a) the elements of any two distinct subgroups K_λ are permutable,

(b) every element $\neq 1$ of G has a unique representation (apart from the order of the factors) as a product of a finite number of elements $\neq 1$ chosen one from each of the subgroups K_λ.

We write $G = \operatorname*{Dr}_{\lambda\in\Lambda} K_\lambda$.

4.1.2 <u>Lemma</u> (i) $\mathcal{K}_\lambda(\Omega:R) \cong 1 + J$,

(ii) $\mathcal{K}(\Omega:R) = \operatorname*{Dr}_{\lambda\in\Omega} \mathcal{K}_\lambda(\Omega:R)$.

<u>Proof</u> We notice that every element of $\mathcal{K}_\lambda(\Omega:R)$ is a generator.

Define $\varphi : \mathcal{K}_\lambda(\Omega:R) \longrightarrow 1 + J$

by $\varphi : n_{\lambda\lambda}(a) \longrightarrow 1 + a$.

Clearly φ is one to one and onto. We show that φ is a homomorphism:-

$$\begin{aligned}
\varphi\{n_{\lambda\lambda}(a_1).n_{\lambda\lambda}(a_2)\} &= \varphi\{n_{\lambda\lambda}(a_1+a_2+a_1a_2)\} \\
&= 1+a_1+a_2+a_1a_2 \\
&= (1+a_1)(1+a_2) \\
&= \varphi(n_{\lambda\lambda}(a_1)).\varphi(n_{\lambda\lambda}(a_2)).
\end{aligned}$$

We now have an isomorphism φ from $\mathcal{K}_\lambda(\Omega:R)$ to $1 + J$ and so

$$\mathcal{K}_\lambda(\Omega:R) \cong 1 + J.$$

(ii) To show that $\mathcal{K}(\Omega:R) = \operatorname*{Dr}_{\lambda\in\Omega}\mathcal{K}_\lambda(\Omega:R)$ we check conditions (a) and (b) of the definition of direct product given above.

Now (a) is clear. For, $n_{\lambda\lambda}(a_1)$ and $n_{\mu\mu}(a_2)$ commute when $\lambda \neq \mu$.

Suppose $k \in \mathcal{K}(\Omega:R)$. Then

$$k = n_{\lambda_1\lambda_1}(a_1) \,\ldots\ldots\, n_{\lambda_r\lambda_r}(a_r)\,, \quad r \geq 1.$$

We can assume $a_i \neq 0$ and $\lambda_i \neq \lambda_j$ for $i \neq j$.

To show that an expression of the type given for k is unique it is sufficient to show that any expression of this type is non-trivial, since inverses have the same form.

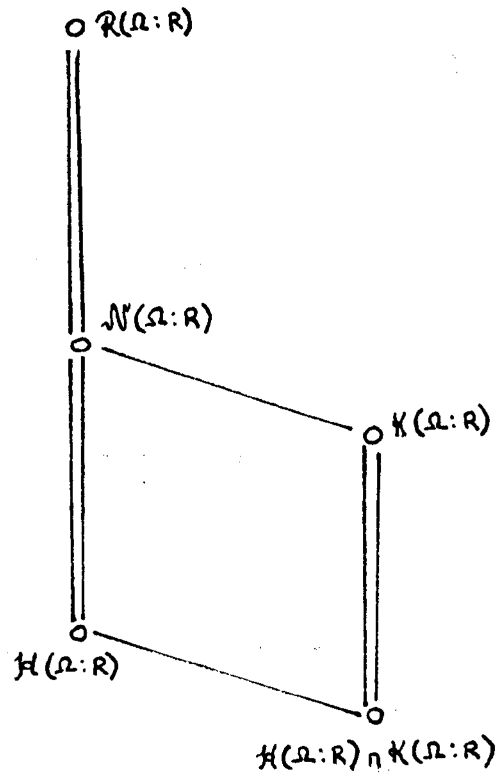
But we can consider k as a diagonal r×r matrix with entries $1+a_1, 1+a_2, \ldots, 1+a_r$. Now this is trivial if and only if $a_i = 0$, $1 \leq i \leq r$; and this is not the case.

Hence $\mathcal{K}(\Omega:R) = \underset{\lambda\in\Omega}{\mathrm{Dr}}\, \mathcal{K}_\lambda(\Omega:R)$.

Notice that combining (i) and (ii) in lemma 4.1.2 gives $(\Omega:R)$ isomorphic to the direct product of Ω copies of $1 + J$, where J is the Jacobson radical of R.

We now study the relations between the various subgroups of $\mathcal{R}(\Omega:R)$ that we have defined.

4.1.3 <u>Theorem</u> Let R be a ring with a 1 and let Ω be any linearly ordered set. Then the subgroups of the group $\mathcal{R}(\Omega:R)$ defined in 4.1.1 satisfy the following inclusion diagram. The double lines indicate normality.



The following properties hold

(i) $\mathcal{R}(\Omega:R)/\mathcal{N}(\Omega:R) \in N\mathfrak{A}$

(ii) $\mathcal{H}(\Omega:R)\mathcal{K}(\Omega:R) = \mathcal{N}(\Omega:R)$

(iii) If the Jacobson radical J of R is commutative then $\mathcal{K}(\Omega:R)$ is Abelian.

<u>Proof</u> That $\mathcal{H}(\Omega:R)$ is normal in $\mathcal{N}(\Omega:R)$ follows

using the commutator relations of 3.1.1.
Now (i) is corollary 3.1.15, and (ii) follows trivially from the definitions.
To prove (iii), let J be commutative. Then $1 + J$ is an Abelian group. Thus $K(\Omega:R)$ is Abelian, being the direct product of $|\Omega|$ copies of $1 + J$ by 4.1.2 (ii).

4.1.4 <u>Definition</u> Let $\mathcal{R}$ be the class of groups defined by:-

$G \in \mathcal{R}$ if and only if $G \cong \mathcal{R}(\Omega:R)$, for some linearly ordered set Ω and some ring R.

Let $\mathcal{N}$ be the class of groups defined by:-

$G \in \mathcal{N}$ if and only if $G \cong \mathcal{N}(\Omega:R)$, for some linearly ordered set Ω and some ring R.

Let $\mathcal{H}$ be the class of groups defined by:-

$G \in \mathcal{H}$ if and only if $G \cong \mathcal{H}(\Omega:R)$, for some linearly ordered set Ω and some ring R.

Let $\mathfrak{F}$ denote the class of finite groups. Then we have

4.1.5 <u>Lemma</u> (i) $\mathcal{R} \cap \mathfrak{F} < \mathfrak{N}$

(ii) $\mathcal{N} \cap \mathfrak{F} < \mathfrak{N}$

(iii) $\mathcal{H} \cap \mathfrak{F} < \mathfrak{N}$

So all finite $\mathcal{R}$, $\mathcal{N}$, $\mathcal{H}$-groups are nilpotent.

__Proof__ Let $G = \mathcal{R}(\Omega{:}R) \in \mathcal{R} \cap \mathcal{Y}$.

Then the ring R^J must be finite or G has infinitely many distinct generators. Any finite ring must have the d.c.c. and so J, the Jacobson radical of R, must be nilpotent by lemma 3.1.7.

Then by theorem 3.1.11, $G \in N\mathfrak{A}$ and so G is nilpotent since any finite group generated by its subnormal Abelian subgroups is nilpotent.

Hence $G \in \mathfrak{N}$.

(ii) Let $G \in \mathcal{N} \cap \mathcal{Y}$, so $G = \mathcal{N}(\Omega{:}R)$.

Then the ring R^J must be finite and so R^J has the d.c.c.
Hence $G \in \mathfrak{N}$, as in (i).

(iii) If $|\Omega| \neq 1$ $\mathcal{N}(\Omega{:}R)$ is finite if and only if $\mathcal{H}(\Omega{:}R)$ is finite, while $\mathcal{H}(\Omega{:}R)$ is trivial if $|\Omega| = 1$.
Hence $\mathcal{N} \cap \mathcal{Y} < \mathfrak{N}$ gives $\mathcal{H} \cap \mathcal{Y} < \mathfrak{N}$.

4.2 __Simple $\mathcal{R}$, $\mathcal{N}$ and $\mathcal{H}$-groups.__

First we consider the Abelian simple groups in the class $\mathcal{R}$.

4.2.1 __Definition__ Let Ω_n denote the linearly ordered set with n elements and let these elements be $1, 2, \ldots, n$.

We recall that $\mathbb{Z}_p n$ denotes the ring of integers modulo p^n, where p is a prime.

4.2.2 Lemma The class $\mathcal{R}$ contains all cyclic groups of prime order and the infinite cyclic groups.

Proof It is a straightforward matter to check that

$\mathcal{R}(\Omega_2:\mathbb{Z}_p) \cong C_p$, the cyclic group of order p, and

$\mathcal{R}(\Omega_2:\mathbb{Z}) \cong C$, the infinite cyclic group.

This lemma shows that $\mathcal{R}$ contains all Abelian simple groups. ~~We show that this exhausts the simple $\mathcal{R}$-groups~~.

4.2.3 Lemma $\mathcal{R}(\Omega:R)$ is not a non-Abelian simple group if $|\Omega| \geq 2$.

Proof Let G be a simple non-Abelian $\mathcal{R}$-group, if such exist, where ~~Then~~ $G \cong \mathcal{R}(\Omega:R)$ for some linearly ordered set Ω, $|\Omega| \geq 2$ and some ring R. But $\mathcal{R}(\Omega:R)$ has a normal subgroup $\mathcal{N}(\Omega:R)$ so suppose $\mathcal{R}(\Omega:R) = \mathcal{N}(\Omega:R)$. Since R contains a 1 this shows that J, the Jacobson radical of R, must contain 1. This is impossible. Suppose that $\mathcal{N}(\Omega:R) = 1$. Then J = 0, and by corollary 3.1.15 we have

$$\mathcal{R}(\Omega:R) \in N\mathfrak{A}.$$

Hence $\mathcal{R}(\Omega:R)$ cannot be simple for a simple non-Abelian group can have no subnormal Abelian subgroups.

We show now that the class $\mathcal{N}$ also contains all Abelian simple groups. We prove a more general result than this in the next lemma.

4.2.4 Lemma Let p be an odd prime. Then the class $\mathcal{N}$ contains all cyclic groups of p-power order, and also the cyclic group of order two.

Proof It is easy to see that

$$\mathcal{N}(\Omega_1:\mathbb{Z}_4) \cong C_2.$$

Let p be an odd prime. We show that

$$\mathcal{N}(\Omega_1:\mathbb{Z}_{p^n}) \cong C_{p^{n-1}}.$$

$\mathcal{N}(\Omega_1:\mathbb{Z}_{p^n}) \cong 1 + J$, where J is the Jacobson radical of $\mathbb{Z}_{p^n}$. The ring $\mathbb{Z}_{p^n}$ has a unique maximal ideal $p\mathbb{Z}_{p^n}$ and so $J = p\mathbb{Z}_{p^n}$.

To complete the proof, it is sufficient to show that if $x \in \mathbb{Z}_{p^n}$, then

$$(1 + px) = (1 + p)^m, \text{ for some integer } m.$$

This gives

$$\mathcal{N}(\Omega_1:\mathbb{Z}_{p^n}) \cong 1 + p\mathbb{Z}_{p^n} = \langle 1 + p\rangle \cong C_{p^{n-1}}.$$

Now

$$(1 + p)^x = 1 + px + a_1p^2 + a_2p^3 + \ldots\ldots \quad (\text{mod } p^n),$$

$$(1 + p)^{-a_1p} = 1 - a_1p^2 + b_1p^3 + \ldots\ldots\ldots \quad (\text{mod } p^n).$$

Whence

$$(1+p)^{x-a_1p} = 1 + px + c_1p^3 + c_2p^4 + \ldots \quad (\text{mod } p^n).$$

We have found an integer $x-a_1p$ which eliminates the term in p^2 in the expansion of $(1+p)^{x-a_1p}$, giving $1 + px$ as the first two terms.

Now use induction. Suppose we can find integers $x_1, x_2, \ldots, x_i$ so that

$$(1+p)^{x+x_1+x_2+\ldots+x_i} = 1+px+d_1p^{i+2}+d_2p^{i+3}+\ldots. \ (\text{mod } p^n)$$

Let $x_{i+1} = -d_1p^{i+1}$.

Then

$$(1+p)^{x+x_1+x_2+\ldots+x_i+x_{i+1}} = 1+px+e_1p^{i+3}+\ldots.(\text{mod } p^n)$$

Since all the series are finite, we can find some exponent m so that

$$(1 + p)^m = 1 + px.$$

We leave the following question open.

Do non-Abelian simple $\mathcal{N}$-groups exist?

We now investigate the class $\mathcal{H}$ for simple groups and find the opposite situation to that in the class $\mathcal{R}$. There are no Abelian simple $\mathcal{H}$-groups, but non-Abelian simple $\mathcal{H}$-groups do exist.

4.2.5 <u>Lemma</u> The class $\mathcal{H}$ contains no non-trivial simple Abelian groups.

<u>Proof</u> Let $\mathcal{H}(\Omega:R)$ be a non-trivial Abelian simple group. Then $\mathcal{H}(\Omega:R) \cong C_p$ for some prime p.

Now Ω must contain at least two elements, since $\mathcal{H}(\Omega_1:R)$ is trivial for all rings R.

Hence $\mathcal{H}(\Omega_2:R) \leq \mathcal{H}(\Omega:R)$ and so we have

$$\mathcal{H}(\Omega_2:R) \cong C_p.$$

But $\mathcal{H}(\Omega_2:R)$ contains non-trivial generators $n_{12}(r)$ and $n_{21}(r)$. Clearly $n_{12}(r)$ cannot be a power of $n_{21}(r)$, since

$$\{n_{21}(r)\}^m = n_{21}(\bar{r}), \quad \text{for some } \bar{r} \in R.$$

However in a cyclic group of prime order, every element $\neq 1$ generates the group.

This contradiction completes the proof.

Before we prove the existence of non-Abelian simple $\mathcal{H}$-groups we require several lemmas and definitions.

4.2.6 <u>Definition</u> Let R be a ring. We define the <u>left annihilator</u> of R to be the subset of elements of R denoted by $\mathrm{ann}_L R$ where

$$a \in \mathrm{ann}_L R \text{ if and only if } ar = 0 \text{ for all } r \in R.$$

We define the <u>right annihilator</u> of R to be the subset of elements $a \in R$ with

$$ra = 0, \quad \text{for all } r \in R.$$

We denote this subset by $\mathrm{ann}_R R$.

It can easily be checked that $\mathrm{ann}_L R$ and $\mathrm{ann}_R R$ are both two-sided ideals of the ring R.

4.2.7 <u>Lemma</u> Let A and B be nxn and nxm matrices over a ring J, respectively, and let C and D be mxn and mxm matrices over J. Suppose J is a subring of some ring R with a 1 and the left annihilator of J is trivial. Then the $(n+m+1)\times(n+m+1)$ matrix

$$X = \left[\begin{array}{c:c:c} 1+A & 0 & B \\ \hdashline 0 & 1 & 0 \\ \hdashline C & 0 & 1+D \end{array}\right]$$

commutes with all matrices of the form

$$(I + rE_{1,n+1})_{(n+m+1)\times(n+m+1)}\ ,$$

for all i with $1 \leq i \leq n+m+1$, $i \neq n+1$ and for all $r \in J$, if and only if $A = B = C = D = 0$.

<u>Proof</u> If $A = B = C = D = 0$, then X is the $(n+m+1)\times(n+m+1)$ identity matrix, and so all $(n+m+1)\times(n+m+1)$ matrices commute with it.

Let $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times m}$, $C = (c_{ij})_{m\times n}$, $D = (d_{ij})_{m\times m}$.

Then if

$$\left[\begin{array}{c|c|c} 1+A & 0 & B \\ \hline 0 & 1 & 0 \\ \hline C & 0 & 1+D \end{array}\right]$$

commutes with $(I + rE_{1,n+1})_{(n+m+1)\times(n+m+1)}$,

we have

$$\left[\begin{array}{c|c|c} I & \begin{matrix} r \\ \vdots \\ 0 \end{matrix} & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & I \end{array}\right] \left[\begin{array}{c|c|c} 1+A & 0 & B \\ \hline 0 & 1 & 0 \\ \hline C & 0 & 1+D \end{array}\right] \left[\begin{array}{c|c|c} I & \begin{matrix} -r \\ \vdots \\ 0 \end{matrix} & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & I \end{array}\right] = \left[\begin{array}{c|c|c} 1+A & 0 & B \\ \hline 0 & 1 & 0 \\ \hline C & 0 & 1+D \end{array}\right]$$

So

$$\left[\begin{array}{c|c|c} I & \begin{matrix} r \\ 0 \\ \vdots \\ 0 \end{matrix} & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & I \end{array}\right] \left[\begin{array}{c|c|c} 1+A & \begin{matrix} -r-a_{11}r \\ -a_{21}r \\ \vdots \\ -a_{n1}r \end{matrix} & B \\ \hline 0 & 1 & 0 \\ \hline C & \begin{matrix} -c_{11}r \\ \vdots \\ -c_{m1}r \end{matrix} & 1+D \end{array}\right] = \left[\begin{array}{c|c|c} 1+A & 0 & B \\ \hline 0 & 1 & 0 \\ \hline C & 0 & 1+D \end{array}\right]$$

giving

$$\left[\begin{array}{c:c:c} 1+A & \begin{matrix} -a_{11}r \\ -a_{21}r \\ \vdots \\ -a_{n1}r \end{matrix} & B \\ \hdashline 0 & 1 & 0 \\ \hdashline C & \begin{matrix} -c_{11}r \\ -c_{21}r \\ \vdots \\ -c_{m1}r \end{matrix} & 1+D \end{array}\right] = \left[\begin{array}{c:c:c} 1+A & 0 & B \\ \hdashline 0 & 1 & 0 \\ \hdashline C & 0 & 1+D \end{array}\right]$$

Hence $a_{i1}r = 0,\ 1 \leq i \leq n,$

$$c_{i1}r = 0,\ 1 \leq i \leq m,$$

and these equations must hold for all $r \in J$. So

$$a_{i1} \in \operatorname{ann}_L J, \quad 1 \leq i \leq n,$$

$$c_{i1} \in \operatorname{ann}_L J, \quad 1 \leq i \leq m.$$

But the left annihilator of J is trivial and so

$$a_{i1} = 0, \quad 1 \leq i \leq n,$$

$$c_{i1} = 0, \quad 1 \leq i \leq m.$$

Similarly, the condition for X to commute with

$$(\,I + rE_{2,n+1}\,)_{(n+m+1)\times(n+m+1)}, \text{ for all } r \in J,$$

gives
$$a_{i2} = 0, \quad 1 \leq i \leq n,$$

$$c_{i2} = 0, \quad 1 \leq i \leq m.$$

When we have used the conditions that X commutes with

$$(\,I + rE_{i,n+1}\,)_{(n+m+1)\times(n+m+1)}, \quad \text{for } 1 \leq i \leq n$$

and all $r \in J$ we obtain $A = 0$ and $C = 0$.

An identical method will yield $B = 0$ and $D = 0$ using the conditions that

$$(I + rE_{i,n+1})_{(n+m+1)\times(n+m+1)}$$

commutes with X, for $n+2 \leqslant i \leqslant n+m+1$ and for all $r \in J$.

We thus obtain $A = B = C = D = 0$ as required.

4.2.8 Lemma Let R be a ring with a 1 and let J be the Jacobson radical of R. Suppose that the left and right annihilators of J are trivial. If Ω is an infinite set, then every normal subgroup $H \neq 1$ of $\mathcal{H}(\Omega{:}R)$ contains a generator $\neq 1$ of $\mathcal{H}(\Omega{:}R)$.

Proof Let h be a non-trivial element of H. Then

$$h = n_{\lambda_1\mu_1}(a_1)n_{\lambda_2\mu_2}(a_2)\ldots\ldots n_{\lambda_k\mu_k}(a_k),$$

where $\lambda_i , \mu_i \in \Omega$, and $a_i \in J$ with $a_i \neq 0$, $1 \leqslant i \leqslant k$.

Let $\beta \in \Omega$ with $\beta \neq \lambda_i, \mu_i$, for $1 \leqslant i \leqslant k$. This is always possible since Ω is an infinite set.

Now consider

$$[h, n_{\mu_j\beta}(\bar{a})] , \quad \text{for some } j, \ 1 \leqslant j \leqslant k,$$

and some $\bar{a} \in J$.

Using the commutator relation

$$[ab,c] = [a,c][a,c,b][b,c],$$

we obtain, for some $I \leqslant \{1,2,\ldots,k\}$,

$$[h, n_{\mu_j\beta}(\bar{a})] = \prod_{i\in I} n_{\lambda_i\beta}(\bar{a}_i) \; .$$

To see this we use induction on k. Suppose the result holds for elements $\bar{h}$ that are products of k-1 generators. Let $h = n_{\lambda_1\mu_1}(a_1)\bar{h}$

Then $\quad [h, n_{\mu_j\beta}(\bar{a})]$

$$= [n_{\lambda_1\mu_1}(a_1), n_{\mu_j\beta}(\bar{a})][n_{\lambda_1\mu_1}(a_1), n_{\mu_j\beta}(\bar{a}), \bar{h}][\bar{h}, n_{\mu_j\beta}(\bar{a})]$$

Now

$$\begin{aligned} [n_{\lambda_1\mu_1}(a_1), n_{\mu_j\beta}(\bar{a})] &= 1 \quad \text{if} \quad \mu_1 \neq \mu_j \\ &= n_{\lambda_1\beta}(a_1\bar{a}) \quad \text{if} \quad \mu_1 = \mu_j \end{aligned}$$

using 3.1.1.

In either case the result follows by the induction hypothesis.

We now use lemma 4.2.7. For, considering h as the matrix X of this lemma, this shows that we can find a j, with $1 \leqslant j \leqslant k$, and some $\bar{a} \in J$ so that

$$\prod_{i\in I} n_{\lambda_i\beta}(a_i) \neq 1.$$

Notice that the $n_{\lambda_i\beta}(\bar{a}_i)$'s commute with each other, since $\beta \neq \rho_i$ for $i \in I$, the ~~ρ's being a subset of the λ's~~.

Choose $\alpha \in \Omega$ with $\alpha \neq \lambda_i$, $i \in I$ and $\alpha \neq \beta$. We may assume that $\lambda_i \neq \lambda_j$ for $i \neq j$. Let a' be the sum of those $\bar{a}_i$ where $\lambda_i = \lambda_i$. Then for $b \in J$ we have

$$[n_{\alpha\lambda_i}(b), \prod_{i\in I} n_{\lambda_i\beta}(\bar{a}_i)] = n_{\alpha\beta}(ba').$$

We may assume that $a' \neq 0$, since $\prod_{i\in I} n_{\lambda_i\beta}(\bar{a}_i) \neq 1$.

But $ba' = a \neq 0$, for some $b \in J$, since the right annihilator of J is trivial.

Hence $n_{\alpha\beta}(a) \in H$, for $a \neq 0$, and so H contains a generator of $\mathcal{H}(\Omega:R)$.

We require a result on simple rings.

4.2.9 <u>Definition</u> We say that the ring R is <u>simple</u> if R has only two ideals and $R^2 \neq 0$.

4.2.10 <u>Lemma</u> Let J be a simple ring and let a be any non-zero element of J. Then $JaJ = J$.

<u>Proof</u> JaJ is a two-sided ideal of J so it is sufficient to show

$$JaJ \neq 0.$$

Suppose $JaJ = 0$. Then aJ and Ja are two-sided ideals of J. For,

$$J(aJ) = 0 \leq aJ,$$

$$(aJ)J = aJ^2 \leq aJ;$$

and
$$(Ja)J = 0 \leq Ja,$$
$$J(Ja) = J^2a \leq Ja.$$

Hence $Ja = J$ or 0, and $aJ = J$ or 0.
If $Ja = J$, then $0 = JaJ = (Ja)J = J^2$.
If $aJ = J$, then $0 = JaJ = J(aJ) = J^2$.
But $J^2 \neq 0$ since J is simple, so we have

$$Ja = 0 = aJ.$$

Now $\langle a\rangle^+$, the additive subgroup of J generated by a, is a two-sided ideal of J which is non-zero since it contains a.

Hence $\langle a\rangle^+ = J$. But $Ja = 0$ and so $a^2 = 0$. This gives $J^2 = 0$, which is a contradiction.

We now prove the main result on simple $\mathcal{H}$-groups.

4.2.11 <u>Theorem</u> Let R be a ring with a 1 with Jacobson radical J and let Ω be an infinite linearly ordered set. Then $\mathcal{H}(\Omega{:}R)$ is simple if and only if J is a simple ring.

<u>Proof</u> Suppose J is a simple ring. Then since the left and right annihilators of J are two-sided ideals they must be trivial. The conditions of lemma 4.2.8 are satisfied. So if H is a normal non-trivial subgroup

of $\mathcal{H}(\Omega:R)$, then

$$n_{\alpha\beta}(a) \in H, \text{ for } \alpha,\beta \in \Omega,\ \alpha \neq \beta,\ 0 \neq a \in J.$$

Let ρ,σ be arbitrary elements of Ω with $\rho \neq \sigma$, and let x be an arbitrary element of J. We show that

$$n_{\rho\sigma}(x) \in H.$$

Choose $\lambda,\mu \in \Omega$ with $\lambda \neq \alpha,\beta,\rho,\sigma$, and $\mu \neq \alpha, \beta, \rho, \sigma, \lambda$. This is always possible since Ω is an infinite set.

We show that

$$n_{\lambda\mu}(x) \in H.$$

Since $JaJ = J$, by lemma 4.2.10,

$$\sum_{i=1}^{n} a_{1i}\ a\ a_{2i} = x, \text{ for some } a_{1i}, a_{2i} \in J,\ 1<i<n.$$

Then

$$\prod_{i=1}^{n} [n_{\lambda\alpha}(a_{1i}), n_{\alpha\beta}(a), n_{\beta\mu}(a_{2i})] = n_{\lambda\mu}(x) \quad , \quad \text{since}$$

λ and μ are both distinct from α and β.

But $n_{\alpha\beta}(a) \in H$ which is a normal subgroup of $\mathcal{H}(\Omega:R)$, so

$$n_{\lambda\mu}(x) \in H.$$

But $\rho,\sigma \in \Omega$ with ρ and σ distinct from α and β so by the same argument

$$n_{\rho\sigma}(x) \in H.$$

Thus H contains all the generators of $\mathcal{H}(\Omega:R)$ and we have

$$H = \mathcal{H}(\Omega:R)$$

showing that $\mathcal{H}(\Omega:R)$ is simple.

Now suppose $\mathcal{H}(\Omega:R)$ is a simple group. We show that J is a simple ring. If J has an ideal $I < J$ with $I \neq 0$, then the normal closure of the subgroup

$$\langle n_{\lambda\mu}(a) : \lambda,\mu \in \Omega,\ \lambda \neq \mu,\ a \in I \rangle$$

is a proper non-trivial normal subgroup of $\mathcal{H}(\Omega:R)$. Hence J has only two ideals. Also $J^2 \neq 0$. For, if $J^2 = 0$, then $(\Omega:R)$ is Abelian.

But $\mathcal{H}$ contains no Abelian simple groups by lemma 4.2.5 and so J is a simple ring.

To show that the class $\mathcal{H}$ contains non-Abelian simple groups it is sufficient to show that a simple radical ring exists. Suppose J is a simple radical ring. Embed J in a ring R with a 1 using the construction of lemma 2.2.6. Then $\mathcal{H}(\Omega:R)$ is a simple non-Abelian group, provided Ω is an infinite set, using theorem 4.2.11.

For a proof of the existence of simple radical rings see [11]. In §6 we look at an example of a simple $\mathcal{H}$-group, see 6.4.

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§5 $\mathcal{H}$-groups that are $\bar{Z}$-groups.

5.1 Scalar local rings

5.1.1 Definition We call a ring R with a 1 a local ring if R has a unique maximal right ideal I satisfying

$$\bigcap_{i<\omega} I^i = 0.$$

Since I is the unique maximal right ideal of R then I = J, the Jacobson radical of R, and so I is also the maximal two-sided ideal. We show that I is also the unique maximal left ideal of R.

5.1.2 Lemma Let R be a ring with a 1 and Jacobson radical J. Then the following are equivalent:-

(i) R/J is a division ring;

(ii) R has a unique maximal right ideal;

(iii) for every $r \in R$, either r or $1-r$ is a unit.

Proof (i) implies (ii) R/J has no proper right ideals, so J is the unique maximal right ideal of R.

(ii) implies (iii) Let $r \notin I$, the unique maximal right ideal of R, then $rR = R$.

Hence $rx = 1$, for some $x \in R$. Now $x \notin I$. For, if $x \in I$, then $rx \in I$, since I is a two-sided ideal.

Then $1 \in I$, which is a contradiction.
Hence $x \notin I$ and so $xR = R$. Then $xy = 1$, for some $y \in R$, giving $rxy = y$.
This shows that $r = y$, so $xr = 1$, which shows that r is a unit.

Now if $r \in I$, $1-r \notin I$, otherwise $1 \in I$.
Thus if $r \notin I$, r is a unit and if $r \in I$, $1-r$ is a unit.

<u>(iii) implies (i)</u> Suppose that R/J is not a division ring. Then for some $r \in R$, $r+J$ is not a unit of R/J.

Clearly r is not a unit of R, so suppose that r has no right inverse. Then rs has no right inverse for all $s \in R$. For, if

$$(rs)x = 1, \text{ then } r(sx) = 1,$$

which is a contradiction.

If $r \notin J$, then there exists a maximal right ideal M with $r \notin M$. Hence

$$rR + M = R,$$

so that $rs + m = 1$, for some $s \in R$, some $m \in M$.
But $1-rs$ is a unit, by (iii). Thus m is a unit, which is a contradiction.

Hence $r \in J$.
Similarly if r has no left inverse then $r \in J$.

Hence if $r + J$ is not a unit in R/J, then

$r + J = J$, the zero of R/J.

This shows that all non-zero elements of R/J are units and so R/J is a division ring.

5.1.3 <u>Corollary</u> If R has a unique maximal right ideal, then this ideal is the unique maximal left ideal of R.

<u>Proof</u> If R has a unique maximal right ideal I, then R/I is a division ring. But lemma 5.1.2 is clearly true if we replace "right ideal" by "left ideal". Then R/I is a division ring implies that I is the unique maximal left ideal of R.

We shall be interested in the case where this maximal ideal is generated by one element, both as a right and left ideal.

5.1.4 <u>Definition</u> We call a ring R a <u>scalar local ring</u> if R is a local ring with unique maximal right ideal I generated by a single element $x \neq 0$, both as a right ideal and as a left ideal.

5.1.5 <u>Lemma</u> Let R be a scalar local ring with unique maximal right ideal I generated by x. If J is the Jacobson radical of R, and if $I_0 \neq 0$ is any two-sided ideal of R, then $I_0 = J^n$, for some integer $n \geq 0$.

<u>Proof</u> J is the unique maximal two-sided ideal of

R, so, if $I_0 < R$, I_0 must be contained in J.

Since R is a local ring $\bigcap\limits_{i<\omega} J^i = 0$ and so we can find an integer n so that

$$I_0 \leq J^n \text{ but } I_0 \not\leq J^{n+1}.$$

There is an element $a \in I_0$ with $a \in J^n$ but $a \notin J^{n+1}$. Since $a \in J^n$, then $a = x^n u$, for some $u \in R$. But since $a \notin J^{n+1}$, $u \notin J$ and so u is a unit.

Hence $(x^n) \leq (a) \leq I_0$, where (a) denotes the two-sided ideal generated by a.

But $J^n = (x^n)$, and so $J^n \leq I_0$.

But $I_0 \leq J^n$ and so $I_0 = J^n$, for some integer $n \geq 1$.

We now give conditions for a scalar local ring to have zero divisors.

5.1.6 <u>Lemma</u> Let R be a scalar local ring with non-trivial Jacobson radical. Then R has zero divisors if and only if the Jacobson radical J of R is nilpotent. If R has zero radical, then R has no zero divisors.

<u>Proof</u> Suppose $J = 0$. Then since R/J is a division ring by lemma 5.1.2, R can have no zero divisors.

Suppose now that $J \neq 0$. Let $a,b \in R$ with $ab = 0$, and $a \neq 0$. Since $a \neq 0$, either $a \notin J$ or there exists an integer $n \geqslant 1$ with

$$a \in J^n \text{ and } a \notin J^{n+1}.$$

In either case

$$a = x^n u$$

where u is a unit and $n \geqslant 0$.

In the same way

$$b = x^m u',$$

where $m \geqslant 0$ and u' is either a unit or zero.

Then

$$ab = x^n u x^m u'.$$

By definition we have

$$ux^m = x^m u''; \text{ where } u \text{ is a unit.}$$

We show that u'' is a unit. It is sufficient to show that if

$$ux = xr_1$$

where u is a unit, then r_1 is a unit.

Let $ux = xr_1$, and $u^{-1}x = xr_2$.

Then $x = u^{-1}ux = u^{-1}xr_1 = xr_2r_1$, so $x(1 - r_2r_1) = 0$.

If r_1 or $r_2 \in J$, then $r_2r_1 \in J$ and $1 - r_2r_1$ is a unit. Thus $x = 0$, which is a contradiction, for J is non-trivial.

Hence $r_1, r_2 \notin J$ and so r_1, r_2 are units as required.

Then

$$ab = x^{m+n}u''u' = 0,$$

so, either $u' = 0$, or $x^{m+n} = 0$.

If J is not nilpotent, then $x^{m+n} \neq 0$, so $u' = 0$. This gives $b = 0$, and R has no zero divisors.

Conversely, if J is nilpotent, then $x^m = 0$, for some integer $m \geq 2$. (We cannot have $m = 1$ since J is non-trivial) We can assume $x^{m-1} \neq 0$.

Then
$$x^{m-1}.x = 0 \quad \text{with} \quad x^{m-1} \neq 0, \; x \neq 0.$$

In this case R has zero divisors, and they comprise all the non-zero elements of J.

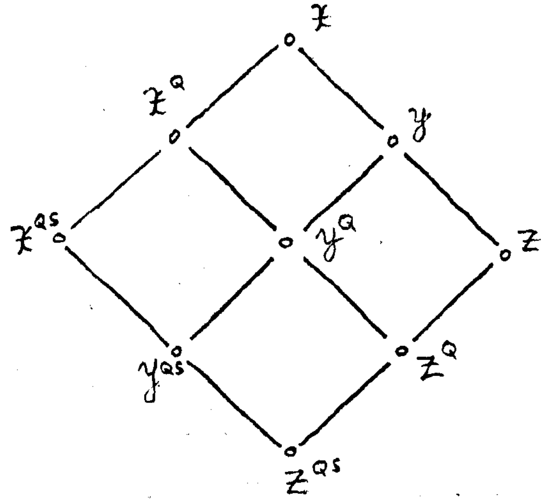
5.1.7 Definition We call a scalar local ring with a non-nilpotent Jacobson radical, a scalar local domain.

5.2 The classes $\overline{SI}$ and $\overline{Z}$.

We have defined the classes $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ in 1.2.3, 1.2.4 and 1.2.6, respectively. Let Q and S be the closure operations defined in 1.1.3. Then Q and S are unary and QS is also a unary closure operation.

We can define classes $\mathfrak{X}^Q$, $\mathfrak{Y}^Q$ and $\mathfrak{Z}^Q$ to be the Q-interiors of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$, respectively. Define $\mathfrak{X}^{QS}$, $\mathfrak{Y}^{QS}$, $\mathfrak{Z}^{QS}$ to be the QS-interiors of $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, respectively.

We have the following inclusion diagram.



5.2.1 <u>Definition</u> We define the classes $\overline{SI}$ and $\overline{Z}$ by:-

$G \in \overline{SI}$ if and only if every invariant series of G can be refined to an invariant $\mathfrak{a}$-series.

$G \in \overline{Z}$ if and only if every invariant series of G can be refined to a central series of G.

5.2.2 <u>Lemma</u> (i) $\overline{SI} = \mathfrak{Y}^Q$;

(ii) $\overline{Z} = \mathfrak{Z}^Q$.

<u>Proof</u> (i) We show that $\overline{SI}$ is Q-closed. Let $G \in \overline{SI}$, and let H be a normal subgroup of G. Take an invariant series of G through H and refine this to an invariant $\mathfrak{a}$-series. Those subgroups lying above H now give, modulo H, an invariant $\mathfrak{a}$-series of G/H.

Hence $$\overline{SI} \leq \mathfrak{Y}^Q.$$

Let $G \in \mathcal{Y}^Q$ and let A/B be a factor of an invariant series of G. Intersect A/B with some invariant $\mathcal{A}$-series of G/B. This gives a refinement of the invariant series of G, with $\mathcal{A}$-factors.

Hence $G \in \overline{SI}$ and $\overline{SI} = \mathcal{Y}^Q$.

(ii) The proof is similar to (i).

In [6] Hall showed that $\mathcal{X}^Q, \mathcal{Y}^Q, \mathcal{Z}^Q$ are not S-closed. Using the notation we have introduced above, Hall's results can be stated as

$$\mathcal{H}(\Omega_2 : R_p) \in \mathcal{Z}^Q,$$

where R_p is the ring of p-adic integers, see 6.2, while $\mathcal{H}(\Omega_2 : R_p)$ has a free subgroup of rank two. In the next section we show that many of the groups $\mathcal{H}(\Omega : R)$ lie in $\mathcal{Z}^Q$ and have subgroups which are not in $\mathcal{X}^Q$.

5.3 <u>The group $\mathcal{H}(\Omega : R)$ when R is a scalar local domain.</u>

5.3.1 <u>Theorem</u> Let Ω be an infinite set and let R be a scalar local domain. Then $\mathcal{H}(\Omega : R) \in \mathcal{Z}^Q$.

<u>Proof</u> By theorem 3.3.1,

$$\mathcal{H}(\Omega : R) \in {}_R\mathfrak{N}_< \mathcal{Z}.$$

We must show that every proper factor group of $\mathcal{H}(\Omega:R)$ belongs to $\mathcal{Z}$.

Let H be any non-trivial normal subgroup of $\mathcal{H}(\Omega:R)$. We show that there exists a normal subgroup K of $\mathcal{H}(\Omega:R)$ with

$$K \leq H \quad \text{and} \quad \mathcal{H}(\Omega:R)/K \in \mathcal{N}.$$

Since $\mathcal{N}$ is Q-closed it will follow that

$$\mathcal{H}(\Omega:R)/H \in \mathcal{N} \leq \mathcal{Z},$$

giving the result.

Let J be the Jacobson radical of R. Since R has no zero divisors, by 5.1.6, J can have no zero divisors. The left and right annihilators of J are therefore trivial. Since Ω is infinite, the conditions of lemma 4.2.8 are satisfied, so

$$n_{\alpha\beta}(a) \in H,$$

for some α, $\beta \in \Omega$ with $\alpha \neq \beta$, and some $a \in J$ with $a \neq 0$.

Let ρ,σ be any elements of Ω with $\rho \neq \sigma$ and let b be any element of the ideal J^2aJ^2.

Then $$b = \sum_{i=1}^{m} x_{1i}x_{2i}ax_{3i}x_{4i}\ , \quad \text{where}$$

$x_{1i}, x_{2i}, x_{3i}, x_{4i} \in J$, for $1 \leq i \leq m$.

Choose $\lambda,\mu \in \Omega$ with $\lambda \neq \alpha,\beta,\rho,\sigma$ and $\mu \neq \alpha,\beta,\rho,\sigma,\lambda$.

This is always possible since Ω is an infinite set. Then

$$[n_{\lambda\alpha}(x_{2i}),n_{\alpha\beta}(a),n_{\beta\mu}(x_{3i})] = n_{\lambda\mu}(x_{2i}ax_{3i}) \text{ ,}$$

since λ,μ,α,β are all distinct.

But $n_{\alpha\beta}(a) \in H$ which is normal in $\mathcal{H}(\Omega:R)$,

so $$n_{\lambda\mu}(x_{2i}ax_{3i}) \in H.$$

But $\rho,\sigma \in \Omega$ with ρ,σ,λ,μ all distinct, and so

$$[n_{\rho\lambda}(x_{1i}),n_{\lambda\mu}(x_{2i}ax_{3i}),n_{\mu\sigma}(x_{4i})] = n_{\rho\sigma}(x_{1i}x_{2i}ax_{3i}x_{4i}).$$

But $n_{\lambda\mu}(x_{2i}ax_{3i}) \in H$ which is normal in $\mathcal{H}(\Omega:R)$

so $n_{\rho\sigma}(x_{1i}x_{2i}ax_{3i}x_{4i}) \in H.$

Thus $$n_{\rho\sigma}(b) \in H.$$

Now J^2aJ^2 is a two-sided ideal of R, and so

$$J^2aJ^2 = J^n, \text{ for some integer } n > 1,$$

by lemma 5.1.5.

Let K be the subgroup generated by

$$\langle n_{\lambda\mu}(a),n_{\lambda\lambda}^{-1}(a_1a_2)n_{\mu\mu}(a_2a_1) \ ; \ \lambda \neq \mu, \ a, \ a_1a_2 \in J^{2n}, \ a_1 \in J\rangle$$

We show $K < H$. We have shown that $n_{\lambda\mu}(a) \in H$, for $\lambda \neq \mu$, $a \in J^n$, so it remains to show $n_{\lambda\lambda}^{-1}(a_1a_2)n_{\mu\mu}(a_2a_1) \in H$.

Now one of a_1,a_2 is in J^n so suppose it is a_1.

Then $n_{\lambda\mu}(a_1) \in H$, so $[n_{\lambda\mu}(a_1), n_{\mu\lambda}(-a_2)] \in H$, since H is normal. Using 3.1.1 (ix), we see that $n_{\lambda\lambda}^{-1}(a_1a_2)n_{\mu\mu}(a_2a_1)$ belongs to H. Hence $K \leq H$.

We now show that K is normal in $\mathcal{H}(\Omega:R)$. Consider

$$[n_{\lambda\mu}(a), n_{\alpha\beta}(b)] \text{ where } a \in J^{2n},\ b \in J.$$

If $n_{\alpha\beta} \neq n_{\mu\lambda}$ then the commutator belongs to K, using 3.1.1. But

$[n_{\lambda\mu}(a), n_{\mu\lambda}(b)]$

$= n_{\lambda\mu}(aba(1-ba)^{-1})n_{\mu\lambda}(-bab(1-ab))n_{\lambda\lambda}^{-1}(-ab)n_{\mu\mu}(-ba) \quad \in \quad K.$

Therefore K is normal in $\mathcal{H}(\Omega:R)$.

We now construct a central series of finite length for $\mathcal{H}(\Omega:R)/K$.

Define $\quad K_1 = \mathcal{H}(\Omega:R)$

For $i > 1$, define

$K_i = \langle n_{\lambda\mu}(a), n_{\lambda\lambda}^{-1}(a_1a_2)n_{\mu\mu}(a_2a_1) \ ;\ \lambda \neq \mu,\ \ a,\ a_1a_2 \in J^i \rangle$.

Then $K_{2n} = K$.

We must show $\quad [K_i, K_1] \leq K_{i+1}$.

This follows from 3.1.1. Therefore $\mathcal{H}(\Omega:R)/K$ is nilpotent and the result is proved.

We now show that $\mathcal{H}(\Omega:R)$ has a subgroup which is either the free product of two cyclic groups of prime order or the free group of rank two, provided R satisfies certain conditions. The first part of the first of these lemmas is proved in [2], but ~~as only an outline of the proof is given there~~, we ~~give~~ also a ~~fuller~~ proof.

5.3.2 <u>Lemma</u> The matrices

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

generate a free group if either

(i) the coefficients are in $\mathbb{Z}$ and $|x| > 1$;

(ii) the coefficients are in $\mathbb{Z}[x]$.

We notice that this lemma applied to the group $\mathcal{H}(\Omega:R)$ gives the following result. Let Ω be a linearly ordered set with $|\Omega| \neq 1$ and let R be a ring with Jacobson radical J. Suppose that J contains an element $x \neq 0$ such that the subring of R generated by $\{1,x\}$ is (i) $\mathbb{Z}$ or (ii) $\mathbb{Z}[x]$. Since $x \neq 1$, for J cannot contain the identity, we see that $\mathcal{H}(\Omega:R)$ contains a subgroup isomorphic to the free group on two generators in both cases.

<u>Proof of 5.3.2</u> Write

$$n_{12} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad n_{21} = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$$

Consider the product

$$y = (n_{21})^{m_1}(n_{12})^{m_2} \cdots\cdots\cdots (n_{21})^{m_k}.$$

where $m_1, m_2, \ldots, m_k$ are integers with $m_i \neq 0$ for $1 < i < k$, and $k \geq 3$.

We show that the product y cannot be trivial. Consider y as a linear map on the $\mathbb{Z}$ module with basis v_1, v_2.

Let
$$y : v_1 \longrightarrow av_1 + bv_2$$
$$y : v_2 \longrightarrow cv_1 + dv_2$$

where $a,b,c,d \in \mathbb{Z}$, where $\mathbb{Z}$ is considered as an ordered ring.

Suppose
$$(n_{21})^{m_1}(n_{12})^{m_2} : v_1 \longrightarrow a^{(1)}v_1 + b^{(1)}v_2$$
$$(n_{21})^{m_1}(n_{12})^{m_2}(n_{21})^{m_3} : v_1 \longrightarrow a^{(2)}v_1 + b^{(2)}v_2$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$

Now $a^{(1)} = 1$, and $|b^{(1)}| = |m_2 x|$ so $|a^{(1)}| < |b^{(1)}|$. Thus $|a^{(2)}| = |a^{(1)} + m_3 x b^{(1)}|$ and $|b^{(2)}| = |b^{(1)}|$, giving $|a^{(2)}| > |b^{(2)}|$, since $|a^{(1)}| < |b^{(1)}|$.

We show by induction that

$$|a^{(i)}| < |b^{(i)}| \text{ if } i \text{ is odd.}$$

$$|a^{(i)}| > |b^{(i)}| \text{ if } i \text{ is even.}$$

Suppose i is odd.

We have $a^{(i+1)} = a^{(i)} + m_{i+2}xb^{(i)}$ and $b^{(i+1)} = b^{(i)}$.

Hence $|a^{(i+1)}| > |m_{i+2}xb^{(i)}| - |a^{(i)}|$

$> 2|b^{(i)}| - |a^{(i)}|$ since $|x| > 1$.

$> |b^{(i)}|$ since $|a^{(i)}| < |b^{(i)}|$ by induction.

$= |b^{(i+1)}|$ since $b^{(i+1)} = b^{(i)}$.

Suppose i is even.

We have $a^{(i+1)} = a^{(i)}$ and $b^{(i+1)} = b^{(i)} + m_{i+2}xa^{(i)}$.

Hence $|b^{(i+1)}| > |m_{i+2}xa^{(i)}| - |b^{(i)}|$

$> 2|a^{(i)}| - |b^{(i)}|$ since $|x| > 1$

$> |a^{(i)}|$ since $|b^{(i)}| < |a^{(i)}|$ by induction

$= |a^{(i+1)}|$ since $a^{(i+1)} = a^{(i)}$.

Also $|a^{(i)}| = |a^{(i-1)}|$ if i is odd;

$|b^{(i)}| = |b^{(i-1)}|$ if i is even.

Hence $|a^{(i)}| > |b^{(i)}| = |b^{(i-1)}| > \ldots\ldots > |a^{(1)}| = 1$, if i is even.

Thus if i is even $|a^{(i)}| > 1$. ———(1)

If i is odd

$$|b^{(i)}| > |a^{(i)}| = |a^{(i-1)}| > 1.$$

Thus if i is odd $|b^{(i)}| > 1$. ———————(2)

Now if y is trivial we must have

$$y : v_1 \longrightarrow v_1$$

This requires $a^{(i)} = 1$ and $b^{(i)} = 0$ for some integer i. This contradicts either (1) or (2) since i must be either even or odd. This completes the proof in the first case.

<u>Case (ii)</u> Consider the product

$$y = (n_{12})^{m_1}(n_{21})^{m_2}\ldots\ldots\ldots (n_{21})^{m_k}$$

We may assume that $n_i \neq 0$, for $1 < i < k$. For, if $n_1 = 0$ or $n_k = 0$, an exactly similar argument holds to the one given below. We show that this product is non-trivial and the result will then follow as in case (i).

Let $\quad y : v_1 \longrightarrow av_1 + bv_2$

$\quad\quad\;\; y : v_2 \longrightarrow cv_1 + dv_2$

where $a,b,c,d \in \mathbb{Z}[x]$.

Suppose

$$(n_{12})^{m_1}(n_{21})^{m_2} : v_1 \longrightarrow f_1(x)v_1 + g_1(x)v_2$$

$$(n_{12})^{m_1}(n_{21})^{m_2}(n_{12})^{m_3} : v_1 \longrightarrow f_2(x)v_1 + g_2(x)v_2$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

Then $f_1(x) = 1 + m_1m_2x^2$; $g_1(x) = m_1x$;

$f_2(x) = 1 + m_1m_2x^2$; $g_2(x) = m_1x + m_3x(1 + m_1m_2x^2)$

We obtain

$$g_{2n}(x) = g_{2n-1}(x) + m_{2n+1}xf_{2n-1}(x),$$

$$f_{2n+1}(x) = f_{2n}(x) + m_{2n+1}xg_{2n}(x).$$

By an induction similar to that used in case (i), we see that $g_{2n}(x)$ is a polynomial in x of degree 2n+1 with last term

$$m_1m_2\ldots\ldots m_{2n+1}x^{2n+1}.$$

Now if $y = 1$, then $a = 1$ and $b = 0$, so in particular

$$g_m(x) = 0, \quad \text{for some integer } m.$$

If $m = 2n$, then $m_1m_2\ldots m_{2n+1} = 0$ which is impossible since $m_i \neq 0$ and m_i are integers.

If $m = 2n+1$, then

$$g_{2n+1}(x) = g_{2n}(x) = 0$$ and, as before, this is impossible.

Hence $y \neq 1$ and the result is proved.

5.3.3 <u>Lemma</u> Let $n_{12} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ and $n_{21} = \begin{bmatrix} 1 & 0 \\ x & 1 \end{bmatrix}$

be matrices with coefficients in GF(p)[x]. Then n_{12} and n_{21} generate

$$C_p * C_p$$

This lemma applied to the group $\mathcal{H}(\Omega:R)$ gives the following result. Let Ω be any linearly ordered set with $|\Omega| \neq 1$, and let R be a ring with Jacobson radical J. Suppose J contains an element x so that the subring of R generated by $\{1, x\}$ is GF(p)[x] for some prime p. Then $\mathcal{H}(\Omega:R)$ contains a subgroup isomorphic to the free product of two cyclic groups of prime order.

Proof of 5.3.3 The proof is essentially the same as lemma 5.3.2 (ii), for we show that

$$\langle n_{12}, n_{21} \rangle \cong \langle n_{12} \rangle * \langle n_{21} \rangle$$

To do this we must show that any product

$$y = (n_{12})^{m_1} (n_{21})^{m_2} \ldots\ldots (n_{21})^{m_k}$$

cannot be trivial, where $m_i \neq 0$, for $1 < i < k$ and

~~Now we can assume~~ $0 < m_i < p$.

Let $\quad y : v_1 \longrightarrow av_1 + bv_2,$

$\quad\quad y : v_2 \longrightarrow cv_1 + dv_2.$

where $a,b,c,d \in GF(p)[x]$.

Suppose that

$$(n_{12})^{m_1}(n_{21})^{m_2} : v_1 \longrightarrow f_1(x)v_1 + g_1(x)v_2$$

$$(n_{12})^{m_1}(n_{21})^{m_2}(n_{12})^{m_3} : v_1 \longrightarrow f_2(x)v_1 + g_2(x)v_2$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

Suppose $y = 1$. Then $a = 1$, $b = 0$ and so $g_m(x) = 0$, for some integer m. As in lemma 5.3.2 this is only possible if

$$m_1 m_2 \cdots\cdots m_{[m]} = 0,$$

where $[m] = m$ if m is odd, $[m] = m+1$ if m is even.

This is clearly impossible since $GF(p)$ has no zero divisors. The result now follows.

We can combine the results of theorem 5.3.1 and lemmas 5.3.2 and 5.3.3 into one theorem.

5.3.4 <u>Theorem</u> Let Ω be any linearly ordered set and let R be a scalar local domain with Jacobson radical J generated as a left and right ideal by x. Suppose that the subring of R generated by $\{1,x\}$ is isomorphic to (i) $\mathbb{Z}$, (ii) $\mathbb{Z}[x]$ or (iii) $GF(p)[x]$, for $p \neq 2,3$. Then $\mathcal{H}(\Omega:R) \in \mathcal{Z}^{Q}$, but $\mathcal{H}(\Omega:R) \notin \mathcal{X}^{QS}$.

<u>Proof</u> By theorem 5.3.1, we have $\mathcal{H}(\Omega:R) \in \mathfrak{Z}^{Q}$. To complete the proof we must show that $\mathcal{H}(\Omega:R) \notin \mathfrak{X}^{QS}$. Let R satisfy the conditions of the theorem and suppose that $\mathcal{H}(\Omega:R) \in \mathfrak{X}^{QS}$.

By lemma 5.3.2, if $Rg\{1,x\} \cong \mathbb{Z}$ or $\mathbb{Z}[x]$, then $\mathcal{H}(\Omega:R)$ contains a subgroup isomorphic to F_2, a free group on two symbols. Then $F_2 \in \mathfrak{X}^{QS}$, since $\mathfrak{X}^{QS}$ is S-closed. But $\mathfrak{X}^{QS}$ is also Q-closed, and so $C_p * C_p \in \mathfrak{X}^{QS}$ for any prime p, every two generator group being a factor group of F_2.

By lemma 5.3.3, $C_p * C_p$ is a subgroup of $\mathcal{H}(\Omega:R)$, when $Rg\{1,x\} \cong GF(p)[x]$. In all cases we have

$$C_p * C_p \in \mathfrak{X}^{QS} \qquad p \neq 2,3,$$

and we show that this is impossible.

The projective special linear group PSL(2,p) is simple, for $p \neq 2,3$, and is generated by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} ,$$

both these matrices having order p.

Hence there is a homomorphism

$$C_p * C_p \longrightarrow PSL(2,p)$$

and so $PSL(2,p) \in \mathfrak{X}^{QS}$.

This is impossible when $p \neq 2,3$ since a non-Abelian finite simple group cannot have an Abelian series. This contradiction completes the proof.

5.3.5 Corollary $\mathfrak{X}^{Q} > \mathfrak{X}^{QS}$, $\mathfrak{Y}^{Q} > \mathfrak{Y}^{QS}$, $\mathfrak{Z}^{Q} > \mathfrak{Z}^{QS}$

The classes $\overline{SI}$ and $\overline{Z}$ are not S-closed.

Proof This is an immediate consequence of the theorem when we have shown that there exist rings R satisfying the conditions of theorem 5.3.4. Examples of such rings are given in the following chapter, see examples 6.2 and 6.3.

§6 Examples

6.1 Example 1 : a locally nilpotent group.

We have given several results concerning the groups $\mathcal{R}(\Omega:R)$ and $\mathcal{N}(\Omega:R)$, when R has a commutative locally nilpotent radical, in section 3.2. It is interesting to look at an example obtained by choosing a specific ring for R, for we have not excluded the possibility that all groups $\mathcal{R}(\Omega:R)$ might be Baer nilgroups or that all groups $\mathcal{N}(\Omega:R)$ might be nilpotent.

We begin by defining a ring $\mathcal{J}$, see [4] example 3. Consider the set A of symbols a_α, where α is any real number $0 < \alpha < 1$.

Let $\mathcal{J}$ be the commutative algebra over GF(p) with basis a_α where $a_\alpha \in A$. Multiplication is defined on the basis elements by:-

$$a_\alpha . a_\beta = a_{\alpha+\beta} \ , \text{ if } \alpha + \beta < 1;$$

$$a_\alpha . a_\beta = 0 \quad , \text{ if } \alpha + \beta \geq 1.$$

Addition is defined formally.

6.1.1 Lemma $\mathcal{J}$ is a commutative locally nilpotent ring.

Proof Every element $\neq 0$ of $\mathcal{J}$ can be written uniquely in the form

$$a = n_1 a_{\alpha_1} + n_2 a_{\alpha_2} + \dots\dots + n_m a_{\alpha_m} \qquad \text{———} (1)$$

$m \geqslant 1$, where $0 \neq n_i \in GF(p)$ and distinct $\alpha_1, \alpha_2, \ldots, \alpha_m \in (0,1)$

Clearly $\underset{\sim}{J}$ is commutative since

$$a_\alpha \cdot a_\beta = a_{\alpha+\beta} = a_{\beta+\alpha} = a_\beta \cdot a_\alpha .$$

Let T be any finite set of elements of $\underset{\sim}{J}$. Every element of T has the form (1), so we can choose a relevant a_α with α minimal for all the elements of T. This is possible since T is a finite set.

Now there exists a least integer n with

$$n\alpha \geqslant 1 .$$

Let S be the subring of $\underset{\sim}{J}$ generated by T. It is easy to see that

$$S^n = 0.$$

Hence every finite set of elements of $\underset{\sim}{J}$ is contained in a nilpotent subring and so $\underset{\sim}{J}$ is locally nilpotent.

We embed $\underset{\sim}{J}$ in a ring $\underset{\sim}{R}$ with a 1 so that $\underset{\sim}{J}$ is the unique maximal ideal of $\underset{\sim}{R}$. We give a general embedding theorem.

6.1.2 <u>Definition</u> We say that a ring R has <u>characteristic</u> p, for some prime p, if $pr = 0$, for all $r \in R$.

6.1.3 <u>Lemma</u> A radical ring J with characteristic p, for some prime p, can be embedded in a ring with a 1 so that the ring, R say, has characteristic p and J is the unique maximal ideal of R.

<u>Proof</u> Let J be radical of characteristic p and let $R = J \times \mathbb{Z}_p$, where addition and multiplication are defined by

$$(r_1,n_1) + (r_2,n_2) = (r_1+r_2,n_1+n_2) ,$$

$$(r_1,n_1)(r_2,n_2) = (r_1r_2+n_1r_2+n_2r_1,n_1n_2) ,$$

where $r_1,r_2 \in J$ and $n_1,n_2 \in \mathbb{Z}_p$.

This is the same multiplication and addition used in 2.2.6.

Now R has a 1, namely the element $(0,1)$ and R has characteristic p since

$$p(r,n) = (pr,pn) = (0,0).$$

Now J is an ideal of R and all the elements of J are quasi-regular, so J is contained in the radical of R. But $R/J \cong GF(p)$, so J is the Jacobson radical of R and J is the unique maximal ideal of R.

Every locally nilpotent ring is radical and the ring $\underset{\sim}{J}$ of lemma 6.1.1 has characteristic p since it is an algebra over GF(p). Embed $\underset{\sim}{J}$ in a ring $\underset{\sim}{R}$ by the construction of lemma 6.1.3. Then $\underset{\sim}{R}$ is a commutative ring with locally nilpotent radical $\underset{\sim}{J}$ and

$$\underset{\sim}{R}/\underset{\sim}{J} \cong GF(p).$$

By 3.2.3 and 3.2.1 we have $\mathcal{N}(\Omega:\underset{\sim}{R}) \in {}_N\mathfrak{A}$ and $\mathcal{R}(\Omega:\underset{\sim}{R}) \in {}_L\mathfrak{N}$. We now consider these groups more closely, proving first that $\mathcal{N}(\Omega:\underset{\sim}{R})$ is not nilpotent. For the rest of this section $\underset{\sim}{R}$ and $\underset{\sim}{J}$ will be the rings defined above.

6.1.4 <u>Lemma</u> Let Ω be any linearly ordered set with $|\Omega| \neq 1$. Then $\mathcal{N}(\Omega:\underset{\sim}{R}) \notin \mathcal{E}_n$ for any n, and so $\mathcal{N}(\Omega:\underset{\sim}{R}) \notin \mathfrak{N}$.

<u>Proof</u> We use the commutator relations of 3.1.1. Suppose $\mathcal{N}(\Omega:\underset{\sim}{R}) \in \mathcal{E}_n$. Choose the positive real number α with $\cancel{n\alpha}^{(n+1)\alpha} < 1$.

Then for $\lambda \neq \mu$ we have

$$[n_{\lambda\mu}(a_\alpha), \underbrace{n_{\mu\mu}(a_\alpha), n_{\mu\mu}(a_\alpha), \ \ldots\ , n_{\mu\mu}(a_\alpha)}_{n}] = n_{\lambda\mu}(a_{(n+1)\alpha}) \neq 1,$$

where $a_\alpha = (a_\alpha, 0) \in \underset{\sim}{R}$.

This contradiction shows that $\mathcal{N}(\Omega:\underset{\sim}{R}) \notin \mathcal{E}_n$ and the proof is complete.

6.1.5 <u>Lemma</u> Let Ω be an infinite linearly ordered set.

Then

(i) $\mathcal{N}(\Omega:\underset{\sim}{R})$ has trivial centre;

(ii) $\mathcal{N}'(\Omega:\underset{\sim}{R}) = \mathcal{N}''(\Omega:\underset{\sim}{R}) = \mathcal{H}(\Omega:\underset{\sim}{R})$;

Hence for infinite Ω, $\mathcal{N}(\Omega:\underset{\sim}{R}) \notin \mathcal{Z}_A$ and $\mathcal{N}(\Omega:\underset{\sim}{R}) \notin \mathcal{Z}_D$.

Proof (i) Now $\underset{\sim}{J}$ has trivial left and right annihilators. For, if

$$a \in \mathrm{ann}_R \underset{\sim}{J} = \mathrm{ann}_L \underset{\sim}{J} ,$$

then $$a = n_1 a_{\alpha_1} + \ldots\ldots \; n_r a_{\alpha_r} \qquad \text{———} (1)$$

Let α_i be maximal among $\alpha_1, \ldots\ldots, \alpha_r$. We can choose $\beta \in (0,1)$ with

$$\alpha_i + \beta < 1, \qquad \text{———} (2)$$

giving $a.a_\beta = a_\beta.a \neq 0$.

It now follows immediately from lemma 4.2.7 that $Z = 1$.

Proof (ii) Let $\mathcal{N}'(\Omega:\underset{\sim}{R})$ be the derived group of $\mathcal{N}(\Omega:\underset{\sim}{R})$. Every generator of $\underset{\sim}{J}$, say a_α, can be expressed in the form

$$a_\alpha = a_{\frac{1}{2}\alpha} \cdot a_{\frac{1}{2}\alpha} .$$

So $\underset{\sim}{J}^2 = \underset{\sim}{J}$. Let a be any element of $\underset{\sim}{J}$. We can write

$$a = \sum_{i=1}^{m} a_{1i}a_{2i} , \quad a_{1i}, a_{2i} \in \underset{\sim}{J},\ 1 < i < m.$$

Then

$$n_{\lambda\mu}(a) = \prod_{i=1}^{m} [n_{\lambda\mu}(a_{1i}), n_{\mu\mu}(a_{2i})]$$

and so $\mathcal{N}'(\Omega:\underset{\sim}{R})$ contains all the generators of the form $n_{\lambda\mu}(a)$, $\lambda\neq\mu$.

Hence $\mathcal{H}(\Omega:\underset{\sim}{R}) < \mathcal{N}'(\Omega:\underset{\sim}{R})$.

But $\mathcal{N}(\Omega:\underset{\sim}{R})/\mathcal{H}(\Omega:\underset{\sim}{R}) \in \mathfrak{A}$, by 4.1.3, since $\underset{\sim}{R}$ is commutative. This gives $\mathcal{H}(\Omega:\underset{\sim}{R}) = \mathcal{N}'(\Omega:\underset{\sim}{R})$.

Let $\lambda,\mu \in \Omega$ with $\lambda \neq \mu$ and choose $\rho \neq \lambda,\mu$. Then for any $a \in \underset{\sim}{J}$,

$$a = \sum_{i=1}^{m} a_{1i}a_{2i}\ , \quad a_{1i}\ , a_{2i} \in \underset{\sim}{J}$$

and $$n_{\lambda\mu}(a) = \prod_{i=1}^{m}[n_{\lambda\rho}(a_{1i}),n_{\rho\mu}(a_{2i})]$$

Hence $\mathcal{N}''(\Omega:\underset{\sim}{R}) = \mathcal{N}'(\Omega:\underset{\sim}{R}) = \mathcal{H}(\Omega:\underset{\sim}{R})$.

We have shown that $\mathcal{N}(\Omega:\underset{\sim}{R})$ need not be a $\mathcal{Z}_A$-group or a $\mathcal{Z}_D$-group, but since $\mathcal{N}(\Omega:\underset{\sim}{R}) \in {}_N\mathfrak{A}$, then it must be a $\mathcal{Z}$-group. In the next lemma we construct a central series for $\mathcal{N}(\Omega:\underset{\sim}{R})$. We introduce some notation.

Let A_α be the set of elements of $\underset{\sim}{J}$ of the form

$$n_1a_{x_1} + n_2a_{x_2} + \dots + n_ra_{x_r}$$

where $x_i > \alpha$, for $1 \leq i \leq r$.

Let $\overline{A}_\alpha$ be the set of elements of $\underset{\sim}{J}$ of the form

$$n_1 a_{x_1} + n_2 a_{x_2} + \cdots\cdots + n_r a_{x_r}$$

where $x_i \geq \alpha$, for $1 \leq i \leq r$.

6.1.6 <u>Lemma</u> The group $\mathcal{N}(\Omega:\underline{R})$ has a central series

$$\{ \Lambda_\alpha, V_\alpha : \alpha \in (0,1) \}$$

where $\Lambda_\alpha = \langle n_{\lambda\mu}(\bar{a}) : \lambda,\mu \in \Omega,\ \bar{a} \in \bar{A}_\alpha \rangle$

$V_\alpha = \langle n_{\lambda\mu}(a) : \lambda,\mu \in \Omega,\ a \in A_\alpha \rangle$

Moreover V_α and Λ_α are nilpotent subgroups of $\mathcal{N}(\Omega:\underline{R})$, of class $\leq (m-1)$, where m is the least integer such that $m\alpha \geq 1$.

<u>Proof</u> Using the commutator relations of 3.1.1, it is easy to see that conditions (i),(ii) and (iii) of definition 1.2.1 are satisfied. Hence

$$\{ \Lambda_\alpha, V_\alpha : \alpha \in (0,1) \}$$

is a series for $\mathcal{N}(\Omega:\underline{R})$.

To show that it is a central series, we need to show that

$$[\Lambda_\alpha, \mathcal{N}(\Omega:\underline{R})] \leq V_\alpha \text{ , for all } \alpha \in (0,1).$$

This is satisfied, since given $\bar{a} \in \bar{A}_\alpha$ and any $a \notin \underline{J}$, we have

$$\bar{a}.a \in A_\alpha \ .$$

To show that Λ_α is nilpotent, we see that

$$\Lambda_\alpha = \overline{H}_1 > \overline{H}_2 > \ldots\ldots > \overline{H}_m = 1$$

is a central series for Λ_α, where

$$\overline{H}_i = \langle n_{\lambda\mu}(\overline{a}) : \lambda,\mu \in \Omega,\ \overline{a} \in \overline{A}_{i\alpha} \rangle \qquad 1 < i < m,$$

Clearly $\overline{H}_m = 1$, since $a_x = 0$ if $x > 1$; and it is easy to see that this is a central series.

The group $\mathcal{N}(\Omega:\underline{R})$ therefore has a normal subgroup Λ_α for every real number $\alpha \in (0,1)$; we investigate the factor group.

6.1.7 <u>Lemma</u> $\mathcal{N}(\Omega:\underline{R})/\Lambda_\alpha \cong \mathcal{N}(\Omega:\underline{R})$.

<u>Proof</u> Let $\underline{R}_\alpha$ be a ring constructed in an identical fashion to the construction of $\underline{R}$ except that the generators a'_β are indexed by $\beta \in (0,\alpha)$ instead of $\beta \in (0,1)$.

We show that

$$\mathcal{N}(\Omega:\underline{R}_\alpha) \cong \mathcal{N}(\Omega:\underline{R}),\ \text{for any } \alpha \in (0,1),$$

by constructing an isomorphism

$$\overline{\varphi}_\alpha : \mathcal{N}(\Omega:\underline{R}_\alpha) \longrightarrow \mathcal{N}(\Omega:\underline{R}).$$

Define φ_α by

$$\varphi_\alpha : a'_\beta \longrightarrow a_{\frac{1}{\alpha}\beta} .$$

Under this map every generator a'_β , $\beta \in (0,\alpha)$, is mapped into $a_{\frac{1}{\alpha}\beta}$ and $\frac{1}{\alpha}\beta \in (0,1)$. Conversely, given any $x \in (0,1)$ then $\alpha x \in (0,\alpha)$, so

$$\varphi_\alpha : a'_{\alpha x} \longrightarrow a_x.$$

φ_α extends naturally to an isomorphism

$$\varphi_\alpha : \underset{\sim}{R}_\alpha \longrightarrow \underset{\sim}{R}.$$

The isomorphism φ_α induces an isomorphism

$$\bar{\varphi}_\alpha : N(\Omega:\underset{\sim}{R}_\alpha) \longrightarrow N(\Omega:\underset{\sim}{R}), \text{ c.f. lemma 3.1.9.}$$

To complete the proof we show that

$$N(\Omega:\underset{\sim}{R})/\Lambda_\alpha \cong N(\Omega:\underset{\sim}{R}_\alpha).$$

We define a homomorphism

$$\bar{\theta}_\alpha : N(\Omega:\underset{\sim}{R}) \longrightarrow N(\Omega:\underset{\sim}{R}_\alpha)$$

First define a homomorphism

$$\theta_\alpha : \underset{\sim}{R} \longrightarrow \underset{\sim}{R}_\alpha$$

by
$$\theta_\alpha : a_\beta \longrightarrow a'_\beta , \quad 0 < \beta < \alpha,$$

$$\theta_\alpha : a_\beta \longrightarrow 0 , \quad \alpha < \beta < 1.$$

Now θ_α extends to a homomorphism $\underset{\sim}{R} \longrightarrow \underset{\sim}{R}_\alpha$ with kernel K where the elements of K are precisely those of the form

$$n_1 a_{x_1} + \ldots\ldots\ldots + n_r a_{x_r} , \quad \alpha \leq x_i < 1 \; ; \; 1 \leq i \leq r.$$

Then θ_α induces a homomorphism

$$\bar{\theta}_\alpha : \mathcal{N}(\Omega:\underset{\sim}{R}) \longrightarrow \mathcal{N}(\Omega:\underset{\sim}{R}_\alpha)$$

with kernel Λ_α.

Hence $\mathcal{N}(\Omega:\underset{\sim}{R})/\Lambda_\alpha \cong \mathcal{N}(\Omega:\underset{\sim}{R})$, for all $\alpha \in (0,1)$.

If we embed $\underset{\sim}{J}$ in a ring $\underset{\sim}{R}^*$ by the construction of 2.2.6 then $\mathcal{R}(\Omega:\underset{\sim}{R}^*) \in {}_L\mathfrak{N}$, by theorem 3.2.1. However it is easy to see that $\underset{\sim}{R}$ is a homomorphic image of $\underset{\sim}{R}^*$, and so $\mathcal{R}(\Omega:\underset{\sim}{R})$ is a homomorphic image of $\mathcal{R}(\Omega:\underset{\sim}{R}^*)$ by 3.1.9. Hence $\mathcal{R}(\Omega:\underset{\sim}{R}^*) \in {}_L\mathfrak{N}$. We show that $\mathcal{R}(\Omega:\underset{\sim}{R}) \notin {}_N\mathfrak{A}$.

6.1.8 <u>Lemma</u> $\mathcal{R}(\Omega:\underset{\sim}{R}) \notin {}_N\mathfrak{A}$, if $|\Omega| > 3$.

<u>Proof</u> Consider the subgroup H of $\mathcal{R}(\Omega:\underset{\sim}{R})$,

$$H = \langle t_{\lambda\mu}(a_\alpha, 1), \; (\Omega:\underset{\sim}{R}) \rangle,$$

where λ, μ are fixed elements of Ω and α is a fixed real number $0 < \alpha < 1$.

To show that $\mathcal{R}(\Omega:\underset{\sim}{R}) \notin {}_N\mathfrak{A}$, it is sufficient to show that $H \notin {}_N\mathfrak{A}$, since the class ${}_N\mathfrak{A}$ is s-closed.

We show that $\langle t_{\lambda\mu}(a_\alpha,1)\rangle^H = H_1$

and $\langle t_{\lambda\mu}(a_\alpha,1)\rangle^{H_1} = H_1$

where $H_1 = \langle t_{\lambda\mu}(a_\alpha,1), \mathcal{H}(\Omega:\underline{R})\rangle$.

Denote $\langle t_{\lambda\mu}(a_\alpha,1)\rangle^H$ by $\langle t\rangle^H$. Choose $\sigma \in \Omega$ with $\sigma \neq \lambda,\mu$.
Now

$[t_{\lambda\mu}(a_\alpha,1), n_{\mu\sigma}(a,0)] \in \langle t\rangle^H$. Let $(a,0) \in \underline{R}$ be an element such that

$$(a_\alpha,1)(a,1) = (0,1).$$

We can certainly find such an a, by the construction of $\underline{R}$.
Then

$$[t_{\lambda\mu}(a_\alpha,1), n_{\mu\sigma}(a,0)] = n_{\lambda\sigma}(a_\alpha a+a) = n_{\lambda\sigma}(-a_\alpha),$$

where we write $(-a_\alpha)$ for $(-a_\alpha,0)$.

Hence $n_{\lambda\sigma}(a_\alpha) \in \langle t\rangle^H$ and so for any $\rho_1,\sigma_1 \in \Omega$ we have $n_{\rho_1\sigma_1}(a_\beta) \in \langle t\rangle^H$, for all $\beta > \alpha$. We use here the fact that $|\Omega| > 3$ and so by taking suitable commutators of $n_{\lambda\sigma}(a_\alpha)$ we can obtain $n_{\rho_1\sigma_1}(a_\beta)$.

But for any $b \in \underline{J}$, and $\sigma \neq \lambda,\mu$,

$$[t_{\lambda\mu}(a_\alpha,1), n_{\mu\sigma}(b,0)] = n_{\lambda\sigma}(a_\alpha b+b) \in \langle t\rangle^H.$$

Hence $n_{\lambda\sigma}(a_\alpha b)n_{\lambda\sigma}(b) \in \langle t\rangle^H$.

But $a_\alpha b$ is an element of $\mathfrak{J}$ which contains only basis elements a_β with $\beta > \alpha$

Hence $n_{\lambda\sigma}(b) \in \langle t\rangle^H$, for all $b \in \mathfrak{J}$. Then taking suitable commutators, we obtain

$$n_{\rho\sigma}(b) \in \langle t\rangle^H, \text{ for all } b \in \mathfrak{J}, \text{ and all } \rho, \sigma \in \Omega$$

with $\rho \neq \sigma$.

Since $(\Omega:\underline{R}) = \langle n_{\rho\sigma}(b) : \rho \neq \sigma \rangle$, we have $\langle t\rangle^H \geqslant H_1$

But H_1 is normal in H, so $H_1 = \langle t\rangle^H$.

However we have only conjugated $t_{\lambda\mu}(a_\alpha, 1)$ by elements of H_1 and so we have

$$\langle t\rangle^{H_1} = H_1.$$

Hence $H \notin N\mathfrak{A}$ and so $\mathfrak{R}(\Omega:\underline{R}) \notin N\mathfrak{A}$.

6.2 Example 2 An $\overline{SI}$-group with a free subgroup of rank 2.

We define the ring of p-adic integers, for p a prime. Define the p-adic integer α to be the infinite series

$$\alpha = a_0 + a_1 p + a_2 p^2 + \ldots. + a_n p^n + \ldots.$$

where $0 \leqslant a_n \leqslant p-1$, for $n = 0, 1, 2, 3, \ldots.$

Let α be as defined above and let β be the p-adic integer given by

$$\beta = b_0 + b_1 p + b_2 p^2 + \dots\dots + b_n p^n + \dots\dots\dots$$

where $0 \leq b_n \leq p-1$, for $n = 0,1,2,3,\dots$

Define

$$\alpha + \beta = c_0 + c_1 p + c_2 p^2 + \dots\dots\dots + c_n p^n + \dots\dots$$

where all the coefficients c_n are residues modulo p and

$$c_0 = a_0 + b_0 - pq_0$$

$$c_n = a_n + b_n + q_{n-1} - pq_n$$

Define

$$\alpha\beta = d_0 + d_1 p + d_2 p^2 + \dots\dots\dots + d_n p^n + \dots\dots\dots$$

where the coefficients d_n are also residues modulo p and

$$d_0 = a_0 b_0 - ps_0$$

$$d_n = \sum_{k+l=n} a_k b_l + s_{n-1} - ps_n$$

From this definition of multiplication it follows immediately that 1 is the unit element and the ring of p-adic integers does not contain divisors of zero.

We denote the ring of p-adic integers by R_p. It is easy to see that R_p is a commutative ring.

The ideal generated by p consists precisely of the non-unit elements of R_p and so (p) is the unique maximal ideal of R_p. Hence J, the Jacobson radical of R_p, is the ideal pR_p.

Now $J^n = p^n R_p$, so $\bigcap_{i<\omega} J^i = 0$.

For, if $\alpha \in \bigcap_{i<\omega} J^i$, then

$$\alpha = a_0 + a_1 p + a_2 p^2 + \ldots. + a_n p^n + \ldots. .$$

Suppose a_j is the first non-zero coefficient.

Then

$$\alpha = a_j p^j + a_{j+1} p^{j+1} + \ldots.. , \quad a_j \neq 0,$$

and so $\alpha \notin J^{j+1}$. Thus $a_j = 0$ for $j = 0,1,2,3,\ldots$, and so $\alpha = 0$.

Therefore $\bigcap_{i<\omega} J^i = 0$.

By 3.3.1 and 3.3.2 we have $\mathcal{N}(\Omega:R_p) \in {}_R\mathfrak{N}_\cap \grave{\aleph} \mathfrak{A}$.

If Ω is infinite the conditions of theorem 5.3.1 are satisfied, so

$\mathcal{H}(\Omega:R_p) \in \mathcal{Z}^a$, if Ω is infinite. However

$Rg\{1,p\} \cong \mathbb{Z} < R_p$ and so the conditions of lemma 5.3.2 are satisfied.

Hence $\mathcal{H}(\Omega:R_p) \notin \mathcal{Z}^{as}$; $|\Omega| \neq 1$

Thus the ring R_p gives the example necessary to complete corollary 5.3.5.

Since the elements of R_p have infinite additive order, it is clear that $\mathcal{H}(\Omega:R_p)$ is generated by elements of infinite order. We now prove

6.2.1 <u>Lemma</u> $\mathcal{N}(\Omega:R_p)$ is generated by elements of infinite order, provided $p \neq 2$.

<u>Proof</u> Since $\mathcal{H}(\Omega:R_p)$ is generated by elements of infinite order it is sufficient to show that

$$n_{\lambda\lambda}(pr) \quad , \quad \lambda \in \Omega \quad , \; r \in R_p \; ,$$

has infinite order when $p \neq 2$.

Suppose $(1 + pr)^q = 1$ where q is some prime $q \neq p$. Let

$$\varphi_m : R_p \longrightarrow R_p/p^mR_p \cong \quad {}_{p^m}.$$

Then $\qquad \varphi_m : (1+pr) \longrightarrow 1 + \varphi_m(pr)$

Hence if $(1+pr)^q = 1$ then $(1+\varphi_m(pr))^q = 1$.

Since $\varphi_m(pr) \in \mathbb{Z}_{p^m}$, this implies that $\varphi_m(pr) = 0$ for all integers m.

Then $r = 0$, so no non-trivial generator $n_{\lambda\lambda}(pr)$ can have order q for $q \neq p$.

Now suppose $n^p_{\lambda\lambda}(pr) = 1$.

Then $$(1+pr)^p = 1.$$

Thus $$1 + p.pr + \frac{p(p-1)}{2}(pr)^2 + \dots + (pr)^p = 1.$$

Hence $$p^2 r\{1 + \frac{p(p-1)}{2} r + \dots + r(pr)^{p-2}\} = 0.$$

Since R_p contains no divisors of zero we have

$$1 + \frac{p(p-1)}{2} r + \dots + r(pr)^{p-2} = 0.$$

Hence $$\{p(p-1)/2 + \dots + (pr)^{p-2}\}r = -1.$$

But -1 is a unit of R_p and so r and

$$p(p-1)/2 + \dots + (pr)^{p-2}$$

must also be units.

If $p \neq 2$ then this gives a contradiction since

$$p(p-1)/2 + \dots + (pr)^{p-2}$$

has a factor p.

Hence if $p > 2$ all the generators have infinite order.

If $p = 2$ then the generator $n_{\lambda\lambda}(-p)$ has order two, for

$$(1-p)^2 = 1 - 2p + p^2 = 1 \quad \text{if } p = 2,$$

so $$\{n_{\lambda\lambda}(-p)\}^2 = n_{\lambda\lambda}(0) = 1.$$

We now investigate the group $\mathcal{H}(\Omega:R_p)$ for locally nilpotent normal subgroups. The Hirsch-Plotkin radical of a group is defined to be the join of the locally nilpotent normal subgroups.

6.2.2 <u>Lemma</u> Let Ω be an infinite linearly ordered set. Then the Hirsch-Plotkin radical of $\mathcal{H}(\Omega:R_p)$ is trivial. In fact no normal subgroup can be an Engel group.

<u>Proof</u> By the proof of theorem 5.3.1 we see that

$$K_1 \leq H$$

for every non-trivial normal subgroup H, where for some integer n,

$$K_1 = \langle n_{\rho\sigma}(b) : \rho,\sigma \in \Omega,\ \rho \neq \sigma,\ b \in J^n \rangle .$$

To complete the proof of this lemma it is sufficient to show that K_1 is not in $\mathcal{E}$, for $\mathcal{E}$ is s-closed.

Let λ,μ,α be distinct elements of Ω. Then

$$[n_{\alpha\mu}(b),n_{\mu\alpha}(b)] \in K_1 \quad \text{for } b \in J^n,\ \ b \neq 0.$$

Putting $-b^2 = x$, and using 3.1.1 (ix) we have

$$n_{\alpha\alpha}^{-1}(x)n_{\mu\mu}(x) \in K_1$$

But $$[n_{\lambda\mu}(x),\underbrace{n_{\alpha\alpha}^{-1}(x)n_{\mu\mu}(x),\ldots\ldots\ ,n_{\alpha\alpha}^{-1}(x)n_{\mu\mu}(x)}_{i}]$$

$$= n_{\lambda\mu}(x^{i+1}).$$

However $x^{i+1} \neq 0$, for any integer i, since R_p is an integral domain and x is non-zero.

Then $n_{\lambda\mu}(x^{i+1}) \neq 1$ and this completes the proof.

6.3 <u>Example 3 Another $\overline{SI}$-group with a subgroup not in $\overline{SI}$</u>.

We define the power series ring in the indetermina~~nt~~te x over the division ring D. Elements are of the form

$$\alpha = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_i x^i + \ldots\ldots$$

$$= \sum_{i<\omega} a_i x^i$$

where $a_i \in D$, for $i = 0,1,2,3,\ldots\ldots$

Addition and multiplication are defined by

$$\sum_{i<\omega} a_i x^i + \sum_{i<\omega} b_i x^i = \sum_{i<\omega} (a_i + b_i) x^i ,$$

$$\sum_{i<\omega} a_i x^i . \sum_{i<\omega} b_i x^i = \sum_{i<\omega} c_i x^i , \text{ where } c_i = \sum_{j+l=i} a_j b_l$$

We denote this ring by $D\{x\}$.

The ideal of $D\{x\}$ generated by x consists precisely of the non-unit elements of $D\{x\}$. Hence $xD\{x\}$ is the

Jacobson radical of $D\{x\}$ and $D\{x\}$ satisfies the conditions of 5.3.1.

Suppose the prime field of D is $\mathbb{Q}$, the rationals. Then

$$Rg\{1,x\} \cong \mathbb{Z}[x]$$

and for Ω an infinite linearly ordered set we have, by theorem 5.3.4,

$$\mathcal{H}(\Omega:D\{x\}) \in \mathcal{Z}^{Q},$$

but

$$\mathcal{H}(\Omega:D\{x\}) \notin \mathcal{X}^{QS}.$$

Suppose the prime field of D is $GF(p)$. Then

$$Rg\{1,x\} \cong GF(p)[x]$$

and for Ω an infinite linearly ordered set and $p \neq 2, 3$, we have

$$\mathcal{H}(\Omega:D\{x\}) \in \mathcal{Z}^{Q}.$$

But by theorem 5.3.4 we have $\mathcal{H}(\Omega:D\{x\}) \notin \mathcal{X}^{QS}$

We have proved the following result.

6.3.1 <u>Lemma</u> Let Ω be an infinite set and let D be a division ring whose prime field is not isomorphic to $GF(p)$ for $p = 2$ or 3.

Then

$$\mathcal{H}(\Omega:D\{x\}) \in \mathcal{Z}^{Q} - \mathcal{Z}^{QS}$$

$$\mathcal{H}(\Omega:D\{x\}) \in \mathcal{Y}^{Q} - \mathcal{Y}^{QS}$$

$$\mathcal{H}(\Omega : D\{x\})' \in \mathfrak{X}^{a} - \mathfrak{X}^{as}$$

6.4 Example 4 An infinite simple group.

We give the construction of the Sasiada simple radical ring, see [11].

Let x and y be two non-commuting indeterminates and let k be a field. Denote by $k\{\{x,y\}\}$ the power series ring in x and y over k.

Let J denote the ideal of $k\{\{x,y\}\}$ consisting of those power series with zero constant term. The elements of J are non-units of $k\{\{x,y\}\}$. Also if $\alpha \in J$ then $1 + \alpha$ is a unit of $k\{\{x,y\}\}$ and so J is a radical ideal of $k\{\{x,y\}\}$.

Let $\alpha \in k\{\{x,y\}\}$. Then $\alpha \in J^n$ if and only if the degrees of all the terms of α are greater than or equal to n.

Hence if $\alpha \in \bigcap_{i<\omega} J^i$, then $\alpha = 0$, giving

$$\bigcap_{i<\omega} J^i = 0.$$

By 3.3.1 we have

$$\mathcal{H}(\Omega : k\{\{x,y\}\}) \in {}_R\mathfrak{N}.$$

In [ ] it is shown that there exists an ideal of J which contains $x - yx^2y$ but not x. Let M be a maximal ideal with respect to containing $x - yx^2y$ and not x. Consider the image of x under the natural homomorphism

$$\varphi : J \longrightarrow J/M.$$

Let S be the ideal of J/M generated by $\varphi(x)$. Then S is a simple non-trivial radical ring. Embed S in a ring R_S with a 1 by the construction of 2.2.6.

For any infinite linearly ordered set Ω, $\mathcal{H}(\Omega : R_S)$ is simple, by theorem 4.2.11.

6.4.1 <u>Lemma</u> Let Ω be an infinite linearly ordered set. Then for each prime p there exists a ring R_S so that $\mathcal{H}(\Omega : R_S)$ is a simple group generated by elements of order p.

<u>Proof</u> Given a prime p, let k = GF(p). Let $k\{\{x,y\}\}$ be the power series ring in two non-commuting indeterminants x and y. Define the simple radical ring S from $k\{\{x,y\}\}$ as above.

Consider $k\{\{x,y\}\}$ as an algebra over GF(p). Then $k\{\{x,y\}\}$ has a unit element and so algebra ideals and ring ideals coincide. Hence S is an algebra over GF(p) since S is obtained from $k\{\{x,y\}\}$ by taking ideals and homomorphic images.

Thus every element of S has additive order p. Let R_S be any ring with a 1 having S as its Jacobson radical.

Let $n_{\lambda\mu}(a)$ $,\lambda,\mu \in \Omega,\ \lambda \neq \mu,\ a \in S,$

be a generator of $\mathcal{H}(\Omega:R_S)$.

Then

$$n^p_{\lambda\mu}(a) = n_{\lambda\mu}(pa) = 1,$$

and all the generators of the simple group $\mathcal{H}(\Omega:R_S)$ have order p.

§7 Generalisation of the Sylow p-subgroup of the symplectic group.

We shall construct a class of groups by generalising the Sylow p-subgroup of the symplectic group over the field GF(p).

7.1 The Sylow p-subgroup of the symplectic group.

In this section we find a set of generators for the Sylow p-subgroup of the symplectic group. In all that follows we suppose p is a prime with $p \neq 2$.

7.1.1 Definition The symplectic group $Sp_{2l}(p)$ is the set of all $2\ell \times 2\ell$ matrices T over GF(p) satisfying

$$TAT' = A,$$

where $A = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, I is the $l \times l$ identity, and

T' denotes the transpose of T.

The order of $Sp_{2l}(p)$ is

$$p^{l^2}(p^{2l}-1)\ldots\ldots\ (p^2-1) \quad ,$$

see [3] page 94, and hence the Sylow p-subgroup of $Sp_{2l}(p)$ have order p^{l^2}.

7.1.2 Definition Let G be the subgroup of $Sp_{2l}(p)$ generated by

$$t_{ij} = \begin{bmatrix} I + E_{ij} & 0 \\ 0 & I - E_{ji} \end{bmatrix} \qquad 0 < i < j \leqslant l$$

$$s_{ij} = \begin{bmatrix} I & 0 \\ E_{ij} + E_{ji} & I \end{bmatrix} \qquad 0 < i \leqslant j \leqslant l$$

where E_{ij} denotes the $l \times l$ matrix whose (i,j) coefficient is 1 and all other coefficients are zero.

We show that G is in fact a Sylow p-subgroup of $Sp_{2l}(p)$ by showing that G is a subgroup of $Sp_{2l}(p)$ of order p^{l^2}.

7.1.3 Theorem G is a Sylow p-subgroup of $Sp_{2l}(p)$.

Proof It is easy to check that $t_{ij} \in Sp_{2l}(p)$ and $s_{ij} \in Sp_{2l}(p)$.

Now $$\begin{bmatrix} I & 0 \\ E_{ij} + E_{ji} & I \end{bmatrix}^n = \begin{bmatrix} I & 0 \\ n(E_{ij}+E_{ji}) & I \end{bmatrix}$$

and so s_{ij} has order p. But all the s_{ij}'s commute and so the product of any number of s_{ij}'s has order p. Let S be the subgroup of $Sp_{2l}(p)$ generated by the s_{ij}'s.

$$S = \langle s_{ij} : 0 < i \leq j \leq l \rangle$$

Then S is elementary Abelian and a p-group of order $p^{l(l+1)/2}$. The t_{ij}'s generate a subgroup T of $Sp_{2l}(p)$ of order $p^{l(l-1)/2}$.

Clearly $T \cap S = 1$.

Also T normalises S. For,

$$\begin{bmatrix} L^{-1} & 0 \\ 0 & N^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ M & I \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & N \end{bmatrix} = \begin{bmatrix} I & 0 \\ N^{-1}ML & I \end{bmatrix}$$

If $\begin{bmatrix} I & 0 \\ M & I \end{bmatrix} \in S$, we have $M' = M$.

If $\begin{bmatrix} L & 0 \\ 0 & N \end{bmatrix} \in T$, we have $LN' = I$.

Hence $(N^{-1}ML)' = L'M'(N^{-1})' = L'M(N^{-1})'$

$$= N^{-1}ML, \text{ since } L' = N^{-1}.$$

This shows that $\begin{bmatrix} I & 0 \\ N^{-1}ML & I \end{bmatrix} \in S.$

Then $G = ST$, the split extension of S by T and

$$|G| = |S||T| = p^{l(l+1)/2}\,p^{l(l-1)/2} = p^{l^2}.$$

This completes the proof.

7.1.4 <u>Corollary</u> The Sylow p-subgroup G of $Sp_{2l}(p)$ is a split extension of an elementary Abelian p-group S of order $p^{l(l+1)/2}$ by an isomorphic copy of the Sylow p-subgroup of the general linear group $GL(l,p)$ of order $p^{l(l-1)/2}$.

7.2 <u>The group $\mathcal{G}(\Omega:p)$</u>

We have considered G as a group of linear transformations on a $2l$ dimensional vector space. We generalise this concept to obtain a group of linear transformations on an infinite dimensional vector space.

7.2.1 <u>Definition</u> Let Ω be any linearly ordered set. Introduce Ω_1, a linearly ordered set order isomorphic to Ω. Define $\bar{\Omega}$ to be the linearly ordered set $\Omega_1 \cup \Omega$, where $\lambda_1 < \lambda$ for any $\lambda_1 \in \Omega_1$ and $\lambda \in \Omega$. From now on we denote the element of Ω_1 corresponding to $\lambda \in \Omega$ under the isomorphism by λ_1.

Let V be a vector space over GF(p), where $p \neq 2$, with basis $v_{\bar{\lambda}}$ for $\bar{\lambda} \in \bar{\Omega}$.

For $\lambda, \mu \in \Omega$ with $\lambda < \mu$, we define the map

$$t_{\lambda\mu} : V \longrightarrow V$$

by

$$t_{\lambda\mu} : v_\lambda \longrightarrow v_\lambda + v_\mu$$

$$t_{\lambda\mu} : v_\nu \longrightarrow v_\nu , \qquad \nu \neq \lambda , \quad \nu \in \Omega$$

$$t_{\lambda\mu} : v_{\mu_1} \longrightarrow v_{\mu_1} - v_{\lambda_1}$$

$$t_{\lambda\mu} : v_{\nu_1} \longrightarrow v_{\nu_1} , \quad \nu_1 \neq \mu_1 , \quad \nu_1 \in \Omega_1 .$$

For $\lambda, \mu \in \Omega$ we define the map

$$s_{\lambda\mu} : V \longrightarrow V$$

by

$$s_{\lambda\mu} : v_\nu \longrightarrow v_\nu , \qquad \nu \in \Omega ,$$

$$s_{\lambda\mu} : v_{\mu_1} \longrightarrow v_{\mu_1} + v_\lambda$$

$$s_{\lambda\mu} : v_{\lambda_1} \longrightarrow v_{\lambda_1} + v_\mu$$

$$s_{\lambda\mu} : v_{\nu_1} \longrightarrow v_{\nu_1} , \quad \nu_1 \neq \lambda_1, \mu_1 , \quad \nu_1 \in \Omega .$$

Let $\mathcal{G}(\Omega : p)$ be the group generated by

$$\langle t_{\lambda\mu}, s_{\alpha\beta} : \lambda < \mu,\ \lambda,\mu,\alpha,\beta \in \Omega \rangle$$

By the definition of $s_{\lambda\mu}$ we see that

$$s_{\lambda\mu} = s_{\mu\lambda} .$$

If $\Omega_1 = \{1,2,\ldots\ldots,l\}$, where $\{1,2,\ldots\ldots,l\}$ is ordered in the natural way, then the group $\mathcal{G}(\Omega_1 : p)$ is isomorphic to the Sylow p-subgroup of the symplectic group. We note however that the generators given here are not identical to those given before since in the definition of the Sylow p-subgroup of the symplectic group

$$s_{11} : v_{1_1} \longrightarrow v_{1_1} + 2v_1 .$$

Since $p \neq 2$ and the vector space V is over GF(p), then the two definitions are equivalent. We now prove a result generalising corollary 7.1.4.

7.2.2 <u>Theorem</u> The group $\mathcal{G}(\Omega : p)$ is a split extension of an elementary Abelian p-group S by McLain's group $T_\Omega(p)$. The action of $T_\Omega(p)$ on S is given by the following six equations.

(i) $t_{\alpha\beta}^{-1} s_{\lambda\mu} t_{\alpha\beta} = s_{\lambda\mu}$ if λ,μ are distinct from α,β.

In the next five equations λ,μ,α,β are distinct elements of Ω.

$$\text{(ii)} \quad t_{\lambda\beta}^{-1}s_{\lambda\mu}t_{\lambda\beta} = s_{\lambda\mu}s_{\mu\beta}$$

$$\text{(iii)} \quad t_{\alpha\mu}^{-1}s_{\lambda\mu}t_{\alpha\mu} = s_{\lambda\mu}$$

$$\text{(iv)} \quad t_{\lambda\mu}^{-1}s_{\lambda\mu}t_{\lambda\mu} = s_{\lambda\mu}s_{\mu\mu}^{2}$$

$$\text{(v)} \quad t_{\lambda\mu}^{-1}s_{\lambda\lambda}t_{\lambda\mu} = s_{\lambda\lambda}s_{\lambda\mu}s_{\mu\mu}$$

$$\text{(vi)} \quad t_{\alpha\lambda}^{-1}s_{\lambda\lambda}t_{\alpha\lambda} = s_{\lambda\lambda}$$

<u>Proof</u> Equations (i) – (vi) are easily verified by considering the action of the elements on the vector space V. Then defining

$$S = \langle s_{\lambda\mu} : \lambda,\mu \in \Omega \rangle ,$$

$$T = \langle t_{\lambda\mu} : \lambda < \mu,\ \lambda,\mu \in \Omega \rangle ,$$

we see that $\mathcal{G}(\Omega : p)$ is an extension of S by T. This extension splits, since clearly $S \cap T = 1$. It is easy to see that the $s_{\lambda\mu}$'s commute and so S is an elementary Abelian subgroup.

To see that T is isomorphic to $T_\Omega(p)$ we define

$$\varphi : T \longrightarrow T_\Omega(p)$$

by $$\varphi : \begin{bmatrix} L & 0 \\ 0 & N \end{bmatrix} \longrightarrow L$$

where L and N are $\Omega\times\Omega$ matrices. Then φ is an isomorphism and the result follows.

All McLain groups $T_\Omega(p)$ are Baer nilgroups and we now investigate the groups $\mathcal{G}(\Omega;p)$ for this property.

7.2.3 <u>Theorem</u> $\mathcal{G}(\Omega;p) \in N\mathfrak{A}$.

<u>Proof</u> Since $\mathcal{G}(\Omega;p)/S \cong T_\Omega(p) \in N\mathfrak{A}$, it is sufficient to prove

$$\langle S, t_{\alpha\beta} \rangle \in N\mathfrak{A},$$

for α, β fixed elements of Ω.

But S is a normal elementary Abelian p-subgroup of $\langle S, t_{\alpha\beta}\rangle$. Hence $\langle S, t_{\alpha\beta}\rangle$ is soluble and by 3.1.13 we need only prove $\langle S, t_{\alpha\beta}\rangle$ is an Engel group of bounded class.

Let $s \in S$ and $t = t_{\alpha\beta}^n$ for some integer n. Then we must show

$$[s, \underbrace{t, t, \ldots\ldots, t}_{m}] = 1 \quad , \text{ for some } m.$$

All other commutators that must be checked to show that $\langle S, t_{\alpha\beta}\rangle$ is an Engel group become trivial after at most two steps.

Consider the endomorphism ring of $\langle S, \cancel{t_{\alpha\beta}}\rangle$. The automorphism t acts by conjugation

$$t : a \longrightarrow t^{-1}at = a^t, \quad \text{for} \quad a \in \langle S, t_{\alpha\beta} \rangle.$$

Then $$s^{t-1} = [s,t] \quad \text{and} \quad [s,\underbrace{t,t,\ldots\ldots,t}_{p}] = s^{(t-1)^p}.$$

But s, t have order p so

$$s^{(t-1)^p} = s^{t^p - 1} = s^0 = 1.$$

Hence $$[s,\underbrace{t,t,\ldots\ldots,t}_{p}] = 1,$$ and the proof is complete.

Let α and β be fixed elements of Ω. Define

$$S_\beta = \langle s_{\lambda\mu} : \mu > \beta, \ \lambda,\mu \in \Omega \rangle$$

$$T_{\alpha\beta} = \langle t_{\lambda\mu} : \lambda < \alpha, \ \mu > \beta, \ \lambda,\mu \in \Omega \rangle.$$

Let $H = \langle t_{\alpha\beta} \rangle^{\mathcal{G}(\Omega:p)}$. Then $\langle S_\beta, T_{\alpha\beta} \rangle = H$, and H is nilpotent of class two. However we only need the following result which will be used in the proof of theorem 7.2.8.

7.2.4 <u>Lemma</u> $S_\beta < \langle t_{\alpha\beta} \rangle^{\mathcal{G}(\Omega:p)}$.

<u>Proof</u> Let $H = \langle t_{\alpha\beta} \rangle^{\mathcal{G}(\Omega:p)}$.

Now $t_{\alpha\beta} \in H$, and so $s_{\lambda\alpha}^{-1} t_{\alpha\beta} s_{\lambda\alpha} \in H$, for $\lambda \neq \alpha, \beta$.

But $(t_{\alpha\beta}^{-1}s_{\lambda\alpha}^{-1}t_{\alpha\beta})s_{\lambda\alpha} = (s_{\lambda\beta}^{-1}s_{\lambda\alpha}^{-1})s_{\lambda\alpha}$, from 7.2.2 (ii).

Thus $$t_{\alpha\beta}s_{\lambda\beta}^{-1} \in H,$$

but since $t_{\alpha\beta} \in H$ we have $s_{\lambda\beta} \in H$, for $\lambda \neq \alpha,\beta$.

To check that $s_{\beta\beta} \in H$ we use 7.2.2 (iii) to obtain

$$(t_{\alpha\beta}^{-1}s_{\alpha\beta}^{-1}t_{\alpha\beta})s_{\alpha\beta} = s_{\beta\beta}^{-2}s_{\alpha\beta}^{-1}s_{\alpha\beta} = s_{\beta\beta}^{-2}.$$

Hence $s_{\beta\beta}^{-2} \in H$ and since $p \neq 2$ this implies that $s_{\beta\beta} \in H$.

Now $$s_{\alpha\alpha}^{-1}t_{\alpha\beta}s_{\alpha\alpha} = t_{\alpha\beta}s_{\alpha\beta}^{-1}s_{\beta\beta}^{-1}$$

and so we have $$s_{\alpha\beta} \in H.$$

But $t_{\beta\mu}^{-1}s_{\lambda\beta}t_{\beta\mu} \in H$, for $\lambda < \beta$, $\mu > \beta$ and so, by 7.2.2 (ii)

$$s_{\lambda\beta}s_{\lambda\mu} \in H.$$

Hence $s_{\lambda\mu} \in H$ for any $\lambda < \beta$, $\mu > \beta$. From what we have already proved this gives

$$s_{\lambda\mu} \in H, \text{ for } \lambda < \beta,\ \mu > \beta.$$

Now $t_{\lambda\gamma}^{-1}s_{\lambda\mu}t_{\lambda\gamma} = s_{\lambda\mu}s_{\mu\gamma}$, from 7.2.2 (ii), and so

$$s_{\mu\gamma} \in H, \text{ for } \mu > \beta,\ \gamma < \lambda.$$

Hence we have shown that $s_{\mu\gamma} \in H$, for $\mu > \beta$ and γ arbitrary, and so $S_\beta \in H$.

We have shown that $\mathcal{G}(\Omega:p) \in {}_N\mathfrak{A}$ and so $\mathcal{G}(\Omega:p) \in {}_L\mathfrak{N}$. We investigate the centre of $\mathcal{G}(\Omega:p)$ in the following lemma.

7.2.5 <u>Lemma</u> The centre of $\mathcal{G}(\Omega:p)$ is trivial if Ω has no greatest element.

<u>Proof</u> Suppose that Ω has no greatest element and suppose the centre of $\mathcal{G}(\Omega:p)$ is Z, where Z is non-trivial.

Let z be a non-trivial element of Z. Then

$$z = st, \text{ where } s \in S \text{ and } t \in T.$$

Then

$$t_{\alpha\beta}^{-1} s t_{\alpha\beta} t_{\alpha\beta}^{-1} t t_{\alpha\beta} = st.$$

But $t_{\alpha\beta}^{-1} s t_{\alpha\beta} = s' \in S$, and $t_{\alpha\beta}^{-1} t t_{\alpha\beta} = t' \in T$.

Thus $st = s't'$ and so $s = s'$, and $t = t'$, since $S \cap T = 1$. Hence t is in the centre of T. But this is trivial when Ω has no greatest element by 1.3.5, and so $t = 1$.

Let $z = s_{i_1 j_1}^{\xi_1} s_{i_2 j_2}^{\xi_2} \cdots\cdots s_{i_n j_n}^{\xi_n}$.

Choose $\lambda > i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_n$. This is always possible since Ω has no greatest element.

We can suppose i_1 is maximal among $i_1, \ldots i_n, j_1, \ldots, j_n$, and that j_1 is the greatest among the sufficies occuring in a term with a i_1.

Then $\quad t_{i_1\lambda}^{-1} z t_{i_1\lambda} = z.$

So $\quad s_{i_1 j_1}^{\xi_1} s_{j_1\lambda}^{\zeta_1} t_{i_1\lambda}^{-1}(s_{i_2 j_2}^{\xi_2} \ldots\ldots\ldots s_{i_n j_n}^{\xi_n}) t_{i_1\lambda} = z.$

But no power of $s_{j_1\lambda}$ can occur in

$$t_{i_1\lambda}^{-1}(s_{i_2 j_2}^{\xi_2} \ldots\ldots s_{i_n j_n}^{\xi_n}) t_{i_1\lambda}$$

by the choice of i_1 and j_1. Hence $z = 1$, and the centre of $\mathcal{G}(\Omega:p)$ is trivial.

We now investigate self-normalising subgroups of $\mathcal{G}(\Omega:p)$.

7.2.6 <u>Theorem</u> If Ω has no greatest element then T is self-normalising in $\mathcal{G}(\Omega:p)$.

<u>Proof</u> Let $g \in \mathcal{G}(\Omega:p)$ normalise T.
Then $g = ts$ and so $s^{-1}t^{-1}Tts = T$, giving $s^{-1}Ts = T$ and so s normalises T.

Then $s^{-1}t_{\alpha\beta}s = t$ and so $t_{\alpha\beta}^{-1}s^{-1}t_{\alpha\beta}s = t'$.

But $t_{\alpha\beta}^{-1}s^{-1}t_{\alpha\beta} = s'$, say, and hence $s's = t'$.

But $S \cap T = 1$, and so $s' = s^{-1}$ and so s commutes with all $t_{\alpha\beta}$.

Hence $s \in Z$, the centre of $\mathcal{G}(\Omega:p)$ and so $s = 1$, since Ω has no greatest element.

Thus T is self-normalising as required.

7.2.7 <u>Corollary</u> If Ω has no greatest element then T is not ascendant in $\mathcal{G}(\Omega:p)$.

If Ω_1 is a finite set, then $\mathcal{G}(\Omega_1:p)$ is a finite p-group and so $\mathcal{G}(\Omega_1:p) \in \mathfrak{N}$. In this case T cannot be self-normalising since T must be subnormal. In general, however, T is not ascendant or descendant in $\mathcal{G}(\Omega:p)$. We now give a result which generalises 7.2.6 and 7.2.7.

7.2.8 <u>Theorem</u> (a) The subgroup T of $\mathcal{G}(\Omega:p)$ is descendant in $\mathcal{G}(\Omega:p)$ if and only if Ω is well ordered. If Ω has no least element the normal closure of T in $\mathcal{G}(\Omega:p)$ is $\mathcal{G}(\Omega:p)$.

(b) The subgroup T of $\mathcal{G}(\Omega:p)$ is ascendant in $\mathcal{G}(\Omega:p)$ if and only if Ω^* is well ordered. If Ω has no greatest element T is self-normalising in $\mathcal{G}(\Omega:p)$.

<u>Proof</u> (a) Suppose Ω is well ordered. We show that

$$\mathcal{G}(\Omega:p) \geqslant H_1 \geqslant H_2 \geqslant \ldots\ldots \geqslant H_\rho \geqslant \ldots\ldots\ldots \geqslant T.$$

is a descending series, where

$$H_\rho = \langle T, s_{\alpha\beta} : \alpha, \beta \in \Omega, \alpha + \beta > \rho \rangle$$

if ρ is not a limit ordinal and

$$H_\rho = \bigcap_{\mu<\rho} H_\mu$$

if ρ is a limit ordinal.

To show that the series is a descending series we must show that

$$H_\rho \text{ is normal in } H_{\rho-1}$$

where ρ is not a limit ordinal.

It is sufficient to show that $s_{\alpha\beta}$ normalises H_ρ where $\alpha + \beta = \rho - 1$.

$$s_{\alpha\beta}^{-1} t_{\alpha\gamma} s_{\alpha\beta} = t_{\alpha\gamma} s_{\beta\gamma}^{-1} \in H_\rho$$

since $\beta + \gamma > \beta + \alpha = \rho - 1$ and so $\beta + \gamma > \rho$.

Checking the other possibilities in a similar manner the result follows.

To prove the converse we first suppose that Ω has no least element. Then

$$S_\beta < \langle T \rangle^{\mathcal{J}(\Omega:p)} \text{, for } \beta \in \Omega \text{, by 7.2.4.}$$

But $$S_\beta = \langle s_{\lambda\mu} : \mu > \beta, \lambda, \mu \in \Omega \rangle$$

and since Ω has no least element

$$S < \langle T \rangle^{\mathcal{G}(\Omega : p)} .$$

So $\langle T \rangle^{\mathcal{G}(\Omega : p)} = \mathcal{G}(\Omega : p)$.

Suppose that T is descendant in $\mathcal{G}(\Omega : p)$. Then Ω must have a least element. If Ω is not well ordered, then there must exist $\alpha \in \Omega$ with no immediate sucessor. Denote all $\lambda \in \Omega$ with $\lambda > \alpha$ by Ω_α. Consider the subgroup $\mathcal{G}(\Omega_\alpha : p)$ of $\mathcal{G}(\Omega : p)$.

Let $G_\alpha = \mathcal{G}(\Omega_\alpha : p)$ and let T_α denote the subgroup T in $\mathcal{G}(\Omega_\alpha : p)$.

Since T is descendant in $\mathcal{G}(\Omega : p)$ then $G_\alpha \cap T$ is descendant in G_α. But $G_\alpha \cap T = T_\alpha$ and T_α cannot be descendant in G_α since $\mathcal{G}(\Omega_\alpha : p) = G_\alpha$ and Ω_α has no least element.

This contradiction completes the proof of (a).

<u>Proof of (b)</u> Suppose Ω^* is well ordered. We show that

$$T < H_1 < H_2 < \ldots\ldots < H_\rho < \ldots\ldots < \mathcal{G}(\Omega : p)$$

is an ascending series, where

$$H_\rho = \langle T, s_{\alpha\beta} : \alpha, \beta \in \Omega, \alpha + \beta > \rho^* \rangle$$

if ρ is not a limit ordinal, while

$$H_\rho = \bigcup_{\lambda<\rho} H_\lambda \quad ,$$

if ρ is a limit ordinal.

It is sufficient to show

$$H_{\rho-1} \text{ is normal in } H_\rho \ ,$$

where ρ is not a limit ordinal. Since $(\rho - 1)^* = \rho^* + 1$, then

$$H_{\rho-1} = \langle T, s_{\alpha\beta} : \alpha, \beta \in \Omega, \ \alpha + \beta \geqslant \rho^* + 1 \rangle$$

The proof can now be completed in exactly the same way as (a).

We now look at $\mathcal{G}(\mathbb{Z}:p)$ where $\mathbb{Z}$ is the set of all integers in their natural order. We show that the subgroup T of $\mathcal{G}(\mathbb{Z}:p)$ is generated by two subgroups which are ascendant in $\mathcal{G}(\mathbb{Z}:p)$, but T is a self-normalising subgroup of $\mathcal{G}(\mathbb{Z}:p)$.

7.2.9 Theorem The subgroups A and B are ascendant Abelian subgroups of $\mathcal{G}(\mathbb{Z}:p)$ where

$$A = \langle t_{2i\ 2i+1} : i \in \mathbb{Z} \rangle$$

$$B = \langle t_{2i-1\ 2i} : i \in \mathbb{Z} \rangle .$$

Proof It is easy to see that A and B are Abelian subgroups. We show that A is ascendant in $\mathcal{G}(\mathbb{Z}:p)$.

We define

$$S_n = \langle s_{ij} : -2n < i < j < 2n-1 \rangle$$

$$T_n = \langle t_{ij} : -2n < i < j < 2n-1 \rangle$$

$$G_n = \langle S_n, T_n \rangle$$

We show firstly that $\langle A, G_n \rangle$ is subnormal in $\langle A, G_{n+1} \rangle$. Now all but a finite number of generators of A centralise G_{n+1} so write

$$A = A_0 A^*$$

where A_0 is a finite p-group and A^* centralises G_{n+1}. But A^* will also centralise G_n since $G_n < G_{n+1}$ and so A^* will centralise all subgroups between G_n and G_{n+1}.

However $\langle A_0, G_n \rangle$ is subnormal in $\langle A_0, G_{n+1} \rangle$ since these are finite p-groups and so are nilpotent.

Let $H_n = \langle A_0, G_n \rangle$ and let $H_{n+1} = \langle A_0, G_{n+1} \rangle$.

Then we have a finite series

$$H_n = H_{n,0} < H_{n,1} < \ldots\ldots < H_{n,m(n)} = H_{n+1}.$$

But A* centralises $H_{n,i}$ for $0 < i < m(n)$ and so we have a series

$$\langle A, G_n \rangle = A^*H_{n,0} < A^*H_{n,1} < \ldots\ldots < A^*H_{n,m(n)} = \langle A, G_{n+1} \rangle$$

Then $\langle A, G_n \rangle$ is subnormal in $\langle A, G_{n+1} \rangle$ as required.

We have the following series of subgroups each being subnormal in the next.

$$A = \langle A, G_0 \rangle < \langle A, G_1 \rangle < \ldots\ldots < \langle A, G_n \rangle < \ldots\ldots\ldots$$

If we take the union of the countable number of groups we have

$$\bigcup_{n<\omega} \langle A, G_n \rangle = \mathcal{G}(\mathbb{Z}:p).$$

We have therefore an ascendant series from A to $\mathcal{G}(\mathbb{Z}:p)$. In the same way B is an ascendant subgroup of $\mathcal{G}(\mathbb{Z}:p)$ and we have completed the proof.

7.2.10 Corollary $\mathcal{G}(\mathbb{Z}:p)$ is the join of two ascendant Abelian subgroups A and B, and an Abelian normal subgroup S.

7.2.11 <u>Corollary</u> The group $\mathcal{J}(\mathbb{Z}:p)$ has two ascendant Abelian subgroups A and B whose join is T, a self-normalising subgroup of $\mathcal{J}(\mathbb{Z}:p)$.

It may be interesting to compute a series from $\langle A, G_n\rangle$ to $\langle A, G_{n+1}\rangle$. We do this in two steps by finding a series from $\langle A, G_n\rangle$ to $\langle A, G_n, S_{n+1}\rangle$ and then from $\langle A, G_n, S_{n+1}\rangle$ to $\langle A, G_n, S_{n+1}, T_{n+1}\rangle = \langle A, G_{n+1}\rangle$.

Define

$$N_0 = 1$$

$$N_i = \langle N_{i-1}, s_{kl} : -2n-2 \leq k \leq l \leq 2n+1,\ k+l = 4n+3-i\rangle$$

$$M_0 = 1$$

$$M_i = \langle M_{i-1}, t_{-2n-3+i\ 2n+1}, t_{-2n-2\ 2n+2-i} : n \in \mathbb{Z}\rangle$$

7.2.12 <u>Lemma</u> The following is a series from $\langle A, G_n\rangle$ to $\langle A, G_{n+1}\rangle$.

$$\langle A, G_n\rangle < \langle A, G_n, N_1\rangle < \ldots < \langle A, G_n, N_{8n+7}\rangle = \langle A, G_n, S_{n+1}\rangle$$

$$< \langle A, G_n, S_{n+1}, M_1\rangle < \langle A, G_n, S_{n+1}, M_2\rangle < \ldots < \langle A, G_n, S_{n+1}, M_{4n+3}\rangle$$

$$= \langle A, G_{n+1}\rangle.$$

<u>Proof</u> It is sufficient to show that

$$\langle A, G_n, N_i \rangle \text{ is normal in } \langle A, G_n, N_{i+1} \rangle$$

and that

$$\langle A, G_n, S_{n+1}, M_i \rangle \text{ is normal in } \langle A, G_n, S_{n+1}, M_{i+1} \rangle.$$

The first of these can be checked from equations 7.2.2 (i)–(vi) while the second is clear since

$$t_{-2n-2+i\ 2n+1} \quad \text{and} \quad t_{-2n-2\ 2n+1-i}$$

normalise $\langle A, G_n, S_{n+1}, M_i \rangle$.

References

1. Baer, R. Nilgruppen. Math. Z., 62 (1955), 402 - 437.

2. Brenner, J. L. Quelque groupes libres de matrices. Comptes Rendus, 241 (1955), 1689 - 1691.

3. Dickson, L. E. "Linear groups" Leipzig, 1901 - Reprint Dover, 1958.

4. Divinsky, N. J. "Rings and Radicals" Allen and Unwin, London, 1965.

5. Gruenberg, K. W. The Engel elements of a soluble group. Illinois J. Math., 3 (1959), 151 - 167.

6. Hall, P. A note on $\overline{SI}$-groups. J. London Math. Soc., 39 (1964), 338 - 344.

7. Hall, P. On non-strictly simple groups. Proc. Cambridge Philos. Soc., 59 (1963), 531 - 553.

8. Jacobson, N. Structure of Rings. Amer. Math. Soc. Colloquium Publications XXXVII.

9. McLain, D. H. A characteristically simple group. Proc. Cambridge Philos. Soc., 58 (1962), 170 - 184.

10. Northcott, D. G. "Ideal Theory" Cambridge University Press, 1960.

11. Sasiada, E. and Cohn, P. M. An example of a simple radical ring. J. Algebra, 5 (1967), 373 - 377.

12. Zassenhaus, H. "The Theory of Groups" 2nd. ed. Chelsea, New York, 1958.

13. Hall, P. and Hartley, B. The stability group of a series of subgroups. Proc. London Math. Soc., (3) 16 (1966), 1-39.

14. Speiser, A. "Die Theorie der Gruppen von endlicher Ordnung" Dover, New York, 1945.