ENTROPY AND ERGODICITY OF SKEW-PRODUCTS OVER SUBSHIFTS OF FINITE TYPE AND CENTRAL LIMIT ASYMPTOTICS

by

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To my parents
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Declaration

The work in Chapters II and III is original as far as I am aware, except when explicitly stated to the contrary. Chapter IV is joint work with W. Parry and is to appear in the *Israel Journal of Maths* Vol. 69, No. 1, (1990).

SUMMARY

We study various aspects of the dynamics of skew-products ($\mathbb{R}$ and $\mathbb{Z}$-extensions) over a subshift of finite type (ssft).

In Chapter I we give the basic definitions and terminology.

Conditions are given in Chapter II to ensure ergodicity of the skew-product defined by a function of summable variation with respect to an invariant measure $\mu \times \lambda$, where μ is an ergodic shift-invariant Borel probability measure which is quasi-invariant under finite coordinate changes in the shift space (or under finite block exchanges), and λ denotes Lebesgue measure on $\mathbb{R}$ or $\mathbb{Z}$ (depending on whether the function is real or integer valued).

In Chapter III we define a topological entropy concept for the skew-product (defined by a Hölder continuous function f), which is given by the growth rate of periodic orbits of bounded f-weights, and we show that this is the minimum value of the pressure function of f.

The asymptotics in the central limit theorem is studied in Chapter IV, for the class of Hölder continuous functions defined on a subshift of finite type endowed with a stationary equilibrium state of another Hölder continuous function.

CONTENTS

INTRODUCTION

As there are already well-known applications of subshifts of finite type (ssft's) and skew-products in ergodic theory and related fields, we shall concentrate on the topics of this thesis, which are the study of various properties concerning skew-products (in our case $\mathbb{R}$ and $\mathbb{Z}$-extensions) over a ssft. In the Concluding Remarks we provide some motivation for this study arising from classification problems of Markov shifts.

The first problem considered in this thesis is to give sufficient conditions to ensure that an ergodic shift-invariant Borel probability measure μ on an irreducible ssft Σ lifts to an ergodic measure $\mu \times \lambda$ for the skew-product S_f: $(x, t) \mapsto (\sigma x, t + f(x))$, where f is a continuous function on Σ taking values in $\mathbb{R}$ or $\mathbb{Z}$, and λ denotes Lebesgue measure on $\mathbb{R}$ or $\mathbb{Z}$, respectively.

When f is a μ-measurable coboundary then the partial sums $f^n(x) = f(x) + f(\sigma x) + \dots + f(\sigma^{n-1} x)$ are bounded in measure, and hence $\mu \times \lambda$ cannot be an ergodic measure for S_f. This observation shows that we should impose the condition that f is not a coboundary and μ is fully supported. Since ergodicity of the skew-product implies recurrence we should also impose the condition $\int f \, d\mu = 0$ (see Proposition I.3.4).

Our approach to this problem relies on the analysis of the asymptotic range of the cocycle defined by the function f for the $\mathbb{Z}$-action of the shift on (Σ, μ). We use the basic tools developed by Schmidt in [ScK.1], concerning the set of essential values of cocycles for general ergodic transformation groups, applied to our case. A geometric interpretation of an essential value $\alpha \neq \infty$ (see definition in Section I.3 of this thesis) is that it is a number in A (where A = $\mathbb{R}$ or $\mathbb{Z}$, depending on which group we are considering), such that the translation R_α on $\Sigma \times A$ given by $(x, t) \mapsto (x, t + \alpha)$ leaves invariant the invariant sets of the skew-product action S_f on $(\Sigma \times A, \mu \times \lambda)$ (by invariant we mean invariant $(\mu \times \lambda)$-modulo 0).

The set of finite essential values form a closed subgroup $E_\sigma(f)$ of A and if $E_\sigma(f) \neq \{0\}$ then the quotient function $f^*: \Sigma \to A/E_\sigma(f)$ given by $f^*(x) = f(x) + E_\sigma(f)$ is a

μ-measurable coboundary and S_f can be regarded as an ergodic $E_\sigma(f)$-extension. Nevertheless, in general, for a given pair of (f, μ) it is not easy to compute $E_\sigma(f)$ or even to decide if $E_\sigma(f) \neq \{0\}$.

In Chapter II we deal with a class of functions f and measures μ with $\int f\, d\mu = 0$, such that $E_\sigma(f)$ can be described by the f-weights of periodic orbits. More precisely, we consider the class of functions with summable variation on Σ, and ergodic shift-invariant measures on Σ which are non-singular with respect to the tail equivalence $\mathscr{R}$ (or the shifted tail equivalence $\mathscr{T}$). These countable equivalence relations are generated by the group of finite coordinate changes Δ, and the group of finite block exchanges $\Gamma \supseteq \Delta$, respectively (see definitions therein). We consider the group action of Δ and Γ on (Σ, μ) (either assuming quasi-invariance of μ with respect to the action of Δ or Γ) and study the asymptotic range (i.e. set of finite essential values) of the Radon-Nikodym cocycle of the unique equilibrium state of f with respect to the Δ or Γ-action. We were able to show the following subtle connection between the ergodicity of certain skew-products over σ and over Γ and Δ:

(i) Assuming quasi-invariance of μ under Δ, then $E_\sigma(f)$ contains the closure of the group generated by the set of 'ratios' of weights, $\{f^n(x)-f^n(y)\colon x, y \in \mathrm{Fix}(n), n > 0\}$, where Fix(n) denotes the periodic points of period n for the shift.

(ii) Assuming quasi-invariance of μ under Γ, then $E_\sigma(f)$ *is* the closure of the group generated by the set of weights, $\{f^n(x)\colon x \in \mathrm{Fix}(n), n > 0\}$.

We show by Example II.2.12 that quasi-invariance under Δ does not imply quasi-invariance under Γ and by means of examples from Markov shifts (see Section II.3) one cannot replace 'contain' in result (i) by an equality. The equilibrium states of functions of summable variation are quasi-invariant under Γ (therefore also under Δ) and hence our results apply in this context.

The above results show that the function f of summable variation is a continuous coboundary if and only if it is a μ-measurable coboundary for some ergodic shift-invariant measure which is quasi-invariant with respect to the action of Δ or Γ. This is proved in [Liv] for equilibrium states of functions of summable variation.

The consideration of the equivalence relations discussed above, although implicitly, appears in the work of Krieger [Kri], where it is shown that the group generated by the values of the Radon-Nikodym cocycle of a Markov measure with respect to the action of the group Δ, is an invariant for finitary isomorphisms of finite expected code length. Parry & Schmidt [PS] use the Radon-Nikodym cocycle of a Markov measure with respect to the actions of Δ and Γ, to derive other invariants for this type of isomorphism. Schmidt [ScK.4] uses the Radon-Nikodym cocycle of the equilibrium state of a function of summable variation with respect to the actions of Δ and Γ, to find obstructions to the existence of hyperbolic isomorphisms between ssft's endowed with equilibrium states of functions of summable variation. We remark that Proposition II.1.4 in this thesis is mentioned in [ScK.4] (without proof). In the survey [ScK.5], Schmidt discusses many applications of the study of general equivalence relations in ergodic theory, which could be of interest to the reader of Chapter II of this thesis.

Whilst Chapter II was in the final stages of preparation we learned that, in a recent paper, Guivarc'h considers the problem of lifting ergodicity to skew-products, where the function f takes values in $G = \mathbb{Z}^k$ or $\mathbb{R}^k$ (cf. [Gui]). In order to describe this work we shall introduce some terminology. Let μ be an arbitrary ergodic shift-invariant measure and let f be a measurable function taking values in G. Let λ denote Haar measure on G. We say that the pair (f, μ) satisfies the *mean-property* if for every $\alpha \in L^1(\mu)$, and every $\xi \in L^1(\lambda)$ of compact support satisfying $\int \xi \, d\lambda = 0$, we have

$$\lim_{n\to\infty} \int \left| \int \alpha(\sigma^n x)\, \xi(t + f^n(x))\, d\mu(x) \right| d\lambda(t) = 0,$$

where $f^n(x)$ denotes $f(x) + \ldots + f(\sigma^{n-1}x)$. Guivarc'h shows that if μ is an equilibrium state of a Hölder continuous function and f is a strictly aperiodic Hölder continuous function(1) such that $\int f \, d\mu = 0$, then (f, μ) satisfies the mean-property. Under these assumptions, it is shown that recurrence is equivalent to ergodicity of the skew-product $(S_f, \mu \times \lambda)$, and the skew-product is recurrent if and only if $k = 1$ or 2.

Therefore, through stronger assumptions on the measure μ, Guivarc'h proves the

(1) f is said to be *strictly aperiodic* if the group generated by $\{f^n(x) - na : x \in \mathrm{Fix}(n), n > 0\}$ is dense in G for every $a \in G$, where G denotes $\mathbb{R}^k$ or $\mathbb{Z}^k$ depending on which group we are considering the skew-product. (In Chapter III we refer to this property as the *strong* non-lattice distribution condition.)

ergodicity of the skew-product in dimensions 1 and 2, and the non-ergodicity for dimensions $k > 2$; while we prove the ergodicity of the skew-product in dimension 1 with weaker assumptions on the measure μ, and show an interplay between the actions of Γ, Δ and σ.

The second problem considered in this thesis is the study of a topological entropy concept for the skew-product S_f (see Chapter III). This concept is defined as follows. Given $\varepsilon > 0$, we count the number of periodic orbits of period n for the shift satisfying $|f^n(x)| < \varepsilon$. Let $h_f(\varepsilon)$ denote the growth rate of these numbers. If the derivative of the pressure function of f vanishes for some t and f is not a coboundary, then this value of t is unique and defines the minimum of the pressure function of f. In this case, we show that $h_f(\varepsilon)$ does not depend on ε and is, in fact, the minimum value of the pressure function of f (cf. Theorem III.1.4). The equilibrium state m_{tf} of the function tf, where t defines the minimum of the pressure function, is the measure of maximal entropy among the shift-invariant probability measures on Σ satisfying $\int f \, d\mu = 0$. The latter is a result of Lanford (cf. [Lan] and Proposition III.1.1 of this thesis).

When f is a function taking values in $\mathbb{Z}$, we show in Section III.2 an application of Theorem III.1.4. In this case, $h_f(\varepsilon)$ is the growth rate of periodic points of period n and f-weight zero, which is the Gurevič topological entropy of a countable ssft canonically associated to the skew-product S_f. This topological entropy is Bowen's topological entropy of the skew-product S_f with respect to a metric on $\Sigma \times \mathbb{Z}$, which is compatible with the one-point compactification of $\mathbb{Z}$. We use Theorem III.1.4 to prove a result which gives examples of entropy increasing factor maps of countable ssft's (cf. Theorem III.2.1). These provide natural examples for some of the questions considered in the paper of Petersen [Pet].

Now let c(f) denote the set $\{c \in \mathbb{R}: \int f \, d\mu = c$, for some shift-invariant probability measure $\mu\}$. This set is a compact interval [a, b], whose interior is the image of the pressure function of f. We may consider an entropy function on [a, b] which is given by $c \mapsto s_c = \sup \{h(\mu): \int f \, d\mu = c\}$, where $h(\mu)$ denotes the measure-theoretic entropy of the shift-invariant probability measure μ. (This construction has been considered by Bohr & Rand in [BR].) Since by Theorem III.1.4 there exist a unique measure m_c such that

$h(m_c) = s_c$ when $a < c < b$, it remains to understand the structure of the measures m for which $h(m) = s_a$ or s_b. We call these measures the *extremal measures* of f associated to the extreme a or b, respectively. Section III.3 is devoted to the consideration of these extreme cases. It is shown that any extremal measure cannot be fully supported and any ergodic component of an extremal measure defines by restriction another extremal measure. We finish Section III.3 showing a connection between the accumulation points of the equilibrium states $m_{t(f-a)}$ (of the function $t(f-a)$) for $t \to -\infty$, in the space of shift-invariant probability measures, and the extremal measures of f associated to the extreme a (similarly to the extreme b, with $t \to \infty$). We believe that there exists a unique accumulation point in each extreme and this measure is the barycentre of the ergodic extremal measures on each extreme, respectively (see Conjecture III.3.8). In the Appendix we provide some simple examples of such measures which give support to the conjecture.

When f is strongly non-lattice distributed Theorem III.1.4 is implicitly proved in the work of Lalley [Laℓ.2], where asymptotic formulas for the number of closed orbits (with constraints on the f-weights) is derived for weak-mixing Axiom A flows.

Chapter IV is a self-contained paper with its own introduction. It is devoted to the study of the speed of convergence and asymptotic expansions in the central limit theorem for the class of Hölder continuous functions on a ssft endowed with a stationary equilibrium state of another Hölder continuous function. The work in this chapter was done jointly with W. Parry and grew out of several informal discussions.

CHAPTER I:
PRELIMINARIES

In this chapter we shall introduce the notation and terminology which will be the basis for the following chapters. In Section 1 we give some general definitions concerning the systems which are the centre of our study, i.e. the subshifts of finite type and their naturally associated measure-theoretic counterpart, the Markov shifts. In Section 2 we give definitions of pressure and equilibrium states, we also discuss some properties of functions with summable variation which shall be used in Chapters II and III. In Section 3 we define cocycles for a countable group action and state some results concerning the skew-product action.

§1. Subshifts of finite type (ssft's) and Markov shifts

Let X(k) denote $\{1, ..., k\}^{\mathbb{Z}}$, i.e. the space of two-sided sequences in k symbols. Giving the discrete topology to {1, ..., k} and considering the product topology on X(k), we make it into a compact metrizable space. The *shift transformation* is the homeomorphism σ: X(k)$\circlearrowleft$ given by $(\sigma x)_n = x_{n+1}$, $\forall n \in \mathbb{Z}$. The pair (X(k), σ) is called the *full shift* on k symbols. Let M be a k×k 0-1 matrix (M is called a *transition matrix*). We define a closed σ-invariant subset of X(k) by

$$\Sigma_M = \{(x_n) \in X(k): \ M(x_n, x_{n+1}) = 1, \forall n \in \mathbb{Z}\}.$$

The restriction of σ to Σ_M shall be denoted by σ_M or also by σ when there is no danger of confusion. The pair (Σ_M, σ) is called a *subshift of finite type* (ssft). The open and closed subsets

$$[i_{-r}, ..., i_s]_m = \{(x_n) \in \Sigma_M: \ x_{j+m} = i_j \ \text{for} \ -r \leq j \leq s\}$$

form a basis for the induced topology on Σ_M and are called *cylinders*.

We can also view Σ_M as the space of two-sided paths in a directed graph $\mathcal{G}$(M) as follows. Define $\mathcal{G}(M)$ by considering k vertices and drawing an edge from vertex i to vertex j if and only if $M(i, j) = 1$ (in this case the *transition* from i to j (denoted $i \to j$) is called *allowable*). $\mathcal{G}(M)$ characterises uniquely Σ_M and is called *the graph* of Σ_M. The *set of followers* of a vertex (or symbol) i is the set $f(i) = \{j: M(i, j) = 1\}$; similarly, we define the *set of predecessors* of a symbol i as the set $p(i) = \{j: M(j, i) = 1\}$.

M is *irreducible* if given any pair of vertices (i, j) of $\mathcal{G}(M)$ there is a path in $\mathcal{G}(M)$ starting in i and ending in j (in other words, when $\mathcal{G}(M)$ is connected). The period d(i) of a vertex i is the highest common factor of $\{n: M^n(i, i) > 0\}$ where M^n denotes the nth power of M. If M is irreducible then $d(i) = d$ does not depend on the chosen symbol and is called the *period* of M. M is called *aperiodic* if $d = 1$.

The dynamical properties of the shift on Σ_M are reflected by the properties of the matrix M as follows. (Σ_M, σ)(2) is topologically transitive if and only if M is irreducible and it is topologically mixing if and only if M is irreducible and $d = 1$ (see [Bow] or [Waℓ.3]). When M is irreducible and $d > 1$ there exists a decomposition of $\{1, ..., k\}$ into d disjoint subsets $S_0, ..., S_{d-1}$ such that $f(S_j) = S_{j+1 (\mathrm{mod}\ d)}$ and $p(S_{j+1(\mathrm{mod}\ d)}) = S_j$, where $f(S) = \cup_{i \in S} f(i)$ (cf. [Sen]). Hence when $d > 1$ there exists a decomposition of Σ_M (the so called *cyclic decomposition* of Σ_M) into disjoint open and closed subsets $U_0, ..., U_{d-1}$ such that $\sigma(U_j) = U_{j+1 (\mathrm{mod}\ d)}$ and $\sigma^d|_{U_j}$ is topologically mixing for every j.

An *allowable* or *admissible block* in Σ_M is a sequence of symbols $[i_0, ..., i_{n-1}]$ such that the transitions $i_0 \to i_1$, $i_1 \to i_2, ..., i_{n-2} \to i_{n-1}$ are allowable. The *n-block system* of Σ_M is the ssft $\Sigma_{M(n)}$ whose symbols are the allowable blocks in Σ_M of length n and transitions given by $[i_0, ..., i_{n-1}] \to [j_0, ..., j_{n-1}]$ if and only if $i_k = j_{k-1}$ for $k = 1, ..., n-1$. It should be clear that the n-block system of Σ_M is topologically conjugate to Σ_M, i.e. there exists a homeomorphism $\varphi: \Sigma_{M(n)} \to \Sigma_M$ which satisfies $\varphi \circ \sigma_{M(n)} = \sigma_M \circ \varphi$.

Let $f: \Sigma_M \to \mathbb{R}$ be a continuous function which depends only on a finite number of coordinates (i.e. there exists $n > 0$ such that $f(x) = f(x_{-n}, ..., x_n)$). The construction of a higher block system allows us to assume that the function f depends only on two coordinates (i.e. $f(x) = f(x_0, x_1)$) when the concept we are studying is invariant under topological conjugacy.

Now let P be a stochastic $k \times k$ matrix (i.e. $P(i, j) \geq 0$ and $\sum_j P(i, j) = 1$). Let P^0 be

(2) In general we shall refer to the pair (Σ_M, σ) as Σ_M.

the associated transition matrix of P, i.e. P^0 is a 0–1 matrix such that $P^0(i, j) = 0$ if and only if $P(i, j) = 0$. P is irreducible or aperiodic if P^0 enjoys the same property. If P is irreducible, by the Perron–Frobenius theorem (see for instance [Sen]), P has a unique strictly positive left eigenvector $p = (p_0, \ldots, p_{k-1})$ associated to the eigenvalue 1 such that $\sum_j p_j = 1$. P defines a Borel probability measure μ_P supported on Σ_{P^0} as follows. Restricted to the semi-algebra of cylinders define μ_P by:

$$\mu_P\Big([i_{-r}, \ldots, i_s]_m\Big) = p_{i_{-r}} P(i_{-r}, i_{-r+1}) \cdots P(i_{s-1}, i_s).$$

The measure μ_P is called the *Markov measure* defined by P. The restriction of σ to Σ_{P^0} is easily seen to preserve μ_P and is called a *Markov shift*. The triple $(\Sigma_{P^0}, \sigma, \mu_P)$ is called a *Markov chain*. μ_P defined in this way is an ergodic measure for σ and if P is aperiodic then μ_P is strongly mixing (see for instance [Bow] or [PT]). (All the measures considered in this thesis will be Borel measures.)

Sometimes the name Markov shift is used also for a chain defined by an irreducible, non-negative matrix Q, not necessarily stochastic, which is defined in a similar way as in the stochastic case: define the associated transition matrix Q^0 as above and consider the ssft (Σ_{Q^0}, σ). By applying the Perron–Frobenius theorem to Q we conclude that there exists a strictly positive right eigenvector $r = (r_0, \ldots, r_{k-1})$ associated to the maximum eigenvalue β of Q. Consider the stochastic matrix P_Q given by

$$P_Q = \left[\frac{Q(i, j)\, r_j}{\beta\, r_i}\right].$$

Define the probability measure μ_Q associated to Q as the Markov measure defined by P_Q. We shall see in the next section that μ_Q has strong ergodic properties from being the equilibrium state of the continuous function $f: \Sigma_{Q^0} \to \mathbb{R}$ defined by $f(x) = \log Q(x_0, x_1)$. When Q is stochastic the measure-theoretic entropy of μ_Q is given by $-\int f \, d\mu_Q$ and when Q coincides with the associated matrix Q^0, the topological entropy h of Σ_Q is $\log \beta$ and the measure μ_Q is the *unique* measure of maximal entropy on Σ_{Q^0} (cf. [Par.1]). This measure is known as the *Parry measure* on Σ_Q.

Now let P be an irreducible non-negative matrix. We shall refer in this thesis to the following additive subgroups Γ_P and Δ_P of $\mathbb{R}$. Γ_P is the group generated by

$$\left\{\log\{P(x_0, x_1)P(x_1, x_2) \cdots P(x_{n-1}, x_0)\}: \ x \in \mathrm{Fix}(n) \text{ and } n > 0\right\}$$

where $\mathrm{Fix}(n) = \{x \in \Sigma_{P^\circ}: \ \sigma^n x = x\}$, and Δ_P is the subgroup of Γ_P given by

$$\left\{\log\left\{\frac{P(x_0, x_1) \ldots P(x_{n-1}, x_0)}{P(y_0, y_1) \ldots P(y_{n-1}, y_0)}\right\}: \ x, y \in \mathrm{Fix}(n) \text{ with } x_0 = y_0 \text{ and } n > 0\right\}.$$

Γ_P and Δ_P are invariants of various isomorphisms between Markov shifts (see the Concluding Remarks for more details on those isomorphisms).

Convention 1.1: Unless otherwise stated we shall omit the defining transition matrix M and denote Σ_M by Σ only.

§2. Equilibrium states

The general reference for this section is the chapter on one dimensional lattice gases of the book of Ruelle [Rue].

Let Σ be an irreducible ssft and consider the class $\mathscr{C}(\Sigma)$ of continuous functions on Σ. $f \in \mathscr{C}(\Sigma)$ is called a *coboundary* if there exists $b \in \mathscr{C}(\Sigma)$ such that $f = b\circ\sigma - b$. b is then called a *cobounding function* for f. Two continuous functions f and g are said to be *cohomologous* if $f-g$ is a coboundary. We shall consider some cohomology invariants which include the set of equilibrium states.

Let $f \in \mathscr{C}(\Sigma)$ be given. We define the *pressure* of f by

$$\mathcal{P}(f) = \sup\, \{h(\mu) + \int f\, d\mu\},$$

where the supremum is taken over all σ-invariant probability measures on Σ and $h(\mu)$ denotes the measure-theoretic entropy of μ. If $\mathcal{P}(f) = h(m) + \int f\ dm$ for some σ-invariant probability measure m, then m is called an *equilibrium state* for f. Since the shift is expansive it follows that the entropy function is upper semi-continuous on the space of σ-invariant probability measures and hence, for any continuous function f given, there exists at least one equilibrium state for f.

Let $f \in \mathscr{C}(\Sigma)$ be given. The nth variation of f is the quantity

$$\mathrm{var}_n(f) = \sup\, \{|f(x) - f(y)| : x_i = y_i \ \text{ for } |i| \le n\}.$$

The function f is said to have *summable variation* if $\sum_{n>0} \mathrm{var}_n(f)$ is finite. If there exists $0<\theta<1$ such that $\sup\{\mathrm{var}_n(f)\cdot\theta^{-n}: n>0\}$ is finite then f is called *Hölder continuous* with Hölder constant θ.

From the definition of pressure we know that if $f-g$ is cohomologous to a constant function then the set of equilibrium states of g and f coincide. When f has summable variation there exists a *unique equilibrium state* for f and we shall refer to it in this thesis as *the* equilibrium state of f and denote it by m_f.

If f takes values in a discrete subset of $\mathbb{R}$ then it is a function of only a finite number of coordinates, i.e. $f(x) = f(x_{-n}, \ldots, x_n)$ for some n and by composing f with a conjugacy we may assume that $f(x) = f(x_0, x_1)$.(3) In this case, the equilibrium state of f is the Markov measure obtained from the non-negative matrix Q defined by $Q(i,j) = Q^0(i,j)\, e^{f(i,j)}$, where the construction is carried out as in the previous section (this result can be found in [PT] or [Rue]).

Let $\mathcal{F}$ denote the space of real Hölder continuous functions on Σ.

Proposition 2.1 [Rue]: *The pressure function* $\mathcal{P}: \mathcal{F} \to \mathbb{R}$ *has the following properties:*

(i) $s \mapsto \mathcal{P}(g+sf)$ *is an analytic function,*

(ii) $d/ds\,\{\mathcal{P}(g+sf)\}|_{s=0} = \int f\, dm_g$,

(iii) $d^2/ds^2\,\{\mathcal{P}(g+sf)\}|_{s=0} = \sigma_g^2(f)$

$$= \lim_{n\to\infty} \frac{1}{n} \int \left(f + f\circ\sigma + \ldots + f\circ\sigma^{n-1} - n\left(\int f\, dm_g\right)\right)^2 dm_g,$$

for every $f, g \in \mathcal{F}$.

$\sigma_g^2(f)$ is the *variance* of the sequence of random variables $\{f\circ\sigma^n\}_{n\geq 0}$ on the probability space (Σ, m_g) (cf. [Feℓ]). It turns out that the pre-image of 0 of the function $\sigma_g^2: \mathcal{F} \to \mathbb{R}$ is the linear subspace defined by functions in $\mathcal{F}$ which are cohomologous to a constant function and hence it does not depend on the choice of the function g.

Later we shall refer to the following characterisation of coboundaries in the space of functions of summable variation and in the space $\mathcal{F}$. Let $f \in \mathcal{C}(\Sigma)$ be given. For

(3) See comments in the previous section concerning the higher block system construction.

$x \in \Sigma$ and $n > 0$, consider the partial sum

$$f^n(x) = \sum_{i=0}^{n-1} f(\sigma^i x). \tag{2.1}$$

Let Fix(n) denote $\{x \in \Sigma\colon \sigma^n x = x\}$. If $x \in \text{Fix}(n)$ and n is the minimal period of x then the number $f^n(x)$ is called the *f-weight* of the closed orbit defined by x or simply the f-weight of x.

Proposition 2.2 [Liv]: *Let* $f \in \mathscr{C}(\Sigma)$ *be a function of summable variation. Then the f-weight of every closed orbit in* Σ *is zero if and only if* f *is a coboundary.*

The proposition above is proved in [Liv] for the case when $f \in \mathscr{F}$, but the proof there can easily be adapted for functions of summable variation.

The general concept of cohomology of dynamical systems involves the study of measurable cohomology, i.e. when the functions are (Borel) measurable functions and the equalities hold μ-a.e. for some (Borel) measure μ on the space under consideration (cf. [MS]). In our case, when dealing with $f, g \in \mathscr{C}(\Sigma)$, there are examples of f and g measurably cohomologous but not continuously cohomologous, i.e. $f-g \in \mathscr{C}(\Sigma)$ can be written as $b\circ\sigma - b$ μ-a.e. with measurable b for some measure μ, but b cannot be replaced by a continuous function. However, due to the work of Livšic [Liv], if $f, g \in \mathscr{C}(\Sigma)$ are continuously cohomologous satisfying $\sum_{n>0} n\cdot\text{var}_n(f) < \infty$ and $\sum_{n>0} n\cdot\text{var}_n(g) < \infty$ then a cobounding function can be chosen to have summable variation. The same property holds for Hölder continuous functions where a Hölder continuous cobounding function can be chosen. (Our convention is whenever we use cohomology between functions, we mean continuous cohomology, unless otherwise stated.)

We say that a function $g \in \mathscr{C}(\Sigma)$ *depends only on future coordinates* if $g(x) = g(y)$ whenever $x_n = y_n$ for $n \geq 0$. The following proposition allows us to reduce the study of any particular cohomology invariant to the case when the function depends only on future coordinates.

Proposition 2.3 ([Rue], [Bow]): *Let* $f \in \mathscr{C}(\Sigma)$ *satisfy* $\sum_{n>0} n\cdot\text{var}_n(f) < \infty$ *(or* $f \in \mathscr{F}$*). There exists a function* g *of summable variation (resp.* $g \in \mathscr{F}$*) depending only on future coordinates such that* g *is cohomologous to* f.

Remark 2.4: When $f \in \mathscr{C}(\Sigma)$ has summable variation the proof of Proposition 2.3 implies the existence of a continuous function g depending only on future coordinates with g cohomologous to f, but g may not have summable variation. We do not know whether, in this case, g can be replaced by a function of summable variation satisfying this property.

§3. Cocycles and skew-products

Since in Chapter III we shall be dealing with a cocycle for a different group action apart from the shift-action, we will discuss cocycles in a rather more general setting. Nevertheless we shall give the definitions assuming that the underlying space is a ssft, which is not a necessary restriction. In this section the definitions make sense for $\mathbb{R}$ replaced by any second countable abelian group, such as the subgroups of $\mathbb{R}$ or the group $\mathbb{Z}^k$ for some k. But the results discussed at the end of this section may not be true in general for these groups.

Let Σ be an irreducible ssft and consider a countable group G acting on Σ by homeomorphisms. A real continuous *cocycle* for the G-action is a map $\alpha\colon G \times \Sigma \to \mathbb{R}$ such that $\alpha(\varphi, \cdot)$ is a continuous function for every fixed $\varphi \in G$ and α satisfies the *cocycle equation*

$$\alpha(\varphi\psi, x) = \alpha(\varphi, \psi x) + \alpha(\psi, x).$$

α is called a *coboundary* if there exists a continuous map $b\colon \Sigma \to \mathbb{R}$ such that $\alpha(\varphi, x) = b(\varphi x) - b(x)$. Two cocycles α and β are said to be *cohomologous* if their difference is a coboundary. For a general consideration of cocycles for ergodic group automorphisms we refer the reader to [ScK.1]. Given a cocycle α we construct the *skew-product* action of G on $\Sigma \times \mathbb{R}$ by

$$T_\varphi^\alpha(x, t) = (\varphi x, \alpha(\varphi, x) + t).$$

In the case of the shift-action on Σ (i.e. the $\mathbb{Z}$-action on Σ given by $n \mapsto \sigma^n$) the concepts are interpreted as follows. A cocycle α for this action is determined by the function $f(x) = \alpha(1, x)$. From the cocycle equation we conclude that

$$\alpha(n, x) = \begin{cases} \sum_{i=0}^{n-1} f(\sigma^i x) & \text{if } n > 0, \\ 0 & \text{if } n = 0, \\ -\sum_{i=n}^{-1} f(\sigma^i x) & \text{if } n < 0. \end{cases}$$

The corresponding skew-product action of the shift on $\Sigma \times \mathbb{R}$ is then generated by the homeomorphism

$$S_f(x, t) = (\sigma x, f(x) + t).$$

S_f is called an *$\mathbb{R}$-extension* of Σ or a *skew-product over* Σ where f is the *skewing function.* Σ will be referred to as the *base* of the skew-product. If the function f takes values in a subgroup H of $\mathbb{R}$, replacing $\mathbb{R}$ by H in the above definition we obtain H*-extensions* of Σ.

Still in the case of the shift-action, it is clear that the concepts for the cohomology of cocycles agree with the concept introduced in the previous section, concerning cohomologous functions.

Let μ be a probability measure on Σ. μ is called *quasi-invariant* with respect to the group action of G on Σ if for every Borel set B we have $\mu(B) = 0$ if and only if $\mu(\cup_{\gamma\in G} \gamma(B)) = 0$. Furthermore the action of G is called *ergodic* with respect to μ if for every Borel set B with $\mu(B\Delta(\cup_{\gamma\in G} \gamma(B))) = 0$ we have $\mu(B) = 0$ or $\mu(\Sigma\backslash B) = 0$.

Let $\bar{\mathbb{R}} = \mathbb{R}\cup\{\infty\}$ denote the one point compactification of $\mathbb{R}$. In the following discussion we suppose that G is a countable group acting on Σ by homeomorphisms and μ is a probability measure on Σ which is quasi-invariant with respect to the G-action. Let α be a cocycle for the G-action. An *essential value* of α is a number $s \in \bar{\mathbb{R}}$ such that for any given neighbourhood N(s) of s in $\bar{\mathbb{R}}$ and any Borel set A with $\mu(A) > 0$ we have

$$\mu\Big(\cup_{\varphi\in G} (A \cap \varphi^{-1}A \cap \{x: \alpha(\varphi, x) \in N(s)\})\Big) > 0 .$$

Since the group G is assumed to be countable, in the above definition there must be an element $\varphi \in G$ with

$$\mu\Big(A \cap \varphi^{-1}A \cap \{x: \alpha(\varphi, x) \in N(s)\}\Big) > 0 .$$

The *full asymptotic range* of α is defined as the set of essential values of α and shall be denoted by $\bar{E}_{G,\mu}(\alpha)$. The *asymptotic range* of α is the set $E_{G,\mu}(\alpha) = \bar{E}_{G,\mu}(\alpha)\setminus\{\infty\}$.

It is clear from the cocycle equation that $\alpha(e, x) = 0$, where e denotes the identity of G, and hence $E_{G,\mu}(\alpha)$ is non-empty since it must contain $0 \in \mathbb{R}$. In [ScK.1] it is shown that $E_{G,\mu}(\alpha)$ is a cohomology invariant (this also appears in the work of Feldman & Moore [FM] where they define proper asymptotic range for orbital cocycles, i.e. cocycles for countable equivalence relations).

The propositions below are translations to our situation of the results Lemma 3.3, Proposition 3.12 and Corollary 5.4 of [ScK.1] respectively in this order, concerning some properties of the asymptotic range $E_{G,\mu}(\alpha)$. We should note that the definitions in [ScK.1] concerning cohomology of cocycles involve (Borel) measurable cocycles and equalities holding μ-a.e. for all group elements. (Our convention is whenever we use the latter definition we attach the word *measurable*, otherwise we mean continuous cohomology.) We shall assume now that the action of the countable group G is ergodic with respect to the measure μ for the following propositions.

Proposition 3.1 [ScK.1]: $E_{G,\mu}(\alpha)$ *is a closed subgroup of* $\mathbb{R}$.

Proposition 3.2 [ScK.1]:(4) *If* $E_{G,\mu}(\alpha) \neq \{0\}$ *then there exists a measurable cocycle* β *for the* G*–action satisfying* $\beta(\varphi, x) \in E_{G,\mu}(\alpha)$ *for every* $\varphi \in G$ *and every* $x \in \Sigma$ *such that* β *is measurably cohomologous to* α.

Proposition 3.3 [ScK.1]: $E_{G,\mu}(\alpha) = \mathbb{R}$ *if and only if the skew-product action* T^{α}_{φ} *of* G *is ergodic with respect to the measure* $\mu \times \lambda$, *where* λ *denotes Lebesgue measure on* $\mathbb{R}$.

Another cohomology invariant which is going to be frequently used is recurrence of cocycles. The cocycle α is called *recurrent* for the non-singular action of the countable group G on (Σ, μ) if for any set of positive measure B and any neighbourhood

(4) The corresponding proposition in [ScK.1] assumes that the cocycle α is *regular*, i.e. when the quotient cocycle $\alpha^*: G\times\Sigma \to \mathbb{R}/E_{G,\mu}(\alpha)$ defined by $\alpha^*(\varphi, x) = \alpha(\varphi, x) + E_{G,\mu}(\alpha)$ satisfies $\bar{E}_{G,\mu}(\alpha^*) = \{0\}$. But in our case the assumption $E_{G,\mu}(\alpha) \neq \{0\}$ guarantees that α is regular since the quotient of $\mathbb{R}$ by $E_{G,\mu}(\alpha)$ is then compact.

$N(0)$ of $0 \in \mathbb{R}$, there exists $\gamma \in G$ such that

$$\mu\Big(B \cap \gamma^{-1}B \cap \{x: \alpha(\gamma, x) \in N(0)\} \cap \{x: \gamma x \neq x\}\Big) > 0 .$$

We shall be particularly interested on the recurrence of cocycles for the shift-action on Σ. In this case, recurrence of α will commonly be referred to as recurrence of the corresponding generating function $f(x) = \alpha(1, x)$. In this setting, recurrence of f means that for any set of positive measure B and any $\varepsilon > 0$ given, there exists $k > 0$ such that

$$\mu\Big(B \cap \sigma^{-k}B \cap \{x: |f^k(x)| < \varepsilon\}\Big) > 0,$$

where $f^k(x)$ is defined by (2.1). The following proposition is an adaptation of a theorem first proved by Atkinson [Atk] (see also Theorem 11.4 of [ScK.1]) to our situation and gives a very useful criterion for checking recurrence of the cocycle defined by a function f.

Proposition 3.4 [Atk]: *Let μ be a non-atomic σ-invariant ergodic probability measure on Σ and $f \in \mathscr{C}(\Sigma)$. Then f defines a recurrent cocycle if and only if $\int f \, d\mu = 0$.*

In Chapter II we will study the properties of the asymptotic range of cocycles for the shift-action on Σ in order to characterise ergodicity of the skew-product. The measure chosen will always be a σ-invariant measure on Σ and in addition we shall consider certain groups (the groups Δ and Γ, see Chapter II) acting non-singularly with respect to the chosen measure.

CHAPTER II:
ERGODICITY OF SKEW-PRODUCTS OVER SSFT's

In this chapter we shall consider the problem of lifting ergodicity to $\mathbb{R}$-extensions or skew-products over a subshift of finite type (ssft) endowed with an ergodic σ-invariant measure. In Section 1 we introduce two countable equivalence relations on the ssft and consider quasi-invariant measures for those equivalence relations. In Section 2 we state and prove the main result of the chapter which is to give necessary and sufficient conditions for ergodicity of the skew-product for the class of functions of summable variation and ergodic σ-invariant measures which are non-singular with respect to one of the equivalence relations discussed in the previous section. In Section 3 we give some examples arising from Markov shifts to illustrate some of the concepts introduced in Section 2.

§1. Tail equivalence and quasi-invariant measures

In this section we shall introduce two countable equivalence relations on a ssft which will be seen to be generated by the action of the group of finite coordinate changes and the group of finite block exchanges. Then we consider non-singular measures for those equivalence relations or equivalently quasi-invariant measures for the action of the corresponding generating group. We shall also state some properties and prove some results for conditional measures on stable σ-algebras of the ssft (i.e. σ-algebras whose atoms are the stable manifolds of each point), these will be used in the proof of the key result of the next section (Lemma 2.6).

Let Σ be an irreducible ssft. In this section we shall always consider an *ergodic σ-invariant probability* measure μ which is *fully supported* on Σ. In addition, as a convention, although we shall be dealing with more than one measure in this section, whenever we consider a set of positive measure we mean positive measure with respect to μ.

We define an equivalence relation $\mathscr{T}$ on Σ by putting $(x, y) \in \mathscr{T}$ if and only if there exist m, n, m′, n′ $\in \mathbb{Z}^+$ such that

$$x_{m+s} = y_{m'+s} \text{ and } x_{-n-s} = y_{-n'-s} \tag{1.1}$$

for all $s \geq 0$. It is easy to see that $\mathscr{T}$ is a countable and measurable equivalence relation, i.e. the set $\mathscr{T}(x) = \{y \in \Sigma: (x, y) \in \mathscr{T}\}$ is countable for every $x \in \Sigma$ and $\mathscr{T}$ is a Borel subset of $\Sigma \times \Sigma$. Taking $m = m'$ and $n = n'$ in the definition (1.1) we define a subrelation $\mathscr{R} \subseteq \mathscr{T}$. $\mathscr{R}$ is sometimes referred to as the *tail equivalence* on Σ and we refer to $\mathscr{T}$ as the *shifted tail equivalence* on Σ. We shall consider explicit generating groups for these equivalence relations as follows.

A *finite coordinate change* on Σ is a homeomorphism $\varphi: \Sigma \circlearrowleft$ such that there exists $N \in \mathbb{Z}^+$ with $(\varphi x)_n = x_n$ for all $|n| \geq N$ and all $x \in \Sigma$. The collection of all finite coordinate changes in Σ is easily seen to be a group under composition and shall be denoted by Δ is this chapter. The group Δ acts canonically on Σ by $(\varphi, x) \mapsto \varphi(x)$. It is easy to check that the Δ-orbit of a point $x \in \Sigma$ is the set $\mathscr{R}(x)$, i.e. Δ generates the equivalence relation $\mathscr{R}$. Similarly we define a *finite block exchange* on Σ by a homeomorphism $\gamma: \Sigma \circlearrowleft$ such that there exist $N \in \mathbb{Z}^+$ and continuous functions $u, t: \Sigma \to \mathbb{Z}$ with $(\gamma x)_n = x_{n-u(x)}$ for $n \leq -N$ and $(\gamma x)_n = x_{n+t(x)}$ for $n \geq N$, for all $x \in \Sigma$. We note that the collection Γ of all finite block exchanges is also a group under composition and is a generating group for $\mathscr{T}$.

The assumption that $(\mathscr{R}, \mu)$ or $(\mathscr{T}, \mu)$ is a non-singular equivalence relation (for definition see for instance [FM]) is equivalent to the assumption that μ is *quasi-invariant* with respect to the Δ-action or Γ-action respectively (see Section I.3). In the following we shall make the latter assumption on μ (either assuming Δ or Γ-quasi-invariance) and hence, equivalently, the logarithms of the Radon-Nikodym derivatives $\log\{d(\mu \circ \varphi)/d\mu\}$ exist and are finite μ-a.e. for all $\varphi \in \Delta$ (resp. $\varphi \in \Gamma$).

Now we shall introduce some terminology on conditional measures which will be used in the next section. Let β be the state partition of Σ and for every m, let $\mathscr{A}(m)$ denote the σ-algebra generated by $\{\sigma^{-k}\beta: m \leq k\}$ (the so called *stable* σ-algebra). Let $\{\upsilon_x^m: x \in \Sigma\}$ be the decomposition of the measure μ with respect to $\mathscr{A}(m)$. Then

$\upsilon_x^m(A) = E(\chi_A | \mathscr{A}(m))(x)$ μ-a.e. for all Borel sets A, where χ_A denotes the indicator function of A and $E(\cdot|\cdot)$ denotes conditional expectation with respect to μ.

We gather some properties of these conditional measures in the next proposition. In this proposition sets are meant to be Borel sets and, when not stated otherwise, equalities are meant to hold μ-a.e.

Proposition 1.1: *Let μ be a Borel σ-invariant probability measure on Σ. The corresponding conditional measures $\{\upsilon_x^m : x \in \Sigma\}$ can be chosen such that the following hold:*

(a) *If $x, y \in \Sigma$ are given such that $x_n = y_n$ for all $n \geq m$ then*

$$\upsilon_x^m(A) = \upsilon_y^m(A)$$

for every set A.

(b) *For every set A and every $m, k \in \mathbb{Z}$ we have*

$$\upsilon_{\sigma^k x}^m(\sigma^k A) = \upsilon_x^{m+k}(A).$$

(c) *For any set A we have*

$$\lim_{m \to -\infty} \upsilon_x^m(A) = \chi_A(x) \quad \mu\text{-a.e.}$$

(d) *If I is a cylinder of the form $[i_0 \dots i_N]_0$ $(N > 0)$ and $x \in I$ is given such that $\upsilon_x(I) \neq 0$, then for any set A we have*

$$\upsilon_x^N(A) = \frac{\upsilon_x(A \cap I)}{\upsilon_x(I)}$$

where υ_x denotes υ_x^0.

Indication of proof: All statements are consequences of the definition of the conditional expectation and (c) is an application of the increasing Martingale theorem (see for instance [Par.2]). □

Proposition 1.2: *Let μ be an ergodic σ-invariant measure on Σ such that μ is quasi-invariant with respect to the action of the group of finite coordinate changes Δ. If μ is ergodic for the Δ-action then (Σ, σ, μ) is a K-automorphism.*

Proof: The state partition β is a generator for the shift on (Σ, μ). Hence it remains to show that the σ-algebra $\cap_{m>0} \mathscr{A}(m)$ is μ-equivalent to the trivial σ-algebra. Fix a set $A \in \cap_{m>0} \mathscr{A}(m)$. Given $\varphi \in \Delta$ there exists $m > 0$ such that φ fixes the coordinates larger than m. Since $A \in \mathscr{A}(m)$ we conclude that $\mu(\varphi(A)\Delta A) = 0$. Therefore A is Δ-invariant. Since the action is ergodic we have $\mu(A)\,(\mu(A) - 1) = 0$. □

Remark 1.3: In particular, on the hypothesis of Proposition 1.2, if μ is ergodic with respect to the Δ-action then μ is a mixing σ-invariant measure. We note that Proposition 1.2 also holds for Γ replacing Δ since $\Delta \subseteq \Gamma$.

Proposition 1.4: *Let Σ be an aperiodic ssft. If μ_f is the equilibrium state of a continuous function* f *on Σ of summable variation then $(\mathscr{R}, \mu_f)$ and $(\mathscr{T}, \mu_f)$ are ergodic non-singular equivalence relations.*

Proof: Let μ denote μ_f in this proof. We note that it suffices to show that $(\mathscr{T}, \mu)$ is non-singular and $(\mathscr{R}, \mu)$ is ergodic.

If $\mathscr{P}(f)$ denotes the pressure of f then there exists a constant $c > 1$ such that for every cylinder of the form $[i_{-n}, \ldots, i_m]_0$ we have

$$\frac{1}{c} < \frac{\mu\left([i_{-n}, \ldots, i_m]_0\right)}{\exp\left\{-(m+n+1)\,\mathscr{P}(f) + \sum_{j=-n}^{m} f(\sigma^j x)\right\}} < c \tag{1.2}$$

for any $x \in [i_{-n}, \ldots, i_m]_0$ (cf. [Rue] or [Bow]). Let $C_{n,m}(x)$ denote the cylinder $[x_{-n}, \ldots, x_m]_0$. Let $\gamma \in \Gamma$ be given and consider the constant N, and the functions u and t satisfying $(\gamma x)_n = x_{n-u}$ for $n \le -N$ and $(\gamma x)_n = x_{n+t}$ for $n \ge N$. For large n we apply (1.2) to the cylinders $C_{n,n}(\gamma x)$ and $C_{n+u,n+t}(x)$ to obtain

$$\left| \log\left\{\frac{\mu(C_{n,n}(\gamma x))}{\mu(C_{n+u,n+t}(x))}\right\} - \left\{ \sum_{j=-n-N}^{0} (f(\sigma^{j-N}\gamma x) - f(\sigma^{j-N-u}x)) \right.\right.$$

$$+ \sum_{-N<j<N} (f(\sigma^{j}\gamma x) - \mathcal{P}(f)) - \sum_{-N-u<j<N+t} (f(\sigma^{j}x) - \mathcal{P}(f))$$

$$\left.\left. + \sum_{j=0}^{n-N} (f(\sigma^{j+N}\gamma x) - f(\sigma^{j+N+t}x)) \right\} \right| < 2\log(c). \tag{1.3}$$

Since f has summable variation, the expression appearing in (1.3) involving the summations of the function f converges to a continuous function of x as $n \to \infty$. Hence the measurable function $g(x) = \lim_n \sup \log\{\mu(C_{n,n}(\gamma x))/\mu(C_{n+u,n+t}(x))\}$ is bounded. Consequently since $\gamma(C_{n+u,n+t}(x)) = C_{n,n}(\gamma x)$ we conclude that $\log\{d(\mu\circ\gamma)/d\mu\}$ exists and is bounded μ-a.e. This shows that $(\mathcal{T}, \mu)$ is non-singular.

Now we prove that $(\mathcal{R}, \mu)$ is ergodic. When $\varphi \in \Delta$ we have $\Sigma_{j\in\mathbf{Z}} | (f(\sigma^j \varphi x) - f(\sigma^j x))| < \infty$ and taking the limit as $n \to \infty$ in (1.3) we obtain

$$\left| \log\left\{\frac{d(\mu\circ\varphi)}{d\mu}\right\}(x) - \sum_{j\in\mathbf{Z}} (f(\sigma^j \varphi x) - f(\sigma^j x)) \right| < 2\log(c) \tag{1.4}$$

for all $\varphi \in \Delta$ and μ-a.e. $x \in \Sigma$. Using (1.4) and the fact that f has summable variation one shows that there exists $K > 0$ (K depending on c of (1.4) and on the sum of the variations of f) such that for any $\varphi \in \Delta$ and any cylinder C we have

$$\left| \frac{d(\mu\circ\varphi)}{d\mu}(x) - \frac{\mu(\varphi(C))}{\mu(C)} \right| < K\,\frac{\mu(\varphi(C))}{\mu(C)} \tag{1.5}$$

for all $x \in C$. Now let A and B be two sets of positive measure. Let $\varepsilon > 0$ be given such that $\varepsilon < (K+2)^{-1}$. Choose cylinders C and D of the form $[i_{-n}, \ldots, i_n]_0$ (with the same length 2n+1) such that $\mu(C\setminus A) < \varepsilon\,\mu(C)$ and $\mu(D\setminus B) < \varepsilon\,\mu(D)$.[5] By the aperiodicity of Σ we may assume that $i_j(C) = i_j(D)$ for $j = -n, n$, where $i_j(C)$ denote the jth coordinate in the block defining C. Let $\psi \in \Delta$ be the homeomorphism which flips C and D.[6] We claim that $\mu(\psi A \cap B) > 0$ and consequently $(\mathcal{R}, \mu)$ is ergodic. It suffices to show that

[5] This is possible by the Lebesgue Density Theorem.

[6] i.e. $(\psi x)_j = x_j$ for $j \notin \{-n+1, \ldots, n-1\}$, $\psi C = D$, $\psi^2 \equiv \mathrm{Id}$ everywhere and $\psi = \mathrm{Id}$ when restricted to $\Sigma\setminus(C\cup D)$, where Id is the identity of Δ.

$\mu(D\backslash\psi A) + \mu(D\backslash B) < \mu(D)$. But $\mu(D\backslash\psi A) = \mu(\psi(C\backslash A))$ and using (1.5) we have

$$\begin{aligned} \mu(\psi(C\backslash A)) &< (1+K)\frac{\mu(D)}{\mu(C)}\,\mu(C\backslash A) \\ &< (1+K)\,\varepsilon\,\mu(D). \end{aligned}$$

Since $\mu(D\backslash B) < \varepsilon\,\mu(D)$, by the choice of ε, we conclude the proof of the claim. □

Remark 1.5: We note that we have in fact

$$\begin{aligned} \log\left\{\frac{d(\mu_f\circ\gamma)}{d\mu_f}\right\}(x) &= \sum_{j\le 0}\left(f(\sigma^{j-N}\gamma x) - f(\sigma^{j-N-u}x)\right) \\ &+ \sum_{-N<j<N}\left(f(\sigma^j\gamma x) - \mathcal{P}(f)\right) - \sum_{-N-u<j<N+t}\left(f(\sigma^j x) - \mathcal{P}(f)\right) \\ &+ \sum_{j\ge 0}\left(f(\sigma^{j+N}\gamma x) - f(\sigma^{j+N+t}x)\right), \end{aligned} \tag{1.6}$$

for all $\gamma \in \Gamma$ and μ_f–a.e. x, where N, u and t are given such that $(\gamma x)_n = x_{n-u}$ for $n \le -N$ and $(\gamma x)_n = x_{n+t}$ for $n \ge N$ (u and t being continuous functions of x). In particular for the group Δ we have

$$\log\left\{\frac{d(\mu_f\circ\varphi)}{d\mu_f}\right\}(x) = \sum_{i\in\mathbb{Z}}\left(f(\sigma^i\varphi x) - f(\sigma^i x)\right). \tag{1.6'}$$

for all $\varphi \in \Delta$ and μ_f–a.e. x. This is proved in the following way. Let $\rho(\varphi, x)$ denote the Radon-Nikodym cocycle of $(\mathcal{T}, \mu_f)$ (i.e. the left-hand side of (1.6)) and let $\widetilde{F}(\gamma, x)$ denote the right-hand side. Taking the limit as $n \to \infty$ in (1.3) we conclude that $\rho - \widetilde{F}$ is a bounded cocycle, therefore it is a measurable coboundary (cf. [MS]). Hence there exists a Borel measurable function b such that

$$\rho(\gamma, x) = \widetilde{F}(\gamma, x) + b(\gamma x) - b(x) \tag{1.7}$$

for all $\gamma \in \Gamma$ and μ_f–a.e. x. Putting $\gamma = \sigma$ in (1.7) we see that b is σ–invariant and hence it must be a constant function μ_f–a.e.

The following propositions will be used in the next section.

Proposition 1.6: *For any set of positive measure* A, $\int_A \upsilon_x^m(A)\, d\mu(x) > 0$ *for every* m.

Proof: We note that $\int E(\chi_A | \mathscr{A}(m))\, d\mu = \mu(A) > 0$ and from the properties of the conditional expectation we have

$$\begin{aligned}\int_A \upsilon_x^m(A)\, d\mu(x) &= \int \chi_A\, E(\chi_A | \mathscr{A}(m))\, d\mu \\ &= \int E(\chi_A\, E(\chi_A | \mathscr{A}(m)) \,|\, \mathscr{A}(m))\, d\mu \\ &= \int (E(\chi_A | \mathscr{A}(m)))^2\, d\mu\,. \ \square\end{aligned}$$

Proposition 1.7: *The following hold:*

(a) *For any set of positive measure* A *we have*

$$\mu\{x \in A:\ \upsilon_x^m(A) = 0\} = 0$$

for every m.

(b) *If* ψ *is a homeomorphism of* Σ *such that* $(\psi x)_n = x_n$ *for* $n \geq m$ *and* $\mu \circ \psi$ *is equivalent to* μ *then*

$$\mu\{x \in A:\ \upsilon_x^m(\psi A) = 0\} = 0.$$

Proof: (a) Define $B = \{x \in A:\ \upsilon_x^m(A) = 0\}$. If $\mu(B) > 0$ then from Proposition 1.6 we have $\int_B \upsilon_x^m(B)\, d\mu(x) > 0$. But since $B \subseteq A$ we conclude that $\upsilon_x^m(B) \leq \upsilon_x^m(A)$, which is then a contradiction.

(b) Let $B = \{x \in A:\ \upsilon_x^m(\psi A) = 0\}$. If $\mu(B) > 0$ then $\mu(\psi B) > 0$ and by Proposition 1.6 we have $\int_{\psi B} \upsilon_x^m(\psi B)\, d\mu(x) > 0$. Since $\upsilon_x^m \equiv \upsilon_{\psi x}^m$ by Proposition 1.1.(a), and $\mu \circ \psi$ is equivalent to μ we conclude that $\int_B \upsilon_x^m(\psi B)\, d\mu(x) > 0$. Now since $B \subseteq A$ we have $\upsilon_x^m(\psi B) \leq \upsilon_x^m(\psi A)$, which is a contradiction. $\square$

Let $\varphi \in \Delta$ and consider the homeomorphism $\varphi_k = \sigma^{-k} \circ \varphi \circ \sigma^k$ for any $k \in \mathbb{Z}$ in the next propositions.

Proposition 1.8: *Let* μ *be quasi-invariant with respect to the action of* Δ. *Then for any set of positive measure* A, $\mu(A \Delta \varphi_k A) \to 0$ *as* $|k| \to \infty$.

Proof: The sequence of Radon-Nikodym derivatives $d(\mu\circ\varphi_k)/d\mu = \{d(\mu\circ\varphi)/d\mu\}\circ\sigma^k$ is uniformly integrable, hence given $\varepsilon > 0$ there exists $\delta > 0$ (choose $\delta < \varepsilon$) such that whenever $\mu(B) < \delta$ then $\mu\circ\varphi_k(B) < \varepsilon$, for every k. Now let F be an open and closed subset of Σ such that $\mu(A\Delta F) < \delta$, then

$$\begin{aligned}\mu(A\Delta\varphi_k(A)) &\le \mu(A\Delta F) + \mu(F\Delta\varphi_k F) + \mu\circ\varphi_k(F\Delta A) \\ &\le \delta + \mu(F\Delta\varphi_k F) + \varepsilon \\ &\le 2\varepsilon + \mu(F\Delta\varphi_k F)\,. \end{aligned} \tag{1.9}$$

Since F is a finite union of cylinders, for $|k|$ sufficiently large we have $\varphi_k F = F$. Thus the proof follows from (1.9). □

Proposition 1.9: *Let* μ *be quasi-invariant with respect to the action of* Δ. *If* A *and* B *are sets of positive measure then there exist infinitely many* $k > 0$ *(and also* $k < 0$*) such that* $\mu(A \cap \varphi_k^{-1}A \cap \sigma^k B) > 0$.

Proof: Applying the Ergodic Theorem we conclude that, for μ-a.e. $x \in A$, the sequence $\{k > 0: x \in \sigma^k B\}$ has density $\mu(B) > 0$ in $\mathbb{Z}^+$. On the other hand, Proposition 1.8 implies that the sequence of functions

$$\frac{1}{k}\sum_{i=0}^{k-1} \chi_A(x)\chi_{\varphi_i^{-1}A}(x)$$

converges in $L^1(\mu)$ to $\chi_A(x)$. Consequently there exists a subsequence which converges to $\chi_A(x)$ almost everywhere. This implies that the upper density in $\mathbb{Z}^+$ of the sequence $\{k > 0: x \in \varphi_k^{-1}A\}$ is 1, for μ-a.e. $x \in A$. Therefore the result follows for positive k. Similarly applying the above to σ^{-1} we obtain the result for negative k. □

Remark 1.10: We note that if γ is a homeomorphism of Σ satisfying $(\gamma x)_n = x_n$ for $n \le 0$ and $\mu\circ\gamma$ is equivalent to μ then $\varphi_k = \sigma^{-k}\circ\gamma\circ\sigma^k$ satisfies Propositions 1.8 and 1.9 for $k > 0$.

§2. Ergodicity of skew-products

In this section we consider the skew-product transformation S_f defined by a continuous function f and we study the problem of ergodicity of S_f with respect to $\mu \times \lambda$, where μ is an ergodic σ-invariant measure on Σ and λ is the Lebesgue measure on $\mathbb{R}$ (see Section I.3 for definitions). Certainly ergodicity of S_f cannot be proved in this generality but we shall prove that, if the measure μ satisfies the regularity condition of quasi-invariance with respect to finite coordinate changes, then any function f of summable variation satisfying the necessary condition $\int f \, d\mu = 0$ defines a cocycle which has cocompact asymptotic range (if it is not a coboundary).

We shall use extensively in this section the terminology and the results of Section I.3. Let Σ be an irreducible ssft and let $f \in \mathscr{C}(\Sigma)$. Consider the set $\mathscr{M}(f)$ of σ-invariant probability measures on Σ such that $\int f \, d\mu = 0$. Let $E_\sigma(f) = E_{\sigma,\mu}(f)$ denote the asymptotic range of the cocycle defined by f for the shift-action.

We say that f is *non-lattice distributed* when the f-weights of periodic orbits generate a dense subgroup of $\mathbb{R}$. In particular f is not cohomologous to any continuous function $g: \Sigma \to \alpha\mathbb{Z}$, for any $\alpha > 0$.

Remark 2.1: We note by Proposition I.3.2 that if $E_{\sigma,\mu}(f) \neq \{0\}$ then f can be regarded as a (measurable) $E_{\sigma,\mu}(f)$-extension which, by Proposition I.3.3, is ergodic with respect to the measure $\mu \times \lambda_E$, where λ_E denotes Lebesgue measure on the group $E_{\sigma,\mu}(f)$.

The idea in what follows is to construct a non-zero essential value for f. However, in order to do so we shall make further assumptions on f and on the ergodic σ-invariant measure μ as follows.

From now on in this section we make the following hypothesis.

STANDING HYPOTHESIS: We assume that $f \in \mathscr{C}(\Sigma)$ is given such that f has *summable variation* and $\mu \in \mathscr{M}(f)$ is an *ergodic σ-invariant* probability measure which is *quasi-invariant* with respect to the action of a group G (where G will be either the group of finite coordinate changes Δ or the group of finite block exchanges Γ, see last section). Furthermore we assume that f *depends only on future coordinates* (see Section I.2).

Since the asymptotic range is a cohomology invariant, by using Proposition I.2.3, we see that the results of this chapter hold for $f \in \mathscr{C}(\Sigma)$ satisfying $\Sigma_n\, n{\cdot}\mathrm{var}_n(f) < \infty$. When $f \in \mathscr{C}(\Sigma)$ has only summable variation then we can only guarantee the existence of a continuous function g depending only on future coordinates with g cohomologous to f, but g may not be of summable variation (see Remark I.2.4). However, the results below still hold for g replacing f when g arises in this way. This follows from the fact that the cocycle $\widetilde{F}$ (see below) defined by f coincides with the corresponding one defined by g and the cocycle F defined by f is cohomologous to the corresponding one defined by g.

Given the function f we can construct a cocycle for the action of Γ and Δ as follows. Define the cocycle $F: \Gamma \times \Sigma \to \mathbb{R}$ by

$$F(\gamma, x) = \sum_{0 \le j < N} f(\sigma^j \gamma x) - \sum_{0 \le j < N+t} f(\sigma^j x) + \sum_{j \ge 0} (f(\sigma^{j+N} \gamma x) - f(\sigma^{j+N+t} x)), \tag{2.1}$$

where N and t are constants satisfying $(\gamma x)_n = x_{n+t}$ for $n \ge N$. Restricted to Δ we have

$$F(\gamma, x) = \sum_{i=0}^{\infty} (f(\sigma^i \gamma x) - f(\sigma^i x)). \tag{2.1'}$$

We note that since f has summable variation, F is a continuous cocycle for Γ and Δ. We shall denote the asymptotic range of the cocycle F for the G-action on Σ ($G = \Delta$ or Γ) by $E_G(F) = E_{G,\mu}(F)$. (Lemma 2.6 below relates $E_\Delta(F)$ to $E_\sigma(f)$.) Here we should remark that we are *not* assuming ergodicity of the action of Δ or Γ on (Σ, μ). One can easily see from the definition of $E_G(F)$ in Section I.3 that $E_G(F)$ is the *intersection* of $E_{G,\mu_B}(F_B)$ where B runs through the ergodic components of the G-action and μ_B, F_B denote the restrictions of μ, F to B respectively. Hence, even in the case of non-ergodicity of the G-action, $E_G(F)$ is a closed subgroup of $\mathbb{R}$.

We introduce also another cocycle for the Γ-action on Σ which is a "two-sided version" of F, i.e. we consider the continuous cocycle $\widetilde{F}: \Gamma \times \Sigma \to \mathbb{R}$ given by

$$\tilde{F}(\gamma, x) = \sum_{j \le 0} (f(\sigma^{j-N}\gamma x) - f(\sigma^{j-N-u}x)) + \sum_{-N<j<N} f(\sigma^j \gamma x)$$
$$- \sum_{-N-u<j<N+t} f(\sigma^j x) + \sum_{j \ge 0} (f(\sigma^{j+N}\gamma x) - f(\sigma^{j+N+t}x)), \qquad (2.2)$$

where N, u and t satisfy $(\gamma x)_n = x_{n-u}$ for $n \le -N$ and $(\gamma x)_n = x_{n+t}$ for $n \ge N$ (where u and t are continuous functions of x).[7] In particular for the group Δ we have

$$\tilde{F}(\varphi, x) = \sum_{i \in \mathbb{Z}} (f(\sigma^i \varphi x) - f(\sigma^i x)), \qquad (2.2')$$

for every $\varphi \in \Delta$ and $x \in \Sigma$. Denote by $E_G(\tilde{F})$ the asymptotic range of the cocycle $\tilde{F}$ for the action of G on (Σ, μ) where $G = \Delta$ or Γ. Since Δ is a subgroup of Γ we note that if μ is quasi-invariant with respect to the Γ-action then $E_\Delta(F) \subseteq E_\Gamma(F)$ and $E_\Delta(\tilde{F}) \subseteq E_\Gamma(\tilde{F})$.

Some relations between the asymptotic ranges of the various cocycles introduced above are exhibited in the following lemmas.

Lemma 2.2: *Let f and μ satisfy the standing hypothesis where $G = \Delta$ and consider the cocycle $\tilde{F}$ for the Δ-action. Then $\tilde{F} \equiv 0$ if and only if f is a coboundary.*

Proof: If f is a coboundary, from the definition of $\tilde{F}$ it is easy to prove that $\tilde{F} \equiv 0$. (In fact if f is a μ-measurable coboundary then $\tilde{F} \equiv 0$ μ-a.e.)

Now, in order to prove the other implication, we assume that f is not a coboundary and show that $\tilde{F} \not\equiv 0$. Let c denote the total variation of f,[8] i.e.

$$c = \sup \{| f^n(\sigma^{-n}x) - f^n(\sigma^{-n}z) | : n \ge 1,\ x, z \in \Sigma \text{ with } x_i = z_i \text{ for } i \le 0\}. \qquad (2.3)$$

By Proposition I.2.2 there must be a periodic orbit $y \in \mathrm{Fix}(n)$ such that $f^n(y) = a \ne 0$. By considering a large period if necessary, we may assume that $|a| > 3c$. Let k be such a period for y and consider the cylinder $C_k = [y_0 y_1 \dots y_{k-1} y_0]_0$. We claim that there exist a period mk of y and a cylinder B_{mk} of the form

(7) From Remark 1.5 we see that $\tilde{F}$ is the Radon-Nikodym cocycle of the equilibrium state of the function $f-\mathcal{P}(f)$ with respect to the Γ-action, where $\mathcal{P}(f)$ is the pressure of f.

(8) We note that if f has summable variation then c is finite (actually $c \le \Sigma \mathrm{var}_n(f)$).

$$[y_0\, y'_1 \dots y'_{km-1}\, y_0]_0$$

such that there exists $x \in B_{mk}$ with $|f^{mk}(x)| < c$. This establishes the result since the homeomorphism $\varphi \in \Delta$ which flips the cylinders B_{mk} and C_{mk}(9) satisfies:

$$|\tilde{F}(\varphi, x) - ma| < 3c. \tag{2.4}$$

Thus, by the assumption on the weight a, (2.4) implies that $\tilde{F}(\varphi, x) \neq 0$.

To prove the claim we apply recurrence of the skew-product defined by f^k for the action of σ^k (cf. Proposition I.3.4) to the cylinder C_k to conclude the existence of $x \in \mathrm{Fix}(mk) \cap C_k$ with $|f^{mk}(x)| < c$, for some m. Now x and the cylinder $B_{mk} = [y_0\, x_1 \dots x_{mk-1}\, y_0]_0$ satisfy the claim. □

Lemma 2.3: *Let* f *and* μ *satisfy the standing hypothesis where* $G = \Delta$. *Then*

$$E_\Delta(\tilde{F}) = \operatorname{clos}\{\tilde{F}(\varphi, x) : \varphi \in \Delta \text{ and } x \in \Sigma\}.$$

Proof: We note that for any $\varphi \in \Delta$ and $x \in \Sigma$ we have

$$\tilde{F}(\sigma^{-n}\varphi\sigma^n, \sigma^{-n}x) = \tilde{F}(\varphi, x) \tag{2.5}$$

for every $n \in \mathbb{Z}$. Fix $\varphi \in \Delta$ and let α denote some value of $\tilde{F}(\varphi, \cdot)$. Then for any $\varepsilon > 0$ given, the set

$$B_{\varepsilon,\varphi}(\alpha) = \{x \in \Sigma : \tilde{F}(\varphi, x) \in (\alpha - \varepsilon, \alpha + \varepsilon)\} \tag{2.6}$$

has positive measure. By Proposition 1.9, given any set of positive measure A, we can find n such that

$$\mu(A \cap \varphi_n^{-1} A \cap \sigma^n B_{\varepsilon,\varphi}(\alpha)) > 0. \tag{2.7}$$

Now the definition of the asymptotic range $E_\Delta(\tilde{F})$ together with (2.5) and (2.7) implies that $\alpha \in E_\Delta(\tilde{F})$. Hence $E_\Delta(\tilde{F}) \supseteq \{\tilde{F}(\varphi, x) : \varphi \in \Delta \text{ and } x \in \Sigma\}$. Since $E_\Delta(\tilde{F})$ is closed, one inclusion is proved. The other inclusion is straight-forward from the definition of $E_\Delta(\tilde{F})$ and the fact that $\tilde{F}$ is a continuous cocycle since μ is fully supported. □

(9) i.e. $(\varphi x)_n = x_n$ for $n \notin \{1, \dots, mk-1\}$, $\varphi B_{mk} = C_{mk}$, $\varphi^2 = \mathrm{Id}$ everywhere and $\varphi = \mathrm{Id}$ when restricted to $\Sigma \setminus (B_k \cup C_k)$, where Id is the identity of Δ.

Lemma 2.4: *Let* f *and* μ *satisfy the standing hypothesis where* $G = \Delta$. *Then*

$$E_\Delta(F) = E_\Delta(\widetilde{F}).$$

Proof: From the definition of the cocycles F and $\widetilde{F}$ we obtain for any $\varphi \in \Delta$ and $x \in \Sigma$,

$$F(\sigma^{-n}\varphi\sigma^n, \sigma^{-n}x) = \widetilde{F}(\sigma^{-n}\varphi\sigma^n, \sigma^{-n}x) - \sum_{i<-n} (f(\sigma^i\varphi x) - f(\sigma^i x))$$

$$= \widetilde{F}(\varphi, x) - \sum_{i<-n} (f(\sigma^i\varphi x) - f(\sigma^i x)) \tag{2.8}$$

where we have used (2.5). Since f has summable variation, for fixed $\varphi \in \Delta$, the summation in (2.8) goes uniformly in x to zero as $n \to \infty$. Hence using Proposition 1.9 applied to any set of positive measure A and the set $B_{\varepsilon,\varphi}(\alpha)$ (see (2.6)) where $\alpha = \widetilde{F}(\varphi, x)$ for some φ and x, we conclude that $\alpha \in E_\Delta(F)$. This proves that $E_\Delta(F) \supseteq E_\Delta(\widetilde{F})$.

Now let $\alpha \in E_\Delta(F)$. Take a long cylinder C, say of length n. By the definition of $E_\Delta(F)$, given any $\varepsilon > 0$ there exists $\varphi = \varphi_{C,\varepsilon} \in \Delta$ such that

$$\mu(C \cap \varphi^{-1}C \cap \{x: F(\varphi, x) \in (\alpha-\varepsilon, \alpha+\varepsilon)\}) > 0.$$

Hence $(\varphi x)_s = x_s$ for $|s| \le n$. Since f depends only on future coordinates φ can be replaced by a finite coordinate change which fixes the coordinates less than n. Thus we have $|F(\varphi, x) - \widetilde{F}(\varphi, x)| \le \Sigma_{j\ge n}\, c_j$, where c_j denotes the jth variation of f. Since f has summable variation, by choosing n large, the latter shows that $\widetilde{F}$ takes values arbitrarily close to α. Therefore $\alpha \in E_\Delta(\widetilde{F})$ by Lemma 2.3. □

Proposition 2.5: *Let* f *and* μ *satisfy the standing hypothesis where* $G = \Delta$. *Let* $\alpha \in E_\Delta(F)$ *be positive. For any set of positive measure* U *and any* $\xi > 0$ *given, there exists* $\varphi \in \Delta$ *and a subset* $A \subseteq U$ *with* $\mu(A) > 0$ *such that:*

(i) $\varphi^2 \equiv \mathrm{Id}$;

(ii) A *is* φ*-invariant;*

(iii) $|F(\varphi, x)| \in (\alpha-\xi, \alpha+\xi)$ *for all* $x \in A$;

(iv) there exists $N > 0$ *such that* $(\varphi x)_n = x_n$ *for* $n \notin \{1, \ldots, N-1\}$.

Proof: By Lemma 2.4 we have $\alpha \in E_\Delta(\widetilde{F})$ and from the definition of $E_\Delta(\widetilde{F})$ there exists $\varphi \in \Delta$ such that $B_{\xi/2,\varphi}(\alpha)$ (defined in (2.6)) has positive measure. For sufficiently large $n > 0$, $\varphi_n = \sigma^{-n}\varphi\sigma^n$ satisfies (iv). Fix n large enough such that the summation in (2.8) is in modulus less than $\xi/2$ and the set $V = U \cap \varphi_n^{-1}U \cap \sigma^n B_{\xi/2,\varphi}(\alpha)$ has positive measure (this is possible by Proposition 1.9). Let $n = N$ satisfy the latter property and put $\tilde{\varphi} = \varphi_N$. From (2.8) we see that for every $x \in V$ we have $F(\tilde{\varphi}, x) \in (\alpha-\xi, \alpha+\xi)$.

Now we shall replace $\tilde{\varphi}$ by a finite coordinate change which leaves invariant a subset of V of positive measure and still satisfies conditions (iii) and (iv) as follows. Let $V' \subseteq V$ be given such that $\mu(V') > 0$ and $V' \cap \tilde{\varphi}V' = \emptyset$. Let C be a cylinder such that $\mu(C \cap V')$ is close to $\mu(C)$[10] and consider the homeomorphism $\psi \in \Delta$ which agrees with $\tilde{\varphi}$ on C and agrees with $\tilde{\varphi}^{-1}$ on $\tilde{\varphi}C$. Now take $A = (C \cap V') \cup \tilde{\varphi}(C \cap V')$. Since F is a cocycle we have $F(\psi, x) = - F(\psi^{-1}, \psi x)$, therefore we conclude that ψ and A satisfy the conditions from (i) to (iv). □

Lemma 2.6: *Let* f *and* μ *satisfy the standing hypothesis where* $G = \Delta$. *Then*

$$E_\Delta(\widetilde{F}) = E_\Delta(F) \subseteq E_\sigma(f).$$

Proof: If $E_\Delta(\widetilde{F}) = \{0\}$ nothing has to be proved. Suppose $\alpha \in E_\Delta(\widetilde{F})$, $\alpha \neq 0$ exists. We may assume $\alpha > 0$. Given any set of positive measure U and any $\xi > 0$, fix the subset A of U and the homeomorphism φ satisfying the conditions from (i) to (iv) of Proposition 2.5 in the course of the proof. Fix also $N > 0$ satisfying (iv).

Let $c = \int_A \upsilon^0_{\varphi x}(A)\, d\mu(x)$. From Proposition 1.5 and the fact that μ is quasi-invariant with respect to the action of Δ we conclude that $c > 0$. Hence the set

$$D = \{x \in A: \upsilon^0_{\varphi x}(A) > c/2\}$$

has positive measure. Let $\mathscr{C}(N)$ denote the set of cylinders in Σ of the form $[i_0 \dots i_N]_0$. For $I \in \mathscr{C}(N)$ and $\delta > 0$ define

$$B_\delta(I) = \{x \in A \cap I: \upsilon^N_x(I) > \delta \text{ and } \upsilon^N_x(\varphi I) > \delta\}.$$

(10) Here we are again using the Lebesgue Density Theorem.

Putting $\delta = 1/n$ and applying Proposition 1.7 to the set I, we obtain

$$\lim_{n\to\infty} \mu(B_{1/n}(I)) = \mu(A\cap I)$$

for any $I \in \mathscr{C}(N)$. Hence we can choose $\delta > 0$ and a cylinder $I \in \mathscr{C}(N)$ such that

$$B = D \cap B_\delta(I)$$

has positive measure. Choose $0 < \varepsilon < \min\{c.\delta/2;\ \mu(B)\}$ and for every $m \le 0$, consider the set

$$C_m = \{x \in A:\ \upsilon_x^t(A) > 1 - \varepsilon,\ \text{for all } t < m\}.$$

Proposition 1.1.(c) implies that there exists $M > 0$ such that $\mu(C_{-M}) > \mu(A) - \varepsilon$, and from the choice of ε we conclude that the set

$$\begin{aligned} E &= B \cap C_{-M} \\ &= D \cap B_\delta(I) \cap C_{-M} \end{aligned}$$

has positive measure.

Recurrence of the skew-product defined by f (cf. Proposition I.3.4) applied to the set E implies that given ξ (from the beginning of proof) there exists $k > M+N$ such that

$$\mu\Big(E \cap \sigma^{-k}E \cap \{x: |f^k(x)| < \xi\} \Big) > 0. \tag{2.9}$$

We claim that for any $x \in E^{ret} = E \cap \sigma^{-k}E \cap \{x: |f^k(x)| < \xi\}$ there exists $y \in A$ with $x_n = y_n$ for all $n \ge 0$ (thus, in particular, $f^k(y) = f^k(x)$ since f depends only on future coordinates) such that

$$\varphi y \in A\cap\sigma^{-k}A \ \text{ and } \ |f^k(\varphi y)| \in (\alpha-2\xi, \alpha+2\xi).$$

We remark that the latter statement proves the lemma since it implies that

$$\mu\Big(U \cap \sigma^{-k}U \cap \{x: |f^k(x)| \in (\alpha-2\xi, \alpha+2\xi)\} \Big) > 0,$$

where U and ξ were arbitrarily chosen.

For the proof of the claim we shall use the properties of the conditional measures of Proposition 1.1 together with the properties of the set E as follows. Let $x \in E^{ret}$. Since

$x \in D$ we have $\upsilon^0_{\varphi x}(A) > c/2$. Since $\sigma^k x \in C_{-M}$ we have $\upsilon^t_{\sigma^k x}(A) > 1 - \varepsilon$, for all $t < -M$. In particular, since $k > M+N$, this is the case for $t = N-k$. Therefore from Proposition 1.1.(b) we get

$$\upsilon^N_x(\sigma^{-k}A) > 1 - \varepsilon. \tag{2.10}$$

Let I denote the cylinder $[(\varphi x)_0 \dots (\varphi x)_N]_0$. Since $\varphi^2 \equiv \mathrm{Id}$ we have $x \in B_\delta(\varphi I)$ and then $\upsilon^N_x(I) > \delta$. Recall the following elementary inequality

$$\upsilon^N_x(Y \cap I) \geq \upsilon^N_x(I) - \upsilon^N_x(\Sigma \setminus Y)$$

where Y is arbitrary. Take $Y = \sigma^{-k}A$ in the above to obtain

$$\frac{\upsilon^N_x(\sigma^{-k}A \cap I)}{\upsilon^N_x(I)} \geq 1 - \frac{\upsilon^N_x(X \setminus \sigma^{-k}A)}{\upsilon^N_x(I)}.$$

Now applying Proposition 1.1.(a) and (d) together with (2.10) we obtain

$$\upsilon^0_{\varphi x}(\sigma^{-k}A) = \frac{\upsilon^N_x(\sigma^{-k}A \cap I)}{\upsilon^N_x(I)} \geq 1 - \frac{\varepsilon}{\delta}.$$

By the choice of ε ($\varepsilon < c.\delta/2$) we conclude that $\upsilon^0_{\varphi x}(\sigma^{-k}A) > 1 - c/2$. Hence $\upsilon^0_{\varphi x}(A \cap \sigma^{-k}A) > 0$. Therefore there exists $y \in A$ with $\varphi y \in A \cap \sigma^{-k}A$ and $x_n = y_n$ for all $n \geq 0$. Hence $|f^k(\varphi y)| \in (\alpha - 2\xi, \alpha + 2\xi)$ since

$$|F(\varphi, y)| \in (\alpha - \xi, \alpha + \xi) \text{ and } |f^k(y)| = |f^k(x)| < \xi.$$

Thus the claim is proved. □

Lemma 2.7: *Let* f *and* μ *satisfy the standing hypothesis where* $G = \Delta$. *Then*

$$E_\Delta(\widetilde{F}) \supseteq \{f^n(x) - f^n(y) \colon x, y \in \mathrm{Fix}(n), n > 0\}.$$

Furthermore if μ *satisfies the standing hypothesis where* $G = \Gamma$ *then*

$$E_\Gamma(\widetilde{F}) \supseteq \{f^n(x) \colon x \in \mathrm{Fix}(n), n > 0\}.$$

Proof: Assume f and μ satisfy the standing hypothesis with $G = \Gamma$. Let $\alpha = f^m(x)$ be the f-weight of the periodic orbit $x \in \mathrm{Fix}(m)$, where m is the minimal period of x. Consider the cylinder $[x_0 x_1 \ldots x_{km-1} x_0]_0$ with large k. By using a higher block representation of Σ, we may assume that there exists a symbol $i \neq x_1$ such that $x_0 \to i$ is allowable. Now take $C = [x_0 x_1 \ldots x_{km-1} x_0 i]_0$. Consider the homeomorphism γ which flips the blocks $[x_0 x_1 \ldots x_{km-1} x_0 i]_0$ and $[x_0 x_1 \ldots x_{(k-1)m-1} x_0 i]_0$. By Remark 1.10 we conclude that given any set of positive measure A, there exists $n > 0$ such that

$$\mu(A \cap \gamma_n^{-1} A \cap \sigma^n C) > 0,$$

where $\gamma_n = \sigma^{-n}\gamma\sigma^n$. By the definition of $\widetilde{F}$ we have $\widetilde{F}(\gamma, z) = \widetilde{F}(\sigma^{-n}\gamma\sigma^n, \sigma^{-n}z)$ for every $z \in \Sigma$ and hence for any $z \in A \cap \gamma_n^{-1} A \cap \sigma^n C$ we obtain

$$|\widetilde{F}(\gamma, z) - \alpha| \leq \sum_{j=k-1}^{\infty} c_j,$$

where c_j denotes the jth variation of f. Taking k large we conclude that $f^m(x) \in E_\Gamma(\widetilde{F})$.

Now assume f and μ satisfy the standing hypothesis where $G = \Delta$. Let again α denote $f^m(x)$ and let β denote $f^m(y)$, where $x, y \in \mathrm{Fix}(m)$. Choose an allowable path in the graph of Σ joining x_0 to y_0, i.e. let i_j be given such that $[x_0 i_1 \ldots i_{p-1} y_0]_0$ is a cylinder of Σ. Consider for large k the cylinder

$$C = [x_0 x_1 \ldots x_{km-1} x_0 i_1 \ldots i_{p-1} y_0 y_1 \ldots y_{km-1} y_0]_0 .$$

Now take the homeomorphism φ which flips the blocks C and

$$D = [x_0 x_1 \ldots x_{(k-1)m-1} x_0 i_1 \ldots i_{p-1} y_0 y_1 \ldots y_{(k+1)m-1} y_0]_0 .$$

By Proposition 1.9, given any set of positive measure A, there exists $n > 0$ such that

$$\mu(A \cap \varphi_n^{-1} A \cap \sigma^n C) > 0,$$

where $\varphi_n = \sigma^{-n}\varphi\sigma^n$. Now a similar computation shows that

$$|\widetilde{F}(\varphi, z) - \alpha + \beta| \leq 2 \sum_{j=k-1}^{\infty} c_j,$$

for any $z \in A \cap \varphi_n^{-1} A \cap \sigma^n C$. Taking k large we conclude that $f^m(x) - f^m(y) \in E_\Gamma(\widetilde{F})$. □

Now we come to the main results of this chapter.

Theorem 2.8: *Let* $f \in \mathscr{C}(\Sigma)$ *be a function of summable variation and let* $\mu \in \mathscr{M}(f)$ *be an ergodic measure which is quasi-invariant with respect to the action of* Δ *(i.e. with respect to finite coordinate changes). Then* $E_\Delta(\widetilde{F})$ *is the closure of the group generated by* $\{f^n(x)-f^n(y)\colon\ x, y \in \mathrm{Fix}(n), n > 0\}$. *Furthermore if* f *is not a coboundary then the following are the only possibilities for* $E_\sigma(f)$:

(i) *If* $\{f^n(x)-f^n(y)\colon\ x, y \in \mathrm{Fix}(n), n > 0\}$ *generates a dense subgroup of* $\mathbb{R}$ *then* $E_\sigma(f) = \mathbb{R}$ *and* $(S_f, \mu \times \lambda)$ *is an ergodic skew-product.*

(ii) *Otherwise,* $E_\Delta(\widetilde{F}) = \beta\mathbb{Z}$ *for some* $\beta > 0$ *and then* $E_\sigma(f) = (\beta/n)\mathbb{Z}$ *for some* $n > 0$ *or* $E_\sigma(f) = \mathbb{R}$.

Proof: By Lemma 2.7 we conclude that $E_\Delta(\widetilde{F})$ contains the closure of the group generated by $\{f^n(x)-f^n(y)\colon\ x, y \in \mathrm{Fix}(n), n > 0\}$. Now let $\alpha \in E_\Delta(\widetilde{F})$ be given. By Lemma 2.3 there exist $\varphi \in \Delta$ and $z \in \Sigma$ such that $\widetilde{F}(\varphi, z)$ is close to α. We may assume that for any given cylinder C there exists a large $k > 0$ with $z \in \sigma^{-k}C \cap \sigma^{k}C$. Therefore there are periodic orbits of period $2k+1$ with difference of f-weights close to $\widetilde{F}(\varphi, z)$. This proves the other inclusion.

Now when f is not a coboundary, Lemma 2.2 shows that $\widetilde{F} \not\equiv 0$, and hence by Lemma 2.3, $E_\Delta(\widetilde{F}) \neq \{0\}$. From Lemma 2.6 we have $E_\Delta(\widetilde{F}) \subseteq E_\sigma(f)$ and the assertions follow from Lemma 2.7 and Proposition I.3.3. □

Remark 2.9: Under the hypothesis of Theorem 2.8 we obtain the following result: f is a (continuous) coboundary if and only if f is a μ-measurable coboundary, which is an improvement on a result of Livšic [Liv] on measurable coefficients.

Theorem 2.10: *Let* $f \in \mathscr{C}(\Sigma)$ *be a function of summable variation and let* $\mu \in \mathscr{M}(f)$ *be an ergodic measure which is quasi-invariant with respect to the action of* Γ *(i.e. with respect to finite block exchanges). If* f *is not a coboundary then* $E_\sigma(f) = E_\Gamma(\widetilde{F})$ *is the closure of the group generated by* $\{f^n(x)\colon\ x \in \mathrm{Fix}(n), n > 0\}$. *(Therefore if f is non-lattice distributed then* $(S_f, \mu \times \lambda)$ *is an ergodic skew-product.)*

Proof: By Theorem 2.8, we know that $E_\sigma(f) \neq \{0\}$. Using the definition of $E_\sigma(f)$ applied to cylinders we conclude that $E_\sigma(f)$ is contained in the closure of the group generated by the f-weights of periodic orbits. Hence by Lemma 2.7, it suffices to show that $E_\Gamma(\widetilde{F}) \subseteq E_\sigma(f)$. However, by Proposition I.3.2, f is measurably cohomologous to a measurable function g taking values in $E_\sigma(f)$. Since f and g define the same cocycle $\widetilde{F}$ for the Γ-action (μ-a.e.), we conclude that $E_\Gamma(\widetilde{F}) \subseteq E_\sigma(f)$. □

Corollary 2.11: *Under the hypotheses of Theorem 2.10, if the group generated by the f-weights of periodic orbits is $\alpha\mathbb{Z}$, then the corresponding $\alpha\mathbb{Z}$-extension of Σ is ergodic with respect to the measure $\mu \times \ell$, where ℓ denotes counting measure on $\alpha\mathbb{Z}$.*

The next example shows that Corollary 2.11 is false if we assume only quasi-invariance under the action of the group Δ. Hence it also provides an example of an ergodic σ-invariant measure which is quasi-invariant under finite coordinate changes but is not quasi-invariant under finite block exchanges.

Example 2.12: Consider the full two-shift $X_2 = \{0, 1\}^{\mathbb{Z}}$. Let σ denote the shift on X_2. Consider a probability measure υ on X_2, such that υ is a fully supported Bernoulli measure for σ^2, regarded as a measure on the four-shift $(\{0, 1\} \times \{0, 1\})^{\mathbb{Z}}$, and υ is not a σ-invariant measure. Therefore υ and $\upsilon\circ\sigma$ are ergodic σ^2-invariant measures on X_2, which are also quasi-invariant under finite coordinate changes. Since the measure $\upsilon\circ\sigma$ is singular with respect to υ, there exists a measurable set B such that $\upsilon(B) = 1$ and $\upsilon(\sigma B) = 0$. Consider the measure μ on X_2 given by $\frac{1}{2}(\upsilon + \upsilon\circ\sigma)$. Then μ is σ-ergodic and quasi-invariant under the action of the group Δ (we note that μ is not mixing and hence the Δ-action is not ergodic). Now we show that the constant function $b = 1/2$ is a μ-measurable coboundary when viewed as a cocycle in the circle $\mathbb{R}/\mathbb{Z}$. Let g be the measurable function defined by

$$g(x) = \begin{cases} 0 \bmod(1) & \text{if } x \in B \text{ and} \\ 1/2 \bmod(1) & \text{if } x \in \sigma B. \end{cases}$$

Then it follows from the definitions that

$$(g(x) - g(\sigma x)) \bmod(1) = 1/2 = b(x) \quad \mu\text{-a.e. } x.$$

Now choosing a function f on X_4 taking values in $(1/2)\mathbb{Z}$, such that $f(x) = 1/2 \bmod(1)$ for all x and $\int f\, d\mu = 0$, we conclude that f does not define an ergodic $(1/2)\mathbb{Z}$-extension with respect to μ. (An explicit choice of υ and f is given by $\upsilon = \omega^{\mathbb{Z}}$, where $\omega(00) = \omega(11) = 1/4$, $\omega(01) = 1/8$, $\omega(10) = 3/8$ and f is a function of the form $f(x) = f(x_0, x_1)$ defined by $f(0, 0) = f(1, 1) = 1/2$ and $f(0, 1) = f(1, 0) = -1/2$.)

The following corollary to Theorem 2.10 appears in the recent work on infinite group extensions of dynamical systems by Guivarc'h [Gui]. (See the Introduction of this thesis for some comments on this work.)

Corollary 2.13: *Let Σ be an aperiodic ssft. Let f be a function on Σ of summable variation such that $\int f\, d\mu_g = 0$, where μ_g is the equilibrium state of a function g of summable variation. If f is non-lattice distributed then $(S_f, \mu_g \times \lambda)$ is ergodic.*

Proof: By Proposition 1.4, the equilibrium state μ_g is non-singular with respect to the action of the group of finite block exchanges. Hence f and μ_g satisfy the hypotheses of Theorem 2.10. □

§3. Examples from Markov shifts

There is a natural correspondence between Markov shifts Σ_P defined by a non-negative irreducible matrix P and pairs (Σ, f) of irreducible ssft's Σ with functions $f: \Sigma \to \mathbb{R}$ depending only on a finite number of coordinates as follows. Given P take P^0 as the transition matrix associated to P (see Section I.1) and consider f given by $f(x_0, x_1) = \log P(x_0, x_1)$; conversely if M is the transition matrix of Σ, given f depending on a finite number of variables, by considering a higher-block system (see Section I.1), we may assume that it is of the form $f(x) = f(x_0, x_1)$ and hence P is the matrix defined by $P(i, j) = e^{f(i, j)}$ if $M(i, j) = 1$ and $P(i, j) = 0$ otherwise.

Now let f be a function on Σ depending only on a finite number of coordinates. Consider the subgroup Γ_f of $\mathbb{R}$ generated by the f-weights of periodic orbits, i.e. the group generated by $\{f^n(x)\colon x \in \mathrm{Fix}(n),\ n > 0\}$; and consider the subgroup $\Delta_f = \{f^n(x)-f^n(y)\colon x, y \in \mathrm{Fix}(n), x_0 = y_0, n > 0\}$. When f is of the form $f(x) = f(x_0, x_1)$ we see that $\Delta_f = \Delta_P$ and $\Gamma_f = \Gamma_P$, where (P^0, f) and P are related as in the discussion above.

Remark 3.1: Theorems 2.8 and 2.10 together with Corollary 2.13 imply the following. If μ is any Markov measure[11] fully supported on Σ such that $\int f\, d\mu = 0$. Then

$$E_{\Delta,\mu}(F) = E_{\Delta,\mu}(\widetilde{F}) = \mathrm{clos}\{\Delta_P\} \text{ and } E_{\Gamma,\mu}(\widetilde{F}) = E_{\sigma,\mu}(f) = \mathrm{clos}\{\Gamma_P\}.$$

Remark 3.1 provides easy examples for the concepts considered in this chapter.

Example 3.2: Consider the full two-shift $\Sigma = \{0, 1\}^{\mathbb{Z}}$ endowed with the Bernoulli measure given by the stationary vector (1/2, 1/2). Let f be a function of the form $f(x) = f(x_0, x_1)$ defined by $f(0, 0) = f(1, 1) = \alpha$ and $f(0, 1) = f(1, 0) = -\alpha$. Then $E_\Gamma(\widetilde{F}) = E_\sigma(f) = \alpha\mathbb{Z}$ and $E_\Delta(F) = E_\Delta(\widetilde{F}) = 2\alpha\mathbb{Z}$. (Compare this with Example 2.12 when $\alpha = 1/2$.)

Example 3.3: Let P be the non-negative matrix $\begin{pmatrix} 1 & p \\ q & 1 \end{pmatrix}$ where $p, q > 0$ are given such that $\log(p)$ and $\log(q)$ are rationally independent. Then we have $\Delta_P = \log(pq)\mathbb{Z}$ and $\Gamma_P = \langle \log(p), \log(q) \rangle$. Consider the function f defined on $\Sigma = \{0, 1\}^{\mathbb{Z}}$ arising from P as in the beginning of this section. Let μ be any fully supported Markov measure on $\Sigma = \{0, 1\}^{\mathbb{Z}}$ and take the function $g = f - \int f\, d\mu$. By the definition of Δ_g we conclude that $\Delta_g = \Delta_P$. Hence by Remark 3.1 we have $E_{\Delta,\mu}(F) = E_{\Delta,\mu}(\widetilde{F}) = \log(pq)\mathbb{Z}$ with f replaced by g. Depending on the choice of μ we can have various possibilities for $\mathrm{clos}\{\Gamma_g\}$ as follows. Let λ_μ denote $\int f\, d\mu$. Then we have the following cases:

---

(11) Here we mean some n-step Markov measure, i.e. an induced measure on Σ which comes from a Markov measure supported on the n-block system of Σ.

(1) $\Gamma_g = \log(pq)\mathbb{Z}$ if $\lambda_\mu \in \log(pq)\mathbb{Z}$;

(2) $\Gamma_g = (1/n)\log(pq)\mathbb{Z}$ for some $n \in \mathbb{N}$ if λ_μ and $\log(pq)$ are rationally dependent.

(3) $\mathrm{clos}\{\Gamma_g\} = \mathbb{R}$ if λ_μ and $\log(pq)$ are rationally independent.

When μ is chosen to be the measure of maximal entropy on Σ then (1) occurs, and when μ is the Markov measure defined by P then (2) occurs with $n = 2$. Since it is easy to produce examples of Markov measures for which (3) occurs, we conclude by Remark 3.1 that we can have any of the three possibilities for $E_{\sigma,\mu}(g)$ varying μ, while $E_{\Delta,\mu}(F) = E_{\Delta,\mu}(\widetilde{F}) = \log(pq)\mathbb{Z}$.

CHAPTER III:
AN ENTROPY CONCEPT FOR SKEW-PRODUCTS OVER SSFT's

In this chapter we introduce a concept of topological entropy for skew-products over a subshift of finite type (ssft) defined by the growth rate of the number of periodic orbits with bounded weights. In Section 1 we relate this concept to the pressure of the corresponding skewing function. In Section 2 we apply the main result of Section 1 to give examples of entropy increasing factor maps of a countable ssft associated canonically with the skew-product, when the skewing function takes values in $\mathbb{Z}$. In Section 3 we discuss a problem related to the measures in the boundary of the interval c(f), which is the range of $\int f\,d\mu$ when μ varies through the σ-invariant probability measures on the shift space.

§1. Topological entropy of a skew-product

In this section we study the growth rate of periodic orbits with bounded weights. We show that this growth rate is related to the pressure of the associated skewing function, and equals the supremum over the measure-theoretic entropies of the σ-invariant measures with respect to which the skewing function has integral zero.

Let Σ be an irreducible ssft. Let $\mathcal{F}$ denote the space of real Hölder continuous functions on Σ. We shall denote by m_f the unique equilibrium state of $f \in \mathcal{F}$. Throughout this chapter we assume that a function $f \in \mathcal{F}$ is given such that f *is not cohomologous to a constant*. This assumption is equivalent to the requirement that the pressure function $t \mapsto \mathcal{P}(tf)$ satisfies

$$\mathcal{P}''(tf) = \frac{d^2}{ds^2}\,\mathcal{P}((t+s)f)\,|_{s=0} > 0 \tag{1.1}$$

for all t (see Proposition I.2.1(iii) and the comments thereafter).

Let $\mathcal{M}$ denote the space of Borel σ-invariant probability measures on Σ, then $\mathcal{M}$ is a compact metrizable space in the weak*-topology. Let $\mathcal{M}(f)$ be the compact subspace of $\mathcal{M}$ consisting of measures μ such that $\int f \, d\mu = 0$. Define

$$s(f) = \sup \{h(\mu) : \mu \in \mathcal{M}(f)\}. \tag{1.2}$$

Since the entropy function on $\mathcal{M}$ is upper semi-continuous, provided that $\mathcal{M}(f)$ is non-empty, there exists a measure m such that $s(f) = h(m)$. In fact more is known,

Proposition 1.1 [Lan]: *If* $\mathcal{P}'(tf) = 0$ *for* $t = \lambda$ *then* $m = m_{\lambda f}$ *(the equilibrium state of the function* λf*) is the unique measure in* $\mathcal{M}(f)$ *satisfying* $s(f) = h(m)$.

Proof: From the definition of pressure (see Section I.2) and the uniqueness of the equilibrium state m_{tf} of the function tf for every fixed t, we have

$$\mathcal{P}(tf) = h(m_{tf}) + t \int f \, dm_{tf} \;\geq\; h(\mu) + t \int f \, d\mu \tag{1.3}$$

for every $\mu \in \mathcal{M}$ with equality if and only if $\mu = m_{tf}$.

In (1.3) if we restrict to measures $\mu \in \mathcal{M}(f)$ and apply Proposition I.2.1(ii), we obtain

$$h(m_{tf}) + t\,\mathcal{P}'(tf) \;\geq\; h(\mu)\,. \tag{1.4}$$

Therefore if $\mathcal{P}'(tf) = 0$ for $t = \lambda$ then $m_{\lambda f} \in \mathcal{M}(f)$ and since $\mathcal{P}''(tf) = \sigma^2_{tf}(f) > 0$ for all t, we conclude that λ is unique. □

If $\mathcal{M}(f)$ is non-empty but $\mathcal{P}'(tf)$ does not vanish then there are examples of non-uniqueness for measures solving the equation $s(f) = h(m)$. In this case any measure $m \in \mathcal{M}(f)$ solving $s(f) = h(m)$ is called an *extremal measure*. We shall come back to this topic in Section 3 (see also the Appendix for examples of these measures).

Let $\varepsilon > 0$ be given and consider the cardinality $N_f(n, \varepsilon)$ of the set

$$\{x \in \mathrm{Fix}(n) : |f^n(x)| < \varepsilon\}.$$

Define

$$h_f(\varepsilon) = \lim_n \sup \frac{1}{n} \log N_f(n, \varepsilon). \tag{1.5}$$

We shall see in Theorem 1.4 that $h_f(\varepsilon) = h_f$ does not depend on ε and coincides with s(f) when $\mathcal{M}$(f) is non-empty. This number will be referred to as the *topological entropy of the skew-product* S_f. Since the weights of periodic orbits are cohomology invariants we conclude that $h_f(\varepsilon)$ is a cohomology invariant.

The following shall be used in the proof of Theorem 1.4. Let $g \in \mathscr{C}(\Sigma)$ and consider the sequence of *orbital measures* μ_n defined implicitly by

$$\int u \, d\mu_n = \frac{\sum_{x \in \mathrm{Fix}(n)} u(x) \exp\{g^n(x)\}}{\sum_{x \in \mathrm{Fix}(n)} \exp\{g^n(x)\}} . \tag{1.6}$$

It is shown in [Rue] Theorem 7.20(b) that if $g \in \mathscr{F}$ then μ_n converges weakly to m_g the equilibrium state of g. Consider the following sequence of subsets of Σ,

$$A_n(\varepsilon) = \{x \in \Sigma: |f^n(x)| < \varepsilon\}. \tag{1.7}$$

Lemma 1.2: *Suppose* $m_g \in \mathcal{M}(f)$ *for some* $g \in \mathscr{F}$. *Then for every* $\varepsilon > 0$ *we have*

$$\lim_n \sup \frac{1}{n} \log m_g(A_n(\varepsilon)) = 0. \tag{1.8}$$

Proof: Fix $\varepsilon > 0$. If there exists $c < 0$ such that for sufficiently large n we have $m_g(A_n(\varepsilon)) \le \exp\{cn\}$ then $\sum_{n>0} m_g(A_n(\varepsilon))$ is finite. Hence the set

$$B = \left\{x \in \Sigma: \sum_{n>0} \chi_{A_n(\varepsilon)}(x) \text{ is finite}\right\}$$

has positive m_g-measure, where $\chi_{A_n(\varepsilon)}(x)$ is the indicator function of $A_n(\varepsilon)$. Now consider for every $s \ge 0$ and $\ell > 0$ the sets

$$B(s, \ell) = \left\{x \in \Sigma: x \in A_s(\varepsilon-(1/\ell)) \text{ and } x \notin A_n(\varepsilon) \text{ for all } n > s\right\}.$$

It is clear that $B = \cup_{s\ge0,\, \ell>0} B(s, \ell)$. Therefore for some s and ℓ the set $B(s, \ell)$ has positive m_g-measure. Since $\int f \, dm_g = 0$, we can use the recurrence of the skew-product $(S_f, m_g \times \lambda)$ (cf. Proposition I.3.4) applied to the set $B(s, \ell)$ to conclude that there exists $k > 0$ such that

$$m_g\left(B(s,\ell) \cap \sigma^{-k}B(s,\ell) \cap \left\{x \in \Sigma : |f^k(x)| < 1/\ell\right\}\right) > 0.$$

Hence there exists a subset D of $B(s,\ell)$ with $m_g(D) > 0$ such that $\sigma^k x \in B(s,\ell)$ and $|f^k(x)| < 1/\ell$ for all $x \in D$. This implies that $|f^{s+k}(x)| < \varepsilon$ for all $x \in D$. The latter contradicts the definition of the set $B(s,\ell)$. Thus the lemma follows since the constant $c < 0$ cannot be chosen. □

Recall for the next result that $f \in \mathscr{F}$ is said to be *strongly non-lattice distributed*(12) if the values $f^n(x)+na$, for periodic points x of period n, generate a dense subgroup of $\mathbb{R}$, for all $a \in \mathbb{R}$; in particular, if f satisfies this condition, f is not cohomologous to a function taking values in a lattice.

Lemma 1.3: *Suppose* $m_g \in \mathscr{M}(f)$ *for some* $g \in \mathscr{F}$. *Consider the orbital measures* μ_n *of* (1.6) *and the subsets* $A_n(\varepsilon)$ *of* (1.7). *If* f *defines a non-lattice distribution then for every* $\varepsilon > 0$ *we have*

$$\lim_n\sup \frac{1}{n} \log \mu_n(A_n(\varepsilon)) = 0. \tag{1.9}$$

Proof: By the uniform convergence in the central limit theorem for orbital measures (see Section IV.6) we have:

$$\mu_n(A_n(\varepsilon)) = \mu_n\left\{x \in \Sigma_A : \frac{f^n(x)}{\sqrt{n}} < \frac{\varepsilon}{\sqrt{n}}\right\} - \mu_n\left\{x \in \Sigma_A : \frac{f^n(x)}{\sqrt{n}} < \frac{-\varepsilon}{\sqrt{n}}\right\}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\varepsilon/\sqrt{n}}^{\varepsilon/\sqrt{n}} \exp\{-u^2/2\sigma^2\}\, du + \beta_n(\varepsilon). \tag{1.10}$$

where σ^2 is the variance of f (which is strictly positive since f is not a coboundary, see (1.1) and Proposition I.2.1(iii)) and $\beta_n(\varepsilon)$ satisfies: $n^{1/2}\,\beta_n(\varepsilon)$ is bounded. After a change of variables in (1.10) we get:

(12) We note that f is strongly non-lattice distributed if f-a is non-lattice distributed (in the definition given in Chapter II) for all $a \in \mathbb{R}$. However, in Chapter IV, we shall use the name non-lattice distributed for the concept introduced in this chapter.

$$\log\{\sqrt{n}\,\mu_n(A_n(\varepsilon))\} = \log\left|\frac{1}{\sigma\sqrt{2\pi}}\int_{-\varepsilon}^{\varepsilon}\exp\{-u^2/2n\sigma^2\}\,du + \sqrt{n}\,\beta_n(\varepsilon)\right|. \qquad (1.11)$$

Since f is non-lattice distributed Theorem IV.3 applied in the case of orbital measures (cf. Section IV.6) implies that $\lim_{n\to\infty} n^{1/2}\beta_n(\varepsilon) = 0$ for every $\varepsilon > 0$. Hence (1.9) follows from (1.11). □

The next result is the main result of this chapter. It has appeared implicitly in the work of Lalley [Laℓ.2] for the case when f defines a non-lattice distribution.

Theorem 1.4: *If* $\mathcal{P}'(tf) = 0$ *for some* t *then* $h_f(\varepsilon) = s(f) = \mathcal{P}(tf)$ *for every* $\varepsilon > 0$.

Proof: Firstly we consider the case when Σ is aperiodic. Let $\mu = m_{tf}$ be the equilibrium state of tf. Then by Proposition 1.1 we know that $s(f) = \mathcal{P}(tf) = h(m_{tf})$. Consider the subsets $A_n(\varepsilon)$ defined in (1.7) and let $\chi_n(x)$ denote the indicator function of $A_n(\varepsilon)$, then it is clear that

$$N_f(n,\varepsilon) = \sum_{x\in Fix(n)} \chi_n(x)\,. \qquad (1.12)$$

Let μ_n be the sequence of orbital measures associated to the function tf (see (1.6)). Computing (1.6) for the function χ_n and using the fact that $\chi_n(x) = 0$ if $|f^n(x)| > \varepsilon$ we obtain

$$\left\{\frac{\sum_{x\in Fix(n)}\chi_n(x)}{\sum_{x\in Fix(n)}\exp\{tf^n(x)\}}\right\}\exp\{-\varepsilon|t|\} \le \mu_n(A_n(\varepsilon)) \le \left\{\frac{\sum_{x\in Fix(n)}\chi_n(x)}{\sum_{x\in Fix(n)}\exp\{tf^n(x)\}}\right\}\exp\{\varepsilon|t|\}\,.$$

Since $\left(\sum_{x\in Fix(n)}\exp\{tf^n(x)\}\right)^{1/n}$ converges to $\exp\{\mathcal{P}(tf)\}$ (cf. Theorem 7.20(a) of [Rue]) and $\mathcal{P}(tf) = s(f)$, we may use (1.12) and take $\lim_n\sup\,(1/n)\log$ of the expression above to get

$$h_f(\varepsilon) - s(f) = \lim_n\sup\,\frac{1}{n}\log\mu_n(A_n(\varepsilon))\,. \qquad (1.13)$$

Since $\mu \in \mathscr{M}(f)$, when f is non-lattice distributed we may apply Lemma 1.3 to (1.13) to conclude that $h_f(\varepsilon) = s(f)$.

When f is cohomologous to a function taking values in a lattice we shall give a direct proof. We may assume that f takes values in $\alpha\mathbb{Z}$ for some constant $\alpha > 0$ (choose α maximal satisfying this property). Hence f depends only on a finite number of coordinates and by considering a higher block system we may assume that $f(x) = f(x_0, x_1)$ (see Section I.1). Let M be the transition matrix of Σ. The equilibrium state μ of tf is then the Markov measure defined by the stochastic matrix

$$P = \left[\frac{e^{tf(i,j)} r_j(t) M(i,j)}{e^{s(f)} r_i(t)} \right]$$

where $(r_1(t), \ldots, r_k(t))$ is a right eigenvector of $[e^{t\, f(i,j)} M(i,j)]$ associated to $e^{s(f)}$. Let $x \in \Sigma$, then

$$\mu([x_0 \ldots x_{n-1}\, x_n]_0) = p_{x_0} \frac{r_{x_n}(t)}{r_{x_0}(t)} \exp\{-n\, s(f) + t\, f^n(x)\}, \tag{1.14}$$

where $p = (p_1, \ldots, p_k)$ is the left eigenvector of P satisfying $\sum_i p_i = 1$ and $pP = p$. Choosing $\varepsilon < \alpha$ we conclude that if $x \in \Sigma$ satisfies $|f^n(x)| < \varepsilon$ then $f^n(x) = 0$. Thus we conclude from (1.14) that there exist $a > 1$ such that

$$a^{-1} \exp\{-n\, s(f)\} \leq \mu([x_0 \ldots x_{n-1}\, x_n]_0) \leq a \exp\{-n\, s(f)\} \tag{1.15}$$

whenever $x \in \Sigma$ satisfies $|f^n(x)| < \varepsilon$. Let $\widetilde{N}_f(n, \varepsilon)$ denote the number of allowable blocks B of length n satisfying $|f^n(x)| < \varepsilon$ whenever $x \in B$. We note that the growth rates of $\widetilde{N}_f(n, \varepsilon)$ and $N_f(n, \varepsilon)$ coincide. (To see this we use recurrence of the skew-product defined by f (as an $\alpha\mathbb{Z}$-extension) to conclude that, given any pair of symbols i and j, there is a path from i to j with f-weight zero.)

Now summing over all allowable n-blocks B with $|f^n(x)| < \varepsilon$ whenever $x \in B$, we obtain from (1.15)

$$a^{-1} \widetilde{N}_f(n, \varepsilon) \exp\{-n\, s(f)\} \leq \mu(A_n(\varepsilon)) \leq a\, \widetilde{N}_f(n, \varepsilon) \exp\{-n\, s(f)\}. \tag{1.16}$$

Taking $\lim_n \sup\, (1/n)\log(\cdot)$ in the above and applying Lemma 1.2 we conclude that

$$s(f) = \lim_n \sup \frac{1}{n} \log \tilde{N}_f(n, \varepsilon),$$

which completes the proof of the theorem when Σ is aperiodic.

If Σ has period d, then we consider the cyclic decomposition of Σ, i.e. the sets $\{\Sigma^{(i)}\}_{i=0,\dots,d-1}$ where $\sigma^d|_{\Sigma(i)}$ is an aperiodic shift (see Section I.1). Then replacing the function f by f^d and applying the previous argument to f^d we conclude the proof for the function f. □

§2. An application in the lattice case

In this section we apply the main result of last section to give examples of entropy increasing factor maps of countable ssft's. We shall be brief and for details we refer the reader to the paper of Petersen [Pet].

We shall assume throughout this section that $f:\Sigma \to \mathbb{Z}$ is continuous. Consider the skew-product transformation S_f on $\Sigma \times \mathbb{Z}$. Since f depends on a finite number of coordinates, by relabelling the vertices of the graph of Σ, we may suppose that $f(x) = f(x_0, x_1)$. Let L denote the states of Σ. S_f can be viewed as the shift on a countable state ssft as follows. Consider a chain Σ_f with states $L \times \mathbb{Z}$ and transitions given by:

$$(i, n) \to (j, m) \text{ in } \Sigma_f \text{ iff } i \to j \text{ in } \Sigma \text{ and } m = n + f(i, j).$$

It is clear that S_f is topologically conjugate to the shift on Σ_f. The canonical projection $\pi:\Sigma_f \to \Sigma$ ($\pi(i, n) = i$) is a natural example of a continuous factor map built out of a finite labelling of the chain Σ_f (i.e. $\tilde{\pi}((i, n), (j, m)) = i$).

Petersen [Pet] uses finite labellings of chains to produce examples of entropy increasing factor maps. As a consequence of Theorem 1.5 we shall be able to show that the natural projection π of a class of skew-products has this property.

Assume that Σ_f is irreducible. Let $h_G(\Sigma_f)$ denote Gurevič's topological entropy of Σ_f, i.e. the growth rate of the number of cycles based on a fixed symbol (i, n) (see [Gur.1] for definition).

Theorem 2.1: *If Σ_f is irreducible then the natural projection $\pi: \Sigma_f \to \Sigma$ increases entropy strictly if and only if $\int f\,dm \neq 0$ where* m *denotes the Parry measure on* Σ.

Proof: Since cycles based on a fixed symbol in Σ_f correspond to periodic orbits of Σ with f-weight zero, we conclude that $h_G(\Sigma_f) = h_f(\varepsilon)$ for any $0 < \varepsilon < 1$. Irreducibility of Σ_f implies that there are closed orbits with positive and negative f-weight. Hence, by Proposition 3.4 below, $\mathcal{P}'(tf) = 0$ for some t. Now applying Theorem 1.4 we conclude that $h_G(\Sigma_f) = s(f) = \sup\{h(\mu): \mu \in \mathcal{M}(f)\}$. Since m is the unique measure of maximal entropy on Σ it follows that $m \in \mathcal{M}(f)$ if and only if $h(m) = s(f)$, otherwise $h(m) > s(f)$. □

Proposition 2.2: *If Σ_f is irreducible then:*

$$s(f) = \sup\{h(\mu): \mu \in \mathcal{M} \text{ and } f = G\circ\sigma - G \ \mu\text{-a.e. for some integrable } G\}.$$

Proof: As in the proof of Theorem 2.1, s(f) is the Gurevič topological entropy of Σ_f. The main result of [Gur.1] implies that s(f) is the supremum over the entropy of the σ-invariant probability measures supported on finite subsystems of Σ_f. But on each finite subsystem S of Σ_f each cycle corresponds to a periodic orbit on the base Σ with f-weight zero. Therefore we conclude that there exists a subsystem S′ on the base Σ with the same topological entropy of S and such that every periodic orbit of S′ has f-weight zero. Hence by Proposition I.2.2 we conclude that f is a coboundary when restricted to S′. □

§3. Extremal measures

In this section we try to understand the measures of maximal entropy as in Proposition 1.1 but when the derivative of pressure does not vanish. In this case there are examples of non-uniqueness of the measures m solving the equation $h(m) = s(f)$.

Consider the set $c(f) = \{c \in \mathbb{R}: \mathcal{M}(f-c) \text{ is non-empty}\}$ where $f \in \mathcal{F}$.

Proposition 3.1: *If $\mathcal{P}'(tf) \neq 0$ for every* t *then $\mathcal{M}(f)$ is non-empty if and only if $\mathcal{P}(tf) > 0$ for every* t.

Proof: Assume $\mathcal{M}(f)$ is non-empty. Let $\mu \in \mathcal{M}(f)$. Then by the definition of $\mathcal{P}(tf)$ we have

$$\mathcal{P}(tf) \geq h(\mu) + t\int f\, d\mu = h(\mu). \tag{3.1}$$

Hence $\mathcal{P}(tf) \geq 0$ for all t. If $\mathcal{P}(tf) = 0$ for $t = \lambda$ then $\mathcal{P}'(\lambda f) = 0$ since $t \mapsto \mathcal{P}''(tf)$ is a strictly positive function. This proves the forward implication.

Now say $\mathcal{P}(tf) > 0$ for all t. Since the entropy function is bounded on $\mathcal{M}$, the equality

$$\mathcal{P}(tf) = h(m_{tf}) + t\int f\, dm_{tf} = h(m_{tf}) + t\ \mathcal{P}'(tf)$$

implies that either $\lim_{t\to-\infty} \mathcal{P}'(tf) = 0$ or $\lim_{t\to\infty} \mathcal{P}'(tf) = 0$. Hence by the compactness of $\mathcal{M}$ there exists a sequence $t_k \to -\infty$ such that $\mu = \lim_{k\to\infty} m_{t_k f} \in \mathcal{M}(f)$. □

Proposition 3.2 [BR]: c(f) *is a compact interval. The interior of* c(f) *is the image of the derivative* $t \mapsto \mathcal{P}'(tf)$ *of the pressure function.*

Proof: c(f) is easily seen to be bounded since $c(f) \subseteq [\inf f(\cdot), \sup f(\cdot)]$. We note that the equilibrium states of tf and of t(f−c) coincide for every t and c. Whenever $\mathcal{P}'(t(f-c)) = 0$ we have $m_{t(f-c)} = m_{tf} \in \mathcal{M}(f-c)$, and since $\mathcal{P}'(t(f-c)) = \mathcal{P}'(tf) - c$ we conclude that c(f) contains the image of the function $t \mapsto \mathcal{P}'(tf)$.

Now if $c \in c(f)$ is given such that $\mathcal{P}'(t(f-c)) \neq 0$ for every t, then by the previous proposition $\mathcal{P}(t(f-c)) > 0$ for every t. Hence by the definition of pressure, either $\lim_{t\to\infty} \mathcal{P}'(t(f-c)) = 0$ or $\lim_{t\to-\infty} \mathcal{P}'(t(f-c)) = 0$. The latter implies that c belongs to the boundary of the image of the function $t \mapsto \mathcal{P}'(tf)$ since $\mathcal{P}'(t(f-c)) = \mathcal{P}'(tf) - c$. □

Given $f \in \mathcal{F}$ consider the associated interval $c(f) = [a, b]$. Since $\mathcal{P}''(tf) > 0$ for all t, the function $t \mapsto \mathcal{P}'(tf)$ is invertible on the open interval (a, b). Hence the inverse $c \mapsto \alpha(c)$ is well-defined and analytic. Consider the following entropy function defined on c(f), $c \mapsto s(f-c)$ (see definition of s(f) in (1.2)). From Proposition 1.1 we know that there exists a unique measure μ_c such that $h(\mu_c) = s(f-c)$ if $c \in (a, b)$. Actually $\mu_c = m_{\alpha(c).(f-c)}$ the equilibrium state of the function $\alpha(c)(f-c)$. (This construction was carried out before by Bohr & Rand [BR] on the study of measures on attractors in one-dimensional dynamics.)

A measure m is called an *extremal measure* for $f \in \mathcal{F}$ if $h(m) = s(f-a)$ and $\int f\, dm = a$, or $h(m) = s(f-b)$ and $\int f\, dm = b$, where $c(f) = [a, b]$. In the Appendix one can find some examples of such measures, including examples of non-uniqueness for any fixed extreme of c(f). In the following we shall prove that the extremal measures cannot be fully supported and each ergodic component of such a measure defines by restriction an ergodic extremal measure.

Proposition 3.3: *If* $\mathcal{M}(f)$ *contains a fully supported measure* m *such that* $\int_B f\, dm = 0$ *for every ergodic component* B *of* m *then there exist periodic orbits with positive and negative f-weight.*

Proof: We may assume that f depends only on future coordinates since the result is unchanged if we replace f by adding a coboundary (see Proposition I.2.3). Let C denote the total variation of f,[(13)] i.e.

$$C = \sup\{|f^n(\sigma^{-n}x) - f^n(\sigma^{-n}z)| : n \geq 1,\ x, z \in \Sigma \text{ with } x_i = z_i \text{ for } i \leq 0\}. \qquad (3.2)$$

If the f-weight of every periodic orbit is zero then f is a coboundary (cf. Proposition I.2.2). Since we are assuming this is not the case, there must be a periodic orbit with positive f-weight (otherwise take -f). Let $y \in Fix(n)$ be given such that $a = f^n(y) > 0$. By taking a larger period if necessary, we may assume that $a > 3C$. Let D denote the cylinder $[y_0\, y_1 \dots y_{n-1}\, y_0]_0$. Since f has integral zero on any ergodic component of a fully supported measure, we apply twice the recurrence of the skew-product defined by f (cf. Proposition I.3.4) to the cylinder D to conclude the existence of a periodic point $z \in Fix(s)$ where $z \in D \cap \sigma^{-k}D \cap \sigma^{-s}D$ such that $|f^s(z)| < C$, for some $k > n$ and $s > k+n$. Now define $x \in Fix(s-n)$ by $x_i = z_i$ for $0 \leq i < k$ and $x_i = z_{i+n}$ for $k \leq i \leq s-n$, i.e. we have deleted from z the block $[z_k\, z_{k+1} \dots z_{k+n-1}]$ periodically with period s. Finally, comparing the f-weights of x and z one concludes that

$$|f^{s-n}(x) + a - f^s(z)| < 2C$$

(13) We note that f has summable variation and then C is finite.

which implies,

$$f^{s-n}(x) < 2C - a + C < 0. \ \square$$

Proposition 3.4: *Let* $f \in \mathcal{F}$. *The following are equivalent:*

(a) $\mathcal{P}'(tf) = 0$ *for some* t.

(b) There exist periodic orbits with positive and negative f-weight.

(c) $\mathcal{M}(f)$ *contains a fully supported measure.*

Proof: (a) ⇒ (c) Follows from Proposition 1.1 since m_{tf} is such a measure.

(b) ⇒ (a) Let x and y be periodic points whose periodic orbits have positive and negative f-weight respectively. Let μ_x and μ_y be the measures concentrated on the corresponding orbits of x and y. Then by the definition of pressure we conclude that $\lim_{t\to\pm\infty} \mathcal{P}(tf) = \infty$. Hence $\mathcal{P}'(tf) = 0$ for some t.

(c) ⇒ (b) Suppose $m \in \mathcal{M}(f)$ is fully supported. If $\int_B f\, dm = 0$ for every ergodic component B of m then the result follows from Proposition 3.3. Otherwise, there are two ergodic components on which f has positive and negative integral respectively. Let $\mu_>$ and $\mu_<$ be the restrictions of m on the corresponding ergodic components. Then again by the definition of pressure we conclude that $\lim_{t\to\pm\infty} \mathcal{P}(tf) = \infty$ and hence $\mathcal{P}'(tf) = 0$ for some t. Now by Proposition 1.1 we obtain an ergodic and fully supported measure in $\mathcal{M}(f)$ and then we apply Proposition 3.3. □

As a consequence of the latter proposition we have the following corollary regarding the extremal measures.

Corollary 3.5: *Let* $f \in \mathcal{F}$. *Any extremal measure of* f *cannot be fully supported on* Σ.

Proof: If $m \in \mathcal{M}(f-c)$ is fully supported for some $c \in c(f) = [a, b]$ then by Proposition 3.4 we conclude that there exists t such that $\mathcal{P}'(t(f-c)) = 0$ and hence c is in the image of the function $t \mapsto \mathcal{P}'(tf)$. Hence c belongs to the open interval (a, b) by Proposition 3.2. □

We shall end this section by showing a connection between some extremal measures and the equilibrium states $m_{t(f-a)}$ (or $m_{t(f-b)}$ depending on which extreme we are considering the extremal measure) where $c(f) = [a, b]$. First we note from Proposition 3.1 that $a = 0$ or $b = 0$ if and only if the pressure function $t \mapsto \mathcal{P}(tf)$ and its derivative never vanish. Furthermore we see from Proposition 3.2 that $a = 0$ implies $t \mapsto \mathcal{P}'(tf)$ is strictly positive and $b = 0$ implies $t \mapsto \mathcal{P}'(tf)$ is strictly negative. If $c(f) = [a, b]$ it follows from Proposition 3.2 that $c(-f) = [-b, -a]$. Hence the consideration of extremal measures can be carried out in one extreme of $c(f)$ and in the other extreme one applies the same arguments replacing the function f by $-f$.

For the next proposition consider the sets

$$L_f^a = \{\mu \in \mathcal{M}(f-a): \exists\, t_k \to -\infty \text{ with } m_{t_k(f-a)} \to \mu \text{ weakly}\} \tag{3.3}$$

and

$$L_f^b = \{\mu \in \mathcal{M}(f-b): \exists\, t_k \to \infty \text{ with } m_{t_k(f-b)} \to \mu \text{ weakly}\}. \tag{3.4}$$

Proposition 3.6: *Let* $f \in \mathcal{F}$ *and consider the corresponding interval* $c(f) = [a, b]$. *Any measure belonging to* L_f^a *or* L_f^b *is an extremal measure associated to the extreme* a *or* b *respectively.*

Proof: We shall give only the proof for the extreme a since the proof is similar for the extreme b. Clearly we may assume $a = 0$. Hence by Proposition 3.2 it follows that $\lim_{t\to-\infty} \mathcal{P}'(tf) = 0$. We shall prove that in fact $\lim_{t\to-\infty} t\,\mathcal{P}'(tf) = 0$. If $t\,\mathcal{P}'(tf) > c$ for sufficiently large t and some constant $c > 0$ then $\mathcal{P}(tf) > c\log(|t|) + \bar{c}$ for sufficiently large t and some other constant $\bar{c}$. This implies that $\lim_{t\to\pm\infty} \mathcal{P}(tf) = \infty$ which contradicts the fact that the derivative of the pressure function never vanishes.

Now by the upper semi-continuity of the entropy function on $\mathcal{M}$ we conclude that given a sequence $t_k \to -\infty$ with $m_{t_k f} \to m$ weakly, we have:

$$h(m) \geq \lim_{k\to\infty} h(m_{t_k f}).$$

By the definition of pressure we have $\mathcal{P}(tf) = h(m_{tf}) + t\,\mathcal{P}'(tf) \geq h(\mu)$ for any $\mu \in \mathcal{M}(f)$. Hence since $m \in \mathcal{M}(f)$ and $\lim_{t\to-\infty} t\,\mathcal{P}'(tf) = 0$ we conclude that $\lim_{k\to\infty} h(m_{t_k f}) = h(m)$ and $h(m) \geq h(\mu)$ for any $\mu \in \mathcal{M}(f)$. □

Proposition 3.7: *If* m *is an extremal measure for* $f \in \mathcal{F}$ *then the restriction of* m *to any of its ergodic components defines an ergodic extremal measure.*

Proof: Let B be an ergodic component of m and denote by m_B the restriction of m to B. We assume that the extreme of c(f) which corresponds to the measure m is zero. We note that we must have $\int_B f\,dm = 0$ since if say $\int_B f\,dm > 0$ then there should be another ergodic component B′ of m such that $\int_{B'} f\,dm < 0$ and repeating the argument used in the proof of Proposition 3.4 we conclude that the function $t \mapsto \mathcal{P}'(tf)$ should vanish, which is then a contradiction to Proposition 3.2. Therefore $m_B \in \mathcal{M}(f)$.

Now if B^c denotes the complement of B and m_{B^c} denotes the corresponding restriction of m to B^c we have $m_{B^c} \in \mathcal{M}(f)$. Since $m = m(B)\,m_B + m(B^c)\,m_{B^c}$ and the entropy function is affine (cf. for instance [Waℓ.3]) we obtain

$$h(m) = m(B)\,h(m_B) + m(B^c)\,h(m_{B^c}).$$

Therefore if $h(m_B) < h(m)$ then from the above we should have necessarily $h(m_{B^c}) > h(m)$ which is a contradiction to the fact that $h(m) \geq h(\mu)$ for all $\mu \in \mathcal{M}(f)$. □

We have been unable to solve the following problem which is indicated to be true by some examples of extremal measures exhibited in the Appendix (the examples are given when the function f depends only on a finite number of coordinates).

Conjecture 3.8: Consider $f \in \mathcal{F}$ and the corresponding interval $c(f) = [a, b]$. The equilibrium states $m_{t(f-a)}$ converge for $t \to -\infty$ to a *unique* measure μ_a. The number of ergodic extremal measures of f is *finite* and

$$\mu_a = \frac{1}{n}(\mu_1 + \ldots + \mu_n),$$

where μ_i denote the ergodic extremal measures for the extreme a. (A similar statement made for the extreme b considering the equilibrium states $m_{t(f-b)}$ as $t \to \infty$.)

CHAPTER IV:

CENTRAL LIMIT ASYMPTOTICS FOR SHIFTS OF FINITE TYPE

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Dedicated to Horst Michel and his family.

Abstract: We study the rate of convergence and asymptotic expansions in the central limit theorem for the class of Hölder continuous functions on a shift of finite type endowed with a stationary equilibrium state. It is shown that the rate of convergence in the theorem is $O(n^{-1/2})$ and when the function defines a non-lattice distribution an asymptotic expansion to the order of $o(n^{-1/2})$ is given. Higher-order expansions can be obtained for a subclass of functions. We also make a remark on the central limit theorem for (closed) orbital measures.

0. Introduction

We consider the central limit theorem for a process $\{f \circ \sigma^n\}$ where σ is a shift of finite type endowed with a stationary equilibrium state m. We assume that f and the potential g defining m are Hölder continuous. For such a process we first prove the theorem with $O(n^{-1/2})$ rate of convergence and when f is not lattice distributed the rate is improved to $o(n^{-1/2})$. These results conform with the classical asymptotics of the central limit theorem for independent processes. However, we stress that the underlying stationary process we are considering has only a finite number of states which limits, for example, the entropy of the process $\{f \circ \sigma^n\}$. This limitation also inhibits us from imposing the classical condition used for higher asymptotics for independent processes. Nevertheless we find a reasonable substitute and exhibit a class of finite state independent processes satisfying this modified condition.

Related results of various strengths have previously appeared in connection with maps of the interval, with Axiom A diffeomorphisms, and indeed with shifts of finite type (cf. [DP], [Keℓ], [Rat], [Sin], [Won]). The paper [DP] is concerned with the approximation by Brownian motion of real-time processes which arise from Hölder suspensions of shifts of finite type. The central limit theorem is obtained as a corollary but without significant

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asymptotics. Keller's work [Keℓ] on piecewise monotonic maps of the unit interval involves Hölder continuous functions together with Perron-Frobenius operators and the author is able to obtain the central limit theorem with $O(n^{-\theta})$ rate of convergence, where $\theta > 0$ is unspecified. The emphasis in our work, apart from an exceptionally simple presentation of the central limit theorem, is on the rate of convergence and asymptotic expansions. Under appropriate conditions we obtain an asymptotic expansion to the order of $o(n^{-1/2})$ and under further conditions this convergence rate is sharpened to give higher-order expansions. (However see our remarks in the last section.) In view of Bowen's modelling theory for Axiom A diffeomorphisms by shifts of finite type our results, of course, apply to this context. We omit the details and refer the reader to Bowen's paper [Bow]. We conclude with remarks on the central limit theorem for (closed) orbital measures. This topic was recently considered by Lalley [Laℓ.2].

The connection between the Ruelle operator and the central limit theorem was first indicated in an exercise in [Rue] - without asymptotics, however. Rousseau-Egèle [R-E] and Lalley [Laℓ.1] also exploit this theme. This paper pursues a similar line of thought. The main ingredients of the method presented here are as follows:

(i) $\int \exp\{ it\,(f + \dots + f\circ\sigma^{n-1})\, n^{-1/2} \}\, dm$ can be written as

$$\int L^n_{g+itf/\sqrt{n}}\, 1\, dm$$

where $L_{g+itf/\sqrt{n}}$ is a Ruelle operator which is a small perturbation of L_g;

(ii) If $\exp\{P(g+itf/\sqrt{n})\}$ is the maximum (simple, isolated) eigenvalue of $L_{g+itf/\sqrt{n}}$ then $\exp\{nP(g+itf/\sqrt{n})\}$ converges to $e^{-t^2\sigma^2/2}$, the Fourier transform of the normal distribution with variance σ^2.

Our basic theme is the exploitation of these observations. Readers are referred to [Rue] for the unproven statements in the next section.

1. Pressure and equilibrium states

Throughout we assume that A is a 0-1 k×k aperiodic matrix and we define

$$\Sigma_A^+ = \left\{ x \in \prod_{n=0}^{\infty} \{1, 2, \dots, k\} : A(x_n, x_{n+1}) = 1, \text{ for all } n \in \mathbb{Z}^+ \right\}.$$

The space Σ_A^+ is a compact metrizable space with the Tychonov topology (generated by the discrete topology on $\{1, 2, ..., k\}$). The shift σ (of finite type) is defined on Σ_A^+ by $(\sigma x)_n = x_{n+1}$, $n \in \mathbb{Z}^+$, which is a continuous surjective map.

With $\mathrm{var}_n f = \sup \{ | f(x) - f(y) | : x_i = y_i, \ i \leq n \}$ and $0 < \theta < 1$, we define $| f |_\theta = \sup (\mathrm{var}_n f / \theta^n)$ and $F_\theta^+ = \{ f \in C(\Sigma_A^+) : | f |_\theta < \infty \}$. F_θ^+ is a Banach space when endowed with the norm $\| f \|_\theta = | f |_\infty + | f |_\theta$, where $| \, . \, |_\infty$ is the uniform norm.

Suppose that $f, g \in F_\theta^+$ are real valued functions and define the Ruelle operators L_{g+sf} on F_θ^+ by $(L_{g+sf} w)(x) = \sum_{\sigma y = x} \exp\{ g(y) + s f(y) \} w(y)$. L_g has a maximum positive eigenvalue, denoted $e^{P(g)}$, which is simple and the rest of the spectrum is contained in a disc of radius strictly less than $e^{P(g)}$. $P(g)$ is called the *pressure* of g. There is a unique σ-invariant probability measure m such that $P(g) = h(m) + \int g \, dm$ which maximizes $h(\mu) + \int g \, d\mu$ for σ-invariant probabilities μ. The measure m is the *equilibrium state* of g. An eigenfunction w of L_g corresponding to $e^{P(g)}$ may be taken to be strictly positive and written $w = e^u$. If we replace g by $g' = g - P(g) + u - u \circ \sigma$ then we obtain $L_{g'} 1 = 1$ and $P(g') = 0$. In this case we say that g' is normalised. It is easy to see that g and g' have the same equilibrium state m. *We shall always suppose that g is normalised and* $\int f \, dm = 0$.

Since the maximum eigenvalue 1 of L_g is isolated and simple, for $|s|$ sufficiently small each L_{g+sf} has a maximum eigenvalue (in modulus) $e^{P(g+sf)}$ which is also simple and the rest of the spectrum of L_{g+sf} is contained in a disc of radius strictly less than $| e^{P(g+sf)} |$. $P(g+sf)$ is an analytic function in, say, $|s| < \varepsilon$. In fact there exists an analytic projection valued function $Q(s)$ in $|s| < \varepsilon$, commuting with L_{g+sf} , so that $Q(s)1$ is 'the' eigenvector of L_{g+sf} corresponding to $e^{P(g+sf)}$ satisfying:

(i) $| 1 - e^{P(g+sf)} | < \eta$ and

(ii) L_{g+sf} restricted to $(Id - Q(s))F_\theta^+$ has spectral radius less than ρ where $\rho < 1 - \eta$.

Let $w(s)$ denote $Q(s)1$. Differentiating the equation $L_{g+sf} \, w(s) = e^{P(g+sf)} w(s)$ with respect to s and using the fact that $L_{g+sf}(\cdot) = L_g (e^{sf} \cdot)$, we deduce that

$$L_{g+sf}\Big(f \, w(s) + w'(s) \Big) = e^{P(g+sf)} \Big(P'(g+sf) \, w(s) + w'(s) \Big) \qquad (1)$$

and at s=0 we have

$$L_g\Big(f + w'(0) \Big) = P'(0) + w'(0) \qquad (2)$$

integrating with respect to m and using the fact that L_g^* fixes the measure m we obtain

$$P'(0) = \int f \, dm = 0. \tag{3}$$

A further differentiation of (1) yields

$$\begin{aligned} L_{g+sf} \; & \big(f^2\, w(s) + 2\, f\, w'(s) + w''(s)\big) = \\ & e^{P(g+sf)}\, P'(g+sf)\, \{P'(g+sf)\, w(s) + w'(s)\} + \\ & e^{P(g+sf)} \{P''(g+sf)\, w(s) + P'(g+sf)\, w'(s) + w''(s)\} \end{aligned} \tag{4}$$

and at s=0,

$$L_g \; \big(f^2 + 2\, f\, w'(0) + w''(0)\big) \; = \; e^{P(g)} \, \big(P''(0) + w''(0)\big) \tag{5}$$

Integrating (5) with respect to m we obtain

$$\int f^2 \, dm + 2 \int f\, w'(0) \, dm = P''(0) \tag{6}$$

Applying the steps (1)-(6) to the equation $(L_{g+sf})^n\, w(s) \; = \; e^{nP(g+sf)} \; w(s)$ and noting that $(L_{g+sf})^n\, (\cdot\,) = (L_g)^n\, (e^{sf^n} \cdot\,)$ where $f^n(x) = f(x) + f(\sigma x) + \ldots + f(\sigma^{n-1} x)$ we get

$$\int (f^n)^2 \, dm + 2 \int f^n \, w'(0) \, dm = \; n\, P''(0) \tag{7}$$

The Ergodic Theorem shows that $(1/n)\, f^n \to 0$ m a.e., therefore we have

$$P''(0) = \; \lim_{n\to\infty} (1/n) \int (f^n)^2 \, dm, \tag{8}$$

this quantity will be denoted in the sequel by σ^2 *and we shall always assume* $\sigma^2 \neq 0$. This condition is equivalent to the assumption that f *cannot* be written as $f = F\sigma - F$, with F continuous.

Differentiating once more we obtain the following expression for the third derivative of P,

$$\begin{aligned} P'''(0) &= \; \lim_{n\to\infty} (1/n) \int (f^n - w'(0))^3 \, dm \\ &= \; \lim_{n\to\infty} (1/n) \int \big((f^n)^3 - 3\, (f^n)^2\, w'(0)\big) \, dm \; . \end{aligned}$$

For later reference we gather these observations:

Lemma 1: For $|s| < \varepsilon$, $P(g+sf)$ has an expression

$$P(g+sf) = \frac{\sigma^2 s^2}{2} + \frac{P'''(0)s^3}{6} + s^4 \varphi(s) \tag{9}$$

where $\varphi(s)$ is analytic.

2. Fourier transforms

In the whole of this section we will fix $\varepsilon > 0$ such that P is well-defined and analytic in $|s| < \varepsilon$; the projection $Q(s)$ of the last section is analytic in $|s| < \varepsilon$ and satisfies (i) and (ii); and ε satisfies

$$\frac{\sigma^2}{2} > \varepsilon\ \{\frac{|P'''(0)|}{6} + |s\,\varphi(s)|\} \quad \text{and} \quad \frac{\sigma^2}{2} > \varepsilon\,|\varphi(s)| \tag{10}$$

when $|s| < \varepsilon$. For the next result, let $\chi_n(t)$ denote $\int \exp\{itn^{-1/2}f^n\}dm$ where m is the equilibrium state of g with $g, f \in F_\theta^+$.

Lemma 2: If ε satisfies the conditions above then

$$\int_0^{\varepsilon\sqrt{n}} \frac{1}{t}\,|\,\chi_n(t) - \exp\{nP(g+itf/\sqrt{n})\}\,|\,dt = O(\frac{1}{n}) \tag{11}$$

where the implied constant depends only on ε.

Proof: We split the vector 1 analytically on the subspaces $Q(s)F_\theta^+$ and $(Id-Q(s))F_\theta^+$, and since $1 - Q(s)1$ vanishes at $s=0$, we obtain

$$1 = w(s) + s\,v(s), \tag{12}$$

where $w(s) = Q(s)1$, for some analytic function $v(s)$ in $|s| < \varepsilon$. Since the vector $s\,v(s) \in (Id-Q(s))F_\theta^+$, we conclude that $v(s)$ is fixed by the projection $Id-Q(s)$ for $s\neq 0$. Therefore by continuity, $(Id-Q(0))\,v(0) = v(0)$. Integrating with respect to m we deduce that $\int v(0)\,dm = 0$. Hence

$$\int v(s)\,dm = s\,\beta(s) \tag{13}$$

for some analytic function β in $|s| < \varepsilon$.

The integrand in (11) can be written

$$\frac{1}{t}\left| \int L^n_{g+itf/\sqrt{n}} 1\, dm - \exp\{nP(g+itf/\sqrt{n})\} \right| =$$

$$\frac{1}{\sqrt{n}} \left| \int \left(L^n_{g+itf/\sqrt{n}} - \exp\{nP(g+itf/\sqrt{n})\}\right) v(it/\sqrt{n})\, dm \right| .$$

The spectral radius of L_{g+sf} restricted to $(Id-Q(s))F_\theta^+$ is less than ρ for each $|s| < \varepsilon$. From this we conclude that there exist K, ρ' ($\rho < \rho' < 1$) such that

$$\| (L_{g+sf})^n v(s) \|_\theta \leq K (\rho')^n \tag{14}$$

for all $|s| < \varepsilon$. In particular,

$$\left| \int (L_{g+sf})^n v(s)\, dm \right| \leq K (\rho')^n ,$$

and

$$\int_0^{\varepsilon\sqrt{n}} \left| \int L^n_{g+itf/\sqrt{n}} v(it/\sqrt{n})\, dm \right| dt \leq K (\rho')^n \varepsilon\sqrt{n} . \tag{15}$$

Now from the choice of ε in (10) and the expression (9) of the pressure, we see that

$$| \exp\{ n P(g+itf/\sqrt{n})\} | \leq e^{-ct^2}$$

for a positive constant $c = c(\varepsilon)$ and all t, n satisfying $|t| \leq \varepsilon\sqrt{n}$.

Since the function β defined in (13) is analytic in $|s| < \varepsilon$, it is clear that there exists a constant $K' = K'(\varepsilon)$ such that

$$\int_0^{\varepsilon\sqrt{n}} | \exp\{ n P(g+itf/\sqrt{n})\}\, t\beta(it/\sqrt{n}) |\, dt \leq K'$$

for all $n > 0$, and hence

$$\int_0^{\varepsilon\sqrt{n}} \left| \exp\{n P(g+itf/\sqrt{n})\} \left(\int v(it/\sqrt{n})\, dm\right) \right| dt \leq \frac{K'}{\sqrt{n}} . \tag{16}$$

Combining (15) and (16) we have proved

$$\int_0^{\varepsilon\sqrt{n}} \left| \int \left(L^n_{g+itf/\sqrt{n}} - \exp\{nP(g+itf/\sqrt{n})\}\right) v(it/\sqrt{n})\, dm \right| dt = O\left(\frac{1}{\sqrt{n}}\right) \tag{17}$$

where the implied constant depends only on ε, and the lemma follows.

Theorem 1: If ε satisfies the conditions above then

$$\int_0^{\varepsilon\sqrt{n}} \frac{1}{t} \left| \chi_n(t) - e^{-\sigma^2 t^2/2}\left(1 - \frac{it^3 P'''(0)}{6\sqrt{n}}\right)\right| dt = O\left(\frac{1}{n}\right) \tag{18}$$

where the implied constant depends only on ε.

Proof: Using the elementary inequality

$$\left| e^{z+ib} - (1+ib) \right| \le |z|\, e^{|z|} + \frac{b^2}{2}$$

when b is real, we obtain from (9)

$$\exp\{nP(g+itf/\sqrt{n})\} = e^{-\sigma^2 t^2/2} \exp\left\{ -\frac{it^3 P'''(0)}{6\sqrt{n}} + \frac{t^4}{n}\varphi\left(\frac{it}{\sqrt{n}}\right)\right\}$$
$$= e^{-\sigma^2 t^2/2}\, e^{z+ib}$$

where we have defined $z = (t^4/n)\,\varphi(it/\sqrt{n})$ and $b = -t^3 P'''(0)/6\sqrt{n}$.

Hence

$$\left| \exp\{nP(g+itf/\sqrt{n})\} - e^{-\sigma^2 t^2/2}(1+ib)\right| \le e^{-\sigma^2 t^2/2}\left| e^{z+ib} - (1+ib)\right|$$

$$\le e^{-\sigma^2 t^2/2}\left(\frac{t^4}{n}|\varphi| \exp\left\{\frac{t^4}{n}|\varphi|\right\} + \frac{t^6}{72n}|P'''(0)|^2\right). \tag{19}$$

From the choice of ε in (10) we see that $\varepsilon|\varphi(it/\sqrt{n})| < \sigma^2/2$ for $t < \varepsilon\sqrt{n}$. Therefore (19) implies

$$\int_0^{\varepsilon\sqrt{n}} \frac{1}{t} \left| \exp\{nP(g+itf/\sqrt{n})\} - e^{-\sigma^2 t^2/2}\left(1 - \frac{it^3 P'''(0)}{6\sqrt{n}}\right) \right| dt \leq \frac{K}{n}$$

for some K depending only on ε. Combining this inequality with lemma 2 we have proved the theorem.

3. Refinements in the central limit theorem

Consider $f,g \in F_\theta^+$ and m the equilibrium state of g. We assume $\int f\, dm=0$ and consider the distribution function $F_n(x) = m\left\{y\in\Sigma_A^+ : n^{-1/2} f^n(y) < x\right\}$. N denotes the normal distribution with zero mean and variance $\sigma^2 \neq 0$. The Central Limit Theorem states that $|F_n(x) - N(x)| = o(1)$, uniformly in x. We are interested in the asymptotic behaviour of this convergence.

The function $\chi_n(t)$ given by $\int \exp\{itn^{-1/2}f^n\}\, dm$ is the Fourier transform of $F_n(x)$. If $\gamma(t)$ is the Fourier transform of the distribution G(x) then a well known 'basic inequality' (cf. [Feℓ]) asserts that

$$\| F_n(x) - G(x) \| \leq \frac{1}{2\pi} \int_0^T \frac{1}{t} |\chi_n(t) - \gamma(t)|\, dt + \frac{24\, M}{\pi T} \tag{19}$$

where M is the maximum value of the derivative G′ of G and T is arbitrary.

Theorem 2: $\| F_n(x) - N(x) \| = O(\frac{1}{\sqrt{n}})$.

Proof: Recall that $e^{-\sigma^2 t^2/2}$ is the Fourier transform of N(x). Taking G as the normal distribution and setting $T = \varepsilon\, n^{1/2}$ in (19), where ε satisfies the conditions in (10), by applying theorem 1 of last section we obtain the result.

We say that f defines a *non-lattice distribution* when, for all $a \in \mathbb{R}$, the values $f^n(x) + na$ for points x of period n generate a dense subgroup of $\mathbb{R}$. The assumption that f defines a non-lattice distribution is equivalent to the condition that whenever $\varphi\circ\sigma = \alpha \exp\{itf\}\, \varphi$ (for a continuous, or even measurable φ) then t=0 and φ is constant. Alternatively, the condition is equivalent to the assumption that L_{g+itf} has spectral radius strictly less than 1, when $t\neq 0$ (cf. [Poℓ.1]). In the special case when $\{f\circ\sigma^n\}$ is an independent sequence, the above concept is implied by the standard notion which is defined, for example, in [Feℓ].

Theorem 3: If f defines a non-lattice distribution then

$$F_n(x) - N(x) = \frac{P'''(0)}{6\sigma^3\sqrt{2\pi n}}\,(1 - \frac{x^2}{\sigma^2})\; e^{-x^2/2\sigma^2} + o(\frac{1}{\sqrt{n}}).$$

Proof: Define $G_n(x) = N(x) + \frac{P'''(0)}{6\sigma^3\sqrt{2\pi n}}\,(1 - \frac{x^2}{\sigma^2})\; e^{-x^2/2\sigma^2}$. Its Fourier transform

is given by

$$\gamma_n(t) = e^{-\sigma^2 t^2/2}\left(1 + \frac{P'''(0)(it)^3}{6\sqrt{n}}\right).$$

If we prove that for any fixed $\varepsilon > 0$ and $\alpha > \varepsilon$

$$\int_{\varepsilon\sqrt{n}}^{\alpha\sqrt{n}} \frac{1}{t}\Big(\chi_n(t) - \gamma_n(t)\Big)\, dt \to 0$$

exponentially fast as $n \to \infty$, then considering the basic inequality (19) with $T = \alpha n^{1/2}$, $G(x) = N(x)$ and $F_n(x)$ replaced by $F_n(x) - G_n(x) + N(x)$, and using the fact that the second term of the right hand side of (19) can be made smaller than any pre-assigned quantity (by choosing a large α), we can apply theorem 1 to establish the result.

Hence it remains to prove that

$$\int_{\varepsilon\sqrt{n}}^{\alpha\sqrt{n}} \frac{1}{t}\,|\chi_n(t)|\; dt \to 0$$

at an exponential rate for any fixed pair ε,α strictly positive. The latter integral is equal to (after a change of variables $y = t\, n^{-1/2}$)

$$\int_\varepsilon^\alpha \Big| \int \exp\{iyf^n\}\, dm\, \Big| \frac{dy}{y} = \int_\varepsilon^\alpha \Big| \int L^n_{g+iyf} 1\, dm\, \Big| \frac{dy}{y}.$$

Since the function f defines a non-lattice distribution, the operator L_{g+iyf} has spectral radius strictly less than 1 for each real $y \neq 0$, therefore the latter integral converges to zero exponentially fast as $n \to \infty$, since y is bounded away from zero.

Remark: The above theorem remains valid if we consider a two-sided shift in place of the one-sided shift we have been considering. This is because the function f may be replaced by $f' = f + F\sigma - F$ with F continuous, without altering the asymptotic. However, we see no reason which would justify a similar claim for higher order asymptotic assertions.

4. Higher order asymptotics

In this section we indicate how the previous results may be improved to an $o(n^{-r/2+1})$ asymptotic. Since the method does not involve anything new, we shall concentrate on one or two of the main features.

For $r \geq 3$ we define a polynomial $\psi_r(s)$ of degree r-2, by

$$P(g+sf) - \frac{s^2\sigma^2}{2} = s^2\psi_r(s) + o(s^r)$$

where $\psi_r(0) = 0$.

Then

$$\left| \exp\left\{ n\,P(g+sf) - \frac{n s^2\sigma^2}{2} \right\} - \sum_{k=0}^{r-2} \frac{(n s^2\psi_r(s))^k}{k!} \right|$$

$$\leq n\,o(s^r)\,e^\gamma + \left| \frac{n s^2\psi_r(s)}{(r-1)!} \right|^{r-1}$$

where $\gamma \leq n\,|s|^3\,K$ for some $K = K(\varepsilon)$ and $|s| < \varepsilon$, when $\varepsilon > 0$ is suitably small.

With $s = it/\sqrt{n}$ ($|t| < \varepsilon\sqrt{n}$) we have

$$\int_0^{\varepsilon\sqrt{n}} \frac{1}{t} \left| \exp\{nP(g+itf/\sqrt{n})\} - e^{-\sigma^2 t^2/2} \sum_{k=0}^{r-2} \frac{(-t^2\psi_r(it/\sqrt{n}))^k}{k!} \right| dt$$

$$\leq \frac{\delta A}{n^{r/2-1}} + \frac{B}{n^{r/2-1/2}} \tag{20}$$

where A, B are absolute constants (depending on r) and $\delta = \delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

As before the upper limit of the integral in (20) may be increased without harm to

$a\sqrt{n}$ for any positive a. But to apply the basic inequality and obtain an $o(n^{-r/2+1})$ asymptotic we have to ensure, say, that

$$\int_{an^{1/2}}^{an^{r/2-1}} \left(\int \exp\{ itf^n/\sqrt{n} \}\, dm \right) \frac{dt}{t} \tag{21}$$

converges to zero rapidly.

A standard condition imposed when $\{f \circ \sigma^n\}$ is an independent process is that lim sup $|\chi(t)| < 1$ where $\chi(t) = \int e^{itf}\, dm$, for this implies that

$$\int e^{itf^n}\, dm \to 0$$

exponentially fast, uniformly for large t. However, this is an inappropriate condition for our situation. (Indeed, it may never be satisfied when f depends on only one variable.)

The following condition is sufficient

$$H_r: \quad \left| \int e^{itf^n}\, dm \right| \le K\left(1 - \frac{c}{t^\alpha}\right)^n$$

for constants c, K, α with $\alpha\,(r-3)/2 < 1$ and all sufficiently large t.

With this condition we have

$$\left| \int_{an^{1/2}}^{an^{r/2-1}} \left(\int \exp\{itf^n/\sqrt{n}\}\, dm \right) \frac{dt}{t} \right|$$

$$= \left| \int_{a}^{an^{\frac{r-3}{2}}} \left(\int \exp\{itf^n\}\, dm \right) \frac{dt}{t} \right|$$

$$\le \left(n^{\frac{r-3}{2}} - 1\right) K \left(1 - \frac{c}{n^{\alpha\left(\frac{r-3}{2}\right)}}\right)^n$$

which tends to zero faster than the reciprocal of any polynomial.

We therefore have

Theorem 4: If condition H_r is satisfied ($r \ge 3$) then there exist polynomials R_k depending only on the first r derivatives of P(g+sf) at s=0 and on the first r–2 derivatives of

$\int v(s)\, dm$ at $s=0$ such that

$$F_n(x) - N(x) = e^{-x^2/2\sigma^2} \sum_{k=3}^{r} n^{-k/2+1} R_k(x) + o(n^{-r/2+1}).$$

5. Arithmetical Bernoulli processes

Here we wish to show that the conditions H_r are non-empty by constructing certain independent processes where $f(x) = f(x_0)$ has its range with specific arithmetical properties. These examples were previously studied by Pollicott [Poℓ.2] in connection with the extension domain of meromorphy of the dynamical zeta function.

First we note the following elementary

Lemma 3: Let $w_0, ..., w_d$ be complex numbers of modulus 1. Let $(p_0, ..., p_d)$ be a probability vector with $p_j > 0$ for all j. Then

$$\left| \sum_{i=0}^{d} p_i w_i \right| \le 1 - \frac{\mu^2}{4\left(\frac{1}{p_i} + \frac{1}{p_j}\right)}$$

where $\mu = | w_i - w_j |$ is the maximum difference.

Proof: Let $\mu = | w_i - w_j |$ be the maximum difference. Clearly

$$\left| \sum_{k=0}^{d} p_k w_k \right| \le 1 - (p_i + p_j) + | p_i w_i + p_j w_j |.$$

Hence it suffices to show

$$\left| \frac{p_i w_i + p_j w_j}{p_i + p_j} \right| \le 1 - \frac{\mu^2 p_i p_j}{4 (p_i + p_j)^2},$$

which follows from the lemma for d=1. Therefore it remains to prove that given a, p, q ($p > 0$, $p+q=1$) then

$$| p + q e^{ia} | \le 1 - \frac{|1 - e^{ia}|^2 pq}{4},$$

which is then a straight-forward computation.

Lemma 4: With $w_j = e^{ia_jt}$ where, say, $a_1-a_0, \dots, a_d-a_0$ are algebraic numbers which are rationally independent (this condition is invariant under permutations), for each $\delta > 0$, we have

$$\left| \sum_{j=0}^{d} p_j e^{ia_jt} \right| \le 1 - \frac{c}{t^{\{2/(d-1)\}+2\delta}}$$

for some constant $c > 0$ and all $t > t_0$ (t_0 depending on δ), where $(p_0, \dots, p_d)$ is a probability vector with $p_j > 0$ for all j.

Proof: Since $a_1-a_0, \dots, a_d-a_0$ are rationally independent, so are $1, \frac{a_2-a_0}{a_1-a_0}, \dots, \frac{a_d-a_0}{a_1-a_0}$.

By the celebrated generalisation of the Thue-Siegel-Roth theorem due to W. Schmidt [ScW] there exist c' and $n_0 = n_0(\delta)$ such that for all $n > n_0$ at least one of the inequalities

$$\left\| n\left(\frac{a_j-a_0}{a_1-a_0}\right)\right\| \ge \frac{c'}{n^{\{1/(d-1)\}+\delta}}$$

holds ($j = 2, \dots, d$). Here $\|x\| = \inf\{|x+m| : m \in \mathbb{Z}\}$.

From this we see that

$$\max_{1\le j\le d} \left| e^{2\pi i t(a_j-a_0)} - 1 \right| \ge \frac{c}{t^{\{1/(d-1)\}+\delta}}$$

for all $t > t_0$ (and some constant $c > 0$). Hence by the previous lemma

$$\left| \sum_{i=0}^{d} p_j e^{ia_jt} \right| \le 1 - \frac{c^2 p}{8t^{\{2/(d-1)\}+2\delta}}$$

where p is the least p_j.

Now suppose that f' is defined on the full d+1 shift endowed with the Bernoulli measure m generated by $(p_0, \dots, p_d)$. We suppose f' is a function of one variable, in fact $f'(j) = a_j$ where $a_1-a_0, \dots, a_d-a_0$ are rationally independent algebraic numbers. Finally we define $f = f' - \int f'\,dm$. Clearly f enjoys the same arithmetical properties. Hence, using lemma 4,

$$\begin{aligned} \left| \int e^{itf^n} dm \right| &= \left| \int e^{itf} dm \right|^n \\ &\le \left(1 - \frac{c}{t^{2/(d-1)+2\delta}}\right)^n \end{aligned}$$

Thus if $(r-3)/(d-1) < 1$ then f satisfies H_r.

6. Remark on orbital measures

As a final remark we consider the central limit theorem for $f \in F_\theta^+$ ($\int f\, dm=0$) with the equilibrium state m replaced by the sequence of orbital measures m_n,

$$\int k\, dm_n = \frac{\sum_{\mathrm{Fix}(n)} k(x)\exp\{g^n(x)\}}{\sum_{\mathrm{Fix}(n)} \exp\{g^n(x)\}}$$

where $\mathrm{Fix}(n) = \{x: \sigma^n(x)=x\}$.

For a proof that $m_n \to m$ weakly cf. [Rue].

The central limit theorem for this sequence states

Theorem 5: $m_n\left\{x: f^n(x)/\sqrt{n} < y\right\} \to N(y)$, as $n \to \infty$.

To see that the refinements of earlier sections apply to this theorem also, we note the following lemma (cf. [Rue]).

Lemma 5: There exists $\varepsilon > 0$ such that for $|s| < \varepsilon$,

$$\lim_n\sup \left| \sum_{\mathrm{Fix}(n)} e^{g^n(x)+sf^n(x)} - e^{nP(g+sf)} \right|^{1/n} < 1.$$

As a consequence we have

$$\left| \int \exp\{itf^n/\sqrt{n}\}dm_n - \exp\{nP(g+itf/\sqrt{n})\} \right|$$

$$= \left| \frac{\sum_{\mathrm{Fix}(n)} e^{g^n(x)+itf^n(x)/\sqrt{n}}}{\sum_{\mathrm{Fix}(n)} e^{g^n(x)}} - \exp\{nP(g+itf/\sqrt{n})\} \right| \le K\rho^n$$

for constants $K = K(\varepsilon)$, ρ $(0 < \rho < 1)$ for all $|t| < \varepsilon\sqrt{n}$.

7. Conclusion

The referee has kindly raised two problems which follow from Theorems 1 and 4. At the present time we are unable to solve them but it seems to us that they are well worth proposing as open problems:

(i) Theorem 1 is a version of the Berry-Essen theorem (cf. [Feℓ]) but without the implied constant being specified. Can a reasonable constant be found, depending on the first few derivatives of the pressure function, and perhaps the first few derivatives of w(s) ?

(ii) How restrictive is the condition H_r in Theorem 4 ? The same question, of course, applies to any reasonable substitute for H_r.

CONCLUDING REMARKS

This note is devoted to discussing the relationship between some classification problems of Markov shifts and equivalence of skew-products over the associated subshifts of finite type. This interplay was the primary motivation for studying the problems considered in this thesis. The initial idea was to reduce the study of isomorphisms preserving the measure-theoretic structure of the shifts, to the study of isomorphisms preserving the topological structure of the corresponding skew-products. Nevertheless the difficulty in the classification of Markov shifts is still present in this transition since one has to consider topological classification of systems in non-compact spaces (due to the skew-product construction). For some comments along these lines we refer the reader to the survey of Parry [Par.5].

Let A and B be irreducible 0-1 matrices and consider the ssft's Σ_A and Σ_B. Denote by Ω_A the set of points in Σ_A which have dense forward and backward shift-orbits. Ω_A is called the set of *doubly transitive points*. A map φ from Σ_A to Σ_B is called an *almost conjugate factor map* if φ is continuous, surjective, satisfies $\varphi\circ\sigma_A = \sigma_B\circ\varphi$ (i.e. φ is *shift-commuting*) and such that the restriction $\varphi|_{\Omega_A}$ is a bijection from Ω_A to Ω_B. From the theory of ssft's we know that if $\varphi\colon \Sigma_A \to \Sigma_B$ is continuous, shift-commuting and finite-to-one (i.e. the pre-image of each point is a finite set) then the topological entropies h_A and h_B coincide if and only if φ is surjective (cf. [CP]). Consequently, if φ is continuous, surjective, shift-commuting and finite-to-one then it preserves the measures of maximal entropy (i.e. $\mu_A\circ\varphi^{-1} = \mu_B$). Since any Borel σ-invariant probability measure on Σ_A with full support gives full measure to Ω_A, we conclude that if φ is an almost conjugate factor map from Σ_A to Σ_B then φ is a measure-preserving isomorphism between the Markov chains $(\Sigma_A, \sigma_A, \mu_A)$ and $(\Sigma_B, \sigma_B, \mu_B)$.

Two ssft's Σ_A and Σ_B are said to be *almost topologically conjugate* if there exists a third ssft Σ_C and almost conjugate factor maps $\varphi\colon \Sigma_C \to \Sigma_A$ and $\theta\colon \Sigma_C \to \Sigma_B$. This concept defines an equivalence relation among ssft's (cf. [AM]). The classification of ssft's under this equivalence is complete by the work of Adler & Marcus [AM]. The following is the main result in their work.

Proposition C.1 [AM]: *Let* A *and* B *be irreducible 0-1 matrices. Then* Σ_A *and* Σ_B *are almost topologically conjugate if and only if they have the same topological entropy and the matrices* A *and* B *have the same period.*

When we are dealing with Markov shifts the equivalences must take into account the measures involved. Due to the rich structure of the shift space, Markov shifts admit a great variety of isomorphisms ranging from finitary isomorphisms, block isomorphisms, almost block isomorphisms, finitary isomorphisms with finite expected code length to hyperbolic isomorphisms (cf. for instance [Kri], [KS], [Par.3], [PS], [PT], [ScK.3,4], [Tun]). For a recent survey on finitary isomorphisms(15) see [Par.5]. We shall concentrate our attention on only a few types of isomorphisms in order to draw the connection with skew-products.

In the classification of Markov shifts there is a measure-theoretic concept of almost topological conjugacy which we shall call Adler-Marcus equivalence. Given two irreducible Markov shifts Σ_P and Σ_Q they are called *Adler-Marcus equivalent* if there exist a third Markov shift Σ_W and continuous, surjective, shift-commuting maps ψ: $\Sigma_W \rightarrow \Sigma_P$ and θ: $\Sigma_W \rightarrow \Sigma_Q$ which are also measure-preserving and one-to-one a.e. The map $\theta \circ \psi^{-1}$: $\Sigma_P \rightarrow \Sigma_Q$ is a finitary isomorphism with nice coding properties (for instance it has finite expected code length and consequently preserves the hyperbolic structure of the shift, for definitions see [PS] and [ScK.3,4]). The classification of Markov shifts under finitary isomorphisms is complete by the work of Keane & Smorodinsky [KS]. The following is the main result in that work.

Proposition C.2 [KS]: *Let* P *and* Q *be irreducible non-negative matrices then* Σ_P *and* Σ_Q *are finitarily isomorphic if and only if* $h(\mu_P) = h(\mu_Q)$ *and the matrices* P *and* Q *have the same period.*

---

(15) A *finitary isomorphism* is a measure-preserving isomorphism which is a homeomorphism when restricted to some subset of full measure endowed with the induced topology.

The classification of Markov shifts under Adler-Marcus equivalence is still an open problem but there are many known invariants, i.e. the groups Γ_P and Δ_P defined in Section I.1, the distinguished generator $c_P^d\Delta_P$ of Γ_P/Δ_P (where d is the period of P) and Tuncel's beta function β_P (for details cf. [Kri], [PS], [ScK.3,4]). The latter two invariants are defined as follows. In [PS] it is proved that Γ_P/Δ_P is cyclic and there is a distinguished generator $c_P^d\Delta_P$ which is also an invariant. Tuncel's *beta function* $\beta_P(t)$ is the maximum positive eigenvalue of the matrix P^t, i.e. the matrix obtained from P by raising every non-zero entry of P to the power t. The following expression holds (cf. [Tun])

$$\beta_P(t) = \lim_{n\to\infty} \left\{ \sum_{x\in Fix(n)} P^t(x_0, x_1) \dots P^t(x_{n-1}, x_0) \right\}^{1/n}.$$

If f is the function $f(x) = \log P(x_0, x_1)$ defined on Σ_{P^0} then $\log \beta_P(t) = \mathcal{P}(tf)$, where $\mathcal{P}(f)$ denotes the pressure of f (cf. [Tun]).

The groups Δ_P and Γ_P are easily seen to be finitely generated. When $\mathrm{rank}(\Delta_P) = 1$ it is shown in [Par.5] and [MT] that the above invariants (i.e. Δ_P, Γ_P, $c_P^d\Delta_P$ and β_P) are a complete set of invariants for Adler-Marcus equivalence. When $\mathrm{rank}(\Delta_P) > 1$ there are counter-examples for this fact since Marcus & Tuncel found new invariants for the equivalence (cf. [MT]).

We now introduce the concept of almost topological conjugacy of skew-products which is a direct translation of the definition made previously for irreducible ssft's. Let $S_f: \Sigma_A \times \mathbb{R} \circlearrowleft$ and $S_g: \Sigma_B \times \mathbb{R} \circlearrowleft$ be topologically transitive skew-products, a *skew almost conjugate factor map* $\varphi: \Sigma_A \times \mathbb{R} \to \Sigma_B \times \mathbb{R}$ is a continuous, surjective map such that $\varphi \circ S_f = S_g \circ \varphi$, φ is *fibre-preserving* (i.e. if $\varphi(x, t) = (y, s)$ then $\varphi(x, t+w) = (y, s+w)$ for every $w \in \mathbb{R}$) and φ is one-to-one when restricted to the set of doubly transitive points of S_f. $(\Sigma_A \times \mathbb{R}, S_f)$ and $(\Sigma_B \times \mathbb{R}, S_g)$ are said to be *almost topologically conjugate* if there exist a third skew-product $S_w: \Sigma_C \times \mathbb{R} \circlearrowleft$ and skew almost conjugate factor maps $\psi: \Sigma_C \times \mathbb{R} \to \Sigma_A \times \mathbb{R}$ and $\theta: \Sigma_C \times \mathbb{R} \to \Sigma_B \times \mathbb{R}$.

Proposition C.3: *Let* $(\Sigma_A \times \mathbb{R}, S_f)$ *and* $(\Sigma_B \times \mathbb{R}, S_g)$ *be skew-products with irreducible base.* $\varphi: \Sigma_A \times \mathbb{R} \to \Sigma_B \times \mathbb{R}$ *is a skew almost conjugate factor map if and only if* φ *is of* *the form* $\varphi(x, t) = (\bar{\varphi}(x), c(x) + t)$ *where* $\bar{\varphi}: \Sigma_A \to \Sigma_B$ *is an almost conjugate factor map and* $c: \Sigma_A \to \mathbb{R}$ *is a continuous function such that* $f(x) - (g \circ \bar{\varphi})(x) = c(\sigma_A x) - c(x)$ *for all* $x \in \Sigma_A$.

Proof: Define $\bar{\varphi}$ and c by the equation $(\bar{\varphi}(x), c(x)) = \varphi(x, 0)$. Since φ is fibre-preserving we have $\varphi(x, t) = (\bar{\varphi}(x), c(x) + t)$ for every $t \in \mathbb{R}$. Now use the commuting property of φ to derive the other assertions. □

Proposition C.4 below shows that the classification of group extensions of ssft's under almost topological conjugacy (in our case $\mathbb{R}$-extensions) is related to the classification of Markov shifts under Adler-Marcus equivalence. (For this type of approach see also the sections on topological walks of [Par.4,5].)

Extending the concept of Adler-Marcus equivalence between Markov shifts to ssft's endowed with equilibrium states (in the obvious way) we have the following.

Proposition C.4: *Let* $(\Sigma_A \times \mathbb{R}, S_f)$ *and* $(\Sigma_B \times \mathbb{R}, S_g)$, *where* f *and* g *are functions with summable variation, be two topologically transitive skew-products then they are skew almost topologically conjugate if and only if* $(\Sigma_A, \sigma_A, \mu_f)$ *and* $(\Sigma_B, \sigma_B, \mu_g)$ *are Adler-Marcus equivalent, where* μ_f *denotes the equilibrium state of* f.

The above provide some motivation for the study of invariants of almost topological conjugacy of skew-products.

APPENDIX:
SOME EXTREMAL MEASURES

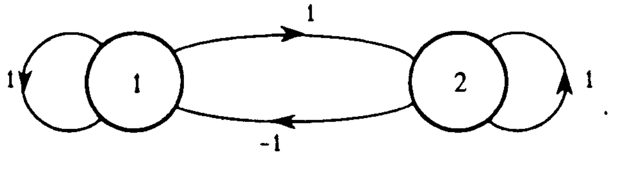
In this appendix we shall consider some simple examples of the extremal measures introduced in Section III.3. We shall see that they give support to the Conjecture III.3.8.

The examples come from the theory of Markov shifts. Here we use the correspondence between Markov shifts and the skew-products over subshifts of finite type (ssft's) defined by functions depending only on a finite number of coordinates (see Section II.3). We will always consider an irreducible ssft Σ and a function $f: \Sigma \to \mathbb{R}$ of the form $f(x) = f(x_0, x_1)$. (We note that the latter condition can always be assumed by considering a higher-block system.) In this case, the pair (Σ, f) is determined uniquely by a connected directed graph with edges labelled by real numbers, i.e. the edges of the graph define the allowable transitions in Σ and $f(i, j)$ is the label of the edge $i \to j$.

In this context, f-weights of periodic orbits of Σ are sums of labels on the cycles of the graph. We shall refer to each of these sums as the *weight* of the corresponding cycle. By Propositions III.3.4 and III.3.2 we see that f is already in one of the extremes of the interval $c(f) = [a, b]$ if all the cycles of the graph have non-negative weights ($a = 0$) or non-positive weights ($b = 0$) and there are cycles of weight zero. (The necessity for the latter condition follows by recurrence of the skew-product.)

The measures considered in this appendix will always be probability measures.

Example A.1: Let (Σ, f) be given by



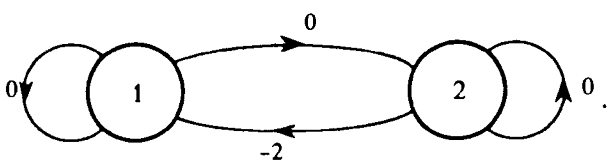
Since for any value of c between 0 and 1 we can find a fully supported one-step Markov measure μ with $\int (f-c)\, d\mu = 0$ and since for any value of $c \notin [0, 1]$ there is no σ-invariant measure with this property, we conclude that $c(f) = [0, 1]$. Another way of seeing this is computing the pressure function and taking the closure of the range of its derivative (by Proposition III.3.2). Hence we find $\mathcal{P}(tf) = \log\{1+e^t\}$ and then $\mathcal{P}'(tf) = e^t/(1+e^t)$ has range $(0, 1)$. Since $s(f) = 0$ we conclude that any σ-invariant measure μ with $\int f\, d\mu = 0$ is an extremal measure for the extreme $a = 0$. However, applying recurrence of the skew-product, there is a *unique ergodic* σ-invariant measure ρ with this property, and ρ is the atomic measure supported on the periodic orbit of $x = (...121212...)$.

Now the equilibrium state m_{tf} of tf is the Markov measure defined by the stochastic matrix

$$\frac{1}{e^{2t}+(1+e^t)^2}\begin{pmatrix} e^{2t} & (1+e^t)^2 \\ (1+e^t)^2 & e^{2t} \end{pmatrix}$$

and hence the unique accumulation point as $t \to -\infty$ is the Markov measure defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is ρ.

In the extreme $b = 1$ we have the following graph for $g = f-1$,

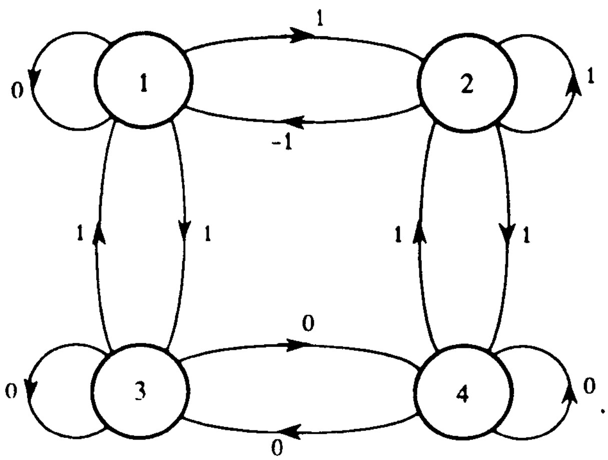


Now $\mathcal{P}(tg) = \log\{1+e^{-t}\}$ and $s(g) = 0$. There are only *two ergodic* σ-invariant measures μ_1 and μ_2 with $\int g\, d\mu_i = 0$ $(i = 1, 2)$, namely μ_1 concentrated on the fixed point $x^1 = (...111...)$ and μ_2 concentrated on the fixed point $x^2 = (...222...)$. The equilibrium state m_{tg} is the Markov measure defined by the stochastic matrix

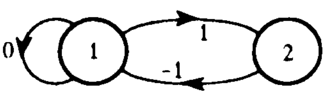
$$\frac{1}{1+e^{-t}}\begin{pmatrix} 1 & e^{-t} \\ e^{-t} & 1 \end{pmatrix},$$

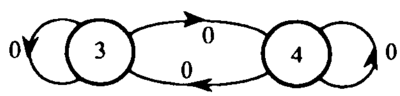
which has stationary vector $(\frac{1}{2}, \frac{1}{2})$, for every t. Hence the unique accumulation point as $t \to \infty$ is the non-ergodic Markov measure defined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ with stationary vector $(\frac{1}{2}, \frac{1}{2})$, i.e. the measure $\frac{1}{2}(\mu_1+\mu_2)$.

Example A.2: Consider (Σ, f) given by



Since for any value of $c \notin [0, 1]$ there is no σ-invariant measure μ with $\int(f-c)\,d\mu = 0$ and since for any $c \in (0, 1)$ we can find one-step Markov measures satisfying this property, we deduce that $c(f) = [0, 1]$.

In the extreme $a = 0$, by analysing the graph of (Σ, f) we conclude that if μ is any ergodic σ-invariant measure with $\int f\,d\mu = 0$ then μ must have its support on one of the subsystems Σ_1 = 

or Σ_2 = 

. (See proof of Proposition A.3 below.) By Proposition III.3.7 any extremal measure for f (in the extreme $a = 0$) must have its support on $\Sigma_1 \cup \Sigma_2$. Therefore since the measure m, supported on Σ_2 and defined by the maximal entropy measure on Σ_2, satisfies $\int f\,dm = 0$, we conclude that $s(f) = \log(2)$. (The latter follows from the fact that any other σ-invariant measure supported on $\Sigma_1 \cup \Sigma_2$ will have entropy strictly less than $\log(2)$.) Therefore there is a unique extremal measure for f in the extreme $a = 0$ and it is the measure m. Hence there is a unique accumulation point for the equilibrium states m_{tf} as $t \to -\infty$ and then $\lim_{t \to -\infty} m_{tf} = m$.

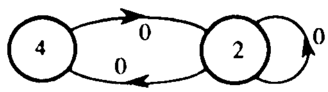
More generally we have the following situation. Let (Σ, f) be as in the discussion at the beginning of the Appendix. Suppose $a = 0$ where $c(f) = [a, b]$. Then equivalently, the weights of all the cycles in the graph of (Σ, f) are non-negative and there are cycles of weight zero. Since there are a finite number of primitive cycles of weight zero we may consider the subsystem Σ' of Σ which consists of points in Σ obtained by concatenation of these primitive cycles. We do not know if Σ' defined in this way is in general a subshift of finite type. However, if f is cohomologous to a *non-negative* function g, depending only on a finite number of coordinates, then Σ' is indeed a ssft. Hence in this case Σ', which may be reducible, will have only a finite number of irreducible components. We shall prove the following result.

Proposition A.3: In the above hypothesis, if Σ' is a ssft then the entropy s(f) is the maximum over the topological entropies of the irreducible components of Σ'. A measure m is an ergodic extremal measure for f (in the extreme a = 0) if and only if m has its support in some irreducible component of Σ' of topological entropy s(f) and restricted to that component, m is the measure of maximal entropy.

Proof: Let B be a cylinder of the form $[x_0 \dots x_{n-1}\, x_0]_\ell$ where $f(x_0, x_1) + \dots + f(x_{n-1}, x_0) = \alpha > 0$. Let μ be any ergodic σ-invariant measure with $\int f\, d\mu = 0$. Then using recurrence of the skew-product $(S_f, \mu \times \lambda)$ we conclude that $\mu(B) = 0$, since otherwise, we would be able to return to the cylinder B with f-weight close to zero and this implies the existence of periodic orbits with negative f-weight, which is a contradiction. Hence if Σ_i $(i = 1, \dots, k)$ are the irreducible components of Σ', then μ must have its support in Σ_i for some i.

Now by Proposition III.3.7 any extremal measure for f (in the extreme a = 0) must have its support on Σ'. Since the topological entropy of Σ' is the maximum over the topological entropies of the irreducible components Σ_i, and is also the maximum over the entropies of the σ-invariant measures with support contained in Σ', we conclude that s(f) is the topological entropy of Σ'. The remaining conclusion follows from the uniqueness of the measures of maximal entropy on each irreducible component of Σ'. □

Remark A.4: Propositions A.3 and III.3.7 imply the following result. If (Σ, f) is a pair defining a Markov shift (see the beginning of the Appendix) such that the subsystem Σ' is a ssft then there are a finite number of ergodic extremal measures for f and any extremal measure is a convex combination of the ergodic ones. Therefore in order to complete the proof of Conjecture III.3.8 in this setting, we need to show that the accumulation sets L_f^a and L_f^b each consist of only one measure and this measure is the barycentre of the ergodic extremal measures in the corresponding extreme. If there is only one ergodic extremal measure (in the extreme a), or equivalently, if Σ' defined by $f-a$ is a ssft and has only one irreducible component of maximal entropy, then the accumulation set L_f^a consists of this ergodic extremal measure, which is the measure of maximal entropy on the above component.

In Example A.2 (where $c(f) = [0, 1]$) we analysed the extreme $a = 0$. From Remark A.4, for the extreme $b = 1$, we conclude that $s(f-1) = \log\{(1+\sqrt{5})/2\}$ (the golden-mean entropy) and $\lim_{t\to\infty} m_{t(f-1)} = m$, where m is the measure of maximal entropy supported on the subsystem 

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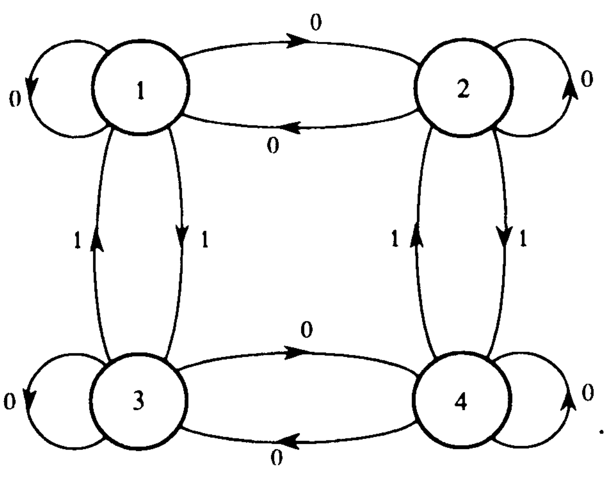
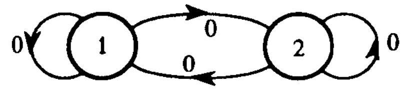
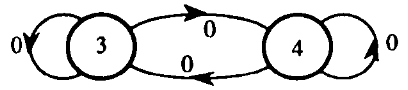
Example A.5: Consider (Σ, f) given by



As in the previous examples we conclude that $c(f) = [0, 1]$. By Proposition A.3 and Remark A.4 we conclude that in the extreme $a = 0$ we have $s(f) = \log(2)$ and in the extreme $b = 1$ we have $s(f-1) = 0$.

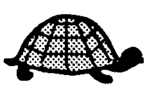
Any extremal measure in the extreme a = 0 is given by $p\mu_1+q\mu_2$ $(p+q = 1)$, where μ_1 is the maximal entropy measure supported on 

and μ_2 is the maximal entropy measure supported on 

. Now the equilibrium state m_{tf} of tf is the Markov measure defined by the stochastic matrix

$$\frac{1}{2+e^t}\begin{pmatrix} 1 & 1 & e^t & 0 \\ 1 & 1 & 0 & e^t \\ e^t & 0 & 1 & 1 \\ 0 & e^t & 1 & 1 \end{pmatrix},$$

which has stationary vector $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, for every t. Hence the unique accumulation point as $t \to -\infty$ is the non-ergodic Markov measure $\frac{1}{2}(\mu_1+\mu_2)$.

 . . .

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