Investment-Consumption Model with Infinite Transaction Costs

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Department of Statistics

September 2014

THE UNIVERSITY OF WARWICK

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Acknowledgments

Foremost, I would like to express my sincere gratitude to my supervisor Prof. David Hobson for the continuous support of my PhD study and research, for his patience, motivation and immense knowledge. I could not have imagined having a better supervisor for my PhD study.

Besides my supervisor, my thanks go to the people at the department of statistics and the University of Warwick for providing help, guidance and facilities.

I am also grateful to the Department of Statistics for the financial support I have received.

Last but not least, I would like to thank my family: my father Taofu Zhu, my mother Lijun Chen and my wife Jingjing Zhou, for their love, kindness, encouragement and tolerance which support me spiritually throughout my life.

Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted for a degree at another university.

The work presented in this thesis was conducted by myself, except where stated. Parts of this thesis have been accepted or submitted for publication as follows.

The work in Chapter 3 and Chapter 4 has been submitted to the *Siam Journal on Financial mathematics* under the title "Optimal consumption and sale strategies for a risk averse agent" with my supervisor, Prof. David Hobson, as a co-author.

The work in Chapter 5 and and some in Chapter 6 has been submitted to the *Annals of Applied Probability* under the title "Multi-asset consumption-investment problems with infinite transaction costs" with my supervisor, Prof. David Hobson, as a co-author.

Abstract

This thesis considers optimal intertemporal consumption and investment problems in which the transaction costs on purchases of the risky asset are infinite. Equivalently, the problems can be classified as (infinitely divisible) asset sale problems with the restriction that the asset cannot be (re)-purchased.

We will first present the classical Merton [41] model which comprises an agent with constant relative risk aversion (CRRA) who wishes to maximise the expected utility of consumption over an infinite horizon. Further, we introduce the extension of the single-asset Merton model with proportional transaction costs by Davis and Norman [13].

After discussing two preliminary optimal consumption and asset sale problems, we consider the special case of the Davis and Norman model, in which the transaction costs on purchase are infinite. Effectively, the asset cannot be purchased but only be sold. We manage to provide a complete and thorough analysis of the problem with rigorous proofs by a new solution technique, which reduces the problem into a first crossing problem.

Based on the new solution technique, we conduct the comparative statics to analyse the optimal strategies and the indifference price, especially their dependance on model parameters. Some surprising results are found and are further discussed.

We then consider the optimal consumption and investment problem with multiple risky assets and with infinite transaction costs. We manage to make significant progress towards an analytical solution and completely characterise the different possible behaviours of the agent by understanding the existence and finiteness of a first crossing problem. The monotonicity of the indifference price in model parameters is proved and a comparative statics is conducted.

Chapter 1

Introduction

1.1 Introduction

The majority of this thesis is concerned with optimal consumption and investment problems with transaction costs. In our model the transaction costs take a special form in that transaction costs on purchases of the endowed asset are infinite. Expressed differently, the endowed asset is assumed to not be available for dynamic trading, either to the market as a whole, or to the agent. It may be because of legal reasons, or simply because it is difficult for individuals to trade particular stocks actively. Instead the assumption is that this asset can only be sold: (re)-purchases are not allowed.

Since trading in the endowed asset is not allowed, the agent faces an incomplete financial market as the risk arising from fluctuations in the value of the asset cannot be fully hedged. Even the agent is allowed to have access to the market and can invest in a riskless bond and trade in multiple risky assets, whose prices are correlated with that of the endowed asset, she still faces the unhedgeable or idiosyncratic part of the risk. For this reason, there is a potential trade-off to consider: selling the endowed asset mitigates the idiosyncratic risk, but holding the asset longer may increase its value. The main objective of this thesis is to consider such a problem in the single-asset and multi-asset settings.

Our agent is assumed to have a power utility function with constant relative risk aversion (CRRA) and is endowed with an initial quantity of a risky asset, whose price follows a geometric Brownian motion. The costs to purchase the endowed asset are infinite. In the single-asset setting, the objective of the agent is to choose optimal consumption and sale strategies (C, Θ), including when and how many to sell units of the infinitely divisible asset over time, so as to maximise the expected discounted utility

over an infinite horizon as measured by the quantity

$$\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{1-R}}{1-R}dt\right]. \tag{1.1}$$

Here β is a positive discount factor and $R \in (0,1) \cup (1,\infty)$ is the risk aversion of the agent. In the multi-asset setting, the agent can trade freely in other risky assets whose prices are correlated with the price of the endowed asset. Apart from consumption and sale strategies, the agent seeks optimal investment strategy Π so as to maximise (1.1).

Our model is an extension of the Merton model and is closely related to the Davis and Norman model. We will introduce these two models further in the following sections.

1.2 Merton's investment-consumption model with multiple assets

The classical portfolio optimisation problem dates back to Merton [41] who considered portfolio optimisation and consumption in a continuous-time stochastic model, with an investment opportunity set comprising a risk-free bond and a risky asset with constant return and volatility. Merton chose to study these issues by first understanding the behaviour of a single agent acting as a price-taker. Under an assumption of constant relative risk aversion (CRRA) he obtained a closed form solution to the problem and the optimal strategy in his model consists of trading continuously in order to keep the fraction of wealth invested in the risky security equal to a constant. As we shall see in the following chapters, Merton's investment-consumption problem is essential to understand the fundamentals of dynamic asset allocation and stochastic control problems. It is also remarkable that this is one of the few nonlinear stochastic control problems that can be solved with an explicit solution.

In this section, we present the Merton's investment-consumption model in a multi-asset financial market and introduce further the Merton line and the Merton proportion.

Let $\left(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}\right)$ be a filtered probability space, such that the filtration satisfies the usual conditions and is generated by a N-dimensional standard, uncorrelated Brownian motion $B = \left(B_t^{(1)}, B_t^{(2)}, B_t^{(3)}, \cdots, B_t^{(N)}\right)^T_{t\geq 0}$. , Here the superscript T represents transposition.

The financial market comprises one risk-free and N risky assets available for investment. The risk-free bond continuously pays a constant rate of interest r. The price

of the bond R satisfies $R_t = e^{rt}R_0$ or in the differential form,

$$dR_t = rR_t dt. \tag{1.2}$$

Suppose the price processes of N risky assets satisfy

$$\frac{dP_t^{(i)}}{P_t^{(i)}} = \mu^{(i)}dt + \sum_{j=1}^{N} \sigma^{(ij)} dB_t^{(j)}, \tag{1.3}$$

for $i = 1, 2, 3, \cdots, N$. Denote by $\mu^{(i)}$ and $\sigma^{(ij)}$ the constant mean rate of return and volatility. The price processes are correlated through the diffusion terms. Let $\mu = \left[\mu^{(1)}, \mu^{(2)}, \mu^{(3)}, \cdots, \mu^{(N)}\right]$, $\sigma = (\sigma^{(ij)})_{1\leq i,j\leq N}$ be the matrix of volatilities, which is assumed to be invertible, and $\lambda = \sigma^{-1}(\mu - r\mathbf{e}_N)$ be the Sharpe ratio matrix, where $\mathbf{e}_N$ is a N-dimensional vector with every component equal to 1.

Definition 1.2.1 (Consumption Process)**.** *A consumption process is an $\{\mathcal{F}_t\}$-progressively measurable, nonnegative process C satisfying $\int_0^t C_s ds < \infty$ almost surely for every $t \in [0, \infty)$.*

Definition 1.2.2 (Portfolio Process)**.** *A portfolio process is an $\{\mathcal{F}_t\}$-progressively measurable, $\mathbb{R}^N$ valued process $\Pi = \left(\Pi^{(1)}, \Pi^{(2)}, \Pi^{(3)}, \cdots, \Pi^{(N)}\right)^T$ such that $\int_0^t \left|\left|\Pi_s\right|\right|^2 ds < \infty$ almost surely for every $t \in [0, \infty)$.*

Let $C = (C_t)_{t\geq 0}$ be the consumption process of the individual and Π be the cash amount invested in the risky assets P, where $\Pi = \left(\Pi_t^{(1)}, \Pi_t^{(2)}, \Pi_t^{(3)}, \cdots, \Pi_t^{(N)}\right)_{t\geq 0}^T$. We denote by $X = (X_t)_{t\geq 0}$ the wealth process of the agent, and suppose that the initial wealth is x. Provided the changes to agent's wealth occur from either consumption or investment, X evolves according to

$$dX_t = \sum_{i,j=1}^{N} \Pi_t^{(i)} \sigma^{(ij)} \left(dB_t^{(j)} + \lambda^{(j)} dt\right) + rX_t dt - C_t dt,$$

which can be written in a more compact notation, $dX_t = \Pi_t^T(\mu - r\mathbf{e}_N)dt + rX_t dt - C_t dt + \Pi_t^T \sigma dB_t$. In particular, X is a process given by

$$X_t = x + \int_0^t \left[\Pi_s^T(\mu - r\mathbf{e}_N) + rX_s - C_s\right] ds + \int_0^t \Pi_s^T \sigma dB_s. \tag{1.4}$$

Remark 1.2.3. It follows from Karatzas and Shreve [35, p373] that the definitions of

portfolio and consumption processes ensure the stochastic differential equation of the associated wealth process has a unique strong solution.

Definition 1.2.4 (Admissible Strategies)**.** *A consumption and portfolio process pair* (C, Π) *is admissible at* x*, and write* $(C, \Pi) \in \mathcal{A}(x)$*, if the corresponding wealth process* $X^{x,C,\Pi}$ *corresponding to* x*,* C*,* Π*, satisfies* $X_t^{x,C,\Pi} \geq 0$ *for* $x \geq 0$ *and* $t \in [0, \infty)$*, almost surely. For* $x < 0$*, we have* $\mathcal{A}(x) = \emptyset$*.*

The objective of the agent is to maximise over admissible strategies the discounted expected utility of consumption over an infinite horizon. Here, the utility function is an increasing concave function of the rate of consumption, in the form

$$U(C_t) = \frac{C_t^{1-R}}{1-R}, \tag{1.5}$$

where R is a constant in $(0, 1) \cup (1, \infty)$. This utility function is equivalent to the CRRA (constant relative risk aversion) class, since $-xU''(x)/U'(x) = R$. In particular, the goal is to find

$$\sup_{(C,\Pi)\in\mathcal{A}(x)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{1-R}}{1-R}dt\right].$$

Here β is a positive discount factor, also known as the 'impatience factor', which measures the 'satisfaction' of the utility function and describes the willingness of the agent to consume now rather than later.

The following theorem characterises the optimal consumption, investment strategies and the corresponding value function of the Merton problem. It will be proved in the following section by a verification theorem.

Theorem 1.2.5. *Suppose the following condition holds*

$$\beta > r(1-R) + \frac{1-R}{2R}||\lambda||^2.$$

Then the value function V *is given by*

$$V(x,t) = e^{-\beta t}\frac{x^{1-R}}{1-R}\left\{\frac{1}{R}\left[\beta - r(1-R) - \frac{1-R}{2R}||\lambda||^2\right]\right\}^{-R}.$$

The optimal portfolio process Π_t^**, the optimal consumption process* C_t^* *and the resulting wealth process are given by*

$$\Pi_t^* = \frac{1}{R}(\sigma\sigma^T)^{-1}(\mu - r\mathbf{e}_N)X_t^*, \qquad C_t^* = \frac{1}{R}\left[\beta - r(1-R) - \frac{1-R}{2R}||\lambda||^2\right]X_t^*, \tag{1.6}$$

$$X_t^* = x \exp\left\{\left(-\frac{\beta}{R} + \frac{r}{R} + \frac{||\lambda||^2}{2R}\right)t + \frac{\lambda^T}{R}B_t\right\}.$$

Remark 1.2.6. In the case where there is only one risky asset (that is, $N = 1$), it follows from Theorem 1.2.5 that $\Pi_t^* = \frac{\lambda}{\sigma R}X_t^*$. Define π^* as the ratio of the money amount invested in the risky asset and total wealth. We then have $\pi^* = \Pi_t^*/X_t^* = \frac{\lambda}{\sigma R}$ and this is the Merton proportion. Denote by $(X_t^* - \Pi_t^*, \Pi_t^*)$ the optimal amount of money that the agent invests in the bond and the risky asset, respectively. Since π^* does not depend on time, the stochastic process $(X_t^* - \Pi_t^*, \Pi_t^*)$ lies in the straight line through the origin with slope $\pi^*/(1-\pi^*)$ for any instant in time, which is the Merton line.

Further, it follows immediately that π^* increases as mean rate of return μ increases and decreases as volatility σ increases or risk aversion R increases. A higher mean rate of return makes the asset more valuable and therefore the asset is worth more investment in. Similarly, a larger σ implies that a long position in the asset involves additional risk and less investment should be made to mitigate such risk. Finally, as risk aversion of the investor increases, she is less tolerant of the risk of the asset and hence more inclined to hold a smaller position in the asset.

For $\pi^* \in (0,1)$, we have $r < \mu < r + \sigma^2 R$. This implies hedging is optimal by investing in both risky asset and bond so as to mitigate risk. If $\mu > r + \sigma^2 R$ then $\pi^* > 1$ and money are borrowed from bank to invest in the risky asset, implying that leverage is optimal. Similarly, when $\mu < r$ then $\pi^* < 0$ and short selling is optimal. Finally if $\mu = r$, we have $\pi^* = 0$ and the optimal strategy is to consume the initial wealth x in the bond without any investment.

1.3 A verification theorem to the Merton problem

In this section, we provide the verification argument for Theorem 1.2.5. The key idea is to construct a candidate value function from the HJB (Hamilton-Jacobi-Bellman) equation and prove that this candidate value function is indeed the value function.

The verification theorem is essential in the following chapters, by which we prove the optimality of the proposed strategies and the validity of the candidate value function. Understanding the verification theorem in the fundamental Merton problem provides an insight into the proofs in the following chapters.

Let $V(x,t)$ be the forward starting value function such that

$$V(x,t) = \sup_{(C,\Pi)\in\mathcal{A}(x,t)} \mathbb{E}\left[\int_t^\infty e^{-\beta t}U(C_t)dt\Big| X_t = x\right]. \tag{1.7}$$

Define the candidate value function G by

$$G(x,t) = e^{-\beta t}\frac{x^{1-R}}{1-R}\left\{\frac{1}{R}\left[\beta - r(1-R) - \frac{1-R}{2R}||\lambda||^2\right]\right\}^{-R}. \tag{1.8}$$

For $F = F(x,t) \in \mathbb{C}^{2,1}$ define operator $\mathcal{L}$ by

$$\begin{aligned}\mathcal{L}F &= F_t + \sup_{(c>0,\pi\in\mathbb{R}^N)}\left\{e^{-\beta t}U(c) + [rx - c + \pi^T(\mu - r\mathbf{e}_N)]F_x + \frac{1}{2}\pi^T\sigma\sigma^T\pi F_{xx}\right\}\\ &= \frac{R}{1-R}e^{-\frac{\beta}{R}t}F_x^{\frac{R-1}{R}} + rxF_x + F_t - \frac{1}{2}||\lambda||^2\frac{F_x^2}{F_{xx}}.\end{aligned}$$

Lemma 1.3.1. *Suppose the following condition holds*

$$\beta > r(1-R) + \frac{1-R}{2R}||\lambda||^2. \tag{1.9}$$

Then for $x \geq 0$ we have $\mathcal{L}G = 0$.

Proof. Given the form of the candidate value function in (1.8), we define constant K by

$$K = \frac{1}{R}\left[\beta - r(1-R) - \frac{1-R}{2R}||\lambda||^2\right]. \tag{1.10}$$

The definition of K then implies

$$\mathcal{L}G = e^{-\beta t}\frac{x^{1-R}}{1-R}K^{-R}\left\{RK + r(1-R) - \beta + \frac{1-R}{2R}||\lambda||^2\right\} = 0.$$

□

We now prove Theorem 1.2.5 by the following verification argument.

Proof. (the verification argument for Theorem 1.2.5) Under the proposed strategies in (1.6), the corresponding wealth process $(X_t^*)_{t\geq 0}$ evolves as $dX_t^* = \Pi_t^{*T}(\mu - r\mathbf{e}_N)dt + rX_t^*dt - C_t^*dt + \Pi_t^{*T}\sigma dB_t$. This gives

$$X_t^* = x\exp\left\{\left(-\frac{\beta}{R} + \frac{r}{R} + \frac{||\lambda||^2}{2R}\right)t + \frac{\lambda^T}{R}B_t\right\}.$$

The value function under the strategies proposed in (1.6) is

$$
\begin{aligned}
& \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t^*) dt\right] \\
= & \frac{K^{1-R} x^{1-R}}{1-R} \int_0^\infty \exp\left\{\left[-\frac{\beta(1-R)}{R} + \frac{r(1-R)}{R} + \frac{1-R}{2R}||\lambda||^2 + \frac{(1-R)^2}{2R^2}||\lambda||^2 - \beta\right] t\right\} dt \\
= & \frac{K^{1-R} x^{1-R}}{1-R} \int_0^\infty \exp\{-Kt\} dt \\
= & K^{-R} \frac{x^{1-R}}{1-R} = G(x,0),
\end{aligned}
$$

where K is defined in (1.10). Hence we have $V(x,0) \geq G(x,0)$.

Now, consider general admissible strategies. Suppose first $R < 1$. Define the process $M = (M_t)_{t \geq 0}$ by

$$
M_t = \int_0^t e^{-\beta s} \frac{C_s^{1-R}}{1-R} ds + G(X_t, t). \tag{1.11}
$$

Applying Itô's formula to M_t leads to

$$
\begin{aligned}
M_t - M_0 &= \int_0^t \left\{ e^{-\beta s} U(C_s) + \left[rX_s - C_s + {\Pi_s}^T(\mu - r\mathbf{e}_N)\right] G_x + G_s + \frac{1}{2} {\Pi_s}^T \sigma\sigma^T \Pi_s G_{xx} \right\} dt \\
&\quad + \int_0^t {\Pi_s}^T \sigma G_x dB_s \\
&=: N_t^1 + N_t^2.
\end{aligned} \tag{1.12}
$$

Lemma 1.3.1 shows that $\mathcal{L}G = 0$, which implies $N_t^1 \leq 0$. Provided the assumption $R < 1$, we have $0 \leq M_t \leq M_0 + N_t^2$ and N_t^2 is a local martingale bounded from below and hence a supermartingale. By taking expectations we find $\mathbb{E}(M_t) \leq M_0 = G(x,0)$, which gives

$$
G(x,0) \geq \mathbb{E}\int_0^t e^{-\beta s} U(C_s) ds + \mathbb{E}[G(X_t,t)] \geq \mathbb{E}\int_0^t e^{-\beta s} U(C_s) ds,
$$

where the last inequality follows since $G(X_t,t) \geq 0$ for $R \in (0,1)$. Letting $t \to \infty$, we have

$$
G(x,0) \geq \mathbb{E}\left(\int_0^\infty e^{-\beta t} U(C_t) dt\right),
$$

and taking a supremum over admissible strategies leads to $G(x,0) \geq V(x,0)$.

Now consider $R > 1$. We take the idea of the proof from Davis and Norman [13] and show that the candidate value function is an upper bound on the value function.

For the candidate value function defined in (1.8), consider for $\varepsilon > 0$,

$$\tilde{V}_\varepsilon(x,t) = \tilde{V}(x,t) = G(x+\varepsilon,t) \tag{1.13}$$

and $\tilde{M}_t = \tilde{M}_t(C,\Pi)$ given by

$$\tilde{M}_t = \int_0^t e^{-\beta s} U(C_s)ds + \tilde{V}\left(X_t,t\right).$$

Then we have by Itô's formula

$$\begin{aligned}
\tilde{M}_t - \tilde{M}_0 &= \int_0^t \left\{ e^{-\beta s} U\left(C_s\right) + \left[rX_s - C_s + {\Pi_s}^T(\mu - r\mathbf{e}_N)\right]\tilde{V}_x + \tilde{V}_s + \frac{1}{2}{\Pi_s}^T\sigma\sigma^T\Pi_s\tilde{V}_{xx}\right\} dt \\
&\quad + \int_0^t {\Pi_s}^T\sigma\tilde{V}_x dB_s \\
&=: \tilde{N}_t^1 + \tilde{N}_t^2.
\end{aligned}$$

Lemma 1.3.1 implies $\tilde{N}_t^1 \le 0$. Now define stopping times $\tau_n = \inf\left\{t \ge 0 : \int_0^t \left({\Pi_s}^T\sigma\tilde{V}_x\right)^2 ds \ge n\right\}$. It follows immediately from (1.6) and (1.13) that ${\Pi_s}^T\sigma\tilde{V}_x$ is bounded and hence $\tau_n \uparrow \infty$. Then the local martingale $(\tilde{N}^2_{t\wedge\tau_n})_{t\ge 0}$ is a martingale and taking expectations we have $\mathbb{E}\left(\tilde{M}_{t\wedge\tau_n}\right) \le \tilde{M}_0$, and hence

$$\mathbb{E}\left(\int_0^{t\wedge\tau_n} e^{-\beta s}U(C_s)ds + \widetilde{V}\left(X_{t\wedge\tau_n}, t\wedge\tau_n\right)\right) \le \tilde{V}\left(x,0\right).$$

It then follows from (1.8) and (1.13) that

$$\begin{aligned}
\tilde{V}\left(x,t\right) &= e^{-\beta t}\frac{(x+\varepsilon)^{1-R}}{1-R}\left\{\frac{1}{R}\left[\beta - r(1-R) - \frac{1-R}{2R}||\lambda||^2\right]\right\}^{-R} \\
&\ge \frac{\varepsilon^{1-R}}{1-R}\left\{\frac{1}{R}\left[\beta - r(1-R) - \frac{1-R}{2R}||\lambda||^2\right]\right\}^{-R}.
\end{aligned}$$

Thus $\tilde{V}$ is bounded, $\lim_{n\to\infty}\mathbb{E}\tilde{V}\left(X_{t\wedge\tau_n}, t\wedge\tau_n\right) = \mathbb{E}\left[\tilde{V}\left(X_t,t\right)\right]$, and

$$\tilde{V}\left(x_0,0\right) \ge \mathbb{E}\left(\int_0^t e^{-\beta s}U(C_s)ds\right) + \mathbb{E}\left[\tilde{V}\left(X_t,t\right)\right].$$

Similarly,

$$\tilde{V}(x,t) \ge e^{-\beta t}\frac{\varepsilon^{1-R}}{1-R}\left\{\frac{1}{R}\left[\beta - r(1-R) - \frac{1-R}{2R}||\lambda||^2\right]\right\}^{-R}$$

and hence $\lim_{t\to\infty} \mathbb{E}\left[\tilde{V}\left(X_t, t\right)\right] \to 0$. Then letting $t \to \infty$ and applying the monotone convergence theorem, we have

$$\tilde{V}_\varepsilon\left(x, 0\right) = \tilde{V}\left(x, 0\right) \geq \mathbb{E}\left(\int_0^\infty e^{-\beta s} U(C_s) ds\right).$$

Finally let $\varepsilon \to 0$. Then $V \leq \lim_{\varepsilon\downarrow 0} \widetilde{V} = G$. Hence, we have $V(x, 0) \leq G(x, 0)$.

□

Remark 1.3.2. The condition (1.9) is a necessary and sufficient condition to ensure the finiteness of the value function. It implies that the discount factor cannot be sufficiently large, otherwise an infinite value function can be obtained by prolonged investment followed by massive consumption. We will prove this remark in Section 3.2.2.

Remark 1.3.3. In this section, the Merton problem is solved via a dynamic programming approach. Since the set-up has a Markovian structure, we expect the value function to be function of current wealth as defined in (1.7). We also expect the process M defined in (1.11) to be a martingale under optimal strategies and a supermartingale under general strategies. Applying Itô's formula to M, we expect the finite variation part to be zero thanks to its martingale nature. This gives the following HJB (Hamilton-Jacobi-Bellman) equation

$$\sup_{(c\geq 0, \pi\in\mathbb{R}^N)} \left\{ e^{-\beta t} U(c) + [rx - c + \pi^T(\mu - r\mathbf{e}_N)] G_x + G_t + \frac{1}{2}\pi^T \sigma\sigma^T \pi G_{xx} \right\} = 0.$$

First order conditions then imply the supremum is attained when

$$c = e^{-\frac{\beta}{R}t} G_x^{-\frac{1}{R}} \qquad \pi = -(\sigma\sigma^T)^{-1}(\mu - r\mathbf{e}_N)\frac{G_x}{G_{xx}}. \tag{1.14}$$

By substituting (1.14) into the HJB equation, we can solve for the candidate value function G. Given the construction of the candidate value function G, the final step is to prove that G is indeed the value function by a verification theorem. In particular, we need to prove that M is a martingale given the candidate optimal strategies and a supermartingale under general strategies.

Apart from the dynamic programming approach, stochastic control problems can also be solved via a dual approach [36, chapter 3], which utilises a Lagrange multiplier so that the constrained objective function with the budget constraint is converted into an unconstrained problem. By Legendre-Frenchel transformation, the maximisation problem in terms of consumption and portfolio processes converts into a minimisation problem

in terms of a Lagrange multiplier.

Remark 1.3.4. Another variation of the Merton model is to consider the problem with income. Merton [41] firstly considers an agent endowed with a constant rate of income I and the wealth process (1.4) evolves as

$$dX_t = \sum_{i,j=1}^{N} \Pi_t^{(i)} \sigma^{(ij)} \left(dB_t^{(j)} + \lambda^{(j)} dt \right) + rX_t dt - C_t dt + I dt.$$

Merton solves this problem under the assumption that the agent can borrow from future income, so that the admissible set (C, Π) allows wealth to be negative. In this sense, all future income can be discounted into an initial amount $\int_0^\infty e^{-rt} I dt = I/r$, and this is identical to the one-dimensional problem in Section 1.2 by letting $X_0 = x + I/r$. Merton shows that $V_I(x) = V(x + I/r)$, where V_I and V represent the value functions with and without income.

1.4 The Davis and Norman model with proportional transaction costs

In the absence of transaction costs, Section 1.2 shows that it is optimal for the agent to continuously rebalance portfolio in the risky asset so as to keep the fraction of the money amount invested in the risky asset to wealth to be the Merton proportion. However, any attempt to apply such optimal strategy in the presence of transaction costs would result in ruin, since incessant trading would be necessary to hold the portfolio on the Merton line. Therefore, it is necessary to consider a different strategy in the presence of transaction costs.

Davis and Norman [13] firstly incorporated proportional transaction costs in the one-dimensional Merton model with a precise mathematical formulation. They showed that under optimal behaviour the no transaction region is of a wedge shape and that the optimal buying and selling strategies are local times at boundaries chosen to keep the process inside the wedge. In the transaction region, transactions take place at infinite speed and except for the initial transaction, all transactions take place at the boundaries. They obtained their results by writing down the non-linear, second order Hamilton-Jacobi-Bellman (HJB) equation with free boundary conditions and then reduced the problem to a non-linear second order ordinary differential equations with free boundary conditions.

In this section, we introduce the model and the key result taken from Davis and

Norman [13]. Our work in the following chapters are motivated by the results here.

1.4.1 The model

Consider a financial market with one bond and one risky asset with prices modelled by (1.2) and (1.3) in Section 1.2 with $N = 1$. The transaction costs are assumed to be proportional to the amount invested. In particular, the investor pays fractions κ and ν of the amount either on purchase or sell the risky asset respectively. The transaction costs and the consumption are assumed to be deducted from the bond holdings.

We denote by X and Y the holdings in bond and stock respectively. Then the stochastic processes of the holdings solve

$$dX_t = (rX_t - C_t)dt - (1+\kappa)dL_t + (1-\nu)dU_t, \tag{1.15}$$

$$dY_t = \mu Y_t dt + \sigma Y_t dB_t + dL_t - dU_t, \tag{1.16}$$

where the initial holdings are $X_0 = x$ and $Y_0 = y$. The cumulative purchase and sale of the risky asset are represented by L and U. We can see that purchase of dL units of the risky asset requires a payment of $(1+\kappa)dL$ units of bond whence sale of dU units of stock generates only $(1-\nu)dU$ units of bond.

Now consider the solvency region where selling of all units of the risky asset or closing of the short position in the risky asset leads to non-negative holdings after paying transaction costs. Define the solvency region by

$$\mathscr{J} = \left\{(x,y) \in \mathbb{R}^2 : x + (1-\nu)y \geq 0, x + (1+\kappa)y \geq 0\right\}.$$

Remark 1.4.1. Doléans-Dade [14] prove that (1.15) and (1.16) have a unique strong solution at least up the first time τ when the processes X, Y leave the region $\mathscr{J}$, i.e. the bankruptcy time, where $\tau = \inf\{t \geq 0 : (X_t, Y_t) \notin \mathscr{J}\}$.

The objective of the investor is to find admissible optimal strategies $(C^*, L^*, U^*) \in \mathcal{A}(x,y)$ so as to maximise the expected utility of consumption over an infinite horizon. The investor's risk preference is measured by the power utility function (1.5) and the value function is defined by

$$V(x,y) = \sup_{(C,L,U)\in\mathcal{A}(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} U(C_t)dt\right].$$

In this problem C is a classical stochastic control as seen in Section 1.2, and L and

U are singular stochastic controls [19, chapter 8]. Davis and Norman use a dynamic programming approach and obtain the following main result.

Prior to the main theorem, we define the auxiliary constants β_1, β_2 and β_3 by

$$\begin{aligned}
\beta_1 &= -\frac{1}{2}\sigma^2 R(1-R) + \mu(1-R) - \beta, \\
\beta_2 &= \sigma^2 R + r - \mu, \\
\beta_3 &= \frac{1}{2}\sigma^2.
\end{aligned}$$

1.4.2 A main theorem

Theorem 1.4.2. *[Davis and Norman(1990)] Suppose the following condition is satisfied.*

$$\beta > r(1-R) + \frac{(1-R)(\mu - r)^2}{2R\sigma^2}. \tag{1.17}$$

Suppose that there exist constants K_1, K_2, $s_{\min}$, $s_{\max}$, and a function $\psi \in \mathbb{C}^2 : [-(1-\nu), \infty) \to \mathbb{R}$ such that

(i) $s_{\min}$, $s_{\max}$ satisfy

$$0 < s_{\min} < s_{\max} < \infty. \tag{1.18}$$

(ii) ψ is increasing for $s \in (-(1-\nu), \infty)$. For $s \leq s_{\min}$, ψ has the following expression

$$\psi(s) = \frac{1}{1-R} K_1 (s + 1 - \nu)^{1-R}. \tag{1.19}$$

(iii) For $s_{\min} \leq s \leq s_{\max}$, ψ is a solution to the following ordinary differential equation

$$\beta_3 s^2 \psi''(s) + \beta_2 s \psi'(s) + \beta_1 \psi(s) + \frac{R}{1-R}[\psi'(s)]^{\frac{R-1}{R}} = 0. \tag{1.20}$$

(iv) For $s \geq s_{\max}$, ψ has the following expression

$$\psi(s) = \frac{1}{1-R} K_2 (s + 1 + \kappa)^{1-R}. \tag{1.21}$$

Now define the buying region, selling region and no-transaction region by

$$\begin{aligned}
\mathcal{BR} &= \{(x,y) \in \mathscr{J} : y \leq s_{\max}^{-1} x\}, \\
\mathcal{SR} &= \{(x,y) \in \mathscr{J} : y \geq s_{\min}^{-1} x\}, \\
\mathcal{NT} &= \{(x,y) \in \mathscr{J} : s_{\max}^{-1} x \leq y \leq s_{\min}^{-1} x\}.
\end{aligned}$$

Then for every $(x, y) \in \mathcal{NT} \backslash \{(0,0)\}$ the optimal consumption process is given in feedback form via

$$C^*(x, y) = y \left[\psi' \left(\frac{x}{y} \right) \right]^{-\frac{1}{R}}.$$

Define $\hat{C}_t = C^*(X_t, Y_t)$. Then the corresponding processes $\hat{X}$, $\hat{Y}$, $\hat{L}$, $\hat{U}$ given by

$$\begin{aligned}
d\hat{X}_t &= [r\hat{X}_t - C^*(X_t, Y_t)]dt - (1+\kappa)d\hat{L}_t + (1-\nu)d\hat{U}_t, \\
d\hat{Y}_t &= \mu\hat{Y}_t dt + \sigma\hat{Y}_t dB_t + d\hat{L}_t - d\hat{U}_t, \\
\hat{L}_t &= \int_0^t \mathbf{1}_{\{(\hat{X}_r, \hat{Y}_r) \in \partial\mathcal{BR}\}} d\hat{L}_r, \\
\hat{U}_t &= \int_0^t \mathbf{1}_{\{(\hat{X}_r, \hat{Y}_r) \in \partial\mathcal{SR}\}} d\hat{U}_r,
\end{aligned}$$

exist and $(\hat{C}, \hat{L}, \hat{U})$ is optimal. Here $\mathbf{1}$ stands for the indicator function and $\partial\mathcal{BR}$ represents the boundary of the buying region. Finally, the value function is given by

$$V(x, y) = y^{1-R} \psi \left(\frac{x}{y} \right).$$

Remark 1.4.3. Note that condition (1.17) is the same as the condition (1.9) in Section 1.2, which requires the discount factor to be large enough so that the value function in the Merton model is finite.

Remark 1.4.4. Theorem 1.4.2 implies that the solvency region $\mathscr{J}$ is divided into three regions, the buying region $\mathcal{BR}$, the selling region $\mathcal{SR}$ and the no-transaction region $\mathcal{NT}$, and each region is a wedge. See Figure 1.1. If the pair (X_t, Y_t) is in either $\mathcal{BR}$ or $\mathcal{SR}$ at $t = 0$, then the investor should make an instantaneous transaction to the boundary of $\mathcal{NT}$. While $(X_t, Y_t) \in \mathcal{NT}$, the investor consumes but does not make any transactions. As long as the pair (X_t, Y_t) reaches the boundary between $\mathcal{BR}$ and $\mathcal{NT}$, the investor purchases the risky asset so as to keep the pair (X_t, Y_t) inside $\mathcal{NT}$. Similarly, when the pair (X_t, Y_t) reaches the boundary between $\mathcal{SR}$ and $\mathcal{NT}$, the investor sells certain units of the risky asset in order to keep (X_t, Y_t) inside $\mathcal{NT}$. Except for the initial transaction, all further transactions take place at the boundaries and the transaction strategies are of 'local time' type, which is to do minimal trading to keep the fraction of wealth invested in the risky asset $\hat{\pi}_t = Y_t/(X_t + Y_t)$ in the no transaction interval $[(1+s_{\max})^{-1}, (1+s_{\min})^{-1}]$.

Remark 1.4.5. Davis and Norman [13] use a traditional dynamic programming approach as introduced in Remark 1.3.3 in Section 1.2. There are also different approaches in the

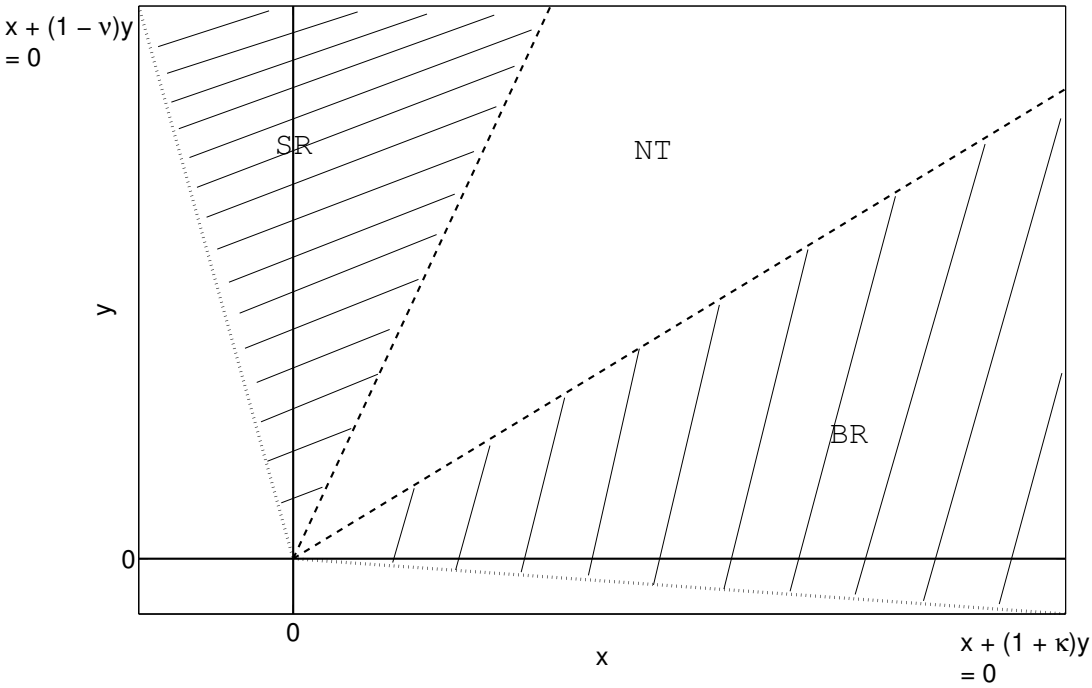

Figure 1.1: Stylised plot of the solvency region $\mathscr{J}$ and its divisions $\mathcal{BR}$, $\mathcal{SR}$ and $\mathcal{NT}$.

existing literature to tackle the Merton problem with proportional transaction costs, for instance, a viscosity solution approach [52] whose key idea is to prove that the value function is a viscosity solution to the HJB equation by maximum principle. Under the assumption of small transaction costs, a perturbation approach [57] is available whose key idea is to take an asymptotic expansion of the value function in terms of small transaction costs.

Remark 1.4.6. Solving the equations (1.19), (1.20) and (1.21) reduced from the HJB equation is a free boundary problem. Here, the $\mathbb{C}^2$ function ψ is governed by simple functions for $s \in [-(1-\nu), s_{\min}]$ and $s \in [s_{\max}, \infty)$ with unknown constants $s_{\min}$, $s_{\max}$, K_1 and K_2 whence ψ solves a second order ODE for $s \in [s_{\min}, s_{\max}]$. Since $\psi \in \mathbb{C}^2$, we expect smooth fit conditions to be satisfied at boundaries if such function ψ exists. In particular, we expect ψ, ψ', ψ'' are continuous at $s_{\min}$ and $s_{\max}$. At any given point $s_{\min} \in [-(1-\nu), \infty)$, equations (1.19) and (1.20) give two different expressions for ψ''. We then have $f(K_1, s_{\min}) = 0$ by equating these two equations for some function f. Similarly, equating (1.20) and (1.21) gives $g(K_2, s_{\max}) = 0$ for some function g. Hence we observe that K_1 and K_2 can be determined by solving f and g once $s_{\min}$ and $s_{\max}$ are determined.

Apart from their main result presented in Theorem 1.4.2, Davis and Norman also provide a rigorous proof of the verification theorem in [13] in order to show the candidate value function is indeed the value function. In order to ensure the conditions in Theorem 1.4.2 are satisfied so that there exists a solution to the free boundary problem (1.19), (1.20) and (1.21), they posed an additional sufficient condition. Although they cannot provide a mathematical proof of the necessity of this condition and Theorem 1.4.2 under this condition, their numerical results convince them of the result. Finally, they provide a numerical algorithm to solve the free boundary problem (1.19), (1.20) and (1.21). In particular, they search $s_{\min}$ and $s_{\max}$ when there is a value matching. They provide a numerical example in which the no-transaction interval $[(1+s_{\max})^{-1}, (1+s_{\min})^{-1}]$ contains the Merton proportion $\hat{\pi}^*$.

Despite the pioneering work by Davis and Norman, there remain some challenges. For instance, the additional condition posed cannot be proved and it is also hard to follow the derivation of this condition in [13]. Further, Theorem 1.4.2 shows that the optimal strategies and the value function are characterised by a free boundary problem. In both $\mathcal{BR}$ and $\mathcal{SR}$, there are undetermined coefficients K_1 and K_2, while in $\mathcal{NT}$ there is a non-linear second order differential equation on an unknown interval $[s_{\min}, s_{\max}]$. This implies that lots of hard work is required in order to understand the problem in a more detailed

fashion, i.e. the dependence of optimal strategies on market parameters, and it is even harder to apply their methodology into a multi-asset problem. Finally, it is complicated to conduct comparative statics of quantities of interest based on the numerical algorithm in [13], which requires a search for $s_{\min}$ and $s_{\max}$ iteratively until the solution to a system of first order differential equations crosses the boundary of some region.

1.5 Literature review

There is a vast literature on the extensions of the Merton model. As introduced in Section 1.4, Davis and Norman [13] extend the Merton model into an incomplete financial market with proportional transaction costs. Apart from that, the Merton model is also extended into other two mainstream directions, namely, non-traded assets and asset liquidation.

1.5.1 Non-traded assets

Whilst active trading in financial assets are allowed in the Merton problem, in other contexts dynamic trading is not possible. Svensson and Werner [54] were the first to consider the problem of pricing non-traded assets in the Merton model. In their model, an agent endowed with units of an asset can sell the asset, but may not make purchases. In the simplest case the agent is endowed with a single unit of an indivisible asset which cannot be traded and the problem reduces to an optimal sale problem for an asset. Evans *et al* [16], see also Henderson and Hobson [28, 30], consider an agent with power utility function who owns an indivisible, non-traded asset and wishes to choose the optimal time to sell the asset in order to maximise the expected utility of terminal wealth

$$\sup_{\tau} \sup_{\pi \in \mathcal{A}_\tau} \mathbb{E} U(X_\tau, \tau),$$

where τ is a stopping time. Their results show that the optimal criterion for the sale of the asset is to sell the first time the value of the non-traded asset exceeds a certain proportion of the agent's trading wealth and this critical threshold is governed by a transcendental equation. Henderson and Hobson [26] also study the problem in the context of real options, where the investor with power utility function, has a claim on units of non-traded assets correlated with the risky asset and wishes to find

$$\sup_{\pi_t, 0 \le t \le T} \mathbb{E} U(X_T),$$

in a finite horizon. Monoyios [44] considers a similar problem in which the investor has an exponential utility function and the non-traded asset is the underlying of an European option. An explicit solution is obtained for put option by perturbation techniques. Miao and Wang [42] consider a risk averse entrepreneur with exponential utility function who is endowed with a non-traded investment project and has access to another correlated risky asset. The project generates cash flow governed by an arithmetic Brownian motion and the price of the risky asset follows a geometric Brownian motion. By a similar approach in [26], they show that risk aversion delays investment and lowers the value of the project.

1.5.2 Asset liquidation

In a separate strand of literature, Merton's model is generalised into the context of asset liquidation. Rogers and Singh [49] model illiquidity of a portfolio in the way that investors with large trading volumes have to pay an inflated price. In contrast, Bank and Baum [6] consider a large trader whose trading behaviour impacts the market price, governed by a family of semi-martingales. Under the assumption that there exists a universal martingale measure for all price processes, they prove the no arbitrage condition in the financial market comprising such a large trader. Henderson and Hobson [29] consider the problem of a risk averse investor who wishes to liquidate a portfolio of infinitely divisible American style options. Longstaff [40] models the illiquidity of portfolio by constraining trading strategies to be of bounded variation so that a trader cannot extricate himself from a position immediately. Schied and Schöneborn [50] introduce two price impacts in order to model illiquidity, one permanent price impact which accumulates over time and one temporary impact which is only affected by the instantaneous change in the number of shares.

1.5.3 Incomplete financial market with transaction costs

Another extension of the Merton's model involves incomplete financial market setting where perfect hedging is no longer possible. Constantinides and Magill [12] (see also Constantinides [11]) were the first to introduce proportional transaction costs to the Merton model and considered an investor whose aim is to maximise the expected utility of consumption over an infinite horizon under power utility. They conjectured the existence of a 'no-transaction' region, and that it is optimal to keep the proportion of wealth invested in the risky asset within some interval.

Subsequently Davis and Norman [13] gave a precise formulation of the problem in

[12] and their work is a landmark in the study of proportional transaction cost problems. Motivated by Davis and Norman's work, Shreve and Soner [52] studied the same problem but with an approach via viscosity solutions. They recover the results from Davis and Norman [13] without imposing all of the conditions of [13]. These approaches remain the main methods for solving portfolio optimisation problems with transaction costs, although recently a different technique based on shadow prices has been proposed, see Guasoni and Muhle-Karbe [22] for a users' guide.

In related work, Duffie and Sun [15], Liu [39] and Korn [38] study the problem when there are fixed (as opposed to proportional) transaction costs. Liu modelled the fixed transaction cost by a constant brokerage fee $F > 0$ and the holdings in stock evolves as

$$dX_t = (rX_t - C_t)dt - F\left(\mathbf{1}_{(dL_t>0)} + \mathbf{1}_{(dU_t>0)}\right),$$

Liu used a dynamic programming approach, deriving an ordinary differential equation to characterise the value function and solving it numerically. He found that if there is only a fixed transaction cost, the optimal trading strategy is to trade to a certain target amount as soon as the fraction of wealth in stock goes outside a certain range. Korn [38] solved a similar problem by an impulse control and optimal stopping approach. He proved the Bellman principle and solved for the reward function by an iteration procedure under the assumption that the value function is finite.

In the literature of multi-asset problems with transaction costs, however, there are relatively limited results. In the multi-asset case, and on the computational side, Muthuraman and Kumar [45] use a process of policy improvement to construct a numerical solution for the value function and the associated no-transaction region, and Collings and Haussman [10] derive a numerical solution via a Markov chain approximation, for which they prove the convergence. On the theoretical front Akian *et al* [2] show that the value function is the unique viscosity solution of the HJB equation (and provide some numerical results in the two-asset case) under the assumption that the price processes of the risky assets are uncorrelated. Chen and Dai [8] identify and prove the shape of the no-transaction region in the two-asset case. Explicit solutions of the general problem remain very rare. One situation when an explicit solution is possible is the rather special case of uncorrelated risky assets, and an agent with constant absolute risk aversion. In that case the problem decouples into a family of optimisation problems, one for each risky asset, see Liu [39]. Another setting for which some progress has been made is the problem with small transaction costs, see Atkinson *et al* [3], Whalley and Willmott [57], and for a more recent analysis Soner and Touzi [53]. Whalley and Willmott use an expansion

method to provide asymptotic formulae for the optimal strategy.

1.6 Overview of thesis

Two main extensions from the Merton model, as introduced in Section 1.5, are non-traded asset sale problems and investment-consumption problems with transaction costs. For a non-traded asset sale problem, the primary challenge is to find the optimal timing for sale, and optimal quantities for sale if the asset is infinitely divisible. Apart from that, it is also crucial to understand the optimal investment and consumption strategies and provide an indifference price for the non-traded asset.

For an investment-consumption problem with transaction costs, as stated in Section 1.4, the challenges are to find new solution techniques to be able to understand the one-dimensional problem in a more detailed fashion and to further generalise to a multi-dimensional problem with transaction costs. Additionally, a more reliable numerical algorithm is also a challenge so that comparative statics can be conducted.

This thesis is concerned with above challenges in the literature of both non-traded asset sale problems and investment-consumption problems with transaction costs. In Chapter 2, we consider two motivation examples of the optimal consumption problem. On example is formulated in a discrete two-period setting, in which explicit solutions of the optimal consumption and sale strategies are obtained. Subsequently, we consider another problem in a continuous-time setting, in which we discover some key features of the optimal consumption process and the utility indifference price.

Chapter 3 considers an (infinitely divisible) asset sale problem under the Merton model, in which we assume the endowed asset can only be sold but not bought and the agent does not have other investment opportunities. This problem can also be considered as a special case of Davis and Norman model in which the cost of purchases of the endowed asset is infinite. In this special setting new solution techniques are available, and we are able to completely classify the different types of optimal strategies and the parameter ranges over which they apply. Further, we can simplify the problem of solving for the value function, especially when compared with direct approaches for solving the HJB equation as a free boundary problem via smooth fit.

Chapter 4 is concerned with the comparative statics of the single-asset problem in Chapter 3. In the comparative statics, we analyse the dependence of the optimal threshold, optimal strategies, the value function and indifference price on market parameters and arrive at some surprising results.

In Chapter 5, we consider an optimal consumption and investment problem with

multiple risky assets, and with infinite transaction costs on purchases of one of the risky assets, the endowed asset. Equivalently, the endowed asset can only be sold. For an agent with CRRA utility function we completely characterise the different possible behaviours of the agent and classify different regimes under market parameters. Further, we prove the monotonicity of the optimal exercise ratio and the indifference price in market parameters.

The comparative statics of Chapter 5 is presented in Chapter 6. Apart from recovering the results in the single-asset case in Chapter 4, we also find that under certain combination of parameters, it is never optimal to cease investment in the risky assets even if liquid wealth hits zero.

Chapter 2

Two preliminary asset sale and consumption problems

In this chapter, we consider two consumption optimisation problems in simple settings. The simplicity of the problems makes the explicit solutions of optimal strategies available so that the key features of consumption optimisation problems are illustrated.

An asset sale problem in a two-period ($t \in \{0, 1\}$) consumption model is considered in Section 2.1, in which an agent endowed with an infinitely divisible asset wishes to choose the optimal consumption and sale strategy in order to maximise the expected utility over two time steps, today and tomorrow. We find explicit solutions of the optimal strategies in a binomial model. Further, we are able to understand different possible behaviours of the agent, which includes always selling the entire holdings of the endowed asset initially, selling a fraction of the entire holdings initially, and never selling the asset till terminal time $t = 1$.

In Section 2.2, we consider a consumption optimisation problem in an infinite horizon. Instead of introducing an endowed asset, we suppose that the agent receives an one-off payment at a specific time during her life time with probability q. The problem is discussed in different scenarios, depending on whether or not the payment is received. Explicit solutions are obtained in each scenario and we show that the optimal consumption process is not necessarily continuous over time and the utility indifference price is not necessarily monotone in risk aversion.

The reason to include these two problems in the thesis is to build intuition for the full problems in the following chapters. By understanding the key features in Section 2.1 and Section 2.2, we find it helpful to better understand and analyse the problems in more complicated settings.

2.1 A two-period divisible asset sale problem

In this section we consider an asset sale problem in a two-period consumption model. An agent with CRRA utility function is endowed with an infinitely divisible asset and her objective is to: (1) find the optimal quantities of the endowed asset to sell in order to better finance consumption; (2) find the optimal consumption strategy which maximises the expected utility of consumption over $t = 0$ and $t = 1$, representing 'today' and 'tomorrow'.

We find that explicit solutions of the optimal strategies are difficult to obtain under the assumption that the price of the endowed asset satisfies a general continuous distribution function. Instead we consider the problem in a binomial model, which simplifies the problem and brings explicit solutions.

2.1.1 The general setting

We consider an agent who is endowed with an initial quantity of θ_0 units of asset Y and the endowed asset is assumed to be only allowed for sale. The price of Y satisfies $Y_0 = y_0$ and $Y_1 = Z$, where Z is some non-negative continuous random variable with a distribution function $F(z)$.

Denote by $C = (C_0, C_1)$ the consumption rate of the the agent at $t = 0$ and $t = 1$ respectively, and Φ the holding of asset Y over time period $(0, 1)$ such that $0 \leq \Phi \leq \theta_0$. In particular, $\Phi = \theta_0$ means no units of Y are sold at $t = 0$, instead all θ_0 units of Y are sold at $t = 1$. Let x_0 be the initial wealth of the agent and X be the wealth at time $t = 1$ such that

$$X = x_0 - C_0 + y_0(\theta_0 - \Phi) + \Phi Z.$$

Here we have x_0, y_0, θ_0 are all constants.

The investor is assumed to have CRRA utility function,

$$U(C) = \frac{C^{1-R}}{1-R}, \tag{2.1}$$

where the relative risk aversion $R \in (0, \infty) \setminus 1$.

The objective of the agent is to determine the quantities of Y to be sold at $t = 0$ and $t = 1$, represented by $(\theta_0 - \Phi)$ and Φ, and the optimal consumption C in order to find

$$\max_{(C,\Phi)} \left\{ \mathbb{E}\left[U(C_0) + e^{-\beta} U(C_1) \right] \right\}, \tag{2.2}$$

subject to $0 \leq C_0 \leq x_0 + (\theta_0 - \Phi)y_0$,

$$C_1 = X = x_0 - C_0 + y_0(\theta_0 - \Phi) + \Phi Z, \tag{2.3}$$

and $0 \leq \Phi \leq \theta_0$. Here $e^{-\beta}$ is the discount factor at $t = 1$ and β is a positive constant. Note that $\beta > 0$ satisfies the condition (1.9) given that $r = 0$ and $\lambda = 0$ in this case.

Note that the right hand side of (2.3) is the remaining wealth before $t = 1$. Here $y_0(\theta_0 - \Phi)$ is the money amount generated by the sale of $(\theta_0 - \Phi)$ units of Y, and ΦZ is the money amount generated by the sale of remaining Φ units of Y. (Any remaining Y after $t = 0$ will be sold at $t = 1$ to finance consumption C_1.)

Provided the constraint (2.3), the objective function in (2.2) can be rewritten into

$$\begin{aligned} & \mathbb{E}\left[U(C_0) + e^{-\beta}U(C_1)\right] \\ = \; & \mathbb{E}(U(C_0)) + e^{-\beta}\mathbb{E}\left[U\left(x_0 - C_0 + y_0(\theta_0 - \Phi) + \Phi Z\right)\right] \\ = \; & \frac{C_0^{1-R}}{1-R} + e^{-\beta}\int_0^\infty \frac{[x_0 - C_0 + y_0(\theta_0 - \Phi) + \Phi z]^{1-R}}{1-R} F(dz). \end{aligned} \tag{2.4}$$

Applying first order conditions to (2.4) gives the optimal consumption and sale strategy, c^* and ϕ^*, if exist, should solve

$$c^{*-R} - e^{-\beta}\int_0^\infty [x_0 - c^* + y_0(\theta_0 - \phi^*) + \phi^* z]^{-R} F(dz) = 0, \tag{2.5}$$

$$e^{-\beta}\int_0^\infty (z - y_0)[x_0 - c^* + y_0(\theta_0 - \phi^*) + \phi^* z]^{-R} F(dz) = 0, \tag{2.6}$$

which are equivalent to $U'(C_0^*) = \mathbb{E}[U_1'(C_1^*)]$ and $y_0 U_1'(C_0^*) = \mathbb{E}[Y_1 U_1'(C_0^*)]$. Here U_1 corresponds to the discounted utility function at $t = 1$, e.g. $U_1(\cdot) = e^{-\beta}U(\cdot)$.

Note that the optimal consumption and sale strategies c^* and ϕ^* should satisfy both (2.5) and (2.6), which gives

$$\int_0^\infty z[x_0 - c^* + y_0(\theta_0 - \phi^*) + \phi^* z]^{-R} F(dz) = e^{\beta} y_0 c^{*-R}. \tag{2.7}$$

The next step is to find expressions for c^* and ϕ^* by solving (2.5) and (2.6).

Remark 2.1.1. We are unable to obtain explicit solutions of c^* and ϕ^* by solving (2.5) and (2.6) given a general function $F(\cdot)$ satisfying $\int_0^\infty F(dz) = 1$. Even if we assume a specific distribution function $F(z)$, i.e. Z follows a log-normal distribution, it is difficult to calculate the integrals in (2.5) and (2.6).

One possible way is to rely on numerical method. Define functions $g_1(\cdot,\cdot)$, $g_2(\cdot,\cdot)$

where

$$
\begin{aligned}
g_1(c,\phi) &= c^{-R} - e^{-\beta}\int_0^\infty [x_0 - c + y_0(\theta_0-\phi) + \phi z]^{-R} F(dz), \\
g_2(c,\phi) &= e^{-\beta}\int_0^\infty (z-y_0)[x_0 - c + y_0(\theta_0-\phi) + \phi z]^{-R} F(dz).
\end{aligned}
$$

The optimal pair (c^*, ϕ^*) can be determined by looking for the intersection/intersections between the trajectories in the $c-\phi$ plane by setting $g_1 = 0$ and $g_2 = 0$.

In the following section, we consider the problem in a binomial model so that (2.5) and (2.6) are simplified and we are able to obtain explicit solutions of c^* and ϕ^*.

2.1.2 An explicit solution in a binomial model

Recall the model setting in Section 2.1.1. Now suppose Z, representing the price of Y at $t=1$, satisfies $\mathbb{P}(Z=\overline{y}) = p$ and $\mathbb{P}(Z=\underline{y}) = 1-p$. Here $\underline{y}$ and $\overline{y}$ are constants such that $\underline{y} < y_0 < \overline{y}$. Then equation (2.4) becomes

$$
\begin{aligned}
\mathbb{E}\left[U(C_0) + e^{-\beta}U(C_1)\right] &= U(C_0) + pe^{-\beta}U(x_0 + (\theta_0-\Phi)y_0 - C_0 + \Phi\overline{y}) \\
&\quad + (1-p)e^{-\beta}U(x_0 + (\theta_0-\Phi)y_0 - C_0 + \Phi\underline{y}) \qquad (2.8)
\end{aligned}
$$

Applying first order conditions to (2.8) gives the optimal consumption and sale strategy, c^* and ϕ^*, if exist, should solve

$$
c^{*-R} - pe^{-\beta}[x_0 + (\theta_0-\phi)y_0 - c^* + \phi\overline{y}]^{-R} - (1-p)e^{-\beta}[x_0 + (\theta_0-\phi)y_0 - c^* + \phi\underline{y}]^{-R} = 0, \quad (2.9)
$$

$$
p(\overline{y}-y_0)e^{-\beta}[x_0 + (\theta_0-\phi^*)y_0 - c + \phi^*\overline{y}]^{-R} - (1-p)(y_0-\underline{y})e^{-\beta}[x_0 + (\theta_0-\phi^*)y_0 - c + \phi^*\underline{y}]^{-R} = 0, \quad (2.10)
$$

which are equivalent to $U'(C_0^*) = \mathbb{E}[U_1'(C_1^*)]$ and $y_0U_1'(C_0^*) = \mathbb{E}[Y_1U_1'(C_0^*)]$. Note that (2.9) and (2.10) can be rewritten into

$$
\left[\frac{c^*}{x_0 + (\theta_0-\phi)y_0 - c^* + \phi\underline{y}}\right]^{-R} = pe^{-\beta}\left[\frac{x_0 + (\theta_0-\phi)y_0 - c^* + \phi\overline{y}}{x_0 + (\theta_0-\phi)y_0 - c^* + \phi\underline{y}}\right]^{-R} + (1-p)e^{-\beta}, \quad (2.11)
$$

$$
\left[\frac{x_0 + (\theta_0-\phi^*)y_0 - c + \phi^*\overline{y}}{x_0 + (\theta_0-\phi^*)y_0 - c + \phi^*\underline{y}}\right]^{-R} = \frac{(1-p)(y_0-\underline{y})}{p(\overline{y}-y_0)}. \quad (2.12)
$$

Since the optimal c^* and ϕ^* should satisfy both (2.11) and (2.12), we substitute (2.12) into (2.11) and have the following equation for c^* and ϕ^*

$$c^* = \frac{\left\{e^{-\beta}(1-p)\frac{\overline{y}-\underline{y}}{\overline{y}-y_0}\right\}^{-\frac{1}{R}}}{1+\left\{e^{-\beta}(1-p)\frac{\overline{y}-\underline{y}}{\overline{y}-y_0}\right\}^{-\frac{1}{R}}}(x_0+y_0\theta_0) - \frac{\left\{e^{-\beta}(1-p)\frac{\overline{y}-\underline{y}}{\overline{y}-y_0}\right\}^{-\frac{1}{R}}}{1+\left\{e^{-\beta}(1-p)\frac{\overline{y}-\underline{y}}{\overline{y}-y_0}\right\}^{-\frac{1}{R}}}(y_0-\underline{y})\phi^*. \quad (2.13)$$

Then, substituting (2.13) into (2.11) and (2.12) respectively gives the following expressions for c^* and ϕ^*

$$c^* = K_1(x_0+y_0\theta_0)\frac{\overline{y}-\underline{y}}{(1+K_1)(\overline{y}-\underline{y})+K_2(y_0-\underline{y})}, \quad (2.14)$$

$$\phi^* = K_2(x_0+y_0\theta_0)\frac{1}{(1+K_1)(\overline{y}-\underline{y})+K_2(y_0-\underline{y})}, \quad (2.15)$$

where K_1 and K_2 are defined as

$$K_1 = \left[e^{-\beta}(1-p)\frac{\overline{y}-\underline{y}}{\overline{y}-y_0}\right]^{-\frac{1}{R}}, \qquad K_2 = \left[\frac{(1-p)(y_0-\underline{y})}{p(\overline{y}-y_0)}\right]^{-\frac{1}{R}} - 1. \quad (2.16)$$

Substituting (2.14) and (2.15) into (2.8) gives the maximum of the expected utility in (2.4),

$$\begin{aligned} &\max_{(C,\Phi)}\left\{\mathbb{E}\left[U(C_0)+e^{-\beta}U(C_1)\right]\right\} \\ = \;& \frac{x_0^{1-R}}{1-R}\left[\frac{(\overline{y}-\underline{y})\left(1+\frac{y_0\theta_0}{x_0}\right)}{(1+K_1)(\overline{y}-\underline{y}+K_2(y_0-\underline{y}))}\right]^{1-R}\left\{K_1^{1-R}+e^{-\beta}p(1+K_2)^{1-R}+e^{-\beta}(1-p)\right\}. \end{aligned}$$

Now we consider some key features of this asset sale problem given the explicit expressions of c^* and ϕ^*. In particular, we want to understand different possible behaviours of the agent by analysing the different possible values of ϕ^*. It is sufficient to discuss the following two special scenarios.

1. Consider the case $\phi^* = 0$. This implies the agent always sells all units of the endowed asset Y initially. From (2.15) and the monotonicity of (2.8) in Φ if the feasibility constraint $\Phi \geq 0$ is violated, we have $\phi^* = 0$ if and only if

$$K_2 = \left[\frac{(1-p)(y_0-\underline{y})}{p(\overline{y}-y_0)}\right]^{-\frac{1}{R}} - 1 \leq 0,$$

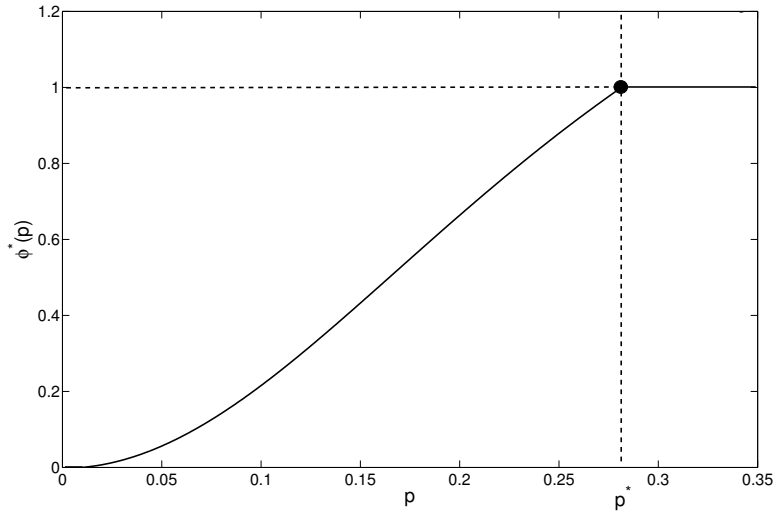

Figure 2.1: Plot of ϕ^* as a function of p. Here parameters are $\underline{y} = 0$, $\overline{y} = 100$, $x_0 = y_0 = \theta_0 = 1$, $\beta = 1$, $R = 0.5$ and $p^* = 0.28$. From the graph, ϕ^* is increasing in p. For $p \geq p^*$, we have $\phi^* = \theta_0 = 1$.

which gives $p\overline{y} + (1-p)\underline{y} \leq y_0$, or equivalently, $\mathbb{E}[Y_1] \leq y_0$.

This implies the following feature of the asset sale problem. If the endowed asset is depreciating over time, then the optimal strategy is to sell the entire asset immediately. Since the endowed asset has a non-positive return but additional risk, the agent always sells immediately so as to remove such risk.

This feature of the asset sale problem will appear again as one of the scenarios in the full problems in the following chapters.

2. Consider the case $\phi^* = \theta_0$. This implies the agent always keeps the entire asset Y initially and sells at terminal time $t = 1$. From (2.15) and the monotonicity of (2.8) in Φ if the feasibility constraint $\Phi \leq \theta_0$ is violated, we have $\phi^* = \theta_0$ if and only if $(1+K_1)(\overline{y}-\underline{y})\theta_0 \leq K_2(x_0+\underline{y}\theta_0)$. This expression, even back to the original parameters, is hard to interpret in an expectation-variance framework. However, if we consider ϕ^* as a function of probability p by fixing other parameters, it follows that $\partial\phi^*/\partial p > 0$ after some lengthy algebra with

$$\lim_{p\to 0} \phi^*(p) = -\frac{x_0 + y_0\theta_0}{K_1(\overline{y}-\underline{y}) + \overline{y} - y_0} < 0, \qquad \lim_{p\to 1} \phi^*(p) = \frac{x_0 + y_0\theta_0}{y_0 - \underline{y}} > \theta_0.$$

Since ϕ^* is continuous in p from (2.15), there exists some critical value $p^* \in (0,1)$ such that $\phi^*(p^*) = \theta_0$. For $p < p^*$, we have $\phi^* < \theta_0$ whence $\phi^* = \theta_0$ for $p \geq p^*$. In particular, p^* is a solution to $(1 + K_1(p))(\overline{y} - \underline{y})\theta_0 = K_2(p)(x_0 + \underline{y}\theta_0)$. See Figure 2.1.

Provided that $\overline{y} > y_0$, a higher probability p implies a better return on Y (This is equivalent if we fix p and consider ϕ^* as a function of $\overline{y}$). This implies another feature of the asset sale problem. The more valuable the endowed asset will be, the more holdings should the agent keep for a better return. Further, when the expected return of the endowed asset reaches certain value, the agent should always hold the asset. The optimal sale strategy Φ^* is governed by a complicated equation. We will observe these features again in the full problems in the following chapters, in which the optimal sale threshold solves a stochastic differential equation.

In the following chapters, we consider this asset sale problem over an infinite horizon under the Merton model. We manage to classify all types of behaviours of the agent, including the ones introduced here, and rigorous proofs are provided in a singular stochastic control framework. Further, the features we find in this simple example are explored further both analytically and numerically.

2.2 Indifference pricing in an optimal consumption problem

In this section we consider an agent with CRRA utility function who wishes to maximise the expected utility of consumption over an infinite horizon. Apart from initial wealth, the agent receives a one-off payment Γ with probability q at a specific time during her life time. Given the randomness of the payment, the agent is not allowed to borrow from future income. The problem is to find the optimal consumption strategy and the value of the uncertain payment.

We classify the problem into three scenarios, (1) $q = 0$ so that the agent does not receive Γ, (2) $q = 1$ so that the agent receives Γ, (3) $0 < q < 1$ so that the agent receives Γ with probability q. By the Lagrangian method, the original problem with constraints is converted into an unconstrained problem. Provided the explicit solutions, we show that the optimal consumption process is not necessarily continuous over time.

Further, we price the value of the uncertain payment Γ via a utility indifference approach. Hodges and Neuberger [34] were the first to adapt the static certainty equivalence concept to a dynamic setting. See Henderson and Hobson [27], and Monoyios [43] for a more detailed introduction. The utility indifference price is the price at which the

agent is indifferent between receiving the payment Γ at a specific time in the future and not having Γ, but receiving some cash amount p now. Our numerical results reveal a surprising feature: the utility indifference price is not necessarily monotone decreasing in risk aversion.

2.2.1 The model setting

Let $C = (C_t)_{t\geq 0}$ and $X = (X_t)_{t\geq 0}$ be the consumption and wealth process of the agent. Denote by $X_0 = x_0$ the initial wealth of the agent with $x_0 \geq 0$.

Suppose the agent receives cash amount Γ with probability q at time $t = 1$. Then the wealth process is

$$X_t = x_0 - \int_0^t C_s ds + \Gamma \mathbf{1}_{(t\geq 1)} \mathbf{1}_{(Z_1 \leq q)}, \tag{2.17}$$

where the random variable $Z_1 \in \mathcal{F}_1$ and follows a uniform distribution on $[0, 1]$. Given there is no interest rate in the model, we do not take into account the discount factor of Γ.

The agent is assumed to have CRRA utility function in (2.1) and the objective of the agent is to choose consumption process $C \in \mathcal{A}(x_0)$ in order to find

$$\max_{C\in\mathcal{A}(x_0)} \mathbb{E}\left[\int_0^\infty e^{-s} \frac{C_s^{1-R}}{1-R} ds\right], \tag{2.18}$$

and the indifference price p for Γ.

Depending on the different values of q, the problem can be classified into three different scenarios. Under each scenario, the agent has a different type of behaviour and the indifference price of Γ has a different feature.

2.2.2 Three scenarios

We classify the problem with different values of q which determine whether or not the agent receives Γ. In particular, we consider three cases with $q = 0$, $q = 1$ and $0 < q < 1$.

(1) Suppose $q = 0$.

In this scenario we have $\mathbf{1}_{(Z_1\leq q)} = 0$, implying that the agent will not receive Γ at time 1. Then the problem becomes deterministic and the wealth process in (2.17) becomes $X_t = x_0 - \int_0^t C_s ds$. This is the Merton problem without any risky assets and we know from Theorem 1.2.5 in Section 1.2 that the optimal consumption process, the

value function, and the indifference price are

$$C_t^* = e^{-\frac{1}{R}t}\frac{x_0}{R}, \qquad V(x_0) = \frac{x_0^{1-R}}{1-R}R^R, \qquad p(x_0) = 0.$$

Provided that the agent does not receive Γ and consumption is financed only by initial wealth, the optimal strategy is to consume averagely over time and the indifference price of Γ is zero.

(2) Suppose $q = 1$.

In this scenario we have $\mathbf{1}_{(Z_1 \leq q)} = 1$, implying that the agent will receive Γ at time 1. Then the problem is deterministic and the wealth process in (2.17) becomes $X_t = x_0 - \int_0^t C_s ds + \Gamma \mathbf{1}_{(t \geq 1)}$. The goal is to find (2.18) subject to the constraints,

$$\int_0^1 C_s ds \leq x_0, \qquad \text{and} \qquad \int_0^\infty C_s ds \leq x_0 + \Gamma. \tag{2.19}$$

Consider first the maximisation problem (2.18) with the second constraint in (2.19). Equivalently, we assume that consumption is continuous over time. Introducing the Lagrange multiplier $\lambda > 0$ for the constraint, then the Lagrangian for the optimisation problem (2.18) is

$$\mathcal{L}(C;\lambda) = \int_0^\infty e^{-t}\frac{C_t^{1-R}}{1-R}dt + \lambda\left\{x_0 + \Gamma - \int_0^\infty C_t dt\right\} = \int_0^\infty \left[e^{-t}\frac{C_t^{1-R}}{1-R} - \lambda C_t\right]dt + \lambda x_0 + \lambda\Gamma,$$

and the Euler-Lagrange equation gives $c^* = \left(e^t\lambda^*\right)^{-1/R}$. Substituting c^* in terms of λ^* into the second constraint in (2.19) gives λ^*, the optimal consumption C^* and the value function as

$$\lambda^* = \left(\frac{x_0+\Gamma}{R}\right)^{-R}, \qquad C_t^* = e^{-\frac{1}{R}t}\frac{x_0+\Gamma}{R}, \qquad V(x_0) = \frac{(x_0+\Gamma)^{1-R}}{1-R}R^R. \tag{2.20}$$

We know from the first scenario that the value function without receiving Γ is $V(x_0, 0) = R^R x_0^{1-R}/(1-R)$. Denote by the value function in (2.20) $V(x_0, \Gamma)$ to emphasise that the agent receives Γ. The indifference price p is then a solution to

$$V(x_0 + p, 0) = V(x_0, \Gamma),$$

which gives $p(x_0) = \Gamma$.

Now we check the first constraint in (2.19) given the expression of C^* in (2.20).

This gives

$$\int_0^1 C_s^* ds = \frac{x_0+\Gamma}{R}\int_0^1 e^{-\frac{1}{R}t}dt = (x_0+\Gamma)(1-e^{-\frac{1}{R}}).$$

Note that the first constraint holds if and only if $(x_0+\Gamma)(1-e^{-1/R}) \leq x_0$, or equivalently $R \geq [\ln(1+x_0/\Gamma)]^{-1}$. However, for $R < [\ln(1+x_0/\Gamma)]^{-1}$, the optimal consumption expressed in (2.20) is not feasible.

Now suppose $R < [\ln(1+x_0/\Gamma)]^{-1}$. Followed by the argument above, we define different consumption rates $C_t^{(1)}$ and $C_t^{(2)}$ for $t \in [0,1]$ and $[1,\infty)$,

$$C_t = \begin{cases} C_t^{(1)}, & \text{if } t \in [0,1]; \\ C_t^{(2)}, & \text{if } t \in [1,\infty). \end{cases}$$

Surely it is now clear that

$$\int_0^1 C_t^{(1)} dt = x_0, \qquad \text{and} \qquad \int_1^\infty C_t^{(2)} dt = \Gamma.$$

Introducing Lagrange multipliers $\lambda > 0$ and $\mu > 0$ for the two constraints, the Lagrangian for the maximisation problem (2.18) is then

$$\begin{aligned} & \mathcal{L}(C;\lambda,\mu) \\ = & \int_0^\infty e^{-t}U(C_t)dt + \lambda\left(x_0 - \int_0^1 C_t^{(1)}dt\right) + \mu\left(x_0+\Gamma - \int_0^1 C_t^{(1)}dt - \int_1^\infty C_t^{(2)}dt\right) \\ = & \int_0^1 \left[e^{-t}U(C^{(1)})_t - \lambda C_t^{(1)} - \mu C_t^{(1)}\right]dt + \int_1^\infty \left[e^{-t}U(C_t^{(2)}) - \mu C_t^{(2)}\right]dt \\ & +\lambda x_0 + \mu(x_0+\Gamma). \end{aligned}$$

Maximising over $c^{(1)}$ and $c^{(2)}$ gives $c^{(1)^*} = e^{-\frac{1}{R}t}(\lambda^*+\mu^*)^{-\frac{1}{R}}$ and $c^{(2)^*} = e^{-\frac{1}{R}t}\mu^{*-\frac{1}{R}}$, whence minimising the two constrains gives $(\lambda^*+\mu^*)^{-\frac{1}{R}} = (1-e^{-\frac{1}{R}})^{-1}\frac{x_0}{R}$ and $\mu^{*-\frac{1}{R}} = e^{\frac{1}{R}}\frac{\Gamma}{R}$. By solving λ^* and μ^*, we have the optimal consumption process C^*,

$$C_t^{(1)^*} = e^{-\frac{1}{R}t}\left(1-e^{-\frac{1}{R}}\right)^{-1}\frac{x_0}{R}, \qquad C_t^{(2)^*} = e^{-\frac{1}{R}(t-1)}\frac{\Gamma}{R}.$$

The corresponding wealth process for $t \in [0,1]$ is

$$X_t^* = x_0 - \int_0^t C_s^{(1)^*}ds = 1 - \left(1-e^{-\frac{1}{R}t}\right)\left(1-e^{-\frac{1}{R}}\right)^{-1},$$

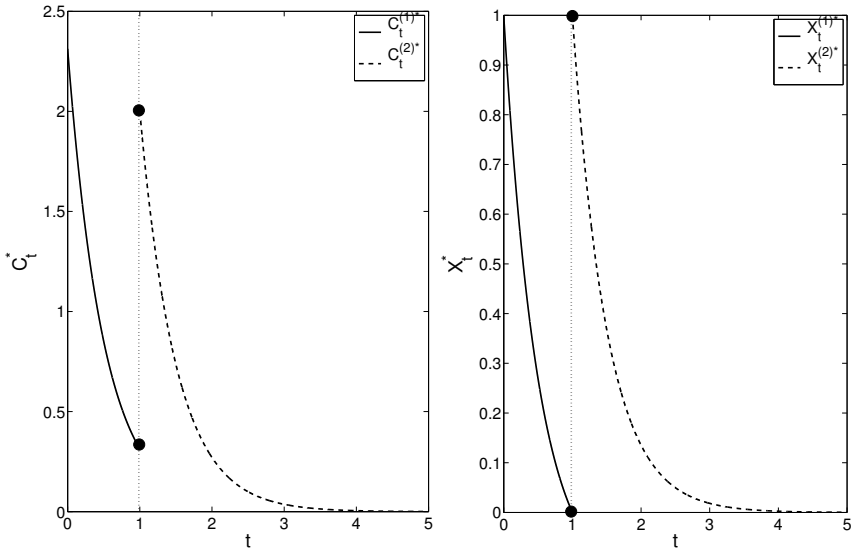

Figure 2.2: Plot of the optimal consumption process C^* and the corresponding wealth process X^* for $R < [\ln(1 + x_0/\Gamma)]^{-1}$. Here parameters are $x_0 = \Gamma = 1$, $q = 1$, $R = 2$. Both C^* and X^* are discontinuous at $t = 1$. In particular, $X^*_{1-} = 0$ and $X^*_{1+} = \Gamma$. Further, C^* and X^* are decreasing for $t \in [0, 1]$ and $t \in [1, \infty)$ respectively.

which implies $X_{1-} = 0$. Given $X_{1+} = X_{1-} + \Gamma = \Gamma$, the wealth process, for $t \in [1, \infty)$, is

$$X_t^* = X_{1+} - \int_1^t C_s^{(2)*} ds = \Gamma - e^{\frac{1}{R}} \frac{\Gamma}{R} \int_1^t e^{-\frac{1}{R}s} ds = e^{-\frac{1}{R}(t-1)}\Gamma.$$

Note that the optimal consumption and the corresponding wealth process are discontinuous at $t = 1$ due to the payment Γ. See Figure 2.2.

Finally, the value function is

$$\begin{aligned} V(x_0) &= \int_0^1 e^{-t} U\left(C_t^{(1)}\right) dt + \int_1^\infty e^{-t} U\left(C_t^{(2)}\right) dt \\ &= R^R \frac{x_0^{1-R}}{1-R} \left[\left(1 - e^{-\frac{1}{R}}\right)^R + e^{-1} \left(\frac{\Gamma}{x_0}\right)^{1-R} \right]. \end{aligned} \tag{2.21}$$

Denote by $V(x_0, \Gamma)$ the value function in (2.21). Then the indifference price is a solution to $V(x_0 + p, 0) = V(x_0, \Gamma)$, or equivalently

$$R^R \frac{(x_0 + p)^{1-R}}{1-R} = R^R \frac{x_0^{1-R}}{1-R} \left[\left(1 - e^{-\frac{1}{R}}\right)^R + e^{-1} \left(\frac{\Gamma}{x_0}\right)^{1-R} \right],$$

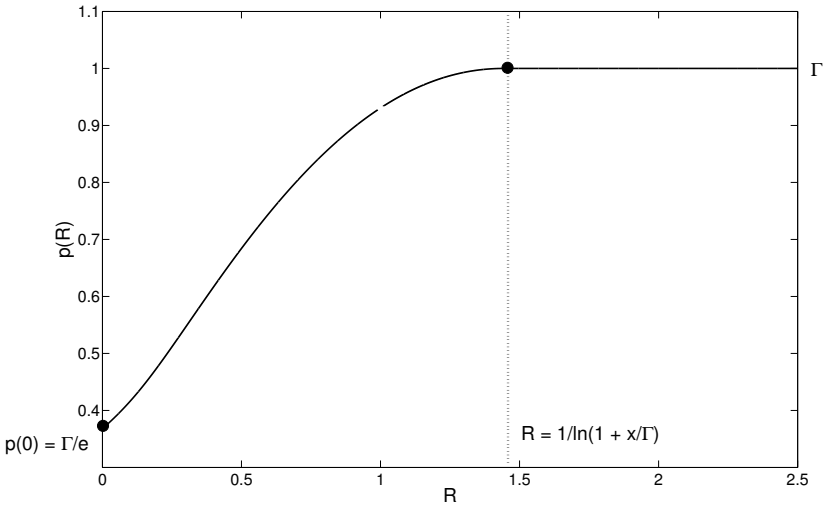

Figure 2.3: Plot of the indifference price p as a function of R. Here parameters are $x_0 = \Gamma = q = 1$. The indifference price p is continuous and increasing in R with $p(0) = e^{-1}\Gamma$ and $\lim_{R\to\infty} p(R) = \Gamma$.

which gives,

$$p(x_0) = x_0 \left\{ \left[\left(1 - e^{-\frac{1}{R}}\right)^R + e^{-1} \left(\frac{\Gamma}{x_0}\right)^{1-R} \right]^{\frac{1}{1-R}} - 1 \right\}.$$

Therefore the indifference price for $q = 1$ is

$$p(x_0) = \begin{cases} \Gamma, & \text{if } R \geq [\ln(1 + x_0/\Gamma)]^{-1}, \\ x_0 \left\{ \left[\left(1 - e^{-\frac{1}{R}}\right)^R + e^{-1} \left(\frac{\Gamma}{x_0}\right)^{1-R} \right]^{\frac{1}{1-R}} - 1 \right\}, & \text{if } R < [\ln(1 + x_0/\Gamma)]^{-1}. \end{cases} \tag{2.22}$$

Remark 2.2.1. Consider the indifference price p as a function of x_0 and R. From (2.22), we have $p(x_0, 0) = e^{-1}\Gamma$, which is the discounted payment of Γ. Define the critical value $R^* = [\ln(1 + x_0/\Gamma)]^{-1}$. Then we have $\lim_{R\to R^*-} p(x_0, R) = \lim_{R\to R^*+} p(x_0, R) = \Gamma$, implying p is continuous at R^*. See Figure 2.3.

For $R < R^*$ and $t \in [0, 1]$, the consumption is financed by initial wealth x_0 and the agent consumes in such a way that $X_{1-} = 0$. For $t \in [1, \infty)$, the consumption is then financed by the payment Γ received. For $R \geq R^*$ and $t \in [0, \infty)$, the consumption

is financed by all future cash flows, $(x_0 + \Gamma)$.

Remark 2.2.2. If the agent is allowed to borrow from future income, then wealth can be negative. Effectively, the first constraint in (2.19) no longer exists. Simple calculations give that the indifference price $p(x_0) = \Gamma$.

(3) Suppose $0 < q < 1$.

In this scenario, the agent receives cash amount Γ with probability q at $t = 1$. The objective in (2.18) can be rewritten as

$$\max_{C \in \mathcal{A}(x_0)} \mathbb{E}\left[\int_0^\infty e^{-s} U(C_s) ds\right] = \max_{C \in \mathcal{A}(x_0)} \mathbb{E}\left\{\int_0^1 e^{-s} U(C_s) ds + \int_1^\infty e^{-s} U(C_s) ds\right\}.$$

Define functions H and G by

$$H(x) = \max_{C \in \mathcal{A}(x_0)} \mathbb{E}\left[\int_1^\infty e^{-s} U(C_s) ds\right], \tag{2.23}$$

$$G(x) = \max_{C \in \mathcal{A}(x_0)} \mathbb{E}\left[\int_1^\infty e^{-s} U(C_s) ds \middle| X_{1-} = x, X_{1+} = x + \tilde{\Gamma}\right], \tag{2.24}$$

where the random variable $\tilde{\Gamma}$ satisfies $\mathbb{P}(\tilde{\Gamma} = 0) = 1 - q$ and $\mathbb{P}(\tilde{\Gamma} = \Gamma) = q$. Then we have

$$G(x) = \mathbb{E}[H(x + \tilde{\Gamma})] = qH(x + \Gamma) + (1 - q)H(x).$$

Consider first the maximisation problem (2.23) subject to $\int_1^\infty C_t dt \leq x$. Introducing the multiplier $\lambda > 0$, the Lagrangian for (2.23) is then

$$\int_1^\infty e^{-s} \frac{C_s^{1-R}}{1-R} ds + \lambda \left(x - \int_1^\infty C_s ds\right).$$

By the same approach in the last two scenarios, we have $C_t^* = e^{-(t-1)/R} x/R$, which is the same as the optimal consumption process in scenario (1), where the agent does not receive cash amount Γ. The corresponding wealth process is $X_t^* = e^{-(t-1)/R} x$ and

$$H(x) = \int_1^\infty e^{-s} \frac{C_s^{*1-R}}{1-R} ds = e^{-1} \frac{x^{1-R}}{1-R} R^R. \tag{2.25}$$

Now consider $H(x+\Gamma)$. Followed by the same argument, we have $C_t^* = e^{-(t-1)/R}(x+\Gamma)/R$, which is the same as the optimal consumption process in scenario (2) for $R \geq [\ln(1 + x_0/\Gamma)]^{-1}$, where the agent receives cash amount Γ. The corresponding wealth

process is $X_t^* = e^{-(t-1)/R}(x+\Gamma)$, and

$$H(x+\Gamma) = e^{-1}\frac{(x+\Gamma)^{1-R}}{1-R}R^R. \tag{2.26}$$

Provided $H(x)$ and $H(x+\Gamma)$ in (2.25) and (2.26), we have

$$G(x) = qH(x+\Gamma) + (1-q)H(x) = e^{-1}\frac{R^R}{1-R}\left[q(x+\Gamma)^{1-R} + (1-q)x^{1-R}\right].$$

Then from (2.24), the objective in (2.18) becomes

$$\max_{C\in\mathcal{A}(x_0)}\left\{\int_0^1 e^{-s}\frac{C_s^{1-R}}{1-R}ds + G(X_1)\right\}, \tag{2.27}$$

subject to $X_1 = x_0 - \int_0^1 C_s ds$.

Introducing the multiplier $\lambda > 0$, the Lagrangian for the maximisation problem (2.27) is

$$\mathcal{L}(C, X_1; \lambda) = \int_0^1 \left[e^{-s}\frac{C_s^{1-R}}{1-R} - \lambda C_s\right]ds + G(X_1) - \lambda X_1 + \lambda x_0.$$

Maximising over c and x_1, we have $G'(x_1^*) = \lambda^*$ which gives

$$G'(x_1^*) = e^{-1}R^R\left\{q(x_1^*+\Gamma)^{-R} + (1-q)x_1^{*-R}\right\} = \lambda^*, \tag{2.28}$$

and $c^* = e^{-t/R}\lambda^{*-1/R}$ which gives, by substituting into the constraint,

$$x_0 - x_1^* = \lambda^{*-\frac{1}{R}}R(1-e^{-\frac{1}{R}}). \tag{2.29}$$

The next step is to find x_1^* and λ^* by solving (2.28) and (2.29), and the optimal C^* can be represented via λ^*. Let $\gamma^* = \lambda^{*1/R}$. Substituting (2.29) into (2.28), we then have γ^* is a solution to

$$q\left[e^{\frac{1}{R}}\gamma\frac{x_0+\Gamma}{R} - (e^{\frac{1}{R}}-1)\right]^{-R} + (1-q)\left[e^{\frac{1}{R}}\gamma\frac{x_0}{R} - (e^{\frac{1}{R}}-1)\right]^{-R} = 1. \tag{2.30}$$

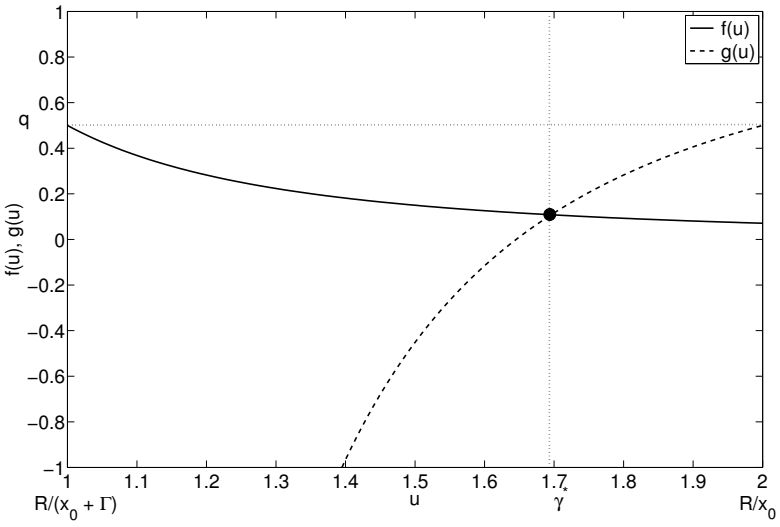

Figure 2.4: Plot of functions f, g and their intersection γ^*. Here parameters are $x_0 = \Gamma = 1$, $q = 0.5$, $R = 0.5$. From the graph, f is decreasing and g is increasing on $[R/(x_0+\Gamma), R/x_0]$ with boundary conditions $f(R/(x_0+\Gamma)) = g(R/x_0) = q$.

Define functions f and g such that

$$\begin{aligned} f(u) &= q\left[\frac{e^{\frac{1}{R}}(x_0+\Gamma)}{R}u - (e^{\frac{1}{R}}-1)\right]^{-R}, \\ g(u) &= 1-(1-q)\left[e^{\frac{1}{R}}\frac{x_0}{R}u - (e^{\frac{1}{R}}-1)\right]^{-R}. \end{aligned}$$

Then γ^* is the intersection between $f(u)$ and $g(u)$. Recall that $\gamma^* = \lambda^{*1/R} = e^{-t/R}/c^*$ and $c^* \in [e^{-t/R}x_0/R, e^{-t/R}(x_0+\Gamma)/R]$, where the upper and lower bounds correspond to the optimal consumptions in the scenarios with and without income Γ. This implies that $\gamma^* \in [R/(x_0+\Gamma), R/x_0]$.

Since f is decreasing in u and g is increasing in u with the following boundary conditions

$$f\left(\frac{R}{x_0+\Gamma}\right) = q, \qquad g\left(\frac{R}{x_0+\Gamma}\right) < q, \qquad f\left(\frac{R}{x_0}\right) < q, \qquad g\left(\frac{R}{x_0}\right) = q,$$

we then have f and g always intersects, or equivalently there always exists some γ^* as a solution to (2.30) on $[R/(x_0+\Gamma), R/x_0]$. See Figure 2.4.

Hence for $t \in [0,1]$, the optimal consumption process is $C_t^* = e^{-t/R}/\gamma^*$ and the

corresponding wealth process is $X_t^* = x_0 - (1 - e^{-t/R})R/\gamma^*$. The value function is

$$\begin{aligned}\int_0^1 e^{-s}\frac{C_s^{*1-R}}{1-R}ds + G(X_1^*) &= \frac{R\gamma^{*R-1}}{1-R}\Bigg\{1 - e^{-\frac{1}{R}} + qe^{-1}\left[\frac{x_0+\Gamma}{R}\gamma^* - \left(1 - e^{-\frac{1}{R}}\right)\right]^{1-R} \\ &\quad + e^{-1}(1-q)\left[\frac{x_0}{R}\gamma^* - \left(1 - e^{-\frac{1}{R}}\right)\right]^{1-R}\Bigg\}.\end{aligned}$$

Denote by $V(x_0, \Gamma)$ the value function above. Equating the value functions $V(x_0 + p, 0)$ and $V(x_0, \Gamma)$, we have the indifference price p solve,

$$\begin{aligned}\frac{(x_0+p)^{1-R}}{1-R}R^R &= \frac{R\gamma^{*R-1}}{1-R}\Bigg\{1 - e^{-\frac{1}{R}} + qe^{-1}\left[\frac{x_0+\Gamma}{R}\gamma^* - \left(1 - e^{-\frac{1}{R}}\right)\right]^{1-R} \\ &\quad + e^{-1}(1-q)\left[\frac{x_0}{R}\gamma^* - \left(1 - e^{-\frac{1}{R}}\right)\right]^{1-R}\Bigg\},\end{aligned}$$

which gives the expression of the indifference price p

$$\begin{aligned}p(x_0) &= \frac{R}{\gamma^*}\Bigg\{1 - e^{-\frac{1}{R}} + qe^{-1}\left[\frac{x_0+\Gamma}{R}\gamma^* - \left(1 - e^{-\frac{1}{R}}\right)\right]^{1-R} \\ &\quad + e^{-1}(1-q)\left[\frac{x_0}{R}\gamma^* - \left(1 - e^{-\frac{1}{R}}\right)\right]^{1-R}\Bigg\}^{\frac{1}{1-R}} - x_0. \qquad (2.31)\end{aligned}$$

Note that (2.31) is the same formula of the indifference price (2.22) in the case $q = 1$. For $q = 1$, we have $\gamma^* = R/(x_0 + \Gamma)$ simply by solving (2.30) with $q = 1$. Then the indifference price by (2.31) gives $p(x_0) = \Gamma$. Further, if borrowing is allowed, then wealth is allowed to be negative. The indifference price by (2.31) also gives $p(x_0) = \Gamma$.

The optimal consumption process and the corresponding wealth process are

$$C_t^* = \begin{cases} e^{-\frac{1}{R}}\frac{1}{\gamma^*}, & \text{if } t \in [0,1]; \\ e^{-\frac{1}{R}(t-1)}\frac{x_0 - R(1-e^{-\frac{1}{R}})\gamma^{*-1}}{R}, & \text{if } t \in [1,\infty) \text{ and without } \Gamma; \\ e^{-\frac{1}{R}(t-1)}\frac{x_0 + \Gamma - R(1-e^{-\frac{1}{R}})\gamma^{*-1}}{R}, & \text{if } t \in [1,\infty) \text{ and with } \Gamma, \end{cases}$$

$$X_t^* = \begin{cases} x_0 - (1 - e^{-\frac{1}{R}t})\frac{R}{\gamma^*}, & \text{if } t \in [0,1]; \\ e^{-\frac{1}{R}(t-1)}\left\{x_0 - R\left(1 - e^{-\frac{1}{R}}\right)\gamma^{*-1}\right\}, & \text{if } t \in [1,\infty) \text{ and without } \Gamma; \\ e^{-\frac{1}{R}(t-1)}\left\{x_0 + \Gamma - R\left(1 - e^{-\frac{1}{R}}\right)\gamma^{*-1}\right\}, & \text{if } t \in [1,\infty) \text{ and with } \Gamma. \end{cases}$$

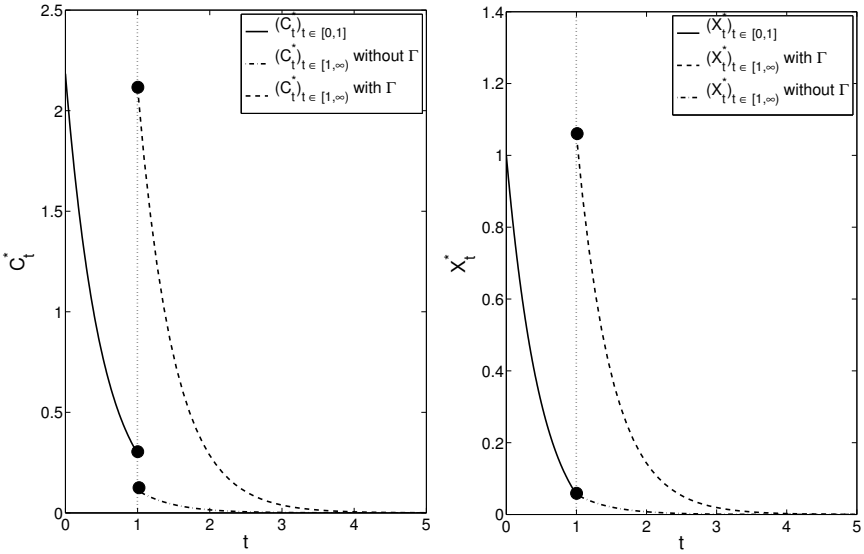

Figure 2.5: Plot of the optimal consumption process C^* and the corresponding wealth process X^* for $0 < q < 1$. Here parameters are $x_0 = \Gamma = 1$, $q = 0.5$, $R = 0.5$. From the graph, if the payment Γ is not received then X^* is continuous at $t = 1$. Otherwise, both C^* and X^* are discontinuous at $t = 1$. Further, C^* and X^* are decreasing on $[0, 1]$ and $[1, \infty)$ respectively.

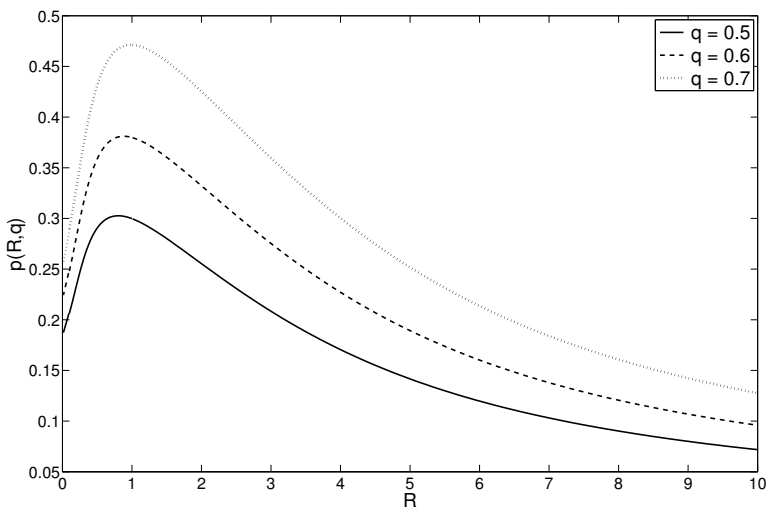

Figure 2.6: Plot of the indifference price p as a function of R. Here parameters are $x_0 = \Gamma = 1$. By fixing q, the indifference price p is initially increasing, but then decreasing in R with $p(0, q) = e^{-1}q\Gamma$. By fixing R, the indifference price p is increasing in q.

2.2.3 Interpretations of the results

Similar to the deterministic case when $q = 1$ and $R < [\ln(1 + x_0/\Gamma)]^{-1}$ in scenario 2, we have the optimal consumption and the corresponding wealth process are not necessarily continuous for $0 < q < 1$ in scenario 3. See Figure 2.2 and 2.5. In particular, there is a jump in the sale time $t = 1$. This is resulted from the uncertainty for $t \in [0, 1)$ brought about by the one-off payment Γ. The corresponding wealth process varies depending on whether or not Γ is received, and hence the optimal consumption financed by wealth is not necessarily continuous. Further, since there is no income process in the model except for a possible Γ cash amount at $t = 1$, the wealth process is decreasing for $t \in [0, 1]$ and $t \in [1, \infty)$ respectively.

The results in scenario 2 and scenario 3 are an example to show that even in the deterministic case ($q = 1$ in scenario 2), the optimal consumption and the corresponding wealth processes are not necessarily continuous and can have a jump in the sale time. However, for models in the following chapters, there is no such uncertainty as the setting for Γ here and the optimal consumption and the resulting wealth processes are continuous over time.

The indifference price p is plotted in Figure 2.3 and 2.6 as a function of R with $q = 1$ and $0 < q < 1$. Naively we might expect the price to be monotone decreasing in risk aversion, since a more risk averse agent will assign a lower value to a risky asset. However, the results show that this is not the case. In the deterministic case ($q = 1$ in scenario 2) as plotted in Figure 2.3, the indifference price is increasing in risk aversion. By fixing $q \in (0, 1)$, as plotted in Figure 2.6, the indifference price is increasing in R for small risk aversions and decreasing for large risk aversions.

Here is an explanation of the non-monotonicity illustrated in Figure 2.3. Recall the optimal strategy in scenario 1 ($q = 0$), in which the agent does not receive Γ at $t = 1$. The optimal consumption rate of the agent is $C^* = x_0/R$; in particular, as the parameter R increases, the agent consumes more slowly. If $R = 0$, the utility function is a linear function of consumption and the discount factor dominates risk aversion, which makes the agent consume all wealth immediately. The indifference price of the payment Γ is then the expected discounted amount from time $t = 1$, which is $e^{-1}\Gamma$. As R increases, the agent postpones consumption to mitigate risk and hence the value of the payment Γ increases due to the discount factor. A similar argument explains the phenomenon in Figure 2.6. For $R = 0$, the indifference price of Γ is the expected discounted amount $e^{-1}q\Gamma$. If $R \ll 0$, the agent consumes now and there is no opportunity to make consumption depend on whether or not the agent receives Γ or not. As R increases, the agent worries about

the worst case - the payment Γ is not received at $t = 1$. Hence, the agent postpones consumption in order to mitigate this risk and the discount factor makes the value of Γ increase. As R becomes large ($R \gg 1$), risk aversion dominates the discount factor and an agent with a large risk aversion worries about the worst situation only and hence the value of the payment Γ decreases.

Figure 2.6 also shows that p is monotone increasing in q. As q increases, the agent is more likely to receive the payment and hence Γ is more valuable.

The result that the indifference price is not monotone decreasing in risk aversion is surprising. However, we are able to show that it is not monotone in risk aversion in scenario 3 as plotted in Figure 2.6 and it can even be increasing in risk aversion in a deterministic problem (scenario 2) as plotted in Figure 2.3. This surprising result will appear again in Figure 4.17 and 4.18 in Chapter 4 and Figure 6.9 in Chapter 6, in which we consider the full problems.

Chapter 3

Optimal consumption/sale strategies for a risk averse agent

This chapter considers an individual who is endowed with cash and units of an infinitely divisible asset, which can be sold but not dynamically traded, and who aims to maximise the expected discounted utility of consumption over an infinite horizon. The problem facing the individual is to choose the optimal strategy for the liquidation of the endowed asset portfolio, and an optimal consumption process chosen to keep cash wealth non-negative. The price process of the endowed asset is assumed to follow an exponential Brownian motion and the agent is assumed to have constant relative risk aversion. As such this problem is a modification of the Merton [41] optimal consumption/optimal portfolio problem which is discussed in Section 1.2 with one risky asset ($N = 1$).

The constraint that the asset can be sold but not bought is equivalent to an assumption of no transaction costs on sales, and an infinite transaction cost on purchases. The assumption of no transaction cost on sales can easily be relaxed to a proportional transaction cost on sales by working with a process representing the post-transaction-cost price rather than the pre-cost price. In this sense the problem we consider can be interpreted as a special case of the Davis-Norman problem under Merton's model with transaction costs in which the transaction cost associated with buying the endowed asset is infinite.

There are at least three main reasons for considering this special case. Firstly, there are often situations whereby agents are endowed with units of assets which they may sell but may not repurchase, whether for legal reasons or because of other trading restrictions. In this sense the problem is interesting in its own right, and as we show, the model has some counter-intuitive features. Secondly, several papers provide insights into

this special case of the Davis-Norman problem (infinite transaction cost to purchase and zero transaction cost to sell the risky asset) but without formal mathematical results. Øksendal *et al* [20] consider this special case as one topic of future research and provide economic intuitions of the solutions. Shreve and Soner [52] also provide their insights into this special case in their remark. Thirdly, relative to the Constantanides-Magill-Davis-Norman model new solution techniques become available which we are able to exploit to give a more complete solution to the problem. With this more complete solution we can investigate the comparative statics of the problem.

Our main results are of three types. Firstly we are able to completely classify the different types of optimal strategies and the parameter ranges over which they apply. Secondly, we can simplify the problem of solving for the value function, especially when compared with direct approaches for solving the HJB equation via smooth fit. Thirdly, we can perform comparative statics on quantities of interest, and uncover some surprising implications of the model. The comparative statics of this problem will be discussed separately in the following chapter.

Some of our main results are as follows.

Result 1. *If the endowed asset is depreciating over time then the investor should sell immediately. Conversely, if the mean return is too strong and the coefficient of relative risk aversion is less than unity, then the problem is ill-posed, and provided the initial holding of the endowed asset is positive the value function is infinite.*

Otherwise, there are two cases. For small and positive mean return there exists a finite critical ratio and the optimal sale strategy for the endowed asset is to sell just enough to keep the ratio of wealth held in the endowed asset to cash wealth below this critical ratio. For larger returns it is optimal to first consume all cash wealth, and once this cash wealth is exhausted to finance consumption through sales of the endowed asset.

Result 2. *In the case where the critical ratio is finite then it is given via the solution of a first crossing problem for a first-order initial-value ordinary differential equation (ODE). Other quantities of interest can be expressed in terms of the solution of this ODE. In the case where the critical ratio is infinite, the value function can again be expressed in terms of the solution of a first-order ODE.*

We work with bond as numéraire (so that interest rate effects can be ignored) and then the relevant parameters are the discount parameter and the relative risk aversion of the agent, and the drift and volatility of the price process of the risky asset. In the non-degenerate parameter cases the agent faces a conflict between the incentive to keep a large holding in the risky asset (since it has a positive return) and the incentive to sell

in order to minimise risk exposure. From the homothetic property we expect decisions to depend on the ratio between the value of the holdings of risky asset and cash wealth.

The HJB equation for our problem is second order, non-linear and subject to value matching and smooth fit of the first and second derivatives at an unknown free-boundary. One of our key contributions is to show that the problem can be reduced to a crossing problem for the solution of a first order ODE. This big simplification is useful both when considering analytical properties of the solution, and when trying to construct a solution numerically. We classify the parameter combinations which lead to different types of solutions and provide a thorough analysis of the existence and finiteness of the critical ratio, and the corresponding optimal strategies. In particular, there are two types of degenerate solution (in one case it is always optimal to liquidate all units of the risky asset immediately, and in the other the value function is infinite and the problem is ill-posed). In addition there are two different types of non-degenerate behaviour (in one case the agent sells units of asset in order to keep the proportion of wealth held in the risky asset below a certain level, and in the other the agent exhausts all her cash reserves before selling any units of the risky asset.) In the case of a finite and positive critical ratio we show how the solution to the problem can be characterised by an autonomous one-dimensional diffusion process with reflection and its local time.

The structure of this chapter is as follows. In Section 3.1, we give a precise description of the model and then a statement of the main results. In Section 3.2, we prove the properties of the value functions in different scenarios and the corresponding verification arguments. Section 3.3 considers the properties of the solution to the first order ODE and the existence and finiteness of the critical ratio. In Section 3.4 and Section 3.5 we further consider the martingale properties of the value function and extend the proofs to the case $R > 1$.

3.1 The model and main results

We work on a filtered probability space $\left(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}\right)$ such that the filtration satisfies the usual conditions and is generated by a standard Brownian motion $B = (B_t)_{t\geq 0}$. The price process $Y = (Y_t)_{t\geq 0}$ of the endowed asset is assumed to be given by

$$Y_t = y_0 \exp\left[\left(\alpha - \frac{\eta^2}{2}\right) t + \eta B_t\right], \tag{3.1}$$

where α and $\eta > 0$ are the constant mean return and volatility of the non-traded asset, and y_0 is the initial price.

Let $C = (C_t)_{t\geq 0}$ denote the consumption rate of the individual and let $\Theta = (\Theta_t)_{t\geq 0}$ denote the number of units of the endowed asset held by the investor. The process Θ is required to be progressively measurable, right-continuous with left limits (RCLL) and non-increasing to reflect the fact that the non-traded asset is only allowed for sale. We assume the initial number of shares held by the investor is θ_0. Since we allow for an initial transaction at time 0 we may have $\Theta_0 < \theta_0$. We write $\Theta_{0-} = \theta_0$. This is consistent with our convention that Θ is right-continuous.

We denote by $X = (X_t)_{t\geq 0}$ the wealth process of the individual, and suppose that the initial wealth is x_0 where $x_0 \geq 0$. Provided the only changes to wealth occur from either consumption or from the sale of the endowed asset, X evolves according to

$$dX_t = -C_t dt - Y_t d\Theta_t, \tag{3.2}$$

subject to $X_{0-} = x_0$, and $X_0 = x_0 + y_0(\theta_0 - \Theta_0)$. We say a consumption/sale strategy pair is admissible if the components satisfy the requirements listed above and if the resulting cash wealth process X is non-negative for all time. Let $\mathcal{A}(x_0, y_0, \theta_0)$ denote the set of admissible strategies for initial setup $(X_{0-} = x_0, Y_0 = y_0, \Theta_{0-} = \theta_0)$.

Remark 3.1.1. The setup of the problem coincides with the framework by Davis and Norman introduced in Section 1.4. Denote $\hat{Y} = Y\Theta$ by the wealth in the risky asset. Then from (1.15) and (3.2), we have $dU_t = -Y_t d\Theta_t$. Further, by Itô's formula, we have

$$\begin{aligned} d\hat{Y}_t &= d(Y_t\Theta_t) \\ &= \Theta_t dY_t + Y_t d\Theta_t \\ &= \Theta_t Y_t \frac{dY_t}{Y_t} - dU_t \\ &= \hat{Y}(\alpha dt + \eta dB_t) - dU_t, \end{aligned}$$

which coincides with (1.16) in the special case of infinite transaction costs.

The objective of the agent is to maximise over admissible strategies the discounted expected utility of consumption over the infinite horizon, where the discount factor is β and the utility function of the agent is assumed to be CRRA with relative risk aversion $R \in (0, \infty) \setminus 1$. In particular, the goal is to find

$$\sup_{(C,\Theta)\in\mathcal{A}(x_0,y_0,\theta_0)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{1-R}}{1-R}dt\right]. \tag{3.3}$$

Since the set-up has a Markovian structure, we expect the value function, opti-

mal consumption and optimal sale strategy to be functions of the current wealth and endowment of the agent and of the price of the risky asset. Let $V = V(x, y, \theta, t)$ be the forward starting value function for the problem so that

$$V(x, y, \theta, t) = \sup_{(C,\Theta)\in\mathcal{A}(x,y,\theta,t)} \mathbb{E}\left[\int_t^\infty e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds \middle| X_{t-} = x, Y_t = y, \Theta_{t-} = \theta\right]. \quad (3.4)$$

Here the space of forward starting, admissible strategies $\mathcal{A}(x, y, \theta, t)$ is such that $C = (C_s)_{s\geq t}$ is a non-negative progressively measurable process, $\Theta = (\Theta_s)_{s\geq t}$ is a right-continuous, decreasing and progressively measurable process and satisfies $\Theta_t - (\Delta\Theta)_t = \theta$, and X given by $X_s = x - \int_t^s C_u du - \int_{[t,s]} Y_u d\Theta_u$ is non-negative.

Define the certainty equivalent value $p = p(x, y, \theta, t)$ of the holdings of the risky asset to be the solution to

$$V(x + p, y, 0, t) = V(x, y, \theta, t). \quad (3.5)$$

In fact, by the scalings of the problem it will turn out that p is independent of time (and henceforth we write $p = p(x, y, \theta)$), and depends on the price y of the risky asset and the quantity θ of the holdings in the risky asset, only through the product $y\theta$.

Our goal is to characterise the value function, the optimal consumption and sale strategies, and the certainty equivalent price p.

The key to the form of the solution to the problem is contained in the following proposition, which concerns the solution of an ODE on $[0, 1)$ and which is proved in Section 3.3. There is a one-to-one correspondence between the four cases in the proposition and the four types of solution to the optimal sale problem.

Let $\epsilon = \alpha/\beta$ and $\delta^2 = \eta^2/\beta$.

Proposition 3.1.2. *For $q \in [0, 1]$ define $m(q) = 1 - \epsilon(1 - R)q + \frac{\delta^2}{2}R(1 - R)q^2$ and $\ell(q) = 1 + \left(\frac{\delta^2}{2} - \epsilon\right)(1 - R)q - \frac{\delta^2}{2}(1 - R)^2q^2 = m(q) + q(1 - q)\frac{\delta^2}{2}(1 - R)$. Let $n = n(q)$ solve*

$$\frac{n'(q)}{n(q)} = \frac{1-R}{R(1-q)} - \frac{\delta^2}{2}\frac{(1-R)^2}{R}\frac{q}{\ell(q) - n(q)} \quad (3.6)$$

subject to $n(0) = 1$ and $\frac{n'(0)}{1-R} < \frac{\ell'(0)}{1-R} = \frac{\delta^2}{2} - \epsilon$. Suppose that if n hits zero, then 0 is absorbing for n. See Figure 3.1.

For $R < 1$, let $q^* = \inf\{q > 0 : n(q) \leq m(q)\}$. For $R > 1$, let $q^* = \inf\{q > 0 : n(q) \geq m(q)\}$. For $j \in \{\ell, m, n\}$ let $q_j = \inf\{q > 0 : j(q) = 0\} \wedge 1$.

1. *Suppose $\epsilon \leq 0$. Then $q^* = 0$.*

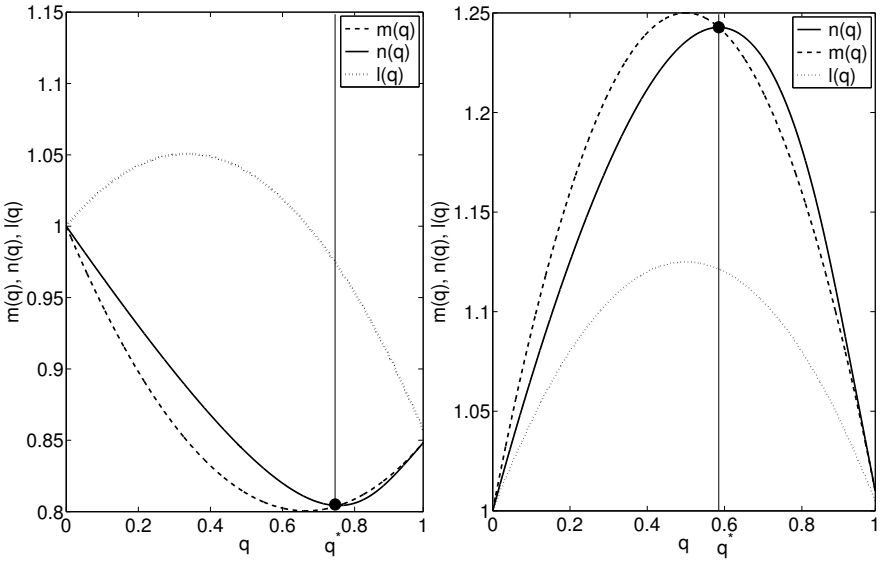

Figure 3.1: Stylised plot of $m(q)$, $n(q)$, $\ell(q)$ and q^*. Parameters are chosen to satisfy the conditions in the second case of Proposition 1 so that $q^* \in (0,1)$. The left figure is in the case $R < 1$ and the right figure $R > 1$.

2. *Suppose* $0 < \epsilon < \delta^2 R$ *and if* $R < 1$, *suppose in addition that* $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$. *Then* $0 < q^* < 1$.

3. *Suppose* $\epsilon \geq \delta^2 R$ *and if* $R < 1$, $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$. *Then* $q^* = 1 = q_\ell = q_n = q_m$.

4. *Suppose* $R < 1$ *and* $\epsilon > \frac{\delta^2}{2}R + \frac{1}{1-R}$. *Then* $q_m < q_n = q_\ell < 1$. *If* $R < 1$, $\epsilon = \frac{\delta^2}{2}R + \frac{1}{1-R}$ *and* $\epsilon < \delta^2 R$ *then* $q_m < q_n = q_\ell = 1$. *If* $R < 1$, $\epsilon = \frac{\delta^2}{2}R + \frac{1}{1-R}$ *and* $\epsilon \geq \delta^2 R$ *then* $q^* = 1 = q_\ell = q_n = q_m$.

Remark 3.1.3. Note that the condition $\epsilon < \delta^2 R$ is equivalent to $(1-R)m'(1) > 0$. Further, if $R < 1$, then the condition $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$ is equivalent to $m(1) > 0$. Also, n has a turning point at $q^* < 1$ if and only if $n(q^*) = m(q^*)$. See Figure 3.1. In particular, if m is monotone (and $\epsilon > 0$) then $q^* = 1$. Then, if $R < 1$, $0 < \epsilon < \delta^2 R$ and $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$, we have $q_\ell = q_n = 1$.

Remark 3.1.4. It is easy to see that $(1-R)n$ is decreasing in ϵ. In fact it can also be shown that over parameter ranges where $0 < q^* < 1$ then q^* is increasing in ϵ.

Theorem 3.1.5 divides the parameter space into the four distinct regions. In particular, it distinguishes the degenerate cases, and it gives necessary and sufficient conditions for the two different regimes in the non-degenerate case.

Theorem 3.1.5. *1. Suppose $\epsilon \leq 0$. Then it is always optimal to sell the entire holding of the endowed asset immediately, so that $\Theta_t = 0$ for $t \geq 0$. The value function for the problem is $V(x, y, \theta, t) = (R/\beta)^R e^{-\beta t}(x + y\theta)^{1-R}/1 - R$; and the certainty equivalent value of the holdings of the asset is $p(x, y, \theta) = y\theta$.*

2. Suppose $0 < \epsilon < \delta^2 R$ and $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$ if $R < 1$. Then there exists a positive and finite critical ratio z^* and the optimal behaviour is to sell the smallest possible quantity of the risky asset which is sufficient to keep the ratio of wealth in the risky asset to cash wealth at or below the critical ratio. If $\theta > 0$ then $p(x, y, \theta) > y\theta$.

3. Suppose $\epsilon \geq \delta^2 R$ and $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$ if $R < 1$. Then the optimal consumption and sale strategy is first to consume liquid (cash) wealth, and then when this liquid wealth is exhausted, to finance further consumption from sales of the illiquid asset. If $\theta > 0$ then $p(x, y, \theta) > y\theta$.

4. Suppose $R < 1$ and $\epsilon \geq \frac{\delta^2}{2}R + \frac{1}{1-R}$. Then the problem is degenerate, and provided θ is positive, the value function $V = V(x, y, \theta, t)$ is infinite. There is no unique optimal strategy, and the certainty equivalent value p is not defined.

Remark 3.1.6. In light of Proposition 3.1.2 there is one fewer case for $R > 1$. The fourth case in the theorem does not happen for $R > 1$ since the value function is always finite, as in Merton's problem introduced in Chapter 1.

Similarly, when $R < 1$, if $\delta^2 \geq 2/(R(1 - R))$ then the third case above does not happen. In that case, as ϵ increases we move directly from $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$ and a finite value function and z^* to $\epsilon \geq \frac{\delta^2}{2}R + \frac{1}{1-R}$ and an infinite value function.

The second and third cases above are non-degenerate and they are further characterised in Theorem 3.1.7 and Theorem 3.1.11. In Theorem 3.1.7 the solution is expressed in terms of a one-dimensional autonomous reflecting stochastic process J and its local time at zero L, see (3.13).

For $0 \leq q \leq q^*$ define $N(q) = n(q)^{-R}(1 - q)^{R-1}$ where n is the solution to (3.6). Assuming that N is monotonic, let W be inverse to N. Let $h^* = N(q^*)$. Then $W(h^*) = q^*$, and $h^*(1 - q^*)^{1-R} = m(q^*)^{-R}$.

Theorem 3.1.7. *i) Suppose $R < 1$. Suppose $0 < \epsilon < \delta^2 R$ and $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$ so that $0 < q^* < 1$. Then N as defined above is increasing, and W is well defined.*

Let z^* be given by

$$z^* = (1 - q^*)^{-1} - 1 = \frac{q^*}{1 - q^*} \in (0, \infty). \tag{3.7}$$

On $[1, h^*]$ *let* h *be the solution of*

$$u^* - u = \int_h^{h^*} \frac{1}{(1-R) f W(f)} df, \tag{3.8}$$

where $u^* = \ln z^*$. *Let* g *be given by*

$$g(z) = \begin{cases} \left(\frac{R}{\beta}\right)^R m(q^*)^{-R} (1+z)^{1-R} & z \in [z^*, \infty); \\ \left(\frac{R}{\beta}\right)^R h(\ln z) & z \in (0, z^*]. \end{cases} \tag{3.9}$$

Then, the value function V *is given by*

$$V(x, y, \theta, t) = e^{-\beta t} \frac{x^{1-R}}{1-R} g\left(\frac{y\theta}{x}\right), \qquad x > 0, \theta > 0 \tag{3.10}$$

and we can extend this to $x = 0$ *and* $\theta = 0$ *by continuity to give*

$$V(x, y, 0, t) = e^{-\beta t} \frac{x^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R \tag{3.11}$$

$$V(0, y, \theta, t) = e^{-\beta t} \frac{y^{1-R}\theta^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R m(q^*)^{-R} \tag{3.12}$$

Fix $z_0 = y_0\theta_0/x_0$. *Let* $(J, L) = (J_t, L_t)_{t \geq 0}$ *be the unique pair such that*

(a) J *is positive,*

(b) L *is increasing, continuous,* $L_0 = 0$, *and* dL_t *is carried by the set* $\{t : J_t = 0\}$,

(c) J *solves*

$$J_t = (z^* - z_0)^+ - \int_0^t \tilde{\Lambda}(J_s) ds - \int_0^t \tilde{\Gamma}(J_s) dB_s + L_t, \tag{3.13}$$

where $\Lambda(z) = \alpha z + z\left(g(z) - \frac{1}{1-R} z g'(z)\right)^{-1/R}$, $\Gamma(z) = \eta z$, $\tilde{\Lambda}(j) = \Lambda(z^* - j)$ *and* $\tilde{\Gamma}(j) = \Gamma(z^* - j)$.

For such a pair $0 \leq J_t \leq z^*$.

If $z_0 \leq z^*$ *then set* $\Theta_0^* = \theta_0$ *and* $X_0^* = x_0$; *else if* $z_0 > z^*$ *then set*

$$\Theta_0^* = \theta_0 \frac{z^*}{(1+z^*)} \frac{(1+z_0)}{z_0}$$

and $X_0^* = x_0 + y_0(\theta_0 - \Theta_0)$. *This corresponds to the sale of a positive quantity* $\theta_0 - \Theta_0$ *of units of the endowed asset at time* 0.

Then, the optimal holdings Θ_t^* of the endowed asset, the optimal consumption process $C_t^* = C(X_t^*, Y_t, \Theta_t^*)$, the resulting wealth process and the certainty equivalent value are given by

$$\Theta_t^* = \Theta_0^* \exp\left\{-\frac{1}{z^*(1+z^*)} L_t\right\}; \tag{3.14}$$

$$X_t^* = \frac{Y_t \Theta_t^*}{(z^* - J_t)}; \tag{3.15}$$

$$C(x, y, \theta) = x\left[g\left(\frac{y\theta}{x}\right) - \frac{1}{1-R}\frac{y\theta}{x} g'\left(\frac{y\theta}{x}\right)\right]^{-\frac{1}{R}}; \tag{3.16}$$

$$p(x, y, \theta) = x\left[\frac{g\left(\frac{y\theta}{x}\right)}{g(0)}\right]^{\frac{1}{1-R}} - x. \tag{3.17}$$

ii) Now suppose $R > 1$ and $0 < \epsilon < \delta^2 R$ so that $0 < q^* < 1$. Let all quantities be defined as before. Then N is decreasing. On $(h^*, 1)$ h is defined via

$$u^* - u = \int_{h^*}^{h} \frac{1}{(R-1) f W(f)} df.$$

The value function, the optimal holdings Θ^*, the optimal consumption process C^*, the resulting wealth process X^* and the certainty equivalent value p are the same as before.

Remark 3.1.8. In order to derive functions m and n, we firstly consider the variable $z = y\theta/x$ and construct the value function as $V = e^{-\beta t} x^{1-R} g(z)/(1-R)$. It then gets down to a nonlinear, non-autonomous second-order ordinary differential equation of a single variable z. By letting $u = \ln z$, we have an autonomous differential equation of a new function h of u. Further, setting $w(h) = dh/du$ simplifies the second-order differential equation into a first-order one. Denote Q by the function of smooth fit points and $W(h) = w(h)/((1-R)h)$. In order to examine the crossings between W and Q, we look for the crossings between the inverse functions $N = W^{-1}$ and $M = Q^{-1}$ thanks to the monotonicity of W and Q. After eliminating scaling factors, functions m and n are constructed.

Remark 3.1.9. Recall that n solves the first order differential equation (3.6), and $q^* \in (0, 1)$ is the solution of a first crossing problem for n. Once we have constructed n and determined q^*, numerically if appropriate, expressions for all other quantities can be derived by solving a further integral equation, which can be re-expressed as a first order

differential equation. This two-stage procedure is significantly simpler than solving the HJB equation directly, as this equation is second order and non-linear, and subject to second-order smooth fit at an unknown free boundary.

Remark 3.1.10. In the corresponding Merton problem for the unconstrained agent who may both buy and sell the risky asset at zero transaction cost, it follows from Theorem 1.2.5 that the optimal behaviour for the agent is to hold a fixed proportion $q^M = \alpha/\eta^2 R = \epsilon/\delta^2 R$ of total wealth in the risky asset. This corresponds to keeping $Q_t = Y_t\Theta_t/(X_t + Y_t\Theta_t) = q^M$ or equivalently $Z_t = Y_t\Theta_t/X_t = z^M := q^M/(1-q^M) = \epsilon/(\delta^2 R - \epsilon)$. In Lemma 3.3.1 below we show that $q^* \geq \epsilon/\delta^2 R = q^M$ so that optimal behaviour for the agent who cannot buy units of the risky asset is to keep the ratio of money invested in the risky asset to cash wealth in in interval $[0, q^*]$ where $q^M \in (0, q^*)$.

The following theorem characterises the solution to the problem in the second non-degenerate case (the third case in Theorem 3.1.5). In this case, the optimal strategy is to first hold the endowed asset and finance consumption with initial wealth. When liquid wealth is exhausted, consumption is further financed by the sale of endowed asset. Here, the critical threshold $z^* = \infty$.

Theorem 3.1.11. *Suppose $\epsilon \geq \delta^2 R$ and if $R < 1$, $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$.*

Let n solve (3.6) on $[0,1]$. Then for the given parameter combinations we have $q^* = 1$. As in Theorem 3.1.7, let $N(q) = n(q)^{-R}(1-q)^{R-1}$. Then N is monotonic.

Let W be inverse to N. For $R < 1$ define $\gamma : (1,\infty) \mapsto \mathbb{R}$ by

$$\gamma(v) = \frac{\ln v}{1-R} + \frac{R}{1-R}\ln m(1) - \frac{1}{1-R}\int_v^\infty \frac{(1-W(s))}{sW(s)}ds. \tag{3.18}$$

If $R > 1$ define $\gamma : (0,1) \mapsto \mathbb{R}$ by

$$\gamma(v) = -\frac{\ln v}{R-1} - \frac{R}{R-1}\ln m(1) - \frac{1}{R-1}\int_0^v \frac{(1-W(s))}{sW(s)}ds. \tag{3.19}$$

Let h be inverse to γ and let $g(z) = (R/\beta)^R h(\ln z)$.

Then, the value function V is given by

$$V(x,y,\theta,t) = e^{-\beta t}\frac{x^{1-R}}{1-R}g\left(\frac{y\theta}{x}\right), \qquad x > 0, \theta > 0 \tag{3.20}$$

which can be extended by continuity to give

$$V(x,y,0,t) = e^{-\beta t}\frac{x^{1-R}}{1-R}\left(\frac{R}{\beta}\right)^R, \tag{3.21}$$

$$V(0,y,\theta,t) = e^{-\beta t}\frac{y^{1-R}\theta^{1-R}}{1-R}\left(\frac{R}{\beta}\right)^R m(1)^{-R}. \tag{3.22}$$

The optimal consumption process C^* is given by $C_t^* = C(X_t^*, Y_t, \Theta_t^*)$ where $C(x,y,\theta)$ is as in (3.16) and the optimal holdings Θ_t^* of the endowed asset and the resulting wealth process are given by

$$\Theta_t^* = \begin{cases} \theta_0 & t \le \tau \\ \theta_0 e^{-\frac{\beta}{R}m(1)(t-\tau)} & t > \tau \end{cases}, \qquad X_t^* = \begin{cases} x_0 - \int_0^t C(X_s^*, Y_s, \theta_0)ds & t \le \tau \\ 0 & t > \tau \end{cases}, \tag{3.23}$$

where $\tau = \inf\{t \ge 0 : X_t^* = 0\}$. Finally the certainty equivalent value is given by (3.17).

Remark 3.1.12. Note that $\lim_{z\uparrow\infty} \frac{1}{z}(g(z) - \frac{zg'(z)}{1-R})^{-1/R} = \beta m(1)/R$ and hence by continuity we may set $C(0,y,\theta) = y\theta\beta m(1)/R$. Then for $t > \tau$ we have that

$$C_t^* = C(0, Y_t, \Theta_t^*) = \frac{\beta}{R}m(1)Y_t\Theta_t^*.$$

3.2 Proofs and verification arguments

For $F = F(x,y,\theta,t) \in C^{1,2,1,1}$ such that $F_x > 0$ define operators $\mathcal{L}$ and $\mathcal{M}$ by

$$\begin{aligned} \mathcal{L}F &= \sup_{c>0}\left\{e^{-\beta t}\frac{c^{1-R}}{1-R} - cF_x\right\} + \alpha y F_y + F_t + \frac{1}{2}\eta^2 y^2 F_{yy} \\ &= \frac{R}{1-R}e^{-\frac{\beta}{R}t}F_x^{1-1/R} + \alpha y F_y + F_t + \frac{1}{2}\eta^2 y^2 F_{yy}, \\ \mathcal{M}F &= F_\theta - yF_x. \end{aligned}$$

Remark 3.2.1. The state space of (X_t, Y_t, Θ_t, t) is $[0,\infty) \times (0,\infty) \times [0,\infty) \times [0,\infty)$, and we want to define $\mathcal{L}$ and $\mathcal{M}$ on this region including at the boundary. In practice, all the functions to which we apply the operators are of the form $F(x,y,\theta,t) = e^{-\beta t}\overline{F}(x,y,\theta)$ for some function $\overline{F}$ which is independent of t in which case $F_t = -\beta F$, and this latter form is well defined at $t = 0$. Also, we typically need $\mathcal{M}F$ only for $\theta > 0$. Then, given F defined for $x > 0$ we can define F at $x = 0$ by continuity, and then $\mathcal{M}F|_{x=0}$ is also well defined. $\mathcal{L}F$ at $\theta = 0$ can be defined similarly, by first defining F at $\theta = 0$ by continuity. In order to define $\mathcal{L}F$ at $x = 0$ for $\theta > 0$ we extend the domain of F to $x > -\theta y$ and

then show that F_x and the other derivatives of F are continuous across $x = 0$ with this extension.

3.2.1 The Verification Lemma in the case of a depreciating asset

Suppose $\epsilon \leq 0$. Our goal is to show that the conclusions of Theorem 3.1.5(1) hold.

From Proposition 3.1.2 we know $q^* = 0$. Define the candidate value function via

$$G(x, y, \theta, t) = e^{-\beta t} \left(\frac{R}{\beta}\right)^R \frac{(x + y\theta)^{1-R}}{1 - R} \qquad x \geq 0, \theta \geq 0. \tag{3.24}$$

The candidate optimal strategy is to sell all units of the risky asset immediately. The domain of G can be extended to $-\theta y < x < 0$ for $\theta > 0$, using the same functional form as in (3.24).

Prior to the proof of the theorem, we need the following lemma.

Lemma 3.2.2. *Suppose $\epsilon \leq 0$. Consider the candidate value function constructed in (3.24). Then on $(x \geq 0, \theta > 0)$ we have $\mathcal{M}G = 0$, and on $(x \geq 0, \theta \geq 0)$ we have $\mathcal{L}G \leq 0$ with equality at $\theta = 0$.*

Proof. Given the form of the candidate value function in (3.24), we have

$$\mathcal{M}G = e^{-\beta t} \left(\frac{R}{\beta}\right)^R y(x + y\theta)^{-R} - e^{-\beta t} \left(\frac{R}{\beta}\right)^R y(x + y\theta)^{-R} = 0.$$

On the other hand, writing $z = y\theta/x$, provided $x > 0$

$$\mathcal{L}G = \beta \left(\frac{R}{\beta}\right)^R e^{-\beta t} \frac{(x + y\theta)^{1-R}}{1 - R} \left[\epsilon(1 - R)\frac{z}{1 + z} - \frac{1}{2}\delta^2 R(1 - R)\left(\frac{z}{1 + z}\right)^2\right] \leq 0,$$

with equality at $z = 0$. If $x = 0$ then $\mathcal{L}G = \beta G(1 - R)[\epsilon - \frac{\delta^2 R}{2}] < 0$. □

Theorem 3.2.3. *Suppose $\epsilon \leq 0$. Then the value function is*

$$V(x, y, \theta, t) = e^{-\beta t} \left(\frac{R}{\beta}\right)^R \frac{(x + y\theta)^{1-R}}{1 - R}, \tag{3.25}$$

and the optimal holdings Θ_t^* of the endowed asset, the optimal consumption process C_t^* and the resulting wealth process are given by

$$(\triangle\Theta^*)_{t=0} = -\theta_0, \qquad C_t^* = \frac{\beta}{R}(x_0 + y_0\theta_0)e^{-\frac{\beta}{R}t}, \qquad X_t^* = (x_0 + y_0\theta_0)e^{-\frac{\beta}{R}t}. \tag{3.26}$$

Proof. Note that candidate optimal strategy given in (3.26) is to sell the entire holding of the risky asset at time zero (which gives $X_0^* = x_0 + y_0\theta_0$) and thereafter to finance consumption from liquid wealth, whence the wealth process $(X_t^*)_{t\geq 0}$ is deterministic and evolves as $dX_t^* = -C_t^* dt$. This gives $X_t^* = (x_0 + y_0\theta_0)e^{-\frac{\beta}{R}t}$. It follows that the candidate optimal strategy is admissible.

The value function under the strategy proposed in (3.26) is

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{*1-R}}{1-R}dt\right] &= \int_0^\infty e^{-\beta t}\left(\frac{\beta}{R}\right)^{1-R}\frac{\left(e^{-\frac{\beta}{R}t}(x_0+y_0\theta_0)\right)^{1-R}}{1-R}dt \\ &= \left(\frac{R}{\beta}\right)^R\frac{(x_0+y_0\theta_0)^{1-R}}{1-R} = G(x_0, y_0, \theta_0, 0).\end{aligned}$$

Hence $V \geq G$.

Now, consider general admissible strategies. Suppose first that $R < 1$. Define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \int_0^t e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds + G\left(X_t, Y_t, \Theta_t, t\right). \tag{3.27}$$

Applying the generalised Itô's formula [23, Section 4.7] to M_t and suppressing the argument $(X_{s-}, Y_s, \Theta_{s-}, s)$ in derivatives of G, leads to

$$\begin{aligned}M_t - M_0 &= \int_0^t\left[e^{-\beta s}\frac{C_s^{1-R}}{1-R} - C_sG_x + \alpha Y_sG_y + \frac{1}{2}\eta^2Y_s^2G_{yy} + G_s\right]ds \\ &\quad + \int_0^t(G_\theta - Y_sG_x)d\Theta_s \\ &\quad + \sum_{0\leq s\leq t}\left[G(X_s, Y_s, \Theta_s, s) - G(X_{s-}, Y_{s-}, \Theta_{s-}, s) - G_x(\triangle X)_s - G_\theta(\triangle\Theta)_s\right] \\ &\quad + \int_0^t \eta Y_sG_ydB_s \\ &= N_t^1 + N_t^2 + N_t^3 + N_t^4.\end{aligned} \tag{3.28}$$

(Note that in the sum we allow for a portfolio rebalancing at $s = 0$.)

Lemma 3.2.2 implies that $\mathcal{L}G \leq 0$ and $\mathcal{M}G = 0$, which leads to $N_t^1 \leq 0$ and $N_t^2 = 0$. Using the fact that $(\Delta X)_s = -Y_s(\Delta\Theta)_s$ and writing $\theta = \Theta_{s-}$, $x = X_{s-}$, $\chi = -(\Delta\Theta)_s$ each non-zero jump in N^3 is of the form

$$(\Delta N^3)_s = G(x + y\chi, y, \theta - \chi, s) - G(x, y, \theta, s) + \chi\left[G_\theta(x, y, \theta, s) - yG_x(x, y, \theta, s)\right].$$

Given the form of the candidate value function in (3.24), it is easy to see that $\psi(\phi) = G(x + y\phi, y, \theta - \phi, s)$ is constant in ϕ, which gives $\psi(\chi) = \psi(0)$ and $yG_x = G_\theta$ whence $(\Delta N^3) = 0$. Then, since $R < 1$, we have $0 \leq M_t \leq M_0 + N_t^4$, and the local martingale N_t^4 is bounded from below and hence a supermartingale. Taking expectations we find $\mathbb{E}(M_t) \leq M_0 = G(x_0, y_0, \theta_0, 0)$, which gives

$$G(x_0, y_0, \theta_0, 0) \geq \mathbb{E}\int_0^t e^{-\beta s}\frac{{C_s}^{1-R}}{1-R}ds + \mathbb{E}G(X_t, Y_t, \Theta_t, t) \geq \mathbb{E}\int_0^t e^{-\beta s}\frac{{C_s}^{1-R}}{1-R}ds, \quad (3.29)$$

where the last inequality follows since $G(X_t, Y_t, \Theta_t, t) \geq 0$ for $R \in (0, 1)$. Letting $t \to \infty$ in (3.29) leads to

$$G(x_0, y_0, \theta_0, 0) \geq \mathbb{E}\int_0^\infty e^{-\beta t}\frac{{C_t}^{1-R}}{1-R}dt,$$

and taking a supremum over admissible strategies leads to $G \geq V$.

The case $R > 1$ is considered in the Section 3.5.

□

3.2.2 Proof in the ill-posed case of Theorem 3.1.5 (scenario 4)

Recall we are in the case where $R < 1$ and $\epsilon \geq \delta^2 R/2 + 1/(1-R)$.

It is sufficient to give an example of an admissible strategy when $\theta > 0$ for which the expected utility of consumption is infinite. Note that $V(x, y, 0, t) = e^{-\beta t}x^{1-R}R^R\beta^{-R}/(1-R)$ so that the value function is not continuous at $\theta = 0$.

Consider a consumption and sale strategy pair $((\tilde{C})_{t\geq 0}, (\tilde{\Theta})_{t\geq 0})$, given by

$$\tilde{\Theta}_t = \tilde{\Theta}_t(\phi) = e^{-\phi t}\theta_0, \qquad \tilde{C}_t = \tilde{C}_t(\phi) = \phi Y_t\tilde{\Theta}_t = \phi y_0\theta_0 \exp\left\{\beta(\epsilon - \delta^2/2 - \phi/\beta)t + \delta\sqrt{\beta}B_t\right\}, \quad (3.30)$$

where ϕ is some positive constant.

Note first that that such strategies are admissible since the corresponding wealth process satisfies $d\tilde{X}_t = -\phi Y_t\tilde{\Theta}_t dt + Y_t d\tilde{\Theta}_t = 0$, and hence $(\tilde{X}_t)_{t\geq 0} = x_0 > 0$. In particular, consumption is financed by the sale of the endowed asset only.

The expected discounted utility from consumption $\tilde{G} = \tilde{G}(\phi)$ corresponding to

the consumption and sale processes $(\tilde{C}, \tilde{\Theta})$ is given by

$$\begin{aligned}
\tilde{G} &= \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{\tilde{C}_t^{1-R}}{1-R}dt\right] \\
&= \frac{(\phi y_0\theta_0)^{1-R}}{1-R}\mathbb{E}\left[\int_0^\infty \exp\left\{\beta\left[(1-R)\left(\epsilon-\frac{\delta^2}{2}-\frac{\phi}{\beta}\right)-1\right]t+(1-R)\delta\sqrt{\beta}B_t\right\}dt\right] \\
&= \frac{(\phi y_0\theta_0)^{1-R}}{1-R}\int_0^\infty \exp\left\{\beta(1-R)\left[\left(\epsilon-\frac{\delta^2 R}{2}-\frac{1}{1-R}\right)-\frac{\phi}{\beta}\right]t\right\}dt
\end{aligned}$$

Suppose first that $\epsilon > \delta^2 R/2 + 1/(1-R)$. Then for $\lambda \in (0,1)$ and $\phi = \lambda\beta(\epsilon - \delta^2 R/2 - 1/(1-R))$ we have

$$\left(\epsilon-\frac{\delta^2 R}{2}-\frac{1}{1-R}\right)-\frac{\phi}{\beta} = (1-\lambda)\left(\epsilon-\frac{\delta^2 R}{2}-\frac{1}{1-R}\right) > 0,$$

and $\tilde{G}$ is infinite.

Now suppose that $\epsilon = \delta^2 R/2 + 1/(1-R)$. Then

$$\tilde{G}(\phi) = \frac{(\phi y_0\theta_0)^{1-R}}{(1-R)}\frac{1}{\phi(1-R)} = \phi^{-R}\frac{(y_0\theta_0)^{1-R}}{(1-R)^2}$$

and $\tilde{G}(\phi) \uparrow \infty$ as $\phi \downarrow 0$.

3.2.3 The Verification Lemma in the first non-degenerate case with finite critical exercise ratio.

Suppose $0 < \epsilon < \delta^2 R$ and if $R < 1$, $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$. From Proposition 3.1.2 we know $0 < q^* < 1$.

Recall the definition $N(q) = n(q)^{-R}(1-q)^{R-1}$ and that W is inverse to N. We have $h^* = N(q^*)$.

Proposition 3.2.4. *1. For $R < 1$, N is increasing on $[0, q^*]$. W is increasing and $0 < W(v) < q^*$ on $(1, h^*)$. For $R > 1$, N is decreasing on $[0, q^*]$. W is decreasing and $0 < W(v) < q^*$ on $(h^*, 1)$.*

2. Let $w(v) = v(1-R)W(v)$. Then w solves

$$\frac{\delta^2}{2}w(v)w'(v) - v + \left(\epsilon-\frac{\delta^2}{2}\right)w(v) + \left(v-\frac{w(v)}{1-R}\right)^{1-1/R} = 0. \tag{3.31}$$

3. *For $R < 1$ and $1 < v < h^*$, and for $R > 1$ and $h^* < v < 1$ we have $w'(v) < 1 - Rw(v)/((1-R)v)$ with $w'(h^*) = 1 - Rw(h^*)/((1-R)h^*)$.*

The proof of Proposition 3.2.4 is given in Section 3.3.

Now define h on $[1, h^*)$ by $\frac{dh}{du} = w(h) = (1-R)hW(h)$ subject to $h(u^*) = h^*$. Then h solves (3.8) and $w'(h)w(h) = \frac{d^2h}{du^2}$. Let $g(z) = (\frac{R}{\beta})^R h(\ln z)$. Then g solves (3.9).

Lemma 3.2.5. *Let $m(q^*)^{-R}$, z^* and g be as given in Equations (3.7) and (3.9) of Theorem 3.1.7. Then, $g(z)$, $g'(z)$, $g''(z)$ are continuous at $z = z^*$.*

Proof. We have

$$g(z^*+) = \left(\frac{R}{\beta}\right)^R h^*(1-q^*)^{1-R}(1+z^*)^{1-R} = \left(\frac{R}{\beta}\right)^R h^* = \left(\frac{R}{\beta}\right)^R h(u^*) = g(z^*-).$$

For the first derivative we have for $z > z^*$,

$$zg'(z) = (1-R)\left(\frac{zg(z)}{1+z}\right)$$

and then since $\frac{z^*}{1+z^*} = q^*$, $z^*g'(z^*) = (1-R)\left(\frac{R}{\beta}\right)^R h^*q^*$. Meanwhile, for $z < z^*$, and noting that $\frac{dh}{du} = h(1-R)W(h) = w(h)$,

$$zg'(z) = \left(\frac{R}{\beta}\right)^R h'(u) = \left(\frac{R}{\beta}\right)^R w(h)$$

so that $z^*g'(z^*-) = \left(\frac{R}{\beta}\right)^R w(h^*)$ and the result follows from the substitution $w(h^*) = (1-R)h^*W(h^*) = (1-R)h^*q^*$.

Finally, for $z > z^*$

$$z^2g''(z) = -R(1-R)\left(\frac{R}{\beta}\right)^R m(q^*)^{-R}(1+z)^{1-R}\left(\frac{z}{1+z}\right)^2 = -R(1-R)g(z)\left(\frac{z}{1+z}\right)^2 \tag{3.32}$$

and $(z^*)^2g''(z^*+) = -R(1-R)g(z^*)(q^*)^2$. For $z < z^*$,

$$z^2g''(z) = \left(\frac{R}{\beta}\right)^R (h'' - h') = \left(\frac{R}{\beta}\right)^R (w'(h) - 1)w(h) \tag{3.33}$$

and at z^*, $(z^*)^2g''(z^*-) = -R(1-R)\left(\frac{R}{\beta}\right)^R h^*(q^*)^2$ where we use Proposition 3.2.4 (3). □

Proposition 3.2.6. *Suppose $g(z)$ solves (3.9). Then for $R < 1$, g is an increasing concave function such that $g(0) = (\frac{R}{\beta})^R$. Otherwise, for $R > 1$, g is a decreasing convex function such that $g(0) = (R/\beta)^R$ and $g(z) \geq 0$. Further, for all values of R we have that $0 \geq Rg'(z)^2 + (1-R)g(z)g''(z)$ with equality for $z \geq z^*$.*

Proof. Consider first $R < 1$. Since the statements are immediate in the region $z \geq z^*$, and since there is second order smooth fit at z^* the result will follow if $h(-\infty) = 1$, h is increasing and, using (3.33), $w(h)w'(h) - w(h) \leq 0$. The last two properties follow from Proposition 3.2.4 since $w(h) \geq 0$ and $w'(h) < 1$.

To evaluate $h(-\infty)$ note that

$$u^* - u = \int_{h(u)}^{h^*} \frac{df}{(1-R)fW(f)} = \int_{W(h(u))}^{q^*} \frac{N'(q)}{(1-R)N(q)q} dq = \int_{W(h(u))}^{q^*} \frac{\frac{\delta^2}{2}(1-R)}{\ell(q) - n(q)} dq.$$

We have that $\ell(q) - n(q)$ is bounded away from zero when q is bounded away from zero. Further, near $q = 0$ we have $\ell(q) - n(q) \sim Cq$ for some positive constant $C = \ell'(0) - n'(0+)$. Hence $W(h(-\infty)) = 0$ and $h(-\infty) = 1$, since $W(1) = 0$.

For $R > 1$, and $z \geq z^*$, the statement holds immediately. For $z \leq z^*$, Proposition 3.2.4 implies that h is decreasing and $w(h) \leq 0$, $w'(h) < 1$. Together with (3.33), we have g is a decreasing convex function and $g(z) \geq 0$ given that $h \in [0, 1]$.

For the final statement of the proposition, for $z \geq z^*$ the result follows immediately, whereas for $z < z^*$

$$(1-R)gg''z^2 + R(zg')^2 = \left(\frac{R}{\beta}\right)^{2R} \left[(1-R)hw(h)[w'(h) - 1] + Rw(h)^2\right] \leq 0$$

where the final inequality follows from Proposition 3.2.4(3), noting that $(1-R)w(h) \geq 0$. □

Define the candidate value function via

$$G(x, y, \theta, t) = e^{-\beta t} \frac{x^{1-R}}{1-R} g\left(\frac{y\theta}{x}\right) \qquad x > 0, \theta > 0; \tag{3.34}$$

and extend to $x \leq 0$ and $\theta = 0$ using the formulae

$$G(x, y, \theta, t) = e^{-\beta t} \frac{(x + y\theta)^{1-R}}{1-R} m(q^*)^{-R} \qquad -\theta y < x \leq 0, \theta > 0; \tag{3.35}$$

$$G(x, y, 0, t) = e^{-\beta t} \frac{x^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R \qquad x \geq 0, \theta = 0. \tag{3.36}$$

Lemma 3.2.7. *Fix y and t. Then $G = G(x, \theta)$ is concave in x and θ on $[0, \infty) \times [0, \infty)$. In particular, if $\psi(\chi) = G(x - \chi y\phi, y, \theta + \chi\phi, t)$, then ψ is concave in χ.*

Proof. Consider first $R < 1$. In order to show the concavity of the candidate value function it is sufficient to show that $G(x, 0)$ is concave in x, $G(0, \theta)$ is concave in θ and that the Hessian matrix given by

$$H_G = \begin{pmatrix} G_{xx} & G_{x\theta} \\ G_{x\theta} & G_{\theta\theta} \end{pmatrix}.$$

has a positive determinant, and that one of the diagonal entries is non-positive. The conditions on $G(x, 0)$ and $G(0, \theta)$ are trivial to verify.

Direct computation leads to

$$\begin{aligned} G_{xx}(x, y, \theta, y) &= e^{-\beta t} x^{-R-1} \left[-Rg(z) + \frac{2R}{1-R} z g'(z) + \frac{1}{1-R} z^2 g''(z) \right], \\ G_{x\theta}(x, y, \theta, t) &= -e^{-\beta t} x^{-R-1} \frac{y}{1-R} \left[Rg'(z) + z g''(z) \right], \\ G_{\theta\theta}(x, y, \theta, t) &= e^{-\beta t} x^{-R-1} \frac{y^2}{1-R} g''(z), \end{aligned}$$

and the determinant of the Hessian matrix is

$$G_{xx}G_{\theta\theta} - (G_{x\theta})^2 = -e^{-2\beta t} x^{-2R} \theta^{-2} \frac{R}{(1-R)^2} \left[(1-R) g(z) z^2 g''(z) + R \left(z g'(z) \right)^2 \right] \tag{3.37}$$

which is non-negative by Proposition 3.2.6. Further, since g is concave we have that $G_{\theta\theta} \leq 0$.

In order to show the concavity of ψ in χ, it is equivalent to examine the sign of $\frac{d^2\psi}{d\chi^2}$. But

$$\frac{d^2\psi}{d\chi^2} = \phi^2 \left[y^2 G_{xx} + G_{\theta\theta} - 2y G_{x\theta} \right] = \phi^2 (y, 1) \det(H_G) (y, 1)^T \leq 0.$$

For $R > 1$ the argument is similar, except that $G_{\theta\theta} \leq 0$ is now implied by the convexity of g. □

Lemma 3.2.8. *Consider the candidate value function constructed in (3.34).*

(a) For $\theta > 0$ and $0 \leq x \leq y\theta/z^*$, $\mathcal{M}G = 0$ and $\mathcal{L}G \leq 0$.

(b) For $\theta > 0$ and $x \geq y\theta/z^*$, $\mathcal{M}G \geq 0$. For $\theta \geq 0$ and $x \geq y\theta/z^*$, $\mathcal{L}G = 0$.

Proof. (a) For $z \geq z^*$, $\mathcal{M}G = 0$ is immediate from the definition of G. For $0 < x \leq y\theta/z^*$ $\mathcal{L}G$ we have that $G(x,y,\theta,t) = \left(\frac{R}{\beta}\right)^R m(q^*)^{-R} e^{-\beta t} \frac{x^{1-R}}{1-R} (1+z)^{1-R}$ and then

$$\begin{aligned}
\mathcal{L}G &= \beta G \left[m(q^*) - 1 + \epsilon (1-R) \frac{z}{1+z} - \frac{1}{2}\delta^2 R (1-R) \frac{z^2}{(1+z)^2} \right], \\
&= \beta G \left[m(q^*) - m\left(\frac{z}{1+z}\right) \right].
\end{aligned}$$

The required inequality follows from Part (5) of Lemma 3.3.1 in Section 3.3 and the fact that $m(q)/(1-R)$ is increasing on $(q^*, 1)$. At $x = 0$ using both (3.34) and (3.35) we have $\mathcal{L}G|_{x=0+} = \mathcal{L}G|_{x=0-} \beta G[m(q^*) - m(1)] < 0$.

(b) In order to prove $\mathcal{L}G = 0$ for $\theta > 0$ we calculate

$$\begin{aligned}
\mathcal{L}G(x,y,\theta,t) &= e^{-\beta t} \frac{x^{1-R}}{1-R} \left[R \left(g - \frac{zg'(z)}{1-R} \right)^{1-1/R} - \beta g + \alpha z g'(z) + \frac{\eta^2}{2} z^2 g''(z) \right] \\
&= \beta e^{-\beta t} \frac{x^{1-R}}{1-R} \left[h^{1-1/R} \left(1 - \frac{w(h)}{(1-R)h} \right) - h + \left(\epsilon - \frac{\delta^2}{2} \right) w(h) + \frac{\delta^2}{2} w'(h) w(h) \right]
\end{aligned}$$

and the result follows from Proposition 3.2.4. For $\theta = 0$, $\mathcal{L}G = 0$ is a simple calculation.

Now consider $\mathcal{M}G$. We have

$$\mathcal{M}G = e^{-\beta t} x^{-R} y \left[\frac{(1+z)}{1-R} g'(z) - g(z) \right]. \tag{3.38}$$

Hence for $R < 1$, it is sufficient to show that $\psi(z) \geq 0$ on $(0, z^*]$ where

$$\psi(z) = \frac{1+z}{1-R} - \frac{g(z)}{g'(z)}.$$

By value matching and smooth fit $g(z^*) = m(q^*)^{-R} (1+z^*)^{1-R}$ and $z^* g'(z^*) = m(q^*)^{-R}(1-R)(1+z^*)^{-R}$. Hence $\psi(z^*) = 0$ and it is sufficient to show that ψ is decreasing. But

$$\begin{aligned}
\psi'(z) &= \frac{R}{1-R} + \frac{g(z) g''(z)}{g'(z)^2} \\
&= \frac{R}{1-R} + \frac{h[w(h) w'(h) - w(h)]}{w(h)^2} \\
&\leq 0
\end{aligned} \tag{3.39}$$

where the last inequality follows from Proposition 3.2.4. Similarly, for $R > 1$, provided that g is decreasing by Proposition 3.2.6, it is sufficient to show that ψ is increasing. But

Proposition 3.2.4 gives

$$\psi'(z) = \frac{R}{1-R} + \frac{g(z)\,g''(z)}{g'(z)^2} = \frac{R}{1-R} + \frac{h\left[w(h)\,w'(h) - w(h)\right]}{w(h)^2} \geq 0.$$

□

Proposition 3.2.9. *Let X^*, Θ^* and C^* be as defined in Theorem 3.1.7. Then they correspond to an admissible wealth process. Moreover $Z_t^* = Y_t\Theta^*/X_t^*$ satisfies $0 \leq Z_t^* \leq z^*$.*

Proof. Note that if $y_0\theta_0/x_0 > z^*$ then the optimal strategy includes a sale of the endowed asset at time zero, and the effect of the sale is to move to new state variables $(X_0^*, y_0, \Theta_0^*, 0)$ with the property that $Z_0^* = y_0\Theta_0^*/X_0^* = z^*$.

Recall the definitions of $\tilde{\Lambda}$ and $\tilde{\Gamma}$ and set $\Sigma(z) = z(1+z)$ and $\tilde{\Sigma}(j) = \Sigma(z^* - j)$.

Consider the equation

$$\hat{J}_t = \hat{J}_0 - \int_0^t \tilde{\Lambda}\left(\hat{J}_s\right) ds - \int_0^t \tilde{\Gamma}\left(\hat{J}_s\right) dB_s + \hat{L}_t \tag{3.40}$$

with initial condition $\hat{J}_0 = (z^* - z_0)^+$. This equation is associated with a stochastic differential equation with reflection (Revuz and Yor [48, p385]) and has a unique solution (J, L) for which (J, L) is adapted, $J \geq 0$, $L_0 = 0$ and L only increases when J is zero.

Note that $\tilde{\Lambda}(z^*) = \Lambda(0) = 0 = \Gamma(0) = \tilde{\Gamma}(z^*)$ and hence J is bounded above by z^*.

Recall that $\Theta_t^* = \Theta_0^* \exp(-L_t/\tilde{\Sigma}(0))$. Then Θ_t^* is adapted, continuous and hence progressively measurable (Karatzas and Shreve [35, p5]). Θ_t^* is also decreasing and $d\Theta_t^* = -\Theta_t^* dL_t/\tilde{\Sigma}(0) = -\Theta_t^* dL_t/\tilde{\Sigma}(J_t)$ since L only grows when $J = 0$.

Then let $Z_t^* = z^* - J_t$, $X_t^* = \Theta_t^* Y_t/Z_t^*$ and $C_t^* = X_t^*(g(Z_t^*) - Z_t^* g'(Z_t^*)/(1 - R))^{-1/R}$. Then X^* and C^* are positive and progressively measurable. It remains to show that X is the wealth process arising from the consumption and sale strategy (C^*, Θ^*). But, from (3.40) and using, for example $\tilde{\Lambda}(J_t) = \Lambda(Z_t^*)$,

$$dZ_t^* = \Lambda\left(Z_t^*\right) dt + \Gamma\left(Z_t^*\right) dB_t + \Sigma\left(Z_t^*\right) \frac{d\Theta_t^*}{\Theta_t^*}.$$

and then

$$\begin{aligned}
dX_t^* &= \frac{\Theta_t^* Y_t}{Z_t^*}\left[\frac{d\Theta_t^*}{\Theta_t^*} + \frac{dY_t}{Y_t} - \frac{dZ_t^*}{Z_t^*} + \left(\frac{dZ_t^*}{Z_t^*}\right)^2 - \frac{dY_t}{Y_t}\frac{dZ_t^*}{Z_t^*}\right] \\
&= X_t^*\left[\left(\eta - \frac{\Gamma(Z_t^*)}{Z_t^*}\right)dB_t + \left(\alpha - \frac{\Lambda(Z_t^*)}{Z_t^*} + \frac{\Gamma(Z_t^*)^2}{(Z_t^*)^2} - \eta\frac{\Gamma(Z_t^*)}{Z_t^*}\right)dt\right] + \left(\frac{Y_t}{Z_t^*} - \frac{Y_t}{Z_t^*}\frac{\Sigma(Z_t^*)}{Z_t^*}\right)d\Theta_t^* \\
&= -C_t^* dt - Y_t d\Theta_t^*
\end{aligned}$$

as required, where we use the definitions of Λ, Γ and Σ for the final equality. □

Proof of Theorem 3.1.7. First we show that there is a strategy such that the candidate value function is attained, and hence that $V \geq G$.

Observe first that if $y_0\theta_0/x_0 > z^*$ then

$$\theta_0 - \Theta_0^* = \theta_0\left(1 - \frac{z^*}{1+z^*}\frac{1+z_0}{z_0}\right)$$

and

$$X_0^* = x_0 + y_0(\theta_0 - \Theta_0^*) = x_0\frac{(1+z_0)}{(1+z^*)}$$

Then, since $g(z^*)/g(z_0) = (1+z^*)^{1-R}/(1+z_0)^{1-R}$ for $z_0 > z^*$,

$$G(X_0^*, y_0, \Theta_0^*, 0) = \frac{(X_0^*)^{1-R}}{1-R}g(z^*) = \frac{x_0^{1-R}}{1-R}g(z_0) = G(x_0, y_0, \theta_0, 0).$$

For a general admissible strategy define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \int_0^t e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds + G\left(X_t, Y_t, \Theta_t, t\right). \tag{3.41}$$

Write M^* for the corresponding process under the proposed optimal strategy. Then $M_0^* = G(X_0^*, y_0, \Theta_0^*, 0) = G(x_0, y_0, \theta_0, 0)$ so there is no jump of M^* at $t = 0$. Further, although the optimal strategy may include the sale of a positive quantity of the risky asset at time zero, it follows from Proposition 3.2.9 that thereafter the process Θ^* is continuous and such that $Z_t^* = Y_t\Theta_t^*/X_t^* \leq z^*$.

From the form of the candidate value function and the definition of g given in (3.9), we know that G is $C^{1,2,1,1}$. Then applying Itô's formula to M_t, using the continuity

of X^* and Θ^* for $t > 0$, and writing $G.$ as shorthand for $G.(X_s^*, Y_s, \Theta_s^*, s)$ we have

$$\begin{aligned} M_t^* - M_0 &= \int_0^t \left[e^{-\beta s} \frac{(C_s^*)^{1-R}}{1-R} - C_s^* G_x + \alpha Y_s G_y + \frac{1}{2}\eta^2 Y_s^2 G_{yy} + G_t \right] ds \\ &\quad + \int_{(0,t]} (G_\theta - Y_s G_x)\, d\Theta_s^* \\ &\quad + \int_0^t \eta Y_s G_y dB_s \\ &=: N_t^1 + N_t^2 + N_t^3 \end{aligned} \tag{3.42}$$

Since $Z_t^* \leq z^*$, and since $C_t^* = e^{-\beta s/R} G_x^{-1/R}$ and $\mathcal{L}G = 0$ for $z \leq z^*$ we have $N_t^1 = 0$. Further, $d\Theta_s \neq 0$ if and only if $Z_t^* = z^*$ and then $\mathcal{M}G = 0$, so that $N_t^2 = 0$.

To complete the proof of the theorem we need the following lemma which is proved in Section 3.4.

Lemma 3.2.10. *1. N^3 given by $N_t^3 = \int_0^t \eta Y_s G_y(X_s^*, Y_s, \Theta_s^*, s) dB_s$ is a martingale.*

2. $\lim_{t\uparrow\infty} \mathbb{E}[G(X_t^*, Y_t, \Theta_t^*, t)] = 0$.

Returning to the proof of the theorem, and taking expectations on both sides of (3.42), we have $\mathbb{E}\left[M_t^*\right] = M_0$, which leads to

$$G\left(x_0, y_0, \theta_0, 0\right) = \mathbb{E}\left(\int_0^t e^{-\beta s} \frac{(C_s^*)^{*1-R}}{1-R} ds \right) + \mathbb{E}\left[G\left(X_t^*, y, \Theta_t^*, t\right)\right].$$

Using the second part of Lemma 3.2.10 and applying the monotone convergence theorem, we have

$$G\left(x_0, y_0, \theta_0, 0\right) = \mathbb{E}\left(\int_0^\infty e^{-\beta s} \frac{C_s^{*1-R}}{1-R} ds \right)$$

and hence $V \geq G$.

Now we consider general admissible strategies. Applying the generalised Itô's formula [23, Section 4.7] to M_t leads to the same expression as in (3.28). Lemma 3.2.8 implies that under general admissible strategies, $N_t^1 \leq 0$, $N_t^2 \leq 0$. Consider the jump term,

$$N_t^3 = \sum_{0 \leq s \leq t} \left[G\left(X_s, Y_s, \Theta_s, s\right) - G\left(X_{s-}, Y_s, \Theta_{s-}, s\right) - G_x(\Delta X)_s - G_\theta(\Delta\Theta)_s \right]. \tag{3.43}$$

Using the fact that $(\Delta X)_s = -Y_s(\Delta\Theta)_s$ and writing $\theta = \Theta_{s-}$, $x = X_{s-}$, $\chi = -(\Delta\Theta)_s$

each non-zero jump in N^3 is of the form

$$(\Delta N^3)_s = G(x + y\chi, y, \theta - \chi, s) - G(x, y, \theta, s) + \chi \left[G_\theta(x, y, \theta, s) - yG_x(x, y, \theta, s) \right].$$

But, by Lemma 3.2.7, $G(x + y\chi, y, \theta - \chi, s)$ is concave in χ and hence $(\Delta N^3) \leq 0$.

For $R < 1$ the rest of the proof is exactly as in Theorem 3.2.3. The case of $R > 1$ is covered in Section 3.5.

□

3.2.4 The Verification Lemma in the second non-degenerate case with no finite critical exercise ratio.

Throughout this section we suppose that $\epsilon \geq \delta^2 R$ and that if $R < 1$ then $0 < \epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$. It follows that $q^* = 1$ and $z^* = \infty$, and that $n(1) = m(1) > 0$.

Recall the definition of n in (3.6) and the subsequent definitions of N by $N(q) = n(q)^{-R}(1-q)^{R-1}$ and $W = N^{-1}$. Suppose $R < 1$ and define γ as in (5.12) by

$$\gamma(v) = \frac{1}{1-R} \ln v + \frac{R}{1-R} \ln m(1) - \frac{1}{1-R} \int_v^\infty \frac{1 - W(s)}{sW(s)} ds.$$

In the case $R > 1$ define γ via (5.16) so that

$$\gamma(v) = -\frac{1}{R-1} \ln v - \frac{R}{R-1} \ln m(1) - \frac{1}{R-1} \int_0^v \frac{1 - W(s)}{sW(s)} ds.$$

For all R define also $\tilde{\gamma}$ by

$$\tilde{\gamma}(v) = \frac{\ln v}{1-R} - \gamma(v).$$

Let h be inverse to γ and set $g(z) = (R/\beta)^R h(\ln z)$.

Lemma 3.2.11. *1. Suppose $R < 1$. Then $\gamma : (1, \infty) \mapsto (-\infty, \infty)$ is well defined, increasing, continuous and onto. Furthermore,*

$$\lim_{v \uparrow \infty} \tilde{\gamma}(v) = \frac{-R}{1-R} \ln m(1) \qquad \textit{and} \qquad \lim_{v \uparrow \infty} (1 - W(v)) e^{\gamma(v)} = 1.$$

Suppose $R > 1$. Then $\gamma : (0, 1) \mapsto (-\infty, \infty)$ is well defined, decreasing, continuous and onto. Furthermore,

$$\lim_{v \downarrow 0} \tilde{\gamma}(v) = \frac{R}{R-1} \ln m(1) \qquad \textit{and} \qquad \lim_{v \downarrow 0} (1 - W(v)) e^{\gamma(v)} = 1.$$

2. h solves $h' = (1-R)hW(h)$, and $h(-\infty) = 1$.

Proof. Suppose $R < 1$, the proof for $R > 1$ being similar. First we want to show that

$$\int^{\infty} \frac{1-W(s)}{sW(s)} ds < \infty, \qquad \text{and} \qquad \int_{1+} \frac{1-W(s)}{sW(s)} ds = \infty,$$

which, given $\lim_{s\uparrow\infty} W(s) = 1$ and $\lim_{s\downarrow 1} W(s) = 0$ is equivalent to

$$\int^{\infty} \frac{1-W(s)}{s} ds < \infty; \qquad \int_{1+} \frac{1}{W(s)} ds = \infty.$$

But $(1-q)N(q)^{1/(1-R)} \xrightarrow{q\uparrow 1} n(1)^{-R/(1-R)}$ and so $(1-W(s)) \sim n(1)^{-R/(1-R)} s^{-1/(1-R)}$ for large s and the first integral is finite. Conversely, since $N'(0+) = \kappa$ for some $\kappa \in (0,\infty)$ we have $W'(1+) = \kappa^{-1}$ and $W(s) \sim (s-1)\kappa^{-1}$ for s near 1. Since $1/(s-1)$ is not integrable near 1, the second integral explodes.

It follows that γ is onto; the fact that γ is increasing follows on differentiation. Indeed $\gamma'(v) = 1/((1-R)vW(v))$ and hence $h' = (1-R)hW(h)$. Also $h(-\infty) := \lim_{u\downarrow-\infty} h(u) = 1$.

The first limit result for $\tilde{\gamma}$ follows immediately from the definition. For the second,

$$\begin{aligned}
\lim_{v\uparrow\infty} e^{\gamma(v)}(1-W(v)) &= \lim_{v\uparrow\infty} e^{-\tilde{\gamma}(v)} v^{1/(1-R)}(1-W(v)) = \lim_{v\uparrow\infty} e^{-\tilde{\gamma}(v)} \lim_{q\uparrow 1} N(q)^{1/(1-R)}(1-q) \\
&= m(1)^{R/(1-R)} \lim_{q\uparrow 1} n(q)^{-R/(1-R)} = 1.
\end{aligned}$$

□

Define the candidate value function via

$$G(x,y,\theta,t) = e^{-\beta t} \frac{x^{1-R}}{1-R} g\left(\frac{y\theta}{x}\right), \qquad x > 0, \theta > 0 \tag{3.44}$$

and extend the definition to $\theta = 0$ and $-\theta y < x \leq 0$ by

$$G(x,y,\theta,t) = e^{-\beta t} \frac{(x+y\theta)^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R m(1)^{-R} \qquad -\theta y < x \leq 0, \theta > 0; \tag{3.45}$$

$$G(x,y,0,t) = e^{-\beta t} \frac{x^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R \qquad x \geq 0, \theta = 0. \tag{3.46}$$

Here continuity of G at $x = 0$ follows from the identity

$$\lim_{z\uparrow\infty} z^{R-1}g(z) = \lim_{u\uparrow\infty} e^{-(1-R)u}h(u) = \lim_{v} e^{-(1-R)\gamma(v)}v = \lim_{v} e^{-(1-R)\tilde{\gamma}(v)} = m(1)^{-R}. \tag{3.47}$$

Lemma 3.2.12. *Fix y and t. Then $G = G(x, \theta)$ is concave in x and θ on $[0, \infty) \times [0, \infty)$. In particular, if $\psi(\chi) = G(x - \chi y, y, \theta + \chi, t)$, then ψ is concave in χ.*

Proof. The proof follows similarly to the proof of Lemma 3.2.7, and makes use of the fact $dh/du = (1 - R)hW(h)$ proved in Lemma 3.2.11. □

Lemma 3.2.13. *Consider the candidate function constructed in (3.44)–(3.46). Then for $x > 0, \theta > 0$, $\mathcal{L}G = 0$, and $\mathcal{M}G \geq 0$. Further, $\mathcal{M}G = 0$ at $(x = 0, \theta > 0)$ and $\mathcal{L}G = 0$ at $x = 0$ and at $\theta = 0$.*

Proof. The majority of the lemma follows exactly as in Lemma 3.2.8.

For $\mathcal{M}G|_{x=0}$, note that $G_\theta|_{x=0} = yG(1 - R)/(x + y\theta)|_{x=0} = (1 - R)G/\theta$. Then, $yG_x|_{x=0-} = yG(1 - R)/(x + y\theta)|_{x=0-} = (1 - R)G/\theta$, whereas for $x > 0$,

$$yG_x = \frac{y(1-R)G}{x} - \frac{g'}{g}\frac{y^2\theta}{x^2}G = \frac{(1-R)G}{\theta}\left[z - \frac{z^2 g'(z)}{(1-R)g(z)}\right],$$

and then for fixed (y, θ)

$$\lim_{x\downarrow 0}\left[z - \frac{z^2 g'(z)}{(1-R)g(z)}\right] = \lim_{u\uparrow\infty} e^u\left(1 - \frac{h'(u)}{(1-R)h(u)}\right) = \lim_{v} e^{\gamma(v)}\left(1 - W(v)\right) = 1.$$

□

Proof of Theorem 3.1.11. For an admissible strategy $(C, \Theta) = (C_t, \Theta_t)_{t\geq 0}$ define the process $M(C, \Theta) = (M_t)_{t\geq 0}$ via

$$M_t = \int_0^t e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds + G\left(X_t, Y_t, 0, t\right). \tag{3.48}$$

where G is as given in (3.44)—(3.46).

Case 1: $\theta_0 = 0$ and $x_0 > 0$: we show $V = G$. For these initial values the agent does not own any units of asset for sale and consumption can only be financed from liquid (cash) wealth. Then $(\Theta_t)_{t\geq 0} = 0$, $dX_t = -C_t dt$ and the problem is non-stochastic. The candidate optimal consumption function is $C(x, y, 0) = \beta x/R$ and the associated consumption process is $C_t^* = \frac{\beta}{R}x_0 e^{-\frac{\beta}{R}t}$ with resulting wealth process $X_t^* = x_0 e^{-\frac{\beta}{R}t}$.

Then the value function is

$$\begin{aligned}\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{*1-R}}{1-R}dt\right] &= \int_0^\infty e^{-\beta t}\left(\frac{\beta}{R}\right)^{1-R}\frac{\left(e^{-\frac{\beta}{R}t}x_0\right)^{1-R}}{1-R}dt \\ &= \left(\frac{R}{\beta}\right)^R\frac{x_0^{1-R}}{1-R} = G(x_0,y_0,0,0),\end{aligned}$$

where the last equality follows from (3.46). Hence, we have $V \geq G$.

Now consider general admissible strategies. Let M^0 be given by $M_t^0 = M_t(C_t, 0)$. Applying Itô's formula to M^0, we get

$$\begin{aligned}M_t^0 - M_0^0 &= \int_0^t\left[e^{-\beta s}\frac{C_s^{1-R}}{1-R} - C_sG_x + \alpha Y_sG_y + \frac{1}{2}\eta^2Y_s^2G_{yy} + G_s\right]ds \\ &\quad + \int_0^t \eta Y_sG_ydB_s \\ &= N_t^1 + N_t^3.\end{aligned}$$

Lemma 3.2.13 implies that $\mathcal{L}G = 0$ and hence $N_t^1 = 0$.

Suppose $R < 1$. Then we have $0 \leq M_t^0 \leq M_0^0 + N_t^3$, and the local martingale N_t^3 is now bounded from below and hence a supermartingale. Taking expectations we conclude $\mathbb{E}(M_t^0) \leq M_0^0 = G(x_0, y_0, 0, 0)$, and hence

$$G(x_0,y_0,0,0) \geq \mathbb{E}\int_0^t e^{-\beta s}\frac{{C_s}^{1-R}}{1-R}ds + \mathbb{E}G(X_t,Y_t,0,t) \geq \mathbb{E}\int_0^t e^{-\beta s}\frac{{C_s}^{1-R}}{1-R}ds, \tag{3.49}$$

Letting $t \to \infty$, (3.49) we conclude

$$G(x_0,y_0,0,0) \geq \mathbb{E}\int_0^\infty e^{-\beta t}\frac{{C_t}^{1-R}}{1-R}dt.$$

and taking a supremum over admissible strategies we have $G \geq V$, and hence $G = V$.

For $R > 1$, a modification of the proof of Theorem 3.2.3 applies here also and $G = V$.

Case 2: $x_0 = 0$ and $\theta_0 > 0$: we show $V \geq G$. Under the candidate optimal strategy defined in Theorem 3.1.11 the consumption and sale processes evolve according to $C_tdt = -Y_td\Theta_t$, meaning that the investor finances consumption only from the sales of the endowed asset and wealth stays constant and identically zero. In this case, the proposed

strategies in (3.23) become

$$\Theta_t^* = \theta_0 e^{-\frac{\beta}{R}\phi t}, \qquad C_t^* = \frac{\beta}{R}\phi Y_t \Theta_t^* = \frac{\beta}{R}\phi y_0 \theta_0 \exp\left\{\beta(\epsilon - \delta^2/2 - \phi/R)t + \delta\sqrt{\beta}B_t\right\}.$$

where temporarily we write $\phi = m(1) = \delta^2 R(1-R)/2 - \epsilon(1-R) + 1 > 0$.

The corresponding value function is

$$\begin{aligned}
G^* &= \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{*1-R}}{1-R}dt\right] \\
&= \left(\frac{\beta}{R}\right)^{1-R}\frac{(\phi y_0\theta_0)^{1-R}}{1-R}\mathbb{E}\left[\int_0^\infty e^{-\beta t}e^{(1-R)\beta(\epsilon-\frac{\delta^2}{2}-\frac{\phi}{R})t+\delta\sqrt{\beta}(1-R)B_t}dt\right] \\
&= \left(\frac{\beta}{R}\right)^{1-R}\frac{(\phi y_0\theta_0)^{1-R}}{1-R}\int_0^\infty e^{\{(\epsilon(1-R)-\frac{\delta^2}{2}R(1-R)-1)-\frac{(1-R)}{R}\phi\}\beta t}dt \\
&= \left(\frac{R}{\beta}\right)^{1-R}\frac{(\phi y_0\theta_0)^{1-R}}{1-R}\int_0^\infty e^{-(\beta\phi/R)t}dt = \left(\frac{R}{\beta}\right)^{R}\frac{(y_0\theta_0)^{1-R}}{1-R}\phi^{-R} = G(0,y_0,\theta_0,0).
\end{aligned}$$

Then, under the candidate optimal strategy,

$$G(0,y_0,\theta_0,0) = \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{(C_t^*)^{1-R}}{1-R}dt\right],$$

and we have $G(0,y_0,\theta_0,0) \le V(0,y_0,\theta_0,0)$.
Case 3: $x_0 > 0$ and $\theta_0 > 0$: we show $V \ge G$. Let $M^* = M(C^*,\Theta^*)$ for the candidate optimal strategies in Theorem 3.1.11.

From the form of the candidate value function we know that G is $C^{1,2,1,1}$. Then applying Itô's formula to M^*, we have

$$\begin{aligned}
M_t^* - M_0^* &= \int_0^t\left[e^{-\beta s}\frac{(C_s^*)^{1-R}}{1-R} - C_s^* G_x + \alpha Y_s G_y + \frac{1}{2}\eta^2 Y_s^2 G_{yy} + G_t\right]ds \\
&\quad + \int_{(0,t]}(G_\theta - Y_s G_x)\,d\Theta_s \qquad (3.50)\\
&\quad + \int_0^t \eta Y_s G_y dB_s \\
&=: N_t^1 + N_t^2 + N_t^3.
\end{aligned}$$

Since $C_s^* = G_x^{-1/R}e^{\beta s/R}$ is optimal and, by Lemma 3.2.13, $\mathcal{L}G = 0$, we have $N_t^1 = 0$. Further, under the proposed strategies in (3.23), $d\Theta_t \neq 0$ if and only if $X_t = 0$. Then, by Lemma 3.2.13, $\mathcal{M}G|_{x=0} = 0$ and $N_t^2 = 0$.

The following Lemma is proved in the Section 3.4.

Lemma 3.2.14. *(1) N^3 given by $N_t^3 = \int_0^t \eta Y_s G_y(X_s^*, Y_s, \Theta_s^*, s) dB_s$ is a martingale.*
(2) $\lim_{t\uparrow\infty} \mathbb{E}[G(X_t^*, Y_t, \Theta_t^*, t)] = 0$

The conclusion that $V \geq G$ now follows exactly as in the proof of Theorem 3.1.7 but using Lemma 3.2.14 in place of Lemma 3.2.10.
Case 4: $x_0 \geq 0$ and $\theta_0 > 0$: $V \leq G$. To complete the proof of the theorem, it remains to show for $\theta_0 > 0$ and general admissible strategies, we have $V(x_0, y_0, \theta_0, 0) \leq G(x_0, y_0, \theta_0, 0)$. Recall the definition of M in (3.48).

Applying the generalised Itô's formula [23, Section 4.7] to M_t leads to the expression in (3.28) and

$$M_t - M_0 = N_t^1 + N_t^2 + N_t^3 + N_t^4.$$

Lemma 3.2.13 implies that under general admissible strategies, $N_t^1 \leq 0$, and $N_t^2 \leq 0$ with equality at $x = 0$. Consider the jump term,

$$N_t^3 = \sum_{0\leq s\leq t} \left[G\left(X_s, Y_s, \Theta_s, s\right) - G\left(X_{s-}, Y_s, \Theta_{s-}, s\right) - G_x(\Delta X)_s - G_\theta(\Delta\Theta)_s\right] \tag{3.51}$$

Using the fact that $(\Delta X)_s = -Y_s(\Delta\Theta)_s$ and writing $\theta = \Theta_{s-}$, $x = X_{s-}$, $\chi = -(\Delta\Theta)_s$ each non-zero jump in N^3 is of the form

$$(\Delta N^3)_s = G(x + y\chi, y, \theta - \chi, s) - G(x, y, \theta, s) + \chi\left[G_\theta(x, y, \theta, s) - yG_x(x, y, \theta, s)\right].$$

Note that by Lemma 3.2.12, $G(x + y\chi, y, \theta - \chi, s)$ is concave in χ and hence $(\Delta N^3) \leq 0$.

For the case $R < 1$ the remainder of the proof follows as in the proof of Theorem 3.2.3. The case $R > 1$ for general admissible strategies is covered in Section 3.5.

□

3.3 Properties of n

Recall the definitions of m and ℓ and the differential equation (3.6) for n, and also the definitions of q_ℓ, q_m, q_n and q^*. Define $\tilde{q} = \inf\{q > 0 : (1 - R)n(q) \geq (1 - R)\ell(q)\} \wedge 1$. Note that $m(0) = 1 = \ell(0)$ and $m(1) = 1 - \epsilon(1 - R) + \delta^2 R(1 - R)/2 = \ell(1)$. The concave function ℓ is positive on $(0, 1)$ if $\ell(1) = 1 - \epsilon(1 - R) + \delta^2 R(1 - R)/2 \geq 0$.

Lemma 3.3.1. *1. Define Φ via*

$$\Phi(\chi) = \chi^2 - (1 - R)\left(\frac{\delta^2}{2} - \epsilon + \frac{1}{R}\right)\chi - \epsilon\frac{(1 - R)^2}{R}.$$

Then for $R \in (0,1)$, $n'(0)$ is the smaller root of $\Phi(\chi) = 0$ and for $R \in (1,\infty)$, $n'(0)$ is the larger root.

2. *For $q \in (0, q_n \wedge \tilde{q})$, $n'(q) > 0$ if and only if $n(q) < m(q)$, similarly $n'(q) = 0$ if and only if $n(q) = m(q)$.*

3. *If $\ell(1) \geq 0$ then $\tilde{q} = q_n = q_\ell = 1$.*

4. *If $\ell(1) < 0$ then $\tilde{q} = q_n = q_\ell < q^*$.*

5. *If $0 \leq q^* < 1$ then $q^* > \epsilon/\delta^2 R$ and $(1-R)m$ is increasing on $(q^*, 1)$.*

Proof. (1) From the expression (3.6) and l'Hôpital's rule, $n'(0) = \chi$ solves

$$\chi = \frac{1-R}{R} - \frac{\delta^2}{2}\frac{(1-R)^2}{R}\frac{1}{(1-R)(\frac{\delta^2}{2} - \epsilon) - \chi},$$

or equivalently $\Phi(\chi) = 0$. Further $\ell'(0) = (1-R)\left(\frac{\delta^2}{2} - \epsilon\right)$ and

$$\Phi\left((1-R)\left(\frac{\delta^2}{2} - \epsilon\right)\right) = -\frac{\delta^2}{2}\frac{(1-R)^2}{R} < 0.$$

For $R < 1$, we have $n'(0) < \ell'(0)$ by hypothesis, so that $n'(0)$ is the smaller root of Φ. For $R > 1$, we have $n'(0) > \ell'(0)$ by hypothesis and $n'(0)$ is the larger root of Φ.

(2) This follows immediately from the expression for $n'(q)$.

(3) Suppose $R < 1$. Since $n'(0) < \ell'(0)$ we have $\tilde{q} > 0$. Notice that if $0 < n(q) < \ell(q)$ and $\ell(q) - n(q)$ is sufficiently small, then $n'(q) < \ell'(q)$. Hence $\tilde{q} \geq q_n$. Further, if $n(q) < \ell(q) - \phi$ for some $\phi > 0$ on some interval $\left[\underline{q}, \overline{q}\right] \subset (0,1)$, then $n'(q)/n(q)$ is bounded below by a constant on that interval and provided $n\left(\underline{q}\right) > 0$ it follows that $n(\overline{q}) > 0$ also. Hence, if ℓ is positive on $[0,1)$ then so is n and $q_n = 1$. For $R > 1$, we have $n'(0) > \ell'(0)$ and the result follows via a similar argument.

(4) Suppose $R < 1$. The same argument as above gives that $\tilde{q} = q_n = q_\ell$ and now these quantities are less than one. Clearly $q_m < q_\ell$, and m is decreasing on $(0, q_m)$. We cannot have $q^* \leq q_m$ for then $n'(q^*) - m'(q^*) > 0$ and $n(q^*) - m(q^*) = 0$ contradicting the minimality of q^*, nor can we have $q_m < q^* \leq q_\ell$ for on this region $m < 0 \leq n$.

(5) We can only have $q^* < 1$ if $m(1) > 0$ and $(1-R)m'(1) > 0$. For $R < 1$ we must have $n'(q^*) = 0 < m'(q^*)$. But m has a minimum at $\epsilon/\delta^2 R$, so $q^* > \epsilon/\delta^2 R$. For $R > 1$, we must have $n'(q^*) = 0 > m'(q^*)$. But m has a maximum at $\epsilon/\delta^2 R$, so $q^* > \epsilon/\delta^2 R$.

□

Proof of Proposition 3.1.2. (1) Note that $\Phi(m'(0)) = (1-R)^2\delta^2\epsilon/2$. Then, if $\epsilon < 0$ we have $n'(0) < m'(0)$ for $R < 1$ and $q^* = 0$. Otherwise, for $R > 1$, we have $n'(0) > m'(0)$ and $q^* = 0$. If $\epsilon = 0$ then $n'(0) = m'(0)$ and more care is needed.

Consider $R < 1$. Since $\epsilon \leq 0$, m is increasing. Suppose $n(\hat{q}) > m(\hat{q})$ for some $\hat{q}$ in $[0,1]$. Let $\underline{q} = \sup\{q < \hat{q} : n(q) = m(q)\}$. Then on $(\underline{q}, \hat{q})$ we have $n'(q) < 0 < m'(q)$ and $m(\hat{q}) - n(\hat{q}) = m(\underline{q}) - n(\underline{q}) + \int_{\underline{q}}^{\hat{q}}[m'(y) - n'(y)]dy > 0$, a contradiction.

For $R > 1$, the only difference is that m is decreasing given $\epsilon \leq 0$ and $n'(0) > m'(0)$.

(2) Consider first $R < 1$ and suppose that $0 < \epsilon < \min\{\delta^2 R, \frac{\delta^2}{2}R + \frac{1}{1-R}\}$. Then $m'(1) > 0$ and $m(1) > 0$. Since $\epsilon > 0$ we have $n'(0) > m'(0)$ and $n - m$ is positive at least initially. Write $n(q) = m(q) + \delta^2(1-R)qb(q)/2$. Then $n(q) \leq \ell(q)$ implies $b(q) \leq 1 - q$.

Suppose $b(q) > 0$ for all $q \in (0,1)$. Then $n(q) \geq m(q)$ and $n'(q) < 0$ so that $n(q) \geq n(1) = m(1)$ and

$$\begin{aligned} m(1) &= m(q) - (1-q)(1-R)\left(\epsilon - \delta^2 R\right) - (1-q)^2\delta^2 R(1-R)/2 \\ &> m(q) + \phi(1-q)\delta^2(1-R)q/2, \end{aligned}$$

for $q > \epsilon/\delta^2 R$ and $\phi < (\delta^2 R - \epsilon)\min\{\frac{2}{\delta^2}, \frac{R}{\epsilon}\}$. For such q, $b(q) > \phi(1-q)$. Hence

$$\frac{n'(q)}{n(q)} = -\frac{1-R}{R}\frac{b(q)}{(1-q)(1-q-b(q))} \leq -\frac{1-R}{R}\frac{\phi}{(1-q)(1-\phi)}$$

and we must have $n'(1-) = -\infty$ contradicting the fact that $n(q) \leq \ell(q)$. It follows that we must have $b(q) = 0$ for some $q \in (0,1)$. At this point n crosses m. Note that this crossing point is unique: at any crossing point $m'(q) > 0 = n'(q)$, so that all crossings of 0 in $(0,1)$ by $n - m$ are from above to below.

For $R > 1$, we have $m'(1) < 0$ and $m(1) > 0$. Since $\epsilon > 0$, we have $n'(0) < m'(0)$ and $n - m$ is negative initially. Let $n(q) = m(q) + \delta^2(1-R)qb(q)/2$. Then $n(q) \geq \ell(q)$ implies $b(q) \leq 1 - q$. Suppose $b(q) > 0$ for all $q \in (0,1)$, then it leads to the same contradiction for $R < 1$. It follows that $b(q) = 0$ for some $q \in (0,1)$, where n crosses m. At any crossing point $m'(q) < 0 = n'(q)$, so that n crosses m from below.

(3) $\epsilon \geq \delta^2 R$ and if $R < 1$, $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$.

Consider first $R < 1$. Since $\epsilon > 0$ we have that $n'(0) > m'(0)$ and $n > m$ in a neighbourhood to the right of zero. Further, m is decreasing and there are no solutions of $n = m$ since at any solution we must have that $0 = n' < m' < 0$.

For $R > 1$, we have m is increasing and $n'(0) < m'(0)$. There are no solutions of

$n = m$ in that at any solution we should have $0 = n' > m' > 0$.

(4) $R < 1$ and $\epsilon \geq \frac{\delta^2}{2} R + \frac{1}{1-R}$

Then $m(1) \leq 0$. Since m is decreasing at least until it hits zero, and since $n' = 0$ at a crossing point we cannot have that n crosses m before it hits zero. □

Proof of Proposition 3.2.4. (1) N solves

$$N'(q) = \frac{\frac{1}{2}\delta^2 (1-R)^2 qN(q)}{\ell(q) - N(q)^{-1/R}(1-q)^{1-1/R}}$$

and N is strictly increasing for $R < 1$. Otherwise, it is decreasing for $R > 1$. W solves

$$W'(v) = \frac{\ell(W(v)) - v^{-1/R}(1-W(v))^{1-1/R}}{\frac{1}{2}\delta^2 (1-R)^2 vW(v)} \tag{3.52}$$

(2) Follows from (3.8) and (3.52).

(3) Consider first $R < 1$. On $(0, q^*)$ we have $n(q) > m(q)$ and then $\ell(q) - n(q) < \ell(q) - m(q) = q(1-q)\delta^2(1-R)/2$. Then $v^{-1/R}(1-W(v))^{1-1/R} = n(W(v))$ and

$$v(1-R)W'(v) = \frac{\ell(W(v)) - n(W(v))}{\frac{\delta^2}{2}(1-R)W(v)} < 1 - W(v)$$

It follows that $w'(v) = (1-R)W(v) + v(1-R)W'(v) < 1 - RW(v)$. At q^*, $n(q^*) = m(q^*)$ and the inequality becomes an equality throughout.

For $R > 1$, we have $n(q) < m(q)$ on $(0, q^*)$ and $\ell(q) - n(q) > \ell(q) - m(q) = q(1-q)\delta^2(1-R)/2$. Then again $v(1-R)W'(v) < 1 - W(v)$ and $w'(v) < 1 - RW(v)$ with equality at h^*.

Note that since W is non-negative, $1 - RW(h) \leq 1$. □

3.4 The martingale property of the value function

Proof of Lemma 3.2.10. First we want to show the the local martingale

$$N_t^3 = \int_0^t \eta Y_s G_y(X_s^*, Y_s, \Theta_s^*, s) dB_s$$

is a martingale. This will follow if, for example,

$$\mathbb{E} \int_0^t (Y_s G_y(X_s^*, Y_s, \Theta_s^*, s))^2 ds < \infty \tag{3.53}$$

for each $t > 0$. From the form of the value function (3.34), we have

$$yG_y(x,y,\theta,s) = e^{-\beta t}\frac{x^{1-R}}{1-R}zg'(z) = G(x,y,\theta,t)\frac{zg'(z)}{g(z)} \leq (1-R)G(x,y,\theta,t) \quad (3.54)$$

where we use that $\frac{zg'(z)}{g(z)} = \frac{w(h)}{h} = (1-R)W(h)$ and $0 \leq W(h) \leq 1$.

Define a process $(D_t)_{t\geq 0}$ by $D_t = \ln G(X_t^*, Y_t, \Theta_t^*, t)$. Then D solves

$$\begin{aligned} D_t - D_0 &= \int_0^t \frac{1}{G}\left(G_t - C_s^* G_x + \alpha Y_s G_y + \frac{1}{2}\eta^2 Y_s^2 G_{yy}\right) ds \\ &\quad + \int_0^t \frac{1}{G}(G_\theta - Y_s G_x)\, d\Theta_s + \int_0^t \frac{1}{G}\eta Y_s G_y dB_s - \int_0^t \frac{1}{2G^2}\eta^2 Y_s^2 G_y^2 ds \\ &= -\int_0^t \frac{e^{-\frac{\beta}{R}s}}{1-R}\frac{1}{G}G_x^{\frac{R-1}{R}} ds + \int_0^t \frac{1}{G}\eta Y_s G_y dB_s - \int_0^t \frac{1}{2G^2}\eta^2 Y_s^2 G_y^2 ds. \end{aligned}$$

It follows that the candidate value function along the optimal trajectory has the representation

$$G(X_t^*, Y_t, \Theta_t^*, t) = G(X_0^*, y_0, \Theta_0^*, 0)\exp\left\{-\int_0^t \frac{e^{-\frac{1}{R}\beta s}}{1-R}\frac{1}{G}G_x^{\frac{R-1}{R}} ds\right\} H_t \quad (3.55)$$

where $H = (H_t)_{t\geq 0}$ is the exponential martingale

$$H_t = \mathcal{E}\left(\frac{\eta Y_s G_y}{G} \circ B\right)_t := \exp\left\{\int_0^t \frac{1}{G}\eta Y_s G_y dB_s - \int_0^t \frac{1}{2G^2}\eta^2 Y_s^2 G_y^2 ds\right\}.$$

Note that (3.54) implies $\frac{1}{G}\eta y G_y \leq \eta(1-R)$, so that H is indeed a martingale, and not merely a local martingale.

From (3.54) and (3.55), we have

$$\begin{aligned} (yG_y)^2 &= G(X_0, y_0, \Theta_0, 0)^2 \left(\frac{zg'(z)}{g(z)}\right)^2 \times \exp\left\{-2\int_0^t \frac{e^{-\frac{1}{R}\beta s}}{(1-R)}\frac{1}{G}G_x^{\frac{R-1}{R}} ds\right\} H_t^2 \\ &\leq G(X_0, y_0, \Theta_0, 0)^2 (1-R)^2 H_t^2. \end{aligned}$$

But

$$H_t^2 = \mathcal{E}\left(\frac{2}{G}\eta Y_s G_y \circ B\right)_t \exp\left\{\int_0^t \frac{1}{G^2}\eta^2 Y_s^2 G_y^2 ds\right\} \leq \mathcal{E}\left(\frac{2}{G}\eta Y_s G_y \circ B\right)_t e^{(1-R)^2\eta^2 t}.$$

Hence $\mathbb{E}[H_t^2] \leq e^{(1-R)^2\eta^2 t}$ and it follows that (3.53) holds for every t, and hence that the

local martingale $N_t^3 = \int_0^t \eta y G_y dB_s$ is a martingale under the optimal strategy.

(ii) Consider $\int_0^t \frac{e^{-\frac{1}{R}\beta s}}{1-R} \frac{1}{G} G_x^{\frac{R-1}{R}} ds$. To date we have merely argued that this function is increasing in t. Now we want to argue that it grows to infinity at least linearly. By (3.34), we have

$$\begin{aligned}
\frac{e^{-\frac{1}{R}\beta t}}{1-R}\frac{1}{G}G_x^{\frac{R-1}{R}} &= \frac{\left[g(z) - \frac{1}{1-R} z g'(z)\right]^{\frac{R-1}{R}}}{g(z)} = \frac{\left[h - \frac{1}{1-R} w(h)\right]^{\frac{R-1}{R}}}{h} \\
&= (1-W(h))^{1-1/R} h^{-1/R} = n(W(h)) \geq \min\{1, n(W(h^*))\} > 0.
\end{aligned}$$

Hence from (3.55) there exists a constant $k > 0$ such that

$$0 \leq (1-R)G(X_t^*, Y_t, \Theta_t^*, t) \leq (1-R)G(x_0, y_0, \theta_0, 0)e^{-kt}H_t \to 0$$

and then $G \to 0$ in L^1, as required. □

Proof of Lemma 3.2.14. This follows exactly as in the proof of Lemma 3.2.10.

□

3.5 Extension to $R > 1$

Verification Lemmas for the case $R > 1$. It remains to extend the proofs of the verification lemmas to the case $R > 1$. In particular we need to show that the candidate value function is an upper bound on the value function. The main idea is taken from Davis and Norman [13].

Suppose $G(x, y, \theta, t)$ is the candidate value function. Consider for $\varepsilon > 0$,

$$\widetilde{V}_\varepsilon(x, y, \theta, t) = \widetilde{V}(x, y, \theta, t) = G(x + \varepsilon, y, \theta, t) \tag{3.56}$$

and $\widetilde{M}_t = \widetilde{M}_t(C, \Theta)$ given by

$$\widetilde{M}_t = \int_0^t e^{-\beta s} \frac{C_s^{1-R}}{1-R} ds + \widetilde{V}(X_t, Y_t, \Theta_t, t),$$

Then,

$$\begin{aligned}
\widetilde{M}_t - \widetilde{M}_0 &= \int_0^t \left[e^{-\beta s} \frac{C_s^{1-R}}{1-R} - C_s \widetilde{V}_x + \alpha Y_s \widetilde{V}_y + \frac{1}{2}\eta^2 Y_s^2 \widetilde{V}_{yy} + \widetilde{V}_t \right] ds \\
&\quad + \int_0^t \left(\widetilde{V}_\theta - Y_s \widetilde{V}_x \right) d\Theta_s \\
&\quad + \sum_{0 \le s \le t} \left[\widetilde{V}(X_s, Y_s, \Theta_s, s) - \widetilde{V}(X_{s-}, Y_{s-}, \Theta_{s-}, s-) - \widetilde{V}_x (\triangle X)_s - \widetilde{V}_\theta (\triangle \Theta)_s \right] \\
&\quad + \int_0^t \eta Y_s \widetilde{V}_y dB_s \\
&= \widetilde{N}_t^1 + \widetilde{N}_t^2 + \widetilde{N}_t^3 + \widetilde{N}_t^4.
\end{aligned}$$

Lemma 3.2.2 (in the case $\epsilon \le 0$ and otherwise Lemma 3.2.8 or Lemma 3.2.13) implies $\widetilde{N}_t^1 \le 0$ and $\widetilde{N}_t^2 \le 0$. The concavity of $\widetilde{V}(x + y\chi, y, \theta - \chi, s)$ in χ (either directly if $\epsilon \le 0$, or using Lemma 3.2.7 or Lemma 3.2.12) implies $(\Delta \widetilde{N}^3) \le 0$.

Now define stopping times $\tau_n = \inf \left\{ t \ge 0 : \int_0^t \eta^2 Y_s^2 \widetilde{V}_y^2 ds \ge n \right\}$. It follows from (3.54) that $y\widetilde{V}_y$ is bounded and hence $\tau_n \uparrow \infty$. Then the local martingale $(\widetilde{N}^4_{t\wedge\tau_n})_{t\ge 0}$ is a martingale and taking expectations we have $\mathbb{E}\left(\widetilde{M}_{t\wedge\tau_n}\right) \le \widetilde{M}_0$, and hence

$$\mathbb{E}\left(\int_0^{t\wedge\tau_n} e^{-\beta s} \frac{C_s^{1-R}}{1-R} ds + \widetilde{V}\left(X_{t\wedge\tau_n}, Y_{t\wedge\tau_n}, \Theta_{t\wedge\tau_n}, t\wedge\tau_n\right) \right) \le \widetilde{V}\left(x_0, y_0, \theta_0, 0\right).$$

In the case $\epsilon \le 0$, (3.24) and (3.56) imply

$$\begin{aligned}
\widetilde{V}\left(x, y, \theta, t\right) &= e^{-\beta t} \frac{(x+\varepsilon)^{1-R}}{1-R} \left(1 + \frac{y\theta}{x+\varepsilon}\right)^{1-R} \left(\frac{R}{\beta}\right)^R \\
&\ge e^{-\beta t} \frac{(x+\varepsilon)^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R \ge \frac{\varepsilon^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R.
\end{aligned}$$

Thus $\widetilde{V}$ is bounded, $\lim_{n\to\infty} \mathbb{E}\widetilde{V}\left(X_{t\wedge\tau_n}, Y_{t\wedge\tau_n}, \Theta_{t\wedge\tau_n}, t\wedge\tau_n\right) = \mathbb{E}\left[\widetilde{V}\left(X_t, Y_t, \theta_t, t\right)\right]$, and

$$\widetilde{V}\left(x_0, y_0, \theta_0, 0\right) \ge \mathbb{E}\left(\int_0^t e^{-\beta s} \frac{C_s^{1-R}}{1-R} ds \right) + \mathbb{E}\left[\widetilde{V}\left(X_t, Y_t, \Theta_t, t\right)\right].$$

Similarly,

$$\widetilde{V}(x, y, \theta, t) \ge e^{-\beta t} \frac{\varepsilon^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R$$

and hence $\mathbb{E}\left[\widetilde{V}\left(X_t, Y_t, \Theta_t, t\right)\right] \to 0$. Then letting $t \to \infty$ and applying the monotone

convergence theorem, we have

$$\widetilde{V}_{\varepsilon}\left(x_0, y_0, \theta_0, 0\right) = \widetilde{V}\left(x_0, y_0, \theta_0, 0\right) \geq \mathbb{E}\left(\int_0^{\infty} e^{-\beta s} \frac{C_s^{1-R}}{1-R} ds\right)$$

Finally let $\varepsilon \to 0$. Then $V \leq \lim_{\varepsilon \downarrow 0} \widetilde{V} = G$. Hence, we have $V \leq G$.

The two non-degenerate cases are very similar, except that now from (3.34) and (3.56),

$$\widetilde{V}\left(x, y, \theta, t\right) = e^{-\beta t} \frac{(x+\varepsilon)^{1-R}}{1-R} g\left(\frac{y\theta}{x+\varepsilon}\right) \geq e^{-\beta t} \frac{\varepsilon^{1-R}}{1-R} \left(\frac{R}{\beta}\right)^R.$$

where we use that for $R > 1$, g is decreasing with $g\left(0\right) = (\frac{R}{\beta})^R > 0$. Hence $\widetilde{V}$ is bounded, and the argument proceeds as before.

□

Chapter 4

Comparative statics of the single-asset problem

In this chapter, we provide comparative statics describing how the outputs of the model depend on market parameters based on the analysis in Chapter 3. Some of the surprising implications are uncovered in this chapter. In particular, we give three sample conclusions from the comparative statics:

Result 3. *1. The optimal consumption process is not monotone in the drift of the endowed asset. Although we might expect that the higher the drift, the more the agent would consume, sometimes the agent's consumption is a decreasing function of the drift.*

2. The certainty equivalent value of the holdings of the risky asset is not monotone in risk aversion. For small quantities of endowed asset, the certainty equivalent value is increasing in risk aversion, while for larger quantities, it is decreasing.

3. The cost of illiquidity (see Definition 4.5.1 below), representing the loss in welfare of the agent when compared with an otherwise identical agent who can buy and sell the risky asset with zero transaction costs, is a U-shaped function of the size of the endowment in the risky asset.

This chapter consists of five sections, analysis of the optimal threshold z^*, the value function g, the optimal consumption $C(x, y, \theta)$, the utility indifference price $p(x, y, \theta)$, and the cost of illiquidity $p^*(x, y, \theta)$. The cost of illiquidity, defined in (4.3) below represents the loss in cash terms faced by our agent when compared with an otherwise identical

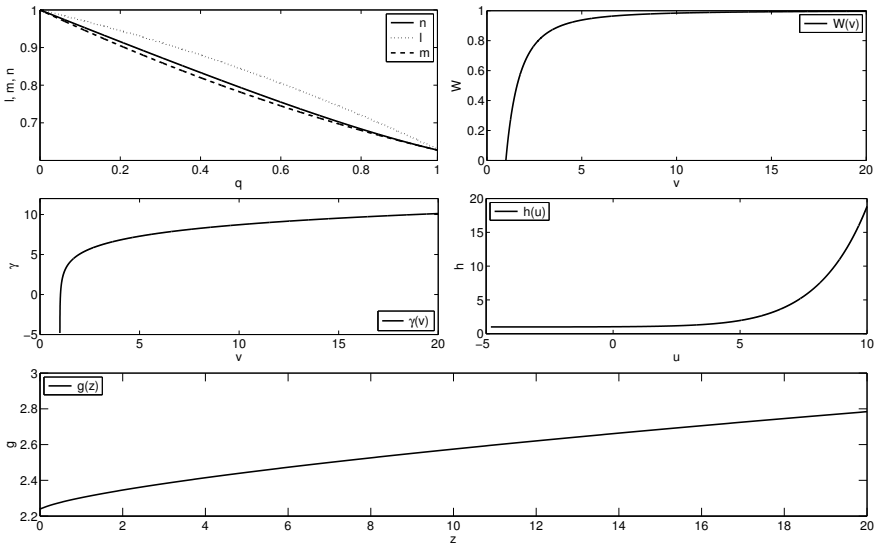

Figure 4.1: Transformations from m, n, ℓ to $W(v)$ to $\gamma(v)$ to $h(u)$ and $g(z)$ in the second non-degenerate scenario in the case $R < 1$. Parameters are $\epsilon = 1$ $\delta = 1$, $\beta = 0.1$ and $R = 0.5$. For these parameters m is monotonic decreasing.

agent with the same initial portfolio who is able to adjust her portfolio of the risky asset in either direction at zero cost.

The equations describing the function n and the first crossing of m are simple to implement in MATLAB, and then it also proved straightforward to calculate h or γ and thence the value function in the non-degenerate cases. Figures 4.1 and 4.2 are generic plots of the various functions used in the construction of the value function. The parameter values are such that we are in the second non-degenerate case ($\epsilon \geq \delta^2 R$ and $\epsilon < \frac{\delta^2 R}{2} + \frac{1}{1-R}$ if $R < 1$), but the figures would be similar for the first non-degenerate case ($0 < \epsilon < \delta^2 R$ and $\epsilon < \frac{\delta^2 R}{2} + \frac{1}{1-R}$ if $R < 1$). The two figures cover the cases $R < 1$ and $R > 1$ respectively. For $R < 1$, as plotted in Figure 4.1, m and n are monotone decreasing and W is increasing on $[1, \infty)$ with $\lim_{v \to 1} W(v) = 0$ and $\lim_{v \to \infty} W(v) = 1$. Further, we have $\gamma(v)$ is increasing on $[1, \infty)$ and g is concave and increasing. For $R > 1$, as plotted in Figure 4.2, m and n are monotone increasing and W is decreasing on $(0, 1]$ with $\lim_{v \to 0} W(v) = 1$ and $\lim_{v \to 1} W(v) = 0$. Finally, we have $\gamma(v)$ is decreasing on $(0, 1]$ and g is convex decreasing and convergent to zero as z tends to infinity.

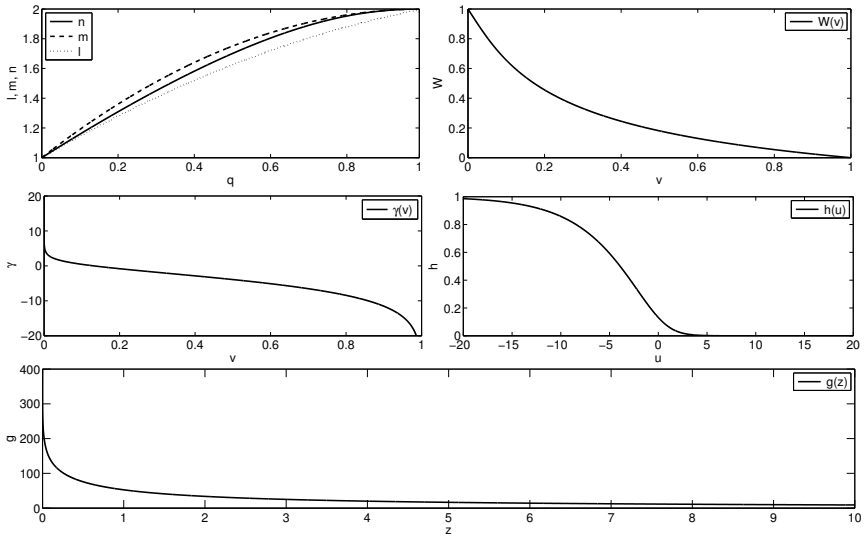

Figure 4.2: Transformations from m, n, ℓ to $W(v)$ to $\gamma(v)$ to $h(u)$ and $g(z)$ in the second non-degenerate scenario in the case $R > 1$. Parameters are $\epsilon = 3$ $\delta = 1$, $\beta = 0.1$ and $R = 2$.

4.1 Dependence of the critical threshold z^* on model parameters

Figures 4.3 and 4.4 show that z^* increases as mean return ϵ increases and decreases as volatility δ increases or risk aversion R increases. As ϵ increases, the non-traded asset Y becomes more valuable and it is optimal for the investor to wait longer to sell Y for a higher return. For $\epsilon = 0$, when the endowed asset has zero return but with additional risk, the optimal strategy is to sell immediately to remove the risk. Similarly, as δ increases, the level of z^* decreases as holding Y involves additional risk. Hence, it is optimal for the investor to sell units of Y sooner in order to mitigate this risk. As the risk aversion of the investor increases, she is less tolerant to the risk of the endowed asset and hence more inclined to sell Y earlier. As $R \to 0$, (provided $\epsilon > 0$) we have $z^* \to \infty$, which implies the optimal strategy is never to sell the asset. In the limit the investor is not concerned about the risk of holding the risky asset. Conversely, as $R \to \infty$, we have $z^* \to 0$. In this case, the investor cannot tolerate any risks and it is therefore optimal to sell the asset immediately to arrive at a safe position.

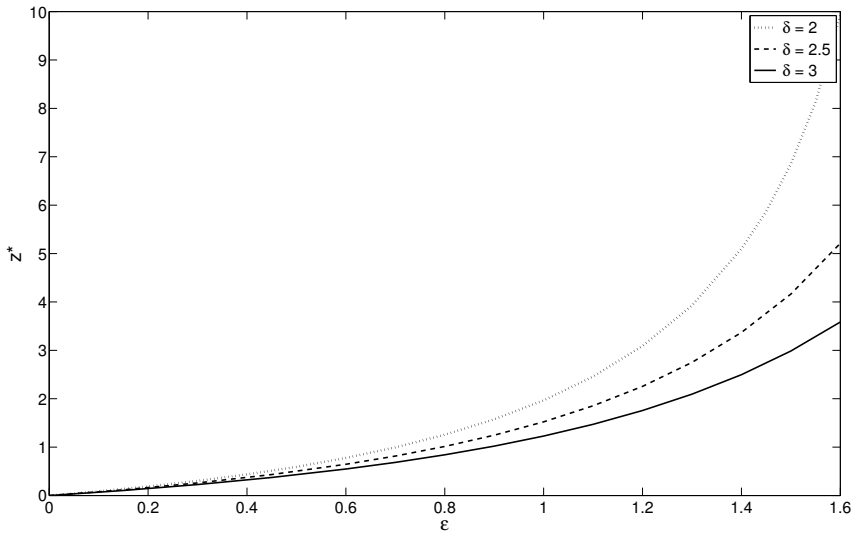

Figure 4.3: z^* increases as ϵ increases or as δ increases. Here $\beta = 0.1$ and $R = 0.5$.

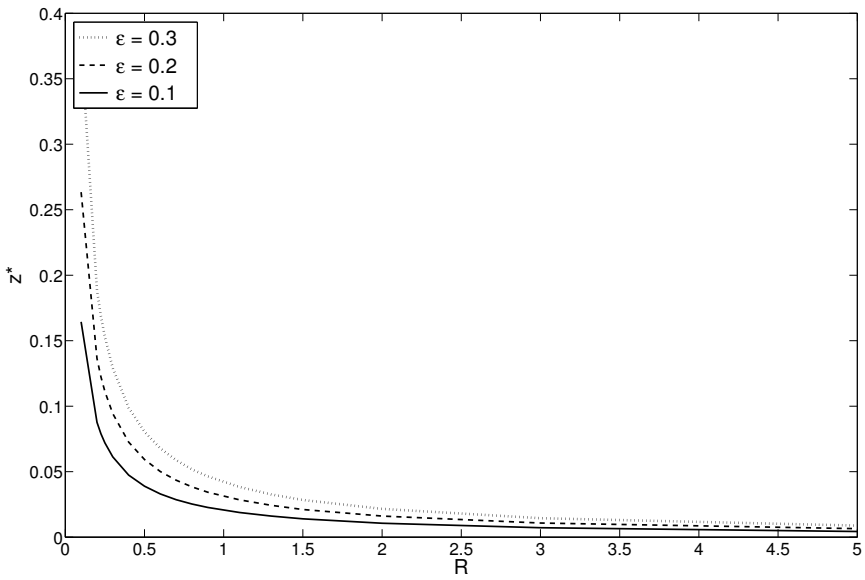

Figure 4.4: z^* decreases as R increases or as ϵ decreases. Here $\delta = 3$ and $\beta = 0.1$.

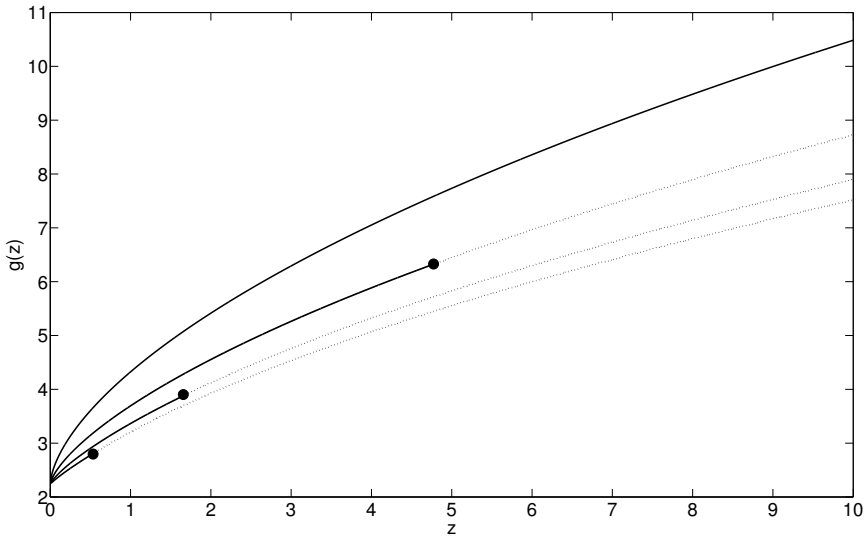

Figure 4.5: $g(z)$ with different ϵ in the first and second non-degenerate scenarios. Dotted line: $z \geq z^*$, solid line: $z \leq z^*$ and dots represent z^*. ϵ varies from top to bottom as 2, 1.5, 1, 0.5, with fixed parameters $\delta = 2$, $\beta = 0.1$ and $R = 0.5$. The top line is the value function g in the second non-degenerate scenario given $\epsilon = \delta^2 R = 2$ and z^* is infinite.

4.2 Dependence of the value function g on model parameters

The value function as expressed via g in non-degenerate cases is plotted in Figures 4.5, 4.6 and 4.7 under different drifts, volatilities and risk aversions. These figures show that g is increasing in drift and decreasing in volatility, while g has no monotonicity in risk aversion. As the non-traded asset becomes more valuable, the investor can choose optimal sale and consumption strategies which lead to a larger value function. Further, as the asset becomes more risky, the additional risk makes the value function smaller. Meanwhile, as ϵ increases, z^* in Figure 4.5 is decreasing and as δ increases, z^* in Figure 4.6 is increasing. (These results are consistent with the results in described in the previous paragraph.) At $z = z^*$, smooth fit conditions are satisfied. Observe also that for different values of drift and volatilities, we have g starts at the same point. This corresponds to the value function when $\theta_0 = 0$ whereby consumption is only financed by initial wealth and this deterministic problem is the Merton model without any risky assets. In this case, Theorem 1.2.5 implies $g(0) = (R/\beta)^R$.

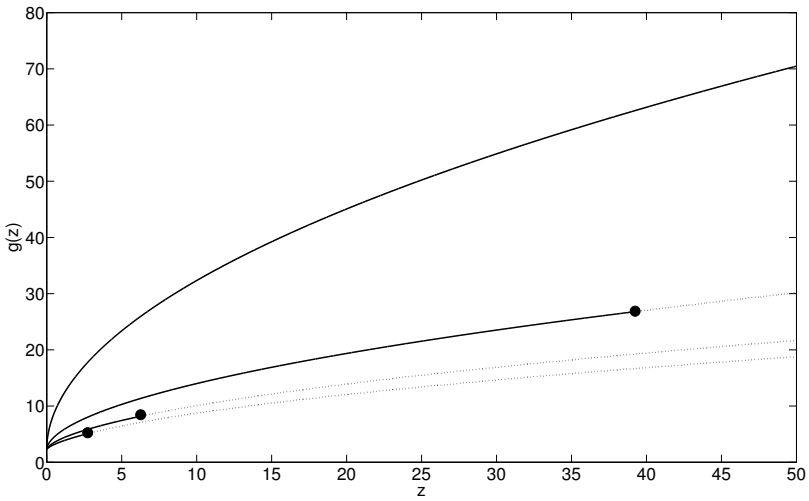

Figure 4.6: $g(z)$ with different δ in the first and second non-degenerate scenarios. Dotted line: $z \geq z^*$, solid line: $z \leq z^*$ and dots represent z^*. δ varies from top to bottom as 2.1, 2.5, 3, 3.5 with fixed parameters $\epsilon = 3$, $\beta = 0.1$ and $R = 0.5$. The top line is the value function g in the second non-degenerate scenario given $\delta = 2.1 < \sqrt{\epsilon/R} = 2.45$ and z^* is infinite.

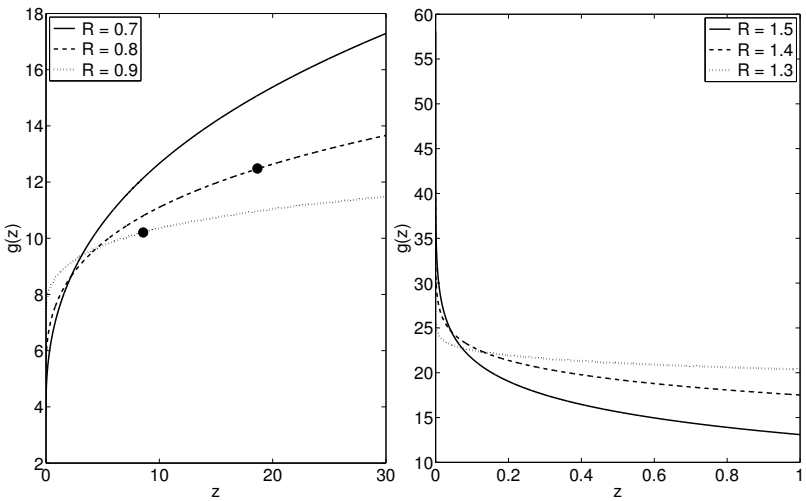

Figure 4.7: $g(z)$ with different risk aversion R in the first and second non-degenerate scenarios. In the left graph, R takes values in 0.7, 0.8 and 0.9. The rest of the parameters are $\epsilon = 3$, $\delta = 2$, $\beta = 0.1$. The critical risk aversion is $R = \epsilon/\delta^2 = 0.75$. The dots represent finite z^* and the solid line is the value function g in the second non-degenerate scenario with infinite z^*. In the right graph, R takes values in 1.3, 1.4 and 1.5 and the rest of the parameters are $\epsilon = 6$, $\delta = 2$ and $\beta = 0.1$.

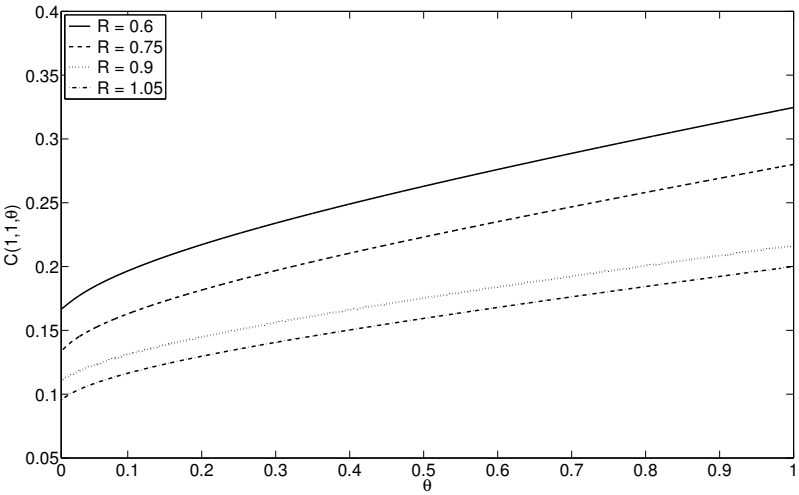

Figure 4.8: Optimal consumption $C(1,1,\theta)$ as R varies. R takes values in 0.6, 0.75, 0.9, 1.05 with parameters $\epsilon = 3$, $\delta = 2$, $\beta = 0.1$ and $\theta \in [0,1]$. The critical risk aversion is $R = \epsilon/\delta^2 = 0.75$. The top two lines correspond to the optimal consumption in the second non-degenerate scenario where z^* is infinite under the condition that $\epsilon \geq \delta^2 R$. The bottom two lines correspond to the first non-degenerate case with finite z^*.

4.3 Dependence of the optimal consumption C^* on model parameters

Optimal consumption $C(x,y,\theta)$ is considered in Figures 4.8—4.12. Figure 4.8 plots the optimal consumption $C(1,1,\theta)$ as a function of endowed units θ and shows that the optimal consumption increases in θ: as the size of the holdings of the non-traded asset Y increases, the agent feels richer and hence consumes at a faster rate. For $\theta = 0$, Theorem 1.2.5 implies the optimal consumption is $C(x,y,0) = xg(0)^{-\frac{1}{R}} = \frac{\beta}{R}x$, which is strictly positive and is financed from cash wealth. Figure 4.8 also suggests that the optimal consumption $C(1,1,\theta)$ decreases in risk aversion. Given the set of parameters the critical risk aversion (i.e. the boundary between the two non-degenerate cases) is at $R = \epsilon/\delta^2 = 0.75$. For the bottom two lines in Figure 4.8 with $R > 0.75$, we have $\epsilon < \delta^2 R$ and this falls into the first non-degenerate case with finite z^*. For $R \leq 0.75$, we have $\epsilon \geq \delta^2 R$, which is the second non-degenerate case with infinite z^*. As we see, there is no discontinuity in consumption with respect to risk aversion at either $R = 0.75$ or $R = 1$. The optimal consumptions for different risk aversions differ primarily in the levels, and the dominant factor is the optimal consumption for $\theta = 0$. As argued above $C(x,y,0) = \beta x/R$ is decreasing in R.

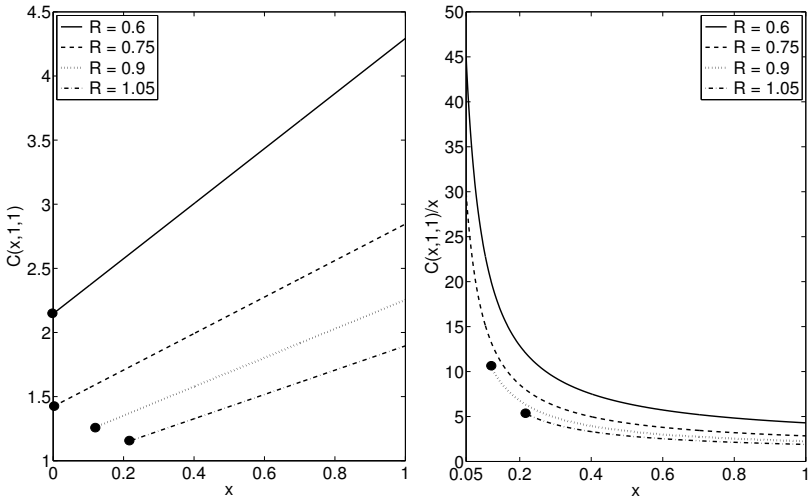

Figure 4.9: Optimal consumption $C(x,1,1)$ and $C(x,1,1)/x$ as R varies. R takes values in 0.6, 0.75, 0.9 and 1.05 with parameters $\epsilon = 3$, $\delta = 2$, $y_0 = 1$ and $\theta_0 = 1$. The dots represent $x^* = 1/z^*$ and the critical risk aversion is $R = \epsilon/\delta^2 = 0.75$. In both graphs, the top two lines correspond to the optimal consumptions in the second non-degenerate case with $x^* = 0$. The bottom two lines are the optimal consumptions in the first non-degenerate case with finite z^*, or equivalently, $x^* > 0$.

Figure 4.9 plots both consumption as a function of wealth $C(x,1,1)$ and the ratio of consumption to wealth $C(x,1,1)/x$ as a function of x with different risk aversions. Note that this can only be shown for $x > y\theta/z^* = 1/z^*$ since if $x < 1/z^*$ the agent makes an immediate sale of units of risky asset. The critical value of the risk aversion is $R = \epsilon/\delta^2 = 0.75$. For $R > 0.75$, we have $z^* < \infty$ and $x^* = 1/z^* > 0$ while for $R \leq 0.75$, $z^* = \infty$ and $x^* = 1/z^* = 0$. The results show that the optimal rate of consumption is an increasing function of wealth but that consumption per unit wealth is a decreasing function of wealth. (In the standard Merton problem, consumption is proportional to wealth.) As the agent becomes richer, she consumes more, but the fraction of wealth that she consumes becomes smaller. The explanation is that her endowed wealth is being held constant. By scaling we have that if both x and θ are increased by the same factor, then consumption would also rise by the same factor, but here x is increasing, but θ (and y) are held constant, and hence consumption increases more slowly than wealth. In the limit $x \to \infty$ we have $\lim_{x\to\infty} C(x,1,1) = \infty$ and $\lim_{x\to\infty} C(x,y,\theta)/x = g(0)^{-\frac{1}{R}} = \beta/R$.

Figures 4.10 and 4.11 plot the optimal consumption $C(1,1,\theta)$ as a function of θ, ϵ and θ, δ. Here we find a first surprising result: we might expect the optimal consumption $C(x,y,\theta)$ to be increasing in the drift, but this is not the case for large

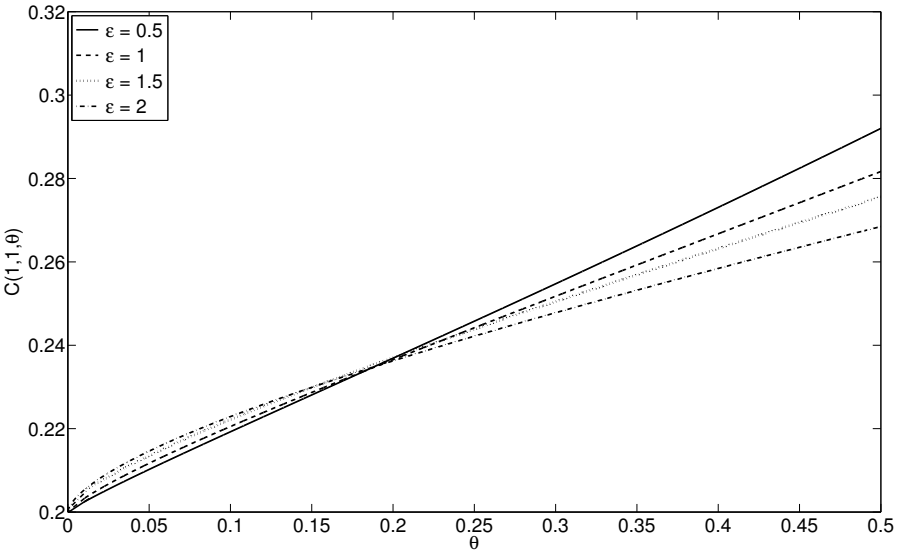

Figure 4.10: Optimal consumption $C(1, 1, \theta)$ as ϵ varies. ϵ takes values in 0.5, 1, 1.5 and 2 with parameters $\delta = 2$, $\beta = 0.1$, $R = 0.5$, $x_0 = 1$ and $y_0 = 1$. The critical mean return is $\epsilon = \delta^2 R = 2$. When $\epsilon = 2$ we are in the second non-degenerate case.

θ. For an explanation of this phenomena, recall that the optimal exercise ratio z^* is increasing in the drift. As the drift increases, the asset has a more promising return on average which makes the agent feel richer and consume at a higher rate. However, a larger drift also implies a larger z^*, indicating that the agent should postpone the sale of the risky asset. Hence, a larger drift involves more risk, and in order to mitigate this risk, the agent consumes less in the short term. Hence, the optimal consumption decreases in the drift for large θ. For a similar reason, optimal consumption is not necessarily decreasing in volatility and optimal consumption as a function of x is not necessarily increasing in the drift either as shown in Figure 4.12.

4.4 Dependence of the indifference price p on model parameters

Figures 4.13—4.19 plot the utility indifference price or certainty equivalence value $p(x, y, \theta)$. Recall that in the second and third cases of Theorem 3.1.5 the certainty equivalent value

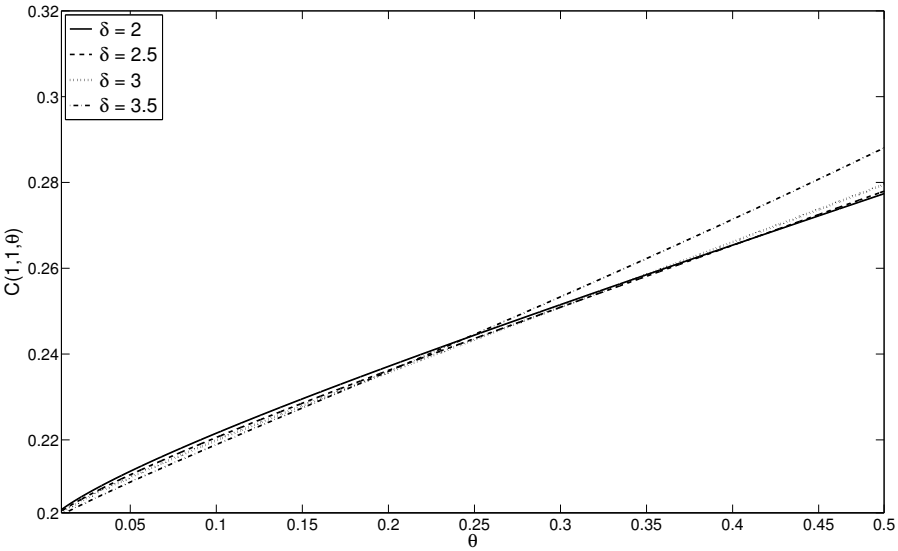

Figure 4.11: Optimal consumption $C(1,1,\theta)$ as δ varies. δ takes values in 2, 2.5, 3 and 3.5 with parameters $\epsilon = 3$, $\beta = 0.1$, $R = 0.5$, $x_0 = 1$ and $y_0 = 1$. The critical volatility is $\delta = \sqrt{\epsilon/R} = 2.45$. The solid line corresponds to the optimal consumption in the second non-degenerate case.

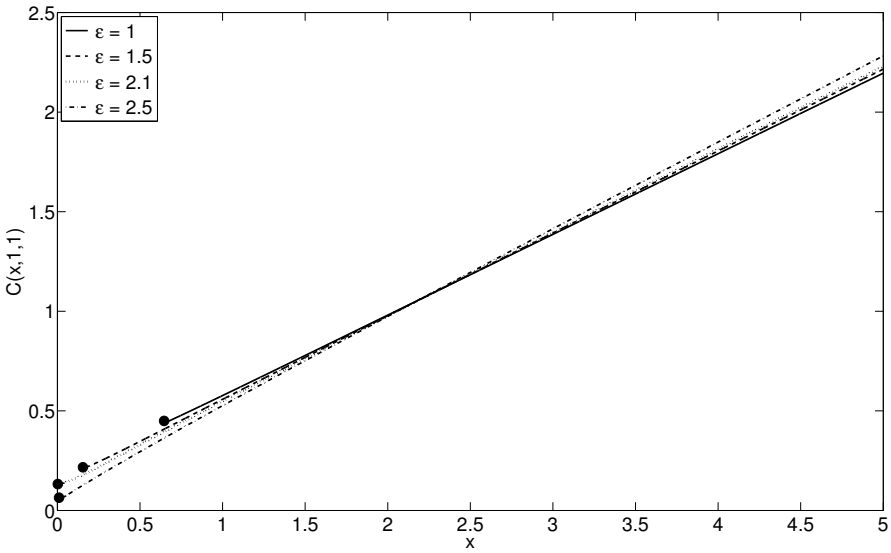

Figure 4.12: Optimal consumption $C(x,1,1)$ as ϵ varies. ϵ takes values in 1, 1.5, 2.1 and 2.5 with parameters $\delta = 2$, $\beta = 0.1$, $R = 0.5$, $y_0 = 1$ and $\theta_0 = 1$. The dots represent $x^* = 1/z^*$ and the critical mean return is $\epsilon = \delta^2 R = 2$. The lines with $x^* = 0$ corresponds to the optimal consumption in the second non-degenerate case and the lines with $x^* > 0$ corresponds to the first non-degenerate case.

of the non-traded asset is given

$$p(x,y,\theta) = x\left[\frac{g\left(\frac{y\theta}{x}\right)}{g(0)}\right]^{\frac{1}{1-R}} - x$$

Figures 4.13 and 4.14 consider the indifference price as a function of wealth. Dots in figures represent the optimal exercise ratio $z^* = y\theta/x$. In each of the figures we choose a range of parameter values such that sometimes we are in the first non-degenerate case, and sometimes in the second non-degenerate case. In Figure 4.13, for $\epsilon < 2$, we have $z^* < \infty$ and $x^* = 1/z^* > 0$, and for $\epsilon \geq 2$, we have $z^* = \infty$ and $x^* = 0$. We can see $p(x,1,1)$ is concave and increasing in x. It follows from Theorem 3.1.7 that $g(z) = (R/\beta)^R m(q^*)^{-R}(1+z)^{1-R}$ for $z \geq z^*$. Further, under the condition that $0 < \epsilon < \delta^2 R$ and $\epsilon < \frac{\delta^2}{2}R + \frac{1}{1-R}$, which ensures a finite exercise ratio,

$$\begin{aligned}\lim_{x\to 0} p(x,y,\theta) &= \lim_{x\to 0} x\left\{\left[\frac{g\left(\frac{y\theta}{x}\right)}{g(0)}\right]^{\frac{1}{1-R}} - 1\right\} \\ &= \lim_{x\to 0}\left\{m(q^*)^{\frac{R}{R-1}}(x+y\theta) - x\right\} = m(q^*)^{\frac{R}{R-1}}y\theta > y\theta.\end{aligned}$$

In that case, for $x = 0$, where no initial wealth is available to finance consumption, it is optimal for the investor to sell some units of the endowed asset Y immediately so as to keep the ratio of the wealth invested in the endowed asset to liquid wealth below z^*, i.e. from the initial portfolio ($x = 0$, $\theta = \Theta_{0-}$) the agent moves to ($x = X_{0+}$, $\theta = \Theta_{0+}$), where $\Theta_{0+} = \frac{z^*}{1+z^*}\Theta_{0-}$ and $X_{0+} = \frac{1}{1+z^*}y\Theta_{0-}$. The monotonicity of $p(x,1,1)$ in ϵ and δ is also illustrated in Figures 4.13 and 4.14: a higher mean return adds value to the asset, while the increasing volatility makes Y more risky and reduces value. Also observe that for the drift larger than the critical value, the change in drift does not move the dot while for the drift smaller than the critical value, the dot (representing the critical ratio) moves rightwards as drift increases. To the left of the dot, the agent should sell the endowed asset initially, while to the right of the dot, the agent should wait. As drift increases, the agent should wait longer for a higher return when selling the asset.

Figure 4.15 considers the indifference price $p(1,1,\theta)$ and unit indifference price $p(1,1,\theta)/\theta$ as a function of θ. We see that $p(1,1,\theta)$ is increasing in θ and for $\theta = 0$, $p(1,1,0) = 0$, reflecting the fact that a null holding is worth nothing. We also have the unit price $p(1,1,\theta)/\theta$ is decreasing in the units of asset θ. For small holdings, the marginal price $\lim_{\theta\to 0} p(1,1,\theta)/\theta$ is infinite. As $\theta \to \infty$, the figures imply that the unit

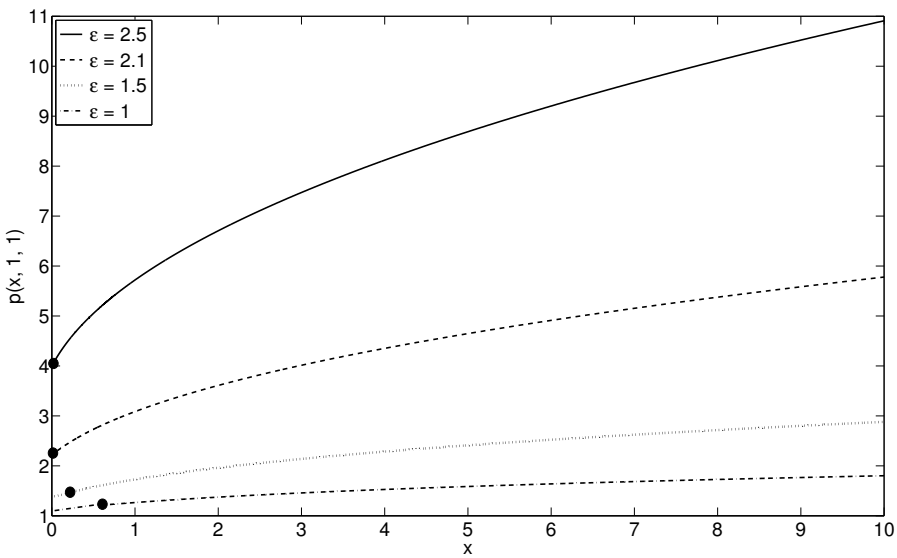

Figure 4.13: Indifference price $p(x, 1, 1)$ as ϵ varies. ϵ varies from top to bottom as 2.5, 2.1, 1.5, 1 with fixed parameters $\delta = 2$, $\beta = 0.1$, $R = 0.5$, $\theta_0 = 1$ and $y_0 = 1$. The dots represent $x^* = 1/z^*$ and the critical mean return is $\epsilon = \delta^2 R = 2$.

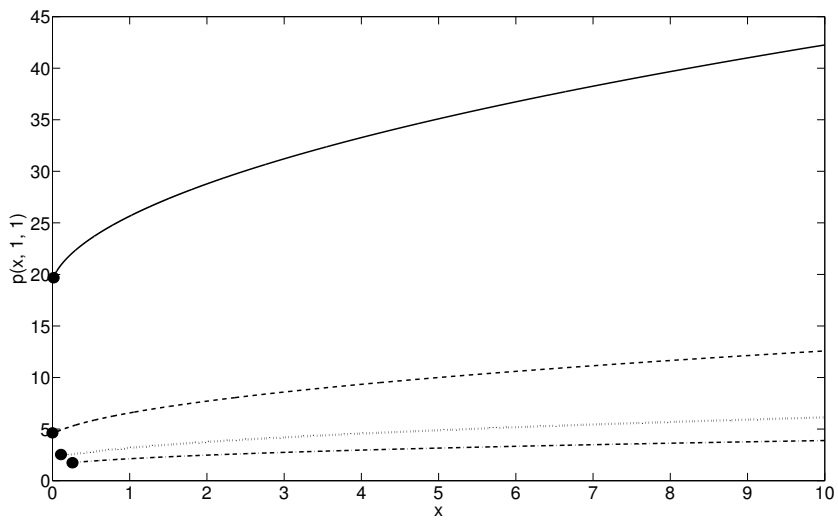

Figure 4.14: Indifference price $p(x, 1, 1)$. δ varies from top to bottom as 2.1, 2.4, 2.8 and 3.2 with fixed parameters $\epsilon = 3$, $\beta = 0.1$, $R = 0.5$, $\theta_0 = 1$ and $y_0 = 1$. The dots represent $x^* = 1/z^*$ and the critical volatility is $\delta = \sqrt{\epsilon/R} = 2.45$. The top two lines correspond to the indifference prices in the second non-degenerate case with $x^* = 0$. The bottom two lines are indifference prices in the first non-degenerate case with $x^* > 0$.

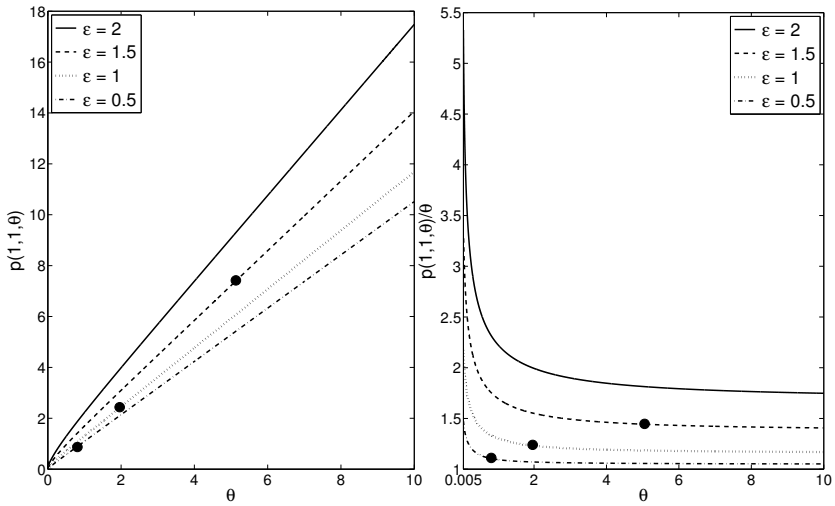

Figure 4.15: Indifference price $p(1,1,\theta)$ and unit price $p(1,1,\theta)/\theta$. ϵ varies from top to bottom as 2, 1.5, 1, 0.5 with fixed parameters $\delta = 2$, $\beta = 0.1$, $R = 0.5$, $x_0 = 1$ and $y_0 = 1$. The dots represent $\theta^* = z^*$ and the critical mean return is $\epsilon = \delta^2 R = 2$. The top line corresponds to the indifference price in the second non-degenerate case with infinite z^*.

price $p(1,1,\theta)/\theta$ tends to some constant larger than the unit price y of Y:

$$\lim_{\theta\to\infty} \frac{p(x,y,\theta)}{\theta} = \lim_{\theta\to\infty} \frac{x\left[\frac{g\left(\frac{y\theta}{x}\right)}{g(0)}\right]^{\frac{1}{1-R}} - x}{\theta} = \lim_{\theta\to\infty} \frac{m(q^*)^{\frac{R}{R-1}}(x+y\theta) - x}{\theta} = m(q^*)^{\frac{R}{R-1}} y > y,$$

where the second equality follows since for $z \geq z^*$, we have

$$g(z) = (R/\beta)^R m(q^*)^{-R}(1+z)^{1-R}.$$

Figure 4.15 also illustrates the monotonicity of p in the drift parameter ϵ and we have $p(1,1,\theta)$ and $p(1,1,\theta)/\theta$ both increase in the drift. Similarly, it can be shown that p is decreasing in δ as shown in Figure 4.16, reflecting the increased riskiness of positions as volatility increases.

Figure 4.17 plots the indifference price as a function of cash wealth for different risk aversions. The results show that the indifference price is not monotone decreasing in risk aversion, and for large wealths the utility indifference price is increasing in R. (If we fix wealth x and consider the certainty equivalent value as a function of quantity θ then we find a similar reversal, and the certainty equivalent value is increasing in R for small θ.) This recovers the results in Figure 2.3 and 2.6 in Section 2.2.2 in which we consider

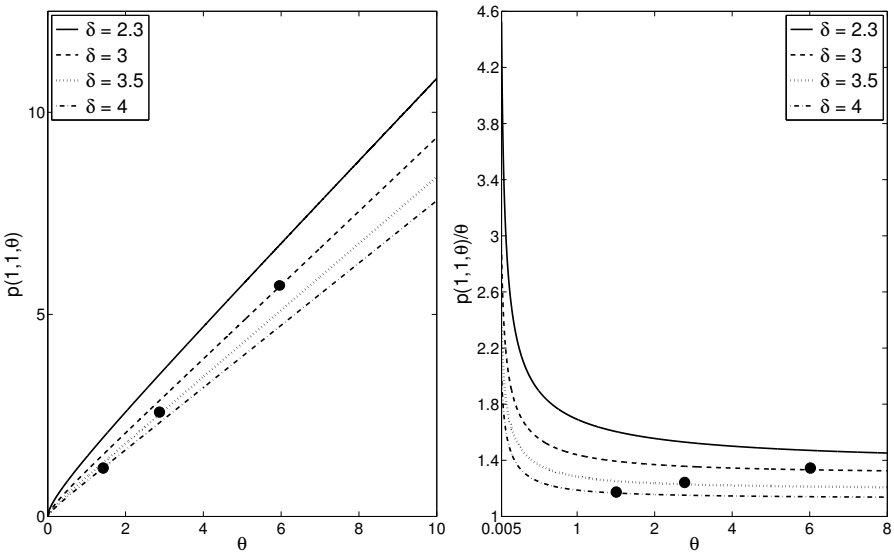

Figure 4.16: Indifference price $p(1,1,\theta)$ and unit price $p(1,1,\theta)/\theta$. δ varies from top to bottom as 2.3, 3, 3.5, 4 with fixed parameters $\epsilon = 3$, $\beta = 0.1$, $R = 0.5$, $x_0 = 1$ and $y_0 = 1$. The critical volatility is $\delta = \sqrt{\epsilon/R} = 2.45$.

a much simpler problem.

Here is an explanation of this phenomena followed by a similar argument in Section 2.2.3. Consider an agent with positive cash wealth and zero endowment of the risky asset. This agent consumes at rate $\beta x/R$; in particular, as the parameter R increases, the agent consumes more slowly. The introduction of a small endowment will not change this result, and in general, an increase in the parameter R postpones the time at which the critical ratio reaches z^*. (Although z^* depends on R also, this is a secondary effect.) Since the endowed asset is appreciating, on average, by the time the agent chooses to start selling the asset, it will be worth more. The total effect is to make the indifference price increasing in R. Similarly, the indifference price $p(1,1,\theta)$ and the unit indifference price $p(1,1,\theta)/\theta$ as functions of θ are not necessarily monotone in risk aversion as shown in Figures 4.18 and 4.19.

4.5 The cost of illiquidity

Finally, we consider the impact of the illiquidity assumption. We do this by considering the value function of our agent who cannot buy the endowed asset and comparing it with the value function of an otherwise identical agent, but who can both buy and sell the

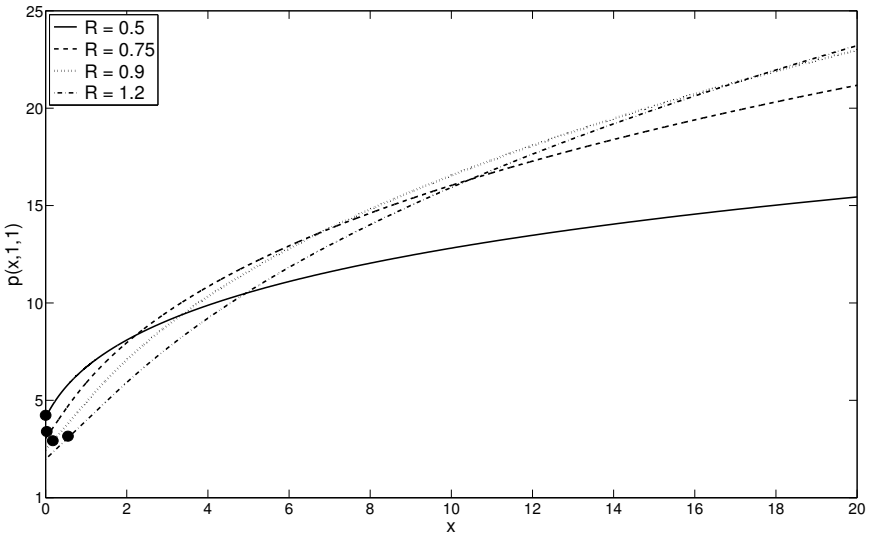

Figure 4.17: Indifference price $p(x, 1, 1)$. R takes values in 0.5, 0.75, 0.9 and 1.2 with fixed parameters $\epsilon = 3$, $\delta = 2$, $\beta = 0.1$, $y_0 = 1$ and $\theta_0 = 1$. The dots represent $x^* = 1/z^*$ and the critical risk aversion is $R = \epsilon/\delta^2 = 0.75$. The top two lines for $x \in [0, 1]$ correspond to the indifference prices in the second non-degenerate case with $x^* = 0$. The bottom two lines are indifference prices in the first non-degenerate case with $x^* > 0$.

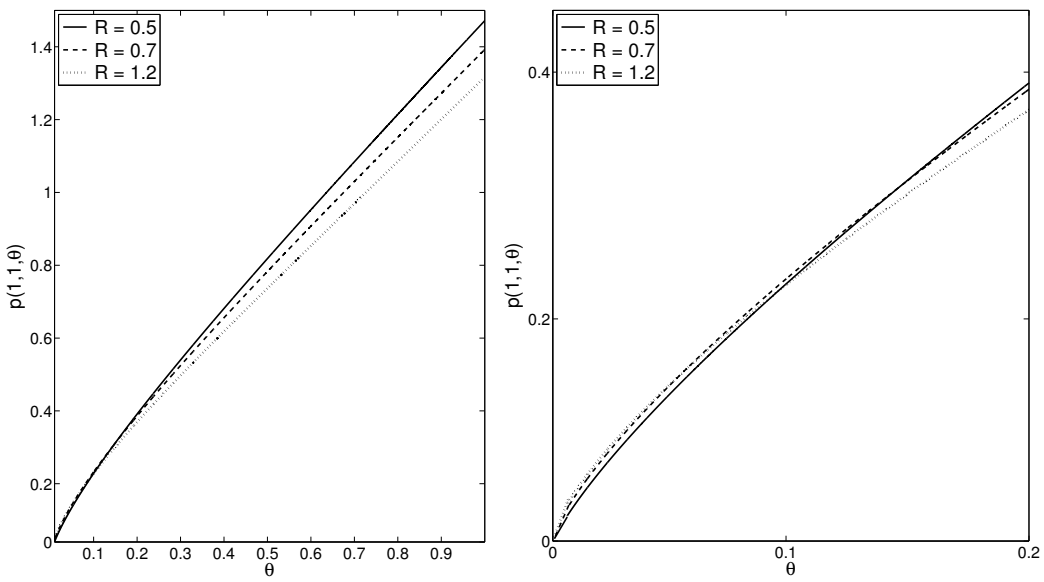

Figure 4.18: Indifference price $p(1, 1, \theta)$ for $\theta \in [0, 1]$(left picture) and an enlarged picture near $\theta = 0$. R takes values in 0.5, 0.7 and 1.2 with fixed parameters $\epsilon = 1$ and $\delta = 2$.

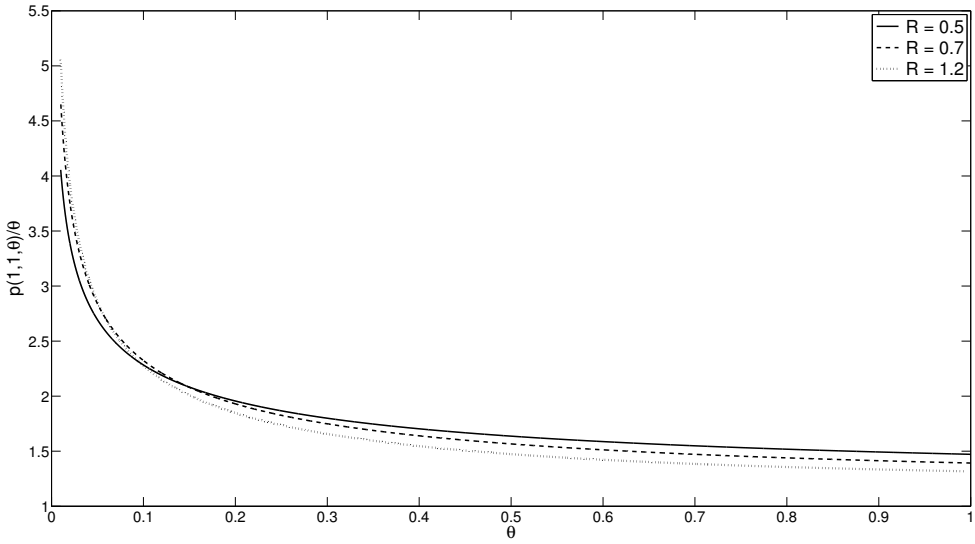

Figure 4.19: Unit price $p(1,1,\theta)/\theta$. R takes values in 0.5, 0.7 and 1.2 with fixed parameters $\epsilon = 1$ and $\delta = 2$. Here, number of holdings $\theta \in [0,1]$.

endowed asset with zero transaction costs. Suppose parameters are such that we are in the second case of Theorem 3.1.5. In the illiquid market, where Y is only allowed for sale, Theorem 3.1.7 proves the value function is

$$V_I(x,y,\theta,0) = \frac{x^{1-R}}{1-R} g\left(\frac{y\theta}{x}\right) = \sup_{(C,\Theta)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} \frac{C_t^{1-R}}{1-R} dt\right], \tag{4.1}$$

where the newly introduced subscript I stands for the value function in the illiquid market, in which the asset can only be sold.

In a liquid market such that Y can be dynamically traded, wealth evolves as $dX_t = -C_t dt + \Pi_t dY_t / Y_t$. Here $(\Pi)_{t\geq 0}$ represents the portfolio process. We suppose the agent is endowed with Θ_0 units of Y initially and is constrained to keep X positive. This is the Merton model and we know from Theorem 1.2.5 that the optimal strategy is to keep a constant fraction of wealth in the risky asset. The initial endowment therefore only changes initial wealth and the value function is

$$V_L(x,y,\theta,0) = \sup_{(C,\Pi)} \mathbb{E}\left[\int_0^\infty e^{-\beta t} \frac{C_t^{1-R}}{1-R} dt\right] = \frac{(x+y\theta)^{1-R}}{1-R}\left[\frac{\beta}{R} - \frac{\alpha^2(1-R)}{2\sigma^2 R^2}\right]^{-R}, \tag{4.2}$$

where the subscript L stands for the value function in the liquid market.

Now we consider the cost of illiquidity.

Definition 4.5.1. *The cost of illiquidity, denoted $p^* = p^*(x, y, \theta)$ is the solution to*

$$V_L(x - P^*, y, \theta, t) = V_I(x, y, \theta, t). \tag{4.3}$$

and represents the amount of cash wealth the agent who can only sell the risky asset would be prepared to forgo, in order to be able to trade the risky asset with zero transaction costs.

Equating (4.1) and (4.2), we can solve for p^* to obtain

$$p^*(x, y, \theta) = x \left[1 + \frac{y\theta}{x} - g \left(\frac{y\theta}{x} \right)^{\frac{1}{1-R}} \left(\frac{\beta}{R} - \frac{\alpha^2(1-R)}{2\sigma^2 R^2} \right)^{\frac{R}{1-R}} \right]. \tag{4.4}$$

Consider (4.4) when $\theta = 0$, where the investor is not endowed any units of Y initially, we have

$$p^*(x, y, 0) = x \left[1 - \left(\frac{\beta}{R} - \frac{\alpha^2(1-R)}{2\sigma^2 R^2} \right)^{\frac{R}{1-R}} g(0)^{\frac{1}{1-R}} \right] = x \left[1 - \left(1 - \frac{\epsilon^2(1-R)}{2\delta^2 R} \right)^{\frac{R}{1-R}} \right] > 0.$$

Suppose $R < 1$, $0 < \epsilon < \frac{\delta^2}{2} R + \frac{1}{1-R}$ and $\epsilon < \delta^2 R$, so that z^* is finite. Figure 4.20 plots $p^*(1, 1, \theta)$ for $\theta \in [0, 10]$. Notice that p^* decreases initially, has a strictly positive minimum near 0.95 and then increases, before becoming linear beyond $\theta = z^*$. Clearly, whatever the initial endowment of the agent, she has a smaller set of admissible strategies than an agent who can trade dynamically, and the cost of liquidity is strictly positive. For small initial endowments the agent would like to increase the size of her portfolio of the risky asset, and the smaller her initial endowment the more she would like to purchase at time zero. Hence the cost of illiquidity is decreasing in θ for small θ. However, for large θ, the agent would like to make an initial transaction (to reduce the ratio of wealth held in the risky asset to cash wealth to below z^*), and indeed since she is free to do so, her optimal strategy involves such a transaction at time zero. Hence for large wealth the cost of liquidity is proportional to $(x + y\theta)$, and hence is increasing in θ. For this reason, the cost of illiquidity is a U-shaped function of θ.

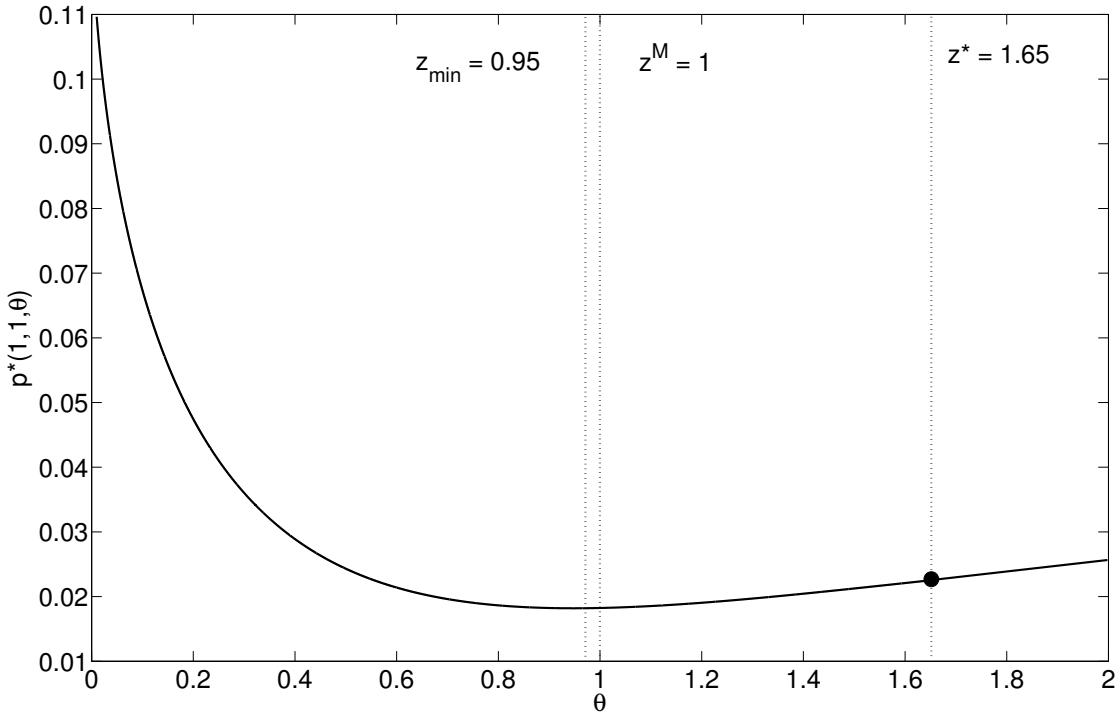

Figure 4.20: Cost of illiquidity $p^*(1,1,\theta)$ as θ varies. Parameters are $\epsilon = 1$, $\delta = 2$ and $R = 0.5$. Here, we fix $x_0 = y_0 = 1$ and $\theta \in [0,1]$. From the graph, we have $z^* > z^M > z_{\min}$ and p^* is a U-shaped function of θ.

Chapter 5

Multi-asset consumption-investment problems with infinite transaction costs

This chapter considers an optimal consumption and portfolio selection problem with transaction costs and with multiple correlated risky assets. In our model the transaction costs take a special form in that transaction costs on purchases of one of the risky assets, the endowed asset, are infinite, and transaction costs involving the other risky assets are zero. Expressed differently, one of the risky assets is assumed to not be available for dynamic trading. Instead the assumption is that this asset can only be sold: (re)-purchases are not allowed. Our agent is endowed with an initial quantity of this asset, and her strategies include when and how many to sell units of this infinitely divisible asset over time. The assumption that there are infinite transaction costs on purchases of endowed asset is complemented by an assumption that sales and purchases of the other risky assets are permissible, and incur zero transaction costs.

In general, multi-asset optimal consumption/portfolio problems are very challenging and there is relatively limited literature on the multiple risky asset case. Recall that the Davis-Norman approach is limited to a single risky asset as introduced in Section 1.5. It is then of great interest to understand how these results generalise to multiple risky assets. For example, in his survey article on consumption/investment problems with transaction costs Cadenillas [7, page 65] says that 'most results in this survey are limited to the case of only one bond and only one stock. It is then important to see if these results can be extended to cover a realistic number of stocks'. Although there has been some progress since that paper was published, similar sentiments are echoed in recent

papers by Chen and Dai [8, page 2]: 'most of the existing theoretical characterisations of the optimal strategy are for the single risky-asset case. In contrast there is a relatively limited literature on the multiple risky-asset case' and Guasoni and Muhle-Karbe [22, page 194]: 'In sharp contrast to frictionless models, passing from one to several risky assets is far from trivial with transaction costs ... multiple assets introduce novel effects, which defy the one-dimensional intuition.' In summary therefore, there is great interest in both theoretical and numerical results on the multi-asset case, and this paper can be considered as a contribution to that literature.

The set-up of our problem in which one asset is identified as a non-traded asset is similar to that in the real options literature (Miao and Wang [42], Henderson [25], Henderson and Hobson [26]) in which an agent with the option to invest in a project (or sell an asset) chooses the optimal sale time. The difference with respect to that literature is that we assume that the non-traded asset is infinitely divisible, whereas in the real options literature it is typically assumed to be indivisible.

Our model then consists of an agent who is endowed with units of an infinitely divisible risky asset which may be sold, but not bought, and whose opportunity set includes investment in a risk-free bond, and investment in other risky assets, to which a zero transaction cost applies. The risky assets follow correlated exponential Brownian motions and the objective of the agent is to choose a consumption rate and an investment strategy (including a sale strategy for the endowed asset) so as to maximise the discounted expected utility of consumption.

This chapter is an extension of Chapter 3 which considers a similar problem with an endowed asset but with no other risky assets. Many of the techniques in Chapter 3 carry over to the wider setting of the problem in this chapter, however, since there are fewer parameters when there are no investments beyond the endowed asset, the problem in Chapter 3 is significantly simpler and much more amenable to a comparative statics analysis. In contrast, this chapter treats the multi-asset problem which has proved so difficult to analyse in full generality, albeit in a rather special case. The multi-asset setting brings new challenges and complicates the analysis.

For an agent with CRRA utility we completely characterise the different possible behaviours. These include always selling the entire holdings of the endowed asset immediately, selling the endowed asset whenever the ratio of the value of the holdings of the endowed asset to other wealth gets above a critical ratio, and selling the endowed asset only when other wealth is zero. This characterisation is in terms of solutions of a boundary crossing problem for a first order ODE, which has four types of solution ('crossing' at zero, crossing in $(0, 1)$, no crossing in $[0, 1]$ and hits zero before crossing). Each different

type of solution is identified with a different type of solution to the optimisation problem; for example the first type of solution corresponds to a strategy of immediately selling all units of the endowed asset. It is relatively straightforward to identify the parameter combinations which lead to different types of solution, even if explicit solutions of the first order ODE are not available. Then we can relate the optimal wealth process, consumption strategy, sale strategy and investment strategy of the agent path-wise to the solution of a Skorokhod-type problem.

The technical contribution is to show that the problem of solving the HJB equation, which is a second order, non-linear PDE subject to smooth fit at an unknown free boundary, can be reduced to this much simpler problem involving an explicit first order ODE. This technical contribution is at the heart of our analytical and numerical results, and allows us to prove monotonicity of the critical exercise threshold and the certainty equivalent value in the model parameters.

The remainder of this chapter is constructed as follows. In Section 5.1, we give a precise description of the model and then a statement of the main results. Section 5.2 considers the degenerate cases, in which we prove the properties of the value functions and the verification theorems. In Section 5.3 and Section 5.4, we consider different characterisations of the first order ODE and prove the existence and finiteness of the critical ratio. Further, we prove the monotonicity of the critical ratio and the indifference price in model parameters. Section 5.5 and Section 5.6 prove the verification arguments in the two non-degenerate cases which has a finite/infinite critical exercise ratio. We further consider the martingale properties of the value function in Section 5.7 and extend the proofs to the case $R > 1$ in Section 5.8. Finally, the limiting case when the assets are completely correlated is considered in Section 5.9.

5.1 The model and main results

5.1.1 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space, such that the filtration satisfies the usual conditions and is generated by a two-dimensional standard Brownian motion $\left(B_t^1, B_t^\perp\right)_{t\geq 0}$. Set $B_t^2 = \rho B_t^1 + \rho^\perp B_t^\perp$, where $(\rho^\perp)^2 = 1 - \rho^2$.

The financial market is modelled with three stochastic processes on this space, a bond paying a constant rate of interest r, a financial (or hedging) asset with price process $P = (P_t)_{t\geq 0}$ and a non-traded (or endowed) asset with price process $Y = (Y_t)_{t\geq 0}$. Assume

the price processes of the risky assets satisfy

$$P_t = p_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t^1\right\},$$

where μ and $\sigma > 0$ are the constant mean return and volatility of the financial asset, and p_0 is the initial price, and

$$Y_t = y_0 \exp\left\{\left(\alpha - \frac{\eta^2}{2}\right)t + \eta B_t^2\right\},$$

where α and $\eta > 0$ are the constant mean return and volatility of the non-traded asset, and y_0 is the initial price. Let $\lambda = (\mu - r)/\sigma$ and let $\zeta = (\alpha - r)/\eta$ be the Sharpe ratios of the hedging and endowed assets respectively.

Let $C = (C_t)_{t\geq 0}$ denote the consumption rate of the individual, let $\Theta = (\Theta_t)_{t\geq 0}$ denote the number of units of the endowed asset held by the investor and let $\Pi = (\Pi_t)_{t\geq 0}$ denote the cash amount invested in the hedging asset P. The process Θ is required to be progressively measureable, right-continuous with left limits and non-increasing to reflect the fact that the non-traded asset is only allowed for sale. We assume the initial number of shares held by the investor is θ_0. Since we allow for an initial transaction at time 0 we may have $\Theta_0 < \theta_0$. We write $\Theta_{0-} = \theta_0$. This is consistent with our convention that Θ is right-continuous.

We denote by $X = (X_t)_{t\geq 0}$ the wealth process of the individual, and suppose that the initial wealth is x_0. Provided the changes to wealth occur from either consumption, investment or from the sale of the endowed asset, X evolves according to

$$dX_t = \Pi_t \frac{dP_t}{P_t} + (X_t - \Pi_t)\, r dt - C_t dt - Y_t d\Theta_t = \sigma \Pi_t dB_t^1 + \{(\mu - r)\Pi_t + rX_t - C_t\}\, dt - Y_t d\Theta_t,$$

subject to $X_{0-} = x_0$, and $X_0 = x_0 + y_0(\theta_0 - \Theta_0)$. We say a consumption/investment/sale strategy triple is admissible if the components satisfy the requirements listed above and if the resulting cash wealth process X is non-negative for all time. Let $\mathcal{A}(x, y, \theta)$ denote the set of admissible strategies for initial setup $(X_{0-} = x, Y_0 = y, \Theta_{0-} = \theta)$.

The objective of the agent is to maximise over admissible strategies the discounted expected utility of consumption over the infinite horizon, where the utility function of the agent is assumed to have constant relative risk aversion, with parameter $R \in (0, \infty) \setminus 1$

and discount factor β. In particular, the goal is to find $\mathcal{V} = \mathcal{V}(x_0, y_0, \theta_0)$ where

$$\mathcal{V}(x, y, \theta) = \sup_{(C,\Pi,\Theta)\in\mathcal{A}(x,y,\theta)} \mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{1-R}}{1-R}dt\right]. \tag{5.1}$$

Remark 5.1.1. We can consider the case of n Brownian motions, and $(n-1)$ financial assets and a single endowed asset, but the situation reduces to the case considered here. In particular, the $(n-1)$ financial assets reduce to a single mutual fund as introduced in Section 1.2.

Remark 5.1.2. The techniques extend to the case $R = 1$ and logarithmic utility, but we will not consider that case here. However, many of the results can be obtained simply by setting $R = 1$ in the various formulae.

Since the set-up has a Markovian structure, we expect the optimal consumption, optimal investment and optimal sale strategy to be of feedback form and to be functions of the current wealth and endowment of the agent and of the price of the risky asset.

Let $V(x, y, \theta, t)$ be the forward starting value function for the problem so that

$$V(x, y, \theta, t) = \sup_{(C,\Pi,\Theta)\in\mathcal{A}(x,y,\theta,t)} \mathbb{E}\left[\int_t^\infty e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds\,\middle|\, X_{t-} = x, Y_t = y, \Theta_{t-} = \theta\right]. \tag{5.2}$$

Here the space of admissible strategies $\mathcal{A}(x, y, \theta, t)$ is such that $C = (C_s)_{s\geq t}$ is a non-negative, progressively measureable process, $\Pi = (\Pi_s)_{s\geq t}$ is progressively measurable, $\Theta = (\Theta_s)_{s\geq t}$ is a right-continuous, non-increasing, progressively measureable process and satisfies $\Theta_t - (\Delta\Theta)_t = \theta$, and $X = (X_s)_{s\geq t}$ is a non-negative process given by

$$X_s = x + \int_t^s (\lambda\sigma\Pi_u + rX_u - C_u)\,du + \int_t^s \sigma\Pi_u dB_u^1 - \int_{[t,s]} Y_u d\Theta_u.$$

It is clear that $V(x, y, \theta, t) = e^{-\beta t}V(x, y, \theta, 0) = e^{-\beta t}\mathcal{V}(x, y, \theta)$. Define $V_0(x, t) = V(x, y, 0, t)$ (note V_0 will not depend on y, since the agent has no units of Y) and $\mathcal{V}_0(x) = \mathcal{V}(x, y, 0)$, and define the certainty equivalent price $p = p(x, y, \theta, t)$ as the solution to

$$V_0(x + p, t) = V(x, y, \theta, t).$$

Note that $p = p(x, y, \theta)$ does not depend on t since p solves $\mathcal{V}_0(x + p) = \mathcal{V}(x, y, \theta)$.

The parameters of the problem are r, β, μ, σ, α, η, ρ and R which we assume to all be constants. We will assume that $\rho \in (-1, 1)$ (the limiting cases $\rho = \pm 1$ are analysed

in Section 5.9 by similar techniques). Define auxiliary parameters $(b_i)_{1\le i\le 4}$,

$$b_1 = \frac{2}{\eta^2(1-\rho^2)}\left[\beta - r(1-R) - \frac{\lambda^2(1-R)}{2R}\right], \quad b_2 = \frac{\lambda^2 - 2R\eta\rho\lambda + \eta^2R^2}{\eta^2R^2(1-\rho^2)}, \quad b_3 = \frac{2(\zeta-\lambda\rho)}{\eta(1-\rho^2)},$$

and $b_4 = \left(\frac{1}{2}\eta^2(1-\rho^2)\right)^{-1}$. Note that

$$b_2 = 1 + \frac{1}{1-\rho^2}\left(\frac{\lambda}{\eta R} - \rho\right)^2 \ge 1.$$

It will turn out that the optimal selling and investment problem depends on the original parameters only through these auxiliary parameters and the risk aversion R.

We will see in later sections that b_1 plays the role of a 'normalised discount factor'. The parameter b_3 is the 'effective Sharpe ratio, per unit of idiosyncratic volatility' of the endowed asset. The parameter b_2 is the hardest to interpret: essentially it is a nonlinearity factor which arises from the multi-dimensional structure of the problem. The case $b_2 = 1$ is rather special and will be excluded to a certain extent from our analysis. (One scenario in which we naturally find $b_2 = 1$ is if $\lambda = 0 = \rho$. In this case there is neither a hedging motive, nor an investment motive for holding the financial asset. Essentially then, the investor can ignore the presence of the financial asset, reducing the dimensionality of the problem. This is the problem considered in Chapter 3.

Our goal is to solve for V, and hence for the certainty equivalent price p. As might be expected, V solves a variational principle, and can be characterised by a second-order, nonlinear partial differential equation in the four variables (x, y, θ, t) subject to value matching and smooth fit (of the first and second derivatives) at an unknown free boundary. In fact various simplifications can be expected from the inherent scalings of the problem. Nonetheless, the remarkable fact on which this paper is based is that expressions for V and a characterisation for the optimal solution all follow from the study of a boundary crossing problem for a single first order ordinary differential equation.

If $R < 1$ and $b_1 \le 0$ then the value function $V_0(x,t)$ is infinite for the Merton problem (in the absence of the endowed asset), and *a fortiori* the value function with a positive endowment of the non-traded asset is also infinite. In this case it is not possible to define a certainty equivalent price. If $R > 1$ and $b_1 \le 0$, then for every admissible strategy with zero initial endowment of the risky asset the expected discounted utility of consumption equals $-\infty$, and again it is not possible to define a certainty equivalent price for units of the endowed asset. To exclude these cases we make the following non-degeneracy assumption:

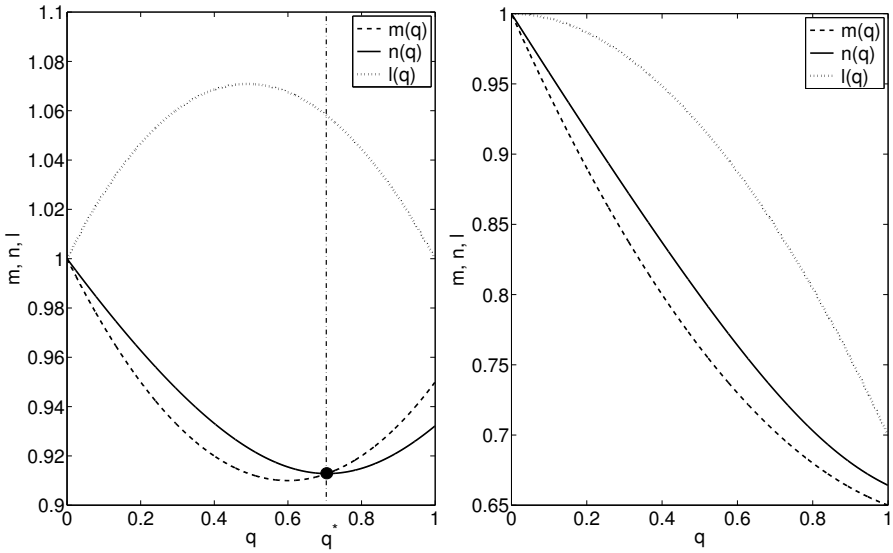

Figure 5.1: Stylised plot of $m(q)$, $n(q)$, $\ell(q)$ and $R \in (0,1)$. Parameters are such that $q^* \in (0,1)$ (left figure) and $q^* = 1$ (right figure).

Standing Assumption 1. Throughout the chapter we assume that $b_1 > 0$.

5.1.2 Main results

The key to our analysis are solutions to the first order differential equation (5.3) the properties of which are stated in Proposition 5.1.3, the proof of which is given in Section 5.4.

Proposition 5.1.3. *For $q \in [0,1]$ define $m(q) = \frac{(1-R)R}{b_1}q^2 - \frac{b_3(1-R)}{b_1}q + 1$ and $\ell(q) = m(q) + \frac{1-R}{b_1}q(1-q) + \frac{(b_2-1)R(1-R)}{b_1}\frac{q}{[(1-R)q+R]}$. Let $n = n(q)$ solve*

$$\frac{n'(q)}{n(q)} = \frac{1-R}{R(1-q)} - \frac{(1-R)^2}{b_1 R}\frac{q}{\ell(q)-n(q)} + \frac{(1-R)q}{2b_1R(1-q)[(1-R)q+R]}\frac{\upsilon(q,n(q))}{\ell(q)-n(q)} \quad (5.3)$$

subject to $n(0) = 1$ and $\frac{n'(0)}{1-R} < \frac{\ell'(0)}{1-R} = (b_2 - b_3)/b_1$, where

$$\upsilon(q,n) = \varphi(q,n) - \operatorname{sgn}(1-R)\sqrt{\varphi(q,n)^2 + 4R^2(1-R)^2(b_2-1)(1-q)^2},$$

and $\varphi(q,n) = b_1 n + (1-R)(b_3 - 2R)q + 2R(1-R) - b_1 - b_2R(1-R)$.

Suppose that if n hits zero, then 0 is absorbing for n.

For $R < 1$, let $q^* = \inf\{q > 0 : n(q) \le m(q)\}$, see Figure 5.1. *For $R > 1$, let*

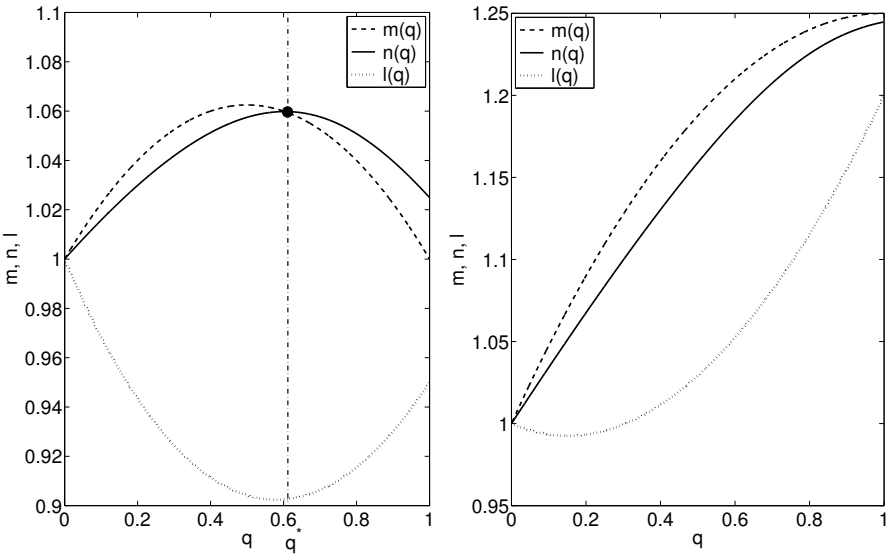

Figure 5.2: Stylised plot of $m(q)$, $n(q)$, $\ell(q)$ and $R \in (1, \infty)$. Parameters are such that $q^* \in (0, 1)$ (left figure) and $q^* = 1$ (right figure).

$q^* = \inf\{q > 0 : n(q) \geq m(q)\}$, *see* Figure 5.2. *For* $j \in \{\ell, m, n\}$ *let* $q_j = \inf\{q > 0 : j(q) = 0\} \wedge 1$.

Set $\bar{b}_3 = 2R$ *if* $R > 1$ *and* $\bar{b}_3 = \min\{2R, R + \frac{b_1}{1-R}\}$ *if* $R < 1$. *For fixed* b_1, b_2 *and* R, *there exists some critical value* $b_{3,crit}(b_1, b_2, R)$, *with* $R < b_{3,crit} \leq \bar{b}_3$, *and such that*

1. *if* $b_3 \leq 0$, *then* $q^* = 0$;
2. *if* $0 < b_3 < b_{3,crit}$ *then* $0 < q^* < 1$;
3. *if* $R > 1$ *and* $b_3 \geq b_{3,crit}(b_1, b_2, R)$ *then* $q^* = 1$; *if* $R < 1$ *and* $b_{3,crit}(b_1, b_2, R) \leq b_3 < \frac{b_1}{1-R} + b_2 R$, *then* $q^* = 1$;
4. *if* $R < 1$, $b_2 = 1$ *and* $b_3 = \frac{b_1}{1-R} + R \geq 2R$, *then* $q_m = q_n = q_\ell = q^* = 1$; *if* $R < 1$, $b_2 = 1$ *and* $b_3 = \frac{b_1}{1-R} + R < 2R$, *then* $q_m < q_n = q_\ell = q^* = 1$; *if* $R < 1$, $b_2 > 1$ *and* $b_3 = \frac{b_1}{1-R} + b_2 R$, *then* $q_m < q_n = q_\ell = q^* = 1$; *if* $R < 1$ *and* $b_3 > \frac{b_1}{1-R} + b_2 R$, *then* $q_m < q_n = q_\ell < q^* = 1$.

Remark 5.1.4. The condition $b_3 < 2R$ is equivalent to $m'(1) > 0$.

If $R < 1$, then the condition $b_3 \leq \frac{b_1}{1-R} + b_2 R$ is equivalent to $\ell(1) \geq 0$. (Note that $\ell(1) \geq 0$ is a necessary condition for $q_n = 1$.) Then, if $R < 1$, $0 < b_3 < 2R$ and $b_3 < \frac{b_1}{1-R} + b_2 R$, we have $q_\ell = q_n = 1$.

We will show in Lemma 5.3.3 that n has a turning point at $q^* \in (0,1)$ if and only if $n(q^*) = m(q^*)$. In particular, if m is monotone, then $q^* = 1$.

Remark 5.1.5. Suppose $b_2 = 1$. Then $\ell(1) = m(1)$. Moreover $\varphi(q,m) = R(1-R)(1-q)^2$. If $R < 1$, then if $n \geq m$ we have $\varphi(q,n) \geq \varphi(q,m) > 0$ and $\upsilon(q,n) = 0$. Conversely, if $R > 1$ then if $n \leq m$, $\varphi(q,n) \leq \varphi(q,m) < 0$ and $\upsilon(q,n) = 0$. Hence the expression in (5.3) for n' simplifies greatly if $b_2 = 1$, and is seen to reduce to the equation for the variable of the same name in (3.6).

Remark 5.1.6. We show below in Lemma 5.3.4 that $n(q)$, q^* and $n(q^*)$ are each monotonic in the parameter b_1, b_2 and b_3 for $q \leq q^*$. In particular, q^* is an increasing function of b_3. It follows that there exists a critical parameter $b_{3,crit}$ and $q^* < 1$ if and only if $b_3 < b_{3,crit}$. Although we can conclude that $R < b_{3,crit} \leq \min\{2R, \frac{b_1}{1-R} + R\}$, we do not have an explicit expression for $b_{3,crit}$.

From the scalings of the problem, it is clear that a key variable is the ratio of wealth in the endowed asset to liquid wealth. (Here we define liquid wealth to be the sum of cash wealth and wealth invested in the hedging asset.) We denote this ratio by Z so that $Z_t = Y_t\Theta_t/X_t \in [0,\infty]$. Under optimal behaviour, consumption and investment rates are functions of liquid wealth and Z.

One of the key contributions of this article is to identify the different types of solutions to the optimisation problem with different classes of solutions to the first crossing problem studied in Proposition 5.1.3.

Theorem 5.1.7. *1. Suppose $b_3 \leq 0$. Then it is always optimal to sell the entire holding of the endowed asset immediately, so that $\Theta_t = 0$ for $t \geq 0$. The value function for the problem is $V(x,y,\theta,t) = \left(\frac{b_1}{b_4 R}\right)^{-R} e^{-\beta t}(x+y\theta)^{1-R}/1-R$; and the certainty equivalent value of the holdings of the asset is $p(x_0,y_0,\theta_0,0) = y_0\theta_0$.*

2. Suppose $0 < b_3 < b_{3,crit}(b_1,b_2,R)$. Then there exists a positive and finite critical ratio z^* and the optimal behaviour is to sell sufficient units of the risky asset so as to keep the ratio of wealth in the risky asset to cash wealth below the critical ratio. If $\theta > 0$ then $p(x,y,\theta,t) > y\theta$.

3. Suppose $b_3 \geq b_{3,crit}(b_1,b_2,R)$ and $b_3 < \frac{b_1}{1-R} + b_2 R$ if $R < 1$. Then the critical ratio z^* is infinite and the optimal behaviour is first to consume liquid wealth and invest in the risky asset, and then when this liquid wealth is exhausted, to finance further consumption and investment in the risky asset from sales of the endowed asset.

4. Suppose $b_3 \geq \frac{b_1}{1-R} + b_2 R$ if $R < 1$. Then the problem is degenerate, and provided

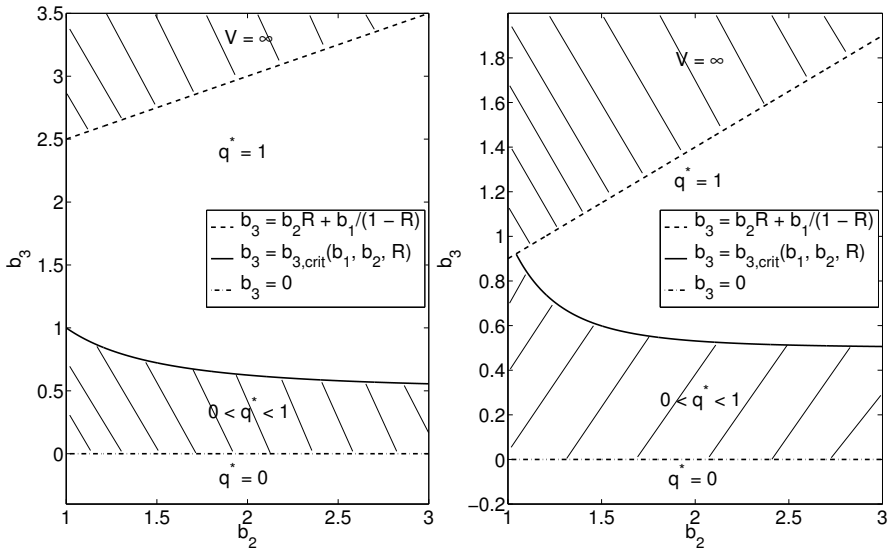

Figure 5.3: Plot of the boundaries of different scenarios as a function of b_2 and b_3 for $R < 1$. In the left graph, parameters are $b_1 = 1$ and $R = 0.5$. In the right graph, parameters are $b_1 = 0.2$ and $R = 0.5$ so that $b_1 < R(1 - R)$ and there is a crossing between the solid and dotted lines.

> *θ_0 is positive, the value function $V = V(x, y, \theta, t)$ is infinite. There is no unique optimal strategy, and the certainty equivalent value p is not defined.*

Remark 5.1.8. This theorem emphasises the role played by the parameter b_3, the 'effective Sharpe ratio' to distinguish between the different scenarios. When b_3 is negative, the endowed asset is a bad investment and it is optimal to sell it immediately. For small and positive b_3, there exists a finite critical ratio and sales of the nontraded asset occur to keep the fraction of wealth held in the nontraded asset below a critical value. As b_3 becomes larger, the endowed asset is more valuable and the agent waits longer for a better return from the endowed asset. For sufficiently large b_3 she does not make any sales of the endowed asset until cash wealth is exhausted. Finally, if $R < 1$ and b_3 becomes too large, the value function is infinite, and the problem with the endowment is ill-posed.

The most interesting cases of Theorem 5.1.7 are the middle two non-degenerate cases, and these two cases we study in more detail in the next two theorems. Recall that we suppose we have constructed the solution n to the differential equation in (5.3). Define $N(q) = n(q)^{-R}(1 - q)^{R-1}$, and let W be inverse to N. Let $h^* = N(q^*)$.

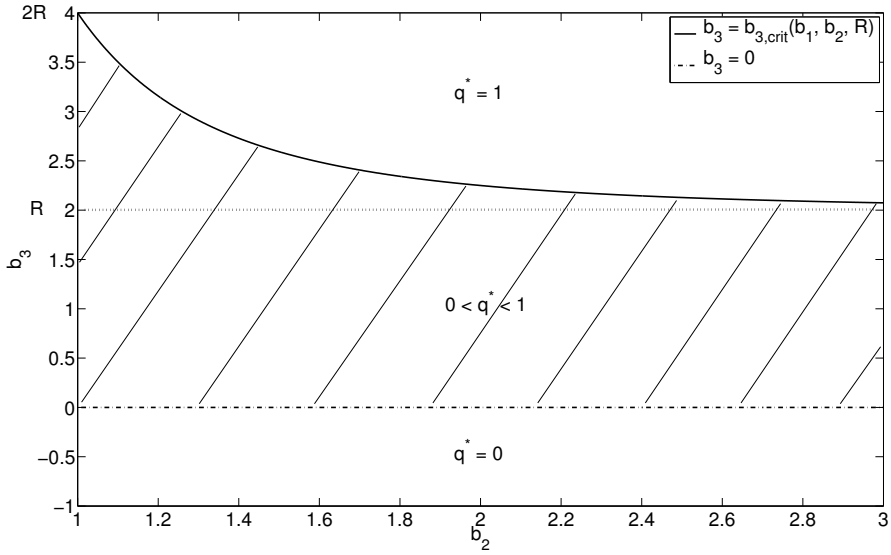

Figure 5.4: Plot of the boundaries of different scenarios as a function of b_2 and b_3 for $R > 1$. Parameters are $b_1 = 1$ and $R = 2$. Note that when $R > 1$ there is no region where the problem is ill-defined.

For a twice differentiable function f define $\Psi_f(z)$ by

$$\Psi_f(z) = \frac{(1-R)f(z) - (1+\frac{\eta\rho R}{\lambda})zf'(z) - \frac{\eta\rho}{\lambda}z^2 f''(z)}{R(1-R)f(z) - 2Rzf'(z) - z^2 f''(z)}. \tag{5.4}$$

Theorem 5.1.9. *Suppose $R < 1$, and suppose $0 < b_3 < b_{3,crit}(b_1, b_2, R)$, so that $0 < q^* < 1$.*

Then $N : [0, q^*] \mapsto [1, h^*]$ is increasing, and $W : [1, h^*] \mapsto [0, q^*]$ is well-defined and increasing. Moreover $n(q^*)^{-R} = h^*(1-q^*)^{1-R}$.

Let z^* be given by

$$z^* = (1-q^*)^{-1} - 1 = \frac{q^*}{1-q^*} \in (0, \infty). \tag{5.5}$$

and let $u^* = e^{z^*}$. On $[1, h^*]$ let h be the solution of

$$u^* - u = \int_h^{h^*} \frac{1}{(1-R)fW(f)} df. \tag{5.6}$$

It follows that $h(-\infty) := \lim_{u \downarrow -\infty} h(u) = 1$.

Let g be given by

$$g(z) = \begin{cases} \left(\frac{b_1}{b_4 R}\right)^{-R} n(q^*)^{-R} (1+z)^{1-R} & z \in (z^*, \infty); \\ \left(\frac{b_1}{b_4 R}\right)^{-R} h(\ln z) & z \in [0, z^*]. \end{cases} \tag{5.7}$$

Then, the value function V is given by

$$V(x, y, \theta, t) = e^{-\beta t} \frac{x^{1-R}}{1-R} g\left(\frac{y\theta}{x}\right). \qquad x > 0, y > 0, \theta \geq 0 \tag{5.8}$$

which by continuity extends to $x = 0$ via

$$V(0, y, \theta, t) = e^{-\beta t} \frac{y^{1-R}\theta^{1-R}}{1-R} \left(\frac{b_1}{b_4 R}\right)^{-R} n(q^*)^{-R}.$$

Let $(J, L) = (J_t, L_t)_{t \geq 0}$ be the unique pair such that

1. *J is positive,*
2. *L is increasing, continuous, $L_0 = 0$, and dL_t is carried by the set $\{t : J_t = 0\}$,*
3. *J solves*

$$J_t = (z^* - z_0)^+ - \int_0^t \tilde{\Lambda}(J_s) ds - \int_0^t \tilde{\Sigma}(J_s) dB_s^1 \int_0^t \tilde{\Gamma}(J_s) dB_s^2 + L_t,$$

where $\tilde{\Lambda}(j) = \Lambda(z^* - j)$, $\tilde{\Sigma}(j) = \Sigma(z^* - j)$ and $\tilde{\Gamma}(j) = \Gamma(z^* - j)$, where in turn

$$\Lambda(z) = z\left(g(z) - \frac{1}{1-R} z g'(z)\right)^{-1/R} - \lambda(\lambda + \eta\rho) z \Psi_g(z) + \lambda^2 z \Psi_g(z)^2 + \zeta\eta z,$$

$\Gamma(z) = \eta z$ and $\Sigma(z) = -\sigma z \Psi_g(z)$.

For such a pair $0 \leq J_t \leq z^*$.

Let $z_0 = y_0\theta_0/x_0$. If $z_0 \leq z^*$ then set $\Theta_0^* = \theta_0$ and $X_0^* = x_0$; else $z_0 > z^*$ and for the optimal strategy there is a sale of a positive quantity $\theta_0 - \Theta_0$ of units at time 0 such that

$$\Theta_0^* = \theta_0 \frac{z^*}{(1+z^*)} \frac{(1+z_0)}{z_0} \leq \theta_0$$

and $X_0^* = x_0 + y_0(\theta_0 - \Theta_0)$.

Then, the optimal holdings Θ_t^* of the endowed asset, and the resulting wealth

process are given by

$$\begin{aligned}\Theta_t^* &= \exp\left\{-\frac{1}{z^*(1+z^*)}L_t\right\};\\ X_t^* &= \frac{Y_t\Theta_t^*}{(z^*-J_t)};\end{aligned}$$

and the optimal consumption process $C_t^* = C(X_t^*, Y_t, \Theta_t^*)$*, the optimal portfolio process* $\Pi_t^* = \Pi(X_t^*, Y_t, \Theta_t^*)$*, and the certainty equivalent value* p *are given in feedback form via*

$$C(x,y,\theta) = x\left[g\left(\frac{y\theta}{x}\right) - \frac{1}{1-R}\frac{y\theta}{x}g'\left(\frac{y\theta}{x}\right)\right]^{-\frac{1}{R}} \tag{5.9}$$

$$\Pi(x,y,\theta) = \frac{\lambda}{\sigma}x\Psi_g\left(\frac{y\theta}{x}\right) \tag{5.10}$$

$$p(x,y,\theta) = x\left[\frac{g\left(\frac{y\theta}{x}\right)}{g(0)}\right]^{\frac{1}{1-R}} - x. \tag{5.11}$$

Now suppose $R > 1$ *and* $0 < b_3 < b_{3,crit}(b_1, b_2, R)$ *so that* $0 < q^* < 1$*. Let all quantities be defined as before. Then* $N : [0, x^*] \mapsto [h^*, 1]$ *is decreasing and* $W : (h^*, 1) \mapsto [0, q^*]$ *is well defined and decreasing. On* $(h^*, 1)$ h *is defined via*

$$u^* - u = \int_{h^*}^{h} \frac{1}{(R-1)fW(f)}df.$$

The value function V*, the optimal holdings* Θ^**, the optimal consumption process* C^**, the optimal portfolio process* Π^**, the resulting wealth process* X^* *and the certainty equivalent value* p *are the same as before.*

Remark 5.1.10. Given n and the first crossing point q^* the construction of N, W, h, g and hence V and p is immediate and straightforward.

Further, given realisations of the price processes P and Y (or equivalently paths of the Brownian motions B^1 and B^2 (or B^1 and $B^\perp$) then J and L arise from a pathwise solution of a Skorokhod problem [48, Lemma VI.2.1]. The optimal endowed asset holdings Θ^* and then also the optimal cash wealth process X^* are given explicitly in terms of the the solution of the Skorokhod problem; the optimal consumption and investment are then given in feedback form as functions of these primary quantities.

Theorem 5.1.11. *Suppose* $R < 1$ *and suppose* $b_{3,crit}(b_1, b_2, R) \le b_3 < \frac{b_1}{1-R} + b_2R$*.*

Let n *solve* (5.3) *on* $[0, 1]$*. Then for the given parameter combinations we have*

$q^* = 1$. *Then* N *is increasing and* W *is well defined. Define* $\gamma : (1, \infty) \mapsto \mathbb{R}$ *by*

$$\gamma(v) = \frac{\ln v}{1-R} + \frac{R}{1-R} \ln n(1) - \frac{1}{1-R} \int_v^\infty \frac{(1 - W(s))}{sW(s)} ds. \tag{5.12}$$

Let h *be inverse to* γ *and let* $g(z) = (b_4 R/b_1)^R h(\ln z)$.

Then, the value function V *is given by*

$$V(x, y, \theta, t) = e^{-\beta t} \frac{x^{1-R}}{1-R} g\left(\frac{y\theta}{x}\right), \qquad x > 0, y > 0, \theta \geq 0 \tag{5.13}$$

and $V(0, y, \theta, t) = e^{-\beta t} \frac{y^{1-R}\theta^{1-R}}{1-R} n(1)^{-R}$.

Suppose $\theta_0 > 0$. *Let* $K_0 = x_0/(y_0\theta_0) \in [0, \infty)$. *Let* $(K, L) = (K_t, L_t)_{t \geq 0}$ *be the unique pair such that*

1. K *is positive,*
2. L *is increasing, continuous,* $L_0 = 0$, *and* dL_t *is carried by the set* $\{t : K_t = 0\}$,
3. K *solves*

$$K_t = K_0 + \int_0^t \hat{\Lambda}(K_s) ds + \int_0^t \hat{\Sigma}(K_s) dB_s^1 + \int_0^t \hat{\Gamma}(K_s) dB_s^2 + L_t,$$

where

$$\hat{\Lambda}(k) = (\eta - \zeta)\eta k + \lambda(\lambda - \eta\rho) k \Psi_g(1/k) - k \left[g(1/k) - \frac{g'(1/k)}{k(1-R)} \right]^{-1/R},$$

$\hat{\Sigma}(k) = \lambda k \Psi_g(1/k)$ *and* $\hat{\Gamma}(k) = -\eta k$.

Then, the optimal holdings Θ_t^* *of the endowed asset, and the optimal wealth process are given by*

$$\Theta_t^* = \theta_0 \exp\{-L_t\}, \tag{5.14}$$

$$X_t^* = Y_t \Theta_t^* K_t \tag{5.15}$$

The optimal consumption process $C_t^* = C(X_t^*, Y_t, \Theta_t^*)$, *the optimal portfolio process* $\Pi_t^* = \Pi(X_t^*, Y_t, \Theta_t^*)$, *and the certainty equivalent value* $p = p(X_t^*, Y_t, \Theta_t^*)$ *are given in feedback form by the expressions in (5.9), (5.10) and (5.11).*

Now suppose $R > 1$ and $b_3 \geq b_{3,crit}$. Then N is decreasing. Define

$$\gamma(v) = -\frac{\ln v}{R-1} - \frac{R}{R-1}\ln n(1) - \frac{1}{R-1}\int_0^v \frac{(1-W(s))}{sW(s)}ds. \tag{5.16}$$

Let h be inverse to γ and define g and the value function as in the case $R < 1$. Then the optimal holdings, optimal consumption, optimal investment in the financial asset, optimal wealth process and the certainty equivalent value of the holdings are all as given in the case $R < 1$.

Remark 5.1.12. Here is one, perhaps surprising, consequence of Theorem 5.1.11 which holds for $b_2 > 1$. Define the stopping time $\tau = \inf\{t; X_t^* = 0\}$. Then, for the parameter combinations studied in Theorem 5.1.11 and under optimal behaviour, the investments in the risky asset are such that $\Pi_\tau^* = \Pi_\tau(0, Y_\tau, \Theta_\tau^*) \neq 0$. This implies that even when liquid wealth is zero, it is not optimal to invest zero amount in the financial asset. Adverse movements in the price of the financial asset have a negative impact on liquid wealth, and must be financed through sales of the endowed asset. Conversely, beneficial movements in the price of the financial asset will generate positive liquid wealth for the agent. In particular, $X^* = 0$ is not an absorbing state.

In contrast, for $b_2 = 1$ and for the parameter combinations where Theorem 5.1.11 applies, no sales of the endowed asset take place until liquid wealth has been exhausted, but once liquid wealth is zero, there are no investments in the financial asset, cash wealth is maintained at zero, and consumption is financed through sales of the endowed asset, continuously over time.

5.2 Proofs and verification arguments: the degenerate cases

For $\mathcal{H} = \mathcal{H}(x, y, \theta) : [0, \infty) \times (0, \infty) \times [0, \infty) \mapsto \mathbb{R}$ with $\mathcal{H} \in C^{2,2,1}$ and such that $\mathcal{H}_x > 0$ define operators

$$\begin{aligned}
\mathcal{L}\mathcal{H} &= \sup_{(c>0,\pi)} \left\{ \frac{c^{1-R}}{1-R} - c\mathcal{H}_x + \alpha y \mathcal{H}_y + \lambda\sigma\pi\mathcal{H}_x + rx\mathcal{H}_x + \frac{1}{2}\sigma^2\pi^2\mathcal{H}_{xx} + \frac{1}{2}\eta^2 y^2 \mathcal{H}_{yy} + \sigma\eta\rho y\pi\mathcal{H}_{xy} \right\} \\
&= \mathcal{H}_x^{1-1/R}\frac{R}{1-R} + rx\mathcal{H}_x + \alpha y\mathcal{H}_y + \frac{1}{2}\eta^2 y^2\mathcal{H}_{yy} - \frac{(\eta\rho y\mathcal{H}_{xy} + \lambda\mathcal{H}_x)^2}{2\mathcal{H}_{xx}}, \\
\mathcal{M}\mathcal{H} &= \mathcal{H}_\theta - y\mathcal{H}_x.
\end{aligned}$$

$\mathcal{L}\mathcal{H}$ is defined on $(0, \infty) \times (0, \infty) \times [0, \infty)$. However we can extend the domain of definition of $\mathcal{L}$ to $[0, \infty) \times (0, \infty) \times [0, \infty)$ by extending the definition of $\mathcal{H}$ to the region $-y\theta < x \leq 0$

in such a way that the derivatives of $\mathcal{H}$ are continuous at $x = 0$. $\mathcal{M}\mathcal{H}$ is defined on $(0, \infty) \times (0, \infty) \times (0, \infty)$. Note that we will not need $\mathcal{M}\mathcal{H}$ at $\theta = 0$, but we can extend the domain of definition of $\mathcal{M}$ to $x = 0$ using the same extension of $\mathcal{H}$ to $x \leq 0$.

5.2.1 The Verification Lemma in the case of a depreciating asset.

Suppose $b_3 \leq 0$. Our goal is to show that the conclusions of Theorem 5.1.7(1) hold.

From Proposition 5.1.3 we know $q^* = 0$. Define the candidate value function via $G(x, y, \theta, t) = e^{-\beta t}\mathcal{G}(x, y, \theta)$ where

$$\mathcal{G}(x, y, \theta) = \left(\frac{b_1}{b_4 R}\right)^{-R} \frac{(x + y\theta)^{1-R}}{1 - R}. \tag{5.17}$$

The candidate optimal strategy is to sell all units of the risky asset immediately.

Prior to the proof of the theorem, we need the following lemma.

Lemma 5.2.1. *Suppose $b_3 \leq 0$. Consider the candidate value function constructed in (5.17). Then $\mathcal{M}\mathcal{G} = 0$, and $\mathcal{L}\mathcal{G} - \beta\mathcal{G} \leq 0$ with equality at $\theta = 0$.*

Proof. Given the form of the candidate value function in (5.17), we have

$$\mathcal{M}\mathcal{G} = \left(\frac{b_1}{b_4 R}\right)^{-R} y(x + y\theta)^{-R} - \left(\frac{b_1}{b_4 R}\right)^{-R} y(x + y\theta)^{-R} = 0.$$

On the other hand, writing $z = y\theta/x$,

$$\mathcal{L}\mathcal{G} - \beta\mathcal{G} = x^{1-R} R \left(\frac{b_1}{b_4 R}\right)^{1-R} (1 + z)^{1-R} \left[\frac{b_3}{b_1}\frac{z}{1 + z} - \frac{R}{b_1}\left(\frac{z}{1 + z}\right)^2\right] \leq 0,$$

with equality at $z = 0$, which completes the proof. □

Theorem 5.2.2. *Suppose $b_3 \leq 0$. Then the value function V is given by*

$$V(x, y, \theta, t) = e^{-\beta t}\left(\frac{b_1}{b_4 R}\right)^{-R} \frac{(x + y\theta)^{1-R}}{1 - R} = G(x, y, \theta, t). \tag{5.18}$$

The optimal holdings Θ_t^* of the endowed asst, the optimal consumption process C_t^* and the resulting wealth process are given by

$$(\triangle\Theta^*)_{t=0} = -\theta_0, \qquad C_t^* = \frac{b_1}{b_4 R} X_t^*, \qquad \Pi_t^* = \frac{\lambda}{\sigma R} X_t^*,$$

$$X_t^* = (x_0 + y_0\theta_0)\exp\left\{\left(\frac{\lambda^2}{R} + r - \frac{b_1}{b_4 R} - \frac{\lambda^2}{2R^2}\right)t + \frac{\lambda}{R}B_t^1\right\}. \tag{5.19}$$

The certainty equivalence price is given by $p(x, y, \theta, t) = y\theta$.

Proof. We prove the result at $t = 0$, ie we show that $\mathcal{V}(x, y, \theta) = V(x, y, \theta, 0) = G(x, y, \theta, 0) = \mathcal{G}(x, y, \theta)$; the case of general t follows from the time-homogeneity of the problem.

Note that under proposed strategies in (5.19), the optimal strategy is to sell the entire endowed asset holding immediately, which gives $X_{0+}^* = x_0 + y_0\theta_0$ and to finance investment and consumption from liquid wealth thereafter. Hence, the wealth process $(X_t^*)_{t\geq 0}$ evolves as $dX_t^* = (\lambda\sigma\Pi_t^* + rX_t^* - C_t^*)dt + \sigma\Pi_t^* dB_t^1$. This gives $X_t^* = (x_0 + y\theta_0)\exp\left\{\left(\frac{\lambda^2}{R} + r - \frac{b_1}{b_4 R} - \frac{\lambda^2}{2R^2}\right)t + \frac{\lambda}{R}B_t^1\right\}$.

The value function under strategy proposed in (5.19) is

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\beta t}\frac{C_t^{*1-R}}{1-R}dt\right] \\
&= \left(\frac{b_1}{b_4 R}\right)^{1-R}\frac{(x_0 + y_0\theta_0)^{1-R}}{1-R}\int_0^\infty \exp\left\{\left[\frac{\lambda^2(1-R)}{2R} + r(1-R) - \frac{b_1(1-R)}{b_4 R} - \beta\right]t\right\}dt \\
&= \left(\frac{b_1}{b_4 R}\right)^{1-R}\frac{(x_0 + y_0\theta_0)^{1-R}}{1-R}\int_0^\infty \exp\left\{\left(-\frac{b_1}{b_4 R}\right)t\right\}dt \\
&= \left(\frac{b_1}{b_4 R}\right)^{-R}\frac{(x_0 + y_0\theta_0)^{1-R}}{1-R} = \mathcal{G}(x_0 + y_0\theta_0, y_0, 0).
\end{aligned}$$

Hence $V(x, y, \theta, 0) \geq G(x, y, \theta, 0)$.

Now, consider general admissible strategies. Suppose first that $R < 1$. Define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \int_0^t e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds + e^{-\beta t}\mathcal{G}\left(X_t, Y_t, \Theta_t\right).$$

Applying the generalised Itô's formula [23, Section 4.7] to M_t leads to

$$\begin{aligned}
M_t - M_0 &= \int_0^t e^{-\beta s}\Big[\frac{C_s^{1-R}}{1-R} - C_s\mathcal{G}_x + \alpha Y_s\mathcal{G}_y + \lambda\sigma\Pi_s\mathcal{G}_x + rX_s\mathcal{G}_x \\
&\qquad + \frac{1}{2}\sigma^2\Pi_s^2\mathcal{G}_{xx} + \frac{1}{2}\eta^2 Y_s^2\mathcal{G}_{yy} + \sigma\eta\rho Y_s\Pi_s\mathcal{G}_{xy} - \beta\mathcal{G}\Big]ds \\
&\quad + \int_0^t e^{-\beta s}(\mathcal{G}_\theta - Y_s\mathcal{G}_x)d\Theta_s \\
&\quad + \sum_{0<s\le t} e^{-\beta s}\left[\mathcal{G}(X_s, Y_s, \Theta_s) - \mathcal{G}(X_{s-}, Y_{s-}, \Theta_{s-}) - \mathcal{G}_x(\triangle X)_s - \mathcal{G}_\theta(\triangle\Theta)_s\right] \\
&\quad + \int_0^t e^{-\beta s}\sigma\Pi_s\mathcal{G}_x dB_s^1 \\
&\quad + \int_0^t e^{-\beta s}\eta Y_s\mathcal{G}_y dB_s^2 \\
&= N_t^1 + N_t^2 + N_t^3 + N_t^4 + N_t^5. \qquad (5.20)
\end{aligned}$$

Note that for a general admissible strategy Θ and X do not need to be continuous, so that here the arguments of $\mathcal{G}_\cdot$ are $(X_{s-}, Y_s, \Theta_{s-})$.

Lemma 5.2.1 implies that $\mathcal{L}\mathcal{G} - \beta\mathcal{G} \le 0$ and $\mathcal{M}\mathcal{G} = 0$, which leads to $N_t^1 \le 0$ and $N_t^2 = 0$. Using the fact that $(\Delta X)_s = -Y_s(\Delta\Theta)_s$ and writing $\theta = \Theta_{s-}$, $x = X_{s-}$, $\chi = -(\Delta\Theta)_s$ each non-zero jump in N^3 is of the form

$$(\Delta N^3)_s = e^{-\beta s}\left\{\mathcal{G}(x + y\chi, y, \theta - \chi) - \mathcal{G}(x, y, \theta) + \chi\left[\mathcal{G}_\theta(x, y, \theta) - y\mathcal{G}_x(x, y, \theta)\right]\right\}.$$

Given the form of the candidate value function in (5.17), it is easy to see that $\mathcal{G}(x + y\phi, y, \theta - \phi)$ is constant in ϕ, whence $y\mathcal{G}_x = \mathcal{G}_\theta$ and $(\Delta N^3) = 0$. Then, since $R < 1$, we have $0 \le M_t \le M_0 + N_t^4 + N_t^5$, and $(N^4 + N^5)_{t\ge 0}$, as the sum of two local martingales, is a local martingale and is bounded from below and hence a supermartingale. By taking expectations we find $\mathbb{E}(M_t) \le M_0 = G(x_0, y_0, \theta_0, 0)$, which gives

$$G(x_0, y_0, \theta_0, 0) \ge \mathbb{E}\int_0^t e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds + \mathbb{E}\left[e^{-\beta s}\mathcal{G}(X_t, Y_t, \Theta_t)\right] \ge \mathbb{E}\int_0^t e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds,$$

where the last inequality follows since $\mathcal{G}(X_t, Y_t, \Theta_t) \ge 0$ for $R \in (0, 1)$. Letting $t \to \infty$, we have

$$G(x_0, y_0, \theta_0, 0) \ge \mathbb{E}\left(\int_0^\infty e^{-\beta t}\frac{C_t^{1-R}}{1-R}dt\right),$$

and taking a supremum over admissible strategies leads to $G(x_0, y_0, \theta_0, 0) \ge V(x_0, y_0, \theta_0, 0)$.

The case $R > 1$ will be considered in Section 5.8.

□

5.2.2 Proof of the ill-posed case of Theorem 5.1.7 (scenario 4)

In the case $R < 1$ and $b_3 \geq \frac{b_1}{1-R} + b_2 R$ it is sufficient to give an example of an admissible strategy for which the expected utility of consumption is infinite.

The condition $b_3 \geq \frac{b_1}{1-R} + b_2 R$ can be rewritten in terms of the original parameters as

$$\zeta\eta(1-R) - \frac{1}{2}\eta^2 R(1-R) + r(1-R) - \beta \geq 0. \tag{5.21}$$

Consider the following consumption, investment and sale strategies

$$\tilde{C}_t = \lambda\eta\tilde{X}_t, \qquad \tilde{\Pi}_t = \frac{\eta}{\sigma}\tilde{X}_t, \qquad \frac{d\tilde{\Theta}_t}{\tilde{\Theta}_t} = -\zeta\eta\frac{\tilde{X}_t}{Y_t\tilde{\Theta}_t}dt. \tag{5.22}$$

Note first that $\tilde{\Theta}$ is a non-increasing process. The corresponding wealth process $\tilde{X}$ satisfies $d\tilde{X}_t = (\zeta\eta + r)\tilde{X}_t dt + \eta\tilde{X}_t dB_t^1$, which gives

$$\tilde{X}_t = x_0 \exp\left\{\left(\zeta\eta + r - \frac{1}{2}\eta^2\right)t + \eta B_t^1\right\}.$$

Hence $\tilde{X}_t \geq 0$ and the strategies defined in (5.22) are admissible.

The expected utility from consumption $\tilde{G}$ corresponding to the consumption, sale and investment strategies $(\tilde{C}, \tilde{\Theta}, \tilde{\Pi})$ is given by

$$\begin{aligned}
\tilde{G} &= \mathbb{E}^{(\tilde{c},\tilde{\theta},\tilde{\pi})}\left[\int_0^\infty e^{-\beta t}\frac{\tilde{C}_t^{1-R}}{1-R}dt\right] \\
&= \frac{(\lambda\eta x_0)^{1-R}}{1-R}\int_0^\infty \exp\left\{\left[\zeta\eta(1-R) - \frac{1}{2}\eta^2 R(1-R) + r(1-R) - \beta\right]t\right\}dt \\
&= \infty,
\end{aligned}$$

where the last equality follows from (5.21).

5.3 Some preliminaries on n and q^*.

Recall the definitions of φ and υ:

$$\begin{aligned}
\upsilon(q,n) &= \varphi(q,n) - \operatorname{sgn}(1-R)\sqrt{\varphi(q,n)^2 + E(q)^2}, \\
\varphi(q,n) &= b_1(n-1) + (1-R)(b_3 - 2R)q + (2-b_2)R(1-R);
\end{aligned}$$

where $E(q)^2 = 4R^2(1-R)^2(b_2-1)(1-q)^2 \geq 0$.

Results not proved in the main body of this section are proved in Section 5.4.

Lemma 5.3.1. *For $q \in [0,1]$, $(1-R)q + R > 0$ and*

$$\begin{aligned}
\varphi(q, m(q)) &= R(1-R)\left\{(1-q)^2 - (b_2-1)\right\} \\
\varphi(q, \ell(q)) &= R(1-q)\left\{(1-R)q + R - \frac{(b_2-1)R^2}{(1-R)q+R}\right\} \\
\upsilon(q, m(q)) &= -2R(1-R)(b_2-1) \\
\upsilon(q, \ell(q)) &= \frac{-2R^2(1-R)(1-q)(b_2-1)}{(1-R)q+R}
\end{aligned}$$

Suppose $b_2 > 1$. Then, for $R < 1$, $\upsilon(q,n) < 0$ and for fixed q, $\upsilon(q,n)$ is an increasing, concave function of n. For $R > 1$, $\upsilon(q,n) > 0$ and for fixed q, $\upsilon(q,n)$ is an increasing, convex function of n.

For $b_2 = 1$, $\upsilon(q,n) = 0$.

Recall that n solves (5.3). This expression can be written in several ways.

Lemma 5.3.2. *$n'(q) = F(q, n(q))$ where*

$$F(q,n)$$

$$= n\left(\frac{1-R}{R(1-q)} - \frac{(1-R)^2}{b_1 R}\frac{q}{\ell(q)-n} + \frac{(1-R)q}{2b_1R(1-q)[(1-R)q+R]}\frac{\upsilon(q,n)}{\ell(q)-n}\right) \tag{5.23}$$

$$= \frac{(1-R)n}{R(1-q)} - \frac{2(1-R)^2qn/R}{2(1-R)(1-q)[(1-R)q+R] - \varphi(q,n) - sgn(1-R)\sqrt{\varphi(q,n)^2 + E(q)^2}} \tag{5.24}$$

$$= -\frac{(1-R)n\left[2b_1[(1-R)q+R](n-m(q)) - q(\upsilon(q,n) - \upsilon(q,m(q)))\right]}{2R(1-q)((1-R)q+R)\{S(q) - b_1(n-m(q))\}} \tag{5.25}$$

where $S(q) = b_1(\ell(q) - m(q)) = (1-R)q(1-q) + (b_2-1)R(1-R)q/((1-R)q+R)$.

Lemma 5.3.3. *1. For $R \in (0,1)$, $n'(0)$ is the smaller root of $\Phi(\chi) = 0$, where*

$$\Phi(\chi) = b_1R\chi^2 + R(1-R)(b_3 - b_2 - \frac{b_1}{R})\chi - b_3(1-R)^2,$$

For $R \in (1,\infty)$, $n'(0)$ is the larger root.

2. For $q \in (0, q_n)$, $(1-R)n(q) \leq (1-R)\ell(q)$.

3. For $q \in (0, q_n)$, $n'(q) < 0$ if and only if $n(q) > m(q)$, $n'(q) > 0$ if and only if $n(q) < m(q)$, and $n'(q) = 0$ if and only if $n(q) = m(q)$.

4. *If $R > 1$ then either $q_n = q^* = 1$ or $q_n > q^*$. If $R < 1$ and $\ell(1) \geq 0$ then $q_n = q_\ell = 1$. If $R < 1$ and $\ell(1) < 0$ then $q_n = q_\ell < q^*$.*

5. *If $0 \leq q^* < 1$ then $q^* > \frac{b_3}{2R}$ and $(1-R)m$ is increasing on $(q^*, 1)$.*

Lemma 5.3.4. *$(1-R)n$ is monotone increasing in b_1, monotone increasing in b_2 and monotone decreasing in b_3 for $q \in [0, q^*]$. Moreover q^* is decreasing as a function of b_1, decreasing as a function of b_2 and increasing as a function of b_3. Finally $(1-R)n(q^*)^{-R}$ is decreasing in b_1, decreasing in b_2 and increasing in b_3.*

Corollary 5.3.5. *The certainty equivalent value $p = p(x, y, \theta)$ is decreasing in b_1 and b_2 and increasing in b_3.*

Lemma 5.3.6. 1. *Suppose $b_2 = 1$. Then $b_{3,crit}(b_1, 1, R) = \bar{b}_3$ where recall $\bar{b}_3 = \min\{2R, R + \frac{b_1}{1-R}\}$ if $R < 1$ and $\bar{b}_3 = 2R$ if $R > 1$.*

2. *Let $b_2 \to \infty$. Then $\lim_{b_2 \to \infty} b_{3,crit}(b_1, b_2, R) = R$.*
   Hence $R < b_{3,crit} \leq \bar{b}_3$.

Proof of Proposition 5.1.3. Suppose $R < 1$. (The proof for $R > 1$ is similar, except that the final paragraph is not necessary.)

Recall the definition of Φ in Lemma 5.3.3. Note that $\Phi(m'(0)) = (1-R)^2 R b_2 b_3$. Then, if $b_3 < 0$ we have $n'(0) < m'(0)$ and $q^* = 0$. If $b_3 = 0$ then $n'(0) = m'(0)$ and more care is needed. Since $b_3 \leq 0$, m is increasing. Suppose $n(\hat{x}) > m(\hat{x})$ for some $\hat{x}$ in $[0, 1]$. Let $\underline{x} = \sup\{x < \hat{x} : n(x) = m(x)\}$. Then on $(\underline{x}, \hat{x})$ we have $n'(x) < 0 < m'(x)$ and $m(\hat{x}) - n(\hat{x}) = m(\underline{x}) - n(\underline{x}) + \int_{\underline{x}}^{\hat{x}} [m'(y) - n'(y)] dy > 0$, a contradiction. Hence $q^* = 0$.

Now suppose $b_3 > 0$. Then $n'(0) > m'(0)$ and at least initially $n > m$, and $q^* > 0$. By Lemma 5.3.4, there exists a critical value of b_3, namely $b_{3,crit}$, such that $q^*(b_3) < 1$ if $b_3 < b_{3,crit}$, and $q^* = 1$ for $b_3 \geq b_{3,crit}$.

Note that if $b_3 \geq \frac{b_1}{1-R} + b_2 R$ then $\ell(1) \leq 0$. The results in (4) now follow from Lemma 5.3.3(4). Conversly, if $b_3 \geq 2R$ then m is decreasing and $q^* \geq q_m$.

□

Recall the definition $N(q) = n(q)^{-R}(1-q)^{R-1}$ and that W is inverse to N. We have $h^* = N(q^*)$. Our aim now is to construct the candidate value function from the solution for n, which we know as the solution of a first order differential equation.

Proposition 5.3.7. *(1) For $R < 1$, N is increasing on $[0, q^*]$. Moreover W is increasing and such that $0 < W(v) < q^*$ on $(1, h^*)$. For $R > 1$, N is decreasing on $[0, q^*]$. and W is decreasing such that $0 < W(v) < q^*$ on $(h^*, 1)$.*

(2) Let $w(s) = (1-R)sW(s)$. Then w solves

$$(w(s)w'(s))^2 + \left\{ b_1 \left[s - \frac{1}{1-R} w(s) \right]^{\frac{R-1}{R}} - [b_1 + b_2 R(1-R)]s + (b_3 + 2R - 2)w(s) \right\} w(s)w'(s)$$
$$+[(2R-1)(b_3-1) + R^2(1-b_2)]w(s)^2 + [(1-2R)b_1 + b_2R(1-R) - R(1-R)b_3]sw(s)$$
$$+b_1R(1-R)h^2 + b_1[(2R-1)w(s) - R(1-R)s] \left[s - \frac{1}{1-R} w(s) \right]^{\frac{R-1}{R}} = 0.$$

(3) For $R < 1$ and $1 < s < h^*$, and for $R > 1$ and $h^* < s < 1$ we have $1 - R < w'(s) < 1 - Rw(s)/((1-R)s)$ with $w'(h^*) = 1 - Rw(h^*)/((1-R)h^*)$.

Proof. (1) N solves

$$\frac{N'(q)}{N(q)} = \frac{2(1-R)^2 q(1-q)[(1-R)q + R] - (1-R)q\upsilon\left(q, (1-q)^{1-\frac{1}{R}} N(q)^{-\frac{1}{R}}\right)}{2b_1(1-q)[(1-R)q + R]\left\{\ell(q) - (1-q)^{1-\frac{1}{R}} N(q)^{-\frac{1}{R}}\right\}}$$

Suppose $R < 1$. We have $\upsilon(q,n) \leq 0$ (see Lemma 5.3.1) and, for $q \leq q^*$, $n(q) = (1-q)^{1-1/R} N(q)^{-1/R} \leq \ell(q)$. Hence N is increasing. For $R > 1$, $\upsilon(q, (1-q)^{1-1/R} N(q)) \geq 0$ and $n(q) = (1-q)^{1-1/R} N(q)^{-1/R} \geq \ell(q)$. Hence, N is decreasing.

(2) Note that if $s = N(q)$ then $W'(s) = 1/N'(q)$ and so W solves

$$W'(s) = \frac{2b_1(1-W(s))[(1-R)W(s) + R]\left\{\ell(W(s)) - (1-W(s))^{1-\frac{1}{R}} s^{-\frac{1}{R}}\right\}}{s\left\{2(1-R)^2 W(s)(1-W(s))[(1-R)W(s) + R] - (1-R)W(s)\upsilon\left(W(s), (1-W(s))^{1-\frac{1}{R}} s^{-\frac{1}{R}}\right)\right\}}$$

The expression for w' follows after some lengthy algebra.

(3) We have that $n'(q) = F(q, n(q))$ and then from the representation (5.24) and the definition of N it follows that

$$\frac{N'(q)}{N(q)} = \frac{2(1-R)^2 q}{2(1-R)^2(1-q)\left(q + \frac{R}{1-R}\right) - \varphi(q, n(q)) - \operatorname{sgn}(1-R)\sqrt{\varphi(q, n(q))^2 + E(q)^2}},$$

where $n(q) = (1-q)^{1-\frac{1}{R}} N(q)^{-1/R}$. Then an alternative representation for W' is

$$s(1-R)W'(s) = \frac{2(1-R)(1-W(s))\left((1-R)W(s) + R\right) - \hat{\varphi} - \operatorname{sgn}(1-R)\sqrt{\hat{\varphi}^2 + E(W(s))^2}}{2(1-R)W(s)}.$$

where $\hat{\varphi} = \varphi\left(W(s), (1-W(s))^{1-\frac{1}{R}} s\right)$.

We want to show for $s \in (1, h^*)$, $1 - R < w'(s) < 1 - \frac{Rw(s)}{(1-R)s}$. This is equivalent

to

$$\frac{(1-R)(1-W(s))}{s} < (1-R)W'(s) < \frac{1-W(s)}{s}. \tag{5.26}$$

Consider the second of these inequalities. It can be rewritten as either $\frac{1}{1-R}\frac{N'(q)}{N(q)} > \frac{1}{1-q}$ or $(1-R)\frac{n'(q)}{n(q)} < 0$, the latter of which is immediate since $(1-R)n$ is decreasing on $(0, q^*)$. Now consider the first inequality in (5.26). If $R > 1$ then it is immediate since W is decreasing. It only remains to show that $n'/n > -1/(1-q)$ for $R < 1$.

Suppose $R < 1$ and let $k(q)$ solve $\frac{F(q,k)}{k} = -\frac{1}{1-q}$. (Note that $F(q, m(q)) = 0$, $F(q, \ell(q)) = -\infty$ and $\varphi(q,k) + \sqrt{\varphi(q,k)^2 + E(q)^2}$ is increasing in k so that from (5.24), $F(q,k)/k$ is decreasing in k. In particular, $\frac{F(q,k)}{k} = -\frac{1}{1-q}$ has a unique solution in $(m(q), \ell(q))$.) Then k is the straight line with $k(0) = 1$ and $k(1) = \ell(1)$, so that

$$k(q) = 1 + q\frac{(b_2R - b_3)(1-R)}{b_1}. \tag{5.27}$$

To verify this, note that if k is as given in (5.27) then $\varphi(q, k(q)) = R(1-R)(2-b_2)(1-q)$ and $\upsilon(q, k(q)) = -2R(1-R)(b_2-1)(1-q)$. The desired conclusion $F(q, k(q)) = -k(q)/(1-q)$ follows after some lengthy algebra.

We have that $\Phi(k'(0)) = R(1-R)^3 b_2 \left\{ b_3 - \left(b_2R + \frac{b_1}{1-R} \right) \right\} / b_1$. By assumption $b_3 < b_2R + b_1/(1-R)$ (else we are in a degenerate case), and then $\Phi(k'(0)) < 0$ and $n'(0) < k'(0)$, so that $n < k$ on some interval $(0, q)$ to the right of zero. Suppose now that $n(q) = k(q)$ for some $q \in (0, q^*)$. Then

$$n'(q) = F(q, n(q)) = -\frac{k(q)}{1-q} = -\frac{1 + qk'(q)}{1-q} < k'(q).$$

Let $\tilde{q}$ be the smallest such point, then $\tilde{q}$ is a downcrossing of n over k, contradicting the fact that $n(q) < k(q)$ on $(0, \tilde{q})$. Hence $n(q) < k(q)$ for any $q \in (0, q^*)$. But for $n \le k(q)$, $F(q, n) > -\frac{n}{1-q}$. Thus $\frac{n'}{n} > -\frac{1}{1-q}$ and $w'(s) > 1 - R$.

□

5.4 Properties of n: proofs

Proof of Lemma 5.3.1. The values of φ and υ at $m(q)$ and $\ell(q)$ follow on substitution. If $b_2 > 1$ then it is immediate from the definition that $(1-R)\upsilon(q,n) < 0$. To see that υ

is increasing in n, note that

$$\frac{\partial \upsilon}{\partial n} = \frac{\partial \varphi}{\partial n}\left\{1 - \frac{\operatorname{sgn}(1-R)\varphi}{\sqrt{\varphi^2 + E(q)^2}}\right\} > 0,$$

where we use that $\partial\varphi/\partial n = b_1 > 0$. Finally to see that $(1-R)\upsilon$ is concave in n, note that

$$\operatorname{sgn}(1-R)\frac{\partial^2 \upsilon}{\partial n^2} = -\frac{b_1^2 E(q)^2}{(\varphi^2 + E(q)^2)^{3/2}} < 0.$$

□

Proof of Lemma 5.3.2. First we prove the equivalence of (5.23) and (5.24).

Consider

$$\begin{aligned}
&b_1(\ell(q) - n) + \varphi(q,n) \\
&= R(1-R)q^2 - b_3(1-R)q + b_1 - b_1 n + (1-R)q(1-q) + \frac{(b_2-1)R(1-R)q}{(1-R)q+R} \\
&\quad + b_1 n - b_1 + b_3(1-R)q + R(1-R)[-2q + 2 - b_2] \\
&= R(1-R)\left[(1-q)^2 - (b_2-1) + \frac{(b_2-1)q}{(1-R)q+R}\right] + (1-R)q(1-q) \\
&= (1-R)(1-q)[R(1-q)+q] - \frac{(b_2-1)R^2(1-R)}{(1-R)q+R}(1-q).
\end{aligned}$$

Then, noting that $(1-R)q + R = R(1-q) + q$,

$$b_1[(1-R)q+R](\ell(q)-n) = (1-R)(1-q)[R(1-q)+q]^2 - R^2(1-R)(b_2-1)(1-q) - \varphi(q,n)[R(1-q)+q],$$

and multiplying by $4(1-R)(1-q)$,

$$\begin{aligned}
&4b_1(1-R)(1-q)[(1-R)q+R](\ell(q)-n) \\
&= 4(1-R)^2(1-q)^2[R(1-q)+q]^2 - 4\varphi(q,n)(1-R)(1-q)[R(1-q)+q] + \varphi(q,n)^2 \\
&\quad -\{\operatorname{sgn}(1-R)\}^2\left(\varphi(q,n)^2 + 4R^2(1-R)^2(b_2-1)(1-q)^2\right) \\
&= \{2(1-R)(1-q)[R(1-q)+q] - \varphi(q,n)\}^2 - \{\operatorname{sgn}(1-R)\}^2\left\{\varphi(q,n)^2 + E(q)^2\right\}.
\end{aligned}$$

Writing this last expression as the difference of two squares we find

$$\begin{aligned}
&\frac{4b_1(1-R)(1-q)[(1-R)q+R](\ell(q)-n)}{2(1-R)(1-q)[(1-R)q+R] - \varphi(q,n) - \operatorname{sgn}(1-R)\sqrt{\varphi(q,n)^2 + E(q)^2}} \\
&= 2(1-R)(1-q)[R(1-q)+q] - \upsilon(q,n),
\end{aligned}$$

from which the result follows, on dividing by $2b_1R(1-q)[(1-R)q+R](\ell(q)-n)/((1-R)q)$.

Now consider the equivalence of (5.23) and (5.25). We have, starting with (5.23),

$$\begin{aligned}
&\frac{(1-R)n}{R(1-q)}\left\{1-\frac{(1-R)q(1-q)}{b_1(\ell(q)-n)}+\frac{q\upsilon(q,n)}{2b_1[(1-R)q+R](\ell(q)-n)}\right\}\\
&= \frac{(1-R)n\left\{2b_1(\ell(q)-n)[(1-R)q+R]-2[(1-R)q+R](1-R)q(1-q)+q\upsilon(q,n)\right\}}{2b_1R[(1-R)q+R](1-q)(\ell(q)-n)}\\
&= \frac{(1-R)n\left\{2b_1[(1-R)q+R]\left[(\ell(q)-m(q))-(n-m(q))-\frac{(1-R)q(1-q)}{b_1}\right]+q\upsilon(q,n)\right\}}{2R(1-q)[(1-R)q+R]\left\{S(q)-b_1(n-m(q))\right\}}.
\end{aligned}$$

The result then follows since

$$2b_1[(1-R)q+R]\left\{\ell(q)-m(q)-\frac{(1-R)q(1-q)}{b_1}\right\}=2R(1-R)(b_2-1)q=-q\upsilon(q,m).$$

□

Proof of Lemma 5.3.3. (1) From the expression (5.3) and l'Hôpital's rule, $n'(0)=\chi$ solves

$$\chi=\frac{1-R}{R}-\frac{(1-R)^2}{b_1R}\frac{1}{\ell'(0)-\chi}+\frac{(1-R)}{2b_1R^2}\frac{\upsilon(0,1)}{\ell'(0)-\chi},$$

where we have

$$\upsilon(0,1)=2R(1-R)-b_2R(1-R)-\operatorname{sgn}(1-R)b_2R|1-R|=2R(1-R)(1-b_2),$$

and $\ell'(0)=(b_2-b_3)(1-R)/b_1$. This gives

$$\chi=\frac{1-R}{R}-\frac{b_2(1-R)^2}{R\left[(b_2-b_3)(1-R)-b_1\chi\right]},$$

or equivalently, we have that χ solves $\Phi(\chi)=0$. Further,

$$\Phi\left(\ell'(0)\right)=\Phi\left(\frac{(b_2-b_3)(1-R)}{b_1}\right)=-(1-R)^2b_2<0.$$

For $R<1$, we have $n'(0)<\ell'(0)$ by hypothesis, so that $n'(0)$ is the smaller root of Φ. For $R>1$, we have $n'(0)>\ell'(0)$ by hypothesis and $n'(0)$ is the larger root of Φ.

(2) For $R<1$, $n'(0)<\ell'(0)$ so that initially $n<\ell$. Then, from (5.25), $\lim_{n\uparrow\ell(q)}F(q,n)=-\infty$. Hence $n(q)<\ell(q)$, at least until $q=1$ or ℓ hits zero. The argument for $R>1$ is similar.

(3) It is clear from (5.25) that $F(q,m(q))=0$. Also, for $q\le q_n$ so that $(1-R)(\ell-$

$n) > 0$, the sign of $F(q, n(q))$ is opposite to the sign of the factor $D = D(q, m(q), n(q))$ where

$$D(q, m(q), n) = 2b_1[(1-R)q + R](n - m(q)) - qv(q, n) + qv(q, m(q)).$$

But $\partial\varphi/\partial n = b_1$, and so

$$\frac{\partial D}{\partial n} = 2b_1[(1-R)q+R] - qb_1 \left[1 - \frac{\operatorname{sgn}(1-R)\varphi(q,n)}{\sqrt{\varphi(q,n)^2 + E(q)^2}}\right] > 2b_1[(1-R)q+R] - 2qb_1 = 2R(1-q)b_1 > 0.$$

Hence, D is increasing in n and $D(q, m(q), n) > 0$ if and only if $n(q) > m(q)$.

(4) If $R > 1$ then $n(q)$ is increasing on $[0, q^*]$. In particular, $q_n > q^*$ unless $q^* = 1$ whence $q_n = 1 = q^*$. If $R < 1$, then $n(q) \leq \ell(q)$ on $(0, 1)$. But from (5.23) we see that n cannot hit zero strictly before ℓ. The result follows since ℓ is concave, so $q_\ell < 1$ if and only if $\ell(1) < 0$.

(5) We can only have $q^* < 1$ if $(1-R)m'(1) > 0$. For $R < 1$, we must have $n'(q^*) = 0 < m'(q^*)$. But m has a minimum at $b_3/2R$, so $q^* > b_3/2R$. For $R > 1$, we must have $n'(q^*) = 0 > m'(q^*)$. But m has a maximum at $b_3/2R$, so $q^* > b_3/2R$.

□

Proof of Lemma 5.3.4. Consider first the monotonicity of $(1-R)n$ on $(0, q^*)$ in b_2.

Suppose $b_3 > 0$ else there is nothing to prove. For fixed $b_1 > 0$, $b_3 > 0$ and R, write $n = n(\cdot; b_2)$ to emphasise the dependence of the solution on b_2, with similar notation for other functions.

The first step is to consider the monotonicity of $\phi(b_2)$ where $\phi(b_2) = n'(0; b_2)$. Differentiating Φ with respect to b_2 we find

$$0 = 2b_1\phi\frac{\partial\phi}{\partial b_2} + (1-R)\left(b_3 - b_2 - \frac{b_1}{R}\right)\frac{\partial\phi}{\partial b_2} - (1-R)\phi,$$

and hence

$$\frac{\partial\phi}{\partial b_2}\left[\phi + \frac{1-R}{2b_1}\left(b_3 - b_2 - \frac{b_1}{R}\right)\right] = \frac{1-R}{2b_1}\phi.$$

But

$$\Phi\left(-\frac{1-R}{2b_1}\left(b_3 - b_2 - \frac{b_1}{R}\right)\right) = -\frac{R(1-R)^2}{4b_1}\left(b_3 - b_2 - \frac{b_1}{R}\right)^2 - b_3(1-R)^2 < 0. \quad (5.28)$$

Hence, if $R < 1$, $\phi < -\frac{(1-R)}{2b_1}\left(b_3 - b_2 - \frac{b_1}{R}\right)$ and since $\phi < 0$, $\partial\phi/\partial b_2 > 0$. If $R > 1$, then

$\phi > -\frac{(1-R)}{2b_1}\left(b_3 - b_2 - \frac{b_1}{R}\right)$ and since $\phi > 0$, $\partial\phi/\partial b_2 < 0$.

Consider now $n'(q;b_2) = F(q, n(q;b_2); b_2)$, and use the representation in (5.24). Fix $q < q^*(b_2)$ and $n(q)$, and suppose they take values $\tilde{q}$, $\tilde{n}$. Differentiating (5.24) with respect to b_2 gives

$$\frac{\partial}{\partial b_2} F(q,n;b_2)\bigg|_{q=\tilde{q},n=\tilde{n}} = \frac{2(1-R)^2\tilde{q}\tilde{n}}{RD(\tilde{q},\tilde{n};b_2)^2}\frac{\partial}{\partial b_2}D(\tilde{q},\tilde{n};b_2),$$

where $D(q,n;b_2) = 2(1-R)(1-q)[(1-R)q+R] - \varphi(q,n;b_2) - \mathrm{sgn}(1-R)\sqrt{\varphi(q,n;b_2)^2 + E(q;b_2)^2}$. Then

$$\begin{aligned}\frac{\partial D}{\partial b_2} &= \frac{\mathrm{sgn}(1-R)}{\sqrt{\varphi(q,n;b_2)^2 + E(q;b_2)^2}} \\ &\quad \times \left\{ -\frac{\partial\varphi}{\partial b_2}\left[\mathrm{sgn}(1-R)\sqrt{\varphi(q,n;b_2)^2 + E(q;b_2)^2} + \varphi(q,n;b_2)\right] - \frac{1}{2}\frac{\partial}{\partial b_2}\left(E(q;b_2)^2\right)\right\},\end{aligned}$$

and using $\frac{1}{2}\frac{\partial}{\partial b_2}\left(E(q;b_2)^2\right) = 2R^2(1-R)^2(1-q)^2$, $\partial\varphi/\partial b_2 = -R(1-R)$ and the fact that $\mathrm{sgn}(1-R)\sqrt{\varphi(q,n;b_2)^2 + E(q;b_2)^2} + \varphi(q,n;b_2)$ is increasing in n we conclude that, for $0 < q < q^*$,

$$\begin{aligned}&R(1-R)\left[\varphi(q,n) + \mathrm{sgn}(1-R)\sqrt{\varphi(q,n)^2 + E(q)^2}\right] - 2R^2(1-R)^2(1-q)^2 \\ &> R(1-R)\left[\varphi(q,m) + \mathrm{sgn}(1-R)\sqrt{\varphi(q,m)^2 + E(q)^2}\right] - 2R^2(1-R)^2(1-q)^2 \\ &= 0.\end{aligned}$$

Hence, for $R < 1$, $\partial D/\partial b_2 > 0$ and it follows that $\frac{\partial}{\partial b_2}F(q,n;b_2)|_{q=\tilde{q},n=\tilde{n}} > 0$, whereas for $R > 1$, $\partial D/\partial b_2 < 0$ and $\partial F/\partial b_2 < 0$.

Now suppose $\overline{b}_2 > \underline{b}_2$ and let $\overline{n}(q) = n(q;\overline{b}_2)$ and $\underline{n}(q) = n(q;\underline{b}_2)$. Comparing derivatives at zero we conclude that initially $(1-R)\overline{n} > (1-R)\underline{n}$. Also if $\overline{n}(\cdot)$ and $\underline{n}(\cdot)$ cross at some point $(\tilde{q},\tilde{n})$ with $\tilde{q} < q^*(\overline{b_2}) \wedge q^*(\underline{b_2})$ then it follows from our knowledge of $\frac{\partial}{\partial b_2}F(q,n(q;b_2);b_2)$ that $(1-R)\overline{n}' > (1-R)\underline{n}'$. But at a first crossing, $(1-R)\overline{n}$ must cross $(1-R)\underline{n}$ from above which is a contradiction. Hence $(1-R)n(\cdot;b_2)$ is increasing in b_2 for $q \le q^*$ and since m does not depend on b_2, we conclude $q^*(b_2)$ is increasing.

Secondly we consider monotonicity in b_3. The approach is similar, but slightly more involved since m depends on b_3.

For fixed $b_1 > 0$, $b_2 \ge 1$ and R, write $n(\cdot) = n(\cdot;b_3)$. Consider $\phi(b_3) = n'(0;b_3)$.

Then

$$\frac{\partial \phi}{\partial b_3} = \frac{(1-R)\left[\frac{1-R}{R} - \phi\right]}{2b_1\phi + (1-R)\left(b_3 - b_2 - \frac{b_1}{R}\right)} = -\frac{(1-R)}{b_1}\left\{\frac{\left[\frac{1-R}{R} - \phi\right]}{\left(\frac{1-R}{R} - \phi\right) + b_1\left(\frac{(1-R)(b_2-b_3)}{b_1} - \phi\right)}\right\}.$$

Suppose $R < 1$. Then $\phi < 0$ and $\phi < \ell'(0) = (1-R)(b_2 - b_3)/b_1$, and

$$0 > \frac{\partial \phi}{\partial b_3} > -\frac{1-R}{b_1} = \frac{\partial m'(0)}{\partial b_3}. \tag{5.29}$$

Now suppose $R > 1$. Then $\phi > 0$ and $\phi > \ell'(0) = (1-R)(b_2 - b_3)/b_1$, and $0 < \partial\phi/\partial b_3 < (R-1)/b_1 = \partial m'(0)/\partial b_3$.

Differentiating (5.24) with respect to b_3 we find

$$\frac{\partial}{\partial b_3} F\bigg|_{q=\tilde{q}, n=\tilde{n}} = \frac{2(1-R)^2\tilde{q}\tilde{n}}{RD(\tilde{q},\tilde{n};b_3)}\frac{\partial}{\partial b_3}D(\tilde{q},\tilde{n};b_3),$$

where

$$\frac{\partial}{\partial b_3}D(q,n;b_3) = -\frac{1}{\sqrt{\varphi(q,n;b_3)^2 + E(q)^2}}\frac{\partial \varphi}{\partial b_3}\left\{\sqrt{\varphi(q,n;b_3)^2 + E(q)^2} + \operatorname{sgn}(1-R)\varphi(q,n;b_3)\right\}.$$

Since $\partial\varphi/\partial b_3 = (1-R)q$, we find $\partial F/\partial b_3|_{q=\tilde{q},n=\tilde{n}}$ has the opposite sign to $1-R$.

By the same argument as in the case for b_2, it follows that $(1-R)n$ is decreasing in b_3. However, this alone is not sufficient to make conclusions about $q^*(b_3)$ since m also depends on b_3.

Define $\zeta(q;b_3) = n(q;b_3) - m(q;b_3)$, and let $m_0(q) = 1 + R(1-R)q^2/b_1$ so that $m(q;b_3) = m_0(q) - b_3(1-R)q/b_1$. Then $d\zeta/dq = H(q,\zeta;b_3)$ where

$$H(q,\zeta;b_3) = -\frac{(1-R)B_1(q,\zeta(q;b_3);b_3)B_2(q,\zeta(q;b_3);b_3)}{2R(1-q)[(1-R)q+R]\{S(q) - b_1\zeta\}} - \frac{2R(1-R)q}{b_1} + \frac{b_3(1-R)}{b_1},$$

and

$$\begin{aligned} B_1(q,\zeta;b_3) &= \zeta + m_0(q) - \frac{b_3(1-R)}{b_1}q \\ B_2(q,\zeta;b_3) &= 2b_1[(1-R)q+R]\zeta - q\upsilon\left(q, m_0(q) - \frac{b_3(1-R)}{b_1} + \zeta\right) + q\upsilon\left(q, m_0(q) - \frac{b_3(1-R)}{b_1}\right). \end{aligned}$$

Suppose $R < 1$. Fix q and $\zeta > -m(q)$. Then B_1 is positive and decreasing in b_3, and B_2 is positive and decreasing in b_3, since υ is concave in n. Hence the product B_1B_2 is

decreasing in b_3 and H is increasing in b_3. By the result in (5.29), $\zeta'(0) = \phi - m'(0)$ is increasing in b_3, so that at least initially, ζ is increasing in b_3. Then since H is increasing in b_3, it follows that solutions of $\zeta'(q; b_3) = H(q, \zeta; b_3)$ for different b_3 cannot cross, and hence $\zeta(q; b_3)$ is increasing in b_3. Thus $q^*(b_3)$ is increasing in b_3. Similar arguments apply when $R > 1$.

Thirdly, consider monotonicity in b_1. Let $\phi(b_1) = n'(0; b_1)$. The ϕ solves $\Phi(\phi(b_1)) = 0$ and hence

$$\frac{d\phi}{db_1} = -\frac{\phi\left(\phi - \frac{1-R}{R}\right)}{2b_1\phi + (1-R)\left(b_3 - b_2 - \frac{b_1}{R}\right)}.$$

Note that $\Phi(-\frac{(1-R)}{2b_1}(b_3 - b_2 - \frac{b_1}{R})) < 0$ (recall (5.28)). Hence, if $R < 1$ then $\phi < 0 < \frac{1-R}{R}$ and $2b_1\phi + (1-R)\left(b_3 - b_2 - \frac{b_1}{R}\right) < 0$. It follows that $d\phi/db_1 > 0$. Conversely, if $R > 1$ we find $d\phi/db_1 < 0$. In either case $(1-R)\phi(b_1)$ is increasing in b_1 and it follows that $(1-R)n$ is initially increasing in b_1.

Now consider $F(q, n; b_1)$. From (5.24) we see that the only dependence on b_1 is via $\varphi + \text{sgn}(1-R)\sqrt{\varphi^2 + E^2}$. Then $\partial F/\partial b_1$ has the opposite sign to $\partial\varphi/\partial b_1 \equiv (n-1)$. Since $(n-1)$ has the opposite sign to $(1-R)$ on $(0, q^*)$ we find that $\text{sgn}(1-R)\partial F/\partial b_1 > 0$. Thus $(1-R)n$ is increasing in b_1 for $q \leq q^*(b_1)$.

As in the case for b_3 this result alone is not sufficient to determine the monotonicity of $q^*(b_1)$. Indeed, to study the dependence of q^* on b_1 it is convenient to introduce new functions. Let $\hat{m}(q) = b_1(m(q) - 1)$ and similarly $\hat{n}(q) = b_1(n(q) - 1)$ and $\hat{\ell}(q) = b_1(\ell(q) - 1)$. Note that $\hat{m}(q) = R(1-R)q^2 - b_3(1-R)q$ does not depend on b_1, and also that

$$\varphi(q, n) = \hat{n} + (1-R)(b_3 - 2R)q + 2R(1-R) - b_2R(1-R).$$

Hence, if we consider φ or υ as functions of $\hat{n}$ then they do not depend on b_1. In particular, $\upsilon(q, 1 + \hat{m}/b_1)$ and $\upsilon(q, 1 + \hat{n}/b_1)$ do not depend on b_1. We have, using (5.25)

$$\begin{aligned}
\hat{n}' &= b_1F(q, 1 + \hat{n}/b_1) \\
&= -\frac{(1-R)}{R}(b_1 + \hat{n})\frac{2((1-R)q + R)(\hat{n} - \hat{m}) - q(\upsilon(q, 1 + \hat{n}/b_1) - \upsilon(q, 1 + \hat{m}/b_1))}{(1-q)[(1-R)q + R](\hat{\ell}(q) - \hat{n})}
\end{aligned}$$

The only dependence on b_1 is through the term $(b_1 + \hat{n})$ and as b_1 increases we find $(1-R)\hat{n}'$ decreases.

It follows that $(1-R)\hat{n}$ is decreasing in b_1. Since $q^* = \inf\{q > 0 : (1-R)n(q) \leq (1-R)m(q)\} = \inf\{q > 0 : (1-R)\hat{n}(q) \leq (1-R)\hat{m}(q)\}$ and since $\hat{m}$ does not depend on b_1 we deduce that q^* is decreasing in b_1.

Finally, for $i = 1, 2, 3$ consider $\frac{\partial}{\partial b_i} n(q^*(b_i))$ over the interval where $q^*(b_i) \in (0, 1)$. We have

$$\frac{\partial}{\partial b_i} n(q^*(b_i)) = \left.\frac{\partial n}{\partial b_i}\right|_{q^*(b_i)} + n'(q^*(b_i)) \frac{\partial q^*}{\partial b_i}.$$

But $n'(q^*) = 0$ and hence $n(q^*(b_i))$ is increasing or decreasing according as $n(q)$ is increasing or decreasing. □

Proof of Corollary 5.3.5. In the case where $0 < q^* < 1$ extend the domain of h to $(-\infty, \infty)$ by

$$h(u) = \frac{(1 + e^u)^{1-R}}{(1 + e^{u^*})^{1-R}} h^* \qquad u > u^*.$$

Then $g(z) = (b_4 R/b_1)^R h(\ln z)$ for all $z \in [0, \infty)$. The monotonicity results for p will follow if $(1 - R)h$ is decreasing in b_1 and b_2 and increasing in b_3.

We focus on monotonicity in b_1 for the case $R < 1$; the proof of monotonicity in b_2 and b_3 and for $R > 1$ follows similarly.

Fix b_2 and $b_3 > 0$ and suppose $b_1 > ((1 - R)b_3 - b_2 R)^+$, to ensure that we are not in the case where the value function is infinite.

Given $n(q; b_1)$ defined on $[0, q^*(b_1)]$ extend the domain of definition to $[0, 1]$ by setting $n(q; b_1) = n(q^*(b_1); b_1)$ for $q > q^*(b_1)$. Let $N(q; b_1) = (1 - q)^{-(1-R)} n(q; b_1)^{-R}$ defined on $[0, 1]$ and let $W(\cdot; b_1)$ be inverse to $N(\cdot; b_1)$.

Then, for each $q \in (0, 1]$, $n(q; b_1)$ is decreasing in b_1 and $N(q; b_1)$ is increasing in b_1. It follows that $W(\cdot; b_1)$ and $w(\cdot; b_1)$ are increasing in b_1.

We know that $\lim_{u \uparrow \infty} e^{-(1-R)u} h(u) = n(1)^{-R} = n(q^*)^{-R}$ is decreasing in b_1. We want to argue that h is decreasing in b_1 for all u.

Fix $\underline{b}_1 < \bar{b}_1$. Suppose there exists $u \in (-\infty, \infty)$ such that $h(u; \underline{b}_1) = h(u; \bar{b}_1)$ and let $\tilde{u}$ be the largest such u; set $h(\tilde{u}; \underline{b}_1) = h(\tilde{u}; \bar{b}_1) = \tilde{h}$. Then $h(u; \underline{b}_1) < h(u; \bar{b}_1)$ for all $u > \tilde{u}$, and we must have $\frac{dh}{du}(\tilde{u}; \underline{b}_1) \leq \frac{dh}{du}(\tilde{u}; \bar{b}_1)$, or equivalently $w(\tilde{h}; \underline{b}_1) \leq w(\tilde{h}; \bar{b}_1)$. But w is increasing in b_1 contradicting the hypothesis that h is not decreasing in b_1.

□

Proof of Lemma 5.3.6. If $b_3 > 0$ and m is monotonic then we must have $q^* = 1$. Hence $b_{3,crit} \leq 2R$.

If $b_3 > 0$, $R < 1$ and $\ell(1) \leq 0$ then $q^* = 1$. Hence for $R < 1$ and $\ell(1) \leq 0$ we must have $b_{3,crit} \leq b_2 R + \frac{b_1}{1-R}$.

These arguments show that $b_{3,crit}(b_1, 1, R) \leq \bar{b}_3$. It remains to show that for $b_2 = 1$ and $0 < b_3 < \bar{b}_3$ we have $q^* = 1$.

If $b_2 = 1$, then $\ell(1) = m(1)$, and provided $b_3 < \bar{b}_3$, $\ell(1) > 0$. Suppose $R < 1$, the case $R > 1$ being easier. If $b_3 \leq R$ then $m(1) \geq m(0) = 1$ and since n is decreasing we must have $q^* < 1$.

So suppose $R < b_3 < \bar{b}_3$. By the arguments in Proposition 5.3.7(3) we have $n(q) \leq k(q) = 1 + \frac{q(R-b_3)(1-R)}{b_1}$. Also

$$(1-q)F(q, m(1)) = m(1)\left[\frac{1-R}{R} - \frac{(1-R)^2}{b_1 R} q \frac{(1-q)}{\ell(q) - \ell(1)}\right] \to m(1)\frac{1-R}{R}\left[1 + \frac{(1-R)}{b_1}\frac{1}{\ell'(1)}\right].$$

But $\frac{b_1 \ell'(1)}{1-R} = 2R - b_3 - 1 = (R-1) + (R-b_3) \in (-1, 0)$. Hence $\lim_{q\uparrow 1}(1-q)F(q, m(1)) = \kappa \in (-\infty, 0)$. Since for $n \geq m(1)$ we must have n crosses the horizontal line at height $m(1)$ before $q = 1$. Then also $q^* < 1$.

(2) Note first that

$$\lim_{b_2\to\infty} \frac{\ell(q)}{b_2} = \frac{R(1-R)}{b_1}\frac{q}{(1-R)q + R}, \qquad \lim_{b_2\to\infty} \frac{\upsilon(q,n)}{b_2} = -2R(1-R).$$

It follows immediately by l'Hôpital's rule that

$$\lim_{b_2\to\infty} F(q,n) = n\left\{\frac{1-R}{R(1-q)} + \frac{(1-R)q}{2b_1 R(1-q)[(1-R)q + R]} \lim_{b_2\to\infty} \frac{v(q,n)/b_2}{\ell(q)/b_2}\right\} = 0. \tag{5.30}$$

Then, if $n_\infty(q) = \lim_{b_2\uparrow\infty} n(q; b_2)$ we have $n'_\infty(q) = 0$, which implies $n(q) = n(0) = 1$. It is easy to see that n_∞ crosses m at some $q^* \in (0,1)$ if and only if $(1-R)m(1) > (1-R)m(0) = (1-R)$ which is equivalent to $0 < b_3 < R$. Otherwise, we have $q^* = 1$.

The final statement follows from the monotonicity of q^* with respect to b_2 and b_3.

□

5.5 The verification lemma in the first non-degenerate case with finite critical exercise ratio

Suppose $0 < b_3 < b_{3,crit}(b_1, b_2, R)$, so that $0 < q^* < 1$. Suppose we have constructed n and N on $[0, q^*]$ and W and w on $[1, h^*]$. Set $z^* = q^*/(1-q^*)$ and $u^* = e^{z^*}$. Define h via $h(u^*) = h^*$ and for $-\infty < u < u^*$ set $\frac{dh}{du} = w(h)$. Then also $\frac{d^2h}{du^2} = w'(h)w(h)$. Then h solves (5.6). Define g via (5.7).

Lemma 5.5.1. *Let z^* and g be as given in Equations 5.5 and 5.7 of Theorem 5.1.9. Then, $g(z)$, $g'(z)$, $g''(z)$ are continuous at $z = z^*$.*

Proof. We have

$$g(z^*+) = \left(\frac{b_1}{b_4 R}\right)^{-R} h^*(1-q^*)^{1-R}(1+z^*)^{1-R} = \left(\frac{b_1}{b_4 R}\right)^{-R} h^* = \left(\frac{b_1}{b_4 R}\right)^{-R} h(u^*) = g(z^*-).$$

For the first derivative, with $K = n(q^*)^{-R}$, we have for $z > z^*$,

$$\begin{aligned} zg'(z) &= \left(\frac{b_1}{b_4 R}\right)^{-R} n(q^*)^{-R}(1+z)^{-R} z(1-R) = (1-R)\left(\frac{b_1}{b_4 R}\right)^{-R} n(q^*)^{-R}(1+z)^{1-R}\frac{z}{1+z} \\ &= (1-R)\left(\frac{zg(z)}{1+z}\right), \end{aligned}$$

and then since $\frac{z^*}{1+z^*} = q^*$, $z^* g'(z^*) = (1-R)\left(\frac{b_1}{b_4 R}\right)^{-R} h^* q^*$. Meanwhile, for $z < z^*$, and noting that $\frac{dh}{du} = w(h)$

$$zg'(z) = \left(\frac{b_1}{b_4 R}\right)^{-R} h'(\ln z) = \left(\frac{b_1}{b_4 R}\right)^{-R} w(h)$$

so that $z^* g'(z^*-) = \left(\frac{b_1}{b_4 R}\right)^{-R} w(h^*)$. The continuity of the first derivative follows from the identity $w(h^*) = (1-R)h^* W(h^*) = (1-R)h^* q^*$.

Finally, for $z > z^*$

$$z^2 g''(z) = -R(1-R)\left(\frac{b_1}{b_4 R}\right)^{-R} n(q^*)^{-R}(1+z)^{1-R}\left(\frac{z}{1+z}\right)^2 = -R(1-R)g(z)\left(\frac{z}{1+z}\right)^2 \tag{5.31}$$

and $(z^*)^2 g''(z^*+) = -R(1-R)g(z^*)(q^*)^2$. For $z < z^*$,

$$z^2 g''(z) = \left(\frac{b_1}{b_4 R}\right)^{-R}(h'' - h') = \left(\frac{b_1}{b_4 R}\right)^{-R}(w'(h) - 1)w(h)$$

and at z^*, $(z^*)^2 g''(z^*-) = -R(1-R)\left(\frac{b_1}{b_4 R}\right)^{-R} h^*(q^*)^2$ where we use Proposition 5.3.7 (3). □

Proposition 5.5.2. *Suppose g solves (5.7). Then for $R < 1$, g is an increasing concave function such that $g(0) = \left(\frac{b_1}{b_4 R}\right)^{-R}$. Otherwise, for $R > 1$, g is a decreasing convex function such that $g(z) \geq 0$. Further, for $z \leq z^*$ we have $0 \geq Rg'(z)^2 \geq (1-R)g(z)g''(z)$ with equality at $z = z^*$.*

Proof. Consider first $R < 1$. Since the statements are immediate in the region $z \geq z^*$, and since there is second order smooth fit at z^* the result will follow if $h(-\infty) = 1$, h is

increasing and, using (5.31), $w(h)w'(h) - w(h) \leq 0$. The last two properties follow from Proposition 5.3.7 since $w(h) \geq 0$ and $w'(h) < 1$.

To see that $h(-\infty) = 1$ note that $w(h) = (1-R)hW(h)$ is bounded away from zero for h bounded away from 1. Then since $W'(1)$ is non-zero finite we conclude that $h(-\infty) = 1$.

For $R > 1$, and $z \geq z^*$, the statement holds immediately. For $z \leq z^*$, Proposition 5.3.7 implies that h is decreasing and $w(h) \leq 0$, $w'(h) > 1$. Together with (5.31), we have g is a decreasing convex function and $g(z) \geq 0$ given that $h \in [0,1]$. □

Define the candidate value function at $t = 0$ by

$$\mathcal{G}(x, y, \theta) = \frac{x^{1-R}}{1-R} g\left(\frac{y\theta}{x}\right). \tag{5.32}$$

Lemma 5.5.3. *Fix y. Then $\mathcal{G} = \mathcal{G}(x, \theta)$ is concave in x and θ. In particular, if $\psi(\chi) = \mathcal{G}(x - \chi y\phi, y, \theta + \chi\phi)$, then ψ is concave in χ.*

Proof. In order to show the concavity of the candidate value function it is sufficient to show that the Hessian matrix given by

$$H_{\mathcal{G}} = \begin{pmatrix} \mathcal{G}_{xx} & \mathcal{G}_{x\theta} \\ \mathcal{G}_{x\theta} & \mathcal{G}_{\theta\theta} \end{pmatrix}.$$

has a positive determinant, and that one of the diagonal entries is non-positive.

Direct computation leads to

$$\begin{aligned}
\mathcal{G}_{xx}(x, y, \theta) &= x^{-R-1}\left[-Rg(z) + \frac{2R}{1-R} zg'(z) + \frac{1}{1-R} z^2 g''(z)\right], \\
\mathcal{G}_{x\theta}(x, y, \theta) &= -x^{-R-1}\frac{y}{1-R}\left[Rg'(z) + zg''(z)\right], \\
\mathcal{G}_{\theta\theta}(x, y, \theta, t) &= x^{-R-1}\frac{y^2}{1-R} g''(z),
\end{aligned}$$

and the determinant of Hessian matrix is

$$\mathcal{G}_{xx}\mathcal{G}_{\theta\theta} - (\mathcal{G}_{x\theta})^2 = -x^{-2R}\theta^{-2}\frac{R}{1-R}\left[g(z) z^2 g''(z) + \frac{R}{1-R}\left(zg'(z)\right)^2\right]. \tag{5.33}$$

If $z \geq z^*$ then the expression on the right-hand-side of (5.33) is zero by Proposition 5.5.2. For $z \leq z^*$, Proposition 5.3.7 yields

$$(1-R)g(z) z^2 g''(z) + R\left(zg'(z)\right)^2 = (1-R)h\left[w(h) w'(h) - w(h)\right] + Rw(h)^2 \leq 0 \tag{5.34}$$

with equality at $h = h^*$ by the smooth fit condition. Further, since g is concave we have that $\mathcal{G}_{\theta\theta} \leq 0$.

In order to show the concavity of ψ in χ, it is equivalent to examine the sign of $\frac{d^2\psi}{d\chi^2}$. But

$$\frac{d^2\psi}{d\chi^2} = \phi^2 \left[y^2\mathcal{G}_{xx} + \mathcal{G}_{\theta\theta} - 2y\mathcal{G}_{x\theta}\right] = \phi^2(y,1)\det(H_{\mathcal{G}})(y,1)^T \leq 0$$

.

□

Lemma 5.5.4. *Consider the candidate function constructed in (5.32). Then*

(a) For $0 < x \leq y\theta/z^*$, $\mathcal{M}\mathcal{G} = 0$ and $\mathcal{L}\mathcal{G} - \beta\mathcal{G} \leq 0$.

(b) For $x \geq y\theta/z^*$, $\mathcal{L}\mathcal{G} - \beta\mathcal{G} = 0$ and $\mathcal{M}\mathcal{G} \geq 0$.

Note that the derivatives of $\mathcal{G}$ are well-defined and continuous at $x = 0$, so that the results in (a) hold at $x = 0$.

Proof. (a) For $z \geq z^*$, $\mathcal{M}\mathcal{G} = 0$ is immediate from the definition of $\mathcal{G}$. For $\mathcal{L}\mathcal{G} - \beta\mathcal{G}$ we have that $\mathcal{G}(x,y,\theta) = \left(\frac{b_1}{b_4 R}\right)^{-R} n(q^*)^{-R}\frac{x^{1-R}}{1-R}(1+z)^{1-R}$ and then

$$\begin{aligned}
& \mathcal{L}\mathcal{G} - \beta\mathcal{G} \\
= \; & \frac{x^{1-R}}{1-R}R\left(\frac{b_1}{b_4 R}\right)^{1-R} n(q^*)^{-R}(1+z)^{1-R}\left[n(q^*) - 1 + \frac{b_3}{b_1}(1-R)\frac{z}{1+z} - \frac{R(1-R)}{b_1}\left(\frac{z}{1+z}\right)^2\right] \\
= \; & \frac{x^{1-R}}{1-R}R\left(\frac{b_1}{b_4 R}\right)^{1-R} n(q^*)^{-R}(1+z)^{1-R}\left[m(q^*) - m\left(\frac{z}{1+z}\right)\right],
\end{aligned}$$

where we use the fact that $n(q^*) = m(q^*)$. The required inequality follows from Lemma 5.3.3 and the fact that m is increasing on $(q^*, 1)$.

(b) In order to prove $\mathcal{LG} - \beta\mathcal{G} = 0$ we calculate

$$\begin{aligned}
\mathcal{LG} - \beta\mathcal{G} &= \frac{x^{1-R}}{1-R}\left\{(z^2 g''(z))^2\right. \\
&\quad + \left[b_4 R\left[g - \frac{1}{1-R} z g'(z)\right]^{\frac{R-1}{R}} - [b_1 + b_2 R(1-R)]g + (b_3 + 2R) z g'(z)\right] z^2 g''(z) \\
&\quad + b_4 R[2Rzg'(z) - R(1-R)g]\left[g - \frac{1}{1-R} z g'(z)\right]^{\frac{R-1}{R}} - [2Rb_1 + R(1-R)b_3] g z g'(z) \\
&\quad \left. + \left[2Rb_3 - R^2 b_2 + R^2\right](z g'(z))^2 + b_1 R(1-R) g^2\right\} \\
&= \frac{x^{1-R}}{1-R}\left(\frac{b_1}{b_4 R}\right)^{-2R}\left\{(w(h)w'(h))^2 + \left\{b_1\left[h - \frac{1}{1-R} w(h)\right]^{\frac{R-1}{R}} - [b_1 + b_2 R(1-R)]\, h\right.\right. \\
&\quad \left.\left. + (b_3 + 2R - 2)\, w(h)\right\} w(h) w'(h) + \left[(2R-1)(b_3 - 1) + R^2(1 - b_2)\right] w(h)^2 \right. \\
&\quad + [(1-2R)b_1 + b_2 R(1-R) - R(1-R)b_3]\, h w(h) + b_1 R(1-R) h^2 \\
&\quad \left. + b_1\left[(2R-1)w(h) - R(1-R)h\right]\left[h - \frac{1}{1-R} w(h)\right]^{\frac{R-1}{R}}\right\} \\
&= 0,
\end{aligned}$$

where the last equality follows from Proposition 5.3.7(2). Note that $\mathcal{G}$ and $\mathcal{LG}$ are well defined and continuous at $\theta = 0$.

Now consider $\mathcal{MG}$. We have

$$\mathcal{MG} = x^{-R} y\left[\frac{(1+z)}{1-R} g'(z) - g(z)\right].$$

Hence it is sufficient to show that $\psi(z) \geq 0$ on $(0, z^*]$ where

$$\psi(z) = \frac{1+z}{1-R} - \frac{g(z)}{g'(z)},$$

By value matching and smooth fit $g(z^*) = \left(\frac{b_1}{b_4 R}\right)^{-R} n(q^*)^{-R}(1+z^*)^{1-R}$ and $z^* g'(z^*) = \left(\frac{b_1}{b_4 R}\right)^{-R} n(q^*)^{-R}(1-R)(1+z^*)^{-R}$. Hence $\psi(z^*) = 0$ and it is sufficient to show that

ψ is decreasing. But

$$\begin{aligned}\psi^{'}(z) &= \frac{R}{1-R}+\frac{g(z)\,g^{''}(z)}{g^{'}(z)^{2}}\\ &= \frac{R}{1-R}+\frac{h\left[w(h)\,w^{'}(h)-w(h)\right]}{w(h)^{2}}\\ &\leq 0\end{aligned}$$

where the last inequality follows from the final part of Proposition 5.3.7.

□

Proposition 5.5.5. *Let X^*, Θ^*, C^* and Π^* be as defined in Theorem 5.1.9. Then they correspond to an admissible wealth process. Moreover $Z_t^* = Y_t\Theta^*/X_t^*$ satisfies $0 \leq Z_t^* \leq z^*$.*

Proof. Note that if $y_0\theta_0/x_0 > z^*$ then the optimal strategy includes a sale of the endowed asset at time zero, and the effect of the sale is to move to new state variables $(X_0^*, y_0, \Theta_0^*, 0)$ with the property that $Z_0^* = y_0\Theta_0^*/X_0^* = z^*$. Thus we may assume that $Z_0 = y_0\Theta_0^*/X_0^* \leq z^*$.

Consider the equation

$$\hat{J}_t = \hat{J}_0 - \int_0^t \tilde{\Lambda}(J_s)ds - \int_0^t \tilde{\Sigma}(J_s)dB_s^1 - \int_0^t \tilde{\Gamma}(J_s)dB_s^2 + \hat{L}_t, \tag{5.35}$$

subject to $\hat{J}_0 = (z^* - z_0)^+$. This equation is associated with a stochastic differential equation with reflection (Revuz and Yor [48, p385]) and has a unique solution (J, L) for which (J, L) is adapted, $J \geq 0$, and L is increasing and continuous, $L_0 = 0$ and L only increases when J is zero. Let (J, L) be the solution to (5.35) with these properties.

Note that $\tilde{\Lambda}(z^*) = \Lambda(0) = 0 = \tilde{\Gamma}(z^*) = \tilde{\Sigma}(z^*)$ and hence J is bounded above by z^*.

Now let $Z_t^* = z^* - J_t$, and $\Theta_t^* = \Theta_0^* \exp\{-L_t/(z^*(1+z^*))\}$, and note that whenever L is increasing, (equivalently Θ is decreasing) we have $Z = z^*$. It follows that the dynamics of Z are governed by

$$dZ_t^* = \Lambda(Z_t^*)dt + \Sigma(Z_t^*)dB_t^1 + \Gamma(Z_t^*)dB_t^2 + Z_t^*(1+Z_t^*)\frac{d\Theta_t^*}{\Theta_t^*}.$$

Now set $X_t^* = Y_t\Theta_t^*/Z_t^*$, $C_t^* = X_t^*[g(Z_t^*) - Z_t^* g'(Z_t^*)/(1-R)]^{-1/R}$ and $\Pi_t^* = \frac{\lambda}{\sigma}X_t^*\Psi_g(Z_t^*)$.

Then X^* and C^* are positive and adapted and moreover

$$\begin{aligned}
dX_t^* &= \frac{Y_t\Theta_t^*}{Z_t^*}\left[\frac{d\Theta_t^*}{\Theta_t^*}+\frac{dY_t}{Y_t}-\frac{dZ_t^*}{Z_t^*}+\left(\frac{dZ_t^*}{Z_t^*}\right)^2-\frac{dY_t}{Y_t}\frac{dZ_t^*}{Z_t^*}\right] \\
&= X_t\Bigg\{\left[\alpha-\frac{\Lambda(Z_t^*)}{Z_t^*}+\frac{\Sigma(Z_t^*)^2}{(Z_t^*)^2}+\frac{\Gamma(Z_t^*)^2}{(Z_t^*)^2}+2\rho\frac{\Gamma(Z_t^*)\Sigma(Z_t^*)}{(Z_t^*)^2}-\eta\rho\frac{\Sigma(Z_t^*)}{Z_t^*}-\eta\frac{\Gamma(Z_t^*)}{Z_t^*}\right]dt \\
&\qquad +\left(\eta-\frac{\Gamma(Z_t^*)}{Z_t^*}\right)dB_t^2-\frac{\Sigma(Z_t^*)}{Z_t^*}dB_t^1-Z_t^*\frac{d\Theta_t^*}{\Theta_t^*}\Bigg\} \\
&= \left(\lambda\sigma\Pi_t^*+rX_t^*-C_t^*\right)dt+\sigma\Pi_t^*dB_t^1-Y_td\Theta_t^*,
\end{aligned}$$

where we use the definitions of Λ, Σ, Γ and Ψ_g for the final equality. It follows immediately that X is the wealth process arising from the consumption, portfolio and sale strategy (C^*, Π^*, Θ^*). □

Proof of Theorem 5.1.9. First we show that there is a strategy such that the candidate value function is attained, and hence that $V \geq G$.

Observe first that if $z_0 = y_0\theta_0/x_0 > z^*$ then

$$\theta_0 - \Theta_0^* = \theta_0\left(1-\frac{z^*}{1+z^*}\frac{1+z_0}{z_0}\right)$$

and

$$X_0^* = x_0 + y_0(\theta_0 - \Theta_0^*) = x_0\frac{(1+z_0)}{(1+z^*)}$$

so that $y\Theta_0^*/X_0^* = z^*$. Then, since $g(z^*)/g(z_0) = (1+z^*)^{1-R}/(1+z_0)^{1-R}$, for $z_0 > z^*$ we have

$$G(X_0^*, y_0, \Theta_0^*, 0) = \frac{(X_0^*)^{1-R}}{1-R}g(z^*) = \frac{x_0^{1-R}}{1-R}g(z_0) = \mathcal{G}(x_0, y_0, \theta_0).$$

Hence, without loss of generality we may assume that $z_0 \leq z^*$ since if $z_0 > z^*$ the agent transacts from (x_0, y_0, θ_0) to (X_0^*, y_0, Θ_0^*) at time zero with no change in value function.

For a general admissible strategy define the process $M = (M_t)_{t\geq 0}$ by

$$M_t = \int_0^t e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds + e^{-\beta t}\mathcal{G}\left(X_t, Y_t, \Theta_t\right). \tag{5.36}$$

Write M^* for the corresponding process under the proposed optimal strategy. Then $M_0^* = \mathcal{G}(X_0^*, y_0, \Theta_0^*) = \mathcal{G}(x_0, y_0, \theta_0)$ so there is no jump of M^* at $t = 0$. Further, although the optimal strategy may include the sale of a positive quantity of the risky

asset at time zero, it follows from Proposition 5.5.5 that thereafter the process Θ^* is continuous and such that $Z_t^* = Y_t\Theta_t^*/X_t^* \leq z^*$.

From the form of the candidate value function and the definition of g given in (5.7), we know that $\mathcal{G}$ is $\mathbb{C}^{2,2,1}$. Then applying Itô's formula to M_t, using the continuity of X^* and Θ^* for $t > 0$, and writing $\mathcal{G}_.$ as shorthand for $\mathcal{G}_.(X_s^*, Y_s, \Theta_s^*)$ we have

$$\begin{aligned}
M_t^* - M_0 &= \int_0^t e^{-\beta s}\Big[\frac{(C_s^*)^{1-R}}{1-R} - C_s^*\mathcal{G}_x + \alpha Y_s\mathcal{G}_y + \lambda\sigma\Pi_s^*\mathcal{G}_x + rX_s^*\mathcal{G}_x \\
&\qquad + \frac{1}{2}\sigma^2\Pi_s^{*2}\mathcal{G}_{xx} + \frac{1}{2}\eta^2 Y_s^2\mathcal{G}_{yy} + \sigma\eta\rho Y_s\Pi_s^*\mathcal{G}_{xy} - \beta\mathcal{G}\Big]ds \\
&\quad + \int_0^t e^{-\beta s}(\mathcal{G}_\theta - Y_s\mathcal{G}_x)d\Theta_s^* \\
&\quad + \int_0^t e^{-\beta s}\sigma\Pi_s^*\mathcal{G}_x dB_s^1 \\
&\quad + \int_0^t e^{-\beta s}\eta Y_s\mathcal{G}_y dB_s^2 \\
&= N_t^1 + N_t^2 + N_t^4 + N_t^5.
\end{aligned}$$

Since $Z_t^* \leq z^*$, and since $C_t^* = \mathcal{G}_x^{-1/R}$ and $\mathcal{L}\mathcal{G} - \beta\mathcal{G} = 0$ for $z \leq z^*$ we have $N_t^1 = 0$. Further, $d\Theta_s \neq 0$ if and only if $Z_t^* = z^*$ and then $\mathcal{M}\mathcal{G} = 0$, so that $N_t^2 = 0$.

To complete the proof of the theorem we need the following lemma proved in Section 5.7.

Lemma 5.5.6. *1. N^4 given by $N_t^4 = \int_0^t e^{-\beta s}\sigma\Pi_s^*\mathcal{G}_x dB_s^1$ is a martingale under the optimal strategy.*

2. N^5 given by $N_t^5 = \int_0^t e^{-\beta s}\eta Y_s\mathcal{G}_y dB_s^2$ is a martingale under the optimal strategy.

3. $\lim_{t\to\infty}\mathbb{E}[e^{-\beta t}\mathcal{G}(X_t^*, Y_t, \Theta_t^*)] = 0$.

Returning to the proof of the theorem, and taking expectations on both sides of $M_t^* - M_0$, we have $\mathbb{E}\left[M_t^*\right] = M_0$, which leads to

$$G\left(x_0, y_0, \theta_0, 0\right) = \mathbb{E}\left(\int_0^t e^{-\beta s}\frac{(C_s^*)^{1-R}}{1-R}ds\right) + \mathbb{E}\left[e^{-\beta t}\mathcal{G}\left(X_t^*, Y_t, \Theta_t^*\right)\right]. \tag{5.37}$$

Then using Lemma 5.5.6 and applying monotone convergence theorem, we have

$$G\left(x_0, y_0, \theta_0, 0\right) = \mathbb{E}\left(\int_0^\infty e^{-\beta s}\frac{C_s^{*1-R}}{1-R}ds\right)$$

and hence $V \geq G$.

Now we consider general admissible strategies and show that $V \leq G$. Exactly as in (5.20) and the proof of Theorem 5.2.2 we have

$$M_t - M_0 = N_t^1 + N_t^2 + N_t^3 + N_t^4 + N_t^5.$$

Lemma 5.5.4 implies that under general admissible strategies, $N_t^1 \leq 0,\ N_t^2 \leq 0$. Consider the jump term,

$$N_t^3 = \sum_{0<s\leq t} e^{-\beta s} \left[\mathcal{G}\left(X_s, Y_s, \Theta_s\right) - \mathcal{G}\left(X_{s-}, Y_s, \Theta_{s-}\right) - \mathcal{G}_x(\Delta X)_s - \mathcal{G}_\theta(\Delta\Theta)_s\right]$$

Using the fact that $(\Delta X)_s = -Y_s(\Delta\Theta)_s$ and writing $\theta = \Theta_{s-}$, $y = Y_s$, $x = X_{s-}$, $\chi = -(\Delta\Theta)_s$ each non-zero jump in N^3 is of the form

$$(\Delta N^3)_s = e^{-\beta s}\Big\{\mathcal{G}(x + y\chi, y, \theta - \chi) - \mathcal{G}(x, y, \theta, s) + \chi\left[\mathcal{G}_\theta(x, y, \theta, s) - y\mathcal{G}_x(x, y, \theta, s)\right]\Big\}.$$

Note that by Lemma 5.5.3, $\mathcal{G}(x + y\chi, y, \theta - \chi)$ is concave in χ and hence $(\Delta N^3) \leq 0$.

For $R < 1$ the rest of the proof is exactly as in Theorem 5.2.2. The case of $R > 1$ will be proved in Section 5.8. □

5.6 The Verification Lemma in the second non-degenerate case (scenario 3) with no finite critical exercise ratio.

Throughout this section we suppose that $b_3 \geq b_{3,crit}(b_1, b_2, R)$ and $b_3 < \frac{b_1}{1-R} + b_2 R$ if $R < 1$. It follows that $q^* = 1$ and $z^* = \infty$.

Recall the definition of γ in (5.12) or (5.16), set $h = \gamma^{-1}$ and let g be given by $g(z) = (Rb_4/b_1)^R h(\ln z)$.

Define the candidate value function as $G(x, y, \theta) = e^{-\beta t}\mathcal{G}(x, y, \theta)$ where

$$\mathcal{G}(x, y, \theta) = \frac{x^{1-R}}{1-R} g\left(\frac{y\theta}{x}\right), \qquad x > 0, y > 0, \theta \geq 0. \tag{5.38}$$

We extend the definition to $y\theta < x \leq 0$ via

$$g(x, y, \theta) = \frac{(x + y\theta)^{1-R}}{1-R}\left(\frac{Rb_4}{b_1}\right)^R n(1)^{-R}.$$

Lemma 5.6.1. *Fix y. Then for $x \geq 0$, $\mathcal{G} = \mathcal{G}(x, \theta)$ is concave in x and θ. In particular, if $\psi(\chi) = \mathcal{G}(x - \chi y\phi, y, \theta + \chi\phi)$, then ψ is concave in χ.*

Proof. The proof follows the proof of Lemma 5.5.3. □

Lemma 5.6.2. *Consider the candidate value function constructed in (5.32). Then for $x \geq 0$, $\mathcal{L}\mathcal{G} - \beta\mathcal{G} = 0$, and $\mathcal{M}G \geq 0$ with equality at $x = 0$.*

Proof. The proof follows the proof of Lemma 5.5.4. □

Proof of Theorem 5.1.11. Consider first the following stochastic differential equation with reflection

$$K_t = K_0 + \int_0^t \hat{\Lambda}(K_s)ds + \int_0^t \hat{\Sigma}(K_s)dB_s^1 + \int_0^t \hat{\Gamma}(K_s)dB_s^2 + L_t,$$

for which $K_0 = x_0/(y_0\theta_0)$. By the same argument as in Proposition 5.5.5, this equation has a unique solution (K, L) which is an adapted continuous process for which K is non-negative, $L_0 = 0$ and L only increases when K is zero.

Let $\Theta_t^* = \theta_0 e^{-L_t}$, $X_t^* = Y_t\Theta_t^* K_t$, $C_t^* = X_t^*[g(1/K_t) - \frac{g'(1/K_t)}{K_t(1-R)}]^{-1/R}$ and $\Pi_t^* = \frac{\lambda}{\sigma}X_t^*\Psi_g(1/K_t)$. Then Θ^* is decreasing and $X^* \geq 0$. Then with $dK_t = \hat{\Lambda}(K_t)dt + \hat{\Sigma}(K_t)dB_t^1 + \hat{\Gamma}(K_t)dB_t^2 - \frac{d\Theta_t^*}{\Theta_t^*}$, and, using $K_t dL_t = 0$ and hence $K_t d\Theta_t^* = 0$ also,

$$\begin{aligned}
dX_t^* &= d(Y_t\Theta_t^* K_t) = \Theta_t^* K_t dY_t + Y_t\Theta_t^* dK_t + \Theta_t^* d[Y, K]_t \\
&= X_t^* \left\{ \left[\alpha + \frac{\hat{\Lambda}(K_t)}{K_t} + \frac{\eta\hat{\Gamma}(K_t)}{K_t} + \frac{\eta\rho\hat{\Sigma}(K_t)}{K_t} \right] dt + \frac{\hat{\Sigma}(K_t)}{K_t} dB_t^1 + \left(\eta + \frac{\hat{\Gamma}(K_t)}{K_t} \right) dB_t^2 \right\} - Y_t d\Theta_t^* \\
&= (\lambda\sigma\Pi_t^* + rX_t^* - C_t^*)\, dt + \sigma\Pi_t^* dB_t^1 - Y_t d\Theta_t^*
\end{aligned}$$

where we use the definitions of $\hat{\Lambda}$, $\hat{\Gamma}$ and $\hat{\Sigma}$ for the final equality. It follows immediately that X is the wealth process arising from the consumption and sale strategy (C^*, Π^*, Θ^*), and hence that X is admissible.

From the form of the candidate value function, we know that $\mathcal{G}$ is $C^{2,2,1}$. Then

applying Itô's formula to M^*, we have

$$\begin{aligned}
M_t^* - M_0 &= \int_0^t e^{-\beta s}\Big[U\left(C_s^*\right) - C_s^*\mathcal{G}_x + \alpha Y_s\mathcal{G}_y + \lambda\sigma\Pi_s^*\mathcal{G}_x + rX_s^*\mathcal{G}_x \\
&\qquad + \frac{1}{2}\sigma^2\Pi_s^{*2}\mathcal{G}_{xx} + \frac{1}{2}\eta^2 Y_s^2\mathcal{G}_{yy} + \sigma\eta\rho Y_s\Pi_s^*\mathcal{G}_{xy} - \beta\mathcal{G}\Big]ds \\
&\quad + \int_0^t e^{-\beta s}(\mathcal{G}_\theta - Y_s\mathcal{G}_x)d\Theta_s^* \\
&\quad + \int_0^t e^{-\beta s}\sigma\Pi_s^*\mathcal{G}_x dB_s^1 \\
&\quad + \int_0^t e^{-\beta s}\eta Y_s\mathcal{G}_y dB_s^2 \\
&= N_t^1 + N_t^2 + N_t^4 + N_t^5.
\end{aligned}$$

Since $C_t^* = \mathcal{G}_x^{-1/R}$ is optimal, and by Lemma 5.6.2, $\mathcal{L}\mathcal{G} - \beta\mathcal{G} = 0$, we have $N_t^1 = 0$. Further, for the proposed optimal strategies $d\Theta_t \neq 0$ if and only if $X_t = 0$. Then by Lemma 5.6.2, $\mathcal{M}\mathcal{G}|_{x=0} = 0$ and $N_t^2 = 0$.

The following lemma is proved in Section 5.7.

Lemma 5.6.3. *1. N^4 given by $N_t^4 = \int_0^t e^{-\beta s}\sigma\Pi_s^*\mathcal{G}_x dB_s^1$ is a martingale under the optimal strategy.*

2. N^5 given by $N_t^5 = \int_0^t e^{-\beta s}\eta Y_s\mathcal{G}_y dB_s^2$ is a martingale under the optimal strategy.

3. $\lim_{t\to\infty}\mathbb{E}[e^{-\beta t}\mathcal{G}(X_t^*, Y_t, \Theta_t^*)] = 0$.

It follows from the lemma that $\mathbb{E}\left[M_t^*\right] = M_0$, and then

$$G\left(x_0, y_0, \theta_0, 0\right) = \mathbb{E}\left[\int_0^t e^{-\beta s}\frac{(C_s^*)^{1-R}}{1-R}ds\right] + \mathbb{E}\left[e^{-\beta t}\mathcal{G}\left(X_t^*, Y_t, \Theta_t^*\right)\right].$$

Then applying monotone convergence theorem and using the final part of the lemma, we have

$$G\left(x_0, y_0, \theta_0, 0\right) = \mathbb{E}\left[\int_0^\infty e^{-\beta s}\frac{C_s^{*1-R}}{1-R}ds\right]$$

and hence $V \geq G$.

Now we consider general admissible strategies and show that $V \leq G$. Applying

the generalised Itô's formula [23, Section 4.7] to M_t leads to

$$\begin{aligned}
M_t - M_0 &= \int_0^t e^{-\beta s}\Big[U(C_s) - C_s\mathcal{G}_x + \alpha Y_s\mathcal{G}_y + \lambda\sigma\Pi_s\mathcal{G}_x + rX_s\mathcal{G}_x \\
&\qquad + \frac{1}{2}\sigma^2\Pi_s^2\mathcal{G}_{xx} + \frac{1}{2}\eta^2 Y_s^2\mathcal{G}_{yy} + \sigma\eta\rho Y_s\Pi_s\mathcal{G}_{xy} - \beta\mathcal{G}\Big]ds \\
&\quad + \int_0^t e^{-\beta s}(\mathcal{G}_\theta - Y_s\mathcal{G}_x)d\Theta_s \\
&\quad + \sum_{0<s\le t} e^{-\beta s}\left[\mathcal{G}(X_s, Y_s, \Theta_s) - \mathcal{G}(X_{s-}, Y_{s-}, \Theta_{s-}) - \mathcal{G}_x(\triangle X)_s - \mathcal{G}_\theta(\triangle\Theta)_s\right] \\
&\quad + \int_0^t e^{-\beta s}\sigma\Pi_s\mathcal{G}_x dB_s^1 \\
&\quad + \int_0^t e^{-\beta s}\eta Y_s\mathcal{G}_y dB_s^2 \\
&= N_t^1 + N_t^2 + N_t^3 + N_t^4 + N_t^5.
\end{aligned}$$

Note that for a general admissible strategy Θ and X do not need to be continuous, so that here the arguments of $\mathcal{G}_\cdot$ is $(X_{s-}, Y_s, \Theta_{s-})$.

Lemma 5.6.3 implies that under general admissible strategies, $N_t^1 \le 0,\ N_t^2 \le 0$. Consider the jump term,

$$N_t^3 = \sum_{0<s\le t} e^{-\beta s}\left[\mathcal{G}(X_s, Y_s, \Theta_s) - \mathcal{G}(X_{s-}, Y_s, \Theta_{s-}) - \mathcal{G}_x(\Delta X)_s - \mathcal{G}_\theta(\Delta\Theta)_s\right] \tag{5.39}$$

Using the fact that $(\Delta X)_s = -Y_s(\Delta\Theta)_s$ and writing $\theta = \Theta_{s-}$, $x = X_{s-}$, $\chi = -(\Delta\Theta)_s$ each non-zero jump in N^3 is of the form $(\Delta N^3)_s = e^{-\beta s}\Delta_s$ where

$$\Delta_s = \mathcal{G}(x + y\chi, y, \theta - \chi) - \mathcal{G}(x, y, \theta) + \chi\left[\mathcal{G}_\theta(x, y, \theta) - y\mathcal{G}_x(x, y, \theta)\right].$$

Note that by Lemma 5.6.1, $\mathcal{G}(x + y\chi, y, \theta - \chi)$ is concave in χ and hence $\Delta_s \le 0$.

For $R < 1$ the rest of the proof is exactly as in Theorem 5.2.2. The case of $R > 1$ will be proved in Section 5.8.

□

5.7 The martingale property of the value function

Proof of Lemma 5.5.6. (i) As a first step of the proof, we show that $\sigma\Pi^*\mathcal{G}_x/\mathcal{G}$ is bounded. From the form of the candidate value function (5.32) and the optimal portfolio process

in Theorem 5.1.9 we have

$$
\begin{aligned}
&\sigma\Pi^*(x,y,\theta)\frac{\mathcal{G}_x(x,y,\theta)}{\mathcal{G}(x,y,\theta)} \\
&= \lambda\left[(1-R)-\frac{zg'(z)}{g(z)}\right]\left\{\frac{-(1-R)g(z)+\left(1+\frac{R\eta\rho}{\lambda}\right)zg'(z)+\frac{\eta\rho}{\lambda}z^2g''(z)}{-R(1-R)g(z)+2Rzg'(z)+z^2g''(z)}\right\} \\
&= \frac{(1-R)}{R}\left(1-\frac{w(h)}{(1-R)h}\right)\left(\lambda-\frac{\left(\eta\rho-\frac{\lambda}{R}\right)w(h)[w'(h)-(1-R)]}{(1-R)h-\left(2-\frac{1}{R}\right)w(h)-\frac{1}{R}w(h)w'(h)}\right) \\
&= \frac{(1-R)}{R}(1-W(h))\left(\lambda-\frac{\left(\eta\rho-\frac{\lambda}{R}\right)W(h)[w'(h)-(1-R)]}{1-\left(2-\frac{1}{R}\right)W(h)-\frac{1-R}{R}W(h)^2-\frac{1-R}{R}hW(h)W'(h)}\right)
\end{aligned}
$$

Now use the fact that $(1-R)<w'(h)<1-RW(h)$ to conclude that $0<w'(h)-(1-R)<R(1-W(h))$ and $h(1-R)W'(h)<1-W(h)$ to conclude that

$$
\begin{aligned}
&1-\left(2-\frac{1}{R}\right)W(h)-\frac{1-R}{R}W(h)^2-\frac{1-R}{R}hW(h)W^{'}(h) \\
&\geq 1-\left(2-\frac{1}{R}\right)W(h)-\frac{1-R}{R}W(h)^2-\frac{1}{R}W(h)(1-W(h)) \\
&= 1-2W(h)+W(h)^2.
\end{aligned}
$$

Then

$$
\left|\sigma\Pi^*\frac{\mathcal{G}_x}{\mathcal{G}}\right| \leq \frac{|1-R|}{R}(1-W(h))\left(|\lambda|+\frac{|\eta\rho R-\lambda|W(h)}{1-W(h)}\right) \leq \frac{|1-R|}{R}(|\lambda|+|\eta\rho R-\lambda|) =: K_\pi \tag{5.40}
$$

Now we want to show that the local martingale

$$
N_t^5 = \int_0^t e^{-\beta s}\eta Y_s\mathcal{G}_y(X_s^*,Y_s,\Theta_s^*)dB_s^2
$$

is a martingale. This will follow if, for example,

$$
\mathbb{E}\int_0^t e^{-2\beta s}\left(Y_s\mathcal{G}_y(X_s^*,Y_s,\Theta_s^*)\right)^2 ds < \infty \tag{5.41}
$$

for each $t>0$. From the form of the value function (5.32), we have

$$
e^{-\beta s}yG_y(x,y,\theta) = e^{-\beta s}\frac{x^{1-R}}{1-R}zg^{'}(z) = e^{-\beta s}\mathcal{G}(x,y,\theta,t)\frac{zg^{'}(z)}{g(z)} \leq (1-R)e^{-\beta s}\mathcal{G}(x,y,\theta) \tag{5.42}
$$

where we use that $\frac{zg'(z)}{g(z)} = \frac{w(h)}{h} = (1-R)W(h)$ and $0 \leq W(h) \leq 1$.

Define a process $(D_t)_{t\geq 0}$ by $D_t = \ln G\left(X_t^*, Y_t, \Theta_t^*, t\right) = \ln \mathcal{G}(X_t^*, Y_t, \Theta_t^*) - \beta t$. Then D solves

$$\begin{aligned} D_t - D_0 &= -\int_0^t \frac{1}{1-R}\frac{1}{\mathcal{G}}\mathcal{G}_x^{\frac{R-1}{R}} ds + \int_0^t \frac{1}{\mathcal{G}}\sigma\Pi_s^*\mathcal{G}_x dB_s^1 + \int_0^t \frac{1}{\mathcal{G}}\eta Y_s\mathcal{G}_y dB_s^2 \\ &\quad -\frac{1}{2}\int_0^t \frac{1}{\mathcal{G}^2}\left[\sigma^2\Pi_s^{*2}\mathcal{G}_x^2 + \eta^2 Y_s^2\mathcal{G}_y^2 + 2\sigma\eta\rho Y_s\Pi_s^*\mathcal{G}_x\mathcal{G}_y\right] ds. \end{aligned}$$

It follows that the candidate value function along the optimal trajectory has the representation

$$G\left(X_t^*, Y_t, \Theta_t^*, t\right) = G\left(X_0^*, y_0, \Theta_0^*, 0\right) \exp\left\{-\int_0^t \frac{1}{1-R}\frac{1}{\mathcal{G}}\mathcal{G}_x^{\frac{R-1}{R}} ds\right\} H_t \tag{5.43}$$

where $H = (H_t)_{t\geq 0}$ is the exponential martingale

$$H_t = \mathcal{E}\left(\frac{\sigma\Pi^*\mathcal{G}_x}{\mathcal{G}} \circ B^1 + \frac{\eta Y_s\mathcal{G}_y}{\mathcal{G}} \circ B^2\right)_t,$$

where $\mathcal{E}(A\circ B)_t = e^{\int_0^t A_s dB_s - \frac{1}{2}\int_0^t A_s^2 ds}$. Note that (5.42) implies $0 \leq \frac{1}{\mathcal{G}}\eta y\mathcal{G}_y \leq \eta|1-R|$ and (5.40) implies that $|\frac{1}{\mathcal{G}}\sigma\Pi^*\mathcal{G}_x| \leq K_\pi$, so that H is indeed a martingale, and not merely a local martingale.

From (5.42) and (5.43), we have

$$\begin{aligned} (yG_y)^2 &= G\left(X_0, y_0, \Theta_0, 0\right)^2 \left(\frac{zg'(z)}{g(z)}\right)^2 \exp\left\{-2\int_0^t \frac{1}{1-R}\frac{1}{\mathcal{G}}\mathcal{G}_x^{\frac{R-1}{R}} ds\right\} H_t^2 \\ &\leq G\left(X_0, y_0, \Theta_0, 0\right)^2 (1-R)^2 H_t^2. \end{aligned}$$

But from (5.40) and (5.42)

$$\begin{aligned} H_t^2 &= \mathcal{E}\left(\frac{2}{\mathcal{G}}\sigma\Pi^*\mathcal{G}_x \circ B^1 + \frac{2}{\mathcal{G}}\eta Y_s\mathcal{G}_y \circ B^2\right)_t \exp\left\{\int_0^t \frac{1}{\mathcal{G}^2}\left[\sigma^2(\Pi_s^*)^2\mathcal{G}_x^2 + \eta^2 Y_s^2\mathcal{G}_y^2 + 2\sigma\eta\rho Y_s\Pi_s^*\mathcal{G}_x\mathcal{G}_y\right] ds\right\} \\ &\leq \mathcal{E}\left(\frac{2}{\mathcal{G}}\sigma\Pi^*\mathcal{G}_x \circ B^1 + \frac{2}{\mathcal{G}}\eta Y_s\mathcal{G}_y \circ B^2\right)_t \exp\left\{[\eta^2(1-R)^2 + K_\pi^2 + 2\eta|\rho(1-R)|K_\pi]t\right\} \end{aligned}$$

Hence $\mathbb{E}[H_t^2] \leq \exp\left\{[\eta^2(1-R)^2 + K_\pi^2 + 2\eta|\rho(1-R)|K_\pi]t\right\}$ and it follows that (5.41) holds for every t, and hence that the local martingale $N_t^5 = \int_0^t \eta y V_y dB_s$ is a martingale under the optimal strategy.

(ii) The same reasoning applies to show that the local martingale

$$N_t^4 = \int_0^t e^{-\beta s} \sigma \Pi_s^* \mathcal{G}_x(X_s^*, Y_s, \Theta_s^*) dB_s^1$$

is a martingale. We have $|\sigma \Pi^* G_x / G| \leq K_\pi$ and $(\sigma \Pi^* G_x(x, y, \theta, t))^2 \leq G(X_0, y_0, \Theta_0, 0)^2 K_\pi^2 H_t^2$ which follows from (5.43).

(iii) Consider $\exp\left\{\int_0^t -\frac{1}{1-R}\frac{1}{\mathcal{G}}\mathcal{G}_x^{\frac{R-1}{R}} ds\right\}$. To date we have merely argued that this function is decreasing in t. Now we want to argue that it decreases to zero exponentially quickly. By (5.32), we have

$$\begin{aligned}
\frac{1}{1-R}\frac{1}{\mathcal{G}}\mathcal{G}_x^{\frac{R-1}{R}} &= \frac{1}{g(z)}\left[g(z) - \frac{1}{1-R} z g'(z)\right]^{\frac{R-1}{R}} = \frac{1}{h}\left[h - \frac{1}{1-R} w(h)\right]^{\frac{R-1}{R}} \\
&= h^{-1/R}(1 - W(h))^{1-1/R} = N(q)^{-1/R}(1-q)^{1-1/R} = n(q).
\end{aligned}$$

Then since n is bounded below on $(0, q^*)$ (by 1 if $R > 1$ and by $n(q^*)$ if $R < 1$) we have a lower bound on all the expressions in the above equation.

Hence from (5.43) we have

$$0 \leq (1-R) G(X_t^*, Y_t, \Theta_t^*, t) \leq (1-R) G(x_0, y_0, \theta_0, 0) e^{-\min\{n(q^*), 1\} t} H_t \to 0$$

and then $G \to 0$ in L^1 as required. □

Proof of Lemma 5.6.3. This follows exactly as in the proof of Lemma 5.5.6, noting that the bound (5.40) applies in this case also. □

5.8 Extension to $R > 1$

It remains to extend the proofs of the verification lemmas to the case $R > 1$. In particular we need to show that the candidate value function is an upper bound on the value function. The main idea is taken from Davis and Norman[13].

Suppose $G(x, y, \theta, t) = e^{-\beta t}\mathcal{G}(x, y, \theta)$ is the candidate value function. Consider for $\varepsilon > 0$,

$$\widetilde{V}_\varepsilon(x, y, \theta, t) = \widetilde{V}(x, y, \theta, t) = G(x + \varepsilon, y, \theta, t) \tag{5.44}$$

and suppose $\widetilde{M_t} = \widetilde{M_t}(C, \Theta)$ is given by

$$\widetilde{M_t} = \int_0^t e^{-\beta s} \frac{C_s^{1-R}}{1-R} ds + \widetilde{V}(X_t, Y_t, \Theta_t, t).$$

Then,

$$
\begin{aligned}
\widetilde{M}_t - \widetilde{M}_0 &= \int_0^t \Big[e^{-\beta s} U(C_s) - C_s \widetilde{V}_x + \alpha Y_s \widetilde{V}_y + \widetilde{V}_s + \lambda \sigma \Pi_s \widetilde{V}_x + r X_s \widetilde{V}_x \\
&\qquad + \frac{1}{2}\sigma^2 \Pi_s^2 \widetilde{V}_{xx} + \frac{1}{2}\eta^2 Y_s^2 \widetilde{V}_{yy} + \sigma\eta\rho Y_s \Pi_s \widetilde{V}_{xy} \Big] ds \\
&\quad + \int_0^t (\widetilde{V}_\theta - Y_s \widetilde{V}_x) d\Theta_s \\
&\quad + \sum_{0<s\le t} \Big[\widetilde{V}(X_s, Y_s, \Theta_s, s) - \widetilde{V}(X_{s-}, Y_{s-}, \Theta_{s-}, s-) - \widetilde{V}_x (\triangle X)_s - \widetilde{V}_\theta (\triangle \Theta)_s \Big] \\
&\quad + \int_0^t \sigma \Pi_s \widetilde{V}_x dB_s^1 \\
&\quad + \int_0^t \eta Y_s \widetilde{V}_y dB_s^2 \\
&= N_t^1 + N_t^2 + N_t^3 + N_t^4 + N_t^5.
\end{aligned}
$$

Lemma 5.2.1 (in the case $b_3 \le 0$ and otherwise Lemma 5.5.4 or Lemma 5.6.2) implies $\widetilde{N}_t^1 \le 0$ and $\widetilde{N}_t^2 \le 0$. The concavity of $\widetilde{V}(x + y\chi, y, \theta - \chi)$ in χ (either directly if $b_3 \le 0$, or using Lemma 5.5.3 and Lemma 5.6.1) implies $(\Delta \widetilde{N}^3) \le 0$.

Now define stopping times τ_n^1 and τ_n^2 by

$$
\tau_n^1 = \inf \left\{ t \ge 0 : \int_0^t \sigma^2 \Pi_s^2 \widetilde{V}_x^2 ds \ge n \right\} \qquad \tau_n^2 = \inf \left\{ t \ge 0 : \int_0^t \eta^2 Y_s^2 \widetilde{V}_y^2 ds \ge n \right\}.
$$

Let $\tau_n = \min\left\{\tau_n^1, \tau_n^2\right\}$ and τ_n is a stopping time. It follows from (5.40) and (5.42) that $\Pi_t \widetilde{V}_x$ and $y\tilde{V}_y$ are bounded and hence $\tau_n^1 \to \infty$, $\tau_n^2 \to \infty$ and hence $\tau_n \to \infty$. Then the local martingale $(\widetilde{N}_{t\wedge\tau_n}^4 + \widetilde{N}_{t\wedge\tau_n}^5)_{t\ge 0}$ is a martingale and taking expectations we have $\mathbb{E}\left(\widetilde{M}_{t\wedge\tau_n}\right) \le \widetilde{M}_0$, and hence

$$
\mathbb{E}\left(\int_0^{t\wedge\tau_n} e^{-\beta s} \frac{C_s^{1-R}}{1-R} ds + \widetilde{V}\left(X_{t\wedge\tau_n}, Y_{t\wedge\tau_n}, \theta_{t\wedge\tau_n}, t\wedge\tau_n\right) \right) \le \widetilde{V}\left(x_0, y_0, \theta_0, 0\right).
$$

In the case $b_3 \le 0$, (5.17) and (5.44) imply

$$
\begin{aligned}
\widetilde{V}\left(X_{t\wedge\tau_n}, Y_{t\wedge\tau_n}, \theta_{t\wedge\tau_n}, t\wedge\tau_n\right) &= e^{-\beta(t\wedge\tau_n)} \frac{\left(X_{t\wedge\tau_n} + \varepsilon\right)^{1-R}}{1-R} \left(1 + \frac{Y_{t\wedge\tau_n}\theta_{t\wedge\tau_n}}{X_{t\wedge\tau_n} + \varepsilon}\right)^{1-R} \left(\frac{b_1}{b_4 R}\right)^{-R} \\
&\ge e^{-\beta(t\wedge\tau_n)} \frac{\left(X_{t\wedge\tau_n} + \varepsilon\right)^{1-R}}{1-R} \left(\frac{b_1}{b_4 R}\right)^{-R} \ge \frac{\varepsilon^{1-R}}{1-R} \left(\frac{b_1}{b_4 R}\right)^{-R}.
\end{aligned}
$$

Thus $\widetilde{V}$ is bounded, $\lim_{n\to\infty} \mathbb{E}\widetilde{V}\left(X_{t\wedge\tau_n}, Y_{t\wedge\tau_n}, \theta_{t\wedge\tau_n}, t\wedge\tau_n\right) = \mathbb{E}\left[\widetilde{V}\left(X_t, Y_t, \theta_t, t\right)\right]$, and

$$\widetilde{V}\left(x_0, y_0, \theta_0, 0\right) \geq \mathbb{E}\left(\int_0^t e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds\right) + \mathbb{E}\left[\widetilde{V}\left(X_t, Y_t, \Theta_t, t\right)\right].$$

Similarly,

$$\widetilde{V}(x, y, \theta, t) \geq e^{-\beta t}\frac{\varepsilon^{1-R}}{1-R}\left(\frac{b_1}{b_4 R}\right)^{-R},$$

and hence $\mathbb{E}\left[\widetilde{V}(X_t, Y_t, \Theta_t, t)\right] \to 0$. Then letting $t \to \infty$ and applying the monotone convergence theorem, we have

$$\widetilde{V}_\varepsilon(x_0, y_0, \theta_0, 0) = \widetilde{V}_\varepsilon(x_0, y_0, \theta_0, 0) \geq \mathbb{E}\left(\int_0^\infty e^{-\beta s}\frac{C_s^{1-R}}{1-R}ds\right).$$

Finally let $\varepsilon \to 0$. Then $V \leq \lim_{\varepsilon\to 0}\widetilde{V} = G$. Hence, we have $V \leq G$.

The two non-degenerate cases are very similar, except that now from (5.32) or (5.38) and (5.44),

$$\widetilde{V}(x, y, \theta) = \frac{(x+\varepsilon)^{1-R}}{1-R}g\left(\frac{y\theta}{x+\varepsilon}\right) \geq \frac{\varepsilon^{1-R}}{1-R}\left(\frac{b_1}{b_4 R}\right)^{-R},$$

where we use that for $R > 1$, g is decreasing with $g(0) = \left(\frac{b_1}{b_4 R}\right)^{-R} > 0$. Hence $\widetilde{V}$ is bounded, and the argument proceeds as before.

5.9 The limiting cases $\rho \in \{-1, 1\}$

In this section we consider the special case when ρ takes values in $\{-1, 1\}$ so that the endowed asset Y is completely correlated with the hedging asset P. Under this assumption, explicit solution to the ordinary differential equation of n in (5.3) is obtained. We shall only consider the case for $\rho = 1$, such that the two risky assets are positively correlated. It follows the same argument for $\rho = -1$ except for sign changes.

Define auxiliary constants d_1, d_2, d_3 as the following limits,

$$d_1 = \frac{b_1(\rho)}{b_4(\rho)} = \beta - r(1-R) - \frac{\lambda_1^2(1-R)}{2R}, \quad d_3 = \lim_{\rho\to 1}\frac{b_3(\rho)}{b_1(\rho)} = \frac{1}{d_1}(\lambda_2 - \lambda_1)\eta,$$

$$d_2 = \lim_{\rho\to 1}\frac{b_2(\rho)}{b_1(\rho)} = \frac{1}{d_1}\frac{\lambda_1^2 - 2R\eta\lambda_1 + \eta^2 R^2}{2R^2} = \frac{1}{d_1}\frac{(\lambda_1 - R\eta)^2}{2R^2}.$$

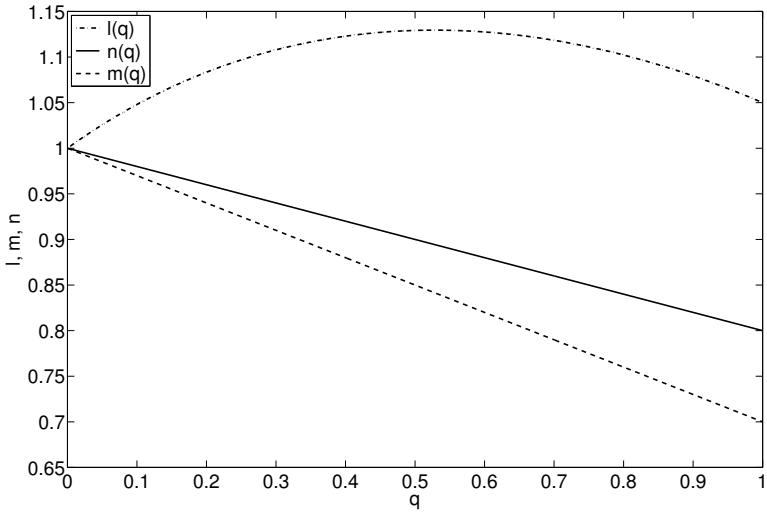

Figure 5.5: Stylised plot of ℓ, m and n in the case $\rho = 1$. Here parameters are chosen to satisfy the condition in the second case of Proposition 5.9.1 so that $q^* = 1$. From the graph, both m and n are straight lines on $[0, 1]$.

For $\rho = -1$, we have instead

$$d_2 = \frac{1}{d_1}\frac{(\lambda_1 + R\eta)^2}{2R^2}, \qquad d_3 = \frac{1}{d_1}(\lambda_2 + \lambda_1)\eta.$$

Note that all the quantities are rescaled by $1/d_1$ which manages to reduce the number of parameters into two, d_2 and d_3. We then have the following proposition for $\rho = 1$ in analog to Proposition 5.1.3.

Proposition 5.9.1. *For $q \in [0, 1]$ define $m(q) = 1 - d_3(1 - R)q$. and*

$$\ell(q) = 1 - d_3(1 - R)q + d_2R + \frac{d_2R^2(1 - q)}{[(1 - R)q^2 + (2R - 1)q - R]} = m(q) + \frac{d_2R(1 - R)q}{(1 - R)q + R}.$$

Let $n = n(q)$ solve

$$\frac{n'(q)}{n(q)} = \frac{1 - R}{R(1 - q)} - \frac{(1 - R)q}{R}\frac{\{n(q) - 1 + d_3(1 - R)q - d_2R(1 - R)\}}{(1 - R)q^2 + (2R - 1)q - R}\frac{1}{\ell(q) - n(q)} \quad (5.45)$$

subject to $n(0) = 1$ and $\frac{n'(0)}{1-R} < \frac{\ell'(0)}{1-R} = d_2 - d_3$. Suppose that if n hits zero, then 0 is absorbing for n. See Figure 5.5.

For $R < 1$, let $q^* = \inf\{q > 0 : n(q) \leq m(q)\}$. For $R > 1$, let $q^* = \inf\{q > 0 : n(q) \geq m(q)\}$. For $j \in \{\ell, m, n\}$ let $q_j = \inf\{q > 0 : j(q) = 0\} \wedge 1$.

Then, $n(q) = [(1-R)q + R]/R$ is a unique solution to (5.45) for $d_2 = 0$. For $d_2 \neq 0$,

$$n(q) = -\frac{1-R}{2}\left\{d_3 - d_2 - \frac{1}{R} + \sqrt{\left(d_3 - d_2 - \frac{1}{R}\right)^2 + \frac{4d_3}{R}}\right\} q + 1 \tag{5.46}$$

is a unique solution to (5.45).

1. *Suppose $d_2 = 0$ or $d_3 \leq 0$. Then $q^* = 0$.*
2. *Suppose $d_3 > 0$ and $1 - d_3(1-R) + d_2R(1-R) > 0$ if $R < 1$. Then $q^* = 1$.*
3. *Suppose $R < 1$ and $1 - d_3(1-R) + d_2R(1-R) \leq 0$. Then $q_m < q_n = q_\ell$. If $R < 1$ and $1 - d_3(1-R) + d_2R(1-R) < 0$ then $q_\ell < 1$.*

Remark 5.9.2. Similar to the case where $|\rho| < 1$, the condition $1-d_3(1-R)+d_2R(1-R) > 0$ is equivalent to $\ell(1) > 0$ and we have $\ell(1) - m(1) = d_2R(1-R) \geq 0$. The key difference here is that m is a straight line here. Thanks to Proposition 5.9.1 and the expression of m, there can not be a crossing $q^* \in (0, 1)$. Instead, there are only two possibilities, $q^* = 0$ or $q^* = 1$, which depends on $n'(0)$ and $m'(0)$, and the 'gap' between ℓ and m does not result in any complications in solving this limiting case.

We have the following theorem in analogy to Theorem 5.1.7. Provided that m and n are straight lines, it follows immediately that $d_{3,crit}(d_2) = 0$ and this observation greatly simplifies the problem.

Theorem 5.9.3. 1. *Suppose $d_2 = 0$ or $d_3 \leq 0$. Then it is always optimal to sell the entire holding of the endowed asset immediately, so that and $\Theta_t = 0$ for $t \geq 0$. The value function for the problem is $V(x, y, \theta, t) = (d_1/R)^{-R} e^{-\beta t}(x + y\theta)^{1-R}/1 - R$; and the certainty equivalent value of the holdings of the asset is $p(x_0, y_0, \theta_0, 0) = y_0\theta_0$.*

2. *Suppose $d_3 > 0$ and $1 - d_3(1-R) + d_2R(1-R) > 0$ if $R < 1$. Then the optimal consumption and sale strategy is first to consume liquid wealth and invest in the risky asset, and then when this liquid wealth is exhausted, to finance further consumption and investment from sales of the illiquid asset. If $\theta > 0$ then $p(x, y, \theta) > y\theta$.*

3. *Suppose* $1 - d_3(1-R) + d_2R(1-R) \leq 0$ *if* $R < 1$. *Then the problem is degenerate, and provided* θ_0 *is positive, the value function* $V = V(x, y, \theta, t)$ *is infinite. There is no unique optimal strategy, and the certainty equivalent value* p *is not defined.*

Remark 5.9.4. The results above are intuitive. Provided that the endowed asset and the risky asset are completely correlated, the agent chooses to either hold the endowed asset until all cash wealth is consumed or sell it immediately. Further, when wealth is exhausted, it is never optimal to invest zero amount in the correlated risky asset since the risks of the endowed asset can be fully hedged by the risky asset. The proof of the theorem above follows the same argument in the proof of Theorem 5.1.7.

Chapter 6

Comparative statics of the multiple-asset problem

In this chapter, we consider the comparative statics of the multi-asset consumption optimisation problem presented in Chapter 5. The key to our results on comparative statics is contained in Lemma 5.3.4 and Corollary 5.3.5. In the section on the problem formulation we showed how the original parameters only affect the solution via four key parameters b_1, b_2, b_3 and b_4. Here b_3 is the effective Sharpe ratio of the endowed asset and measures the excess expected return, net of any expected growth from correlation with the market asset. b_1 is an effective discount parameter, taking account of the investment opportunities in the market. b_4 is a measure of the idiosyncratic risk of the endowed asset, and only affects the solution via a scaling of the value function — more idiosyncratic risk also enters the other parameters b_i. Finally, b_2 is the hardest parameter to interpret, but is a measure of the extent to which the investment motive and the hedging motive cancel with each other.

This chapter is arranged as follows. In Section 6.1, we provide numerical methods to compute the solutions to the differential equations which construct the optimal strategies. Then we analyse the dependence of the critical threshold z^*, the value function g, the optimal consumption C^* and the optimal portfolio Π^* on derived constants b_1, b_2, b_3 and R. Finally we discuss the results from comparative statics in terms of the original parameters, μ, α, r, σ, η, ρ and β.

6.1 Numerical methods

In this section we provide numerical methods to compute n and g, which are introduced in Chapter 5. Computing n and g numerically is essential to construct the value function and the optimal strategies from the analysis in the previous chapter and hence to conduct the comparative statics. Our algorithms and codes are implemented in MATLAB.

Recall the definitions of m, ℓ, υ in Proposition 5.1.3 and the non-linear ordinary differential equation n solves

$$\frac{n^{'}(q)}{n(q)} = \frac{1-R}{R(1-q)} - \frac{(1-R)^2}{b_1 R}\frac{q}{\ell(q)-n(q)} + \frac{(1-R)q}{2b_1 R(1-q)[(1-R)q+R]}\frac{\upsilon(q,n(q))}{\ell(q)-n(q)}, \quad (6.1)$$

subject to $n(0) = 1$ and $\frac{n'(0)}{1-R} < \frac{\ell'(0)}{1-R} = (b_2 - b_3)/b_1$. Solving the initial value problem (6.1) on $[0,1]$ is complicated due to the singularity near $q = 0$. Instead, we consider (6.1) on $[\epsilon, 1]$ for small $\epsilon > 0$. Lemma 5.3.3 provides an explicit expression for $n'(0)$ which by Taylor expansion gives $n(\epsilon) = n(0) + n'(0)\epsilon + O(\epsilon^2)$. Here $O(\epsilon^2)$ denotes terms proportional to ϵ^2 and to higher powers of ϵ.

Then the initial value problem (6.1) is solved by the Runge-Kutta 4th order method on $[\epsilon, 1]$ with initial condition $n(\epsilon) = n(0) + n'(0)\epsilon + O(\epsilon^2)$, whence the solution is represented by the Taylor expansion on $[0, \epsilon]$. In our algorithm, we choose $\epsilon = 1 \times 10^{-5}$. By solving the first crossing problem of m and n, we are able to find numerical values of q^* and $n(q^*)$ given a set of model parameters.

Now we consider the numerical algorithm to solve g with quantities q^* and $n(q^*)$. Recall from Theorem 5.1.9 that g solves the following non-linear second order differential equation on $[0, z^*]$,

$$g'' = \frac{-\psi_1(z,g,g') - \mathrm{sgn}(1-R)\sqrt{\psi_1(z,g,g')^2 - 4\psi_2(z,g,g')}}{2z^2}, \quad (6.2)$$

subject to the conditions at $z = 0$ and $z = z^* = q^*/(1-q^*)$,

$$g(0) = \left(\frac{b_1}{b_4 R}\right)^{-R}, \quad (6.3)$$

$$g(z^*) = \left(\frac{b_1}{b_4 R}\right)^{-R} m(q^*)^{-R}(1+z^*)^{1-R}, \quad (6.4)$$

$$g'(z^*) = (1-R)\left(\frac{b_1}{b_4 R}\right)^{-R} m(q^*)^{-R}(1+z^*)^{-R}. \quad (6.5)$$

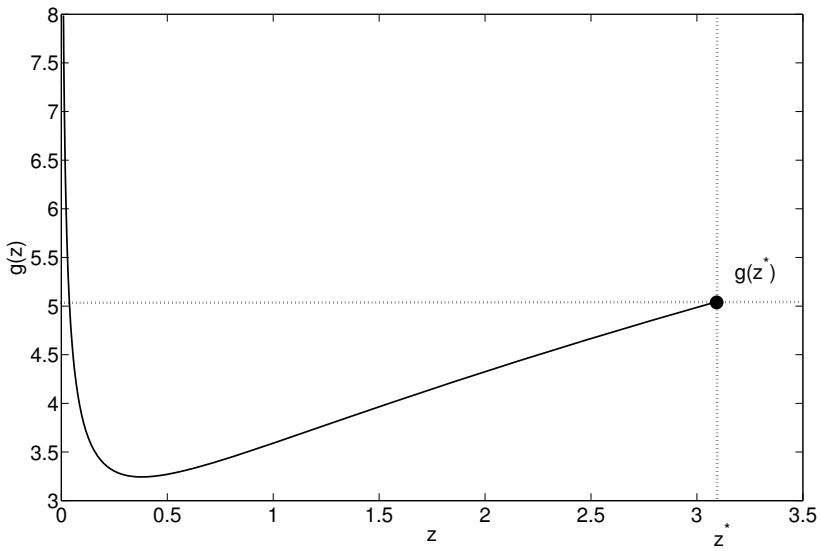

Figure 6.1: Stylised plot of g for $R < 1$ by MATLAB solver ode23tb. From the figure, the solution is diverging as it approaches $z = 0$ and it fails to meet the condition that $g(0) = (b_1/(b_4R))^{-R} < g(z^*)$.

Here ψ_1 and ψ_2 are functions defined by

$$\begin{aligned}
\psi_1(z, g, g') &= b_4R\left[g - \frac{1}{1-R}zg'(z)\right]^{\frac{R-1}{R}} - [b_1 + b_2R(1-R)]g + (b_3 + 2R)zg'(z), \\
\psi_2(z, g, g') &= b_4R[2Rzg'(z) - R(1-R)g]\left[g - \frac{1}{1-R}zg'(z)\right]^{\frac{R-1}{R}} - [2Rb_1 + R(1-R)b_3]gzg'(z) \\
&\quad + \left[2Rb_3 - R^2b_2 + R^2\right](zg'(z))^2 + b_1R(1-R)g^2.
\end{aligned}$$

Note that the ordinary differential equation (6.2) subject to (6.3), (6.4) and (6.5) can be solved either as an (backward) initial value problem or a boundary value problem.

However, MATLAB (solvers for stiff ordinary differential equations, including ode45, ode23, ode23s, ode23tb) fails to provide a convergent solution (with $g(0) = (b_1/(b_4R))^{-R}$) by computing (6.2) backwards with initial conditions (6.4) and (6.5). This is because of the stiffness [47] of (6.2) and the singularity of (6.2) at $z = 0$. See Figure 6.1.

We then consider the problem in (6.2) as a boundary value problem on $[0, z^*]$ with boundary conditions (6.3) and (6.4). In order to avoid the singularity at $z = 0$, we consider the problem on $[\epsilon, z^*]$ instead of $[0, z^*]$ for small $\epsilon > 0$. Since numerical

evidence shows that $g(z^*)$ is increasing in $g'(\epsilon)$, we implement the following algorithm which adopts the idea from the shooting method [51].

1. fix $\epsilon = 1 \times 10^{-7}$ and let $g(\epsilon) = (b_1/(b_4 R))^{-R}$
2. choose a lower bound and an upper bound for $g'(\epsilon)$, denote as κ_l and κ_u, such that given $g'(\epsilon) = \kappa_l$, the solution of (6.2) gives $g(z^*) < \left(\frac{b_1}{b_4 R}\right)^{-R} m(q^*)^{-R}(1+z^*)^{1-R}$; for $g'(\epsilon) = \kappa_u$, the solution of (6.2) gives $g(z^*) > \left(\frac{b_1}{b_4 R}\right)^{-R} m(q^*)^{-R}(1+z^*)^{1-R}$
3. let $[\kappa_l, \kappa_u]$ be our search space and conduct a binary search to find the target value of $g'(\epsilon)$ such that the solution of (6.2) as an initial value problem with initial conditions $(g(\epsilon), g'(\epsilon))$ satisfies

$$\left| g(z^*) - \left(\frac{b_1}{b_4 R}\right)^{-R} m(q^*)^{-R}(1+z^*)^{1-R} \right| \leq 1 \times 10^{-3} \tag{6.6}$$

4. for accepted $g'(\epsilon)$, further examine the following inequality

$$\left| g'(z^*) - (1-R)\left(\frac{b_1}{b_4 R}\right)^{-R} m(q^*)^{-R}(1+z^*)^{-R} \right| \leq 1 \times 10^{-3}. \tag{6.7}$$

   If the inequality holds, then accept the value of $g'(\epsilon)$. Otherwise, decrease the threshold in condition (6.6), e.g.

$$\left| g(z^*) - \left(\frac{b_1}{b_4 R}\right)^{-R} m(q^*)^{-R}(1+z^*)^{1-R} \right| \leq 1 \times 10^{-5},$$

   and repeat from step (2) to start another binary search for $g'(\epsilon)$ with more accuracy

Numerical evidence shows that once the condition (6.6) is satisfied, the other condition (6.7) is satisfied as well. The algorithm is implemented by the stiff solver ode23tb with tolerance 1×10^{-7}.

6.2 dependence of critical threshold z^* on b_1, b_2, b_3 and R

In this section we focus on the critical ratio $z^* = q^*/(1-q^*)$ of endowed wealth to liquid wealth at which sales occur. From Figure 6.2, 6.3 and 6.4, we observe that z^* is decreasing in b_1 and b_2, and increasing in b_3, which is consistent with the results in Lemma 5.3.4. The monotonicity in b_3 is straightforward to interpret, and has a clear

intuition. The greater the effective Sharpe ratio the more valuable the holdings in the endowed asset, and the longer the agent should hold units of it in her portfolio. The dependence on b_1 is also as expected. The greater the effective discount parameter, the greater the incentive to bring forward consumption which needs to be financed by sales of the risky asset — thus the endowed asset is sold sooner. The dependence on b_2 is less easy to interpret, but the lemma and its corollary show that there is monotonicity in this parameter also. Finally, as the risk aversion of the investor increases, she is less tolerant to the risk of the endowed asset and hence more inclined to sell Y earlier.

6.3 dependence of g and p on b_1, b_3 and R

The value function as expressed via g in the first non-degenerate case, where there is a finite exercise ratio, is plotted in Figures 6.5 and 6.6 under different values of b_1 and b_3. These figures show that g is concave and increasing in z for $R < 1$. Further, g is increasing in b_3 and decreasing in b_1. As b_3 increases, the non-traded asset becomes more valuable. Hence, the investor can choose optimal sale and consumption strategies which lead to a larger value function. Similar arguments explain the monotonicity of g in b_1. Observe also that for different values of b_3, we have g starts at the same point. This corresponds to the value function when $\theta_0 = 0$ whereby consumption is financed by initial wealth and investment in the risky asset, and the problem is the Merton model with one risky asset. In this case, Theorem 1.2.5 implies $g(0) = (b_1/(b_4 R))^{-R}$.

The certainty equivalent value $p = p(x, y, \theta)$ of the holdings in the endowed asset is plotted in Figure 6.7, 6.8 and 6.9. From the graphs, the indifference price is increasing in the number of holdings θ, decreasing in b_1 and increasing in b_3 which are consistent with Corollary 5.3.5. The monotonicity in b_3 is as expected. For a larger discount factor b_1, it is more incentive to sell the endowed asset sooner so as to bring forward consumption. Then there is less opportunity for the benefits from the expected growth of the endowed asset to be enjoyed, thus reducing its value. The non-monotonicity of the indifference price in risk aversion is also observed here, which follows from a similar argument in Section 4.4.

6.4 Optimal consumption and portfolio process C^* and Π^*

Figure 6.10 and 6.11 plot the optimal consumption process as a function of the endowed units θ and show that the optimal consumption increases in θ: as the size of the holdings of the non-traded asset Y increases, the agent feels richer and hence consumes at a

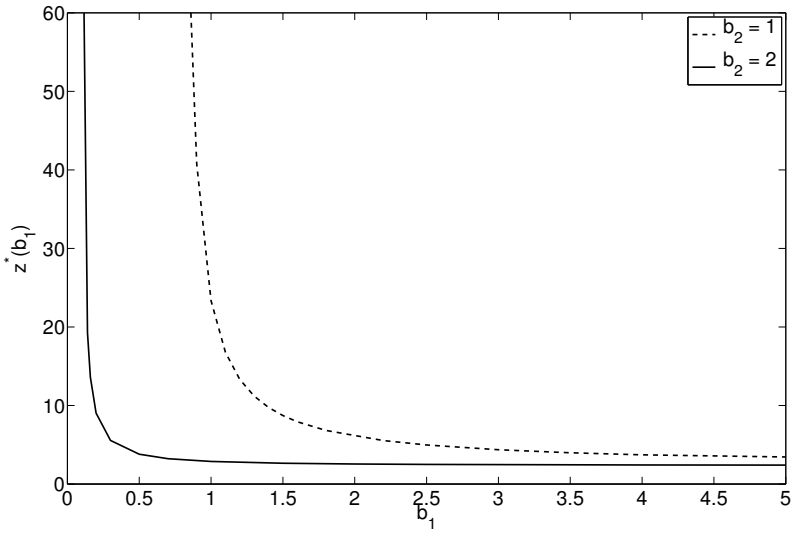

Figure 6.2: Plot of $z^*(b_1)$ with different b_2. Here parameters are $b_3 = 0.7$, $R = 0.5$ and b_2 takes values in 1 and 2. From the graph, z^* is decreasing in b_1 and increasing in b_2.

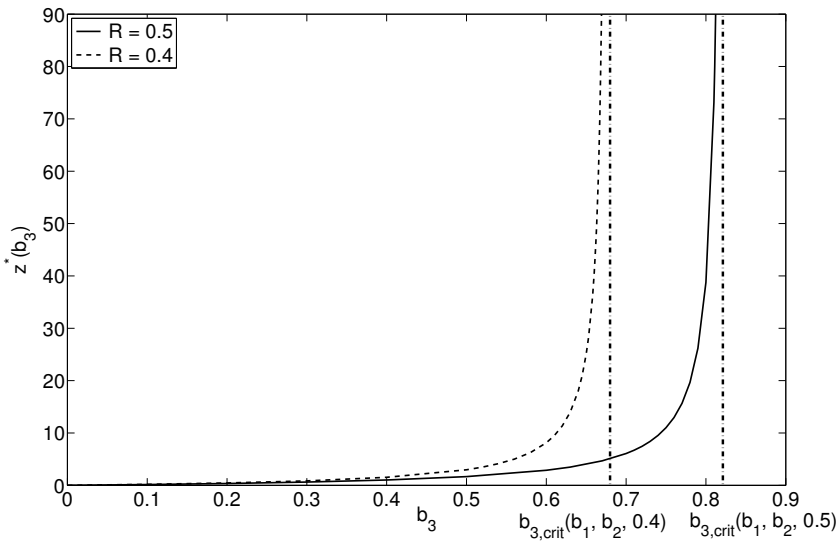

Figure 6.3: Plot of $z^*(b_3)$ with different R. Here parameters are $b_1 = 2$, $b_2 = 2$ and R takes values in 0.4 and 0.5. From the graph, z^* is increasing in b_3 and decreasing in R. As $b_3 \to b_{3,crit}$, the critical threshold $z^* \to \infty$.

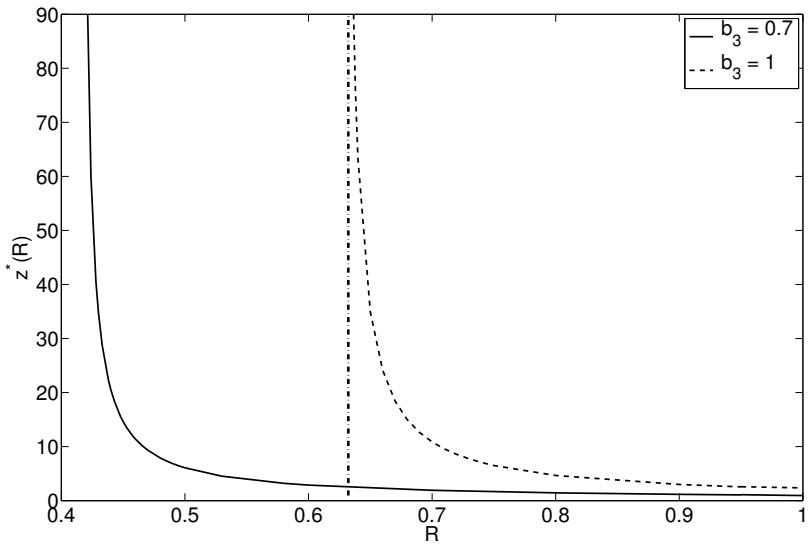

Figure 6.4: Plot of $z^*(R)$ with different b_3. Here parameters are $b_1 = 2$, $b_2 = 2$, and b_3 takes values in 0.7 and 1. From the graph, z^* is decreasing in R.

faster rate. For $\theta = 0$, Theorem 1.2.5 implies the optimal consumption is $C(x, y, 0) = xg(0)^{-\frac{1}{R}} = \frac{b_1}{b_4 R}x$, which is strictly positive. The graphs also show that the optimal consumption C^* is increasing in b_1 and not monotone in b_3. A larger discount factor makes the agent more incentive to consume quicker rather than slower. As a result, the consumption rate is increasing in b_1. The same argument in Section 4.3 explains the non-monotonicity of the optimal consumption in b_3 here.

Figure 6.12 plots the optimal portfolio Π^* as a function of wealth x. Since the optimal portfolio process Π^* is characterised by the original parameters rather than the derived constants b_i, it is hard to analyse the dependence on model parameters. From the graph, we observe that the optimal portfolio process Π^* is increasing in wealth x. The more wealth the agent owns, the more investment she can make into the risky asset. Here parameters are chosen such that we are in the second non-degenerate scenario ($b_3 \geq b_{3,crit}(b_1, b_2, R)$ and $b_3 < \frac{b_1}{1-R} + b_2 R$ if $R < 1$). The picture shows that $\lim_{x\to 0} \Pi^*(x, 1, 1) > 0$, which implies even when liquid wealth is zero, it is not optimal to invest zero amount in the risky asset. Adverse movements in the price of the financial asset have a negative impact on liquid wealth, and must be financed through sales of the endowed asset. Conversely, beneficial movements in the price of the financial asset will generate positive liquid wealth for the agent.

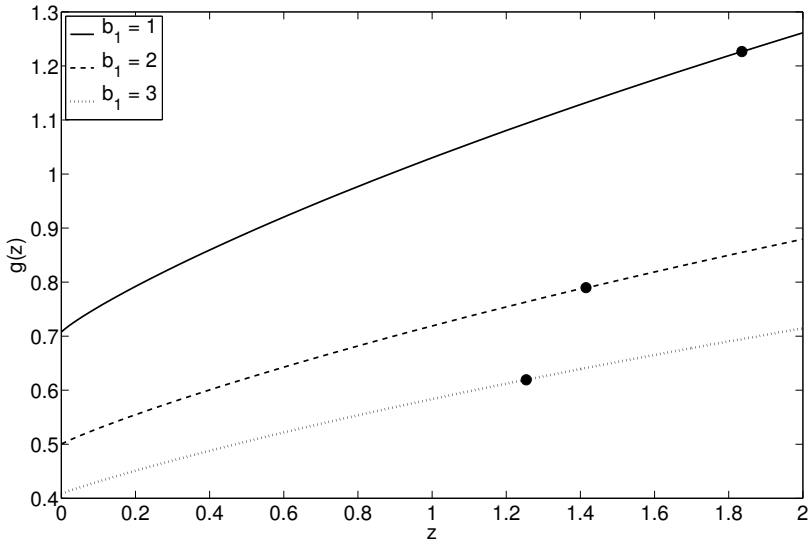

Figure 6.5: Plot of g with different b_1. Parameters are $b_2 = 1.5$, $b_3 = 0.5$, $b_4 = 1$, $R = 0.5$. b_1 takes values in 1, 2, 3. Dots here represent z^*. From the graph, g is concave and increasing in z, and decreasing in b_1.

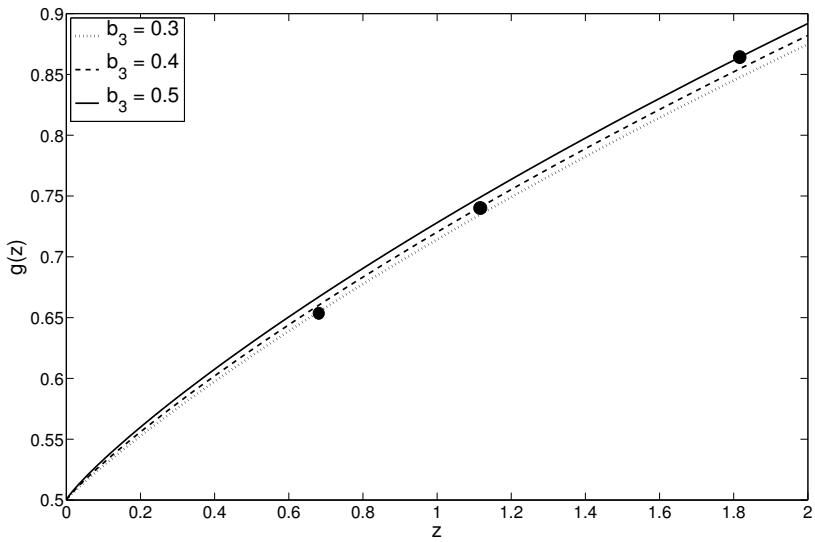

Figure 6.6: Plot of $g(z)$ with different b_3. Parameters are $b_1 = 1$, $b_2 = 1.5$, $b_4 = 0.5$ and $R = 0.5$. b_3 takes values in 0.3, 0.4 and 0.5. Dots here represent z^*. From the graph, g is increasing in b_3.

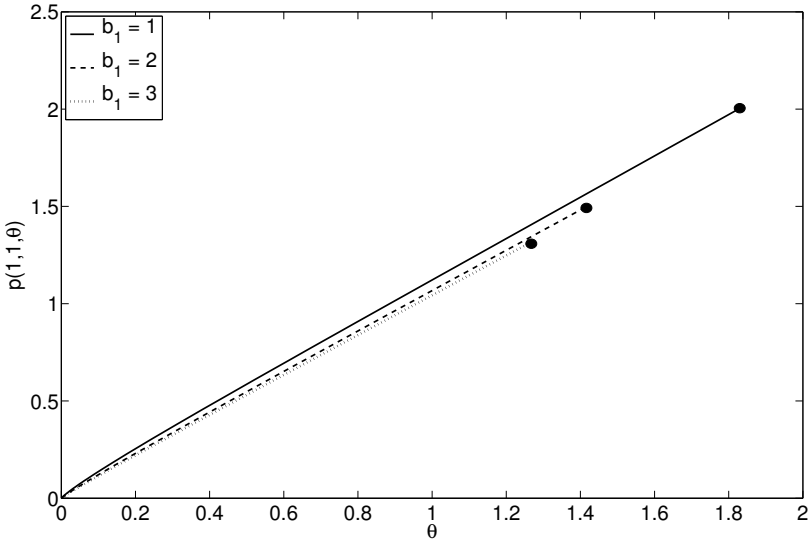

Figure 6.7: Plot of the indifference price $p(1,1,\theta)$ with different b_1. Parameters are $b_2 = 1.5$, $b_3 = 0.5$, $b_4 = 1$, $R = 0.5$. b_1 takes values in 1, 2, 3. Dots here represent $z^* = \theta^*$. From the graph, p is increasing in θ and decreasing in b_1.

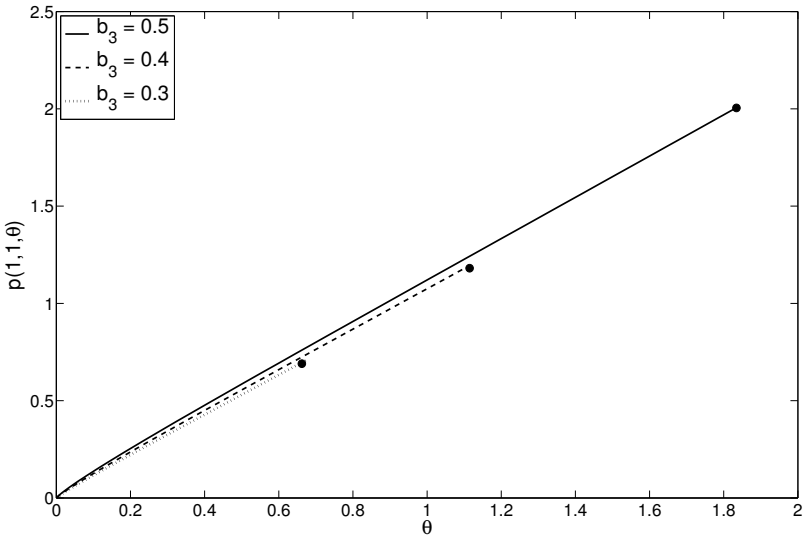

Figure 6.8: Plot of the indifference price $p(1,1,\theta)$ with different b_3. Parameters are $b_1 = 1$, $b_2 = 1.5$, $R = 0.5$. b_3 takes values in 0.3, 0.4 and 0.5. Dots here represent $z^* = \theta^*$. From the graph, p is increasing in b_3.

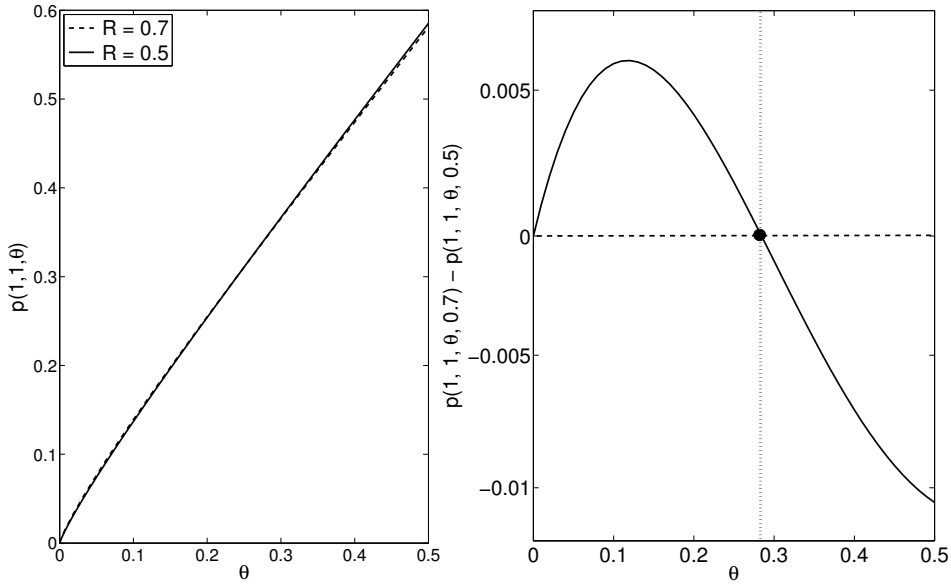

Figure 6.9: Plot of the indifference price $p(1,1,\theta)$ for $\theta \in [0, 0.5]$ (left picture) and the difference between the indifference price with $R = 0.7$ and $R = 0.5$ (right picture). Parameters are $b_1 = 1$, $b_2 = 1.5$, $b_3 = 0.5$. From the graph on the right, the difference $p(1,1,\theta,0.7) - p(1,1,\theta,0.5)$ is initially positive but then becomes negative near $\theta = 0.28$. Hence, the indifference price p is not monotone in R.

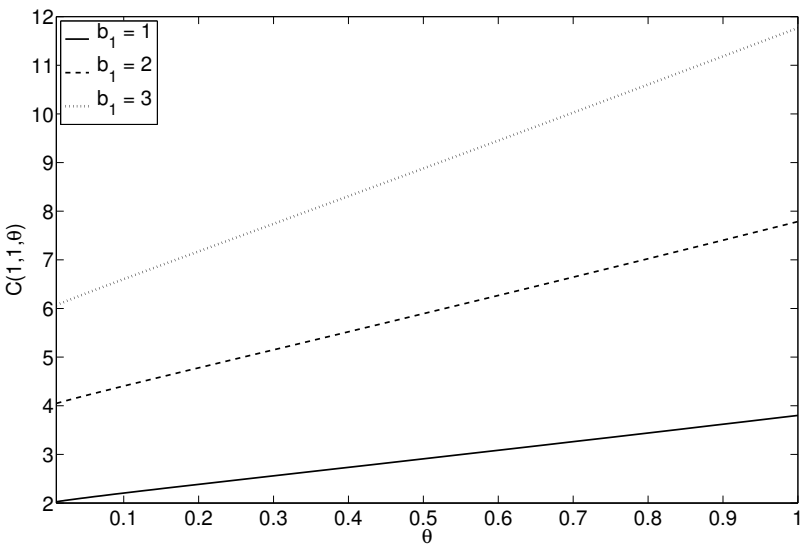

Figure 6.10: Plot of the optimal consumption $C(1,1,\theta)$ with different b_1. Parameters are $b_2 = 1.5$, $b_3 = 0.5$, $b_4 = 1$, $R = 0.5$. b_1 takes values in 1, 2, 3. From the graph, the optimal consumption C is increasing in θ and b_1.

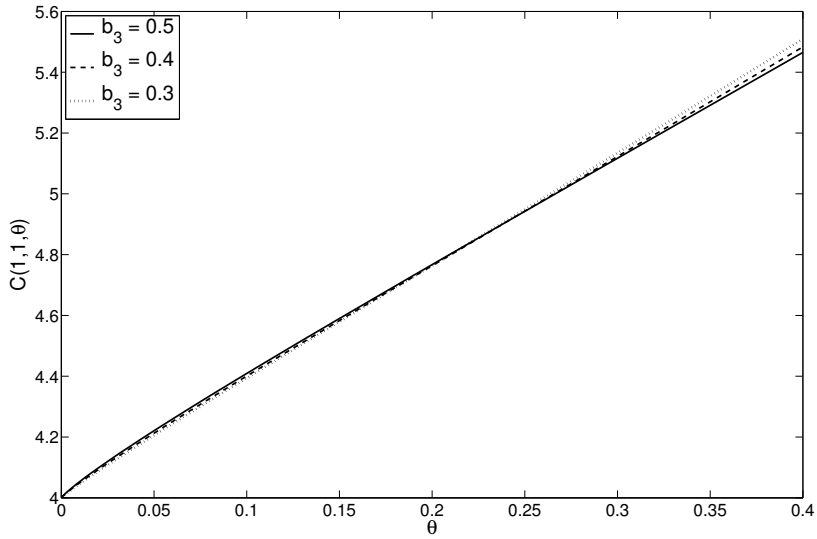

Figure 6.11: Plot of the optimal consumption $C(1,1,\theta)$ with different b_3. Parameters are $b_1 = 1$, $b_2 = 1.5$, $R = 0.5$. b_3 takes values in 0.3, 0.4 and 0.5. From the graph, the optimal consumption C is not monotone in R.

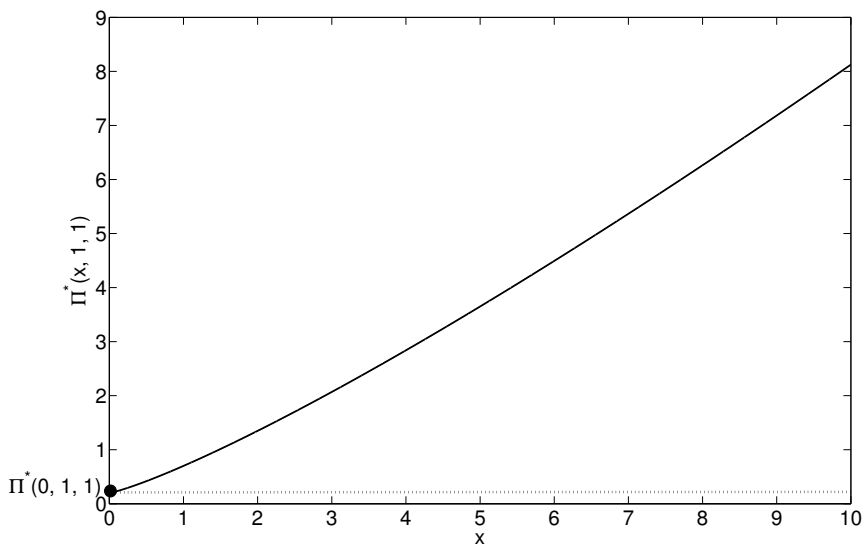

Figure 6.12: Plot of the optimal portfolio $\Pi(x,1,1)$ in the second non-degenerate case. Parameters are $b_1 = 1$, $b_2 = 2$, $b_3 = 0.8$, $R = 0.5$. From the graph, the optimal portfolio process Π^* is increasing in x and $\lim_{x\to 0} \Pi^*(x,1,1) = 0.21 > 0$.

6.5 Discussion in terms of original parameters

In the preceding sections we have discussed the comparative statics in terms of the derived parameters. In order to understand the comparative statics with respect to the original parameters r, β, μ, σ, α, η and ρ we need to consider how the auxiliary parameters depend on these original parameters.

The parameters β (discount rate) and α (mean return on the endowed asset) only affect one of the parameters $(b_i)_{i=1,2,3}$, and hence the comparative statics for these parameters are straightforward. However, the parameters r, μ, σ, η and ρ each enter into the definitions of b_1, b_2 and b_3. Hence the comparative statics with respect to these parameters is more complicated, and in general there is no monotonicity of the critical ratio or the certainty equivalent value with respect to any of these parameters. For example, an increase in the volatility η of the endowed asset decreases b_1 and may increase or decrease the value of b_2 or b_3 depending on the values of other parameters. Thus, the effects of a change in the volatility of the endowed asset on the critical ratio or on the certainty equivalent price are generally mixed.

We restrict our comments on the consumption and investment rate to the following observation about the critical ratio and the Merton line.

Consider an investor who is free to buy and sell units of Y with zero transaction cost. This is the Merton problem with two risky assets and we know from Theorem 1.2.5 that it is optimal for the agent to invest a constant fraction $\frac{(\zeta-\lambda\rho)}{R\eta(1-\rho^2)} = \frac{b_3}{2R}$ of their total wealth in the risky asset.

In contrast, the constrained investor chooses to keep the fraction of his total wealth invested in the endowed asset below q^*, i.e. to choose Θ_t to ensure that $\frac{Y_t\Theta_t}{X_t} \leq z^* = \frac{q^*}{1-q^*} = z^*$, or equivalently $\frac{Y_t\Theta_t}{X_t+Y_t\Theta_t} \leq q^*$. But, it follows from Lemma 5.3.3 that $q^* > \frac{b_3}{2R}$. Hence the 'no-sale' region for the constrained investor contains in its interior the 'Merton line' of portfolio positions for the unconstrained investor.

Chapter 7

Conclusions and future work

7.1 Results achieved

We have studied the optimal consumption and investment problems with infinite transaction costs with both a single risky asset and multiple risky assets. These problems are equivalent to (infinitely divisible) non-traded asset sale problems in which one of the risky assets cannot be purchased.

We introduce a new solution technique that manages to reduce the problem into a boundary crossing problem for the solution of a first-order ordinary differential equation. This first crossing problem has four types of solutions and each type of solution corresponds to a type of optimisation problem. Further, the optimal strategies, the value function and the indifference price of the optimisation problem can all be constructed via the first-order ODE and the solution to the boundary crossing problem.

In the single-asset case, we give a more complete analysis of the problem compared with the results by Davis and Norman [13] thanks to the special structure and the solution technique. We also conduct the comparative statics based on the new solution technique and numerically compute the optimal strategies, indifference price and the value function from the solution of the ODE and the solution to the boundary crossing problem. After analysing the dependance of these quantities on model parameters, we find some surprising results. For instance, the optimal consumption is not monotone in the drift of the endowed asset and the indifference price is not monotone in risk aversion.

In the multiple-asset case where there are limited results in the literature, we manage to completely characterise the different possible behaviours of the agent. Though the multi-dimensional setting brings new challenges, the solution technique allows us to prove the monotonicity of the critical exercise threshold and the indifference price in

model parameters. Finally, we conduct the comparative statics to analyse the dependance on derived constants. The results in the single-asset case are recovered and we also find that different to the single-asset case, the optimal wealth process at zero is not an absorbing state for a certain combination of parameters.

7.2 Future work

It is of great interest to extend our work into the following directions.

1. A comparison between the divisible asset sale problem (this thesis) and the indivisible asset sale problem by Henderson and Hobson [31].

   It would be extremely interesting to compare the conditions (on model parameters) of different types of behaviours, the optimal strategies and certainty equivalent values both analytically and numerically to understand the impact of the indivisibility.

2. Formulating and computing the cost of indivisibility.

   Recall the definition we make in Section 4.5 about the cost of illiquidity. It is natural to extend this definition to the cost of indivisibility by comparing the value functions in the divisible case and the indivisible case. It is also worth efforts conducting the comparative statics of the cost of indivisibility on model parameters.

3. A different set of admissible strategies

   One promising direction of future work is to consider a different set of admissible strategies, which allows wealth to be negative. We would expect similar types of optimal behaviours of the agent. One conjecture is that the 'gap' between ℓ and m in Chapter 5 disappears when borrowing is allowed and we will consider on $[0, \hat{q}]$ instead of $[0, 1]$, for some $\hat{q} > 1$ such that $\ell(\hat{q}) = m(\hat{q})$.

4. Another price process for the endowed asset

   In our problems, the price of the endowed asset is assumed to follow a geometric Brownian motion. It would be interesting to work with a different price dynamics of the endowed asset and compare how the distributions of the endowed asset impact the types of behaviours and the optimal strategies of the agent. For instance, in the literature of non-traded asset, Miao and Wang [42] assume the non-traded asset follows an arithmetic Brownian motion.

Bibliography

[1] R. P. Agarwal, D. O. Regan. Second order initial value problems of Lane-Emden type. *Applied Mathematics Letters*, 20: 1198 - 1205, 2007.

[2] M. Akian, J. L. Menaldi, A. Sulem. Multi-asset portfolio selection problem with transaction costs. *Mathematics and Computers in Simulation*, 38(1-3): 163-172, 1995.

[3] C. Atkinson, S. Mokkhavesa. Multi-asset portfolio optimisation with transaction cost. *Applied Mathematical Finance*, 11(2): 95-123, 2004.

[4] P. B. Bailey, L. F. Shampine, P. E. Waltman. *Nonlinear two point boundary value problems*. Academic Press, New York and London, 1968.

[5] P. Bailey, P. Waltman. Existence and uniqueness of solutions of the first boundary value problems for nonlinear second order differential equations. *Rational Mechanics and Analysis*, 21(4): 310-320, 1966.

[6] P. Bank and D. Baum. Hedging and portfolio optimisation in financial markets with a large trader. *Mathematical Finance*, 14: 1-18, 2004.

[7] A. Cadenillas. Consumption-investment problems with transaction costs: survey and open problems. *Mathematical Methods of Operations Research*, 51: 43-68, 2000.

[8] X. F. Chen, M. Dai. Characterisation of optimal strategy for multi-asset investment and consumption with transaction costs. *SIAM Journal on Financial Mathematics*, 4(1): 857-883, 2013.

[9] C. Chicone. *Ordinary differential equations with applications*. Springer, New York, 1999.

[10] P. Collings, U. G. Haussmann. Optimal portfolio selection with transaction costs. *In: Proceedings of the Conference on Control of Distributed and Stochastic Systems*, Kluwer.

[11] G. M. Constantinides. Capital market equilibrium with transaction costs. *The Journal of Political Economy*, 94(4): 842-862, 1986.

[12] G. M. Constantinides, M. J. P. Magill. Portfolio selection with transaction costs. *Journal of Economic Theory*, 13: 264-271, 1976.

[13] M. H. A. Davis, A. Norman. Portfolio selection with transaction costs. *Mathematics of Operations Research*, 15: 676-713, 1990.

[14] C. Doléans-Dade. On the existence and unicity of solutions of stochastic integral equations. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 36: 93-101, 1976.

[15] D. Duffie, T. S. Sun. Transactions costs and portfolio choice in a discrete-continuous-time setting. *Journal of Economic Dynamics and Control*, 14: 35-51, 1990.

[16] J. D. Evans, V. Henderson, D. Hobson. Optimal timing for an asset sale in an incomplete market. *Mathematical Finance*, 18(4): 545-568, 2008.

[17] U. Elias. Existence of global solutions of some ordinary differential equations. *Journal of Mathematical Analysis and Applications*, 340: 739-745, 2008.

[18] A. G. Farrell, D. O. Regan. Existence results for some initial and boundary-value problems. *Proceedings of the American Mathematical Society*, 110(3): 661-673, 1990.

[19] W. H. Fleming, M. H. Soner *Controlled Markov processes and viscosity solutions, second edition*, Springer, New York, 1993.

[20] N. C. Framstad, B. Øksendal, A. Sulem. Optimal consumption and portfolio in a jump diffusion market with proportional transaction costs. *Journal of Mathematical Economics*, 35: 233-257, 2001.

[21] A. Granas, R. B. Guenther and J.W. Lee. Nonlinear boundary value problems for some classes of ordinary differential equations. *Journal of Mathematics*, 10(1): 35 - 58, 1980.

[22] P. Guasoni, J. Muhle-Karbe. Portfolio choice with transaction costs: a user's guide. *Available at SSRN 2120574*, 2012.

[23] J. M. Harrison. *Brownian Motion and Stochastic Flow Systems*. Wiley, New York, 1985.

[24] P. Hartman, On boundary value problems for systems of ordinary, nonlinear, sec on order differential equations. *Transactions of the American Mathematical Society*, 96(3): 493-509, 1960.

[25] V. Henderson. Valuation of claims on nontraded assets using utility maximisation. *Mathematical Finance*, 12: 351-373, 2002.

[26] V. Henderson, D. Hobson. Real options with constant relative risk aversion. *Journal of Economic Dynamics and Control*, 27: 329-355, 2002.

[27] V. Henderson, D. Hobson. Utility indifference pricing: an overview. *Indifference Pricing: Theory and Applications*, Princeton University Press, 2009.

[28] V. Henderson, D. Hobson. An explicit solution for an optimal stopping/optimal control problem which models an asset sale. *Annals of Applied Probability*, 18(5): 681-1705, 2008.

[29] V. Henderson, D. Hobson. Optimal Liquidation of Derivative Portfolios. *Mathematical Finance*, 21(3): 365-382, 2010.

[30] V. Henderson, D. Hobson. Risk aversion, indivisible timing options, and gambling. *Operations Research*, 61(1): 126-137, 2013.

[31] V. Henderson, D. Hobson. Consumption and investment in a model for an indivisible asset sale. *In Preparation*.

[32] D. Hobson, Y. Zhu. Optimal consumption and sale strategies for a risk averse agent. *Submitted to the SIAM Journal on Financial Mathematics*.

[33] D. Hobson, Y. Zhu. Multi-asset consumption-investment problem with infinite transaction costs. *To be submitted to the Annals of Applied Probability*.

[34] S. D. Hodges, A. Neuberger. Optimal replication of contingent claims under transaction costs. *Review of Futures Markets*, 8: 222-239, 1989.

[35] I. Karatzas, S.E. Shreve. *Brownian motion and stochastic calculus, second edition*, Springer, New York, 1998.

[36] I. Karatzas, S.E. Shreve. *Methods of mathematical finance*, Springer, New York, 1998.

[37] I. T. Kiguradze, B. L. Shekhter. Singular boundary-value problems for ordinary second-order differential equations. *Journal of Soviet Mathematics*, 43(2): 2340 - 2417.

[38] R. Korn. Portfolio optimisation with strictly positive transaction costs and impulse control. *Finance and Stochastics*, 2: 85-114, 1998.

[39] H. Liu. Optimal consumption and investment with fixed and proportional transaction costs. *Available at SSRN 1493722*, 2000.

[40] F. A. Longstaff. Optimal portfolio choice and the valuation of illiquid securities. *The Review of Financial Studies*, 14: 407-431, 2001.

[41] R. C. Merton. Life portfolio selection under uncertainty: the continuous-time case. *The Review of Economics and Statistics*, 51: 247-257, 1969.

[42] J. Miao, N. Wang. Investment, hedging, and consumption smoothing. *Available at SSRN 565601*, 2004.

[43] M. Monoyios. Utility indifference pricing with market incompleteness. *Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing*, Nova Science Publishers, Hauppage, New York, 2008.

[44] M. Monoyios. Performance of utility-based strategies for hedging basis risk. *Quantitative Finance*, 4: 245-255, 2004.

[45] K. Muthuraman, S. Kumar. Multi-dimensional portfolio optimisation with proportional transaction costs. *Mathematical Finance*, 16(2): 301-335, 2006.

[46] A. D. Polyanin, V. F. Zaitsev. *Handbook of exact solutions for ordinary differential equations*. Chapman & Hall/CRC, New York, 2003.

[47] P. Rentrop. A Taylor series method for the numerical solution of two-point boundary value problems. *Numerische Mathematik*, 31(4): 359 - 375, 1978.

[48] D. Revuz, M. Yor. *Continuous Martingales and Brownian Motion, third edition*. Springer, New York, 1999.

[49] L. C. G. Rogers and S. Singh. The cost of illiquidity and its effect on hedging. *Mathematical Finance*, 20: 597-615, 2010.

[50] A. Schied and T. Schöneborn. Risk aversion and the dynamics of optimal liquidation strategies in illiquid markets. *Finance and Stochastics*, 13: 181-204, 2009.

[51] L. F. Shampine. Singular boundary value problems for ODEs. *Applied Mathematics and Computation*, 138(1): 99 - 112, 2003.

[52] S. E. Shreve, H. M. Soner. Optimal investment and consumption with transaction costs. *Annals of Applied Probability*, 4: 609-692, 1994

[53] H. M. Soner, N. Touzi. Homogenization and asymptotics for small transaction costs. *arXiv preprint, arXiv: 1202.6131*, 2013.

[54] L. E. Svensson, I. Werner. Nontraded assets in incomplete markets: pricing and portfolio choices. *European Economic Review*, 37(5): 1149-1168, 1993.

[55] W. Walter. *Ordinary differential equations*. Springer, New York, 1998.

[56] D. Willett. Uniqueness for second order nonlinear boundary value problems with applications to almost periodic solutions. *Annali di Matematica Pura ed Applicata*, 81(1): 77 - 92, 1969.

[57] A. E. Whalley, P. Wilmott. An asymptotic analysis of an optimal hedging model for option pricing with transaction costs. *Mathematical Finance*, 7(3): 307-324, 1997.