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Article Title: SEXTONIONS AND THE MAGIC SQUARE

Year of publication: 2006

Link to published version: <http://dx.doi.org/10.1112/S0024610706022605>

Publisher statement: None

## SEXTONIONS AND THE MAGIC SQUARE

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### ABSTRACT

Associated to any complex simple Lie algebra is a non-reductive complex Lie algebra which we call the intermediate Lie algebra. We propose that these algebras can be included in both the magic square and the magic triangle to give an additional row and column. The extra row and column in the magic square correspond to the sextonions. This is a six-dimensional subalgebra of the split octonions which contains the split quaternions.

### 1. Introduction

The Freudenthal magic square is a  $4 \times 4$  array of complex semisimple Lie algebras. The rows and columns are indexed by the real division algebras and the square is symmetric. This is magic because the row (or column) indexed by the octonions consists of four of the five exceptional simple Lie algebras. There are three constructions which give this square, namely the Tits construction, the Vinberg construction and the triality construction. Each of these constructions can be extended to give a rectangle of Lie algebras. There is an alternative point of view which gives a triangle of Lie algebras.

In this paper we introduce the sextonions as a six-dimensional real alternative algebra intermediate between the split quaternions and the split octonions. Then we argue that the above magic square, magic rectangle and magic triangle should all be extended to include an extra row and column. If the rows or columns are indexed by division algebras then this extra row or column is indexed by the sextonions. In the following extended magic square we give the derived algebras of the intermediate algebras.

$A_1$	$A_2$	$C_3$	$C_3.H_{14}$	$F_4$	(1)
$A_2$	$2A_2$	$A_5$	$A_5.H_{20}$	$E_6$	
$C_3$	$A_5$	$D_6$	$D_6.H_{32}$	$E_7$	
$C_3.H_{14}$	$A_5.H_{20}$	$D_6.H_{32}$	$D_6.H_{32}.H_{44}$	$E_7.H_{56}$	
$F_4$	$E_6$	$E_7$	$E_7.H_{56}$	$E_8$	

The notation in this table is that  $G.H_n$  means that  $G$  has a representation  $V$  of dimension  $n$  with an invariant symplectic form,  $\omega$ . Then  $H_n$  means the Heisenberg algebra of  $(V, \omega)$  and  $G.H_n$  means the semidirect product of  $G$  and  $H_n$ . The entry at the intersection of the additional row and column is the bigraded algebra (20).

These intermediate algebras are also examples of a more general construction. For example, the intermediate algebras for the symplectic algebras are the odd symplectic algebras whose character theory is studied in [19], [20], [21] and [25].

In Section 5.1 of this paper we also extend the exceptional series of Lie algebras to include some Lie superalgebras. Also in Section 6 we construct further dual reductive pairs in the exceptional Lie algebras and hence extend the magic triangle. These two extensions are distinct but both involve the intermediate algebras.

Our two general references are the survey article [2] on the four real division algebras and [3] which gives the three constructions of the magic squares of real Lie algebras and gives isomorphisms between the Lie algebras given by these constructions.

This paper is a revised version of the preprint referred to in [18]. There is some overlap between these two articles.

### 2. Intermediate Lie algebras

Our discussion of intermediate algebras is based on the grading associated to extremal elements. The main application of these has been to the study of simple modular Lie algebras (see, for example, [4]). Another application is in [5].

DEFINITION 2.1. A triple in  $\mathfrak{g}$  is a set of three elements of  $\mathfrak{g}$ ,  $\{E, F, H\}$ , such that

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

DEFINITION 2.2. An element  $e \in \mathfrak{g}$  is extremal if the one-dimensional space with basis  $e$  is an inner ideal. This means that for all  $y \in \mathfrak{g}$ ,  $[e, [e, y]]$  is a scalar multiple of  $e$ . A triple  $(E, H, F)$  is extremal if  $E$  (and therefore  $F$ ) is extremal.

Let  $\mathfrak{g}$  be a complex simple Lie algebra. Then extremal triples can be constructed by choosing a Borel subalgebra and a root  $\alpha$  with the same length as the highest root. Then there is an extremal triple with  $E$  in the root space of  $\alpha$  and  $F$  in the root space of  $-\alpha$  and  $H = [E, F]$ .

Conversely every extremal triple arises this way. Let  $\{E, H, F\}$  be an extremal triple. Let  $\bar{\mathfrak{g}}$  be the centraliser of this triple and let  $\bar{\mathfrak{h}}$  be a Cartan subalgebra of  $\bar{\mathfrak{g}}$ . Then a Cartan subalgebra of  $\mathfrak{g}$  is given by taking the direct sum of  $\bar{\mathfrak{h}}$  with the vector space spanned by  $H$ . Then both  $E$  and  $F$  span root spaces; the roots are of the form  $\pm\alpha$  and have the same length as the highest root.

In particular this shows that extremal triples are unique up to automorphism of  $\mathfrak{g}$ .

Any triple gives a grading on  $\mathfrak{g}$  by taking the eigenspaces of  $H$ . For an extremal triple this grading has the following form.

$$\begin{array}{ccccc}
 \hline
 -2 & -1 & 0 & 1 & 2 \\
 \mathbb{C} & V & \bar{\mathfrak{g}} \oplus \mathbb{C} & V & \mathbb{C} \\
 \hline
 \end{array} \tag{2}$$

Alternatively we can define  $\bar{\mathfrak{g}}$  to be the centraliser of the triple. Then we can consider the adjoint representation of  $\mathfrak{g}$  as a representation of the subalgebra  $\bar{\mathfrak{g}} \oplus \mathfrak{sl}(2)$ . This representation decomposes as

$$(\bar{\mathfrak{g}} \otimes 1) \oplus (1 \otimes \mathfrak{sl}(2)) \oplus (V \otimes A), \tag{3}$$

where  $A$  is the two-dimensional fundamental representation of  $\mathfrak{sl}(2)$ . This shows that  $\overline{\mathfrak{g}}$  and  $\mathfrak{sl}(2)$  are a dual reductive pair.

The intermediate algebra of  $\mathfrak{g}$  is defined to be the non-negative part of the graded algebra (2). This is denoted by  $\mathfrak{g}_P$ . This subalgebra can also be defined as the centraliser of  $E$  in  $\mathfrak{g}$ . The derived subalgebra  $\mathfrak{g}_{P'}$  has codimension 1. The degree two component of these algebras is a one-dimensional ideal. Let the quotient of  $\mathfrak{g}_P$  by this ideal be  $\mathfrak{g}_R$ . This gives the following commutative diagram.

$$\begin{array}{ccc}
 \mathfrak{g}_{P'} = \overline{\mathfrak{g}} \oplus V \oplus \mathbb{C} & \longrightarrow & \mathfrak{g}_P = (\overline{\mathfrak{g}} \oplus \mathbb{C}) \oplus V \oplus \mathbb{C} \\
 \downarrow & & \downarrow \\
 \mathfrak{g}_{R'} = \overline{\mathfrak{g}} \oplus V & \longrightarrow & \mathfrak{g}_R = (\overline{\mathfrak{g}} \oplus \mathbb{C}) \oplus V
 \end{array} \tag{4}$$

The horizontal arrows are inclusions of derived subalgebras and the vertical arrows are surjective with one-dimensional kernel. The homomorphism  $\mathfrak{g}_{P'} \rightarrow \mathfrak{g}_{R'}$  is a universal central extension of the perfect Lie algebra  $\mathfrak{g}_{R'}$ . Thus  $\mathfrak{g}_P$  is constructed from  $\mathfrak{g}_{R'}$  by first taking a central extension and then adjoining a grading operator. This is analogous to the construction of the Kac–Moody algebra from the loop algebra.

The degree zero components of  $\mathfrak{g}_P$  and  $\mathfrak{g}_R$  are the centraliser of  $H$  in  $\mathfrak{g}$  and the degree zero components of  $\mathfrak{g}_{P'}$  and  $\mathfrak{g}_{R'}$  are the centraliser of the triple. Hence in all four cases the degree zero component is reductive. In all four cases the algebra is graded and so the sum of the components with positive degree is a nilpotent ideal. The quotient by this ideal is reductive and so this ideal is the nilpotent radical.

The main reason for considering the two algebras  $\mathfrak{g}_P$  and  $\mathfrak{g}_{P'}$  is that they are both subalgebras of  $\mathfrak{g}$  which contain  $\overline{\mathfrak{g}}$ . The main reason for considering the two quotient algebras is that they arise when considering finite dimensional representations. More precisely, the two vertical arrows in (4) give restriction functors on the categories of finite dimensional representations. These two functors are isomorphisms.

### 2.1. Examples

For the special linear algebras this structure can be seen as follows. Let  $U$  and  $V$  be any two vector spaces. Then  $\mathfrak{gl}(U) \oplus \mathfrak{gl}(V)$  is a subalgebra of  $\mathfrak{g} = \mathfrak{gl}(U \oplus V)$ . Then the restriction of the adjoint representation of  $\mathfrak{g}$  to this subalgebra decomposes as

$$\mathfrak{gl}(U) \oplus \mathfrak{gl}(V) \oplus U \otimes V^* \oplus U^* \otimes V.$$

If we take the special linear group then we get

$$\mathfrak{gl}(U) \oplus \mathfrak{sl}(V) \oplus U \otimes V^* \oplus U^* \otimes V.$$

If we take  $V$  to be two dimensional then  $V$  and  $V^*$  are equivalent representations and so we see that

$$\overline{\mathfrak{sl}(n+2)} = \mathfrak{gl}(n) \tag{5}$$

and the symplectic representation is the sum of the vector representation and its dual.

For the symplectic algebras this structure can be seen as follows. Let  $U$  and  $V$  be symplectic vector spaces. Then  $\mathfrak{sp}(U) \oplus \mathfrak{sp}(V)$  is a subalgebra of  $\mathfrak{g} = \mathfrak{sp}(U \oplus V)$ .

Then the restriction of the adjoint representation of  $\mathfrak{g}$  to this subalgebra decomposes as

$$\mathfrak{sp}(U) \oplus \mathfrak{sp}(V) \oplus U \otimes V. \tag{6}$$

Taking  $V$  to be two dimensional we see that

$$\overline{\mathfrak{sp}(2n + 2)} = \mathfrak{sp}(2n)$$

and the symplectic representation is the vector representation. These are the Lie algebras of Lie groups known as intermediate symplectic groups or odd symplectic groups. The characters and representations of these groups are studied in [20] and [25].

For the special orthogonal algebras this structure can be seen as follows. Let  $U$  and  $V$  be vector spaces with non-degenerate symmetric inner products. Then  $\mathfrak{so}(U) \oplus \mathfrak{so}(V)$  is a subalgebra of  $\mathfrak{g} = \mathfrak{so}(U \oplus V)$ . Then the restriction of the adjoint representation of  $\mathfrak{g}$  to this subalgebra decomposes as

$$\mathfrak{so}(U) \oplus \mathfrak{so}(V) \oplus U \otimes V. \tag{7}$$

Taking  $V$  to be four dimensional and using the isomorphism  $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$  we see that

$$\overline{\mathfrak{so}(n + 4)} = \mathfrak{so}(3) \oplus \mathfrak{so}(n)$$

and the symplectic representation is the tensor product of the two-dimensional representation of  $\mathfrak{so}(3)$  with the vector representation of  $\mathfrak{so}(n)$ .

For the exceptional simple Lie algebras we have the following table.

$\mathfrak{g}$	$G_2$	$F_4$	$E_6$	$E_7$	$E_8$
$\overline{\mathfrak{g}}$	$A_1$	$C_3$	$A_5$	$D_6$	$E_7$
$\dim(V)$	4	14	20	32	56

In all these five cases the representation  $V$  is irreducible.

### 2.2. Structure

In this section we make some observations based on the above examples.

The first observation is that  $\overline{\mathfrak{g}}$  can also be described as the reductive Lie algebra whose rank is one less than the rank of  $\mathfrak{g}$  and where the Dynkin diagram is given by removing the support of the highest root from the Dynkin diagram of  $\mathfrak{g}$ . Also the subalgebra  $\mathfrak{g}_P$  is the parabolic subalgebra associated to the same subset of the simple roots.

Another property that can be observed in these examples is that the representation  $V$  is miniscule. In fact this is given a direct proof in [4, Section 3].

Finally we observe that these Lie algebras have some of the properties of a Kac–Moody algebra or Borcherds algebra. The basis of these properties is the observation that  $\mathfrak{g}_P$  is the semidirect product of a semisimple Lie algebra and a Heisenberg algebra. A semisimple Lie algebra is a Kac–Moody algebra. It is shown in [13, Section 2.8] that a Heisenberg algebra is the Kac–Moody algebra associated to the zero Cartan matrix.

Next we observe that each of these intermediate algebras has a triangular decomposition. Let  $\overline{\mathfrak{h}}$  be a Cartan subalgebra of  $\overline{\mathfrak{g}}$ . Then a Cartan subalgebra of  $\mathfrak{g}_P$  is given

by taking  $\bar{h} \oplus \mathbb{C}$  in degree zero, zero in degree one and  $\mathbb{C}$  in degree two. Note that this is nilpotent but not abelian. This Cartan subalgebra also has the property that any non-zero ideal of  $\mathfrak{g}_P$  contains a non-zero element of the Cartan subalgebra.

Now consider a triangular decomposition of  $\bar{\mathfrak{g}}$ ,

$$\bar{\mathfrak{g}} = n_- \oplus \bar{\mathfrak{h}} \oplus n_+.$$

Then we decompose  $V$  as  $V = V_- \oplus V_+$  where each subspace is a Lagrangian subspace and  $V_-$  (respectively  $V_+$ ) is invariant under  $n_-$  (respectively  $n_+$ ). Then we have the triangular decomposition of  $\mathfrak{g}_P$ :

$$(n_- \oplus V_-) \oplus \bar{\mathfrak{h}} \oplus (n_+ \oplus V_+).$$

In general  $\mathfrak{g}_P$  fails to be a Borchers algebra since there is no non-degenerate invariant symmetric bilinear form. However such a form does exist for the intermediate algebra of  $\mathfrak{g} = \mathfrak{sl}(n + 2)$ . This form is constructed by taking the sum of the Killing form on  $\bar{\mathfrak{g}} = \mathfrak{sl}(n)$  and the form on the complementary Heisenberg algebra constructed in [13, Section 2.8].

### 2.3. Superalgebras

This structure can also be extended to the basic Lie superalgebras by choosing an extremal triple in the even algebra. Our notation for the dimension of a superspace is  $(n | m)$  where  $n$  is the dimension of the even subspace and  $m$  is the dimension of the odd subspace.

For the special linear algebras this structure can be seen as follows. Let  $U$  and  $V$  be any two superspaces. Then  $\mathfrak{gl}(U) \oplus \mathfrak{gl}(V)$  is a subalgebra of  $\mathfrak{g} = \mathfrak{gl}(U \oplus V)$ . Then the restriction of the adjoint representation of  $\mathfrak{g}$  to this subalgebra decomposes as

$$\mathfrak{gl}(U) \oplus \mathfrak{gl}(V) \oplus U \otimes V^* \oplus U^* \otimes V.$$

If we take the special linear group then we get

$$\mathfrak{gl}(U) \oplus \mathfrak{sl}(V) \oplus U \otimes V^* \oplus U^* \otimes V. \tag{8}$$

If we take  $V$  to have dimension  $(2 | 0)$  then  $V$  and  $V^*$  are equivalent representations and so we see that

$$\overline{\mathfrak{sl}(n + 2 | m)} = \mathfrak{gl}(n | m)$$

and the symplectic representation is the sum of the vector representation and its dual. Alternatively, if we take  $V$  to have dimension  $(0 | 2)$  then  $V$  and  $V^*$  are equivalent representations and so we see that

$$\overline{\mathfrak{sl}(n | m + 2)} = \mathfrak{gl}(n | m)$$

and the symplectic representation is the sum of the vector representation and its dual.

For the orthosymplectic algebras this structure can be seen as follows. Let  $U$  and  $V$  be superspaces with non-degenerate symmetric inner products. Then  $\mathfrak{osp}(U) \oplus \mathfrak{osp}(V)$  is a subalgebra of  $\mathfrak{g} = \mathfrak{osp}(U \oplus V)$ . Then the restriction of the adjoint representation of  $\mathfrak{g}$  to this subalgebra decomposes as

$$\mathfrak{osp}(U) \oplus \mathfrak{osp}(V) \oplus U \otimes V. \tag{9}$$

This includes (7) and (6) as special cases. Taking  $V$  to have dimension  $(4 \mid 0)$  and using the isomorphism  $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$  we see that

$$\overline{\mathfrak{osp}(n+4 \mid m)} = \mathfrak{so}(3) \oplus \mathfrak{osp}(n \mid m)$$

and the symplectic representation is the tensor product of the two-dimensional representation of  $\mathfrak{so}(3)$  with the vector representation of  $\mathfrak{so}(n)$ . Alternatively, taking  $V$  to have dimension  $(0 \mid 2)$  we see that

$$\overline{\mathfrak{osp}(n \mid m+2)} = \mathfrak{osp}(n \mid m)$$

and the symplectic representation is the vector representation of  $\mathfrak{osp}(n \mid m)$ .

For the exceptional simple Lie superalgebra  $G(3)$  the even algebra is  $A_1 \oplus G_2$  and the odd part is the tensor product of the two-dimensional fundamental representation of  $A_1$  with the seven-dimensional fundamental representation of  $G_2$ . This means we can take  $\overline{G(3)} = G_2$  and the symplectic representation is the superspace  $(0 \mid V)$  where  $V$  is the seven-dimensional fundamental representation of  $G_2$ .

For the exceptional simple Lie superalgebra  $F(4)$  the even algebra is  $A_1 \oplus B_3$  and the odd part is the tensor product of the two-dimensional fundamental representation of  $A_1$  with the eight-dimensional spin representation of  $B_3$ . This means we can take  $\overline{F(4)} = B_3$  and the symplectic representation is the superspace  $(0 \mid V)$  where  $V$  is the eight-dimensional spin representation of  $B_3$ .

### 3. Sextonions

In this section we construct the sextonions. This is a six-dimensional real algebra. This is a subalgebra of the split octonions which is closed under conjugation. This algebra was explicitly constructed in [15]. This algebra was used in [12] to study the conjugacy classes in  $G_2$  in characteristics other than 2 or 3. The sextonions were also constructed in [22, Theorem 5] and shown to be a maximal subalgebra of the split octonions.

The real normed division algebras are the real numbers, the complex numbers, the quaternions and the octonions. These are denoted by

$$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}.$$

Each algebra is obtained from the previous one by Cayley–Dickson doubling. These can be complexified to give complex algebras. These complex algebras are

$$\mathbb{R} \otimes \mathbb{C} = \mathbb{C}, \quad \mathbb{C} \otimes \mathbb{C} = \mathbb{C} \oplus \mathbb{C}, \quad \mathbb{H} \otimes \mathbb{C} = M_2(\mathbb{C}), \quad \mathbb{O} \otimes \mathbb{C}.$$

The three complex algebras other than  $\mathbb{C}$  have a second real form. These real forms are denoted  $\widetilde{\mathbb{C}}$ ,  $\widetilde{\mathbb{H}}$  and  $\widetilde{\mathbb{O}}$ . There are isomorphisms  $\widetilde{\mathbb{C}} = \mathbb{R} \oplus \mathbb{R}$  and  $\widetilde{\mathbb{H}} = M_2(\mathbb{R})$ . The normed division algebras are called the compact forms and this second real form is called the split real form. These split real forms are composition algebras but are not division algebras. The sextonions are intermediate between the split quaternions and the split octonions.

The sextonions can be constructed as follows. The split quaternions are isomorphic to the algebra of  $2 \times 2$  matrices. The norm is given by the determinant and the conjugate of a matrix is the adjoint matrix. This algebra has a unique alternative bimodule which is not associative. This is the two-dimensional Cayley module. This result is given in [11]. This bimodule can be constructed by taking a simple left module  $M$  with action denoted by juxtaposition and defining new left and right

actions by

$$q \cdot m = \bar{q}m \quad \text{and} \quad m \cdot q = qm$$

for all  $q \in \tilde{\mathbb{H}}$  and all  $m \in M$ .

DEFINITION 3.1. Let  $\tilde{\mathbb{S}}$  be the split null extension of  $\tilde{\mathbb{H}}$  by  $M$ . This means that we put  $\tilde{\mathbb{S}} = \tilde{\mathbb{H}} \oplus M$  and define a *multiplication* by

$$(q_1, m_1)(q_2, m_2) = (q_1q_2, \bar{q}_1m_2 + q_2m_1)$$

for all  $q_1, q_2 \in \tilde{\mathbb{H}}$  and all  $m_1, m_2 \in M$ . The *norm* is given by

$$N(q, m) = \det(q)$$

and if  $x = (q, m)$  then  $\bar{x} = (\bar{q}, -m)$ .

Next we show that this is a subalgebra of the split octonions. The split octonions can be constructed from the split quaternions by the Cayley–Dickson doubling process. Put  $\tilde{\mathbb{O}} = \tilde{\mathbb{H}} \oplus \tilde{\mathbb{H}}$  and define a multiplication by

$$(A_1, B_1)(A_2, B_2) = (A_1A_2 - \varepsilon B_2\bar{B}_1, \bar{A}_1B_2 + A_2B_1) \tag{10}$$

and define  $\overline{(A, B)} = (\bar{A}, -B)$  and  $|(A, B)| = |A| + \varepsilon|B|$ .

If we apply this to  $\mathbb{H}$  and take  $\varepsilon > 0$  then we get the compact octonions and if  $\varepsilon < 0$  then we get the split octonions. If we apply this to the split quaternions then we get the split octonions for all  $\varepsilon \neq 0$ .

Then we see that if we take  $B$  to have zero second column then we obtain the sextonions as a subalgebra.

Note also that we have two commuting actions of  $SL_2(\mathbb{R})$ . Let  $X \in SL_2(\mathbb{R})$  so  $\bar{X} = X^{-1}$ . Then these actions are given by

$$(A, B) \mapsto (XAX^{-1}, XB) \tag{11}$$

and

$$(A, B) \mapsto (A, B\bar{X}). \tag{12}$$

The sextonions are not a division algebra or a composition algebra or a normed algebra since there is a non-trivial radical given by the Cayley module  $M$  and this is the null space for the inner product. However they are a subalgebra of the split octonions which is closed under conjugation. There is a multiplication, a conjugation and an inner product which are given in Definition 3.1. This structure is also given by restriction on the split octonions so any identities which involve this structure and which hold in the split octonions also hold in the sextonions.

The octonions have a 3-step  $\mathbb{Z}$ -grading. The map  $A \mapsto (A, 0)$  is an inclusion of  $\tilde{\mathbb{H}}$  in  $\tilde{\mathbb{O}}$  and we take the image to be the subspace of degree zero. The subspace of pairs of the form  $(0, B)$  is a left module. This has a decomposition as a left module into a subspace  $U_-$  where the second column of  $B$  is zero and  $U_+$  where the first column of  $B$  is zero. Take  $U_-$  to be the subspace of degree  $-1$  and  $U_+$  to be the subspace of degree  $+1$ . Note that the product of two elements of  $U_-$  or  $U_+$  is zero so this is a grading.

$$\begin{array}{ccc}
 \hline
 -1 & 0 & 1 \\
 U_- & \tilde{\mathbb{H}} & U_+ \\
 \hline
 \end{array} \tag{13}$$



Since the multiplication is needed for later calculations we give it here explicitly. This is closely related to the description of the split octonions in [18]:

$$\begin{pmatrix} u_1 \\ A_1 \\ v_1 \end{pmatrix} \begin{pmatrix} u_2 \\ A_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} \overline{A_1}u_2 + A_2u_1 \\ A_1A_2 + (u_2, v_2)(u_1, v_1) \\ \overline{A_1}v_2 + A_2v_1 \end{pmatrix}, \tag{14}$$

where  $(u, v)$  means put the two column vectors  $u$  and  $v$  side by side to form a matrix.

#### 4. Elementary series

There are three simple constructions which associate a Lie algebra to the four normed division algebras. These can all be extended to the sextonions. In this section we show that for each of these constructions we have that  $\mathfrak{g}(\tilde{\mathbb{O}}) = \mathfrak{g}(\tilde{\mathbb{H}})$  and that the intermediate algebra is  $\mathfrak{g}(\tilde{\mathbb{S}})$ .

##### 4.1. Derivations

The first construction is the derivation algebra. The derivation algebras of the composition algebras are as follows.

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
0	0	$A_1$	$G_2$

First we look at the derivation algebra of  $\tilde{\mathbb{O}}$ . The model we take for this is the Cayley–Dickson double of  $\tilde{\mathbb{H}} \cong M_2(\mathbb{R})$  given in (10).

The grading in (13) induces a 5-step  $\mathbb{Z}$ -grading on the derivation algebra. This grading is given by

-2	-1	0	1	2
$\mathbb{R}$	$V$	$\mathfrak{der}(\tilde{\mathbb{H}}) \oplus \mathbb{R}$	$V$	$\mathbb{R}$

(15)

where  $V$  is the four-dimensional irreducible representation of  $\mathfrak{der}(\tilde{\mathbb{H}}) \cong \mathfrak{sl}_2(\mathbb{R})$ .

Let  $E$  and  $F$  be the maps

$$\begin{aligned} E &: (u, A, v) \mapsto (0, 0, u), \\ F &: (u, A, v) \mapsto (v, 0, 0). \end{aligned}$$

Then these can be shown to be derivations by direct calculation. Put  $H = [E, F]$ ; then  $\{E, H, F\}$  is a triple. Note that the Lie subalgebra given by this triple is the Lie algebra of the second action of  $\mathrm{SL}_2(\mathbb{R})$  in (12).

Also the grading on  $\tilde{\mathbb{O}}$  in (13) is also the grading by the eigenvalues of  $H$ . This implies that the grading on the derivation algebra induced by the grading in (13) is also the grading by the eigenvalues of  $H$ .

A direct calculation also shows that any derivation of degree two is a scalar multiple of  $E$ . Hence the triple  $\{E, H, F\}$  is an extremal triple.

Now the derivation algebra of  $\mathbb{O}$  was identified with the Lie algebra  $G_2$  by Elie Cartan in 1915. Take an extremal triple in  $G_2$  with  $E$  in the highest root space

and  $F$  in the lowest root space. Then by inspecting the root diagram we see that the associated 5-step  $\mathbb{Z}$ -grading is given by (15).

Alternatively we can take the construction of the derivation algebra of  $\tilde{\mathbb{O}}$  given in [27]. This construction shows that the derivation algebra has a grading by the cyclic group of order three with components  $W, \text{SL}(W), W^*$  in degrees  $-1, 0, 1$  where  $W$  has dimension three. Then take an extremal triple in  $\text{SL}(W)$ . The gradings on  $W$  and  $W^*$  given by the eigenspaces of  $H$  are both given by taking one-dimensional spaces in dimensions  $-1, 0$  and  $1$ . This shows that the dimensions of the graded components of the derivation algebra are as given in (15).

Next we consider the derivations of the sextonions.

DEFINITION 4.1. Define  $\mathfrak{der}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}})$  to be the subalgebra of derivations of  $\tilde{\mathbb{O}}$  which preserve  $\tilde{\mathbb{S}}$ .

Then the main result of this section is that the restriction homomorphism  $\mathfrak{der}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \rightarrow \mathfrak{der}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{S}})$  is the homomorphism  $\mathfrak{g}_P \rightarrow \mathfrak{g}_R$  in (4) for  $\mathfrak{g} = \mathfrak{der}(\tilde{\mathbb{O}})$ .

It is clear from (15) that we can identify  $\mathfrak{g}_P$  with  $\mathfrak{der}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}})$ . Then this induces a homomorphism of graded Lie algebras  $\mathfrak{g}_{P'} \rightarrow \mathfrak{der}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{S}})$ . Our aim now is to show that this is an isomorphism. It is clear that this is an inclusion and that both graded Lie algebras have non-zero components only in degrees zero and one. The graded Lie algebra  $\mathfrak{g}_{P'}$  has  $\mathfrak{gl}(2)$  in degree zero and a four-dimensional irreducible representation in degree one.

The derivations of  $\tilde{\mathbb{S}}$  of degree zero are a subspace of  $\text{End}(\tilde{\mathbb{H}}) \oplus \text{End}(U)$ . The derivations in  $\text{End}(\tilde{\mathbb{H}})$  are the derivations of  $\tilde{\mathbb{H}}$  which give a Lie algebra isomorphic to  $\mathfrak{sl}(2)$ . A calculation shows that a derivation in  $\text{End}(U)$  is a scalar multiple of the grading operator  $H$ .

The derivations of  $\tilde{\mathbb{S}}$  of degree one are a subspace of  $\text{Hom}(\tilde{\mathbb{H}}, U)$  which has dimension eight. This space has an action of the degree zero derivations and the subspace of derivations is invariant under this action.

A derivation of  $\tilde{\mathbb{S}}$  of degree one is of the form

$$(A, u) \mapsto (0, \psi(A)),$$

where  $\psi: \text{End}(U) \rightarrow U$  is a linear map. The condition on  $\psi$  for this to be a derivation is that

$$\psi(A_1 A_2) = \overline{A_1} \cdot \psi(A_2) + A_2 \cdot \psi(A_1)$$

for all  $A_1, A_2 \in \text{End}(U)$ .

Putting  $A_2 = 1$  shows that  $\psi(1) = 0$ .

Now assume that  $A_1$  and  $A_2$  have zero trace. Then

$$\psi([A_1, A_2]) = -2A_1\psi(A_2) + 2A_2\psi(A_1).$$

Then if we choose a triple this shows that  $\psi(E)$  and  $\psi(F)$  are arbitrary and that these values then determine  $\psi(H)$  and hence  $\psi$ .

#### 4.2. Triality

Let  $\mathbb{A}$  be a composition algebra. Then the triality group  $\text{Tri}(\mathbb{A})$  consists of triples  $(\theta_1, \theta_2, \theta_3)$  in  $\text{SO}(\mathbb{A}) \times \text{SO}(\mathbb{A}) \times \text{SO}(\mathbb{A})$  such that

$$\theta_1(a)\theta_2(b) = \theta_2(ab)$$

for all  $a, b \in \mathbb{A}$ . Let  $\mathfrak{tri}(\mathbb{A})$  be the Lie algebra of  $\text{Tri}(\mathbb{A})$ . The triality algebras of the composition algebras are as follows.

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
0	$T_2$	$3A_1$	$D_4$

The three conditions  $\theta_i(1) = 1$  define three subgroups. These three subgroups are isomorphic and any one of them can be taken as the intermediate group  $\text{Int}(\mathbb{A})$ . Let  $\mathfrak{int}(\mathbb{A})$  be the Lie algebra of  $\text{Int}(\mathbb{A})$ . The intersection of any two of these intermediate subgroups is the automorphism group, whose Lie algebra is  $\mathfrak{det}(\mathbb{A})$ .

The intermediate algebras of the composition algebras are as follows.

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
0	$T_1$	$2A_1$	$B_3$

Let  $O$  be the orbit of  $1 \in \mathbb{A}$  under the action of  $\text{SO}(\mathbb{A})$ . Then we have  $\text{SO}(\mathbb{A})/\text{Int}(\mathbb{A}) \cong O$  and  $\text{Int}(\mathbb{A})/\text{Aut}(\mathbb{A}) \cong O$ . In terms of the Lie algebras we can identify the tangent space of  $1 \in O$  with  $\text{Im}(\mathbb{A})$  and then we have vector space isomorphisms

$$\mathfrak{tri}(\mathbb{A}) = \mathfrak{int}(\mathbb{A}) \oplus \text{Im}(\mathbb{A}) \quad \text{and} \quad \mathfrak{int}(\mathbb{A}) = \mathfrak{det}(\mathbb{A}) \oplus \text{Im}(\mathbb{A}). \tag{16}$$

The definition of  $\text{Tri}(\mathbb{A})$  uses the norm and multiplication on  $\mathbb{A}$ . The same definition makes sense for a degenerate norm and this defines the group  $\text{Tri}(\tilde{\mathbb{S}})$  and the Lie algebra  $\mathfrak{tri}(\tilde{\mathbb{S}})$ . We also define  $\text{Tri}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}})$  to be the subgroup of  $\text{Tri}(\tilde{\mathbb{O}})$  consisting of triples  $(\theta_1, \theta_2, \theta_3)$  such that each  $\theta_i$  preserves  $\tilde{\mathbb{S}} \subset \tilde{\mathbb{O}}$ . Then there is a natural restriction  $\mathfrak{tri}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \rightarrow \mathfrak{tri}(\tilde{\mathbb{S}})$ . Then the Lie algebras  $\mathfrak{int}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}})$  and  $\mathfrak{int}(\tilde{\mathbb{S}})$  are defined similarly together with a natural restriction  $\mathfrak{int}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \rightarrow \mathfrak{int}(\tilde{\mathbb{S}})$ .

Then the grading on the intermediate algebra  $\mathfrak{int}(\tilde{\mathbb{O}})$  is

-2	-1	0	1	2	
$\mathbb{R}$	$V \oplus U$	$\mathfrak{int}(\tilde{\mathbb{H}}) \oplus \mathbb{R}$	$V \oplus U$	$\mathbb{R}$	(17)

where  $V$  is the four-dimensional vector space in (15).

This shows that the restriction homomorphism  $\mathfrak{int}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \rightarrow \mathfrak{int}(\tilde{\mathbb{S}})$  is the homomorphism  $\mathfrak{g}_P \rightarrow \mathfrak{g}_R$  in (4) for  $\mathfrak{g} = \mathfrak{int}(\tilde{\mathbb{O}})$ .

Then the grading on the triality algebra  $\mathfrak{tri}(\tilde{\mathbb{O}})$  is

-2	-1	0	1	2	
$\mathbb{R}$	$V \oplus 2U$	$\mathfrak{tri}(\tilde{\mathbb{H}}) \oplus \mathbb{R}$	$V \oplus 2U$	$\mathbb{R}$	(18)

where  $V$  is the four-dimensional vector space in (15).

This shows that the restriction homomorphism  $\mathfrak{tri}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \rightarrow \mathfrak{tri}(\tilde{\mathbb{S}})$  is the homomorphism  $\mathfrak{g}_P \rightarrow \mathfrak{g}_R$  in (4) for  $\mathfrak{g} = \mathfrak{tri}(\tilde{\mathbb{O}})$ .

### 4.3. Superalgebras

There are also two constructions of Lie superalgebras. These constructions are given in [10], [14] and [26]. One construction is to take  $\mathfrak{g}(\mathbb{A})$  to be the superspace

with even part  $\mathfrak{sl}(A) \oplus \mathfrak{der}(\mathbb{A})$  and odd part  $A \otimes \text{Im}(\mathbb{A})$  where  $A$  is a two-dimensional vector space. This construction gives the following Lie superalgebras.

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$A_1$	$B(0, 1)$	$B(1, 1)$	$G(3)$

The grading on  $\mathfrak{g}(\tilde{\mathbb{O}})$  is the following.

-2	-1	0	1	2
$\mathbb{R}$	$(V \mid U \otimes A)$	$\mathfrak{g}(\tilde{\mathbb{H}}) \oplus \mathbb{R}$	$(V \mid U \otimes A)$	$\mathbb{R}$

This shows that the Lie superalgebra with even part  $\mathfrak{sl}(A) \oplus \mathfrak{der}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}})$  and with odd part  $A \otimes \text{Im}(\tilde{\mathbb{S}})$  where  $A$  is a two-dimensional vector space is the intermediate algebra of  $\mathfrak{g}(\tilde{\mathbb{O}}) = G(3)$ .

A second construction is to take  $\mathfrak{g}(\mathbb{A})$  to be the superspace with even part  $\mathfrak{sl}(A) \oplus \text{int}(\mathbb{A})$  and odd part  $A \otimes \mathbb{A}$  where  $A$  is a two-dimensional vector space. This construction gives the Lie superalgebras as follows.

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$B(0, 1)$	$A(1, 0)$	$D(2, 1; \mu)$	$F(4)$

The grading on  $\mathfrak{g}(\tilde{\mathbb{O}})$  is

-2	-1	0	1	2
$\mathbb{R}$	$(V \oplus U \mid U \otimes A)$	$\mathfrak{g}(\tilde{\mathbb{H}}) \oplus \mathbb{R}$	$(V \oplus U \mid U \otimes A)$	$\mathbb{R}$

where we have used the grading (17).

This shows that the Lie superalgebra with even part  $\mathfrak{sl}(A) \oplus \text{int}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}})$  and odd part  $A \otimes \tilde{\mathbb{S}}$  is the intermediate algebra of  $\mathfrak{g}(\tilde{\mathbb{O}}) = F(4)$ .

### 5. The magic square

There are three constructions of the magic square. All three constructions take a pair of composition algebras  $(\mathbb{A}, \mathbb{B})$  and produce a semisimple Lie algebra  $L(\mathbb{A}, \mathbb{B})$ . The original construction is due to Freudenthal and Tits. Other constructions are the Vinberg and the triality constructions. These constructions are shown to give isomorphic Lie algebras in [3]. In all these cases we can extend the construction to include the sextonions and all constructions give isomorphic Lie algebras. Again we find that the intermediate subalgebra of  $L(\mathbb{A}, \mathbb{O})$  is  $L(\mathbb{A}, \tilde{\mathbb{S}})$ . This Lie algebra is non-negatively graded; the sum of the components with positive degree is the nilpotent radical; the degree zero component is a complement and is the reductive subalgebra  $L(\mathbb{A}, \tilde{\mathbb{H}})$ .

Let  $\mathbb{A}$  be a composition algebra and  $J$  a Jordan algebra. The Tits construction is

$$T(\mathbb{A}, J) = \mathfrak{der}(\mathbb{A}) \oplus \mathfrak{der}(J) \oplus \text{Im}(\mathbb{A}) \otimes \text{Im}(J).$$

Then the grading on  $T(\tilde{\mathbb{O}}, J)$  is as follows.

-2	-1	0	1	2
$\mathbb{R}$	$V \oplus U \otimes \text{Im}(J)$	$T(\tilde{\mathbb{H}}, J) \oplus \mathbb{R}$	$V \oplus U \otimes \text{Im}(J)$	$\mathbb{R}$

For the construction of the magic square we take the Jordan algebra  $J = J(\mathbb{B})$  to be  $H_3(\mathbb{B})$  which consists of  $3 \times 3$  Hermitian matrices with entries in  $\mathbb{B}$ . We can also take  $J = 0$  which gives the derivation algebras.

Then for  $\mathbb{A} = \tilde{\mathbb{S}}$  we get a subalgebra of  $T(\tilde{\mathbb{O}}, J)$  by taking

$$\text{der}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \oplus \text{der}(J) \oplus \text{Im}(\tilde{\mathbb{S}}) \otimes \text{Im}(J).$$

It is clear from the grading on  $T(\tilde{\mathbb{O}}, J)$  that this is the intermediate algebra of  $T(\tilde{\mathbb{O}}, J)$ .

The Vinberg construction is

$$V(\mathbb{A}, \mathbb{B}) = \text{der}(\mathbb{A}) \oplus \text{der}(\mathbb{B}) \oplus A'_3(\mathbb{A} \otimes \mathbb{B}),$$

where  $A'_3(\mathbb{A})$  means trace-free anti-Hermitian  $3 \times 3$  matrices with entries in  $\mathbb{A}$ . Then the grading on  $V(\tilde{\mathbb{O}}, \mathbb{B})$  is the following.

-2	-1	0	1	2
$\mathbb{R}$	$V \oplus A'_3(U \otimes \mathbb{B})$	$V(\tilde{\mathbb{H}}, \mathbb{B}) \oplus \mathbb{R}$	$V \oplus A'_3(U \otimes \mathbb{B})$	$\mathbb{R}$

Since  $U$  is imaginary we can identify  $A'_3(U \otimes \mathbb{B})$  with  $U \otimes H'_3(\mathbb{B})$  where  $H'_3(\mathbb{A})$  means trace-free Hermitian  $3 \times 3$  matrices with entries in  $\mathbb{A}$ . This is also  $\text{Im}(J)$  for  $J = H_3(\mathbb{A})$ .

Then for  $\mathbb{A} = \tilde{\mathbb{S}}$  we get a subalgebra of  $V(\tilde{\mathbb{O}}, \mathbb{B})$  by taking

$$\text{der}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \oplus \text{der}(\mathbb{B}) \oplus A'_3(\tilde{\mathbb{S}} \otimes \mathbb{B}).$$

It is clear from the grading on  $V(\tilde{\mathbb{O}}, \mathbb{B})$  that this is the intermediate algebra of  $V(\tilde{\mathbb{O}}, \mathbb{B})$ .

The triality construction is

$$A(\mathbb{A}, \mathbb{B}) = \text{tri}(\mathbb{A}) \oplus \text{tri}(\mathbb{B}) \oplus 3(\mathbb{A} \otimes \mathbb{B}).$$

Then the grading on  $A(\tilde{\mathbb{O}}, \mathbb{B})$  is

-2	-1	0	1	2
$\mathbb{R}$	$V \oplus 2U \oplus 3U \otimes \mathbb{B}$	$A(\tilde{\mathbb{H}}, \mathbb{B}) \oplus \mathbb{R}$	$V \oplus 2U \oplus 3U \otimes \mathbb{B}$	$\mathbb{R}$

where we have used the grading (18).

Then for  $\mathbb{A} = \tilde{\mathbb{S}}$  we get a subalgebra of  $A(\tilde{\mathbb{O}}, \mathbb{B})$  by taking

$$\text{tri}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \oplus \text{tri}(\mathbb{B}) \oplus 3(\tilde{\mathbb{S}} \otimes \mathbb{B}).$$

It is clear from the grading on  $A(\tilde{\mathbb{O}}, \mathbb{B})$  that this is the intermediate algebra of  $A(\tilde{\mathbb{O}}, \mathbb{B})$ .

Next we describe the algebra given by taking both algebras in the Vinberg or triality constructions to be the sextonions. Let  $V_{56}$  be the 56-dimensional fundamental

representation of  $E_7$ . Then the grading on  $E_8$  has the following components.

-2	-1	0	1	2
$\mathbb{C}$	$V_{56}$	$E_7 \oplus \mathbb{C}$	$V_{56}$	$\mathbb{C}$

Then take an extremal triple in  $E_7$ . Then this triple commutes with the extremal triple in  $E_8$  and so we have a bigrading on  $E_8$  with components

					$\mathbb{C}$						
		$V_{12}$			$S_{32}$			$V_{12}$			
$\mathbb{C}$	$S_{32}$	$D_6 \oplus \mathbb{C} \oplus \mathbb{C}$			$S_{32}$	$\mathbb{C}$					
		$V_{12}$			$S_{32}$			$V_{12}$			
					$\mathbb{C}$						

(19)

where  $S_{32}$  is a spin representation of  $D_6$  of dimension 32 and  $V_{12}$  is the vector representation of dimension 12. This constructs the Lie algebra  $E_8$  as

$$(D_6 \oplus \mathfrak{sl}(A) \oplus \mathfrak{sl}(B)) \oplus (V \otimes A \otimes B) \oplus S \otimes A \oplus S \otimes B,$$

where  $A$  and  $B$  are two-dimensional vector spaces.

Using the Vinberg construction we get the following subalgebra of  $E_8$ :

$$\mathfrak{der}^{\tilde{S}}(\tilde{\mathcal{O}}) \oplus \mathfrak{der}^{\tilde{S}}(\tilde{\mathcal{O}}) \oplus A'_3(\tilde{S} \otimes \tilde{S}).$$

Using the triality construction we get the following subalgebra of  $E_8$ :

$$\mathfrak{tri}^{\tilde{S}}(\tilde{\mathcal{O}}) \oplus \mathfrak{tri}^{\tilde{S}}(\tilde{\mathcal{O}}) \oplus 3(\tilde{S} \otimes \tilde{S}).$$

Both of these constructions give the following bigraded Lie algebra which is the positive and zero part of the bigrading (19):

$D_6 \oplus \mathbb{C} \oplus \mathbb{C}$	$S_{32}$	$\mathbb{C}$		
$S_{32}$	$V_{12}$			
$\mathbb{C}$				

(20)

Note that if we take the total grading in (19) we get the grading with components

-2	-1	0	1	2
$V_{14}$	$S_{64}$	$D_7 \oplus \mathbb{C}$	$S_{64}$	$V_{14}$

where  $S_{64}$  is a spin representation of  $D_7$  of dimension 64 and  $V_{14}$  is the vector representation of dimension 14. The non-negative part of this grading gives a second maximal parabolic subgroup of  $E_8$ . The even part of this grading is isomorphic to  $D_8$ . The odd part is a spin representation of  $D_8$ . This is used in [1] to construct the Lie algebra  $E_8$ .

### 5.1. Exceptional series

In this section we consider the exceptional series introduced in [8] and the subexceptional series. These are a finite series of reductive algebraic groups. Here we consider the corresponding series of Lie algebras. This series includes all five

exceptional simple Lie algebras. Here we take these Lie algebras to be parametrised by  $m$ . Different authors have used other parameters such as the dual Coxeter number. All of these parameters are related to  $m$  by Mobius transformations.

For the Lie algebras in the magic square we get  $L(\mathbb{O}, \mathbb{A})$  in the exceptional series with  $m = \dim(\mathbb{A})$  and  $L(\mathbb{H}, \mathbb{A})$  in the subexceptional series again with  $m = \dim(\mathbb{A})$ . This gives the last three rows of (1) with columns labelled by  $m = 1, 2, 4, 6, 8$ . The exceptional series also includes further columns. Four of these columns are given below:

$m$	$-4/3$	$-1$	$-2/3$	$0$
$\bar{\mathfrak{g}}$	$0$	$T$	$A_1$	$3A_1$
$\mathfrak{g}_{P'}$	$0$	$T.H_2$	$A_1.H_4$	$(3A_1).H_8$
$\mathfrak{g}$	$A_1$	$A_2$	$G_2$	$D_4$

The column with  $m = 0$  contains the triality algebras and the column with  $m = -2/3$  contains the derivation algebras.

In this section we extend the exceptional series to include some simple Lie superalgebras. Let  $\mathfrak{g}(\mathbb{H})$  be a Lie algebra in the subexceptional series and  $\mathfrak{g}(\mathbb{O})$  the corresponding Lie algebra in the exceptional series. Then  $\mathfrak{g}(\mathbb{H})$  has a distinguished representation  $V$  of dimension  $6m + 8$  which has a  $\mathfrak{g}(\mathbb{H})$ -invariant symplectic form. This is the representation  $V$  in (3).

This is consistent with the dimension formulae:

$$\dim(\mathfrak{g}(\mathbb{H})) = 3 \frac{(2m + 3)(3m + 4)}{(m + 4)}, \quad \dim(\mathfrak{g}(\mathbb{O})) = 2 \frac{(3m + 7)(5m + 8)}{(m + 4)}.$$

In these notes we show that this construction also makes sense for some values of  $m$  for which  $6m + 8$  is a negative integer. In this case we take  $V$  to be an odd superspace and apply the same construction to obtain a Lie superalgebra.

$m$	$-3$	$-8/3$	$-5/2$	$-7/3$	$-2$	$-3/2$
$6m + 8$	$-10$	$-8$	$-7$	$-6$	$-4$	$-1$
$\mathfrak{g}(\mathbb{H})$	$D_5$ $\mathfrak{so}(10)$	$B_3$ $\mathfrak{so}(7)$	$G_2$ $G_2$	$A_2 + T$ $\mathfrak{gl}(3)$	$A_1$ $\mathfrak{sl}(2)$	$0$ $0$
$\mathfrak{g}(\mathbb{O})$	$D(5, 1)$ $\mathfrak{osp}(10   2)$	$F(4)$ $F(4)$	$G(3)$ $G(3)$	$A(2, 1)$ $\mathfrak{sl}(3   2)$	$A(1, 1)$ $\mathfrak{sl}(2   2)$	$B(1, 1)$ $\mathfrak{osp}(1   2)$

There is a distinguished representation  $V$  of  $\mathfrak{g}(\mathbb{H})$  dimension  $-6m - 8$ . The structure that these representations have in common is that

$$S^2(V) = 1 \oplus V^2 \quad \text{and} \quad \Lambda^2(V) = \mathfrak{g} \oplus V_2.$$

The representation  $V_2$  is somewhat degenerate.

- (1) For  $m = -3$ ,  $\mathfrak{g}(\mathbb{H}) = \mathfrak{so}(10)$ ,  $V$  is the vector representation and  $V_2 = 0$ .
- (2) For  $m = -8/3$ ,  $\mathfrak{g}(\mathbb{H}) = \mathfrak{so}(7)$ ,  $V$  is the spin representation and  $V_2$  is the vector representation.
- (3) For  $m = -5/2$ ,  $\mathfrak{g}(\mathbb{H}) = G_2$  and  $V$  and  $V_2$  are both the seven-dimensional fundamental representation.

(4) For  $m = -7/3$ ,  $\mathfrak{g}(\mathbb{H}) = \mathfrak{gl}(3)$ ,  $V$  is the sum of the vector representation and its dual and  $V_2$  is the adjoint representation.

(5) For  $m = -3/2$ ,  $\mathfrak{g}(\mathbb{H}) = 0$ ,  $V$  has dimension one and  $V_2 = 0$ .

Note that in some cases we can replace  $\mathfrak{g}(\mathbb{H})$  by a Lie superalgebra and still keep this structure.

$m$	$-3$	$-7/3$	$-2$
$6m + 8$	$-10$	$-6$	$-4$
$\mathfrak{g}(\mathbb{H})$	$D(n + 5, n)$ $\mathfrak{osp}(2n + 10   2n)$	$A(n + 2, n) + T$ $\mathfrak{gl}(n + 3   n)$	$A(n + 1, n)$ $\mathfrak{sl}(n + 2   n)$
$\mathfrak{g}(\mathbb{O})$	$D(n + 5, n + 1)$ $\mathfrak{osp}(2n + 10   2n + 2)$	$A(n + 2, n + 1)$ $\mathfrak{sl}(n + 3   n + 2)$	$A(n + 1, n + 1)$ $\mathfrak{sl}(n + 2   n + 2)$
$m$	$-3/2$	$-4/3$	$-1$
$6m + 8$	$-1$	$0$	$2$
$\mathfrak{g}(\mathbb{H})$	$B(n + 1, n)$ $\mathfrak{osp}(2n + 1   2n)$	$A(n, n) + T$ $\mathfrak{gl}(n   n)$	$A(n + 1, 1) + T$ $\mathfrak{gl}(n + 1   1)$
$\mathfrak{g}(\mathbb{O})$	$B(n + 1, n + 1)$ $\mathfrak{osp}(2n + 1   2n + 2)$	$A(n + 1, n)$ $\mathfrak{sl}(n + 2   n)$	$A(n + 2, 2)$ $\mathfrak{sl}(n + 3   n)$

These follow from the general decompositions in (8) and (9).

The point  $m = -8/5$  on the exceptional line corresponds to the trivial Lie algebra. However there is no corresponding Lie algebra on the subexceptional line.

### 5.2. Magic triangle

There is another approach to the magic square based on dual reductive pairs. This constructs a magic triangle. This magic triangle is given in [7], [9] and [24]. This is also implicit in [6].

The involution which sends  $\mathfrak{g}$  to the centraliser in  $E_8$  corresponds to the involution

$$m \mapsto \frac{-2m}{m + 2}.$$

If we include the Lie algebra  $E_7.H_{56}$  with  $m = 6$  then this suggests that we should also include a Lie algebra for  $m = -3/2$ . This Lie algebra is given as the Lie superalgebra  $\mathfrak{osp}(1 | 2)$ . Taken literally this suggests that  $\mathfrak{osp}(1 | 2)$  and  $E_7.H_{56}$  are a dual reductive pair in  $E_8$ . However  $\mathfrak{osp}(1 | 2)$  is not a subalgebra and  $E_7.H_{56}$  is not reductive.

More generally the decomposition (3) shows that  $A_1$  and  $\bar{\mathfrak{g}}$  are a dual reductive pair in  $\mathfrak{g}$ . Here we do a formal calculation which shows that as characters of  $\bar{\mathfrak{g}} \oplus \mathfrak{sl}(2)$  we have

$$\mathfrak{g} = \mathfrak{g}_P' \otimes 1 \oplus 1 \otimes \mathfrak{osp}(1 | 2) \oplus (V \oplus 1) \otimes A, \tag{21}$$

where  $A$  is the vector representation of dimension  $(2 | 1)$ .

Then we write a super vector space as  $V_+ - V_-$  where  $V_+$  is the even part and  $V_-$  is the odd part. We write  $[n]$  for the irreducible highest weight representation



of  $\mathfrak{sp}(1)$  with highest weight  $n$  (and dimension  $n + 1$ ) and we regard a representation of  $\mathfrak{osp}(1 | 2)$  as a super representation of  $\mathfrak{sp}(1)$ . In particular, the adjoint representation of  $\mathfrak{osp}(1 | 2)$  is written as  $[2] - [1]$  and the representation  $A$  is written as  $[1] - [0]$ . Then the right hand side of (21) is

$$(\bar{\mathfrak{g}} + V + 1) \otimes [0] \oplus 1 \otimes ([2] - [1]) \oplus (V + 1) \otimes ([1] - [0]).$$

Expanding this and cancelling equal terms with opposite signs leaves

$$\bar{\mathfrak{g}} \otimes [0] + 1 \otimes [2] + V \otimes [1],$$

which is (3).

If we apply this to the Lie algebras in the exceptional series then this formal calculation is our justification for including an extra row and column in the magic triangle.

### 6. Adams series

The triality construction constructs the Lie algebra  $L(\mathbb{A}, \mathbb{B})$  with a  $\mathbb{Z}_2 \times \mathbb{Z}_2$ -grading. If we take any one of the three  $\mathbb{Z}_2$ -gradings then in degree zero we get the Lie algebra

$$\mathfrak{t}(\mathbb{A}, \mathbb{B}) = \mathfrak{tri}(\mathbb{A}) \oplus \mathfrak{tri}(\mathbb{B}) \oplus \mathbb{A} \otimes \mathbb{B}$$

and in degree one we get the spin representation  $(\mathbb{A} \otimes \mathbb{B}) \oplus (\mathbb{A} \otimes \mathbb{B})$ . The table for these Lie algebras is given in [17]. Note that  $\mathfrak{t}(\mathbb{A}, \mathbb{B})$  is a subalgebra of equal rank in  $L(\mathbb{A}, \mathbb{B})$ . The corresponding subgroups are studied in [16].

Comparing this construction and (7) we observe that there is a variation on these two constructions. Let  $V$  be a vector space with a non-degenerate symmetric inner product. Then there is a Lie algebra whose underlying vector space is

$$\mathfrak{tri}(\mathbb{A}) \oplus \mathfrak{so}(V) \oplus \mathbb{A} \otimes V. \tag{22}$$

The Lie bracket is defined so that  $\mathfrak{t}(\mathbb{A}) \oplus \mathfrak{so}(V)$  is a subalgebra and  $\mathbb{A} \otimes V$  is the obvious representation. The Lie bracket of two elements of  $\mathbb{A} \otimes V$  is the usual Lie bracket so that  $\mathfrak{so}(\mathbb{A} \oplus V)$  is a subalgebra.

This construction can then be modified to give the following.

DEFINITION 6.1. Let  $V$  be a vector space with a non-degenerate symmetric inner product. Then we define the *Lie algebra*  $\mathfrak{a}(\mathbb{A}, V)$  by

$$\mathfrak{a}(\mathbb{A}, V) = \mathfrak{int}(\mathbb{A}) \oplus \mathfrak{so}(V) \oplus \text{Im}(\mathbb{A}) \otimes V.$$

These Lie algebras have the property that if  $V$  and  $W$  both have a non-degenerate symmetric inner product then there is a natural isomorphism of Lie algebras

$$\mathfrak{a}(\mathbb{A}, V \oplus W) \cong \mathfrak{a}(\mathbb{A}, V) \oplus \mathfrak{so}(W) \oplus (\text{Im}(\mathbb{A}) \oplus V) \otimes W. \tag{23}$$

Then using this, we can identify  $\mathfrak{a}(\mathbb{A}, \mathbb{R})$  with  $\mathfrak{tri}(\mathbb{A})$ ; and more generally we can identify the Lie algebra in (22) with  $\mathfrak{a}(\mathbb{A}, V \oplus \mathbb{R})$ .

Note that we have inclusions  $\mathfrak{so}(\text{Im}(\mathbb{A}) \oplus W) \subset \mathfrak{a}(\mathbb{A}, W)$ .

The grading on  $\mathfrak{a}(\tilde{\mathbb{O}}, W)$  is

-2	-1	0	1	2
$\mathbb{R}$	$V \oplus U \oplus U \otimes W$	$\mathfrak{a}(\tilde{\mathbb{H}}, W) \oplus \mathbb{R}$	$V \oplus U \oplus U \otimes W$	$\mathbb{R}$

where  $V$  is the four-dimensional vector space in (15) where we have used the grading (17).

If we include the sextonions by taking the Lie algebra

$$\mathfrak{int}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \oplus \mathfrak{so}(V) \oplus \tilde{\mathbb{S}} \otimes V$$

then we get the intermediate algebra of  $\mathfrak{a}(\tilde{\mathbb{O}}, V)$ .

The grading on  $\mathfrak{t}(\tilde{\mathbb{O}}, \mathbb{B})$  is

-2	-1	0	1	2
$\mathbb{R}$	$V \oplus 2U \oplus U \otimes \mathbb{B}$	$\mathfrak{t}(\tilde{\mathbb{H}}, \mathbb{B}) \oplus \mathbb{R}$	$V \oplus 2U \oplus U \otimes \mathbb{B}$	$\mathbb{R}$

where we have used the grading (18).

If we include the sextonions by taking the Lie algebra

$$\mathfrak{tri}^{\tilde{\mathbb{S}}}(\tilde{\mathbb{O}}) \oplus \mathfrak{tri}(\mathbb{B}) \oplus \tilde{\mathbb{S}} \otimes \mathbb{B}$$

then we get the intermediate algebra of  $\mathfrak{t}(\tilde{\mathbb{O}}, \mathbb{B})$ .

The important representations of  $\mathfrak{a}(\mathbb{A}, W)$  are the vector representation whose restriction to  $\mathfrak{int}(\mathbb{A}) \oplus \mathfrak{so}(W)$  is the representation  $\text{Im}(\mathbb{A}) \oplus V$  and the spin representations whose restrictions to  $\mathfrak{int}(\mathbb{A}) \oplus \mathfrak{so}(W)$  are  $\mathbb{A} \otimes \Delta$  where  $\Delta$  is a spin representation of  $\mathfrak{so}(W)$ . These can be defined by considering the following push-forward diagram.

$$\begin{array}{ccc} \mathfrak{so}(W) & \longrightarrow & \mathfrak{so}(\text{Im}(\mathbb{A}) \oplus W) \\ \downarrow & & \downarrow \\ \mathfrak{int}(\mathbb{A}) \oplus \mathfrak{so}(W) & \longrightarrow & \mathfrak{a}(\mathbb{A}, W) \end{array}$$

These definitions also apply to the sextonions since  $\text{Im}(\tilde{\mathbb{S}}) \oplus W$  is the non-negative part of  $\text{Im}(\tilde{\mathbb{O}}) \oplus W$  and  $\tilde{\mathbb{S}} \otimes \Delta$  is the non-negative part of  $\tilde{\mathbb{O}} \otimes \Delta$ .

### 6.1. Bigradings

Here we generalise (19). Take an extremal triple in  $L(\mathbb{A}, \mathbb{O})$  with centraliser  $L(\mathbb{A}, \mathbb{H})$  and then take an extremal triple in  $L(\mathbb{A}, \mathbb{H})$ . These two triples commute and so we get a bigrading on  $L(\mathbb{A}, \mathbb{O})$ . Put  $m = \dim(\mathbb{A})$  then this bigrading is given by

		$\mathbb{C}$			
	$V_{m+4}$	$S_{4m}$	$V_{m+4}$		
$\mathbb{C}$	$S_{4m}$	$\mathfrak{a}(\mathbb{A}, W_5) \oplus \mathbb{C} \oplus \mathbb{C}$	$S_{4m}$	$\mathbb{C}$	(24)
	$V_{m+4}$	$S_{4m}$	$V_{m+4}$		

where  $S_{4m}$  is a spin representation of  $\mathfrak{a}(\mathbb{A}, W_5)$  of dimension  $4m$  and  $V_{m+4}$  is the vector representation of dimension  $m + 4$ . This constructs the Lie algebra  $L(\mathbb{A}, \mathbb{O})$  as

$$(\mathfrak{a}(\mathbb{A}, W_5) \oplus \mathfrak{sl}(A) \oplus \mathfrak{sl}(B)) \oplus (V_{m+4} \otimes A \otimes B) \oplus S_{4m} \otimes A \oplus S_{4m} \otimes B,$$

where  $A$  and  $B$  are two-dimensional vector spaces.

Also if we take the total grading in (24) we get the grading with components

-2	-1	0	1	2
$V_{m+6}$	$S_{8m}$	$\mathfrak{a}(\mathbb{A}, W_7) \oplus \mathbb{C}$	$S_{8m}$	$V_{m+6}$

where  $S_{8m}$  is a spin representation of  $\mathfrak{a}(\mathbb{A}, W_7)$  of dimension  $8m$  and  $V_{m+6}$  is the vector representation of dimension  $m + 6$ . The non-negative part of this grading gives a second maximal parabolic subgroup of  $L(\mathbb{A}, \mathbb{O})$ . The even part of this grading is isomorphic to  $\mathfrak{a}(\mathbb{A}, W_9) \cong \mathfrak{t}(\mathbb{A}, \mathbb{O})$ . The odd part is a spin representation of dimension  $16m$ . This is used in [1] to construct the Lie algebra  $L(\mathbb{A}, \mathbb{O})$ .

### 6.2. Dual reductive pairs

Here we show how the Lie algebras in Definition 6.1 give rise to dual reductive pairs in the exceptional Lie algebras. Adams [1] constructs the exceptional Lie algebras  $L(\mathbb{A}, \mathbb{O})$  as

$$L(\mathbb{A}, \mathbb{O}) \cong \mathfrak{a}(\mathbb{A}, W_9) \oplus \Delta_{16m}, \tag{25}$$

where  $m = \dim \mathbb{A}$  and  $\Delta_{16m}$  is a spin representation of dimension  $16m$ .

For  $0 \leq n \leq 4$  this representation of  $\mathfrak{a}(\mathbb{A}, W_9)$  can be restricted to  $\mathfrak{a}(\mathbb{A}, W_{2n}) \oplus \mathfrak{so}(Z_{9-2n})$ . Using (23) this gives the following decomposition of  $L(\mathbb{A}, \mathbb{O})$ :

$$\mathfrak{a}(\mathbb{A}, W_{2n}) \oplus \mathfrak{so}(Z_{9-2n}) \oplus ((\text{Im}(\mathbb{A}) \oplus W_{2n}) \otimes Z_{9-2n}) \oplus (\Delta_{m2^n} \otimes \Delta_{2^{4-n}}).$$

Here  $\Delta_{m2^n}$  is a spin representation of  $\mathfrak{a}(\mathbb{A}, W_{2n})$  of dimension  $m2^n$ . The restriction of this representation to a representation of  $\mathfrak{tri}(\mathbb{A}) \oplus \mathfrak{so}(W_{2n-1})$  is  $\mathbb{A} \otimes \Delta_{2^n}$ .

In particular, for  $n = 0$ , this gives the following construction of the exceptional series of Lie algebras from the intermediate algebras:

$$L(\mathbb{A}, \mathbb{O}) \cong \mathfrak{int}(\mathbb{A}) \oplus \mathfrak{so}(W_9) \oplus (\text{Im}(\mathbb{A}) \otimes W_9) \oplus (\mathbb{A} \otimes \Delta_{16}).$$

The Adams construction (25) and the decomposition (23) also give, for  $0 \leq n \leq 3$ , the following decomposition of  $L(\mathbb{A}, \mathbb{O})$ :

$$\begin{aligned} \mathfrak{a}(\mathbb{A}, W_{2n+1}) \oplus \mathfrak{so}(Z_{8-2n}) \oplus (\text{Im}(\mathbb{A}) \oplus W_{2n+1}) \otimes Z_{8-2n} \\ \oplus (\Delta_{m2^n}^+ \otimes \Delta_{2^{3-n}}^+) \oplus (\Delta_{m2^n}^- \otimes \Delta_{2^{3-n}}^-), \end{aligned}$$

where  $\Delta_{m2^n}^\pm$  are the spin representations of  $\mathfrak{a}(\mathbb{A}, W_{2n+1})$ . The restriction of these representations to  $\mathfrak{tri}(\mathbb{A}) \oplus \mathfrak{so}(W_{2n})$  are  $\mathbb{A} \otimes \Delta_{2^n}^\pm$ . For  $n = 0$  this gives the triality construction.

These two constructions show that, for  $0 \leq p \leq 8$ ,  $\mathfrak{a}(\mathbb{A}, W_p)$  and  $\mathfrak{so}(Z_{9-p})$  are a dual reductive pair in  $L(\mathbb{A}, \mathbb{O})$ . Some of these dual reductive pairs are constructed in [23]. Since these Lie algebras also form a sequence of subalgebras this gives a second magic triangle. We will not consider this second magic triangle. Instead we note that two of these can be inserted in the sequence of subalgebras giving the

original magic triangle as follows:

$$G_2 \rightarrow B_3 \rightarrow D_4 \rightarrow B_4 \rightarrow F_4.$$

In particular this suggests that from the point of view of the magic triangle the exceptional series should be further extended to include  $B_3$  and  $B_4$ . These two cases are not consistent with the numerology of the exceptional series.

From this point of view the magic triangle should be extended to include the inclusions

$$\mathrm{der}(\mathbb{A}) \rightarrow \mathrm{int}(\mathbb{A}) \rightarrow \mathrm{tri}(\mathbb{A}) \rightarrow \mathfrak{a}(\mathbb{A}, \mathbb{R}^2) \rightarrow L(\mathbb{A}, \mathbb{R}).$$

This sequence makes for  $\mathbb{A} = \widetilde{\mathbb{S}}$  and gives the Lie algebras intermediate between the sequence for  $\mathbb{A} = \widetilde{\mathbb{S}}$  and the sequence for  $\mathbb{A} = \widetilde{\mathbb{H}}$ .

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