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Robustness of equilibrium in the Kyle model of informed speculation

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Abstract

We analyze a static Kyle (1983) model in which a risk-neutral informed trader can use arbitrary (linear or non-linear) deterministic strategies, and a finite number of market makers can use arbitrary pricing rules. We establish a strong sense in which the linear Kyle equilibrium is *robust*: the first variation in any agent's expected payoff with respect to a small variation in his conjecture about the strategies of others vanishes at equilibrium. Thus, small errors in a market maker's beliefs about the informed speculator's trading strategy do not reduce his expected payoffs. Therefore, the original equilibrium strategies remain optimal and still constitute an equilibrium (neglecting the higher-order terms.) We also establish that if a non-linear equilibrium exists, then it is not robust.

JEL Classification Numbers: G12, G14, C62.
Keywords: Market microstructure, informed speculation, Bayesian Nash equilibrium, uniqueness, robustness, information.

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1 Introduction

A central feature of information-based models of financial markets is that each strategic market participant considers not only knowledge about asset fundamentals, but also makes tremendously sophisticated and accurate assessments of the strategies that other agents employ. Typically, researchers assume away all errors in those assessments. However, in complicated financial market speculation settings, it seems likely that agents's beliefs about the strategies of others may be slightly mis-specified. Concretely, a market maker may get an informed speculator's strategy slightly wrong.

In this paper, we investigate the robustness of the static Kyle (1983) model of strategic trading to errors of this form. The Kyle (1983) model generalizes the Kyle (1985) model to a setting with a finite number J of market makers who obtain finite expected profits in equilibrium. In principle, Kyle (1983) allows a monopolistically-informed trader to choose a possibly non-linear trading strategy, and market makers to simultaneously choose possibly non-linear supply schedules. He establishes that there is a unique equilibrium in which the trading strategy and pricing rules are linear. With many market makers, $J \to \infty$, Kyle shows that market makers submit competitive supply schedules, and that the pricing rule becomes informationally efficient, i.e., the model reduces to the static Kyle (1985) model. Note that, apart from a particular limiting case of Kyle (1985) model[1], the uniqueness of equilibrium for the Kyle (1983) model has not been established.

Our contribution is twofold. We first establish a remarkable robustness property of the linear Kyle equilibrium: the first variation of any agent's expected payoff with respect to a small variation of her *own* conjectures about the strategies of others vanishes in equilibrium. In particular, small errors in the beliefs of a market maker about the trading strategy of the informed speculator do not reduce the market maker's expected payoff, and the original equilibrium strategies (more precisely, their functional forms) still constitute an equilibrium. In fact, we show that each market participant is also indifferent to small errors in the beliefs that others hold. As slight errors in beliefs do not affect agents' expected profits and corresponding strategies, this means that the linear Nash equilibrium remains an equilibrium; and it follows that considering the limit $J \to \infty$, the standard linear equilibrium of Kyle (1985) model is robust. We next establish that if a non-linear Nash equilibrium to the Kyle (1983) model exists, then it is not similarly robust to slight mis-specifications in beliefs.

Our notion of robustness is even more demanding than that considered by other re-

[1] Boulatov, Kyle, and Livdan (2013) prove uniqueness of equilibrium for the Kyle (1985) model.

searchers, who typically only require that small perturbations of beliefs lead to ε-best responses and ε-equilibria for every nearby perturbation (see Stauber (2006, 2011) or Barelli (2009)). In our case, we require that ε-variations of beliefs may only lead to higher-order $O(\varepsilon^2)$ variations of equilibrium strategies and expected payoffs. Of course, our characterization solely pertains to the Kyle model.

2 The Model

We begin with a review of the Kyle (1983) model. A single risk neutral informed trader, privately observes an asset's liquidation value v, which is drawn from a normal distribution with mean zero and variance σ_0^2. Liquidity traders trade a quantity u, which is drawn independently from a normal distribution with mean zero and variance σ_u^2. After observing the liquidation value v, but not the level of noise trading u, the informed trader chooses a quantity x to trade. This "market order" x does not depend on the equilibrium price, but does depend on the observed liquidation value v. The informed trader's strategy is thus a function $X(\cdot)$ that details for each value of v, the traded quantity $x = X(v)$.

The informed trader and liquidity traders trade in a market with $J \geq 3$ risk-neutral, profit-maximizing market makers. Market makers know the joint distribution of v and u, but do not see either realization. Each market maker $k = 1, \dots J$ submits a limit order described by a non-discriminatory supply schedule $y_k(P)$ that details for each price the quantity it will supply. The equilibrium price clears the market, $Y(P) = \sum_{k=1}^{J} y_k(P) = x + u$: the *pricing rule* $P(\cdot)$ is defined as an inverse function[2] of the aggregate demand, i.e., $P(Y(\xi)) = \xi$, for $\xi \in \mathbb{R}$.

A symmetric Nash equilibrium consists of a trading strategy $X^*(\cdot)$ and a supply schedule $Y^*(\cdot)$ that are mutual best responses, i.e., are profit maximizing for the insider and market makers. Kyle solves for the following linear equilibrium trading strategy and pricing rule:

$$X^*(v) = \beta_* v, \quad \text{and} \quad Y^*(P) = (\Lambda_*)^{-1} P, \tag{1}$$

with

$$\beta_* = \left(\frac{J-2}{J-1}\right)\frac{\sigma_u}{\sigma_v}, \qquad \Lambda_* = \frac{\sigma_v}{2\sigma_u}\left(\frac{J-1}{J-2}\right), \tag{2}$$

which we refer to as the *standard linear solution*. From (1), it follows that the pricing rule $P^*(\cdot)$ is linear and given by

$$P^*(Y) = \Lambda_* Y. \tag{3}$$

[2] We will show that $Y(\cdot)$ is monotonic and therefore invertible.

Note that (1) and (3) reduce to the linear solutions of Kyle (1985) in the limit as $J \to \infty$. In what follows, we normalize both σ_u and σ_v to one, so that the parameters of standard linear solution (1) take the form

$$\beta_* = \left(\frac{J-2}{J-1}\right), \quad \text{and} \quad \Lambda_* = \frac{1}{2}\left(\frac{J-1}{J-2}\right). \tag{4}$$

We focus on *symmetric* Nash equilibria. To examine non-linear trading strategies, it is useful to develop notation that describes the reaction functions of agents to possibly non-linear trading strategies of the others. The notation $y_k(P; X_{M,k}(\cdot))$ indicates that the supply schedule of market maker k depends on both a scalar argument given by the execution price P and the function argument given by market maker k's conjecture about the insider's demand function $X_{M,k}(\cdot)$. Given our focus on symmetric equilibria, beliefs are the same: $X_{M,k}(\cdot) \equiv X_M(\cdot)$, $k = 1, ..J$.

In what follows, we make extensive use of functionals, i.e., functions mapping both scalars and other functions into scalars. To keep notation clear, we follow the above example by placing scalar arguments in front of functional arguments, separating the two types of arguments by a semi-colon, and using a dot to indicate function arguments.

2.1 Parametric example

As a preview, to convey the nature of our finding, we consider a *linear parametric* example. We take the model formulated above, and suppose that both the informed trader and market makers follow linear strategies

$$X(v) = \beta_* v, \quad \text{and} \quad Y(P) = (\Lambda_*)^{-1} P, \tag{5}$$

characterized by the coefficients β_* and Λ_* representing an informed trader's trading intensity and the inverse market depth, respectively. In the symmetric BNE described by Kyle (1983), the conjectures that all agents make about each other's coefficients are correct. We now derive what happens to the payoffs of the market makers and informed trader when their conjectures are slightly wrong, but the conjectures still retain a linear structure.

In particular, we suppose that all J market makers have the same *linear* conjecture $X(v) = \beta_c v$ about the informed agent's trading strategy, a conjecture that can be wrong. Analogously, we suppose that the informed agent's *linear* conjecture about Λ is Λ_c.

Proposition 1 1. If market makers conjecture that the informed trader's trading intensity is β_c, when the trader's actual intensity is $\beta = \beta_*$, the expected payoff of each market maker is

$$\overline{\pi}_M = A\frac{\beta_c}{\beta_c^2 + \beta_*^2}, \tag{6}$$

with $A = \frac{E\left[(\beta_* v + u)^2\right]}{J(J-2)} = \frac{1+\left(\frac{J-2}{J-1}\right)^2}{J(J-2)}$. Importantly, $\bar{\pi}_M$ is a smooth function of β_c and achieves its maximum when the conjecture is correct, $\beta_c = \beta_*$.

2. If the informed trader's conjecture about the inverse market depth is Λ_c when the actual market maker strategy is Λ_*, then the informed trader's expected profit is

$$\overline{\pi}_I = \frac{1}{2\Lambda_*}\xi\left(1 - \frac{\xi}{2}\right), \tag{7}$$

where $\xi = \frac{\Lambda_*}{\Lambda_c}$. Further, $\bar{\pi}_I$ is a smooth function of ξ and therefore of Λ_c, and achieves its maximum when $\xi = 1$, i.e., when the conjecture is correct, $\Lambda_c = \Lambda_*$.

Proof: See the Appendix.

In fact, the expected payoffs (6) and (7) have an interesting *scaling property* with respect to the conjecture errors. With loss of generality, we can *scale* a market makers' conjecture in terms of the true trading intensity β_* as $\beta_c = a\beta_*$, where there is no conjecture error when $a = 1$. Then, it follows from (6) and (7), that the percentage reduction in the market maker's expected profit from a conjecture that deviates from the correct conjecture of β_* to $a\beta_*$ is

$$\epsilon_M = \frac{\overline{\pi}_M^* - \overline{\pi}_M}{\overline{\pi}_M^*} = 1 - \frac{2a}{a^2+1} = \frac{(1-a)^2}{1+a^2}, \tag{8}$$

which does not depend on the level of expected profits. Analogously, when the informed trader's conjecture about pricing is scaled as $\Lambda_c = b\Lambda_*$, the percentage reduction in his profits is

$$\epsilon_I = \frac{\overline{\pi}_I^* - \overline{\pi}_I}{\overline{\pi}_I^*} = 1 - \frac{2}{b}\left(1 - \frac{1}{2b}\right) = \frac{(1-b)^2}{b^2}. \tag{9}$$

Clearly, $\epsilon_M = \epsilon_I = 0$ when $a = b = 1$—there is no drop of expected profits in equilibrium when the conjectues are correct.

The Kyle (1985) model can be viewed as a limiting case of Kyle (1983) as $J \to \infty$ (in which case $\beta_* \to 1$ and $\Lambda_* \to \frac{1}{2}$). Thus, the above result holds for the classical Kyle (1985) model. The proposition reveals that the expected profit $\bar{\pi}_M$ of a market maker and $\bar{\pi}_I$ of the informed trader both achieve their global maxima when their conjectures are correct, i.e., at the standard linear BNE as described by Kyle (1983). Since their expected profits are differentiable, it follows that the first derivatives of their expected payoffs with respect to their conjectures are zero at equilibrium when the conjectures are correct—thus, small errors have no impact on their profits. This property does **not** follow from the first-order conditions, because when agents determine their optimal strategies in a BNE, their conjectures are taken as given and are not part of the optimization. That is, agents do **not** optimize with respect to their conjectures in a BNE. Therefore, it does not follow immediately that the derivative of their expected profits with respect to their conjectures should be zero. In fact, we will show that this is **not true** if the equilibrium around which the agents deviate with different conjectures is **nonlinear**.

In principle, this "insensitivity" or "robustness" of payoffs with respect to small conjecture errors makes a case for linear equilibria in Kyle (1983), even if nonlinear equilibria exist. In a broader context, it may also partially explain why, despite the fact that many behavioral biases have been discovered in the lab, fewer have been identified in the market. Intuitively, one may expect that there may be multiple equilibria, but that robust equilibria should be more likely to survive under small perturbations caused by behavioral biases, and these are the equilibria that are practically not affected by the biases.

When we restrict attention to linear strategies and linear conjectures, we obtain explicit solutions for expected payoffs, regardless of the sizes of the conjecture errors. One can then address: what is a reasonable definition of "small" conjecture errors? From (6) and (7), it becomes clear that if the errors in conjectures are small relative to the true conjectures, they do not affect agents' expected payoffs. One can therefore measure conjecture errors as fractions of the equilibrium strategies. That is, conjecture errors are small if they are small relative to the equilibrium strategies.

We will show that, quite generally, conjecture errors are small if they are small relative to the equilibrium strategies even when we do not require that agents only play linear strategies, and errors in conjectures need not come from a linear family. In particular, this means that regardless of the source of a "small deviation" from the correct conjecture—limited reasoning capacity, over-confidence, and so on—which can each result in distinct, small non-linear errors, the consequences for payoffs are negligible when equilibrium strategies are linear.

2.2 Insider optimization

We consider the Nash equilibrium of Kyle (1983). When the insider observes the realization v, conjectures that market makers adopt strategy $y_I(\cdot)$ with a pricing rule $P_I(\cdot)$ and trades the quantity x, then the insider's expected payoff, $\Pi_I(v, x; P_I(\cdot))$, is

$$\begin{aligned}\Pi_I(v, x; P_I(\cdot)) &= E_u[(v - P_I(x+u))x] \\ &= (v - \overline{P}_I(x))x,\end{aligned} \tag{10}$$

where the expected price $\overline{P}_I(x)$ is defined as

$$\overline{P}_I(x) = E_u[P_I(x+u)]. \tag{11}$$

Define the insider's reaction functional to her conjecture about the market maker's strategy $y_I(\cdot)$ by

$$R_I(v; y_I(\cdot)) = \arg\max_x \ \Pi_I(v, x, P_I(\cdot)). \tag{12}$$

Making use of the above definitions, we obtain the following first-order condition for the informed trader's profit-maximization problem:

Proposition 2 The first-order condition describing the insider's strategy is

$$v = \overline{P}_I(X(v)) + X(v)\overline{P}'_I(X(v)), \tag{13}$$

which must hold pointwise for each v.

Proof: Evaluating the first variation of the payoff (10) with respect to x, yields (13).

The insider's first-order condition (13) has a simple economic interpretation: The marginal value from one more share on the left-hand side is equated to the marginal cost of increasing that position on the right-hand side.

We have the following auxiliary result:

Corollary 1 *The insiders' reaction $R_I(v; y_I(\cdot))$ is monotonically increasing in v.*

Proof: See the Appendix.

Since the reaction functional $R_I(\cdot\,; y(\cdot))$ is monotonic in its first argument, it is invertible. Therefore, we can introduce an inverse reaction functional $V_I(x; y(\cdot))$ such that $V_I(R_I(v; y(\cdot)); y(\cdot)) \equiv v$ and rewrite the first-order condition for the insider, equation (13), as

$$\begin{aligned}V_I(x; y_I(\cdot)) &= \overline{P}_I(x) + x\overline{P}'_I(x) \\ &= \frac{\partial}{\partial x}\left(x\overline{P}_I(x)\right),\end{aligned} \tag{14}$$

which explicitly relates the insider's reaction to the conjectured functional form of the expected pricing rule $\overline{P}_I(\cdot)$.

2.3 Market maker optimization

Suppose that when market maker k conjectures that all other market makers $j \neq k$ submit the schedule $y_{M_k}(\cdot)$ that he supplies $y_k(\cdot)$. For each realization of v and u (and, therefore, the market clearing price P), the total liquidity supply from market makers of $Y = y_k(P) + (J-1)y_{M_k}(P)$ must match the total or net demand of $X(v) + u$. This, in turn, determines the market-clearing price P as a function of y_k:

$$y_k(P) + (J-1)y_{M_k}(P) = X(v) + u = Y. \tag{15}$$

We denote market maker k's expected payoff given his conjectures $y_{M_k}(\cdot)$ and $X_{M,k}(\cdot)$ by:

$$\begin{aligned} \overline{\Pi}_{M,k}\left(y_k(\cdot), y_{M_k}(\cdot), X_{M,k}(\cdot)\right) &= E_{v,u}[y_k(P)(P-v)|y_{M_k}(\cdot), X_{M,k}(\cdot)] \\ &= E_{v,P}[y_k(P)(P-v)|y_{M_k}(\cdot), X_{M,k}(\cdot)] \\ &= E_{v,P}\left[y_k(P)\left(P - P_e\left(P; y_{M_k}(\cdot), X_{M,k}(\cdot)\right)\right)\right], \end{aligned} \tag{16}$$

where $P_e\left(P; y_{M_k}(\cdot), X_{M,k}(\cdot)\right)$ is an informationally-efficient price from the perspective of market maker k:

$$P_e\left(P; y_{M_k}(\cdot), X_{M,k}(\cdot)\right) = E\left[v|P; y_{M_k}(\cdot), X_{M,k}(\cdot)\right]. \tag{17}$$

Note that since the total demand $Y = X(v) + u$ defines the market clearing price by the market clearing condition (15), the informationally-efficient price (17) can be also transformed into a function of the total order flow $\widehat{P}_e\left(Y; X_{M,k}(\cdot)\right) = E\left[v|Y; X_{M,k}(\cdot)\right]$, in which case there remains only functional dependence on the insider's conjectured strategy $X_{M,k}(\cdot)$, but not on the other market makers' strategies. This is because the total order flow $Y = X(v) + u$ contains all information on the realization of the fundamental v available to the market makers. Formally, as it follows from (15), the transition from P to Y already involves the market maker k's conjecture $y_{M_k}(\cdot)$ of the market makers' $j \neq k$ symmetric equilibrium strategies.

Since the expected payoff (16) has to be optimized point-by-point for each realization of the market clearing price P, and since the market maker k submits a schedule $y_k(\cdot)$ which is defined for each realization of P, the optimization problem is equivalent to a point-by-point optimization for each P. Taking into account this and the expected payoff functional (16), we obtain a characterization of the equivalent point-by-point optimization problem, and then derive the first-order condition describing the strategy of market maker $k = 1, ..J$ in a symmetric Nash equilibrium:

Proposition 3 1.The market maker k's problem is equivalent to a point-by-point optimization, for each realization of the price P, with the target functional

$$\begin{aligned} \Pi_{M,k}\left(y_k, P; y_{M_k}\left(\cdot\right), X_{M,k}\left(\cdot\right)\right) &= E_v[y_k(P-v)] \\ &= y_k(P - P_e\left(P; y_{M_k}\left(\cdot\right), X_{M,k}\left(\cdot\right)\right)). \end{aligned} \tag{18}$$

2.The first variation for market maker k's problem is

$$\begin{aligned} 0 &= \delta_{y_k}\Pi_{M,k}\left(y_k, P; y_{M_k}\left(\cdot\right), X_{M,k}\left(\cdot\right)\right) \\ &= \left\{P - P_e\left(P; y_{M_k}\left(\cdot\right), X_{M,k}\left(\cdot\right)\right)\right\}\delta y_k + y_k \delta P, \end{aligned} \tag{19}$$

and the first-order condition describing her strategy is

$$y_k\left(P\right) = \left(J-1\right) y'_{M_k}\left(P\right)\left(P - P_e\left(P; y_{M_k}\left(\cdot\right), X_{M,k}\left(\cdot\right)\right)\right). \tag{20}$$

Proof: Suppose that market maker k supplies y_k. Evaluating the first variation of the payoff functional (18) with respect to y_k, yields (19). Making use of the market-clearing condition (15) and noting that total demand does not change as a result of the deviation, we obtain

$$\delta y_k + (J-1) y'_{M_k}\left(P\right)\delta P = \delta Y = 0, \tag{21}$$

and therefore

$$\delta P = -\frac{1}{\left(J-1\right)}\frac{\delta y_k}{y'_{M_k}\left(P\right)}. \tag{22}$$

This means that the variation of the market-clearing price only depends on the variation of the strategy of market maker k and the functional form of the conjectured strategy $y_{M_k}\left(\cdot\right)$. Substituting (22) into (19) yields (20).

The FOC (20) implicitly defines the reaction function of market maker k,

$$R_{M_k}\left(P; y_{M_k}\left(\cdot\right), X_{M,k}\left(\cdot\right)\right) = \left(J-1\right) y'_{M_k}\left(P\right)\left(P - P_e\left(Y; X_{M,k}\left(\cdot\right)\right)\right), \tag{23}$$

to his conjectures $y_{M_k}\left(\cdot\right)$ and $X_{M,k}\left(\cdot\right)$ about the strategies of other agents.

We next establish that in a symmetric Bayesian Nash equilibrium, market maker strategies are monotonic in price. First, suppose that this is indeed the case. Then, it follows that the aggregate supply is invertible. That is, (20) yields:

$$P - \frac{1}{J-1}\frac{y_k}{y'_{M_k}\left(P\right)} = P_e\left(Y; X_{M,k}\left(\cdot\right)\right). \tag{24}$$

Introducing the pricing rule reaction $R_{MP}(Y; X_M(\cdot))$ as an inverse of the reaction function and using the fact that in a symmetric equilibrium, $y_{M_k}(P) \equiv y(P) = \frac{1}{J}Y$, equation (24) yields

$$R_{MP}(Y; X_M(\cdot)) - \frac{1}{J-1} Y R'_{MP}(Y; X_M(\cdot)) = P_e(Y; X_M(\cdot)). \tag{25}$$

Define $D(Y; X_M(\cdot)) \equiv \frac{\partial}{\partial y} R_{MP}(Y; X_M(\cdot))$ and $D_e(Y; X_M(\cdot)) \equiv \frac{\partial}{\partial y} P_e(Y; X_M(\cdot))$. Differentiating (25) with respect to Y yields

$$D(Y; X_M(\cdot)) - \frac{1}{J-2} Y D'(Y; X_M(\cdot)) = \frac{J-1}{J-2} D_e(Y; X_M(\cdot)), \tag{26}$$

which has an explicit solution

$$D(Y; X_M(\cdot)) = (J-1) Y^{J-2} \int_Y^{+\infty} d\xi \xi^{1-J} Q_e(\xi; X_M(\cdot)). \tag{27}$$

We show in the Appendix that the equilibrium informationally efficient pricing rule $P_e(\cdot; X_M(\cdot))$ is monotonically increasing. Therefore, $D_e(\xi; X_M(\cdot)) \geq 0, \quad \xi \in R$. Then, it follows from (27), that $D(Y; X_M(\cdot)) \geq 0$, or equivalently $\frac{\partial}{\partial y} R_{MP}(Y; X_M(\cdot)) \geq 0$.

Summarizing, we have the following result:

Corollary 2 *The equilibrium market makers' strategies $y^*(\cdot)$ are monotonic, and the equilibrium aggregate supply $Y^*(\cdot)$ is invertible.*

In the competitive limit $J \to \infty$, the second term on the left-hand side in (25) vanishes and the price becomes informationally efficient, which means that the model reduces to Kyle (1985). Since the equilibrium pricing rule is monotonically increasing in its first argument, the pricing rule is steeper than the informationally efficient one for any finite number of market makers J. Solving the ODE (25), yields

$$R_{MP}(Y; X_M(\cdot)) = (J-1) Y^{J-1} \int_Y^{+\infty} d\xi \xi^{-J} P_e(\xi; X_M(\cdot)), \tag{28}$$

which is an explicit characterization of the pricing rule in terms of the market makers' conjecture about the insider's strategy $X_M(\cdot)$ for any finite number of market makers $J \geq 3$. Note that with linear strategies, $P_e(Y; X_M(\cdot)) = \frac{1}{2}Y$, i.e., we reproduce the standard result for the equilibrium pricing rule (4).

3 Nash Equilibrium

The Nash equilibrium strategies of the insider and market makers, denoted $X^*(\cdot)$ and $y^*(\cdot)$ are defined by fixed-point conditions that can be expressed in terms of the reaction functionals $R_I(\cdot\,; y_I(\cdot))$ and $R_M(\cdot\,; y_M(\cdot), X_M(\cdot))$ as

$$\begin{aligned} X^*(\cdot) &= R_I(\cdot\,; y^*(\cdot)), \\ y^*(\cdot) &= R_M(\cdot\,; y^*(\cdot), X^*(\cdot)). \end{aligned} \tag{29}$$

The first condition states that in a Nash equilibrium, given the equilibrium strategy of market makers, the informed trader chooses the trading rule that market makers believe she is going to follow[3]. So, too, the optimal trading strategy of each market maker is the one that she conjectures about the other market makers given their equilibrium beliefs about the insider's strategy. In other words, all agents' conjectures turn out to be correct at equilibrium and represent the optimal reactions of each agent to the strategies of the other agents.

Using (14) and (28), the conditions in (29) can be explicitly expressed in terms of the inverse insider's strategy $V^*(x)$ and the expected pricing rule $\overline{P}^*(x; X^*(\cdot))$ as

$$\begin{aligned} V^*(x) &= \frac{\partial}{\partial x}\left(x\overline{P}^*(x; X^*(\cdot))\right), \\ \overline{P}^*(x; X^*(\cdot)) &= (J-1)\,E_u\left[Y^{J-1}\int_Y^{+\infty} d\xi\xi^{-J}P_e(\xi; X^*(\cdot))\right]. \end{aligned} \tag{30}$$

In the limit $J \to \infty$, the second condition yields $\overline{P}^*(x; X^*(\cdot)) = E_u[P_e(Y; X^*(\cdot))]$ and (30) reduces to the definition of Nash equilibrium in Kyle (1985)[4].

4 Robustness and Linearity

We now analyze the robustness of the Kyle (1983) model of strategic trading. To describe the sensitivity of expected payoffs to variations in the conjectured insider's strategy, we use the notion of *functional differentiation* commonly used in functional analysis.

Definition 1 The functional differential of the functional $F(x; X(\cdot))$ with respect to the strategy $X(\cdot)$ is

$$\delta_X F(x; X(\cdot), \delta X(\cdot)) = \lim_{\varepsilon\to 0}\left\{\frac{F(x; X(\cdot)+\varepsilon\delta X(\cdot)) - F(x; X(\cdot))}{\varepsilon}\right\}, \tag{31}$$

[3]Although our reaction-function notation emphasizes the choice of the function $X(\cdot)$, the condition (29) leads to a definition of Nash equilibrium logically equivalent to that in Kyle (1983) and Kyle (1985). The two definitions are equivalent since the informed trader's optimization problem decomposes into separate state-by-state optimization problems for each realization of v.

[4]This follows from the following representation of the Dirac's delta function, $\delta(\cdot)$:

$$\lim_{J\to\infty}(J-1)\,Y^{J-1}(Y+z)^{-J} = \delta(z).$$

provided that the limit *(31)* exists for every $\delta X(\cdot)$ (from the same functional space), and that it defines a functional, linear and bounded in $\delta X(\cdot)$.

The above definition corresponds to the *Gateaux differential* (see, e.g., Kolmogorov and Fomin (1999)). Note that (31) can be viewed as an extension of the *directional derivative* of functions depending on several variables, familiar from standard calculus, to the case when some arguments can be functions. If the scalar variable x changes as a result of changing the conjecture $X(\cdot)$, then we consider the *full differential* of the functional $F(x; X(\cdot))$ defined as

$$\begin{aligned} \Delta_X F(x; X(\cdot), \delta X(\cdot)) &= \frac{\partial}{\partial x} F(x; X(\cdot))\, dx + \delta_X F(x; X(\cdot), \delta X(\cdot)) \\ &= d_X F(x, dx; X(\cdot)) + \delta_X F(x; X(\cdot), \delta X(\cdot)). \end{aligned} \tag{32}$$

In what follows, the scalar argument of an agent's payoff usually corresponds to his own trading strategy and to market variables such as price P, while the functional arguments are the agent's conjectures about the strategies of others. If the scalar variable x changes as a result of changing an agent's conjectures, one needs to use the notion of a full differential.[5]

We focus on a particularly demanding notion of robustness of equilibrium payoffs to small errors that agents make in their conjectures about the strategies that other agents adopt:

Definition 2 An equilibrium is **robust** if and only if the full differential of an agent's k expected payoffs $\Pi_k(x_k; X_k(\cdot))$ with respect to his or other agents' conjectures about the strategies of others vanishes at equilibrium:

$$\begin{aligned} \Delta_{X_k} \Pi_k(x_k; X_k(\cdot)) &= 0, \\ \Delta_{X_k} \Pi_j(x_j; X_j(\cdot)) &= 0, \quad j \neq k. \end{aligned} \tag{33}$$

We set a demanding criterion for an equilibrium to be robust. It requires that all market participants be indifferent to small errors in agents' beliefs about what others will do: not only must each market participant's payoffs be unaffected by small errors in his or her own beliefs, but they must also be unaffected by small errors in the beliefs that others hold.

The above definition leads to the following important observation. Since the First Order Condition (FOC) is satisfied in the non-distorted equilibrium when all conjectures are correct, and the distortions of strategies are small for small conjecture errors, these distortions

[5] Since the standard notion of a differential is a particular case of the functional one, we sometimes use short hand notation, by analogy with the standard notation of the full derivative $\Delta_X F(x; X(\cdot), \delta X(\cdot)) = \delta_X F(x; X(\cdot), \delta X(\cdot))$, having in mind that scalar arguments x may also be functionals of the conjectures $X(\cdot)$.

of agents' strategies do not affect their expected payoffs in the first-order with respect to the magnitude of the conjecture errors. Therefore, we have the following:

Corollary 3 *If the equilibrium is robust, the equilibrium strategies remain optimal and still constitute an equilibrium, neglecting the higher-order terms with respect to the conjecture errors.*

Proof: See Appendix.

4.1 Robustness

The Nash equilibrium in Kyle (1983) model is constructed as follows. Before observing the realization of the fundamental v, the insider defines his optimal reaction $R_I(\cdot\ ; X_M(\cdot))$ to the market makers' conjecture $X_M(\cdot)$ of her true trading strategy in order to maximize expected payoffs. Simultaneously, each market maker $k = 1, ..J$ makes a conjecture $X_{M_k}(\cdot) \equiv X_M(\cdot)$, of the true strategy $X(\cdot)$ and submits her schedule $y_k(P; y_{M_k}(\cdot), X_{M_k}(\cdot)) \equiv y(P; X_M(\cdot))$ which is a function of the market-clearing price and a functional of the conjecture. The pricing rule is defined by inversion of $y(P\ ; X_M(\cdot))$ and can be viewed as a reaction function to total order flow. In the competitive limit considered in Kyle (1985), market makers submit informationally-efficient pricing rules and earn zero profits at equilibrium.

By definition, equilibrium requires that $X_M(\cdot) = X^*(\cdot)$, i.e., each market maker's conjecture about the insider's strategy is correct. Now, suppose that we found the Nash equilibrium described above and consider a small variation in the conjecture of market maker k:

$$X_{M_k}(\cdot) = X^*(\cdot) + \delta X_{M_k}(\cdot). \tag{34}$$

Economically, this means that the conjecture of one market maker is "slightly off", i.e., there is a small deviation from the constructed Nash equilibrium. As a result of the variation of beliefs, the trading strategy $y_k(P; y_{M_k}(\cdot), X_{M_k}(\cdot))$ shifts by $\delta y_k(P; y_{M_k}(\cdot), X_{M_k}(\cdot))$, which leads to a shift of the market-clearing price δP according to (22). In addition, the market maker's estimate of the informationally efficient price also shifts, which also affects her expected payoffs.

Since market maker k's conjecture turns out to be wrong and the insider actually does not deviate, the total order flow Y remains the same as in the original Nash equilibrium, and the variation in market maker's k supply δy_k is completely absorbed by the adjustment of aggregate supply due to the shift of the market-clearing price P. Because the FOC in the original Nash equilibrium is satisfied, the effects of demand and price shifts exactly offset each other and hence do not affect market maker k's expected payoff. Therefore, a market maker's expected payoffs may only change due to the shift of the estimated informationally efficient price in her information set.

Importantly, the expected payoffs of other market makers $j \neq k$, may change as a result of shifting the market clearing price P, and this effect does not vanish due to the FOC. What we show is that the expected payoffs of all agents (including the insider) remain the same if and only if the original Nash equilibrium is linear.

Our main results are summarized by the following:

Theorem 1 *1.* The standard linear equilibrium of Kyle (1983) is robust with respect to small conjecture errors of the market makers or the informed trader.

2. The only equilibrium of Kyle (1983) that is robust in the sense of Definition 2 is the standard linear equilibrium.

Proof: See Appendix.

The proof proceeds in two steps. First, we prove that the equilibrium expected payoffs of a market maker do not change as a result of a small variation of her own conjecture only if the equilibrium in question is linear. That is, we prove that linearity is a necessary condition for robustness, i.e., no non-linear equilibrium can be robust. We then show that the standard linear equilibrium is indeed robust with respect to small deviations in any agent's conjectures.

Because the Kyle (1985) model can be viewed as a limiting case of Kyle (1983) with $J \to \infty$, the above results apply to Kyle (1985). One may argue that the variation (53) vanishes in the limit $J \to \infty$ and therefore the proposed robustness condition may not be relevant as $J \to \infty$. However, this concern is misplaced. First, (53) is finite for any finite J. One can view Kyle (1985) as the continuous limiting case of Kyle (1983) as J grows infinitely large. Second, we may consider the situation in which a finite fraction $q < 1$ of market makers deviates from the equilibrium conjecture with the same $\delta X_{M,k}(\cdot)$. In this case, all results hold with the change that the factor $\frac{1}{J}$ in the right-hand side of the last expression in (53) is replaced with q, which remains finite in the limit $J \to \infty$.

5 Conclusion

We establish a very strong sense in which the standard linear Nash equilibrium of the Kyle (1983, 1985) model is robust. We say that a Nash equilibrium *robust* if the first variations of each agent's expected payoff with respect to small variations in their conjectures about the strategies of others vanishes at equilibrium. We prove that each market participant is indifferent to small errors in his or her own beliefs and to small errors in the beliefs that others hold. Further, the only robust Nash equilibrium of Kyle (1983) model is the standard linear one.

The notion of robustness that we establish is a particularly appealing one, as action spaces are continuous, and the strategic interactions in this financial market speculation game are especially complex, rendering it implausible that market makers fully understand the nature of the trading strategy that the speculator adopts. Fortunately, we establish that the equilibrium is unaffected when the conjectures that market makers make are slightly wrong.

6 Bibliography

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7 Appendix

Proof of Proposition 1. 1. First, it follows immediately that, when the market makers' strategies are symmetric, the inverse market depth is

$$\Lambda = \Lambda_e \left(\frac{J-1}{J-2} \right), \tag{35}$$

where Λ_e is an informationally efficient market depth given by

$$\Lambda_e = \frac{\beta_c}{\beta_c^2 + \beta^2}. \tag{36}$$

Define the total order flow $Y = \beta_* v + u$. Then the expected payoff for each market maker in a symmetric linear setting is

$$\begin{aligned} \overline{\pi}_M &= E\left[\frac{1}{J} Y \left(\Lambda - \Lambda_e \right) Y \right] = \\ &= \frac{1}{J(J-2)} E\left[Y^2 \right] \Lambda_e = A \frac{\beta_c}{\beta_c^2 + \beta^2}. \quad Q.E.D. \end{aligned} \tag{37}$$

2. Suppose the informed trader conjectures that the market depth is Λ_c when the actual depth is Λ_*. Then, as in Kyle (1983), the informed constructs his own strategy $\beta = (2\Lambda_c)^{-1}$ as an optimal reaction to *conjectured* market makers' strategies. Then the expected payoff of informed trader is

$$\begin{aligned} \overline{\pi}_I &= E\left[\beta v \left(v - \Lambda_* \beta v \right) \right] = \beta \left(1 - \Lambda_* \beta \right) \\ &= \frac{1}{2\Lambda_c} \left(1 - \frac{\Lambda_*}{2\Lambda_c} \right) = \frac{1}{2\Lambda_*} \xi \left(1 - \frac{\xi}{2} \right). \quad Q.E.D. \end{aligned} \tag{38}$$

Proof of Corollary 1. The insider's payoff (10) takes the form

$$\Pi_I \left(v, x; X_c(\cdot) \right) = x \left(v - \overline{P} \left(x; X_c(\cdot) \right) \right). \tag{39}$$

Therefore for a given insider's strategy $X(\cdot)$,

$$\Pi_I \left(v, X(v); X_c(\cdot) \right) = X(v) \left(v - \overline{P} \left(X(v); X_c(\cdot) \right) \right). \tag{40}$$

Now suppose that $X(\cdot)$ corresponds to the insider's reaction functional and therefore according to Proposition 1 it optimizes (40) for each point v. Then, for any two points $v_1 \geq v_2$

$$X(v_1) \left(v_1 - \overline{P} \left(X(v_1); X_c(\cdot) \right) \right) \geq X(v_2) \left(v_1 - \overline{P} \left(X(v_2); X_c(\cdot) \right) \right), \tag{41}$$

and

$$X(v_2)\left(v_2 - \overline{P}(X(v_2); X_c(\cdot))\right) \geq X(v_1)\left(v_2 - \overline{P}(X(v_1); X_c(\cdot))\right). \tag{42}$$

Subtracting the r.h.s. of (42) from (41) and comparing that to the difference between the r.h.s. of (41) and the l.h.s. of (42), we obtain

$$X(v_1)(v_1 - v_2) \geq X(v_2)(v_1 - v_2),$$

and therefore

$$(X(v_1) - X(v_2))(v_1 - v_2) \geq 0, \tag{43}$$

which means that $X(v_1) \geq X(v_2)$ if $v_1 \geq v_2$. Therefore, $X(\cdot)$ is monotonically increasing if it solves the insider's optimization problem. In particular, this is satisfied for the insider's reaction functional $R_I(\cdot\,; Y(\cdot))$ for any admissible conjecture $Y(\cdot)$. *Q.E.D.*

Proof of Corollary 2. We only need to prove that the equilibrium informationally efficient pricing rule $P_e(\cdot; X_M(\cdot))$ is monotonically increasing. The informationally efficient pricing rule is given by

$$P_e(Y; X_M(\cdot)) = E_{v|Y}[v]. \tag{44}$$

Introduce the marginal p.d.f. $Z_P(Y; X(\cdot))$ as

$$Z_P(Y; X(\cdot)) \equiv \int dv' e^{-\frac{(v')^2}{2}} e^{-\frac{(Y - X(v'))^2}{2}}. \tag{45}$$

At a symmetric equilibrium, $P_e(Y; X_{M,k}(\cdot)) \equiv P_e(Y; X_M^*(\cdot))$ and $Z_P(Y; X(\cdot)) \equiv Z_P(Y; X^*(\cdot))$. Since the optimal insider's strategy is fixed, we use the short-hand notation $P_e(Y; X_M^*(\cdot)) = P_e(Y)$ and $Z_P(Y; X^*(\cdot)) = Z_P(Y)$. We have

$$P_e(Y) = \frac{\int dv' e^{-\frac{(v')^2}{2}} e^{-\frac{(Y - X_M(v'))^2}{2}} v'}{Z_P(Y)}. \tag{46}$$

Differentiation with respect to Y and making use of (45) yields

$$\begin{aligned} P_e'(Y) &= \frac{\int dv' e^{-\frac{(v')^2}{2}} e^{-\frac{(Y - X_M(v'))^2}{2}} (v' - P_e(Y))(X_M(v') - Y)}{Z_P(Y)} \\ &= Cov_{v|Y}[v, X_M(v)] \geq 0, \end{aligned}$$

where the last inequality follows in equilibrium from the results of Corollary 1, which says that the equilibrium insider's strategies are monotonically increasing. *Q.E.D.*

Proof of Corollary 3. The intuitive construction of the proof is described in the main text. Consider the full differential of the expected payoff functional for agent k. We have

$$\Delta_{X_k}\Pi_k\left(x_k; X_k\left(\cdot\right)\right) = d_{x_k}\Pi_k\left(x_k; X_k\left(\cdot\right)\right) + \delta_{X_k}\Pi_k\left(x_k; X_k\left(\cdot\right)\right) = 0, \tag{47}$$

where the strategy distortion δx_k occurs due to the distortion of the conjectures $\delta X_k\left(\cdot\right)$. For the small distortions $\delta X_k\left(\cdot\right)$, the strategy distortions δx_k are linear functionals of $\delta X_k\left(\cdot\right)$, and therefore their magnitude is first order with respect to the conjecture errors $\delta X_k\left(\cdot\right)$. For example, in case of the market makers' problem, (20) yields

$$\delta y_k\left(P\right) = -\left(J-1\right) y'_{M_k}\left(P\right)\delta P_e\left(Y; X_{M,k}\left(\cdot\right)\right), \tag{48}$$

which is expressed as a linear functional of $\delta X_{M,k}\left(\cdot\right)$ through $\delta_{X_{M,k}} P_e\left(Y; X_{M,k}\left(\cdot\right)\right)$.

From the FOC, we know that $d_{x_k}\Pi_k\left(x_k; X_k\left(\cdot\right)\right) = 0$, since the first variation of the expected payoffs should vanish for all possible variations of the trading strategies, including those that occur due to the variation of conjectures, in particular given by (48) for the market makers' problem. Therefore, (47) reduces to

$$\Delta_{X_k}\Pi_k\left(x_k; X_k\left(\cdot\right)\right) = \delta_{X_k}\Pi_k\left(x_k; X_k\left(\cdot\right)\right) = 0, \tag{49}$$

which means that the small distortions of agent k strategies resulting from distortion of his conjectures are irrelevant, and the same level of expected utility is achieved if all agents would have followed the non-distorted equilibrium strategies x_k.

Analogous, we have for $j \neq k$

$$\begin{aligned}\Delta_{X_k}\Pi_j\left(x_j; X_j\left(\cdot\right)\right) &= d_{x_k}\Pi_j\left(x_j; X_j\left(\cdot\right)\right) + \delta_{X_k}\Pi_j\left(x_j; X_j\left(\cdot\right)\right) \\ &= d_{x_k}\Pi_j\left(x_j; X_j\left(\cdot\right)\right) = 0,\end{aligned} \tag{50}$$

where the second term on the r.h.s. in the first equation vanishes, since, by assumption, the conjectures of agents $j \neq k$ are not affected by the conjecture of agent k. However, the strategy distortion δx_j still occurs since the distortion of the conjecture $\delta X_k\left(\cdot\right)$ leads to the shift of price by distorting the strategy x_k of the agent k. In this case, the scalar variation does not automatically vanish due to FOC, since it occurs due to the price shift caused by other agent. However, (50) says that the distortion δx_j caused by the distortion of conjecture $\delta X_k\left(\cdot\right)$ does not affect payoffs of any agent $j \neq k$.

Therefore, if the conditions of definition 2 hold, the expected payoffs of all agents are not affected by $\delta X_k\left(\cdot\right)$, regardless whether or not they adjust their trading strategies to the conjecture distortion $\delta X_k(\cdot)$. In particular, the same level of expected payoffs is achieved on the strategies corresponding to the non-distorted equilibrium. *Q.E.D.*

Proof of Theorem 1.

1. Impact of MM k conjecture error on her own expected payoffs.

The expected profits of market maker k are given by the functional

$$\begin{aligned}\overline{\Pi}_{M,k}\left(y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right) &= E_Y\left[\Pi_{M,k}\left(y_k,P;y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right)\right] \\ &= E_Y\left[y_k(P-P_e\left(Y;X_{M,k}\left(\cdot\right)\right))\right].\end{aligned} \tag{51}$$

Evaluating the first variation of (51) with respect to $\delta X_{M_k}(\cdot)$, yields

$$\begin{aligned}&\delta_{X_{M,k}}\overline{\Pi}_{M,k}\left(y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right) \\ =\ & E_Y\left[\delta y_k(P-P_e\left(Y;X_{M,k}\left(\cdot\right)\right))+y_k\delta P-y_k\delta_{X_{M,k}}P_e\left(Y;X_{M,k}\left(\cdot\right)\right)\right].\end{aligned} \tag{52}$$

Taking into account that the actual total demand does not change as a result of a wrong conjecture and making use of the FOC (19) at equilibrium, observe that the first two terms on the right-hand side of (52) cancel and the following "envelope theorem" result holds:

$$\begin{aligned}\delta_{X_{M,k}}\overline{\Pi}_{M,k}\left(y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right) &= -E_Y\left[y\delta_{X_{M,k}}P_e\left(Y;X_{M,k}\left(\cdot\right)\right)\right] \\ &= -\frac{1}{J}E_{u,v}\left[Y\delta_{X_{M,k}}P_e\left(Y;X_{M,k}\left(\cdot\right)\right)\right].\end{aligned} \tag{53}$$

Therefore, the robustness condition reduces to

$$E_{u,v}\left[Y\delta_{X_{M,k}}P_e\left(Y;X_{M,k}\left(\cdot\right)\right)\right]=0. \tag{54}$$

We have

$$\begin{aligned}&E_{u,v}\left[Y\delta_{X_{M,k}}P_e\left(Y;X_M\left(\cdot\right)\right)\right] \\ =\ & E_{u,v}\left[Y\frac{\int dv' e^{-\frac{(v')^2}{2}}e^{-\frac{(Y-X_M(v'))^2}{2}}\left(v'-P_e\left(Y\right)\right)\left(Y-X_M\left(v'\right)\right)\delta X_{M,k}\left(v'\right)}{Z_P\left(Y\right)}\right] \\ =\ & \int dv\int dY Y\delta_{X_{M,k}}P_e\left(Y;X_M\left(\cdot\right)\right).\end{aligned} \tag{55}$$

Changing the order of integration and making use of (45) yields

$$\begin{aligned}
& E_{u,v}\left[Y\delta_{X_{M,k}}P_e\left(Y;X_M\left(\cdot\right)\right)\right] && (56)\\
= \; & \int dY Z_P\left(Y\right)Y\frac{\int dv' e^{-\frac{(v')^2}{2}}e^{-\frac{(Y-X_M(v'))^2}{2}}\left(v'-P_e\left(Y\right)\right)\left(Y-X_M\left(v'\right)\right)\delta X_{M,k}\left(v'\right)}{Z_P\left(Y\right)}\\
= \; & \int dve^{-\frac{v^2}{2}}\delta X_{M,k}\left(v\right)\int dye^{-\frac{(Y-X_M(v))^2}{2}}Y\left(v-P_e\left(Y\right)\right)\left(Y-X_M\left(v'\right)\right)\\
= \; & E_v\left[\delta X_{M,k}\left(v\right)E_u\left[\left(u+X_M\left(v\right)\right)\left(v-P_e\left(Y\right)\right)\left(Y-X_M\left(v\right)\right)\right]\right]\\
= \; & E_v\left[\delta X_{M,k}\left(v\right)E_u\left[u\left(u+X\left(v\right)\right)\left(v-P_e\left(Y\right)\right)\right]\right].
\end{aligned}$$

Now, we have

$$\begin{aligned}
E_u\left[u\left(u+X\left(v\right)\right)\left(v-P_e\left(Y\right)\right)\right] &= E_u\left[\frac{\partial}{\partial Y}\left(Y\left(v-P_e\left(Y\right)\right)\right)\right] && (57)\\
&= v-\overline{P}_e\left(X\left(v\right)\right)-X\left(v\right)\overline{P}'_e\left(X\left(v\right)\right)-\overline{P}''_e\left(X\left(v\right)\right),
\end{aligned}$$

where the last equality is obtained by integrating the noise trade out and using Stein's lemma. Combining the last equality of (57) with the insider's FOC in Proposition 1 and substituting into (56) and (53), reveals that at equilibrium

$$\delta_{X_{M,k}}\overline{\Pi}_{M,k}\left(y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right)=\frac{1}{J}E_v\left[\delta X_{M,k}\left(v\right)\overline{P}''_e\left(X\left(v\right)\right)\right], \qquad (58)$$

which can be viewed as a "double envelope theorem" result since it makes use of envelope properties with respect to optimization by both the insider and market makers.

Taking into account that $\delta X_{M,k}\left(\cdot\right)$ is an arbitrary variation and making use of the basic lemma of Variation Calculus (see, e.g., Kolmogorov and Fomin (1999)), we conclude that the functional variation in (58) vanishes if and only if $\overline{P}''_e\left(X\left(v\right)\right)=0$, $\forall X\left(v\right)\in R$, which means that at equilibrium, the expected payoff of each market maker is insensitive with respect to a small deviation in her own conjecture.

From the above analysis, it follows that the robustness with respect to the conjectures by market makers is equivalent to the linearity of the pricing rule: the above condition says that at equilibrium the market makers' expected profit cannot vary with a small variation of their own conjecture (34) only if the equilibrium is linear. Therefore, linearity is a necessary condition for robustness, i.e., no equilibrium save for a linear one can be robust. We next show that the standard linear equilibrium is indeed robust.

2. Impact of MM k conjecture error on the payoffs of other agents.

Now, suppose that the initial equilibrium is linear, market maker k's conjecture is slightly "off", and let us analyze the equilibrium expected payoffs of market makers $j \neq k$ and the insider. The expected payoffs of market maker $j \neq k$ are given by

$$\overline{\Pi}_{M,j}\left(y\left(\cdot\right),X_{M,k}\left(\cdot\right)\right)=E_{Y}\left[y_{j}(P-P_{e}\left(Y;X_{M,j}\left(\cdot\right)\right))\right]. \tag{59}$$

Evaluating the first variation of (59) with respect to $\delta X_{M_k}(\cdot)$ and using the fact that (3) implies that the linear equilibrium is characterized by a linear relation between the pricing rule and the market efficient price, $P=\frac{J-1}{J-2}P_e\left(Y;X_M\left(\cdot\right)\right)$ we obtain

$$\begin{aligned}\delta_{X_{M,k}}\overline{\Pi}_{M,j}\left(y\left(\cdot\right),X_{M,j}\left(\cdot\right)\right) &= E_Y\left[y_j\delta P\right]\\ &= \frac{1}{J}E_Y\left[Y\delta_{X_{M,k}}P_e\left(Y;X_M\left(\cdot\right)\right)\right]\\ &= \frac{1}{J}E_v\left[\delta X_{M,k}\left(v\right)\overline{P}_e''\left(X\left(v\right)\right)\right]=0,\end{aligned} \tag{60}$$

which establishes that the payoffs of other market makers are insensitive with respect to a small error in the conjecture by market maker k.

The equilibrium expected payoffs of the insider are

$$\overline{\Pi}_I\left(X\left(\cdot\right),P\left(\cdot\right)\right)=E_v[X\left(v\right)\left(v-\overline{P}\left(X\left(v\right)\right)\right)], \tag{61}$$

where $\overline{P}\left(\cdot\right)$ is the equilibrium pricing rule. If market maker k makes an incorrect conjecture, this should affect the pricing rule and hence the insider's expected payoffs. We have

$$\begin{aligned}\delta_{X_{M,k}}\overline{\Pi}_I\left(X\left(\cdot\right),P\left(\cdot\right)\right) &= -E\left[X\left(v\right)\delta_{X_{M,k}}\overline{P}\left(X\left(v\right)\right)\right] \qquad (62)\\ &= -\frac{J-1}{J-2}E_{v,u}\left[X\left(v\right)\delta_{X_{M,k}}P_e\left(Y;X_M\left(\cdot\right)\right)\right].\end{aligned}$$

Now, we have

$$\begin{aligned}\delta_{X_M}P\left(y;X_M\left(\cdot\right)\right) &= E_{v'|y}\left[\left(v'-P\left(y;X_M\left(\cdot\right)\right)\right)\left(y-X_M\left(v'\right)\right)\delta X_M\left(v'\right)\right] \qquad (63)\\ &= \int dv' f\left(v';X_M\left(\cdot\right)|y\right)\left(v'-P\left(y;X_M\left(\cdot\right)\right)\right)\left(y-X_M\left(v'\right)\right)\delta X_M\left(v'\right),\end{aligned}$$

where the conditional p.d.f. $f\left(v;X_M\left(\cdot\right)|y\right)$ is defined by

$$f(v;X\left(\cdot\right)|X\left(v\right)+u=y)=\frac{1}{f\left(y;X\left(\cdot\right)\right)}\exp\left[-\frac{\left(y-X\left(v\right)\right)^2}{2\Sigma_u}\right]\exp\left[-\frac{v^2}{2\Sigma_0}\right], \tag{64}$$

the marginal distribution density function

$$f(y;X\left(\cdot\right))=\int_{-\infty}^{+\infty}dv'\exp\left[-\frac{\left(y-X\left(v'\right)\right)^2}{2\Sigma_u}\right]\exp\left[-\frac{\left(v'\right)^2}{2\Sigma_0}\right], \tag{65}$$

and the parameters normalized to one, $\Sigma_0 = \Sigma_u = 1$. Therefore,

$$\delta_{X_M} E_v \left[X\overline{P}\right] = E_{v,u} X(v)\, \delta_{X_M} P, \tag{66}$$

and

$$\delta_{X_M} P = \frac{\int dv' e^{-\left(\frac{v'^2}{2} + \frac{(y - X_M(v'))^2}{2}\right)} \left(v' - P(y)\right)\left(y - X_M(v')\right) \delta X_M(v')}{\int dv'' e^{-\left(\frac{v''^2}{2} + \frac{(y - X_M(v''))^2}{2}\right)}}, \tag{67}$$

where we dropped the functional arguments to simplify notation. For example, $P(y)$ is short hand for $P(y; X_M(\cdot))$. Making use of (63) and changing the order of integration, yields

$$E_{v,u}\left[X(v)\, \delta_{X_M} P(y; X_M(\cdot))\right] = E_v\left[\delta X_M(v)\, E_u\left[Q(y; X_M(\cdot))\left(v - P(y; X_M(\cdot))\right)\left(y - X_M(v)\right)\right]\right],$$

where

$$Q(y; X_M(\cdot)) = E_{v|y}\left[X(v)\right] = \frac{\int dv e^{-\left(\frac{v^2}{2} + \frac{(y - X_M(v))^2}{2}\right)} X(v)}{\int dv' e^{-\left(\frac{v'^2}{2} + \frac{(y - X_M(v'))^2}{2}\right)}}. \tag{68}$$

In short-hand notation, we have

$$\begin{aligned} \delta_{X_M} E_v\left[X\overline{P}\right] &= E_v\left[\delta X_M(v)\, E_u\left[Q(y)\left(v - P(y)\right)\left(y - X_M(v)\right)\right]\right] \\ &= E_v\left[\delta X_M(v)\, E_u\left[\frac{\partial}{\partial y}\left(Q(y)\left(v - P(y)\right)\right)\right]\right]. \end{aligned} \tag{69}$$

In a linear equilibrium, $Q(y) = \beta P(y) = \lambda\beta y$. Therefore, (69) yields

$$\delta_{X_M} E_v\left[X\overline{P}\right] = \lambda\beta E_v\left[\delta X_M(v)\, E_u\left[\frac{\partial}{\partial y}\left(y\left(v - P(y)\right)\right)\right]\right]. \tag{70}$$

From Stein's lemma, we obtain

$$E_u\left[\frac{\partial}{\partial y}\left(y\left(v - P(y)\right)\right)\right] = v - \overline{P}(X(v)) - X(v)\, \overline{P}'(X(v)) - \overline{P}''(X(v)) = 0,$$

where the last equality follows from the insider's FOC and linearity. Substituting (69) back into (70) and then into (62), we finally obtain $\delta_{X_{M,k}} \overline{\Pi}_I(X(\cdot), P(\cdot)) = 0$, which proves that the insider's expected equilibrium payoffs are insensitive to a small variation in a conjecture by one of the market makers.

3. Impact of Insider's conjecture error on her own expected payoffs.

Suppose the insider's conjecture about some market maker's strategy is slightly "off". As a result, the insider's conjecture of the pricing rule is "off," and the insider incorrectly assumes that the pricing rule $P(\cdot)$ deviates from the Nash equilibrium one $P^*(\cdot)$ to

$P(\cdot) = P^*(\cdot) + \delta P(\cdot)$. The insider reacts to the conjectured pricing rule by deviating from the original equilibrium strategy $X^*(\cdot)$. From the optimization by the insider, we conclude that the shift of insider's demand due to the shift in the pricing rule conjecture $\delta P(\cdot)$ is

$$\delta x = \widehat{O}\delta P(x) = \left(-\frac{1}{2}\right)\frac{1}{\overline{P}'(x)}\left(1 + x\frac{\partial}{\partial x}\right)\delta P(x), \tag{71}$$

which means that the shift of the insider's strategy is commensurate with the insider's conjecture error. The actual pricing rule is unaffected by the insider's conjecture, so the "ex-post" variation of the insider's expected payoffs is given by

$$\begin{aligned} &\delta\overline{\Pi}_I(X(\cdot), P(\cdot)) \\ = \;& E\left[\delta X(v)\left(v - \overline{P}(X(v)) - X(v)\right)\overline{P}'(X(v))\right] = 0, \end{aligned} \tag{72}$$

where the last equality reflects the FOC that holds at the symmetric Nash equilibrium.

4. Impact of Insider's conjecture error on expected payoffs of market makers.

We must evaluate the variation of the market maker's expected equilibrium payoffs. These expected profits shift because the insider's optimal strategy shifts due to her conjecture error, which shifts the demand for liquidity. As a result, both the aggregate liquidity supply and market-clearing price shift, which could shift market maker expected payoffs. Importantly, the market makers do not change the functional form of their strategies and therefore they all supply the same amount $y_k = \frac{1}{J}Y$ and the pricing rule $P(\cdot)$ is the one defined in the symmetric Nash equilibrium. We have

$$\begin{aligned} \overline{\Pi}_{M,k}(y(\cdot), X_M(\cdot)) &= E\left[\frac{1}{J}Y\left(P(Y; X_M(\cdot)) - v\right)\right] \\ &= \frac{1}{J}E_{v,u}\left[(X(v) + u)\left(P(X(v) + u; X_M(\cdot)) - v\right)\right]. \end{aligned} \tag{73}$$

With the use of the Stein's lemma, the last expression yields

$$\overline{\Pi}_{M,k}(y(\cdot), X_M(\cdot)) = \frac{1}{J}E_v\left[X(v)\left(\overline{P}(X(v); X_M(\cdot)) - v\right) + \overline{P}'(X(v); X_M(\cdot))\right], \tag{74}$$

which is simply a normalized-by-$\frac{1}{J}$ difference between the losses of liquidity traders and the profits of informed. Since, as discussed above, the market makers' conjecture about the informed's strategy is unaffected by the informed's error, the functional form of the pricing rule does not change. Therefore, we drop the functional arguments in (74). The first variation of expected market maker's profits takes the form

$$\delta\overline{\Pi}_{M,k}(y(\cdot)) = \frac{1}{J}E_v\left[\delta X(v)\left(\left(\overline{P}(X(v)) + X(v)\overline{P}'(X(v)) - v\right) + \overline{P}''(X(v))\right)\right]. \tag{75}$$

Note that the first three terms on the right-hand side of (75) cancel due to the FOC of the insider's problem, and we finally obtain an envelope theorem result

$$\delta\overline{\Pi}_{M,k}\left(y\left(\cdot\right)\right)=\frac{1}{J}E_{v}\left[\delta X\left(v\right)\overline{P}''\left(X\left(v\right)\right)\right]. \tag{76}$$

Analogous to the first part of the proof, we argue that since $\delta X\left(\cdot\right)$ is an arbitrary variation (defined in an appropriate functional space), the basic lemma of Variation Calculus states that the variation (76) vanishes if and only if $\overline{P}''\left(X\left(v\right)\right)=0$, $\forall X\left(v\right)\in R$. In particular, this means that in a linear equilibrium, the expected payoff of each market maker is insensitive to a small error in the insider's conjecture. *Q.E.D.*