Flexibility vs. Protection from an Unrepresentative Legislative Majority*

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Abstract

We derive the equilibrium institutional design of representative democracy by citizens who first vote on the supermajority required for a new policy to be adopted, and then delegate decision making to a legislature that selects policy given that institutional constraint. A legislature that can freely tailor policy to reflect society's current preferences is good. However, the views of the median legislator or agenda setter may differ from the median citizen's, and an unchecked legislature can implement bad policy. We characterize how the primitives describing the preferences of actors and the status quo policy affect the equilibrium degree of legislative flexibility.

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1 Introduction

The effect of [representative democracy is] to refine and enlarge the public views, by passing them through the medium of a chosen body of citizens, whose wisdom may best discern the true interest of the nation.

James Madison

Citizens' tastes for government policy vary as circumstances change. For example, a financial crisis may sharply alter societal preferences for government spending. In a direct democracy, in which citizens vote directly on policy, the median citizen's preferences would likely prevail. However, more typically, representative democracy reigns: citizens delegate policy choices to a legislature comprised of elected representatives. How easily this legislature can modify existing policy depends on the rules that govern the legislative process.

An unencumbered legislative process will ensure that policy closely reflects the preferences of the *legislative* body. This benefits citizens when legislative preferences mirror society's. However, for reasons such as party affiliation, incumbency advantage, gerrymandered districts, imperfect voter information, time between elections, asynchronous (e.g., Senate) elections, etc., an elected legislature may ill-represent society. To protect society from a radical legislature's desire to implement 'bad' policy, it may be optimal to introduce a degree of inertia or legislative rigidity to the legislative process, as encapsulated in the proportion of legislative votes required to pass a proposal.

Examples of supermajority rules are commonplace: the United States Constitution establishes supermajority rules for a range of decisions such as overriding vetoes, the U.S. Senate requires a three-fifths majority to end a filibuster, and many states require supermajorities to raise taxes, pass spending bills, restrict local communities' regulatory powers, etc. Requiring approval from multiple groups—the House, Senate and President—similarly introduces inertia to the legislative process. Thus, if the House rejects immigration reform passed by the Senate, the status quo prevails.

We develop a model of the legislative process that addresses the tradeoff between flexibility and protection embedded in voting rules. Our model has three building blocks. First, citizens have quadratic preferences over policy outcomes, and the preferred policy e of the median citizen may differ from an established status quo. Second, legislative preferences reflect those of voters, but only imperfectly. While the median legislator's preferred policy m corresponds to the median citizen's preferred policy *in expectation*, their realizations may differ—the median legislator is, in effect, a noisy, imperfect copy of the median citizen. Third, a (possibly probabilistic) rule selects a member of the legislature, p, to propose a policy. This policy is adopted if and only if it wins enough support in the legislature in a vote against the status quo.

We then characterize the optimal extent of legislative inertia, as captured by the share of votes required for a proposal to defeat the status quo. Posed differently, we derive the equilibrium institutional design of representative democracy by citizens who first vote on the supermajority required for a new policy to be adopted, and then delegate decision making to a legislature that selects policy given that institutional constraint.

We address fundamental questions: When is it optimal to have legislative outcomes determined by simple majority? How does increasing the likelihood of 'extreme' legislatures—legislatures with a median politician whose preferred policy is far from the median citizen's—affect the equilibrium supermajority? How will selecting a proposer whose interests are further from the median legislator's, and hence is *ex ante* less representative of society, affect the equilibrium supermajority? What is the impact of bias in the initial status quo policy?

The equilibrium voting rule trades off the benefit of reduced policy variance with the cost of policy that is more biased toward the status quo. The equilibrium voting rule does not weigh proposers who are free from legislative constraint, able to propose their preferred policies and defeat the status quo. So, too, the voting rule does not weigh proposers who are completely blocked, unable to find a winning policy that they prefer to the status quo. Marginal changes in the voting rule have *no* effect on policy outcomes when a proposer is free or blocked. The equilibrium voting rule only weighs proposers who are constrained by the voting rule, able only to move policy partway toward their bliss points. The equilibrium voting rule has a simple characterization: whenever a supermajority is optimal, the equilibrium voting rule is such that conditional on a proposer being constrained, the expected policy outcome equals the expected bliss point of the median citizen. Neither submajority nor unanimity voting rules are ever optimal: submajorities facilitate unrepresentative shifts in policy from the status quo; while large, but not unanimous, supermajorities would approve only policy changes that the median citizen strictly prefers.

If the median legislator lies close to the status quo, so do most legislators, making it difficult for a proposer to identify a policy that enough legislators prefer to the status quo. Conditional on drawing such a "conservative" legislature, the voting rule typically restrains policy movement by too much, i.e., from the median citizen's perspective, new policy is typically biased too close to the status quo. If, instead, the median legislator lies far from the status quo, so do most legislators, thus enlarging the set of policies that enough legislators prefer to the status quo. Conditional on drawing such a "radical" legislature, the voting rule typically restrains policy movement by too little. The equilibrium supermajority rule optimally trades off between these two possibilities.

We first suppose that the proposer is always the median legislator. This selection often emerges in legislative settings with no political parties. This base case scenario lets us identify when supermajorities are optimal even though, ex ante, the proposer's preferences do not deviate in expectation from the median citizen's. A median proposer is never constrained by a simple majority voting rule. Hence, simple majorities raise the median citizen's utility if the distance between his bliss point e and the median legislator's is less than that between e and the status quo. It follows that *if* the legislature is sufficiently moderate *and* the median legislator proposes policy, then a simple majority voting rule is optimal. Posed differently, fixing the representativeness of a legislature, if the median citizen's preferences are sufficiently volatile, so that large policy shifts from the status quo are likely to be desired, then simple majority is optimal.

We then address how unrepresentative a legislature must be before it is optimal to constrain a median proposer via supermajority. We consider three classes of distributions over median citizen and median legislator bliss points: two-point, uniform and normal. In each class, a supermajority becomes optimal even when the volatility in the median citizen's preferences exceeds that in the median legislator's preferences around the median citizen's bliss point. Moreover, as long as supermajorities are optimal, less representative legislatures raise the optimal supermajority. This result reflects the first-order intuition that a more extreme median legislator is more likely both to want to implement a "worse" policy, and to be able to do so, making it optimal to constrain him further.

These results might lead one to conjecture that less representative legislatures should always increase the optimal supermajority. This conjecture is *false*—this first-order intuition misleads: given any single-peaked, symmetric, strictly quasi-concave, absolutely continuous density over the median legislator's bliss point, one can *always* find a symmetric, mean-preserving spread such that the optimal voting rule is *reduced*. For example, one can shift probability mass from moderate median proposers quite close to the median citizen's bliss point and place it on those who are a little further away, leaving unaltered the probability mass on extreme proposers, whom the median citizen would most like to constrain. Moderate proposers are also more likely to be conservative, and hence to be blocked by a supermajority. In contrast, those who are a little less representative are more likely to be heavily-constrained by the supermajority, on average, generating insufficient policy movement. Blocked proposers do not directly affect the optimal voting rule, but increasing the likelihood of heavily-constrained proposers who typically do not move policy far enough means that the optimal voting rule falls. Alternatively, one can replace median proposers who are likely to be constrained, but by too little, by those who are quite extreme, and hence are likely to be far from the status quo: such radical median proposers are free to propose their preferred policy.

So, too, a conjecture that more volatile citizen preferences should always reduce the equilibrium supermajority is *false*. Shifting the median citizen's bliss point away from the status quo by one unit also shifts the distribution of the median legislator's bliss point away by *one* unit; and a proposer can exploit the increased distance from the status quo to shift policy by up to *two* more units. This increases the ratio of (a) constrained proposers who move policy too far to (b) constrained proposers who move policy too little. Hence, even when the status quo becomes less representative of society, *greater* supermajorities can become optimal.

A key contribution of our paper is to provide an explanation for why slight supermajorities are not observed in practice—they may *never* be optimal. The optimal voting rule can be a discontinuous function of the underlying parameters when moving from majority to supermajority. Supermajority voting rules are blunt instruments, restricting radical proposers insufficiently, but excessively restricting conservative proposers. Slight supermajorities incur the costs of disproportionately restricting conservative proposers sharply, but provide limited beneficial restraint on radical proposers. Thus, the issue for the designer of the voting rule becomes (a) should the median legislator be left unchecked via a simple majority voting rule; or (b) is the median legislator more likely to be so unrepresentative that a large supermajority should be required, forsaking gains that a conservative proposer can achieve in order to restrain radical proposers?

We then consider non-median proposers. Non-median proposers arise in multi-party settings when a proposer is from a majority party that is on one ideological side of a legislature; or when the proposer is a president or prime minister, whose preferences do not perfectly mirror the median legislator's. We characterize the impact of proposers who are equally likely to lie u_p to the left or right of the median legislator. An increase in this 'polarity' distance u_p implies, on average, more radical proposers. A setting in which the identity of a proposer is divorced from the median legislator's may reflect a political design with checks and balances. In particular, increased polarity raises the separation between the proposer and the pivotal voter on policy in the legislature, providing an additional measure of control. We address: when does increasing the separation of powers in this way complement control via size of the supermajority, and when does it substitute?

One's intuition might be that more extreme proposers should make greater supermajorities optimal because proposers who are more extreme *within* a legislature also tend to be further from the median citizen. With highly polarized proposers, this intuition is correct: further increases in polarity always lead to complementary increases in the optimal supermajority, in order to protect citizens from very bad, albeit rare, policy proposals from extreme proposers.

However, replacing median proposers with slightly polarized ones can *reduce* the optimal voting rule: when uncertainty is normally distributed, we provide necessary and sufficient conditions for this to occur. We prove that if the legislature is sufficiently unrepresentative of citizens that supermajorities are optimal, then *slight* polarity *reduces* the optimal supermajority—slight separation of powers substitutes for control via the size of the supermajority. The intuition is that *more extreme proposers are more constrained than the median legislator by the necessity of winning approval from moderate legislators.* Conditional on being able to move policy, more polarized proposers are more likely to only be able to move policy in the direction preferred by the median citizen. Thus, the relationship between a proposer's polarity and the optimal supermajority is U-shaped. The optimal supermajority is high when polarity is low, in order to protect against a proposer who finds it easy to move policy in the direction he prefers; and when polarity is high, in order to protect against a rogue legislature that would let highly-polarized proposers implement extreme policies.

Finally, we determine how initial policy bias affects the optimal voting rule, providing insights into how the optimal design changes when past policy is slow to adjust to changes in society's preferences. For example, the status quo on immigration reform may be too far to the right. A conservative right-wing proposer is unlikely to move policy; but a radical left-wing proposer can convert the threat of a moderately bad status quo into an equally bad (or worse) extreme left-wing policy. Reducing the voting rule reduces the costs of inertia associated with the first legislature, but raises the costs of flexibility associated with the second. Intuition might suggest that with a status quo that is less representative of societal preferences, citizens would prefer a reduced supermajority that gives the legislature more flexibility to determine policy. When uncertainty is normally distributed, this is indeed true if the median legislator is the proposer. However, when the legislature tends to be representative of citizens but proposers are sufficiently polarized, this intuition is misplaced: introducing slight initial policy bias *raises* the optimal voting rule. The correct intuition is that with a slight status quo policy bias, the proposers who are more likely to be constrained by a given supermajority are those who would move policy *away* from what the median citizen prefers.

Related Literature. The benefits and costs of delegating authority have been well studied. Papers with results related to ours include Klumpp [2010], Aghion et al. [2004] and Compte and Jehiel [2010b]. Klumpp [2010] shows that in a model of indirect democracy, constituents may select a representative with preferences closer to the status quo. In his model of the legislative process a proposer is randomly selected. Electing a more conservative representative can moderate legislative outcomes, raising voter welfare. As in our model, the equilibrium voting rule generates policy that is, on average, closer to the status quo than a median citizen prefers.

In Aghion et al. [2004], the voting rule determines the power of an ethical leader to institute reform or an unethical leader to expropriate wealth. A larger supermajority increases the probability that a welfare enhancing reform passes, but individuals are also more likely to suffer losses from expropriation. They show that the optimal amount of insulation depends on the size of the aggregate improvement from reform, the aggregate and idiosyncratic uncertainties over reform outcomes, the degree of polarization of society, the availability and efficiency of fiscal transfers, and the degree of protection of property rights against expropriation. In contrast, in our model, the motivation of elected politicians is pure. Uncertainty concerns politicians' preferred policies, not their integrity.

Compte and Jehiel [2010b] show that in a model of collective search, unanimity is undesirable in large committees with sufficiently patient members. Generally speaking, unanimity makes it difficult to find a proposal that is acceptable to all, thereby inducing costly delay. The optimal majority rule solves best the trade-off between speeding up the decision-making process and avoiding the risk of adopting relatively inefficient proposals. In contrast, in our model the optimal voting rule solves best the trade-off between a flexible legislative process that can respond to changes in society's tastes and a rigid legislative process that can guard against radical legislators.

Another strand of research examines optimal voting rules as a way to aggregate information when information is dispersed throughout the electorate. Young [1995] provides a good review. This role for voting is absent in our model since all agents have the same information.

Several papers derive optimal supermajority voting rules in settings where issues of dynamic consistency can arise, and supermajorities commit future governments to behave appropriately (Gradstein [1999], Messner and Polborn [2004], Dal Bo [2006]). Our model has no dynamic consistency problems. Supermajorities arise, not from concern about the damage a future government might do, but rather from uncertainty about what the current government might do. Closely related are papers that view supermajorities as necessary to ensure stability of institutional rules (Barberà and Jackson [2004], Acemoglu et al. [2008]). The equilibrium voting rules we derive are inherently stable since all citizens share the same preferences over voting rules.

Duggan and Kalandrakis [2011] and Duggan and Kalandrakis [forthcoming] develop a very general, infinite horizon legislative bargaining model with an endogenous status quo determined by the previous period's outcome, establish equilibrium existence, and develop numerical methods that could be used to analyze outcomes. However, they do not further characterize such outcomes.

Section 2 presents our model of the legislative process. Section 3 characterizes how uncertainty over the preferences of citizens and the median legislator affect optimal voting rules when the median legislator proposes policy. Section 4 looks at the impact of a proposer's polarity, and Section 5 explores the impact of initial policy bias. Section 6 concludes. Proofs are in the Appendix.

2 The Model

In our model of representative democracy, government policy, S, is defined over the real line. There is an initial government policy, S_0, that we refer to as the status quo. A legislative process that we describe shortly, generates a new policy S_1. A citizen with policy bliss point ω derives utility $U_\omega = -(S_1 - \omega)^2$ from the implemented policy. The initial distribution of citizen bliss points is symmetrically distributed around a median that we normalize to zero. Citizens first vote on the institutional design of the legislature. Next, shocks to the environment shift the preferred policies of citizens and legislators. Finally, the legislature determines policy.

After citizens determine the legislative design, they are subject to a common shock that affect their tastes for government policy. For example, a financial crisis may cause society to prefer increased government spending, or a terrorist attack may lead society to prefer reduced personal freedoms. The shock shifts the preferred policy of each citizen by e away from his initial bliss point. Thus, the median citizen's preferred policy shifts from 0 to e. The distribution over these shocks, F_e, is symmetric about zero, $F_e(-e) = 1 - F_e(e), \forall e > 0$, and the associated density is f_e.

In a representative democracy, citizens delegate policy choice to a legislature whose preferences may not perfectly mirror theirs. To capture this, we assume that the policy bliss point of the median legislator is $m = e + \mu_m$, where μ_m is distributed according to F_m with an associated density f_m that is symmetric around zero. Thus, in expectation, the median legislator's preferences correspond to the median citizen's, but their realizations may differ. A mean-preserving spread in the distribution of μ_m corresponds to a less-representative legislature. This non-alignment of interests may reflect gerrymandering of heterogeneous districts, party affiliation, incumbency advantage, etc. In many political settings, most districts are "safe seats" held by representatives whose preferences are often far from the median citizen's, so that a few swing districts determine the ideology of the median legislator. As a result, the median legislator's preferences can deviate sharply from the median citizen's. The distribution of bliss points in the legislature is described by F_l with an associated density f_l that is symmetric around the median legislator, m.

Next, a legislator is selected to propose a new policy k on which the legislature will vote. The proposer's policy bliss point is given by $p = m + \mu_p$, where $\mu_p \sim F_p$. We assume symmetry, $F_p(-\mu_p) = 1 - F_p(\mu_p)$, for $\mu_p > 0$. Much of our analysis focuses on two particular distributions. We first focus on the case where the median legislator is always the proposer, in which case μ_p is zero. In a legislative setting without political parties (e.g., local city councils, student/faculty senates), the median proposer emerges naturally. We then consider $\mu_p \in \{-u_p, u_p\}$. A non-median proposer can emerge when the proposer is the median from a majority party that tends to be ideologically on one side of the legislature, or when a president or prime minister sets the agenda for the legislature. The polarity parameter u_p—the proposer's distance from the median legislator—captures the effects of less representative proposers in a tractable way.

Finally, legislators simultaneously vote on whether to replace current government policy (the status quo) with the proposed policy k. A legislator is fully characterized by his policy bliss point, and so we refer to a legislator with bliss point b as legislator b. Each legislator, including the proposer, seeks to minimize the distance between his bliss point and the policy the legislature adopts. To defeat the status quo and be adopted, a proposal must garner the required proportion, α, of votes from the legislature. This proportion α is determined by the choice of institutional design initially selected by the citizens. After the vote is taken, policy is implemented and payoffs are realized.

When there is symmetric uncertainty over policy outcomes,[1] the quadratic preferences imply that the ex-ante utility of a citizen whose bliss point is always δ from the median equals that of the median minus the constant δ^2 (see Bernhardt et al. [2009] or Bernhardt et al. [2011]). Hence, maximizing the median citizen's welfare also maximizes the welfare of all citizens. The equilibrium voting rule α (the proportion of votes required to change policy) maximizes the median citizen's ex-ante welfare. When deciding on the voting rule, citizens account for the incentives of the legislative proposers, who, in turn, consider the incentives of fellow legislators.

In summary, the median citizen's policy bliss point e is symmetrically distributed around zero, the median legislator's bliss point m is symmetrically distributed around e, and each legislator's bliss point including the proposer's p is symmetrically distributed around m. Institutional design is chosen by citizens ex ante, while proposals and legislator votes are made ex post.

Equilibrium. An equilibrium is a voting rule $\alpha \in [0, 1]$, policy proposal choices $k(p, m, S_0, \alpha) \to \mathbb{R}$, and legislator voting rules $v(l, k, S_0, \alpha) \to \{0, 1\}$ such that:

- the voting rule α maximizes the median citizen's ex-ante expected utility given the optimal policy choice $k(p, m, S_0, \alpha)$ by each proposer p, and legislator voting rules, $v(l, k, S_0, \alpha)$;
- each proposer p's policy proposal, $k(p, m, S_0, \alpha)$ minimizes $|S_1 - p|$ given median legislator location m, status quo S_0, voting rule α, and legislator voting rules $v(l, k, S_0, \alpha)$;
- each legislator l votes for proposal k if $|k - l| \leq |l - S_0|$, i.e., setting $v(l, k, S_0, \alpha) = 1$, and votes for the status quo alternative, setting $v(l, k, S_0, \alpha) = 0$, otherwise;

[1]One can preserve symmetry with a status quo that differs from the median citizen's initial bliss point by introducing a stage where S_0 is drawn from a mean-zero, symmetric distribution.

and the law of motion for new policy, S_1, evolves according to

$$S_1 = \begin{cases} k(p, m, S_0, \alpha) & \text{if } \int_{-\infty}^{\infty} v(l, k, S_0, \alpha) dF_l(l; m) \geq \alpha \\ S_0 & \text{otherwise.} \end{cases}$$

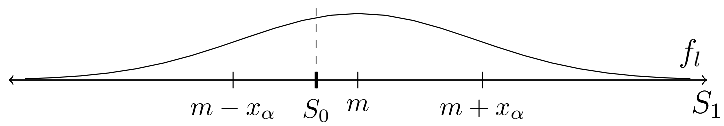
Implicit in the equilibrium definition is the assumption that legislators adopt the weakly dominant strategy of voting for a policy k if and only if they weakly prefer it to the status quo. This assumption rules out uninteresting equilibria, e.g., where a policy k wins because a proposer believes that everyone will vote against any other policy (even though a proportion $\alpha' > \alpha$ of legislators prefer a policy k' that the proposer also prefers to k).

Feasible Policy Set. Legislator l weakly prefers policy k to the status quo of S_0 if and only if l is at least as close to k as he is to the status quo. Thus, legislator $l > S_0$ supports policy k if and only if $k \in [S_0, 2l - S_0]$. It follows that (a) if legislators l and l' support a policy $k \neq S_0$, then so do the legislators located between them (the set of legislators supporting policy k is connected); and (b) legislators on opposite sides of the status quo never both support the same change in policy.

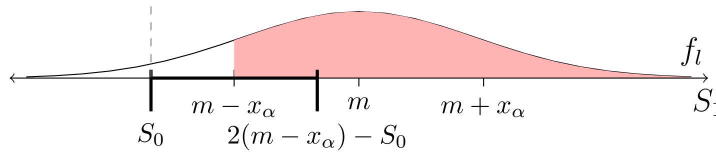
We use $R(m, S_0, \alpha)$ to denote the *feasible policy set*—the set of policies preferred to the status quo by the required proportion α of legislators when the median legislator is located at m and the status quo is S_0. We focus on $\alpha \geq 1/2$, so that at least a simple majority is required to shift legislation. Graham [2011] proves that $\alpha < 1/2$ is never optimal. The intuition for this result is that increasing a minority voting rule reduces the set of possibly successful legislative changes *around* the median legislator. On average, the median legislator is more representative of the populace than a non-median proposer, so increasing a minority voting rule always raises citizen welfare.

We define the *voting rule distance* $x_\alpha \equiv F_l^{-1}(\alpha; m) - m$: $x_\alpha \geq 0$ is the distance from the median legislator that identifies the key legislators who determine $R(m, S_0, \alpha)$. This distance depends only on the voting rule α and the dispersion of legislators around the median, and not on the position of the median legislator. Following Compte and Jehiel [2010a], we refer to the legislators located at $m - x_\alpha$ and $m + x_\alpha$ as *key legislators*. Legislators to the left of both key legislators comprise the "*far left*", legislators to the right of both key legislators comprise the "*far right*". The remaining legislators are "*centrists*".

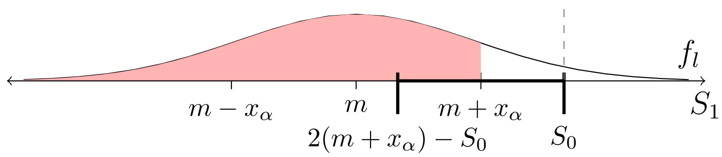
With a simple majority voting rule ($x_\alpha = 0$), a proposal succeeds if and only if the median legislator supports it—simple majority never constrains a median proposer. With a supermajority voting rule ($x_\alpha > 0$), a proposal succeeds if and only if both key legislators, $m - x_\alpha$ and $m + x_\alpha$, support it. For example, suppose $|m - S_0| < x_\alpha$ (Figure 1.1). The key legislator at $m - x_\alpha < S_0$ supports the status quo against policies $k > S_0$, as does the far left, together forming the required minority to block policy shifts to the right. The key legislator at $m + x_\alpha > S_0$ supports the status



1.1: $|m - S_0| < x_\alpha$.



1.2: $S_0 < m - x_\alpha$.



1.3: $m + x_\alpha < S_0$.

Figure 1: In each sub-figure, the thick black line denotes the feasible policy set, $R(m, S_0, \alpha)$. The shaded area represent the required proportion α of legislators who support a movement in policy from the status quo to any point in the feasible policy set.

quo against policies $k < S_0$, as does the far right, forming the required minority to block policy shifts to the left. Thus, when $|m - S_0| < x_\alpha$, the only feasible policy is the status quo.

More generally, if both key legislators support a proposal, so too do both centrists and one of the extreme groups. Together these two groups form the required majority α to pass a proposal. If a key legislator does not support a proposal, then neither does one of the periphal groups. This group forms the required minority to block a proposal. In Figure 1.2 where $S_0 < m - x_\alpha$, proposals to the right of $2(m - x_\alpha) - S_0$ are blocked by the far left; and in Figure 1.3 where $m + x_\alpha < S_0$, proposals to the left of $2(m + x_\alpha) - S_0$ are blocked by the far right.

Summarizing, the feasible policy set from which a proposer selects is

$$R(m, S_0, \alpha) = \begin{cases} [S_0, 2(m - x_\alpha) - S_0] & \text{if } S_0 < m - x_\alpha \\ S_0 & \text{if } |m - S_0| < x_\alpha \\ [2(m + x_\alpha) - S_0, S_0] & \text{if } m + x_\alpha < S_0. \end{cases}$$

Policy Outcomes. Given a voting rule α, p proposes his most preferred feasible policy. Thus, the implemented policy solves

$$S_1 = \min_{k \in R(m, S_0, \alpha)} |p - k|.$$

This policy k is unique since $R(m, S_0, \alpha)$ is closed and the objective function is strictly quasi-concave. If p lies inside the feasible policy set, the proposer can move policy to his bliss point so the new policy is p. Such a proposer is *free*. When a proposer lies on the opposite side of the status quo to the feasible policy set or the feasible policy set only consists of the status quo, then he cannot change policy to raise his utility. Such a proposer is *blocked*.

In all other cases, the proposer is *constrained*: the proposer lies further from the status quo than the boundary of the feasible policy set and can move policy toward his preferred policy, but only to the boundary of $R(m, S_0, \alpha)$. This boundary is defined by the preferences of the key legislator closest to the status quo. For example, when $m - x_\alpha > S_0$ the feasible policy set is $[S_0, 2(m - x_\alpha) - S_0]$. Therefore, a proposer $p > 2(m - x_\alpha) - S_0$ selects $k = 2(m - x_\alpha) - S_0$, the policy that makes the key legislator closest to the status quo, $m - x_\alpha$, indifferent between the proposal and the status quo.

Thus, as an explicit function of m, p, S_0 and x_α, the policy outcome S_1 is given by

$$S_1 = \begin{cases} S_0 & \text{if } |m - S_0| < x_\alpha \text{ or } m + x_\alpha < S_0 < p \text{ or } m - x_\alpha > S_0 > p \\ p & \text{if } (m + x_\alpha < S_0 \text{ and } 2(m + x_\alpha) - S_0 < p < S_0) \\ & \text{or } (m - x_\alpha > S_0 \text{ and } S_0 < p < 2(m - x_\alpha) - S_0) \\ 2(m + x_\alpha) - S_0 & \text{if } m + x_\alpha < S_0 \text{ and } p < 2(m + x_\alpha) - S_0 \\ 2(m - x_\alpha) - S_0 & \text{if } S_0 < m - x_\alpha \text{ and } 2(m - x_\alpha) - S_0 < p. \end{cases}$$

As the required vote share α rises, a proposer is more likely to have to adjust his proposal to win enough legislative support. Further, the status quo prevails for large voting rules that place a key legislator on the opposite side of the status quo from either the other key legislator or the proposer.

When the median legislator proposes policy, he is never constrained by simple majority. For slight supermajorities, only a conservative median proposer, i.e., one close to the status quo is ever constrained. This is because when a median proposer is far from the status quo, then with a slight supermajority, so are the key legislators, who hence prefer the median proposer's bliss point to the status quo. Thus, slight supermajorities only constrain conservative median proposers.

In contrast, simple majority may constrain or block polarized proposers as they must win approval from moderate legislators whose bliss points are far away. One can view an institutional design where the proposer is not the median legislator, as one with checks and balances, since a proposer is less likely to get his way. We will derive how this design affects the optimal supermajority.

Initially, we only require that (a) the densities f_e and f_m be elements of the set Ω of *symmetric, strictly quasi-concave, mean-zero, absolutely continuous* densities, and (b) the status quo is ex-ante unbiased, equal to the median citizen's ex-ante bliss point of zero. We later consider distributions that allow explicit solutions for the optimal voting rule. Section 5 introduces bias in the status quo, i.e., $S_0 \neq 0$. Throughout, we assume there is enough dispersion in the bliss points of legislators that first-order conditions characterize the optimal majority, i.e., the support of f_l is large enough that the optimal x_α is interior.

When there is no ex-ante bias in the status quo, we can write the median citizen's ex-ante expected utility as

$$U = -2\int_{e=-\infty}^{\infty}\left\{\int_{m=0}^{x_\alpha} e^2\, dF_m(m-e) + \int_{m=x_\alpha}^{\infty}\left[\int_{p=-\infty}^{0} e^2\, dF_p(p-m) + \int_{p=0}^{2(m-x_\alpha)} (e-p)^2\, dF_p(p-m) + \int_{p=2(m-x_\alpha)}^{\infty} (e-2(m-x_\alpha))^2\, dF_p(p-m)\right] dF_m(m-e)\right\} dF_e(e), \tag{1}$$

With an unbiased status quo, one can focus on the case where the median is to the right of the status quo, i.e., $m \geq 0$, and multiply by two. The double integral is associated with key legislators who lie on opposite sides of the status quo so that proposers are blocked—the left key legislator votes against policy shifts to the right, and the right key legislator votes against shifts to the left. The first triple integral is associated with proposers who lie to the left of the status quo and thus, cannot change policy. The second triple integral is associated with proposers who lie to the right of the status quo and are close enough to it that they are free. The last triple integral is associated with proposers who are far enough to the right of the status quo that their policy choices are constrained by the necessity of winning the support of the key legislator closest to the status quo.

The relevant primitive is the *voting rule distance* x_α. The distribution of legislative preferences around the median legislator, F_l, only enters citizen payoffs indirectly via this distance. The optimal voting distance is unaffected by changes in the distribution of preferences within the legislature. The optimal voting rule, α^* simply delivers the optimal voting distance. As a result, we have:

Proposition 1. *Consider two legislatures, one with more dispersed legislator ideologies than the other, i.e.,* $F_{l_2}(l;m) > F_{l_1}(l;m)$ *for* $l > m$ *and* $F_{l_1}(l;m) < 1$. *Then* $\alpha^*_{l_2} \geq \alpha^*_{l_1}$. *The inequality is strict if and only if a strict supermajority is optimal, i.e.,* $\alpha^*_{l_2} > 1/2$.

Unanimity Is Never Optimal. We next establish that citizens always prefer a voting rule that allows slight movement in policy to one that blocks all proposers:

Proposition 2. *Let* F_m *have bounded support,* $[-\bar{m}, \bar{m}]$, *and* F_e *have positive variance. Then voting rules that are close enough to unanimity to prevent any shift in policy, are never optimal.*

As the voting rule is reduced from unanimity, the first proposers to be 'unblocked' are on the same side of the status quo as the median citizen, albeit further away. Unblocking these first proposers always moves policy slightly toward the median citizen's bliss point, raising welfare.

Optimal Voting Rule. The first-order condition that characterizes the equilibrium voting rule distance, x_α^*, (i.e., that maximizes expected citizen payoffs in equation (1)) can be written as:

$$\frac{\partial U}{\partial x_\alpha}\bigg|_{x_\alpha^*} = -8 \int_{e=-\infty}^{\infty} \int_{m=x_\alpha^*}^{\infty} (e - 2(m - x_\alpha^*))(1 - F_p(m - 2x_\alpha^*))\, dF_m(m-e)\, dF_e(e) \leq 0, \quad (2)$$

with strict equality if $x_\alpha^* > 1/2$ (recall that equation (1) exploits symmetry of distributions to restrict attention to $m > 0$ and note that the continuity of the policy function with respect to the random variables is continuous so that the differentiation of the Leibnitz terms cancel—the complete derivation can be found in the Appendix). Only constrained proposers enter this first-order condition: the median citizen only cares about realizations of m that are far enough to the right of the status quo, $m > x_\alpha$, that the feasible policy set is non-singleton. Moreover, the median citizen only cares about the *measure* of proposers $1 - F_p(m - 2x_\alpha)$ who lie further to the right of the status quo than the feasible policy set: they all have the same most preferred *feasible* policy, $2(m - x_\alpha)$.[2] The optimal voting rule does not weigh proposers who are so close to the status quo that they lie in the feasible policy set, and hence can freely propose their bliss points—marginal voting rule changes do not affect them. It also does not weigh proposers who are on the opposite side of the status quo from the feasible policy set—these proposers are blocked from implementing policies that they prefer to the status quo, and the size of the voting rule that blocks them is irrelevant. *When $x_\alpha^* > 1/2$, a simple maxim describes the median citizen's optimal voting rule: choose the voting rule so that the expected new policy equals his expected bliss point, where both expectations are conditioned on the proposer being constrained.*

To understand the tradeoffs, recognize that marginally increasing the voting rule affects welfare in two ways. First, raising x_α affects the probability that a proposer is constrained: (a) there are more realizations of the legislature for which the feasible policy set is just the status quo, i.e., the limit of integration over m shrinks; and (b) for those realizations of the legislature for which the feasible set is non-trivial, a larger voting rule reduces the feasible policy set, raising the probability $1 - F_p(m - 2x_\alpha)$ that a proposer is constrained to propose his most preferred feasible policy.

Second, raising x_α affects the realized policy *when* a proposer is constrained. Such a proposer must win approval from the key legislator at $m - x_\alpha$, resulting in a new policy of $2(m - x_\alpha)$. Thus,

[2] $Pr(m + \mu_p > 2(m - x_\alpha)) = Pr(\mu_p > m - 2x_\alpha) = 1 - F_p(m - 2x_\alpha)$.

a marginal increase in x_α moves policy toward the status quo by *twice* the increase in x_α. If the median citizen's bliss point is to the left of the new policy ($e < 2(m - x_\alpha)$), then a marginal increase in x_α raises utility; but if it is to the right ($e > 2(m - x_\alpha)$), then increasing x_α reduces utility.

For any median citizen e, increasing the voting rule trades off the benefit of reduced policy variance with the cost of increased policy bias toward the status quo.

3 Median legislator proposes policy

If the median legislator always proposes policy, then $1 - F_p(m - 2x_\alpha)$ is zero if $m > 2x_\alpha$, and it is one if $m < 2x_\alpha$. The first-order condition that describes the optimal voting rule simplifies to

$$\left.\frac{\partial U}{\partial x_\alpha}\right|_{x_\alpha^*} = -8\int_{e=-\infty}^{\infty}\int_{m=x_\alpha^*}^{2x_\alpha^*} (e - 2(m - x_\alpha^*))\, dF_m(m-e)\, dF_e(e) \leq 0, \tag{3}$$

where $\frac{\partial U}{\partial x_\alpha}|_{x_\alpha^*} = 0$ if the optimal voting rule exceeds simple majority. *Only* intermediate median proposers $m \in [x_\alpha, 2x_\alpha]$ are constrained: a conservative median proposer $m < x_\alpha$ is blocked, and a radical median proposer $m > 2x_\alpha$ is free. On average, proposers close to x_α^* are too constrained—they do not move policy as far as the median citizen usually wants—but proposers closer to $2x_\alpha^*$ tend to move policy excessively—they are less constrained than the median citizen usually prefers.

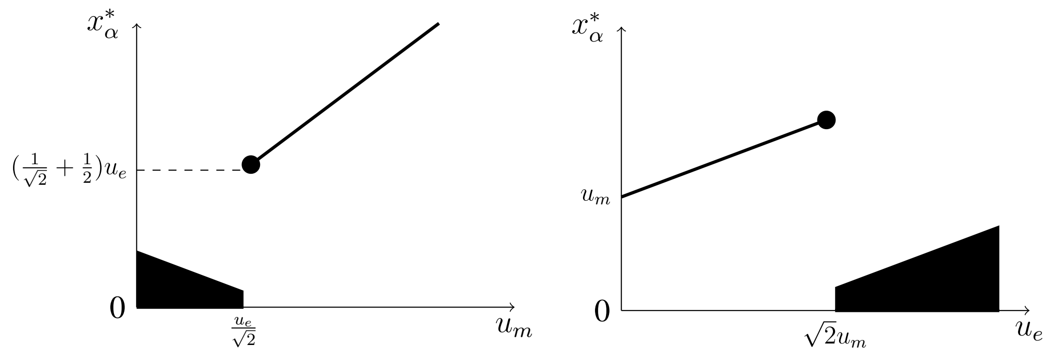
We next characterize how the optimal voting rule varies with primitives for three classes of distributions. For each class, we find that an increase in the dispersion of the median legislator around the median citizen's bliss point increases the optimal voting rule. We then show how the simple conjecture that a less representative legislature—one with greater dispersion in f_m (i.e., a first-order stochastic shift in $f_m(\cdot|m \geq 0)$)—*necessarily* leads to a weakly larger optimal voting rule is *false*.

Two-point uncertainty. Suppose that the bliss points of the median citizen and median legislator are drawn from symmetric distributions with two-point supports. Thus, $e \in \{-u_e, u_e\}$, and $m \in \{e - u_m, e + u_m\}$: u_e measures the movement in the median citizen's bliss point from the status quo, and u_m measures the median legislator's (un)representativeness of society.

Proposition 3 shows that the optimal voting rule is either: (1) simple majority, or (2) the much larger voting rule that blocks the median proposer closest to the status quo and exactly constrains the proposer furthest from the status quo so that the new policy adopted is e. The optimal x_α depends on the extent to which the legislature is representative of citizen preferences:

Proposition 3. *Let $e \in \{-u_e, u_e\}$ and let $m \in \{e - u_m, e + u_m\}$, each with probability 1/2. If $u_m < u_e/\sqrt{2}$, i.e, if the median legislator's preferences are sufficiently less dispersed than the me-*

dian citizen's, then simple majority is optimal. If, instead, $u_m > u_e/\sqrt{2}$, then the optimal voting rule of $x^*_\alpha = u_m + u_e/2$ blocks the median legislator closest to the status quo from changing policy and constrains the median legislator furthest from the status quo to enact new policy exactly equal to the median citizen's bliss point e. When $u_m = u_e/\sqrt{2}$, either policy is optimal.



2.1: x^*_α as a function of u_m given u_e.

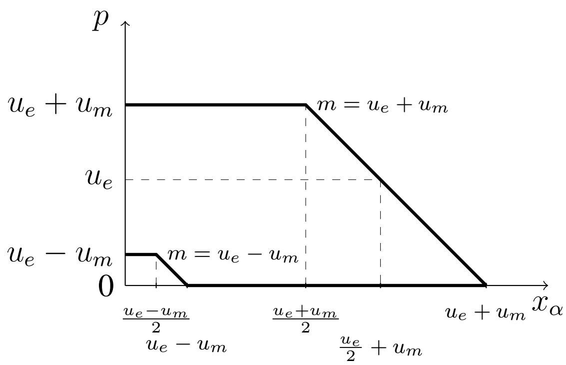
2.2: x^*_α as a function of u_e given u_m.

Figure 2: Equilibrium voting rule distance when e and m are drawn from symmetric distributions with two-point supports, characterized by u_e and u_m respectively.

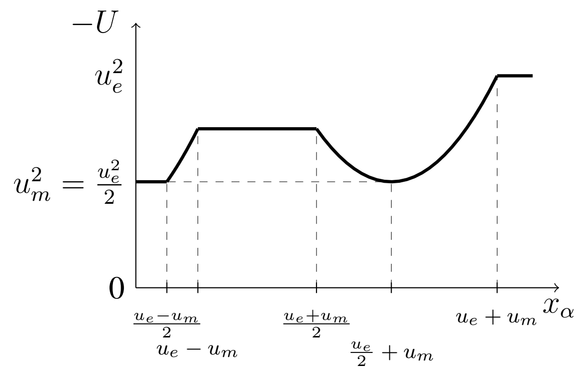
Figure 2 shows how the optimal voting rule varies with u_e and u_m. The shaded "blocks" reflect that not only is simple majority optimal for those parametizations, but so is any voting rule $x^*_\alpha \in [0, \frac{u_e - u_m}{2}]$ that leaves the median legislator unconstrained. As u_m rises, the optimal voting rule (weakly) increases. This result reflects the first-order intuition that increases in u_m make the median legislator less representative of the median citizen. This leads citizens to rely more on the status quo to determine policy.

The median voter's problem boils down to: is it better to have a simple majority voting rule, which leaves both median legislative proposers free to implement their preferred policies; or is it better to focus on a radical proposer, constraining him with a large supermajority to implement e, at the cost of fully blocking a conservative median legislator from shifting policy? As a result, *the optimal voting rule is discontinuous in u_e and u_m going from simple majority to a large supermajority.*

To understand why, consider Figure 3 which shows how changes in the voting rule affect policy and utility when $u_m = u_e/\sqrt{2}$ so that the shift u_e in citizen preferences is high relative to the median legislator's unrepresentativeness u_m. Suppose, as is shown in the figure, that the median citizen's preferred policy shifts to the right, i.e., $+u_e$ is drawn. For small supermajorities ($x_\alpha < \frac{u_e - u_m}{2}$), no proposers are constrained, so increasing x_α has no welfare effects. However, once x_α increases past $\frac{u_e - u_m}{2}$, the conservative proposer (at $u_e - u_m$) who lies between the status quo and the median citizen is constrained to implement a policy that is *even* closer to the status quo, hurting the median citizen; but the radical median legislator further from the status quo (at $u_e + u_m$) remains unconstrained (as the key legislators remain closer to his bliss point than the status quo). Hence, as x_α is raised further, citizen disutility increases, reaching a local maximum when the conservative legislator becomes blocked. Only when increases in x_α start to constrain the radical median legislator, does citizen welfare start to rise, reaching a local maximum when this median legislator is constrained to move policy only to e.



3.1: Policy outcomes as a function of x_α.



3.2: Disutility as a function of x_α.

Figure 3: Policy outcomes and disutility as a function of x_α when $e = u_e > 0$ and $u_m = u_e/\sqrt{2}$.

The comparative statics are straightforward. Increasing the shift u_m in the median legislator's bliss point by ϵ also shifts the key legislators by ϵ, ceteris paribus allowing the median proposer to shift policy 2ϵ further from the status quo. To offset this, the optimal x_α *increases* by ϵ so that he again enacts e. Alternatively, increasing the shift u_e in the median citizen's bliss point by ϵ shifts the radical median legislator by ϵ and hence policy by 2ϵ from the status quo. To offset this, the optimal x_α *increases* by $\epsilon/2$ so that he shifts policy by ϵ to enact $e + \epsilon$.

In the online appendix, we solve for the optimal voting rule when e and m are both uniformly distributed. The qualitative properties of the optimal voting rule are very similar. In particular, a small supermajority is never optimal: Fixing the distribution of e, as the distribution of m grows more dispersed, the optimal voting rule jumps from simple majority to a non-trivial supermajority.

Normal uncertainty. We now characterize optimal voting rules when the median citizen's bliss point and that of the median legislator are normally distributed.

Proposition 4. *Let $e \sim N(0, \sigma_e^2)$ and $m \sim N(e, \sigma_m^2)$. If the legislature is sufficiently unrepresentative of citizen preferences that $\sigma_m^2 > \sigma_e^2/2$, then the optimal voting rule rises with the volatility in* *the representativeness of the legislature, σ_m^2, and falls with the volatility of citizen preferences, σ_e^2. Further, $x_\alpha^*(\gamma\sigma_e, \gamma\sigma_m) = \gamma x_\alpha^*(\sigma_e, \sigma_m)$ where $x_\alpha^*(\sigma_e, \sigma_m)$ is the optimal voting rule distance given σ_e and σ_m. If, instead, $\sigma_m^2 \leq \sigma_e^2/2$, then simple majority is optimal.*

To prove this result, we first show that the optimal voting rule distance, if positive, solves

$$\Delta(\sigma_e^2, \sigma_m^2) \equiv \frac{2\sigma_m^2 + \sigma_e^2}{2\delta^2} = \gamma \frac{[\Phi(2\gamma) - \Phi(\gamma)]}{[\phi(\gamma) - \phi(2\gamma)]}, \tag{4}$$

where $\delta \equiv \sqrt{\sigma_e^2 + \sigma_m^2}$ and $\gamma \equiv x_\alpha/\delta$. Uniqueness of the optimal voting rule is proven by showing that the right-hand side of equation (4) strictly increases in γ.

The comparative statics follow directly. As with two-point or uniform distributions, x_α^* increases as the median legislator becomes less representative of society, in order to constrain the median proposer, who is more likely to be radical, from enacting extreme policies. However, in contrast to the two-point uncertainty settings, the optimal voting rule rises continuously from simple majority. Rather, the analogue of the discontinuity in that setting is that the slope of the optimal voting rule is infinite at the point where it increases from simple majority. Thus, once again, slight supermajorities are (almost) never optimal.

Increased dispersion in $\mathbf{F_m}$. One might conjecture, based on consideration of Propositions 3–5, that when legislative preferences grow less representative of their citizens' then the optimal supermajority should *always* rise to provide citizens increased protection from a rogue legislature. Proposition 5 shows that this conjecture is false. Provided that the legislature is sufficiently unrepresentative that a supermajority is optimal, one can *always* find a less representative distribution of median legislator's preferences such that the optimal voting rule is reduced.

Proposition 5. *Suppose given distributions $F_e(\cdot),\ F_m(\cdot) \in \Omega$ over median citizen and median legislator bliss points that $x_{\alpha F}^* > 0$. Then there exists a more dispersed distribution $G_m(\cdot) \in \Omega$ of median legislator bliss points, i.e., $F_m(\mu_m) \geq G_m(\mu_m)$ for all $\mu_m \geq 0$, yet $x_{\alpha G}^* < x_{\alpha F}^*$.*

The constructive proof reflects that citizens would like to constrain moderately conservative median proposers by less on average, and constrain more radical proposers by more. However, raising x_α has no effect on the margin on proposers m who are so far from the status quo that they are free from legislative constraint—outcomes associated with the most radical proposers do not influence the optimal voting rule. Some spreads of the distribution of proposers *raise* the fraction of *constrained* proposers whom citizens want to constrain by less. The more dispersed distribution G_m that we construct replaces median legislators who are an intermediate distance from e with those who are far away. That is, G_m replaces proposers who are very likely to be constrained, but by too little, with proposers whom citizens want to constrain, but cannot as they are very likely to be free. As a result, conditional on a shift in policy, the expected new policy is too close to the status quo given $x^*_{\alpha F}$, and welfare is raised by relaxing the voting rule to $x^*_{\alpha G}$.

One can analogously find spreads of the distribution of the median citizen's bliss point—so that citizens want greater changes in policy—such that *larger supermajorities are optimal,* i.e., citizens want to further restrain the legislature from changing policy by as much. The intuition and qualitative construction of the proof is identical.

4 More Extreme Legislative Proposers

We now consider proposers other than the median legislator. Such extreme proposers arise in multi-party settings, where a proposer is from the majority party, which is to one ideological side of the legislature; or when the proposer is the president or prime minister, whose preferences do not perfectly coincide with the median legislator's. One is especially interested in how the extent of a proposer's extremism affects the optimal voting rule. In particular, when does increasing the separation of powers by divorcing the identity of a proposer from that of the median legislator complement control via the size of a supermajority, and when does it substitute?

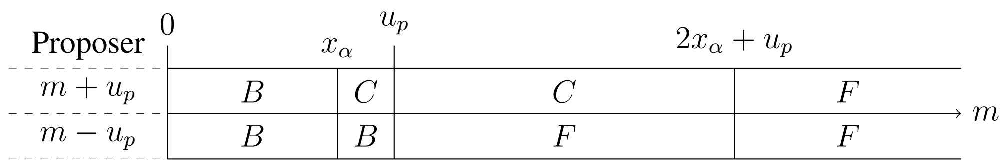
To focus on the "representativeness" of a proposer, we consider randomly-chosen proposers who are equally likely to lie u_p to the left or right of the median legislator, i.e., $F_p(z) = 0$ if $z < -u_p$, $F_p(z) = 1/2$ if $-u_p \leq z < u_p$, and $F_p(z) = 1$ if $u_p \leq z$. We refer to u_p as the proposer's *polarity*, since a higher u_p represents a more polarized proposer within the legislature. Graham [2011] shows that the qualitative impact of a more polarized u_p extends directly (via integration) to settings where u_p has a continuous support and there is a shift in probability mass away from the median legislator.

We first show that if the proposer is so unrepresentative of the legislature that $u_p > x^*_\alpha$, i.e., if the proposer is further from the median legislator than the key legislators, then, as simple intuition might suggest, further increases in a proposer's polarity raise the optimal voting rule, due to the importance of further restraining the proposer. Indexing the optimal voting rule by u_p, we have

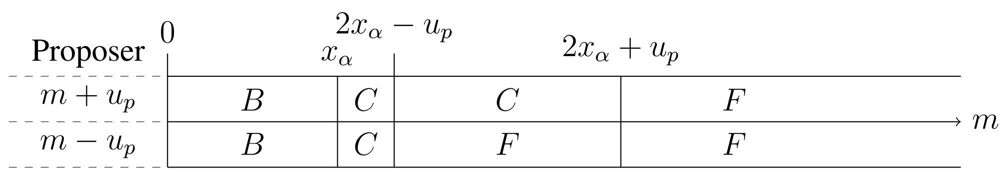
Proposition 6. *If $u_p > x^*_\alpha(u_p) > 0$, then an increase in u_p raises the optimal supermajority.*

Figure 4 divides realizations of median legislators into three states for a given polarity level u_p and voting rule x_α. For realizations of m close to the status quo, the radical proposer is blocked

(B), for realizations far from the status quo, the radical proposer is free (F) and for intermediate realizations, the radical proposer is constrained (C). A marginal change in the voting rule affects utility only via its effect on constrained proposers. As Figure 4 sows, for a proposer to be constrained, the median legislator must be at least x_α from the status quo. Furthermore, as Figure 4.1 shows, when μ_p exceeds x_α, the bliss point of the proposer closest to the status quo is either within the feasible policy set ($m > u_p$)[3] or it is on the opposite side of the status quo to the median legislator ($m < u_p$), i.e., this proposer is either free or blocked, but never constrained. The optimal voting rule ensures that, conditional on the proposer furthest from the status quo being constrained, his expected proposed policy equals the median citizen's expected bliss point.



4.1: $u_p > x_\alpha$.



4.2: $x_\alpha > u_p$.

Figure 4: Realizations of m classified by proposer state: free (F), blocked (B), or constrained (C) for a given voting rule and polarity.

We define the radical proposer associated with median legislator at the point $2(x_\alpha + u_p)$ as the *marginally-free* radical proposer; a further shift of this proposer away from the status quo makes the proposer constrained. The only affect of a marginal increase in polarity is to constrain this marginally-free proposer. This additional constrained policy movement is always further from the status quo than m ; with the quasi-concavity of $f_e(e)$, the expected location of the median citizen is also closer to the status quo. Hence, a further increase in the already substantial polarity u_p of proposers makes a larger supermajority optimal.

The logic behind Proposition 6 extends more generally to distributions that place more weight on such polarized proposers: Graham [2011] shows that if distributions $F_p^1(z)$ and $F_p^2(z)$ agree on $-x_\alpha^{*1} \leq z \leq x_\alpha^{*1}$, but F_p^2 has more dispersed tails so that $F_p^2(z) \leq F_p^1(z)$ for all $z > x_\alpha^{*1}$, strictly for some z and $x_\alpha^{*1} > 0$, then the optimal supermajority is larger for the distribution with more

[3] $0 < m - u_p < m - x_\alpha < 2(m - x_\alpha)$

polarized proposers, $x_\alpha^{*2} > x_\alpha^{*1}$.

However, this logic does *not* extend when $x_\alpha^* > u_p$, so that both radical *and conservative* proposers can be constrained. Figure 4.2 illustrates this case: conservative and radical marginally-free proposers both lie to the right of the status quo—the conservative proposer lies at $2(x_\alpha - u_p) > 0$, while the radical proposer lies further away at $2(x_\alpha + u_p)$. When u_p is small, both proposers move policy too far on average relative to the median citizen's bliss point. However, the radical marginally-free proposer moves policy further away, since he is further from the status quo.

Raising polarity (a) increases the measure of constrained radical proposers, and it *reduces* the measure of constrained *conservative* proposers. The net effect is too much movement in policy. However, (b) the median legislator associated with the conservative marginally-free proposer is closer to the status quo at $2x_\alpha - u_p$ than the median legislator associated with the radical marginally-free proposer at $2x_\alpha + u_p$. This means that conservative marginally-free proposers are more likely to be realized—conservative marginally-free proposers have higher probability weights than radical ones. The question for the design of the voting rule becomes whether and when this likelihood of selection effect dominates the relatively higher measure of radical proposers.

We next characterize conditions for the optimal voting rule to *fall* as the polarity of proposers rises. For an increase in u_p to reduce the optimal voting rule, the dispersion of the median legislator around e must be large relative to the variation in e, i.e., the legislature must be sufficiently unrepresentative. When this dispersion rises, so too does x_α^*, and the greater probability weight placed on a conservative marginally-free proposer who lies closer to the status quo dominates.

Proposition 7. (Normal Uncertainty). *Let $e \sim N(0, \sigma_e^2)$ and $\mu_m \sim N(0, \sigma_m^2)$. Then introducing slight polarization reduces the optimal voting rule if and only if the legislator is sufficiently unrepresentative relative to the variation in citizen preferences: $\frac{\partial x_\alpha}{\partial u_p}\big|_{u_p=0} < 0$ if and only if $\sigma_e^2/\sigma_m^2 < \beta$, where $\beta = 4\eta^2 - 2$ and η solves*

$$\frac{4\eta^2}{8\eta^2 - 2} = \eta \frac{[\Phi(2\eta) - \Phi(\eta)]}{[\phi(\eta) - \phi(2\eta)]} \quad \Rightarrow \quad \beta \approx 1.339.$$

When u_p is zero, i.e., when the median legislator is the proposer, the first-order effect of an increase in u_p on the optimal voting rule is zero (equation (17) in the Appendix). In the limit, radical and conservative marginally-free proposers realize the same policy outcome and are given the same weight since they have the same probability of being realized.

The second-order effect is

$$\left.\frac{\partial^3 U}{\partial x_\alpha \partial u_p^2}\right|_{u_p=0} = 8\int_{-\infty}^{\infty} f_m(2x_\alpha - e) + (2x_\alpha - e) f_m'(2x_\alpha - e)\, dF_e(e). \tag{5}$$

The first term in the integrand of equation (5) represents the net movement in policy away from the status quo conditional on the proposer being constrained due to the increase in radical constrained proposers and reduction in conservative constrained proposers. This effect on the optimal voting rule is always positive. The second term represents the difference in probability weight placed on the conservative marginally-free proposer relative to the radical marginally-free proposer. This term is symmetric around $2x_\alpha$ as a function of e and is negative for all values of e. For the large voting rules implied in this context by the extent of volatility in legislature preferences relative to society, i.e., for $\sigma_e^2/\sigma_m^2 < \beta \approx 1.339$, this negative second effect outweighs the positive first effect.

Graham [2011] proves an analogous result with uniform uncertainty: when $e \sim U[-a, a]$, $\mu_m \sim U[-b, b]$ and $b > a$, the optimal supermajority is *higher* when the proposer is the median than when the proposer has a moderate polarity, $u_p \in (b - a, b - \frac{a}{2})$.

5 Status Quo Policy Bias

We conclude by studying how bias in the status quo policy affects the optimal voting rule. With bias, the status quo no longer equals the median citizen's expected bliss point. Our analysis provides insights into how the optimal voting rule is affected when past policy choices by a legislature are slow to catch up with shifts in citizen preferences. With greater bias, the status quo is less representative of the median citizen's preferences. Intuition might then suggest that this should cause citizens to rely more on the legislature to determine policy by reducing x_α. However, we prove that this conjecture is false whenever legislative preferences tend to be representative of society's, but legislative proposers are sufficiently polarized: whenever this is so, slight policy bias causes citizens to raise the voting rule and rely *less* on the legislature.

To facilitate analysis, we suppose that F_e and F_m are normally distributed. Proposition 8 provides sufficient conditions on the volatility of societal preferences, median legislator preferences and the proposer's polarity for an increase in initial policy bias from zero to either decrease or increase the optimal supermajority. Recall that a supermajority is optimal if (a) there is sufficient volatility in the median legislator's bliss point relative to that of the median citizen's and the median legislator is the proposer, or (b) when the median legislator's preferences are more representative of the median citizen's but the legislative proposer is sufficiently polarized. The proposition first establishes that if the median is the proposer and a positive supermajority is optimal, then introducing slight policy bias *reduces* the optimal voting rule. It then establishes that if, instead, the median legislator is sufficiently representative *and* the proposer's polarity is high enough, then slight bias *raises* the optimal voting rule.

Proposition 8. *Let* $e \sim N(0, \sigma_e^2)$ *and* $m \sim N(e, \sigma_m^2)$.

1. *Suppose that the legislature is unrepresentative of citizens, so that* $\sigma_e^2 < 2\sigma_m^2$*, and the median legislator is the proposer. Then introducing slight bias in the initial policy away from the median citizen's expected bliss point* **reduces** *the optimal voting rule.*

2. *Suppose, instead, that the legislature is more representative of citizen preferences, so that* $\sigma_m^2 < \chi\sigma_e^2$ *where* $\chi = \frac{2(1-\Phi(1))-\phi(1)}{2(\phi(1)+\Phi(1)-1)} \approx 0.452$*, and the legislative proposer is not the median legislator. There exists a* $\bar{u}_p$ *such that when the proposer's polarity exceeds* $\bar{u}_p$*, introducing slight bias in the initial policy* **raises** *the optimal voting rule.*

The first-order effect of slight positive policy bias on the optimal policy rule is zero. Proposers to the left of the median citizen's expected bliss point are less constrained and generate greater shifts in policy away from the status quo, but this is exactly offset by the reduced shifts away from the status quo by proposers to the right who are more constrained.

Hence, we need to understand the second-order effects. These are similar in nature to those for the impact of a marginal increase in the proposer's polarity from zero:

$$\frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2}\Big|_{S_0=0} = 8\int_{-\infty}^{\infty} [(2(x_\alpha + u_p) - e) f'_m(2x_\alpha + u_p - e) + e f'_m(x_\alpha - e)]\, dF_e(e). \quad (6)$$

Slightly increasing policy bias from zero has two second-order effects: those associated with changes in the probabilities that a proposer is (a) marginally *free* vs. (b) marginally *blocked*, where a proposer is marginally blocked when he is exactly x_α from the status quo.

Equation (6) uses symmetry to focus on changes in policy when m and u_p are positive. The first term in (6) is associated with the change in the probability that the proposer is marginally free, weighted by the difference between the policy outcome associated with this proposer and the median citizen's bliss point. The median legislator associated with a marginally-free proposer is located at $2x_\alpha + u_p + S_0$, and, hence, is shifted to the right by a slight positive status quo bias. Since $f'_m(.|\mu_m > 0) < 0$, this rightward shift reduces the probability the marginally-free legislator is chosen if and only if $e < 2x_\alpha + u_p$. Hence, if $e < 2x_\alpha + u_p$ or $e > 2(x_\alpha + u_p)$, this first term is negative, i.e., conditionally-constrained new policy shifts to the left, implying that policy moves too little, suggesting a smaller voting rule is optimal. If, instead, $2x_\alpha + u_p < e < 2(x_\alpha + u_p)$ then this first term is positive, i.e., conditionally-constrained new policy shifts to the right, implying that policy moves too much, suggesting a larger voting rule. The net effect of these two forces is always negative. However, as the polarity u_p increases, this net effect goes to zero: the probabilities associated with marginally-free proposers who are located in the extreme tail of f_m are approximately constant as we introduce slight policy bias.

The second term in (6) is associated with changes in the likelihood of the marginally-blocked proposer, weighted by the difference between the policy outcome associated with this proposer and the median citizen's bliss point. The median legislator associated with the marginally-blocked proposer is located at $x_\alpha + S_0$, and is also shifted to the right by slight positive status quo bias. This shift reduces the probability this legislator is chosen if and only if $e < x_\alpha$. Hence, when $e < 0$ or $e > x_\alpha$ conditionally-constrained new policy moves to the right, implying excessive movement in policy and a larger optimal voting rule. When $0 < e < x_\alpha$ conditionally-constrained new policy moves to the left, implying too little policy movement, suggesting a smaller voting rule is optimal. When the voting rule is small enough, so that x_α is small, the net effect is excessive movement in policy.

Which effect dominates depends on the proposer's polarity and the optimal voting rule distance x_α. When u_p is high, the marginally-blocked proposer effect dominates since the marginally-free proposer effect is approximately zero. The marginally-blocked proposer effect is strictly positive when the optimal voting rule is simple majority. Thus, when u_p is high, and σ_m^2 is small (so that x_α is small), introducing slight initial policy bias causes x_α^* to rise.

6 Conclusion

Ceteris paribus, a legislature that can freely tailor policy to reflect societal preferences is good. However, a legislature's composition will not always reflect society's, so that an unchecked legislature sometimes implements bad policy. This paper characterizes how the primitives describing the preferences of society, the median legislator, and the agenda setter affect the optimal degree of inertia—the required vote share to implement a proposed policy rather than the status quo.

If the legislature and agenda setter are always sufficiently representative of society, then, of course, citizens never want to constrain them—a simple majority voting rule is optimal. However, when the legislature is more likely to be unrepresentative, the optimal/equilibrium voting rule trades off between reducing excessive policy variation and introducing more policy bias. Supermajority voting rules are blunt instruments. A greater supermajority that restrains more radical proposers, overly constrains or completely blocks more conservative proposers from implementing socially beneficial policy changes. We show that as a result, as the legislature grows stochastically less representative of society, the optimal voting rule can jump from simple majority to a large supermajority: mirroring real world practice, slight supermajorities, which tend to restrict moderate proposers disproportionately, may never be chosen.

When the median legislator is the proposer, we identify families of distributions of preferences for which less-representative legislatures make larger supermajorities optimal, reflecting the first-order intuition that less-representative proposers are more likely to *want* to implement policies that harm society. However, this intuition is incomplete—the optimal voting rule only reflects proposers whose policy choices are constrained by the voting rule, weighing neither those who are completely free to implement preferred policies nor those who are completely blocked from implementing any change that they prefer. As a result, the optimal supermajority is smaller for less-representative legislatures that feature reduced likelihoods of drawing proposers who are close to the status quo, and hence are blocked from changing policy, and higher likelihoods of proposers who are moderately further away and tend to be overly-constrained (leaving likelihoods of other proposers unchanged).

We also derive the consequences of having polarized proposers set policy agendas. Proposers who are more extreme within a legislature also tend to be further from the median citizen, making it important to restrain them. However, the intuition that more polarized agenda setters should always make greater supermajorities optimal is misplaced—more extreme proposers, although tending to desire more extreme policies, are also more constrained by the necessity of winning approval from moderate representatives—it is *easier* to restrain extremists from moving policy in a direction that they prefer, away from what society prefers, with the result that extremists often end up doing nothing. In contrast, an unrepresentative median proposer finds it easier to move policy away from what society prefers. As the dispersion of the median legislator around the median citizen's bliss point rises, *conditional on legislative success*, a more polarized proposer is more likely to move policy in the direction society prefers. We characterize conditions under which the relationship between polarity and the optimal supermajority is U-shaped.

We conclude by investigating the impact of initial policy bias. One's intuition may be that policy bias should induce citizens to rely more on the legislature to determine policy. However, we prove that when the variation in society's preferred policy is large relative to that in the representativeness of the median legislator, introducing slight initial policy bias *raises* the optimal voting rule. The intuition is that with slight policy bias, the proposers who are more likely to be constrained by a given supermajority are those who would move policy *away* from the median citizen.

An interesting direction to take our analysis would be to model formally the bicameral system of a House and Senate, where there are two draws of ideologies, one for each branch and the requisite vote shares required to shift policy from the status quo are possibly not equal. Now the correlation structure between the median citizen's preferences and those of the House and Senate are vital for determining the optimal voting rule. This correlation structure may reflect that at each two year cycle, every House seat is up for re-election, but only one third of the Senate; or that the distribution of ideologies in the House may have fatter tails or tend to be more conservative than that for the Senate. It may also be that the degree of accord between the House and Senate tends to be high when they accord with preferences of the median citizen, making a simple majority optimal; but times of disagreement tend to be associated with substantial differences in the m's of the two branches relative to e, making substantial inertia optimal, i.e., the optimal effective supermajority should be large. Thus, the bicameral system may implement an endogenous supermajority depending on the correlation of interests between the two chambers, which, in turn, is correlated with the variation of m relative to e. Interestingly, even with an i.i.d. structure in the House and Senate, it may well be that a supermajority might be optimal in, for example, the Senate, but a simple majority in the House—to achieve the "right" amount of flexibility. This is quite speculative, however.

Also, although our analysis focuses on the political arena, it has broader implications. For example, it provides insights into the design of corporate governance. In corporate settings, shareholder and management interests may be less aligned on some dimensions—management compensation, management entrenchment (e.g., via anti-takeover provisions), or membership in the board of directors—than others. Optimal policy may vary with firm circumstances, and the optimal size of the shareholder vote required to shift policy may vary across dimensions in ways that we describe.

7 Appendix

Derivation of the Optimal Voting Rule first-order condition (2). To see the derivation most clearly, first suppose that the median legislator is the proposer. Then the median voter's utility is:

$$-2\int_{e=-\infty}^{\infty}\left[\int_{\mu_m=-\infty}^{-2x_\alpha-e}\mu_m^2\,dF_m(\mu_m)+\int_{\mu_m=-2x_\alpha-e}^{-x_\alpha-e}(e-2(m+x_\alpha))^2\,dF_m(\mu_m)\right.$$
$$\left.+\int_{\mu_m=-x_\alpha-e}^{-e}e^2\,dF_m(\mu_m)\right]dF_e(e).$$

To express everything in terms of the same random variables, we change the variable of integration of the inner integral from μ_m to $m=e+\mu_m$ to obtain

$$-2\int_{e=-\infty}^{\infty}\left[\int_{m=-\infty}^{-2x_\alpha}(m-e)^2\,dF_m(m-e)+\int_{m=-2x_\alpha}^{-x_\alpha}(e-2(m+x_\alpha))^2\,dF_m(m-e)\right.$$
$$\left.+\int_{m=-x_\alpha}^{0}e^2\,dF_m(m-e)\right]dF_e(e).$$

Differentiating only the Leibnitz terms with respect to x_α gives

$$-2\int_{e=-\infty}^{\infty}\left[(-2x_\alpha-e)^2f_m(-2x_\alpha-e)(-2)+e^2f_m(-x_\alpha-e)(-1)\right.$$
$$\left.-(e+2x_\alpha)^2f_m(-2x_\alpha-e)(-2)-e^2f_m((-x_\alpha-e)(-1)\right]dF_e(e),$$

which equals zero: *the derivatives of the Leibnitz terms cancel*. Thus,

$$\left.\frac{\partial U}{\partial x_\alpha}\right|_{x_\alpha^*}=8\int_{e=-\infty}^{\infty}\left[\int_{m=-2x_\alpha}^{-x_\alpha}(e-2(m+x_\alpha)\,dF_m(m-e)\right]dF_e(e).$$

Converting this into an expression analogous to the one in the text using e and μ_m gives

$$\left.\frac{\partial U}{\partial x_\alpha}\right|_{x_\alpha^*}=8\int_{e=-\infty}^{\infty}\left[\int_{\mu_m=-2x_\alpha-e}^{-x_\alpha-e}(-e-2(\mu_m+x_\alpha))\,dF_m(\mu_m)\right]dF_e(e).\qquad(7)$$

For the derivation of the first-order condition in the more general setting where the median need not be the proposer, we use Liebnitz's rule and express the derivative terms from the outer integration limits inwards. The last term is the derivative with respect to the integrand. Differentiating (1) with respect to x_α gives

$$\frac{\partial U}{\partial x_\alpha} = -2\int_{e=-\infty}^{\infty}\Bigg[e^2 f_m(x_\alpha - e)$$
$$-\left(\int_{p=-\infty}^{0} e^2\, dF_p(p-x_\alpha) + \int_{p=0}^{0}(e-p)^2\, dF_p(p-x_\alpha) + \int_{p=0}^{\infty} e^2\, dF_p(p-x_\alpha)\right) f_m(x_\alpha - e)$$
$$-\int_{m=x_\alpha}^{\infty} 2(e-2(m-x_\alpha))^2 f_p(m-2x_\alpha) - 2(e-2(m-x_\alpha))^2 f_p(m-2x_\alpha)\, dF_m(m-e)$$
$$+\int_{m=x_\alpha}^{\infty}\int_{p=2(m-x_\alpha)}^{\infty} 2(e-2(m+x_\alpha))(-2)(-1)\, dF_p(p-m)\, dF_m(m-e)\Bigg]\, dF_e(e)$$

Once again the derivatives of the Liebnitz terms cancel (the first line cancels with its expansion on the second line, and the third line clearly cancels) leaving only

$$-2\int_{e=-\infty}^{\infty}\left[\int_{m=x_\alpha}^{\infty}\int_{p=2(m-x_\alpha)}^{\infty} 2(e-2(m+x_\alpha))(-2)(-1)\, dF_p(p-m)\, dF_m(m-e)\right] dF_e(e),$$

which can be rewritten as (2).

Proof of Proposition 1: Let x^*_α maximize equation (1). By the definition of x^*_α, $F^{-1}_{l_1}(\alpha^*_{l_1}; m) = F^{-1}_{l_2}(\alpha^*_{l_2}; m)$. Since $F^{-1}_{l_1}(\alpha^*_{l_2}; m) \geq F^{-1}_{l_2}(\alpha^*_{l_2}; m)$ for a given m, this implies $\alpha^*_{l_2} \geq \alpha^*_{l_1}$. If $\alpha^*_{l_2} > 0$ then $F^{-1}_{l_1}(\alpha^*_{l_2}; m) > F^{-1}_{l_2}(\alpha^*_{l_2}; m)$ and $\alpha^*_{l_2} > \alpha^*_{l_1}$. ■

Proof of Proposition 2: Let $x'_\alpha = \bar{m} + \epsilon$, where $\epsilon < \bar{e}$, the support of e (which could be infinite). Ex ante, the probability of a policy change is strictly positive since $e + \bar{m} > \bar{m}, \forall e > \epsilon$ and $e - \bar{m} < \bar{m}, \forall e < -\epsilon$. For any value of μ_m, policy is either unchanged or the policy change moves policy closer to e. To see this let $e \geq 0$ (the analysis for $e < 0$ is similar). For any $m = e + \mu_m < 0$, $S_1 = 0$ since $|m| < \bar{m}$. Movement in policy is only possible if $m = e + \mu_m > \bar{m} + \epsilon$. Fix such an m. The feasible policy set, $R(m, \alpha) = [0, 2(m - \bar{m} - \epsilon)]$. For all m, $2(m - \bar{m} - \epsilon) - e = e - 2(\bar{m} + \epsilon - \mu_m) < e$ since $\mu_m < \bar{m}$. Hence, any movement in policy is toward e and utility under x'_α is greater than under a voting rule where policy never changes. ■

Proof of Proposition 3: If $u_e > u_m$, then

$$U = \begin{cases} -u_m^2 & \text{if } 0 < x_\alpha < \frac{u_e - u_m}{2} \\ -\frac{(u_e - 2(u_m + x_\alpha))^2 + u_m^2}{2} & \text{if } \frac{u_e - u_m}{2} < x_\alpha < \min\{u_e - u_m, \frac{u_e + u_m}{2}\} \\ -\frac{(u_e - 2(u_m + x_\alpha))^2 + (u_e + 2(u_m - x_\alpha))^2}{2} & \text{if } \frac{u_e + u_m}{2} < x_\alpha < u_e - u_m \\ -\frac{u_e^2 + u_m^2}{2} & \text{if } u_e - u_m < x_\alpha < \frac{u_e + u_m}{2} \\ -\frac{u_e^2 + (u_e + 2(u_m - x_\alpha))^2}{2} & \text{if } \max\{u_e - u_m, \frac{u_e + u_m}{2}\} < x_\alpha < u_e + u_m \\ -u_e^2 & \text{if } u_e + u_m < x_\alpha \end{cases}$$

and

$$\frac{\partial U}{\partial x_\alpha} = \begin{cases} 0 & \text{if } 0 < x_\alpha < \frac{u_e - u_m}{2} \\ 2(u_e - 2(u_m + x_\alpha)) & \text{if } \frac{u_e-u_m}{2} < x_\alpha < \min\{u_e - u_m, \frac{u_e+u_m}{2}\} \\ 4(u_e + 2x_\alpha) & \text{if } \frac{u_e+u_m}{2} < x_\alpha < u_e - u_m \\ 0 & \text{if } u_e - u_m < x_\alpha < \frac{u_e+u_m}{2} \\ 2(u_e + 2(u_m - x_\alpha)) & \text{if } \max\{u_e - u_m, \frac{u_e+u_m}{2}\} < x_\alpha < u_e + u_m \\ 0 & \text{if } u_e + u_m < x_\alpha. \end{cases}$$

If $u_e < u_m$ then

$$U = \begin{cases} -u_m^2 & \text{if } 0 \leq \frac{u_m - u_e}{2} \\ -\frac{(u_e - 2(u_m - x_\alpha))^2 + u_m^2}{2} & \text{if } \frac{u_m-u_e}{2} < x_\alpha < \min\{u_m - u_e, \frac{u_m+u_e}{2}\} \\ -\frac{(u_e - 2(u_m - x_\alpha))^2 + (u_e + 2(u_m - x_\alpha))^2}{2} & \text{if } \frac{u_m+u_e}{2} < x_\alpha < u_m - u_e \\ -\frac{u_e^2 + u_m^2}{2} & \text{if } u_m - u_e < x_\alpha < \frac{u_m+u_e}{2} \\ -\frac{u_e^2 + (u_e + 2(u_m - x_\alpha))^2}{2} & \text{if } \max\{u_m - u_e, \frac{u_m+u_e}{2}\} < x_\alpha < u_m + u_e \\ -u_e^2 & \text{if } u_m + u_e < x_\alpha \end{cases}$$

and

$$\frac{\partial U}{\partial x_\alpha} = \begin{cases} 0 & \text{if } 0 \leq \frac{u_m - u_e}{2} \\ -2(u_e - 2(u_m - x_\alpha)) & \text{if } \frac{u_m-u_e}{2} < x_\alpha < \min\{u_m - u_e, \frac{u_m+u_e}{2}\} \\ 8(u_m - x_\alpha) & \text{if } \frac{u_m+u_e}{2} < x_\alpha < u_m - u_e \\ 0 & \text{if } u_m - u_e < x_\alpha < \frac{u_m+u_e}{2} \\ 2(u_e + 2(u_m - x_\alpha)) & \text{if } \max\{u_m - u_e, \frac{u_m+u_e}{2}\} < x_\alpha < u_m + u_e \\ 0 & \text{if } u_m + u_e < x_\alpha. \end{cases}$$

The utility from a voting rule $x_\alpha \in [0, \frac{|u_e - u_m|}{2}]$ is $-\mu_m^2$. Over each interval where utility is a function of x_α, the second-order condition for a maximum is always satisfied. It is easy to show that the solution to the first-order condition over each interval of x_α satisfies the interval constraints only when $\max\{\frac{u_e+u_m}{2}, |u_e - u_m|\} < x_\alpha < u_e + u_m$. The solution to the relevant first-order condition is $x_\alpha^* = u_m + u_e/2$, yielding utility $-u_e^2/2$. This exceeds $-u_m^2$ only if $u_e < \sqrt{2}u_m$. ■

Proof of Proposition 4: Letting $\mu_m = m - e$ in equation (3) and assuming $x_\alpha^* > 0$, we have

$$\frac{\partial U}{\partial x_\alpha} = 8 \int_{-\infty}^{\infty} \int_{x_\alpha - e}^{2x_\alpha - e} (e + 2(\mu_m - x_\alpha))\, dF_m(\mu_m)\, dF_e(e).$$

Expanding this gives

$$\begin{aligned}\frac{\partial U}{\partial x_\alpha} &= 8\int_{-\infty}^{\infty} e(F_m(2x_\alpha - e) - F_m(x_\alpha - e))\, dF_e(e)\\ &+ 16\int_{-\infty}^{\infty} \mu_m(F_m(2x_\alpha - \mu_m) - F_m(x_\alpha - \mu_m))\, dF_m(\mu_m)\\ &- 16x_\alpha\int_{-\infty}^{\infty} (F_m(2x_\alpha - e) - F_m(x_\alpha - e))\, dF_e(e),\end{aligned}$$

where the last term comes from interchanging the order of integration, μ_m and e. Converting the distributions to standard normal distributions gives

$$\begin{aligned}\frac{\partial U}{\partial x_\alpha} &= 8\int_{-\infty}^{\infty} e\left(\Phi\left(\frac{2x_\alpha - e}{\sigma_m}\right) - \Phi\left(\frac{x_\alpha - e}{\sigma_m}\right)\right)\phi\left(\frac{e}{\sigma_e}\right) de/\sigma_e \qquad (8)\\ &+ 16\int_{-\infty}^{\infty} \mu_m\left(\Phi\left(\frac{2x_\alpha - \mu_m}{\sigma_e}\right) - \Phi\left(\frac{x_\alpha - \mu_m}{\sigma_e}\right)\right)\phi\left(\frac{\mu_m}{\sigma_m}\right) d\mu_m/\sigma_m\\ &- 16x_\alpha\int_{-\infty}^{\infty} \left(\Phi\left(\frac{2x_\alpha - e}{\sigma_m}\right) - \Phi\left(\frac{x_\alpha - e}{\sigma_m}\right)\right)\phi\left(\frac{e}{\sigma_e}\right) de/\sigma_e,\end{aligned}$$

where Φ and ϕ are the standard normal distribution function and probability density function, respectively. From Patel and Read [1996],

$$\begin{aligned}\int_{-\infty}^{\infty} \Phi(a+bx)\phi(x)dx &= \Phi(a/\sqrt{1+b^2}) \quad \text{and}\\ \int_{-\infty}^{\infty} x\Phi(a+bx)\phi(x)dx &= \frac{b}{\sqrt{1+b^2}}\phi\left(\frac{a}{\sqrt{1+b^2}}\right).\end{aligned}$$

Defining $\delta = \sqrt{\sigma_e^2 + \sigma_m^2}$, equation (8) simplifies to

$$\frac{\partial U}{\partial x_\alpha} = \frac{8\sigma_e^2}{\delta}\left[\phi\left(\frac{x_\alpha}{\delta}\right) - \phi\left(\frac{2x_\alpha}{\delta}\right)\right] + \frac{16\sigma_m^2}{\delta}\left[\phi\left(\frac{x_\alpha}{\delta}\right) - \phi\left(\frac{2x_\alpha}{\delta}\right)\right] - 16x_\alpha\left[\Phi\left(\frac{2x_\alpha}{\delta}\right) - \Phi\left(\frac{x_\alpha}{\delta}\right)\right].$$

Thus, when the median legislator is always the proposer, and the optimal voting rule distance x_α^* is strictly positive, x_α^* must solve

$$\left(\frac{\sigma_e^2 + 2\sigma_m^2}{2\delta^2}\right)\left[\phi\left(\frac{x_\alpha^*}{\delta}\right) - \phi\left(\frac{2x_\alpha^*}{\delta}\right)\right] - 2\left(\frac{x_\alpha^*}{\delta}\right)\left[\Phi\left(\frac{2x_\alpha^*}{\delta}\right) - \Phi\left(\frac{x_\alpha^*}{\delta}\right)\right] = 0.$$

Define $\Delta(\sigma_e^2, \sigma_m^2) = \frac{\sigma_e^2 + 2\sigma_m^2}{2\delta^2}$, $\gamma(x_\alpha) = \frac{x_\alpha}{\delta}$ and $\Gamma(\gamma) = \gamma\left[\frac{\Phi(2\gamma) - \Phi(\gamma)}{\phi(\gamma) - \phi(2\gamma)}\right]$. Then x_α^* solves

$$\Delta(\sigma_e^2, \sigma_m^2) = \Gamma(\gamma(x_\alpha^*)). \qquad (9)$$

To show that the optimal voting rule is unique it is sufficient to show that $\Gamma(\gamma)$ is strictly increasing in γ. To do this we use the following lemma due to Pinelis [2002].

Lemma 1. *Let $-\infty \leq a < b \leq \infty$ and let f and g be differentiable functions on (a, b). Assume that either $g' > 0$ everywhere on (a, b) or $g' < 0$ on (a, b). Further, suppose that $f(a^+) = g(a^+) = 0$ or $f(b^-) = g(b^-) = 0$ and f'/g' is strictly increasing on (a, b). Then the ratio f/g is strictly increasing on (a, b).*

To use this lemma, let $f(\gamma) = \Phi(2\gamma) - \Phi(\gamma)$ and $g(\gamma) = [\phi(\gamma) - \phi(2\gamma)]/\gamma$. Then

$$f'(\gamma) = 2\phi(2\gamma) - \phi(\gamma) = \frac{e^{-2\gamma^2}}{\sqrt{2\pi}}(2 - e^{\frac{3\gamma^2}{2}}),$$

and

$$\begin{aligned} g'(\gamma) &= \frac{\gamma^2(4\phi(2\gamma) - \phi(\gamma)) - \phi(\gamma) + \phi(2\gamma)}{\gamma^2} \\ &= \frac{e^{-2\gamma^2}}{\gamma^2\sqrt{2\pi}}(1 + 4\gamma^2 - e^{\frac{3\gamma^2}{2}}(1 + \gamma^2)). \end{aligned}$$

Let $\bar{\gamma} \approx 0.654$ be the unique positive root to the transcendental equation $g'(\gamma) = 0$. Then $g'(\gamma) > 0$ on $(0, \bar{\gamma})$, and $g'(\gamma) < 0$ on $(\bar{\gamma}, \infty)$. Also, $f(0) = g(0) = 0$ and $f(\infty) = g(\infty) = 0$. Thus,

$$\frac{f'(\gamma)}{g'(\gamma)} = \frac{\gamma^2\left(2 - e^{\frac{3\gamma^2}{2}}\right)}{1 + 4\gamma^2 - e^{\frac{3\gamma^2}{2}}(1 + \gamma^2)}$$

and

$$\left(\frac{f'(\gamma)}{g'(\gamma)}\right)' = \frac{\gamma\left(4 + 2e^{3\gamma^2} + 3e^{\frac{3\gamma^2}{2}}(-2 + \gamma^2 - 2\gamma^4)\right)}{\left(1 + 4\gamma^2 - e^{\frac{3\gamma^2}{2}}(1 + \gamma^2)\right)^2}. \tag{10}$$

Equation (10) is increasing for all $\gamma \in (0, \infty)\backslash\{\bar{\gamma}\}$ since

$$\frac{\partial}{\partial\gamma}(4 + 2e^{3\gamma^2} - 3e^{\frac{3\gamma^2}{2}}(2 - \gamma^2 + 2\gamma^4)) = 3e^{\frac{3\gamma^2}{2}}\gamma\left(4e^{\frac{3\gamma^2}{2}} - (4 + 5\gamma^2 + 6\gamma^4)\right)$$

and

$$\frac{\partial}{\partial\gamma}\left(4e^{\frac{3\gamma^2}{2}} - 4 - 5\gamma^2 - 6\gamma^4\right) = 2\gamma\left(6e^{\frac{3\gamma^2}{2}} - 12\gamma^2 - 5\right),$$

which is positive for all $\gamma > 0$. Hence, by Lemma 1, Γ is strictly increasing over the intervals $(0, \bar{\gamma})$ and $(\bar{\gamma}, \infty)$. By continuity of the function, Γ is strictly increasing over $[0, \infty)$.

We now use L'Hôpital's rule twice to determine when the first-order condition characterizes x^*_α. Let $\Gamma_N(\gamma) = \gamma(\Phi(2\gamma) - \Phi(\gamma))$ and $\Gamma_D(\gamma) = \phi(\gamma) - \phi(2\gamma)$. Then

$$\lim_{\gamma\to 0}\frac{\Gamma'_N(\gamma)}{\Gamma'_D(\gamma)} = \lim_{\gamma\to 0}\frac{\gamma(2\phi(2\gamma) - \phi(\gamma)) + \Phi(2\gamma) - \Phi(\gamma)}{\gamma(4\phi(2\gamma) - \phi(\gamma))} = \frac{0}{0},$$

where we use the fact that $\phi'(\lambda\gamma) = -\lambda\gamma\phi(\lambda\gamma)$, and

$$\lim_{\gamma\to 0}\frac{\Gamma_N''(\gamma)}{\Gamma_D''(\gamma)} = \lim_{\gamma\to 0}\frac{4\phi(2\gamma)-2\phi(\gamma)+\gamma(4\phi'(2\gamma)-\phi'(\gamma))}{4\phi(2\gamma)-\phi(\gamma)+\gamma(8\phi'(2\gamma)-\phi'(\gamma))} = \frac{2}{3} = \Gamma(0).$$

The optimal voting rule is implicitly given by the solution to equation (9) if $\sigma_e^2 > 2\sigma_m^2$. Otherwise, $\alpha^* = 1/2$.

Next we show that the optimal voting rule increases with σ_m^2. As σ_m^2 increases, the left-hand side of equation (9) increases since

$$\frac{\partial\Delta(\sigma_e^2,\sigma_m^2)}{\partial\sigma_m^2} = \frac{\sigma_e^2}{2(\sigma_e^2+\sigma_m^2)^2} > 0.$$

An increase in σ_m^2 evaluating at $x_\alpha = x_\alpha^*$ leads to decreases in γ and $\Gamma(\gamma)$. Hence, the right-hand side of equation (9) falls at $x_\alpha = x_\alpha^*$. Thus, the optimal voting rule must increase with σ_m^2.

Next, we show that the optimal voting rule is decreasing in the volatility of the median citizen's bliss point. Differentiating equation (9) with respect to σ_e^2 gives

$$\begin{aligned}
\left.\frac{\partial^2 U}{\partial x_\alpha \partial\sigma_e^2}\right|_{x_\alpha = x_\alpha^*} &= -\frac{\sigma_m^2\,(\phi(\gamma)-\phi(2\gamma))}{2(\sigma_e^2+\sigma_m^2)^2} \\
&+ \left(\frac{\gamma(\sigma_e^2+2\sigma_m^2)(4\phi(2\gamma)-\phi(\gamma))}{2(\sigma_e^2+\sigma_m^2)} - (\Phi(\gamma)-\Phi(2\gamma)) - \gamma(2\phi(2\gamma)-\phi(\gamma))\right)\frac{\partial\gamma}{\partial\sigma_e^2} \\
&= \frac{\sigma_e^2(1-\gamma^2)\phi(\gamma)-(\sigma_e^2+4\sigma_m^2\gamma^2)\phi(2\gamma)}{4(\sigma_e^2+\sigma_m^2)^2} \\
&< \sigma_e^2\frac{(1-\gamma^2)\phi(\gamma)-(1+2\gamma^2)\phi(2\gamma)}{4(\sigma_e^2+\sigma_m^2)^2}.
\end{aligned}$$

The second equality follows from using equation (9) to substitute for $\gamma(\Phi(2\gamma)-\Phi(\gamma))$ and substituting $\frac{\partial\gamma}{\partial\sigma_e^2}$ with $-\frac{\gamma}{2(\sigma_e^2+\sigma_m^2)}$. The inequality follows since $2\sigma_m^2 > \sigma_e^2$. Again it is straightforward to verify that this inequality is negative for all γ.

Finally, we show that $x_\alpha^*(\gamma\sigma_e,\gamma\sigma_m) = \gamma x_\alpha^*(\sigma_e,\sigma_m)$. Let $\sigma_m' = \eta\sigma_m$ and $\sigma_e' = \eta\sigma_e$. Then, $\Delta(\sigma_e'^2,\sigma_m'^2) = \Delta(\sigma_e^2,\sigma_m^2)$. Hence, $\gamma({x_\alpha'}^*) = \gamma(x_\alpha^*)$, which implies

$$\frac{{x_\alpha'}^*}{\sqrt{\sigma_m'^2+\sigma_e'^2}} = \frac{x_\alpha^*}{\sqrt{\sigma_m^2+\sigma_e^2}} \Rightarrow {x_\alpha'}^* = \left(\frac{\sqrt{\sigma_m'^2+\sigma_e'^2}}{\sqrt{\sigma_m^2+\sigma_e^2}}\right)x_\alpha^* = \eta x_\alpha^*. \qquad \blacksquare$$

Proof of Proposition 5: Let $x_{\alpha F}^*$ be the optimal voting distance given F_m and F_e. Recall that if $x_{\alpha F}^* > 0$ then it solves (3). From equation (3), substitute $\mu_m = m - e$ for m to obtain

$$-\int_{-\infty}^{\infty}\int_{x_{\alpha F}^*-e}^{2x_{\alpha F}^*-e}(2(x_{\alpha F}^*-\mu_m)-e)\,dF_m(\mu_m)\,dF_e(e) = 0.$$

Next switch the order of integration so that at $x^*_{\alpha F}$

$$-\int_{-\infty}^{\infty}\int_{x^*_{\alpha F}-\mu_m}^{2x^*_{\alpha F}-\mu_m}(2(x^*_{\alpha F}-\mu_m)-e)\,dF_e(e)\,dF_m(\mu_m)=0. \tag{11}$$

Define

$$H(\mu_m)=\int_{x^*_{\alpha F}-\mu_m}^{2x^*_{\alpha F}-\mu_m}(2(x^*_{\alpha F}-\mu_m)-e)\,dF_e(e)+\int_{x^*_{\alpha F}+\mu_m}^{2x^*_{\alpha F}+\mu_m}(2(x^*_{\alpha F}+\mu_m)-e)\,dF_e(e).$$

$H(\mu_m)$ is the sum of the values of the inner integral of the left-hand-side of (11) evaluated at μ_m and $-\mu_m$, for any $\mu_m > 0$. By the absolute continuity of F_e, H is continuous and bounded over $\mathbb{R}$.

We begin with three preliminary lemmas.

Lemma 2. *For any pair of distributions, F_e of $e \in \Omega$ and F_m of $\mu_m \in \Omega$, such that $x^*_{\alpha F} > 0$, there exists an $a > 0$ such that*

$$\int_0^a H(\mu_m)\,dF_m(\mu_m) > 0.$$

Proof of Lemma 2: Denote the supremum of the support of F_e by $\overline{e}_F$. If $x^*_{\alpha F} \geq \overline{e}_F$, then $H(\mu_m) = 0$ for all $\mu_m \in [0, x^*_{\alpha F} - \overline{e}_F]$, since for such μ_m, $F_e(e) = 1$ for all $e \in [x^*_{\alpha F} - \mu_m, 2x^*_{\alpha F} + \mu_m]$. Further, for all $\mu_m \in (x^*_{\alpha F} - \overline{e}_F, x^*_{\alpha F} - \overline{e}_F/2)$, for all $e \in [x^*_{\alpha F} + \mu_m, 2x^*_{\alpha F} + \mu_m]$ $F_e(e) = 1$. Thus

$$\begin{aligned}
H(\mu_m) &= \int_{x^*_{\alpha F}-\mu_m}^{2x^*_{\alpha F}-\mu_m}(2(x^*_{\alpha F}-\mu_m)-e)\,dF_e(e)\\
&= \int_{x^*_{\alpha F}-\mu_m}^{\overline{e}_F}(2(x^*_{\alpha F}-\mu_m)-e)\,dF_e(e)\\
&\geq \int_{x^*_{\alpha F}-\mu_m}^{\overline{e}_F}(2(x^*_{\alpha F}-\mu_m)-\overline{e}_F)\,dF_e(e)\\
&> 0,
\end{aligned}$$

for all $\mu_m \in (x^*_{\alpha F} - e_F, x^*_{\alpha F} - e_F/2)$. Therefore,

$$\int_0^{x^*_{\alpha F}-\overline{e}_F/2} H(\mu_m)\,dF_m(\mu_m) = \int_{x^*_{\alpha F}-\overline{e}_F}^{x^*_{\alpha F}-\overline{e}_F/2} H(\mu_m)\,dF_m(\mu_m) > 0.$$

If $x^*_{\alpha F} < \overline{e}_F$, then at $\mu_m = 0$,

$$H(0) = 2\int_{x^*_{\alpha F}}^{2x^*_{\alpha F}}(2x^*_{\alpha F}-e)\,dF_e(e) > 0,$$

since $x^*_{\alpha F} < \overline{e}_F$. By the continuity of H, there exists an $a > 0$ such that $H(\mu_m) > 0$ for all $0 \leq \mu_m \leq a$. Thus, $\int_0^a H(\mu_m)\,dF_m(\mu_m) > 0$. ■

Lemma 3. *For any distribution F_e of $e \in \Omega$, $\lim_{\mu_m \to \infty} |H(\mu_m)| = 0$.*

Proof of Lemma 3: For all $\mu_m > 2x^*_{\alpha F}$, the first integral that defines $H(\mu_m)$ is negative and the maximum density of e over this integral is $f_e(2x^*_{\alpha F} - \mu_m)$. For all $\mu_m > 0$, the second integral that defines $H(\mu_m)$ is positive and the maximum density of e over this integral is $f_e(x^*_{\alpha F} + \mu_m)$. Thus, for all $\mu_m > 2x^*_{\alpha F}$, substituting $2x^*_{\alpha F} - \mu_m > e$ for e in the first integral and $x^*_{\alpha F} + \mu_m < e$ for e into the second integral, we have

$$
\begin{aligned}
|H(\mu_m)| &\leq \left| \int_{x^*_{\alpha F}-\mu_m}^{2x^*_{\alpha F}-\mu_m} (2(x^*_{\alpha F} - \mu_m) - (2x^*_{\alpha F} - \mu_m))\, dF_e(e) \right| \\
&+ \int_{x^*_{\alpha F}+\mu_m}^{2x^*_{\alpha F}+\mu_m} (2(x^*_{\alpha F} + \mu_m) - (x^*_{\alpha F} + \mu_m))\, dF_e(e) \\
&= \int_{x^*_{\alpha F}-\mu_m}^{2x^*_{\alpha F}-\mu_m} \mu_m\, dF_e(e) + \int_{x^*_{\alpha F}+\mu_m}^{2x^*_{\alpha F}+\mu_m} (x^*_{\alpha F} + \mu_m)\, dF_e(e) \\
&\leq \int_{x^*_{\alpha F}-\mu_m}^{2x^*_{\alpha F}-\mu_m} \mu_m f_e(2x^*_{\alpha F} - \mu_m)\, de + \int_{x^*_{\alpha F}+\mu_m}^{2x^*_{\alpha F}+\mu_m} (x^*_{\alpha F} + \mu_m) f_e(x^*_{\alpha F} + \mu_m)\, de \\
&= x^*_{\alpha F} \left[\mu_m f_e(2x^*_{\alpha F} - \mu_m) + (x^*_{\alpha F} + \mu_m) f_e(x^*_{\alpha F} + \mu_m) \right].
\end{aligned}
$$

By the absolute continuity of F_e, both of these terms converge to zero as $\mu_m \to \infty$.

Lemma 4. *For any pair of distributions, F_e of $e \in \Omega$ and F_m of $\mu_m \in \Omega$, such that $x^*_{\alpha F} > 0$, there exists an interval $[a, b] \subset$ supp(F_m) such that (i) $F_m(b) - F_m(a) > 0$ and (ii) $H(\mu_m) < 0$, for all $\mu_m \in [a, b]$.*

Proof of Lemma 4: From Lemma 2, there exists a c such that

$$
\epsilon \equiv \int_0^c H(\mu_m)\, dF_m(\mu_m) > 0,
$$

and from Lemma 3 and the absolute continuity of F_m, there exists a d such that for all $\mu_m \geq d$,

$$
\int_d^\infty H(\mu_m)\, dF_m(\mu_m) > -\epsilon/2.
$$

From the absolute continuity of F_e we can assume that $d < \sup\{\text{supp}(F_e))\}$. Thus,

$$
\int_c^d H(\mu_m)\, dF_m(\mu_m) \leq -\epsilon/2, \tag{12}
$$

since

$$\begin{aligned} 0 &= \int_0^c H(\mu_m)\, dF_m(\mu_m) + \int_d^c H(\mu_m)\, dF_m(\mu_m) + \int_d^\infty H(\mu_m)\, dF_m(\mu_m) \\ &\geq \epsilon + \int_c^d H(\mu_m)\, dF_m(\mu_m) - \epsilon/2. \end{aligned}$$

From equation (12), the absolute continuity of F_m, and the continuity of H there exists an interval $[a, b] \subset [c, d]$ such that (i) $H(\mu_m) < 0$, for all $\mu \in [a, b]$, and (ii) $F_m(b) - F_m(a) > 0$. ■

We now finish the proof of Proposition 5. By Lemma 4, choose an interval $[a, b] \subset \text{supp}(F_m)$ such that $F_m(b) - F_m(a) > 0$, $f_m(b) > 0$ and $H(\mu_m) < 0$, for all $\mu_m \in [a, b]$. Define

$$\epsilon \equiv -\int_a^b H(\mu_m)[f_m(\mu_m) - f_m(b)]/2\, d\mu_m,$$

and

$$\lambda \equiv \int_a^b [f_m(\mu_m) - f_m(b)]/2\, d\mu_m = [F_m(b) - F_m(a) - (b-a) f_m(b)]/2. \tag{13}$$

Denote the supremum of the support of F_m by $\overline{\mu}_F$.

Case 1: $\overline{\mu}_F = \infty$. By Lemma 2 there exists a c such that for all $\mu_m \geq c$,

$$\int_c^\infty |H(\mu_m)| f_m(b)\, d\mu_m < \epsilon/3.$$

The function

$$h(\mu_m) = \int_c^{\mu_m} (f_m(c) - f_m(y))/2\, dy$$

is a continuous, strictly increasing monotonic function, with domain $[0, \infty)$. Hence, there exists a unique d such that $h(d) = \lambda$.

Claim: The distribution

$$G_m(\mu_m) = \begin{cases} F_m(\mu_m) & \text{for } \mu_m \in [0, a) \\ [F_m(\mu_m) + F_m(a) + (\mu_m - a) f_m(b)]/2 & \text{for } \mu_m \in [a, b) \\ F_m(\mu_m) - \lambda & \text{for } \mu_m \in [b, c) \\ [F_m(\mu_m) + F_m(c) + (\mu_m - c) f_m(c)]/2 - \lambda & \text{for } \mu_m \in [c, d) \\ F_m & \text{for } \mu_m \geq d \end{cases}$$

satisfies the conditions of the proposition and $x^*_{\alpha G} < x^*_{\alpha F}$.

We first show that $F_m(\cdot|\mu_m \geq 0) \geq G_m(\cdot|\mu_m \geq 0)$. For $\mu_m \in [0,a) \cup [d,\infty)$, we have $F_m(\mu_m) - G_m(\mu_m) = 0$. For $\mu_m \in [a,b)$,

$$\begin{aligned} F_m(\mu_m) - G_m(\mu_m) &= (F_m(\mu_m) - F_m(a) - (\mu_m - a)f_m(b))/2 \\ &= \int_a^{\mu_m} f_m(y) - f_m(b)\,dy/2 \\ &\geq 0. \end{aligned}$$

For $\mu_m \in [b,c)$, we have $F_m(\mu_m) - G_m(\mu_m) = \lambda > 0$. For $\mu_m \in [c,d)$,

$$\begin{aligned} F_m(\mu_m) - G_m(\mu_m) &= \lambda + \int_c^{\mu_m} f_m(y) - g_m(y)\,dy \\ &= \lambda - \int_c^{\mu_m} (f_m(c) - f_m(y))/2\,dy \\ &= \lambda - h(\mu_m) \\ &\geq 0, \end{aligned}$$

since $\mu_m < d$. Thus, $F_m(\cdot|\mu_m \geq 0) \geq G_m(\cdot|\mu_m \geq 0)$.

We next show that $g_m(\cdot|\mu_m \geq 0)$ is strictly quasi-concave. For $\mu_m \in [0,a) \cup [b,c) \cup [d,\infty)$, $g_m(\mu_m) = f_m(\mu_m)$ For $\mu_m \in [a,b)$, $g_m(\mu_m) = [f_m(\mu_m) + f_m(b)]/2$. For $\mu_m \in [c,d)$, $g_m(\mu_m) = [f_m(\mu_m) + f_m(c)]/2$. Thus, g_m is strictly quasi-concave.

It remains to show that equation (3) is negative at $x^*_{\alpha F}$. Adding $\int_0^\infty H(\mu_m)\,dF_m(\mu_m)$ to $-\int_0^\infty H(\mu_m)\,dG_m(\mu_m)$ yields

$$\begin{aligned} -\int_0^\infty H(\mu_m)\,dG_m(\mu_m) &= \int_0^\infty H(\mu_m)\,dF_m(\mu_m) - \int_0^\infty H(\mu_m)\,dG_m(\mu_m) \\ &= \int_a^b H(\mu_m)\,dF_m(\mu_m) - \int_a^b H(\mu_m)\,dG_m(\mu_m) \\ &\quad + \int_c^d H(\mu_m)\,dF_m(\mu_m) - \int_c^d H(\mu_m)\,dG_m(\mu_m) \\ &\leq -\epsilon + \int_c^d |H(\mu_m)|\,dF_m(\mu_m) + \int_c^d |H(\mu_m)|\,dG_m(\mu_m) \\ &\leq -\epsilon + \int_c^d |H(\mu_m)| f_m(c)\,d\mu_m + \int_c^d |H(\mu_m)| f_m(c)\,d\mu_m \\ &\leq -\epsilon + \int_c^\infty |H(\mu_m)| f_m(b)\,d\mu_m + \int_c^\infty |H(\mu_m)| f_m(b)\,d\mu_m \\ &\leq -\epsilon + \epsilon/3 + \epsilon/3. \end{aligned}$$

Thus, under G_m, the left hand side of equation (3) evaluated at $x^*_{\alpha F}$ is negative, implying the optimal voting rule must decrease.

Case 2: $\overline{\mu}_F < \infty$.

Since $f_m(b) < \infty$, there exists a c_1 such that $d < c_1 < \overline{\mu}_F$ and for all $\mu_m > c_1$,

$$\int_{\mu_m}^{\overline{\mu}_F} f_m(y)\,dy < (\overline{\mu}_F - \mu_m) f_m(b) < \lambda,$$

and there exists a c_2 such that for all $\mu_m > c_2$,

$$\int_{\mu_m}^{\overline{\mu}_F} |H(y)| f_m(y)\,dy < \int_{\mu_m}^{\overline{\mu}_F} |H(y)| f_m(b)\,dy < \epsilon/4.$$

Define $c = \max\{c_1, c_2\}$ and $\lambda' = \lambda + 1 - F_m(c)$, where λ is given by (13).

By Lemma 3, there exists a d such that for all $\mu_m \geq d$,

$$\int_d^{\infty} |H(\mu_m)| f_m(c)\,d\mu_m < \epsilon/4.$$

Define

$$\bar{H} = \max_{\mu_m \in [c,d]} |H(\mu_m)|.$$

Such a maximum exists since H is a continuous function and $[c, d]$ is compact. Finally, define

$$\overline{\mu}_G = c + \lambda' \max\{8\bar{H}(d-c)/\epsilon, 4/f_m(c)\}.$$

Claim: The distribution

$$G_m(\mu_m) = \begin{cases} F_m(\mu_m) & \text{for } \mu_m \in [0, a) \\ (F_m(\mu_m) + F_m(a) + (\mu_m - a) f_m(b))/2 & \text{for } \mu_m \in [a, b) \\ F_m(\mu_m) - \lambda & \text{for } \mu_m \in [b, c) \\ F_m(c) - \lambda + \frac{\lambda'(\mu_m - c)(2\overline{\mu}_G - \mu_m - c)}{(\overline{\mu}_G - c)^2} & \text{for } \mu_m \in [c, \overline{\mu}_G) \\ 1 & \text{for } \mu_m \geq \overline{\mu}_G \end{cases}$$

satisfies the conditions of the proposition and $x^*_{\alpha G} < x^*_{\alpha F}$.

We first show that the associated probability density function g_m is strictly quasi-concave by showing that $g_m(x) > g_m(y)$ if and only if $|x| > |y|$. We have

$$g_m(\mu_m) = \begin{cases} f_m(\mu_m) & \text{for } \mu_m \in [0, a) \\ (f_m(\mu_m) + f_m(b))/2 & \text{for } \mu_m \in [a, b) \\ f_m(\mu_m) & \text{for } \mu_m \in [b, c) \\ \frac{2\lambda'}{(\overline{\mu}_G - c)^2}(\overline{\mu}_G - \mu_m) & \text{for } \mu_m \in [c, \overline{\mu}_G) \\ 0 & \text{for } \mu_m \geq \overline{\mu}_G. \end{cases}$$

The density f_m is symmetric and strictly quasi-concave by assumption, and for all $\mu_m \in (c, \overline{\mu}_G)$,

$$g'(\mu_m) = -\frac{2\lambda'}{(\overline{\mu}_G - c)^2} < 0.$$

Finally,

$$\lim_{\mu_m \to c^-} g_m(\mu_m) = f_m(c) > f_m(c)/2 \geq 2\lambda'/(\overline{\mu}_G - c) = g_m(c),$$

where the weak inequality holds since $\overline{\mu}_G - c \geq 4\lambda'/f_m(c)$. Thus, g_m is strictly quasi-concave.

We next show that $F_m(\cdot|\mu_m \geq 0) \geq G_m(\cdot|\mu_m \geq 0)$. For $\mu_m \in [0, a)$, $F_m(\mu_m) - G_m(\mu_m) = 0$. For $\mu_m \in [a, b)$,

$$\begin{aligned} F_m(\mu_m) - G_m(\mu_m) &= [F_m(\mu_m) - F_m(a) - (\mu_m - a) f_m(b)]/2 \\ &= \int_a^{\mu_m} f_m(y) - f_m(b)\, dy/2 \\ &\geq 0. \end{aligned}$$

For $\mu_m \in [b, c)$, $F_m(\mu_m) - G_m(\mu_m) = \lambda$. For $\mu_m \in [c, \overline{\mu}_F)$,

$$\begin{aligned} F_m(\mu_m) - G_m(\mu_m) &= \lambda + \int_c^{\mu_m} f_m(y) - g_m(y)\, dy \\ &> \lambda - \int_c^{\mu_m} g_m(y)\, dy \\ &> \lambda - \int_c^{\mu_m} f_m(b)\, dy \\ &\geq \lambda - \int_c^{\overline{\mu}_F} f_m(b)\, dy \\ &\geq 0, \end{aligned}$$

since $c \geq c_1$. For $\mu_m \geq \overline{\mu}_F$, $F_m(\mu_m) - G_m(\mu_m) = 1 - G_m(\mu_m) \geq 0$. Thus, $F_m(\cdot|\mu_m \geq 0) \geq G_m(\cdot|\mu_m \geq 0)$.

Finally, we show that equation (3) is negative at $x^*_{\alpha F}$.

$$\begin{aligned} -\int_0^{\infty} H(\mu_m)\, dG_m(\mu_m) &= \int_0^{\infty} H(\mu_m)\, dF_m(\mu_m) - \int_0^{\infty} H(\mu_m)\, dG_m(\mu_m) \\ &= \int_a^{b} H(\mu_m)\, dF_m(\mu_m) - \int_a^{b} H(\mu_m)\, dG_m(\mu_m) \\ &\quad + \int_c^{\overline{\mu}_F} H(\mu_m)\, dF_m(\mu_m) - \int_c^{\overline{\mu}_G} H(\mu_m)\, dG_m(\mu_m) \\ &= -\epsilon + \int_c^{\overline{\mu}_F} H(\mu_m)\, dF_m(\mu_m) - \int_c^{\overline{\mu}_G} H(\mu_m)\, dG_m(\mu_m), \end{aligned}$$

where the last inequality follows since, by definition,

$$\int_a^b H(\mu_m)\,dF_m(\mu_m) - \int_a^b H(\mu_m)\,dG_m(\mu_m) = -\epsilon.$$

Since $c \geq c_2$,

$$\int_c^{\overline{\mu}_F} H(\mu_m)\,dF_m(\mu_m) \leq \int_c^{\overline{\mu}_F} |H(\mu_m)|\,dF_m(\mu_m) < \epsilon/4.$$

Finally,

$$\begin{aligned} -\int_c^{\overline{\mu}_G} H(\mu_m)\,dG_m(\mu_m) &\leq \int_c^{\overline{\mu}_G} |H(\mu_m)|\,dG_m(\mu_m) \\ &= \int_c^{d} |H(\mu_m)|\,dG_m(\mu_m) + \int_d^{\overline{\mu}_G} |H(\mu_m)|\,dG_m(\mu_m) \\ &\leq \int_c^{d} \bar{H}\,dG_m(\mu_m) + \int_d^{\overline{\mu}_G} |H(\mu_m)|\,dG_m(\mu_m). \end{aligned} \tag{14}$$

Evaluating the first term of equation (14) gives

$$\begin{aligned} \int_c^{d} \bar{H}\,dG_m(\mu_m) &< \bar{H}(d-c)g_m(c) \\ &= \bar{H}(d-c)\frac{2\lambda'}{(\overline{\mu}_G - c)} \\ &\leq \bar{H}(d-c)\frac{2\lambda'}{8\lambda'\bar{H}(d-c)/\epsilon} \\ &= \epsilon/4. \end{aligned}$$

where the last inequality follows since $\overline{\mu}_G - c \geq c + 8\lambda'\bar{H}(d-c)/\epsilon$. Evaluating the second term of equation (14) gives

$$\begin{aligned} \int_d^{\overline{\mu}_G} |H(\mu_m)|\,dG_m(\mu_m) &= \int_d^{\overline{\mu}_G} |H(\mu_m)| g_m(\mu_m)\,d\mu_m \\ &< \int_d^{\overline{\mu}_G} |H(\mu_m)| f_m(c)\,d\mu_m \\ &< \epsilon/4. \end{aligned}$$

Hence,

$$-\int_0^{\infty} H(\mu_m)\,dG_m(\mu_m) \leq -\epsilon + \epsilon/4 + \epsilon/4 + \epsilon/4 < 0.$$

Thus, under G_m, the left hand side of equation (3) evaluated at $x^*_{\alpha F}$ is negative. Therefore, the optimal voting rule must decrease. ■

Proof of Proposition 6: The median citizen's utility when $x_\alpha < u_p$ is

$$U = -\int_{-\infty}^{\infty} \Bigg[\int_{-\infty}^{-u_p - e} (\mu_m + u_p)^2 \, dF_m(\mu_m) + \int_{-u_p-e}^{x_\alpha - e} e^2 \, dF_m(\mu_m) + \int_{x_\alpha - e}^{2x_\alpha + u_p - e} (e + 2(\mu_m - x_\alpha))^2 \, dF_m(\mu_m) + \int_{2x_\alpha + u_p - e}^{\infty} (\mu_m + u_p)^2 \, dF_m(\mu_m) \Bigg] \, dF_e(e).$$

where we use the symmetry of the distributions to simplify the expression. When $x_\alpha < u_p$ then

$$\frac{\partial U}{\partial x_\alpha} = 4 \int_{-\infty}^{\infty} \int_{x_\alpha - e}^{2x_\alpha + u_p - e} (e + 2(\mu_m - x_\alpha)) \, dF_m(\mu_m) \, dF_e(e). \tag{15}$$

Differentiating equation (15) with respect to u_p gives

$$\frac{\partial^2 U}{\partial x_\alpha \partial u_p} = 4 \int_{-\infty}^{\infty} (2x_\alpha + 2u_p - e) f_m(2x_\alpha + u_p - e) \, dF_e(e). \tag{16}$$

As a function of e, $2x_\alpha + 2u_p - e$ is symmetric around $2x_\alpha + 2u_p$ which is more than $2x_\alpha + u_p$. The function $f_m(2x_\alpha + u_p - e)$ is symmetric around $2x_\alpha + u_p > 0$, while the function $f_e(e)$ is symmetric around zero. Hence, equation (16), evaluated at $x_\alpha = x_\alpha^*$, is positive and the optimal voting rule must increase. ■

Proof of Proposition 7: The median citizen's utility when $u_p < x_\alpha$ is

$$U = -\int_{-\infty}^{\infty} \Bigg[\int_{-\infty}^{-2x_\alpha + u_p - e} (\mu_m + u_p)^2 \, dF_m(\mu_m) + \int_{-2x_\alpha + u_p - e}^{-x_\alpha - e} (e + 2(\mu_m + x_\alpha))^2 \, dF_m(\mu_m) + \int_{-x_\alpha - e}^{x_\alpha - e} e^2 \, dF_m(\mu_m) + \int_{x_\alpha - e}^{2x_\alpha + u_p - e} (e + 2(\mu_m - x_\alpha))^2 \, dF_m(\mu_m) + \int_{2x_\alpha + u_p - e}^{\infty} (\mu_m + u_p)^2 \, dF_m(\mu_m) \Bigg] \, dF_e(e),$$

where we use the symmetry of the distributions to simplify the expression. When $u_p < x_\alpha$, the effect on utility from marginally increasing x_α is

$$\frac{\partial U}{\partial x_\alpha} = 4 \int_{-\infty}^{\infty} \int_{x_\alpha - e}^{2x_\alpha + u_p - e} (e + 2(\mu_m - x_\alpha)) dF_m(\mu_m) - \int_{-2x_\alpha + u_p - e}^{-x_\alpha - e} (e + 2(\mu_m + x_\alpha)) dF_m(\mu_m) dF_e(e).$$

We see that

$$\left. \frac{\partial^2 U}{\partial x_\alpha \partial u_p} \right|_{u_p = 0} = 4 \int_{-\infty}^{\infty} (2x_\alpha - e) f_m(2x_\alpha - e) - (2x_\alpha + e) f_m(2x_\alpha + e) \, dF_e(e) = 0, \tag{17}$$

and

$$\left.\frac{\partial^3 U}{\partial x_\alpha \partial u_p^2}\right|_{u_p=0} = 8\int_{-\infty}^{\infty} f_m(2x_\alpha - e) + (2x_\alpha - e)f_m'(2x_\alpha - e)\,dF_e(e). \tag{18}$$

When $e \sim N(0, \sigma_e^2)$ and $\mu_m \sim N(0, \sigma_m^2)$, equation (18) simplifies to

$$\begin{aligned}\left.\frac{\partial^3 U}{\partial x_\alpha \partial u_p}\right|_{u_p=0} &= 8\int_{-\infty}^{\infty}\left(2 - \frac{(e+2x_\alpha)^2}{\sigma_m^2}\right)\frac{\phi(\frac{2x_\alpha+e}{\sigma_m})}{\sigma_m}\frac{\phi(\frac{e}{\sigma_e})}{\sigma_e}\,de \\ &= 8\frac{\phi(\frac{2x_\alpha}{\delta})}{\delta^{5/2}}(\sigma_e^4 + 3\sigma_e^2\sigma_m^2 + 2\sigma_m^2(\sigma_m^2 - 2{x_\alpha}^2)). \end{aligned}\tag{19}$$

The positive root of equation (19) is $\tilde{x}_\alpha^* = \frac{\sqrt{\sigma_e^4+3\sigma_e^2\sigma_m^2+2\sigma_m^4}}{2\sqrt{\sigma_m^2}}$. So if $x_\alpha^* > \tilde{x}_\alpha^*$ then a marginal increase in u_p from zero leads to a reduction in the optimal voting rule. Let $\tilde{\gamma}(\sigma_m^2, \sigma_e^2) = \frac{\sqrt{\sigma_e^2+2\sigma_m^2}}{2\sqrt{\sigma_m^2}}$. Then

$$\Delta(\sigma_e^2, \sigma_m^2) = \Gamma(\tilde{\gamma}(\sigma_m^2, \sigma_e^2)), \tag{20}$$

defines σ_e^2 as an implicit function of σ_m^2, $\tilde{\sigma_e}^2(\sigma_m^2)$, for which the optimal voting rule is exactly $\tilde{x}_\alpha^*$ when $u_p = 0$.

Let (σ_m^2, σ_e^2) solve equation (20) and let $\sigma_m' = \eta\sigma_m$ and $\sigma_e' = \eta\sigma_e$. Then, $\Delta(\sigma_e'^2, \sigma_m'^2) = \Delta(\sigma_e^2, \sigma_m^2)$ and $\tilde{\gamma}(\sigma_m'^2, \sigma_e'^2) = \tilde{\gamma}(\sigma_m^2, \sigma_e^2)$. Hence, $(\sigma_m'^2, \sigma_e'^2)$ also solves (20) and it follows that $\tilde{\sigma_e^2}(\sigma_m^2) = \beta\sigma_m^2$. We solve for β by substituting $\sigma_e^2 = \beta\sigma_m^2$ into (20). That is, β solves

$$\frac{4\eta^2}{8\eta^2 - 2} = \eta\frac{[\Phi(2\eta) - \Phi(\eta)]}{[\phi(\eta) - \phi(2\eta)]},$$

where $\eta = \frac{\sqrt{\beta+2}}{2}$. Solving numerically gives $\beta \approx 1.33872$. Recall from Proposition 4, as σ_e^2 increases, x_α^* decreases. Hence, as σ_e^2 increases above $\tilde{\sigma}_e^2(\sigma_m^2)$, equation (19) becomes positive, implying that the optimal voting rule must increase, and as σ_e^2 decreases below $\tilde{\sigma}_e^2(\sigma_m^2)$, equation (19) becomes negative, implying that the optimal voting rule must decrease. ■

Proposition 8 preliminaries: As a preliminary step, we give a more detailed characterization of the optimal voting rule when the initial policy is unbiased and the legislative proposers are sufficiently extreme.

Lemma 5. *Suppose $S_0 = 0$, $e \sim N(0, \sigma_e^2)$, $m \sim N(e, \sigma_m^2)$, and define $\tilde{u}_p(\sigma_e^2, \sigma_m^2)$ implicitly to be the solution to*

$$\Delta(\sigma_e^2, \sigma_m^2) = \frac{u_p(\Phi(3u_p/\delta) - \Phi(u_p/\delta))}{\delta(\phi(u_p/\delta) - \phi(3u_p/\delta))}.$$

Then if $u_p > \tilde{u}_p(\sigma_e^2, \sigma_m^2)$, the optimal voting rule increases with the volatility σ_m^2 in the representativeness of society by the legislature.

When $u_p > x_\alpha$, the proposer nearest to the status quo is never constrained. An increase in σ_m^2 implies that, on average, a constrained median proposer lies further from e. Therefore, constraining the proposer further by increasing x_α, reduces the feasible policy set, improving welfare.

The function $\tilde{u}_p(\sigma_e^2, \sigma_m^2)$ is the polarity distance, given σ_e^2 and σ_m^2 at which $x_\alpha^*(u_p) = u_p$. Characterizing this function allows us to provide sufficient parametric conditions under which $x_\alpha^*(u_p) \leq u_p$. Since $\tilde{u}_p(\sigma_e^2, \sigma_m^2)$ is homogeneous of degree one, without loss of generality, we normalize $\sigma_e^2 = 1$ and show in Figure 5 how $\tilde{u}_p(1, \sigma_m^2)$ varies with σ_m^2. Consistent with Lemma 5, $\tilde{u}_p(1, \sigma_m^2)$ increases in σ_m^2. As σ_m^2 increases, so does x_α^*, so that $\tilde{u}_p(1, \sigma_m^2)$ must increase to maintain equality with x_α^*.

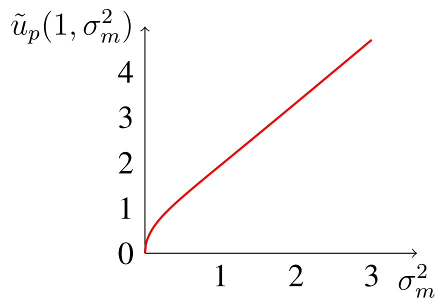


Figure 5: $\tilde{u}_p(1, \sigma_m^2)$ as a function of σ_m^2.

Proof of Lemma 5: When $S_0 = 0$, $e \sim N(0, \sigma_e^2)$, $m \sim N(e, \sigma_m^2)$ and $u_p > x_\alpha$ the optimal voting rule must satisfy equation (15). Define $\psi = \frac{u_p}{\delta}$ and $\Psi(\gamma, \psi) = \gamma \frac{(\Phi(2\gamma+\psi)-\Phi(\gamma))}{(\phi(\gamma)-\phi(2\gamma+\psi))}$. An approach similar to that used in the proof of Proposition 4 shows that at the optimal voting rule distance

$$\Delta(\sigma_e^2, \sigma_m^2) = \Psi(\gamma, \psi). \tag{21}$$

The function Ψ decreases in ψ since $\frac{\Phi(2\gamma+\psi)-\Phi(\gamma)}{\phi(\gamma)-\phi(2\gamma+\psi)}$ is the inverse of the expected value of a truncated standard normal random variable between γ and $2\gamma + \psi$. As u_p increases, ψ increases. By Proposition 6, x_α^* increases in u_p, so γ also increases. Thus, Ψ must increase in γ for (21) to hold.

We now show that there exists a $\tilde{u}_p$ such that for all $u_p \geq \tilde{u}_p$, $x_\alpha^* \leq u_p$. Using the implicit function theorem, we derive how u_p affects x_α^*:

$$\frac{\partial x_\alpha^*}{\partial u_p} = -\frac{\frac{\partial \Psi}{\partial \psi}\frac{\partial \psi}{\partial u_p}}{\frac{\partial \Psi}{\partial \gamma}\frac{\partial \gamma}{\partial x_\alpha^*}} = -\frac{\frac{\partial \Psi}{\partial \psi}}{\frac{\partial \Psi}{\partial \gamma}},$$

$$\frac{\partial \Psi}{\partial \psi} = \frac{\gamma(\phi(2\gamma+\psi)(\phi(\gamma)-\phi(2\gamma+\psi)) - \Phi(2\gamma+\psi) + \Phi(\gamma))}{(\phi(\gamma)-\phi(2\gamma+\psi))^2},$$

and

$$\begin{aligned}\frac{\partial \Psi}{\partial \gamma} &= \Big[(\Phi(2\gamma+\psi) - \Phi(\gamma) + 2\gamma(\phi(2\gamma+\psi) - \phi(\gamma)))(\phi(\gamma) - \phi(2\gamma+\psi)) \\ &+ \gamma(\Phi(2\gamma+\psi) - \Phi(\gamma))(\gamma\phi(\gamma) - 2(2\gamma+\psi)\phi(2\gamma+\psi))\Big]\Big/(\phi(\gamma)-\phi(2\gamma+\psi))^2.\end{aligned}$$

Hence, when $x^*_\alpha = u_p$,

$$\begin{aligned}\left.\frac{\partial x^*_\alpha}{\partial u_p}\right|_{x^*_\alpha = u_p} &= \{\psi\left[\Phi(3\psi) - \Phi(\psi) - \phi(3\psi)\left(\phi(\psi) - \phi(3\psi)\right)\right]\}\Big/ \\ &\quad \{[\Phi(3\psi) - \Phi(\psi) + 2\psi(\phi(3\psi) - \phi(\psi))]\left[\phi(\psi) - \phi(3\psi)\right] \\ &\quad +\psi\left[\Phi(3\psi) - \Phi(\psi)\right]\left[\psi\phi(\psi) - 6\phi(3\psi)\right]\}.\end{aligned} \quad (22)$$

At $u_p = 0$ equation (22) equals one, and a graphical analysis shows that the function is monotonically decreasing (see Figure 6). Hence, the ratio of x^*_α/u_p at $x^*_\alpha = u_p$ is decreasing. Thus, for

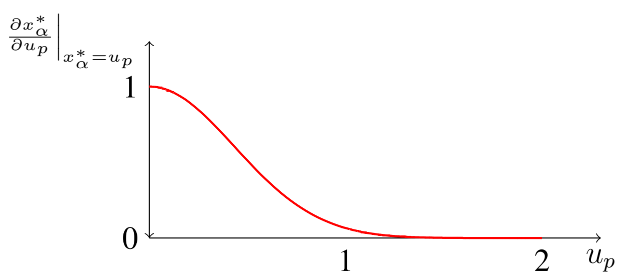


Figure 6: When $x^*_\alpha = u_p$, the increase in x^*_α as u_p increases is always less than 1.

any value of σ_m and σ_e there is a unique u_p, $\tilde{u}_p$, at which $x^*_\alpha = \tilde{u}_p$ and for all $u_p \geq \tilde{u}_p$, $x^*_\alpha \leq u_p$. This critical polarity distance $\tilde{u}_p$ solves $\Delta(\sigma_e^2, \sigma_m^2) = \Psi(\psi, \psi)$. Note that a proportionally-equal increase in ψ and γ increases Ψ.

Now we show that if $u_p > x^*_\alpha$, then x^*_α is increasing in σ_m^2 . As shown in the proof of Proposition 4, the left-hand side of equation (21) is increasing in σ_m^2. As σ_m^2 increases, ψ and γ decrease by the same proportion, which implies that Ψ decreases. Hence, the right-hand side of equation (21) decreases and x^*_α/δ must increase to ensure the equality of equation (21) at the optimum. ■

Proof of Proposition 8: We prove the proposition in order of the claims.

Proof of Claim 1: When the median legislator proposes policy then

$$\begin{aligned} U &= -\int_{-\infty}^{\infty}\int_{-\infty}^{-2x_\alpha+S_0-e} \mu_m^2 dF_m(\mu_m) + \int_{-2x_\alpha+S_0-e}^{-x_\alpha+S_0-e} (e+2(\mu_m+x_\alpha)-S_0)^2 dF_m(\mu_m) \\ &+ \int_{-x_\alpha+S_0-e}^{x_\alpha+S_0-e} (e-S_0)^2\, dF_m(\mu_m) + \int_{x_\alpha+S_0-e}^{2x_\alpha+S_0-e} (e+2(\mu_m-x_\alpha)-S_0)^2\, dF_m(\mu_m) \\ &+ \int_{2x_\alpha+S_0-e}^{\infty} (\mu_m+u_p)^2\, dF_m(\mu_m) \end{aligned}$$

Because the initial policy is biased, we can no longer use symmetry to simplify the expression.

The change in utility from a marginal change in the voting rule distance is

$$\begin{aligned} \frac{\partial U(x_\alpha, S_0)}{\partial x_\alpha} &= 4\int_{-\infty}^{\infty}\Big[\int_{x_\alpha+S_0-e}^{2x_\alpha+S_0-e}\Big(e+2(u_m-x_\alpha)-S_0\Big)\, dF_m(u_m) \\ &- \int_{-2x_\alpha+S_0-e}^{-x_\alpha+S_0-e}\Big(e+2(u_m+x_\alpha)-S_0\Big)\, dF_m(u_m)\Big]\, dF_e(e). \end{aligned} \tag{23}$$

Differentiating equation (23) with respect to S_0 gives

$$\begin{aligned} \frac{\partial^2 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0} &= 4\int_{-\infty}^{\infty}\Big[(e-S_0)\Big(f_m(S_0-x_\alpha-e)+f_m(S_0+x_\alpha-e)\Big) \\ &+ (S_0-2x_\alpha-e)f_m(S_0-2x_\alpha-e)+(S_0+2x_\alpha-e)f_m(S_0+2x_\alpha-e) \\ &+ \int_{S_0-2x_\alpha}^{S_0-x_\alpha} 1\, dF_m(u_m) - \int_{S_0+x_\alpha}^{S_0+2x_\alpha} 1\, dF_m(u_m)\Big]\, dF_e(e). \end{aligned} \tag{24}$$

Evaluated at $S_0 = 0$, equation (24) is zero. Differentiating equation (24) with respect to S_0 gives

$$\begin{aligned} \frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2} &= 4\int_{-\infty}^{\infty}\left[(e-S_0)\left(f'_m(S_0-x_\alpha-e)+f'_m(S_0+x_\alpha-e)\right)\right. \qquad (25) \\ &+ (S_0-2x_\alpha-e)f'_m(S_0-2x_\alpha-e)+(S_0+2x_\alpha-e)f'_m(S_0+2x_\alpha-e)\, dF_e(e). \end{aligned}$$

Evaluated at $S_0 = 0$, equation (25) simplifies to equation (6) where $u_p = 0$. When $u_p = 0, e \sim N(0, \sigma_e^2)$ and $m \sim N(e, \sigma_m^2)$, equation (6) becomes

$$\frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2}\Big|_{S_0=0} = 8\phi\left(\frac{2x_\alpha}{\delta}\right)\frac{\sigma_m^2\sigma_e^2}{\delta^{5/2}}\left(e^{\frac{3x_\alpha{}^2}{2\delta^2}}\sigma_e^2(\delta^2-x_\alpha{}^2)-\sigma_e^2\delta^2-4\sigma_m^2 x_\alpha{}^2\right). \tag{26}$$

At $x_\alpha = 0$, equation (26) equals zero and is decreasing in x_α for all $x_\alpha > 0$. To see this, differentiate the bracketed term in equation (26) with respect to x_α and define $\theta = x_\alpha{}^2/\delta^2$.

$$\begin{aligned} \frac{\partial}{\partial x_\alpha}\left(e^{\frac{3x_\alpha{}^2}{2\delta^2}}\sigma_e^2(\delta^2-x_\alpha{}^2)-\sigma_e^2\delta^2-4\sigma_m^2 x_\alpha{}^2\right) &= -8\sigma_m^2 x_\alpha + e^{\frac{3x_\alpha{}^2}{2\delta^2}}\sigma_e^2 x_\alpha(\delta^2-3x_\alpha{}^2)/\delta^2 \\ &= -x_\alpha\left(8\sigma_m^2+\sigma_e^2 e^{\frac{3}{2}\theta}(3\theta-1)\right) \\ &< -x_\alpha\sigma_e^2\left(4+e^{\frac{3}{2}\theta}(3\theta-1)\right) \\ &< 0, \end{aligned}$$

where the first inequality follows since $\sigma_e^2 < 2\sigma_m^2$.

Since equation (26) is negative for all x_α and $\sigma_e^2 < 2\sigma_m^2$, it must be negative for the optimal voting rule when initial policy is unbiased. Hence, the first-order condition evaluated at the optimal voting rule when initial policy is unbiased becomes negative as S_0 increases.

Proof of Claim 2: Let $y_\alpha = 2x_\alpha + u_p$ and $z_\alpha = 2x_\alpha - u_p$. When $u_p > x_\alpha$ then the median citizen's expected utility is

$$\begin{aligned}
U &= -\int_{-\infty}^{\infty}\int_{-\infty}^{-u_p+S_0-e}(\mu_m+u_p)^2\,dF_m(\mu_m)+\int_{-u_p+S_0-e}^{x_\alpha+S_0-e}(e-S_0)^2\,dF_m(\mu_m)\\
&+ \int_{x_\alpha+S_0-e}^{y_\alpha+S_0-e}(e+2(\mu_m-x_\alpha)-S_0)^2\,dF_m(\mu_m)+\int_{y_\alpha+S_0-e}^{\infty}(\mu_m+u_p)^2\,dF_m(\mu_m)\\
&+ \int_{-\infty}^{-y_\alpha+S_0-e}(\mu_m-u_p)^2\,dF_m(\mu_m)+\int_{-y_\alpha+S_0-e}^{-x_\alpha+S_0-e}(e+2(\mu_m+x_\alpha)-S_0)^2\,dF_m(\mu_m)\\
&+ \int_{-x_\alpha+S_0-e}^{u_p+S_0-e}(e-S_0)^2\,dF_m(\mu_m)+\int_{u_p+S_0-e}^{\infty}(\mu_m-u_p)^2\,dF_m(\mu_m)\,dF_e(e)/2
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial U(x_\alpha,S_0)}{\partial x_\alpha} &= 4\int_{-\infty}^{\infty}\int_{x_\alpha+S_0-e}^{2x_\alpha+u_p+S_0-e}(e+2(\mu_m-x_\alpha)-S_0)\,dF_m(\mu_m) \\
&- \int_{-2x_\alpha-u_p+S_0-e}^{-x_\alpha+S_0-e}(e+2(\mu_m+x_\alpha)-S_0)\,dF_m(\mu_m)\,dF_e(e).
\end{aligned} \tag{27}$$

Differentiating equation (27) with respect to S_0 gives

$$\begin{aligned}
\frac{\partial^2 U(x_\alpha,S_0)}{\partial x_\alpha \partial S_0} &= 4\int_{-\infty}^{\infty}\Big[F_m(-x_\alpha+S_0-e)-F_m(-2x_\alpha-u_p+S_0-e)\\
&- F_m(2x_\alpha+u_p+S_0-e)+F_m(x_\alpha+S_0-e)\\
&+ (-2(x_\alpha+u_p)+S_0-e)f_m(-2x_\alpha-u_p+S_0-e)-(S_0-e)f_m(-x_\alpha+S_0-e)\\
&+ (2(x_\alpha+u_p)+S_0-e)f_m(2x_\alpha+u_p+S_0-e)-(S_0-e)f_m(x_\alpha+S_0-e)\Big]\,dF_e(e).
\end{aligned} \tag{28}$$

Evaluated at $S_0 = 0$, equation (28) is zero. Differentiating equation (28) with respect to S_0 gives

$$\begin{aligned}
\frac{\partial^3 U(x_\alpha,S_0)}{\partial x_\alpha \partial S_0^2} &= 4\int_{-\infty}^{\infty}\Big[(2(x_\alpha+u_p)+S_0-e)f_m'(2x_\alpha+u_p+S_0-e)\\
&+ (-2(x_\alpha+u_p)+S_0-e)f_m'(-2x_\alpha-u_p+S_0-e)\\
&- (S_0-e)(f_m'(x_\alpha+S_0-e)+f_m'(-x_\alpha+S_0-e))\Big]\,dF_e(e).
\end{aligned} \tag{29}$$

Evaluated at $S_0 = 0$, equation (29) simplifies to equation (6). When $e \sim N(0,\sigma_e^2)$ and $m \sim$

$N(e, \sigma_m^2)$, equation (6) becomes

$$\frac{\partial^3 U(x_\alpha, S_0)}{\partial x_\alpha \partial S_0^2}\Big|_{S_0=0} = \frac{\sqrt{8}}{\sqrt{\pi}\delta^5}\Big(e^{-\frac{x_\alpha{}^2}{2\delta^2}}\sigma_e^2\sigma_m^2(\delta^2 - x_\alpha{}^2) - e^{-\frac{(2x_\alpha+u_p)^2}{2\delta^2}}\sigma_m^3(\sigma_e^2\delta^2 + u_p^2(\delta^2+\sigma_m^2) + 2u_p x_\alpha(\delta^2 + 2\sigma_m^2) + 4\sigma_m^2 x_\alpha{}^2)\Big) \quad (30)$$

Suppose $S_0 = 0$. From equation (21) we see that in the limit as $u_p \to \infty$, $x_\alpha^* = \delta$ if

$$\frac{1-\Phi(1)}{\phi(1)} = \frac{\sigma_e^2 + 2\sigma_m^2}{2(\sigma_e^2 + \sigma_m^2)},$$

which holds when $\frac{\sigma_m^2}{\sigma_e^2} = \chi$ where χ is defined in the proposition. By Lemma 5, x_α^*/δ increases with σ_m^2. Also, from Proposition 6, x_α^* increases in u_p. Hence, $x_\alpha^* < \delta$ for all $u_p > \delta$ if $\sigma_m^2 < \chi\sigma_e^2$. Now consider the impact of a marginal increase in initial policy bias. From equation (30), as $u_p \to \infty$, the second term goes to zero, while the first term remains strictly positive. Hence, there exists a $\bar{u}_p$ such that for all $u_p > \bar{u}_p$, the optimal voting rule increases in S_0 when $S_0 = 0$. ■

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