Classes of Maximal-Length Reduced Words in Coxeter Groups

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Declaration

I declare that to the best of my knowledge the material contained in this thesis is original unless explicitly stated otherwise.

Summary

This thesis is concerned with the graph $\mathcal{R}$ of all reduced words for the longest element in the Coxeter groups of classical type, the edges representing braid relations.
In chapter 2 we consider the equivalence relation on the vertices of $\mathcal{R}$ generated by commuting adjacent letters if the corresponding simple reflections commute. An inductive way of describing all of the resulting **commutation classes** is described.
A characterisation of the **quiver-compatible** commutation classes, couched in terms of letter-multiplicities, is presented in chapter 3.
Chapter 4 introduces for each positive root β an equivalence relation on $\mathcal{R}$ whose equivalence classes are connected subgraphs called **β-components.**
It is shown that the β-components are in bijective correspondence with the **root vectors** for β (following Bédard) when β is the highest root α_0; in general there are more β-components.
It happens that the natural quotient graph of β-components is determined up to isomorphism by the length of β; we choose to focus on the α_0-components.
In chapter 5 we show that each α_0-component in type A_l contains a unique quiver-compatible commutation class.
In chapter 6 we count the α_0-components in type B_l by exhibiting explicit representatives which have a natural interpretation as **partial quivers.**
The edges of the graphs of α_0-components in types A_l and B_l are determined by interpreting maximal chains in certain posets as elements of the Coxeter group of type A_{l-2} or B_{l-2}, respectively.
Chapter 7 establishes an isomorphism between the graphs of β-components in types B_l and D_l whenever β is long.

Index of Notation

1. Commutation Classes of Reduced Words

1.1 Finite Coxeter Groups and Root Systems

We begin with a brief overview of basic Coxeter group theory, following [Humphreys]. No originality is claimed for any of the calculations in chapter 1.

Let V be an l-dimensional real Euclidean vector space equipped with positive-definite symmetric bilinear form $\langle\ ,\ \rangle$. For each $\alpha \in V \setminus \{0\}$ define the **reflection** in the hyperplane orthogonal to α by

$$s_\alpha : v \mapsto v - 2\frac{\langle v, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha; \quad v \in V.$$

A **reflection group** is a group generated by reflections.

Any finite subset $\Phi \subseteq V \setminus \{0\}$ satisfying $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ and $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$ is called a **root system**. The members of Φ are **roots**. Let $\mathcal{W}_\Phi := \langle s_\alpha \mid \alpha \in \Phi\rangle$ be the associated reflection group. In fact $\mathcal{W}_\Phi$ is a finite reflection group, and every finite reflection group occurs as some $\mathcal{W}_\Phi$, possibly for many choices of Φ. Now let Φ be a fixed root system and set $\mathcal{W} := \mathcal{W}_\Phi$.

A **positive system** is a subset Π of Φ consisting of all roots which are positive relative to some total order on V. We have $\Phi = \Pi \dot\cup (-\Pi)$ whenever Π is a positive system.

A **simple system** is a subset Δ of Φ such that Δ is a basis of V and, further, each root is a linear combination of Δ with coefficients all ≥ 0 or all ≤ 0. The members of Δ are called **simple roots**.

Each positive system contains a unique simple system and each simple system is contained in a unique positive system. In particular, simple systems exist, as positive systems are easy to construct. By definition, the cardinality of any simple system equals l, and we call this number the **rank** of $\mathcal{W}$.

If Δ is the simple system determined by some positive system Π then for any $w \in \mathcal{W}$, $w(\Delta)$ is another simple system, with corresponding positive system $w(\Pi)$. In fact, all simple systems are conjugate under $\mathcal{W}$, and similarly for positive systems. From now, let $\Delta := \{\alpha_1, \ldots, \alpha_l\}$ be a fixed simple system, and let Φ^+ denote the corresponding positive system containing Δ. Set $\Phi^- := -\Phi^+$. The members of Φ^+ are **positive roots** and those of Φ^- **negative roots**. We have $\Phi = \Phi^+ \dot\cup \Phi^-$.

A **simple reflection** is a reflection s_{α_i} for some simple root α_i. We will denote s_{α_i} by s_i for brevity. It happens that $\mathcal{W}$ is generated by the simple reflections alone. In fact, $\mathcal{W}$ is generated by the simple reflections s_i, and the only relations are of the form $(s_i s_j)^{m_{ij}} = \mathrm{id}$ for certain positive integers $m_{ij} = m_{ji}$ with $m_{ii} = 1$ for all i. Any group with such a presentation is a **Coxeter group**. Therefore $\mathcal{W}$ is a finite Coxeter group. It is well known that the finite Coxeter groups are precisely the finite reflection groups; see [Humphreys, 6.4].

For $w \in \mathcal{W}$, let $\ell(w)$, the **length** of w be the least integer r for which w may be expressed as a product of r simple reflections. There is a geometrical interpretation of $\ell(w)$ as the number of positive roots sent to some negative root by w, that is, $\ell(w) = |\{\alpha \in \Phi^+ \mid w(\alpha) \in \Phi^-\}|$. The relationship between $\ell(w)$ and $\ell(s_i w)$ is as follows.

1.1.1 LEMMA.

(a) We have $\ell(s_i w) = \ell(w) + 1$ if and only if $w^{-1}(\alpha_i) \in \Phi^+$.

(b) We have $\ell(s_i w) = \ell(w) - 1$ if and only if $w^{-1}(\alpha_i) \in \Phi^-$. ∎

Note that, as $\ell(s_i) = 1$ and $s_i(\alpha_i) = -\alpha_i$, it follows that α_i is the unique positive root sent to Φ^- by s_i.

A consequence of the geometrical interpretation of $\ell(w)$ is that the only $w \in \mathcal{W}$ satisfying $w(\Phi^+) = \Phi^+$ is the identity. Now, Φ^+ is the positive system corresponding to some total order on V, so it is clear that Φ^- is the positive system corresponding to the reverse total order. Since positive systems are $\mathcal{W}$-conjugate, there is some $w_0 \in \mathcal{W}$ such that $w_0(\Phi^+) = \Phi^-$. Furthermore, w_0 is the unique element with this property, using the fact that only the identity stabilises Φ^+. We reserve the letter N for $\ell(w_0)$. Here are some elementary properties of w_0.

1.1.2 Lemma.

(a) We have $w_0 = w_0^{-1}$.

(b) The element w_0 is the unique element of length N and no element has greater length.

(c) For all $w \in \mathcal{W}$ there exists some w' such that $w_0 = w'w$ and $\ell(w_0) = \ell(w') + \ell(w)$.

(d) For each simple reflection s_i we have $w_0 s_i w_0 = s_{\bar{\imath}}$ for some $\bar{\imath}$. ∎

By virtue of (c), w_0 is called the **longest element.** In (d), the correspondence $i \mapsto \bar{\imath}$ is an involution (it may be trivial) and so induces involutions of Φ and $\mathcal{W}$ in the obvious manner, both called the **opposition involution.** Clearly, the opposition involution is trivial if and only if w_0 is central.

There is a partial order on $\mathcal{W}$ defined by saying $u \leq w$ if there exists some u' such that $w = u'u$ and $\ell(w) = \ell(u') + \ell(u)$. This is properly called the left-sided weak Bruhat order, but we will call it the **weak order** for brevity. The weak order has bottom element id and top element w_0. In fact, $\mathcal{W}$ is a *ranked* poset with this ordering if we define the rank of an element to be its length. We shall sometimes find it convenient to view $\mathcal{W}$ as the *graph* underlying the Hasse diagram of the weak order.

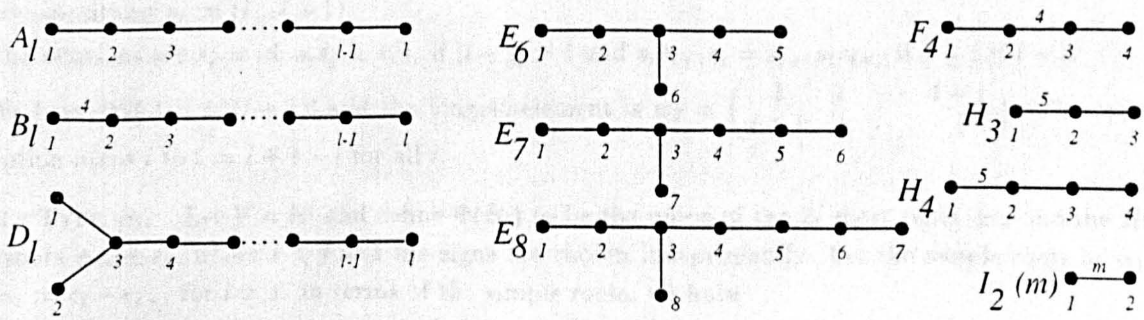
If $I \subseteq \{1, \ldots, l\}$, set $\mathcal{W}_I := \langle s_i \mid i \in I \rangle$, a **parabolic subgroup** of $\mathcal{W}$. Also define $\mathcal{W}^I$ to be the set $\{w \in \mathcal{W} \mid \ell(ws_i) > \ell(w) \text{ for all } i \in I\}$.

1.1.3 Proposition. Given $w \in \mathcal{W}$, there is a unique $u \in \mathcal{W}^I$ and a unique $v \in \mathcal{W}_I$ such that $w = uv$. We have $\ell(w) = \ell(u) + \ell(v)$. Further, u is the unique element of smallest length in $w\mathcal{W}_I$. ∎

We shall now outline the classification of finite Coxeter groups. The **Coxeter graph** of $\mathcal{W}$, also denoted Δ, has vertex set in bijection with the simple roots, with the vertex corresponding to α_i being labelled i in the graph. Vertex i is joined to vertex j by an edge if and only if $m_{ij} \geq 3$. If $m_{ij} \geq 4$ this edge is labelled m_{ij}. If Δ is a connected graph then $\mathcal{W}$ is called **irreducible.** Clearly, every Coxeter group is a direct product of irreducible Coxeter groups.

1.1.4 Theorem. Every irreducible finite Coxeter group $\mathcal{W}$ of rank l has an associated Coxeter graph which is one of the types A_l with $l \geq 1$, B_l with $l \geq 2$, D_l with $l \geq 4$, E_6, E_7, E_8, F_4, H_3, H_4 or $I_2(m)$ where $m \geq 5$. Figure 1.1.5 shows these graphs (with a particular labelling of the vertices). ∎

1.1.5 Figure. The irreducible Coxeter graphs.



The Coxeter groups of types A_l, B_l and D_l are the **classical** types.

If $2\langle \alpha_i, \alpha_j \rangle / \langle \alpha_j, \alpha_j \rangle$ is integral for all simple roots α_i, α_j then Φ is a **crystallographic** root system and $\mathcal{W}$ is called a **Weyl group.** In this situation, each root is an *integral* linear combination of the simple roots. If m_{ij} is the order of $s_i s_j$ in the Weyl group $\mathcal{W}$, it may be shown that $m_{ij} \in \{2, 3, 4, 6\}$ whenever $i \neq j$. It follows that the irreducible Weyl groups are the Coxeter groups of types A_l, B_l, D_l, E_l ($l = 6, 7, 8$), F_4 and $G_2 := I_2(6)$.

We now state two properties of irreducible Weyl groups. Partially order V by saying $v \geq v'$ if $v - v'$ is a non-negative linear combination of Δ. This induces a partial order on the set of positive roots.

1.1.6 Proposition. If $\mathcal{W}$ is an irreducible Weyl group then there exists a unique highest positive root relative to this partial order, denoted α_0. ∎

In crystallographic root systems with irreducible Weyl group, there are at most two lengths of roots. If all edges in the Coxeter graph are unlabelled, all roots have the same length. If only one root length occurs, all roots are called **long**, otherwise we have **short** roots also. The ratio of lengths of long roots to short roots can only be $\sqrt{2}$ or $\sqrt{3}$.

1.1.7 PROPOSITION. If $\mathcal{W}$ is an irreducible Weyl group then all roots of the same length form a single orbit under $\mathcal{W}$. ∎

We shall now summarise how the Weyl group acts on the set of roots. Let i and j be two vertices of the Coxeter graph.

First, we have $s_i(\alpha_i) = -\alpha_i$.

If i, j are not joined by an edge, that is, $m_{ij} = 2$ then $s_i(\alpha_j) = \alpha_j$.

If i, j are joined by an unlabelled edge, that is, $m_{ij} = 3$ then $s_i(\alpha_j) = \alpha_i + \alpha_j$.

Lastly, suppose i, j are joined by an edge labelled 4, that is, $m_{ij} = 4$. Without loss of generality, let α_i be the short root and α_j the long root. We have $s_i(\alpha_j) = 2\alpha_i + \alpha_j$ and $s_j(\alpha_i) = \alpha_i + \alpha_j$.

1.2 The Coxeter Groups of Classical Type

We shall describe the Coxeter groups and root systems of types A_l, B_l and D_l. Refer to Figure 1.1.5 for the numbering of the vertices of the Coxeter graphs.

In these concrete descriptions, since $\mathcal{W}$ acts on the left of V, we always compose permutations from right-to-left. Let $\{e_1, \ldots, e_{l+1}\}$ be the standard basis of R^{l+1}.

1.2.1 TYPE A_l. Let $V \subseteq \mathsf{R}^{l+1}$ be the hyperplane of vectors whose coordinates sum to 0. Define $\Phi(A_l) := \{e_i - e_j \mid 1 \le i \ne j \le l+1\}$ and let the simple roots be $\alpha_i := e_i - e_{i+1}$ for $1 \le i \le l$. In terms of the simple roots, we have

$$\Phi^+(A_l) = \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 1 \le i \le j \le l\}.$$

We have $|\Phi^+(A_l)| = l(l+1)/2$ and highest root $\alpha_0 := \alpha_1 + \cdots + \alpha_l$. The Coxeter group $\mathcal{W}(A_l)$ acts on V by permuting the e_i, and we identify it with the symmetric group on $\{1, \ldots, l+1\}$. The simple reflections are the transpositions $s_i := (i\ ,\ i+1)$.

The relations are $s_i^2 = \mathrm{id}$, $s_i s_j = s_j s_i$ if $|i-j| > 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ if $1 \le i \le l-1$.

We have $|\mathcal{W}(A_l)| = (l+1)!$ and the longest element is $w_0 = \begin{pmatrix} 1 & 2 & \cdots & l+1 \\ l+1 & l & \cdots & 1 \end{pmatrix}$. The opposition involution maps i to $\bar{i} := l+1-i$ for all i.

1.2.2 TYPE B_l. Let $V = \mathsf{R}^l$ and define $\Phi(B_l)$ to be the union of the $2l$ short roots $\pm e_i$ and the $2l(l-1)$ long roots $\pm e_i \pm e_j$, where $i < j$ and the signs are chosen independently. Let the simple roots be $\alpha_1 := e_1$ and $\alpha_i := e_i - e_{i-1}$ for $i > 1$. In terms of the simple roots, we have

$$\begin{aligned}\Phi^+(B_l) &= \{\alpha_1 + \alpha_2 + \cdots + \alpha_i \mid 1 \le i \le l\} \\ &\cup \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 2 \le i \le j \le l\} \\ &\cup \{2(\alpha_1 + \alpha_2 + \cdots + \alpha_i) + \alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_j \mid 1 \le i < j \le l\}.\end{aligned}$$

We have $|\Phi^+(B_l)| = l^2$ and the highest root is $\alpha_0 := 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-1} + \alpha_l$. We identify $\mathcal{W}(B_l)$ with the group of all signed permutations of $\{1, \ldots, l\}$, with simple reflections $s_1 := (1\ ,\ -1)$ and $s_i := (i-1\ ,\ i)$ for $i > 1$. Note that $s_1(\alpha_2) = 2\alpha_1 + \alpha_2$ and $s_2(\alpha_1) = \alpha_1 + \alpha_2$.

The relations are $s_i^2 = \mathrm{id}$, $s_i s_j = s_j s_i$ if $|i-j| > 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ if $2 \le i \le l-1$ and $s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$.

We have $|\mathcal{W}(B_l)| = 2^l l!$ and the longest element is $w_0 = \begin{pmatrix} 1 & 2 & \cdots & l \\ -1 & -2 & \cdots & -l \end{pmatrix}$. The opposition involution is trivial.

1.2.3 Type D_l. Let $V = \mathbb{R}^l$ and define $\Phi(D_l) := \{\pm e_i \pm e_j \mid 1 \le i < j \le l\}$, where the signs are chosen independently. Let the simple roots be $\alpha_1 := e_1 + e_2$ and $\alpha_i := e_i - e_{i-1}$ for $i > 1$. In terms of the simple roots, we have

$$\begin{aligned}\Phi^+(D_l) &= \{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j \mid 3 \le i \le j \le l\} \\ &\cup \{\alpha_1 + \alpha_3 + \alpha_4 + \cdots + \alpha_l,\ \alpha_2 + \alpha_3 + \alpha_4 + \cdots + \alpha_l\} \\ &\cup \{\alpha_1 + \alpha_2 + 2(\alpha_3 + \alpha_4 + \cdots + \alpha_i) + \alpha_{i+1} + \alpha_{i+2} + \cdots + \alpha_j \mid 2 \le i < j \le l\}.\end{aligned}$$

We have $|\Phi^+(D_l)| = l(l-1)$ and the highest root is $\alpha_0 := \alpha_1 + \alpha_2 + 2(\alpha_3 + \cdots + \alpha_{l-1}) + \alpha_l$. Note that $\Phi^+(D_l)$ is the set of all *long* roots in $\Phi^+(B_l)$. To avoid confusion, let α_i^B denote the simple roots of type B_l and α_i^D the simple roots of type D_l. The positive roots of type D_l are precisely the positive roots in type B_l whose coefficient of α_1^B is 0 or 2. We have the relationships $\alpha_1^D = 2\alpha_1^B + \alpha_2^B$ and $\alpha_i^D = \alpha_i^B$ if $i \ne 1$. We identify $\mathcal{W}(D_l)$ with the group of signed permutations of $\{1, \ldots, l\}$ with an *even* number of sign changes. The simple reflections are $s_1 := (1\ ,\ -2)$ and $s_i := (i-1\ ,\ i)$ for $i > 1$.

The relations, which are cumbersome to list explicitly, are self-evident from the labelling of the Coxeter graph D_l.

We have $|\mathcal{W}(D_l)| = 2^{l-1} l!$. If l is even then $w_0 = \begin{pmatrix} 1 & 2 & \cdots & l \\ -1 & -2 & \cdots & -l \end{pmatrix}$ and the opposition involution is trivial. If l is odd then $w_0 = \begin{pmatrix} 1 & 2 & \cdots & l \\ 1 & -2 & \cdots & -l \end{pmatrix}$ and the opposition involution swaps 1 and 2, fixing everything else.

1.3 Reduced Words

A **word** is an element of the free monoid on $\mathbb{N}$ with multiplication defined by concatenation and unit element the **null word**, $\emptyset$. Elements of $\mathbb{N}$ are called **letters**.

Let $\mathbf{i} := i_r \ldots i_2 i_1$ be a word. The **length** of $\mathbf{i}$, denoted $\ell(\mathbf{i})$, is r; the null word has length 0. Let the symbol $s_{\mathbf{i}}$ denote $s_{i_r} \ldots s_{i_2} s_{i_1}$. If $\ell(s_{\mathbf{i}}) = \ell(\mathbf{i})$ then $\mathbf{i}$ is a **reduced word** for $s_{\mathbf{i}}$.

A **subword** of $\mathbf{i}$ is a product of *consecutive* letters from $\mathbf{i}$. Any subword of a reduced word is reduced. From now, all words are reduced unless stated otherwise.

Let m_{ij} be the order of $s_i s_j$ and suppose $\mathbf{i}$ has a subword $\mathbf{b}_{ij} := ijij\ldots$ of length m_{ij}. If $\mathbf{i}'$ is the word obtained by replacing this subword with $\mathbf{b}_{ji}$ then $\mathbf{i}$ and $\mathbf{i}'$ are related by a **braid**. Specifically, if $m_{ij} > 2$ we call this an m_{ij}**-braid** and write $\mathbf{i} \overset{m_{ij}}{\longleftrightarrow} \mathbf{i}'$. If $m_{ij} = 2$, that is, $s_i s_j = s_j s_i$, we call this type of braid a **commutation**. Two reduced words related by some sequence of braids are **braid-equivalent**; this is an equivalence relation. The term **commutation-equivalent** is similarly defined. Let $\equiv$ denote braid-equivalence and $\sim$ denote commutation-equivalence.

The following three Propositions are well known.

1.3.1 Proposition. Any two reduced words for the same element are braid-equivalent. ▮

1.3.2 Proposition. If $\mathbf{i}$ is a reduced word but $j\,\mathbf{i}$ is nonreduced then $\mathbf{i}$ is braid-equivalent to a word with first letter j. (A symmetrical statement holds if $\mathbf{i}\,j$ is nonreduced.) ▮

1.3.3 Proposition. If $\mathbf{i}\omega$ and $\mathbf{j}\omega$ are braid-equivalent reduced words, both ending with the same subword, then $\mathbf{i}$ and $\mathbf{j}$ are braid-equivalent. (The same is true if both words begin with ω.) ▮

A similar, yet trivial, observation will be used often in the sequel.

1.3.4 Lemma. If $\mathbf{i}\omega$ and $\mathbf{j}\omega$ are commutation-equivalent reduced words, both ending with the same subword, then $\mathbf{i}$ and $\mathbf{j}$ are commutation-equivalent. (The same is true if both words begin with ω.) ▮

For $w \in \mathcal{W}$, let Φ_w^+ be those positive roots sent to some negative root by w.

1.3.5 Lemma. Let $\mathbf{i} := i_r \ldots i_1$ be a reduced word for w. The r positive roots Φ_w^+ can be totally ordered as

$$\alpha_{i_1},\ s_{i_1}(\alpha_{i_2}),\ s_{i_1}s_{i_2}(\alpha_{i_3}), \ldots, s_{i_1}s_{i_2}\cdots s_{i_{r-1}}(\alpha_{i_r}).$$

Let $<_{\mathbf{i}}$ denote this total order, the first root above being lowest in the order. Distinct reduced words give rise to distinct total orders. ∎

1.3.6 Proposition. Let $\mathbf{i}$ and $\mathbf{i}'$ be reduced words.

(a) If $\mathbf{i}'$ is obtained from $\mathbf{i}$ by a commutation, the effect on the total order $<_{\mathbf{i}}$ is to replace two consecutive positive roots α, β by β, α, where $\alpha + \beta \notin \Phi^+$.

(b) If $\mathbf{i}'$ is obtained from $\mathbf{i}$ by a 3-braid, then three consecutive positive roots of the form $\alpha, \alpha+\beta, \beta$ are replaced by $\beta, \alpha+\beta, \alpha$.

(c) If $\mathbf{i}'$ is obtained from $\mathbf{i}$ by a 4-braid, and $\mathbf{i}$ contains $ijij$, α_i being the short root and α_j the long root, then four consecutive positive roots of the form $\alpha, \alpha+\beta, \alpha+2\beta, \beta$ are replaced by $\beta, \alpha+2\beta, \alpha+\beta, \alpha$. Here, β is short and α is long. ∎

To illustrate (c), $\mathbf{i} := 1212$ is a reduced word for w_0 in type B_2, and it gives rise to the total order α_2, $\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$, α_1. Applying the 4-braid, $\mathbf{i}' := 2121$ induces the opposite ordering on these roots.

1.3.7 Lemma. If α, β and $\alpha+\beta \in \Phi_w^+$ then for all reduced words $\mathbf{i}$ for w, the root $\alpha+\beta$ lies inbetween α and β with respect to the total order $<_{\mathbf{i}}$. ∎

1.3.8 Note. Consider any reduced longest word $i_N \ldots i_k \ldots i_1$ and suppose that β corresponds to the letter i_k, that is

$$\beta = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k}).$$

We could however 'start from the other end' and instead calculate

$$\bar{\beta} = s_{i_N} \ldots s_{i_{k+1}}(\alpha_{i_k}),$$

provided we undo the effect of the opposition involution. (Verifying this is a simple calculation.) We make this remark because in practise it is often easier to calculate the root corresponding to a letter using one formula rather the the other.

1.4 Commutation Classes

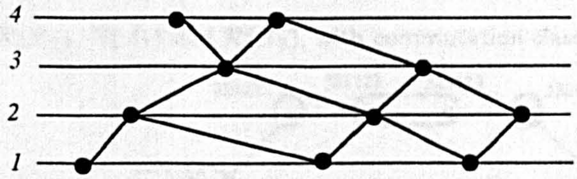
Let $[\mathbf{i}]$ denote the equivalence class of $\mathbf{i}$ under commutation-equivalence.

Following an idea of Roger Carter, commutation classes can be described by certain diagrams which are similar to those found in [Knuth, p.29]. in the context of parallel sorting algorithms. Let $\mathbf{i} := i_r \ldots i_1$ be a reduced word whose letters belong to $\{1, \ldots, l\}$. Draw l parallel horizontal lines, numbered 1 to l from bottom to top. Place r (unlabelled) vertices $v_r, \ldots, v_1$ on these lines, vertex v_j on line i_j. It is convenient to place each vertex v_j slightly to the left of vertex v_{j-1}. Partially order these vertices by saying that v_j precedes v_k if the letter i_j lies to the right of i_k in all words which are commutation-equivalent to $\mathbf{i}$. Thus, vertices which are *minimal* in this partial order are placed towards the *right* of the diagram. The resulting diagram is called the **partial order graph** of $[\mathbf{i}]$.

The words comprising a commutation class C are precisely the **linear extensions** of the partial order graph of C, that is, the total orders which are compatible with the partial order. Thus, different commutation classes have different partial order graphs. For example, if $\mathbf{i} := 1243412312$, which happens to be a reduced word for $w_0 \in \mathcal{W}(A_4)$, then $[\mathbf{i}]$ has the partial order graph shown in Figure 1.4.1.

We now describe another characterisation of commutation classes. Let $\mathbf{i}$ be reduced for w. Define the relation $\leq_{[\mathbf{i}]}$ on Φ_w^+ by saying $\alpha \leq_{[\mathbf{i}]} \beta$ if, for all $\mathbf{j}$ such that $\mathbf{j} \sim \mathbf{i}$, we have $\alpha \leq_{\mathbf{j}} \beta$. This relation is a partial order. Clearly, if we label each vertex v_j of the partial order graph of $[\mathbf{i}]$ with the positive root $s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ and remove the horizontal lines, we obtain the diagram of the partial order $\leq_{[\mathbf{i}]}$.

1.4.1 FIGURE. The partial order graph of [1243412312].



The following two observations will not be required until chapter 5, but it is natural to include them here. In (b) we say that β **covers** α if β is one step higher in the partial order.

1.4.2 LEMMA. Let C be a commutation class of reduced words for $w \in \mathcal{W}$, and let $\alpha, \beta \in \Phi_w^+$.

(a) If β is maximal with respect to $\leq_C$ then there exists some representative $\mathbf{j} \in C$ such that β is the maximum with respect to $<_{\mathbf{j}}$.

(b) If β covers α with respect to $\leq_C$ then there exists some representative $\mathbf{j} \in C$ such that β covers α with respect to $<_{\mathbf{j}}$.

PROOF. Let $\mathbf{i} := i_r \ldots i_1$ be any representative of C.

(a) We have $\beta = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$ for some k. If $k = r$, we can set $\mathbf{j} := \mathbf{i}$. Otherwise, if some letter i_p of $\mathbf{i}$ with $p > k$ fails to commute with i_k then the positive root $\gamma := s_{i_1} \ldots s_{i_{p-1}}(\alpha_{i_p})$ satisfies $\beta <_{\mathbf{j}} \gamma$ for all $\mathbf{j} \in C$. Thus $\beta \leq_C \gamma$, contradicting the maximality of β. Therefore $\mathbf{i} \sim \mathbf{j} := i_k \mathbf{i}'$, where $\mathbf{i}' := i_r \ldots \widehat{i_k} \ldots i_1$, and we have $\beta = s_{\mathbf{i}'}^{-1}(\alpha_{i_k})$, which shows that β is the maximum root relative to $<_{\mathbf{j}}$.

(b) Let $\alpha = s_{i_1} \ldots s_{i_{h-1}}(\alpha_{i_h})$ and $\beta = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$, where, necessarily, $h < k$. For any g satisfying $h < g < k$, the letter i_g must be able to commute to the right of i_h or to the left of i_k, for otherwise the positive root $\gamma := s_{i_1} \ldots s_{i_{h-1}} \ldots s_{i_{g-1}}(\alpha_{i_g})$ would satisfy $\alpha <_C \gamma <_C \beta$ (strict inequalities), which contradicts our hypothesis. Thus, by commuting all letters i_g with $g \neq h, k$ out of the subword $i_k \ldots i_g \ldots i_h$, we obtain a word $\mathbf{j}$ in which α and β are consecutive with respect to $<_{\mathbf{j}}$. ∎

We can now prove that different commutation classes give rise to different partial orders.

1.4.3 LEMMA. Let $\mathbf{i}, \mathbf{j}$ be reduced words for $w \in \mathcal{W}$. If the partial orders $\leq_{[\mathbf{i}]}$ and $\leq_{[\mathbf{j}]}$ on Φ_w^+ are identical then $[\mathbf{i}] = [\mathbf{j}]$.

PROOF. We use induction on $\ell(w)$. If $\ell(w)$ is 0 or 1, the result is immediate, so suppose $\ell(w) > 1$. Let i be the first letter of $\mathbf{i}$, so that $\mathbf{i} = i\mathbf{i}'$, say. Thus, the positive root $\beta := s_{\mathbf{i}'}^{-1}(\alpha_i)$ is maximal with respect to both $\leq_{[\mathbf{i}]}$ and, by hypothesis, $\leq_{[\mathbf{j}]}$. It follows from 1.4.2 (a) that there exists some commutation-equivalence $\mathbf{j} \sim j\mathbf{j}'$ such that $s_{\mathbf{j}'}^{-1}(\alpha_j) = \beta$, whence

$$s_{\mathbf{j}'}^{-1}(\alpha_j) = s_{\mathbf{i}'}^{-1}(\alpha_i).$$

Multiply the left side by $w = s_j s_{\mathbf{j}'}$ and the right side by $w = s_i s_{\mathbf{i}'}$ to deduce that $-\alpha_i = -\alpha_j$, whence $i = j$. Therefore $\mathbf{i} \sim i\mathbf{i}'$ and $\mathbf{j} \sim i\mathbf{j}'$. Set $w' := s_{\mathbf{i}'} = s_{\mathbf{j}'}$, so that $\ell(w') = \ell(w) - 1$ and $\Phi_{w'}^+ = \Phi_w^+ \setminus \{\beta\}$. Since $[\mathbf{i}']$ and $[\mathbf{j}']$ clearly induce the same partial order on $\Phi_{w'}^+$, we have by induction that $\mathbf{i}' \sim \mathbf{j}'$. Therefore $\mathbf{i} \sim \mathbf{j}$. ∎

Let $\mathcal{R}$ denote the graph whose vertices are the reduced words for w_0 and whose edges join pairs of words related by a single braid.

Let $\mathcal{C}$ denote the graph whose vertices are the commutation classes for w_0 and whose edges join pairs of commutation classes which contain representatives related by a single braid.

In [Stanley] the following formula is obtained for the number of reduced words for w_0 in type A_l:

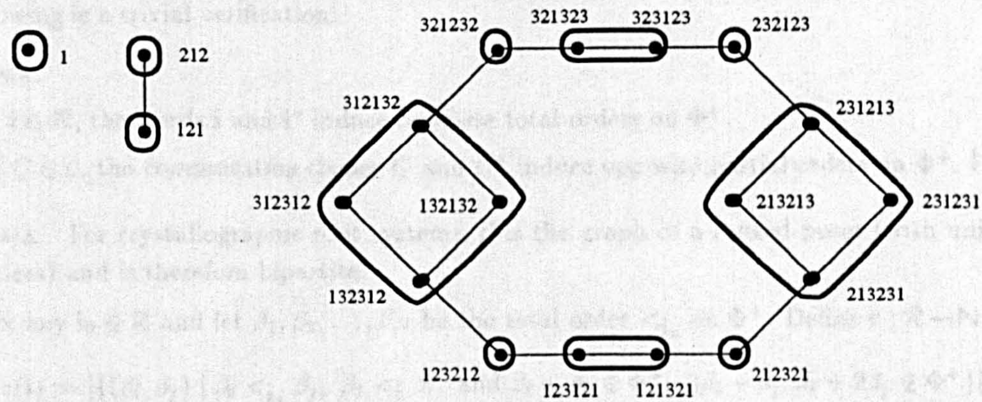
$$|\mathcal{R}(A_l)| = \frac{\binom{l+1}{2}!}{1^l\, 3^{l-1}\, 5^{l-2} \ldots (2l-1)^1}.$$

A formula for $|\mathcal{C}(A_l)|$ is unknown at present. These values have been obtained with the aid of a computer; see [Knuth, p.35].

l	1	2	3	4	5	6	7	8
$\lvert\mathcal{C}(A_l)\rvert$	1	2	8	62	908	24698	1232944	112018190

Figure 1.4.4 shows the graphs $\mathcal{R}(A_1)$, $\mathcal{R}(A_2)$ and $\mathcal{R}(A_3)$. The encircled subgraphs are the commutation classes.

1.4.4 FIGURE. The graphs $\mathcal{R}(A_1)$, $\mathcal{R}(A_2)$ and $\mathcal{R}(A_3)$, with commutation classes circled.



The corresponding graphs $\mathcal{C}(A_1)$, $\mathcal{C}(A_2)$ and $\mathcal{C}(A_3)$ are shown in Figure 1.4.5.

1.4.5 FIGURE. The graphs $\mathcal{C}(A_1)$, $\mathcal{C}(A_2)$ and $\mathcal{C}(A_3)$.

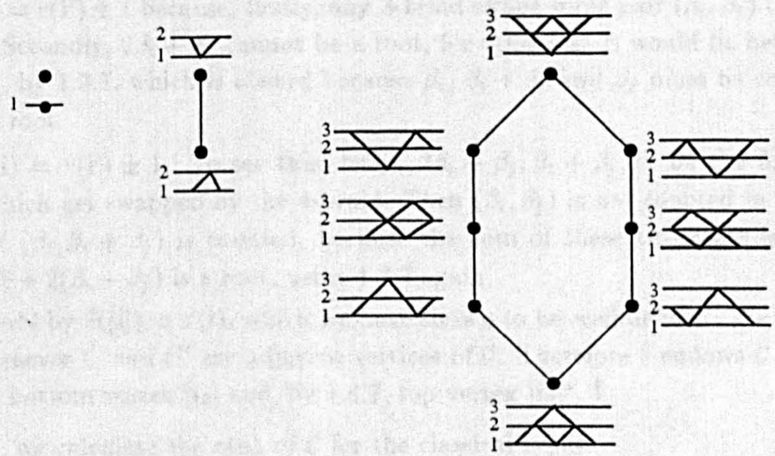
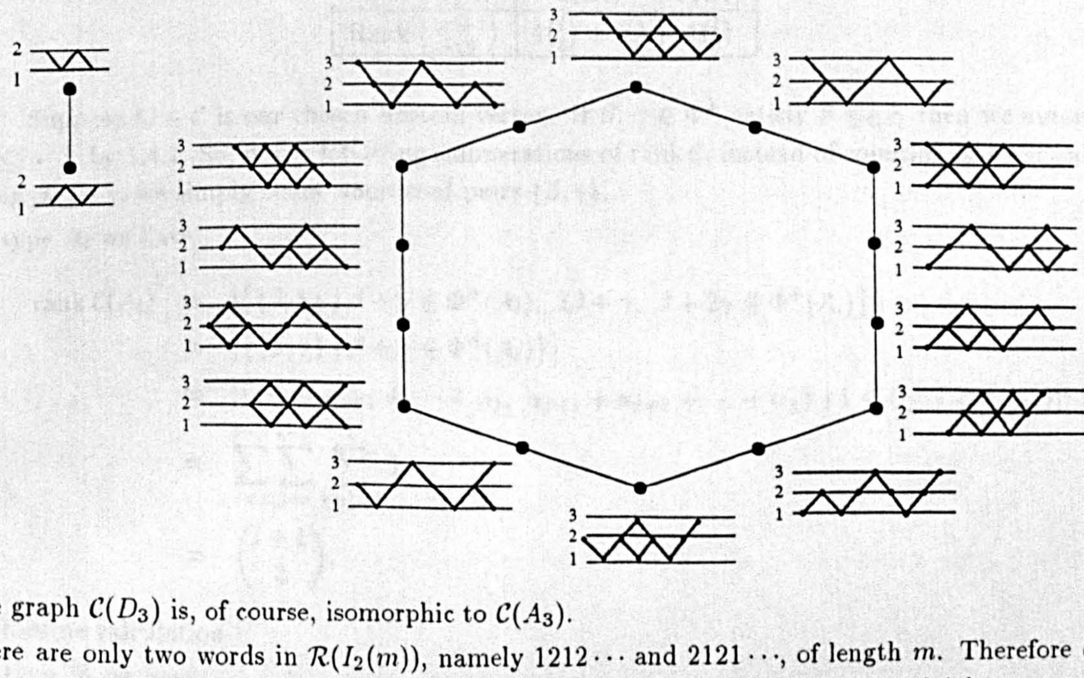


Figure 1.4.6 shows the graphs $\mathcal{C}(B_2)$ and $\mathcal{C}(B_3)$.

1.4.6 FIGURE. The graphs $\mathcal{C}(B_2)$ and $\mathcal{C}(B_3)$.



The graph $\mathcal{C}(D_3)$ is, of course, isomorphic to $\mathcal{C}(A_3)$.

There are only two words in $\mathcal{R}(I_2(m))$, namely $1212\cdots$ and $2121\cdots$, of length m. Therefore $\mathcal{C}(I_2(m))$ consists of just two vertices joined by an edge. We will not be concerned with this trivial case any further.

If $\mathbf{i} = i_r \dots i_1$, define the **reversal** of $\mathbf{i}$ by $\mathbf{i}^{\text{rev}} := i_1 \dots i_r$, and also define $\bar{\mathbf{i}} := \bar{i_r} \dots \bar{i_1}$, applying the opposition involution to each letter. Now define $[\mathbf{i}]^{\text{rev}} := [\mathbf{i}^{\text{rev}}]$ and $\overline{[\mathbf{i}]} := [\bar{\mathbf{i}}]$. Noting that the opposition involution and reversal commute, set $\mathbf{i}^* := \bar{\mathbf{i}}^{\text{rev}}$ and $[\mathbf{i}]^* := [\mathbf{i}^*]$. These definitions are all well defined.

The following is a trivial verification.

1.4.7 Lemma.

(a) For all $\mathbf{i} \in \mathcal{R}$, the words $\mathbf{i}$ and $\mathbf{i}^*$ induce opposite total orders on Φ^+.

(b) For all $C \in \mathcal{C}$, the commutation classes C and C^* induce opposite partial orders on Φ^+. ∎

1.4.8 Lemma. For crystallographic root systems, $\mathcal{C}$ is the graph of a ranked poset (with unique top and bottom vertices) and is therefore bipartite.

Proof. Fix any $\mathbf{i}_0 \in \mathcal{R}$ and let $\beta_1, \beta_2, \dots, \beta_N$ be the total order $<_{\mathbf{i}_0}$ on Φ^+. Define $r : \mathcal{R} \to \mathbb{N}$ by

$$r(\mathbf{i}) := |\{(\beta_i, \beta_j) \mid \beta_i <_{\mathbf{i}_0} \beta_j,\ \beta_j <_{\mathbf{i}} \beta_i, \text{ and } \beta_i + \beta_j \in \Phi^+;\ 2\beta_i + \beta_j, \beta_i + 2\beta_j \notin \Phi^+\}|.$$

By 1.3.6 (a), if $\mathbf{i} \sim \mathbf{i}'$ then $r(\mathbf{i}) = r(\mathbf{i}')$ because any commutation swaps a pair of roots whose sum is *not* a root.

If $\mathbf{i} \xrightarrow{3} \mathbf{i}'$ then $r(\mathbf{i}) = r(\mathbf{i}') \pm 1$ because, firstly, any 3-braid swaps some pair (β_i, β_j) of roots whose sum *is* a root, by 1.3.6 (b). Secondly, $2\beta_i + \beta_j$ cannot be a root, for otherwise it would lie between β_i and $\beta_i + \beta_j$ in the total order $<_{\mathbf{i}}$, by 1.3.7, which is absurd because β_i, $\beta_i + \beta_j$ and β_j must be consecutive. Similarly, $\beta_i + 2\beta_j$ cannot be a root.

If $\mathbf{i} \xrightarrow{4} \mathbf{i}'$ then $r(\mathbf{i}) = r(\mathbf{i}') \pm 1$. To see this, let $\beta_i, 2\beta_i + \beta_j, \beta_i + \beta_j, \beta_j$ be the four consecutive roots with respect to $<_{\mathbf{i}}$ which get swapped by the 4-braid. Then (β_i, β_j) is *not* counted in calculating $r(\mathbf{i})$ since $2\beta_i + \beta_j \in \Phi^+$. But $(\beta_i, \beta_i + \beta_j)$ *is* counted, because the sum of these two roots is a root, and neither $2\beta_i + (\beta_i + \beta_j)$ nor $\beta_i + 2(\beta_i + \beta_j)$ is a root, using 1.3.7 again.

Now define $\hat{r} : \mathcal{C} \to \mathbb{N}$ by $\hat{r}([\mathbf{i}]) := r(\mathbf{i})$, which we have shown to be well defined. We have also shown that $\hat{r}(C) = \hat{r}(C') \pm 1$ whenever C and C' are adjacent vertices of $\mathcal{C}$. Therefore $\hat{r}$ endows $\mathcal{C}$ with the structure of a ranked poset, with bottom vertex $[\mathbf{i}_0]$ and, by 1.4.7, top vertex $[\mathbf{i}_0]^*$. ∎

For completeness, we calculate the rank of $\mathcal{C}$ for the classical types.

1.4.9 Proposition. The rank of the poset $\mathcal{C}$ (with any choice of bottom vertex) is as follows.

Poset	$\mathcal{C}(A_l)$	$\mathcal{C}(B_l)$	$\mathcal{C}(D_l)$
Rank	$\binom{l+1}{3}$	$4\binom{l}{3} + \binom{l}{2}$	$4\binom{l}{3}$

Proof. Suppose $C \in \mathcal{C}$ is our chosen bottom vertex. If $\beta, \gamma \in \Phi^+$ satisfy $\beta \leq_C \gamma$ then we automatically have $\gamma \leq_{C^*} \beta$, by 1.4.7. So, in our following enumerations of rank $\mathcal{C}$, instead of counting *ordered* pairs (β, γ) satisfying $\beta \leq_C \gamma$, we simply count *unordered* pairs $\{\beta, \gamma\}$.

For type A_l we have

$$\begin{aligned}
\operatorname{rank} \mathcal{C}(A_l) &= |\{\{\beta,\gamma\} \mid \beta + \gamma \in \Phi^+(A_l);\ 2\beta + \gamma,\ \beta + 2\gamma \notin \Phi^+(A_l)\}| \\
&= |\{\{\beta,\gamma\} \mid \beta + \gamma \in \Phi^+(A_l)\}| \\
&= |\{\{\alpha_i + \alpha_{i+1} + \cdots + \alpha_j,\ \alpha_{j+1} + \alpha_{j+2} + \cdots + \alpha_k\} \mid 1 \leq i \leq j < k \leq l\}| \\
&= \sum_{i=1}^{l-1} \sum_{j=i}^{l-1} \sum_{k=j+1}^{l} 1 \\
&= \binom{l+1}{3},
\end{aligned}$$

after a routine calculation.

For type B_l we have

$$
\begin{aligned}
\operatorname{rank}\mathcal{C}(B_l) &= \left|\{\{\beta,\gamma\} \mid \beta+\gamma \in \Phi^+(B_l);\ 2\beta+\gamma,\ \beta+2\gamma \notin \Phi^+(B_l)\}\right| \\
&= \left|\{\{\alpha_1+\alpha_2+\cdots+\alpha_i,\ \alpha_1+\alpha_2+\cdots+\alpha_j\} \mid 1 \le i < j \le l\}\right| \\
&\quad +\left|\{\{\alpha_i+\alpha_{i+1}+\cdots+\alpha_j,\ \alpha_{j+1}+\alpha_{j+2}+\cdots+\alpha_k\} \mid 2 \le i \le j < k \le l\}\right| \\
&\quad +\sum_{i=2}^{l}\left|\left\{\{\alpha_i+\alpha_{i+1}+\cdots+\alpha_j,\ \underbrace{(2\alpha_1+\alpha_2+\cdots+\alpha_{i-1})}_{:=2\alpha_1 \text{ if } i=2}+\underbrace{(\alpha_2+\alpha_3+\cdots+\alpha_k)}_{:=0 \text{ if } k=1}\}\right\}\right|,
\end{aligned}
$$

where, in the last line, for fixed i, the index j obeys $i \le j \le l$, (giving $l+1-i$ choices), and, for fixed j as well, the index k obeys $1 \le k \le l$ and $k \ne i-1, j$ (giving $l-2$ choices). Therefore

$$
\begin{aligned}
\operatorname{rank}\mathcal{C}(B_l) &= \binom{l}{2}+\binom{l}{3}+\sum_{i=2}^{l}(l+1-i)(l-2) \\
&= \binom{l}{2}+4\binom{l}{3}.
\end{aligned}
$$

For type D_l, we have

$$
\operatorname{rank}\mathcal{C}(D_l) = \left|\{\{\beta,\gamma\} \mid \beta+\gamma \in \Phi^+(D_l)\}\right|.
$$

If some unordered pair $\{\beta,\gamma\}$ is counted in the enumeration of $\operatorname{rank}\mathcal{C}(D_l)$ then $\beta,\ \gamma,\ \beta+\gamma \in \Phi^+(D_l) \subseteq \Phi^+(B_l)$ and $2\beta+\gamma,\ \beta+2\gamma \notin \Phi^+(D_l)$. For a contradiction, suppose, however, that $2\beta+\gamma \in \Phi^+(B_l)$. Then this root must be of the form $\alpha_1^B+\alpha_2^B+\cdots+\alpha_i^B$ for some $i \ge 1$. However, $2\beta+\gamma$ visibly cannot have all nonzero coefficients equal to 1. This is our required contradiction. By a symmetrical argument we therefore obtain $2\beta+\gamma,\ \beta+2\gamma \notin \Phi^+(B_l)$. Consequently, the pair $\{\beta,\gamma\}$ is also counted in the enumeration of $\operatorname{rank}\mathcal{C}(B_l)$. We can therefore make use of our calculation of $\operatorname{rank}\mathcal{C}(B_l)$. The only pairs $\{\beta,\gamma\}$ of roots in $\Phi^+(B_l)$ that we need to exclude are those of the form $\{\alpha_1^B+\alpha_2^B+\cdots+\alpha_i^B, \alpha_1^B+\alpha_2^B+\cdots+\alpha_j^B\}$, where $1 \le i < j \le l$, because these roots do not belong to $\Phi^+(D_l)$. There are $\binom{l}{2}$ such pairs. So $\operatorname{rank}\mathcal{C}(D_l) = \operatorname{rank}\mathcal{C}(B_l) - \binom{l}{2} = 4\binom{l}{3}$. ∎

1.5 Generation of Fundamental Groups

In this section we make a simple observation about the closed paths in $\mathcal{C}$, although this will not be used elsewhere.

A **morphism** $G \twoheadrightarrow G'$ of connected graphs is a surjective mapping of the vertices of G onto the vertices of G' such that pairs of adjacent vertices of G are mapped to vertices which are either adjacent or coincident. We call G' a **quotient graph** of G.

By 1.3.1, $\mathcal{R}$ and $\mathcal{C}$ are connected graphs, and $\mathcal{C}$ is clearly a quotient graph of $\mathcal{R}$.

Let A be a set of closed paths in the connected graph G. The fundamental group of G is **generated** by A if, given a vertex $v \in G$, the fundamental group of G relative to v is generated by paths of the form $p^{-1}ap$, where p is a path from v to the origin of the loop $a \in A$, and p^{-1} is p traced backwards. (We compose maps from right-to-left.) This property is independent of v, by connectedness of G.

The **support** of a reduced word $\mathbf{i} := i_r \ldots i_1$ is the set $\operatorname{supp}(\mathbf{i}) := \{i_1, i_2, \ldots, i_r\}$.

The following theorem is due to [Tits, §4] and can also be found in [Ronan, Theorem 2.17]. Recall the notation $\mathbf{b}_{ij} = ijij\ldots$ with m_{ij} letters, where s_is_j has order m_{ij}.

1.5.1 THEOREM. The fundamental group of $\mathcal{R}$ is generated by the closed paths of the form

(a) $(\mathbf{i}_0\mathbf{b}_{ij}\mathbf{i},\ \mathbf{i}_1\mathbf{b}_{ij}\mathbf{i},\ \mathbf{i}_1\mathbf{b}_{ji}\mathbf{i},\ \mathbf{i}_0\mathbf{b}_{ji}\mathbf{i},\ \mathbf{i}_0\mathbf{b}_{ij}\mathbf{i})$;

(b) $(\mathbf{i}\mathbf{i}_0\mathbf{i}',\ \mathbf{i}\mathbf{i}_1\mathbf{i}',\ \ldots,\ \mathbf{i}\mathbf{i}_r\mathbf{i}',\ \mathbf{i}\mathbf{i}_0\mathbf{i}')$, where the support of the words $\mathbf{i}_0, \mathbf{i}_1, \ldots, \mathbf{i}_r$ is a set of cardinality three. ∎

1.5.2 Corollary. The fundamental group of $\mathcal{C}$ is generated by

(i) quadrilaterals and octagons in types A_l, D_l and E_6, E_7, E_8;

(ii) quadrilaterals, octagons and 14-gons in types B_l and F_4.

Proof. The fundamental group of $\mathcal{C}$ is generated by the images of the paths (a) and (b) of 1.5.1 under the natural morphism $\mathcal{R} \twoheadrightarrow \mathcal{C}$.

Let p denote the image of a path of type (a). Clearly, p has no more than four vertices. We therefore only have to rule out the possibility that p is a triangle. This, however, follows from 1.4.8, which showed that $\mathcal{C}$ is bipartite.

Now let p denote the image of a path of type (b). Let $\{i, j, k\}$ be the three letters involved in the braids and let Δ' be the subgraph of the Coxeter graph with these three vertices. If Δ' is of type $A_1 \oplus A_1 \oplus A_1$ (disjoint copies of A_1) then p is a single vertex. If Δ is of type $A_2 \oplus A_1$ or $B_2 \oplus A_1$ then p has two vertices. If Δ' is of type A_3 then p is isomorphic to a closed path in $\mathcal{C}(A_3)$, and is therefore an octagon. Similarly, if Δ' is of type B_3 then p has 14 sides. ∎

2. Normal Representatives of Commutation Classes

2.1 Introduction

In this chapter we obtain representatives of all commutation classes, called **normal representatives**, which have a particularly simple form. In general, each commutation class has many normal representatives, but they do not differ widely from one another. We will show how normal representatives can be used to give an inductive method for constructing commutation classes.

In order to describe normal representatives concisely, we introduce the following notation. Let $i,\ j \in \mathbb{N}$. Define words

$$(i \nearrow j) := \begin{cases} i\, i+1\, i+2 \cdots j & \text{if } i \leq j \\ \emptyset & \text{if } i > j, \end{cases}$$

$$(i \searrow j) := \begin{cases} i\, i-1\, i-2 \cdots j & \text{if } i \geq j \\ \emptyset & \text{if } i < j. \end{cases}$$

For example, $(1 \nearrow 4) = 1234$, $(4 \searrow 4) = 4$ and $(4 \searrow 5) = \emptyset$.

If i is a vertex of the Coxeter graph of type A_l or B_l then define adjacent vertices

$$i^+ := i+1\ (i \neq l) \text{ and } i^- := i-1\ (i \neq 1).$$

If i is a vertex of the Coxeter graph of type D_l, define

$$i^+ := i+1\ (i \neq 1, l),\ \ 1^+ := 3,\ \ i^{--} := i^- := i-1\ (i \neq 1,2,3),\ \ 3^- := 2,\ \ 3^{--} := 1.$$

If now $\mathbf{i} = i_r \ldots i_1$, define $\mathbf{i}^+ := i_r^+ \ldots i_1^+$, and similarly for $\mathbf{i}^-$, $\mathbf{i}^{--}$.

Further, in type D_l, let $\mathbf{i}^{\mathrm{aut}}$ denote the word obtained from $\mathbf{i}$ by interchanging 1 and 2, fixing everything else. (This differs from $\bar{\mathbf{i}}$ only in that the opposition involution is trivial if l is even.)

If $\mathbf{i}$ is a reduced word and $t \in \mathbb{N}$ then the **multiplicity** of t in $\mathbf{i}$, denoted $\mathrm{mul}_t(\mathbf{i})$, is the number of occurrences of the letter t. Clearly, commutations leave letter multiplicities unchanged, so we can define the **multiplicity** of t in $[\mathbf{i}]$ to be $\mathrm{mul}_t([\mathbf{i}]) := \mathrm{mul}_t(\mathbf{i})$.

We define the **promotion** operator to be the map

$$\begin{array}{rcl} \partial : \mathcal{R} & \rightarrow & \mathcal{R} \\ i_N \ldots i_2 i_1 & \mapsto & \overline{i_1} i_N \ldots i_2, \end{array}$$

where $\overline{i_1}$ is the effect of the opposition involution; see 1.1.2 (d). This terminology and notation is motivated by [EG], in which a bijection between $\mathcal{R}(A_l)$ and the set of all **balanced tableaux** of shape $(l-1, \ldots, 1)$ is given. The **elementary promotion** operator on the tableaux corresponds to our definition of ∂.

2.2 Normal Representatives in Type A_l

Let $C \in \mathcal{C}(A_l)$. Any representative of C of the form $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$, for certain subwords $\boldsymbol{\lambda}$ and $\boldsymbol{\varrho}$ is called a **normal representative** of C. For example, the commutation class $[1243412312] \in \mathcal{C}(A_4)$, whose partial order graph is Figure 1.4.1, has $121\,4321\,432$ as a possible normal representative. Here, $\boldsymbol{\lambda} = 121$ and $\boldsymbol{\varrho} = 432$.

2.2.1 PROPOSITION. Let $C \in \mathcal{C}(A_l)$ have a normal representative $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$. Each of the following is a normal representative of some $C' \in \mathcal{C}(A_l)$ which is adjacent to C. The possibilities are given intuitive names which suggest how C' is obtained from C. (The restrictions in parentheses will be known to hold automatically once Lemma 2.2.2 is proven.)

'left' $\boldsymbol{\lambda}^\#(l \searrow 1)\boldsymbol{\varrho}$, where $\boldsymbol{\lambda}^\#$ is obtained from $\boldsymbol{\lambda}$ by commutations, followed by a 3-braid.

'right' $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}^\#$, where $\boldsymbol{\varrho}^\#$ is obtained from $\boldsymbol{\varrho}$ by commutations, followed by a 3-braid.

'left to right' $\boldsymbol{\lambda}'(l \searrow 1)i^+\,\boldsymbol{\varrho}$, where $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}' i$, $(i \neq l)$.

'right to left' $\boldsymbol{\lambda}\, i^-(l \searrow 1)\boldsymbol{\varrho}'$, where $\boldsymbol{\varrho} \sim i\boldsymbol{\varrho}'$, $(i \neq 1)$.

PROOF. This is clear for the first two cases.

Consider case 'left to right'. The hypotheses allow us to apply the following sequence of braids:

$$\begin{aligned}\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho} &\sim \boldsymbol{\lambda}' i (l \searrow 1)\boldsymbol{\varrho} \\ &\sim \boldsymbol{\lambda}'(l \searrow i+2)\, i\; i+1\; i\, (i-1 \searrow 1)\boldsymbol{\varrho} \\ &\overset{3}{\leftrightarrow} \boldsymbol{\lambda}'(l \searrow i+2)\, i+1\; i\; i+1\, (i-1 \searrow 1)\boldsymbol{\varrho} \\ &\sim \boldsymbol{\lambda}'(l \searrow 1)\, i^+\, \boldsymbol{\varrho}.\end{aligned}$$

Informally, i hops from left to right over $(l \searrow 1)$ and increases by 1. Since one 3-braid occurs amongst the above sequence of commutations, C' is adjacent to C. The last case is similar. ▮

2.2.2 LEMMA. If $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$ is a normal representative then $l \notin \operatorname{supp}(\boldsymbol{\lambda})$ and $1 \notin \operatorname{supp}(\boldsymbol{\varrho})$.

PROOF. Suppose that $l \in \operatorname{supp}(\boldsymbol{\lambda})$, and consider the rightmost l in $\boldsymbol{\lambda}$. By 2.2.1, we can apply 'left to right' moves to each letter of $\boldsymbol{\lambda}$ lying to the right of this l, leaving a word containing the subword ll, which is nonreduced. This is absurd, so $l \notin \operatorname{supp}(\boldsymbol{\lambda})$. The other part is proved similarly. ▮

We now show that the normal representatives of a commutation class do not differ widely from one another.

2.2.3 LEMMA. Let $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$ and $\boldsymbol{\lambda}'(l \searrow 1)\boldsymbol{\varrho}'$ be normal representatives of C and C', respectively. Then $C = C'$ if and only if $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}'$ and $\boldsymbol{\varrho} \sim \boldsymbol{\varrho}'$.

PROOF. Sufficiency is clear. For the converse we will establish a slightly more general result. Suppose we are given commutation-equivalent reduced words $\mathbf{L}(l \searrow 1)\mathbf{R}$, $\mathbf{L}'(l \searrow 1)\mathbf{R}'$ satisfying $1 \notin \operatorname{supp}(\mathbf{R}), \operatorname{supp}(\mathbf{R}')$ and $l \notin \operatorname{supp}(\mathbf{L}), \operatorname{supp}(\mathbf{L}')$. We will prove that $\mathbf{L} \sim \mathbf{L}'$ and $\mathbf{R} \sim \mathbf{R}'$ by induction upon $\ell(\mathbf{R})$.

If $\ell(\mathbf{R}) = 0$ then $\mathbf{L}(l \searrow 1) \sim \mathbf{L}'(l \searrow 1)\mathbf{R}'$. The letter 1 of $(l \searrow 1)$ in $\mathbf{L}'(l \searrow 1)\mathbf{R}'$ is the rightmost occurrence of 1, so this 1 commutes with $\mathbf{R}'$; in particular, $2 \notin \operatorname{supp}(\mathbf{R}')$. So $\mathbf{L}(l \searrow 1) \sim \mathbf{L}'(l \searrow 2)\mathbf{R}'1$, which implies that $\mathbf{L}(l \searrow 2) \sim \mathbf{L}'(l \searrow 2)\mathbf{R}'$, by 1.3.4.

Now repeat this argument, considering the rightmost $2, 3, \ldots, l-1$ in turn; we see that $\operatorname{supp}(\mathbf{R}')$ contains none of $3, 4, \ldots, l$. So $\mathbf{R}' = \emptyset$. Therefore $\mathbf{L}(l \searrow 1) \sim \mathbf{L}'(l \searrow 1)$, which implies that $\mathbf{L} \sim \mathbf{L}'$, applying 1.3.4.

Now let $\ell(\mathbf{R}) > 0$. Write $\mathbf{R} = \mathbf{R}_1 i$, so that $\mathbf{L}(l \searrow 1)\mathbf{R}_1 i \sim \mathbf{L}'(l \searrow 1)\mathbf{R}'$. Since the rightmost i in the second word commutes to the end, and since $i \neq 1$, we must have $i \in \operatorname{supp}(\mathbf{R}')$. So $\mathbf{R}' \sim \mathbf{R}_1' i$ for some R_1'. So, by 1.3.4 we obtain $\mathbf{L}(l \searrow 1)\mathbf{R}_1 \sim \mathbf{L}'(l \searrow 1)\mathbf{R}_1'$. By induction, $\mathbf{L} \sim \mathbf{L}'$ and $\mathbf{R}_1 \sim \mathbf{R}_1'$, so $\mathbf{R} \sim \mathbf{R}'$. ▮

2.2.4 LEMMA. Let $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$ be a normal representative. Consider the corresponding total order on $\Phi^+(A_l)$; see 1.3.5. The l positive roots corresponding to the letters of the subword $(l \searrow 1)$ are precisely those with summand α_1, namely

$$\alpha_1,\ \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_l,$$

in some order.

PROOF. For each $1 \le i \le l$, the positive root corresponding to the letter i in $(l \searrow 1)$ is

$$s_{\varrho}{}^{-1} s_1 s_2 \ldots s_{i-1}(\alpha_i) = s_{\varrho}{}^{-1}(\alpha_1 + \ldots + \alpha_i).$$

Since, by 2.2.2, $1 \notin \operatorname{supp}(\boldsymbol{\varrho})$, it follows that $s_{\varrho}{}^{-1}$ cannot delete the summand α_1. So i gives rise to a root with summand α_1. So the l letters of $(l \searrow 1)$ must correspond to all l positive roots with summand α_1. ▮

We can now prove the converse of 2.2.1.

2.2.5 PROPOSITION. Let $C \in \mathcal{C}(A_l)$ have a normal representative $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$ and suppose that C' is adjacent to C. Then C' has a normal representative of exactly one of the four types listed in 2.2.1.

PROOF. Any representative of C' may be obtained from $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$ by commutations, a 3-braid, and further commutations. Prior to applying the 3-braid, we track the letters, as well as their corresponding positive roots, throughout the commutations; see 1.3.6 (a). We note whether each letter comes from λ, $(l \searrow 1)$ or ϱ. Let iji be the 3-letter subword to which the 3-braid is applied. Let α, $\alpha+\beta$, β be the corresponding positive roots; see 1.3.6 (b). Visibly, either none or exactly two of these three roots may have summand α_1. So by 2.2.4, either none or exactly two of the letters in iji come from $(l \searrow 1)$. We look at the two cases in turn.

Suppose that no letter of iji comes from $(l \searrow 1)$. It is impossible for iji to be a *mixture* of letters from λ and ϱ because the intervening subword $(l \searrow 1)$ would prevent the letters of iji commuting into adjacent positions. So C' contains a normal representative of type 'left' or 'right' according as iji comes entirely from λ or ϱ.

If exactly two letters of iji come from $(l \searrow 1)$, they must be either the first or last two of iji. If it is the last two, then $j = i+1$ and the first letter, i, comes from λ. It is clear that i may commute to the rightmost letter of λ, that is, $\lambda \sim \lambda' i$, because otherwise it would be impossible to commute the letters of iji into adjacent positions. Therefore C' has a representative of type 'left to right', since, by 2.2.2, we have $i \neq l$.

If the *first* two letters of iji come from $(l \searrow 1)$, we similarly obtain type 'right to left'.

Finally, C' cannot have normal representatives of more than one of the four types, because, using the criterion in 2.2.3, it is easily checked that no two types lie in the same commutation class. ∎

2.2.6 PROPOSITION. Every commutation class in $\mathcal{C}(A_l)$ contains a normal representative.

PROOF. Proposition 2.2.1 shows that if some vertex of the graph $\mathcal{C}(A_l)$ contains a normal representative then so do all adjacent vertices. Since $\mathcal{C}(A_l)$ is connected we therefore only have to exhibit one normal representative. If $w_0^{(l)}$ denotes the longest element of $\mathcal{W}(A_l)$ then a calculation shows that

$$w_0^{(l)} = w_0^{(l-1)} s_l \dots s_2 s_1 \text{ and } \ell(w_0^{(l)}) = \ell(w_0^{(l-1)}) + l$$

for $l > 1$. Therefore

$$1(2 \searrow 1)(3 \searrow 1)\dots(l \searrow 1)$$

belongs to $\mathcal{R}(A_l)$ and is clearly a normal representative. ∎

We now describe an algorithm for constructing all the commutation classes in $\mathcal{C}(A_l)$ from those of $\mathcal{C}(A_{l-1})$.

Let $C \in \mathcal{C}(A_{l-1})$ and let I be an **ideal** of the partial order graph of C, that is, a set of vertices such that if $i \in I$ and j is lower in the partial order, then $j \in I$. Let $\mathbf{i}_1$ be the word corresponding to any linear extension of I, and extend to a reduced longest word, so that

$$C = [\mathbf{i}_2\mathbf{i}_1] \in \mathcal{C}(A_{l-1})$$

for some word $\mathbf{i}_2$. Since $\mathbf{i}_2\mathbf{i}_1 \in \mathcal{R}(A_{l-1})$, we have $\mathbf{i}_2\mathbf{i}_1(l \searrow 1) \in \mathcal{R}(A_l)$. So, by applying $\ell(\mathbf{i}_1)$ 'left to right' moves (see 2.2.1), we obtain $\mathbf{i}_2(l \searrow 1)\mathbf{i}_1^+$. (See 2.1 for the notation $\mathbf{i}_1^+$.) Set

$$C_I := [\mathbf{i}_2(l \searrow 1)\mathbf{i}_1^+] \in \mathcal{C}(A_l).$$

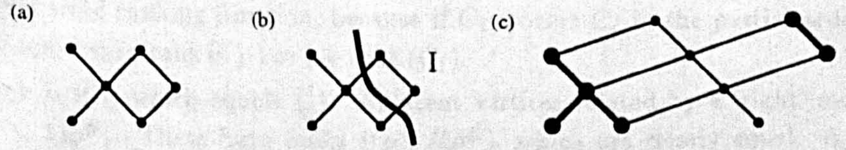
Since all linear extensions of I are commutation-equivalent to one another, C_I is unambiguously defined.

For example, if $C \in \mathcal{C}(A_3)$ has the partial order graph shown in Figure 2.2.7 (a), and if I is the ideal indicated in (b) then C_I is shown in (c).

We shall omit the numbered horizontal lines in the partial order graphs from now onwards; if drawn in, they would be numbered from bottom to top, as usual.

In terms of words, $C = [132132]$, the word 32 is a linear extension of I, and $C_I = [1321\,4321\,43]$.

2.2.7 FIGURE. Constructing C_I.



2.2.8 PROPOSITION. The mapping

$$\begin{array}{rcl}\{(C,I)\mid C\in\mathcal{C}(A_{l-1}), I \text{ is an ideal of } C\} & \to & \mathcal{C}(A_l)\\ (C,I) & \mapsto & C_I\end{array}$$

is bijective.

PROOF. For surjectivity, take any $[\boldsymbol{\lambda}(l\searrow 1)\boldsymbol{\varrho}]\in\mathcal{C}(A_l)$, set $C:=[\boldsymbol{\lambda}\ \boldsymbol{\varrho}^-]\in\mathcal{C}(A_{l-1})$, and let I be the ideal of the partial order graph of C formed by the letters of $\boldsymbol{\varrho}^-$. By definition, (C,I) maps to $[\boldsymbol{\lambda}(l\searrow 1)\boldsymbol{\varrho}]$.

For injectivity, suppose that $C_I=C'_{I'}$, with the obvious notation. Then $[\mathbf{i}_2(l\searrow 1)\mathbf{i}_1^+]=[\mathbf{i}'_2(l\searrow 1)\mathbf{i}'^+_1]$. By 2.2.3 we have $\mathbf{i}_2\sim\mathbf{i}'_2$ and $\mathbf{i}_1^+\sim\mathbf{i}'^+_1$, hence $\mathbf{i}_1\sim\mathbf{i}'_1$. So $C=[\mathbf{i}_2\mathbf{i}_1]=[\mathbf{i}'_2\mathbf{i}'_1]=C'$ and since $\mathbf{i}_1\sim\mathbf{i}'_1$, we have $I=I'$. ∎

Before giving an example, we shall introduce a useful map and list some elementary properties.

2.2.9 PROPOSITION. Define

$$\begin{array}{rcl}\delta:\mathcal{C}(A_l) & \to & \mathcal{C}(A_{l-1})\\ [\boldsymbol{\lambda}(l\searrow 1)\boldsymbol{\varrho}] & \mapsto & [\boldsymbol{\lambda}\ \boldsymbol{\varrho}^-].\end{array}$$

(a) The map δ is a surjective graph morphism. Each fibre is a connected subgraph and any two vertices of the same fibre are related to one another by 'left to right' and 'right to left' moves.

(b) The δ-fibre of $C\in\mathcal{C}(A_{l-1})$ is

$$\{C_I\mid I \text{ is an ideal of } C\}.$$

This is a poset with unique top and bottom by setting

$$C_I\le C_{I'} \text{ if } I\subseteq I'.$$

(c) Each δ-fibre is a ranked poset with rank function

$$\operatorname{rank}(C_I):=|I|$$

and highest rank $\binom{l}{2}$. Adjacent vertices of $\mathcal{C}(A_l)$ related to one another by a 'left' or 'right' move have the same rank (in their respective fibres).

PROOF. First note that δ is well defined, by 2.2.3.

(a) If $C=[\mathbf{i}]\in\mathcal{C}(A_{l-1})$ then $[\mathbf{i}(l\searrow 1)]$ maps to C, so δ is onto. Using 2.2.1, it is easily checked that under 'left' and 'right' moves, the δ-images of adjacent vertices are themselves adjacent. Under 'left to right' and 'right to left' moves, the δ-images coincide. So δ is a morphism.

Now consider a general vertex $[\boldsymbol{\lambda}(l\searrow 1)\boldsymbol{\varrho}]\in\delta^{-1}(C)$. By definition, $\boldsymbol{\lambda}\ \boldsymbol{\varrho}^-\sim\mathbf{i}$. So, by applying $\ell(\boldsymbol{\varrho})$ 'left to right' moves to $[\mathbf{i}(l\searrow 1)]$, we obtain a connected path, lying in $\delta^{-1}(C)$, joining $[\mathbf{i}(l\searrow 1)]$ to $[\boldsymbol{\lambda}(l\searrow 1)\boldsymbol{\varrho}]$. Thus, $\delta^{-1}(C)$ is connected.

(b) This is immediate from the definitions; in the notation of (a), we have

$$\begin{array}{rcl}\delta^{-1}(C) & = & \{[\boldsymbol{\lambda}(l\searrow 1)\boldsymbol{\varrho}]\mid C=[\boldsymbol{\lambda}\ \boldsymbol{\varrho}^-]\}\\ & = & \{C_I\mid C=[\boldsymbol{\lambda}\ \boldsymbol{\varrho}^-] \text{ and } I \text{ is the ideal formed by } \boldsymbol{\varrho}^-\}\\ & = & \{C_I\mid I \text{ is an ideal of } C\}.\end{array}$$

The claimed poset structure is clear.

(c) This is clearly a valid ranking function, because if $C_{I'}$ covers C_I in the partial order then I' has one more vertex than I and hence $\operatorname{rank}(C_{I'})=1+\operatorname{rank}(C_I)$.

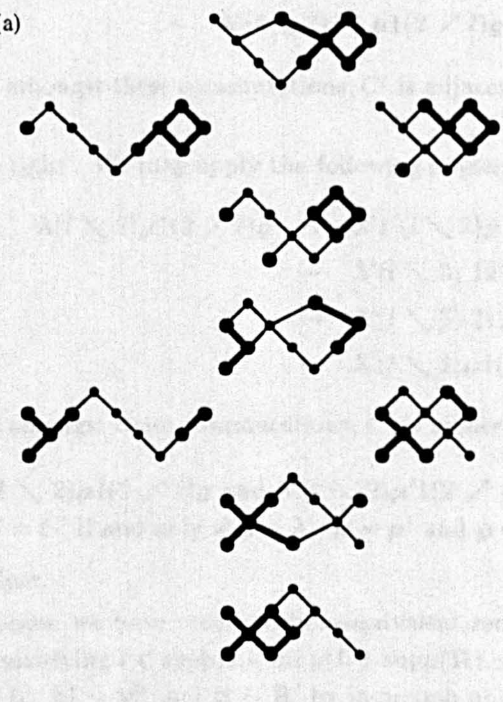
The highest rank is $|C|$, which equals $\binom{l}{2}$. Adjacent vertices related by a 'right' move have the form $[\boldsymbol{\lambda}(l\searrow 1)\boldsymbol{\varrho}]$, $[\boldsymbol{\lambda}(l\searrow 1)\boldsymbol{\varrho}^\#]$. These have ranks $\ell(\boldsymbol{\varrho})$, $\ell(\boldsymbol{\varrho}^\#)$, which are clearly equal. A similar argument applies for 'left' moves. ∎

So, in order to obtain all the commutation classes in $\mathcal{C}(A_l)$, we construct the δ-fibre of each $C \in \mathcal{C}(A_{l-1})$ in turn.

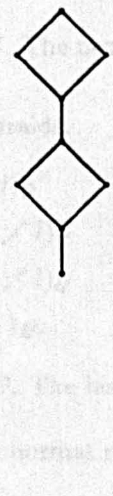
Let $\delta : \mathcal{C}(A_4) \to \mathcal{C}(A_3)$. The δ-fibre of C (see Figure 2.2.7 (a)) is shown in Figure 2.2.10 (a). The way C splits into two parts is highlighted in the heavily drawn parts of the partial order graphs. Figure 2.2.10 (b) shows the poset structure of this δ-fibre. Note that it is a ranked poset with top rank $6 = \binom{4}{2}$, agreeing with 2.2.9 (c).

2.2.10 FIGURE. The δ-fibre of C of Figure 2.2.7 (a).

(a)



(b)



2.3 Normal Representatives in Type B_l

Let $C \in \mathcal{C}(B_l)$. Any representative of C of the form $\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho$ *which satisfies the conditions* $l \notin \mathrm{supp}(\lambda), \mathrm{supp}(\varrho)$ *and* $1, 2 \notin \mathrm{supp}(\mu)$ is called a **normal representative** of C.

2.3.1 NOTES.

(a) In 2.3.7 we will prove that *any* word in $\mathcal{R}(B_l)$ of the form $\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho$ is in fact a normal representative. However, we find it convenient to start with the stronger definition.

(b) Normal representatives appear asymmetrical, but this is misleading since the conditions $1, 2 \notin \mathrm{supp}(\mu)$ imply that $\mu 1 \sim 1\mu$.

2.3.2 PROPOSITION. Let $C \in \mathcal{C}(B_l)$ have a normal representative $\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho$. Each of the following is a normal representative of some C' which is adjacent to C.

'left' $\lambda^{\#}(l \searrow 2)\mu 1(2 \nearrow l)\varrho$, where $\lambda^{\#}$ is obtained from λ by commutations and a 3- or 4-braid.

'middle' $\lambda(l \searrow 2)\mu^{\#} 1(2 \nearrow l)\varrho$, where $\mu^{\#}$ is obtained from μ by commutations and a 3- or 4-braid.

'right' $\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho^{\#}$, where $\varrho^{\#}$ is obtained from ϱ by commutations and a 3- or 4-braid.

'left to middle' $\lambda'(l \searrow 2)\, i^{+}\, \mu 1(2 \nearrow l)\varrho$, where $\lambda \sim \lambda' i$ and $i \neq 1$.

'middle to left' $\lambda\, i^{-}\, (l \searrow 2)\mu' 1(2 \nearrow l)\varrho$, where $\mu \sim i\mu'$.

'right to middle' $\lambda(l \searrow 2)\mu\, i^{+}\, 1(2 \nearrow l)\varrho'$, where $\varrho \sim i\varrho'$ and $i \neq 1$.

'middle to right' $\lambda(l \searrow 2)\mu' 1(2 \nearrow l)\, i^{-}\, \varrho$, where $\mu \sim \mu' i$.

'left to right' $\lambda'(l \searrow 2)\mu 1(2 \nearrow l)\, 1\, \varrho$, where $\lambda \sim \lambda' 1$ and $3 \notin \mathrm{supp}(\mu)$.

'right to left' $\lambda\, 1\, (l \searrow 2)\mu 1(2 \nearrow l)\varrho'$, where $\varrho \sim 1\varrho'$ and $3 \notin \mathrm{supp}(\mu)$.

In the last two cases, C and C' have representatives which are related to one another by a 4-braid.

PROOF. This is clear for the first three cases.

Consider case 'left to middle'. We may apply the following sequence of braids:

$$\begin{aligned}
\lambda(l\searrow 2)\mu 1(2\nearrow l)\varrho &\sim \lambda' i\,(l\searrow 2)\mu 1(2\nearrow l)\varrho\\
&\sim \lambda'(l\searrow i+2)\; i\; i+1\; i\; (i-1\searrow 2)\mu 1(2\nearrow l)\varrho\\
&\overset{3}{\leftrightarrow} \lambda'(l\searrow i+2)\; i+1\; i\; i+1\; (i-1\searrow 2)\mu 1(2\nearrow l)\varrho\\
&\sim \lambda'(l\searrow 2)\; i^{+}\; \mu 1(2\nearrow l)\varrho.
\end{aligned}$$

Since one 3-braid occurs amongst these commutations, C' is adjacent to C. The next three cases are proved similarly.

Consider case 'left to right'. We may apply the following sequence of braids:

$$\begin{aligned}
\lambda(l\searrow 2)\mu 1(2\nearrow l)\varrho &\sim \lambda' 1\,(l\searrow 2)\mu 1(2\nearrow l)\varrho\\
&\sim \lambda'(l\searrow 3)\; 1212\; \mu(3\nearrow l)\varrho\\
&\overset{4}{\leftrightarrow} \lambda'(l\searrow 3)\; 2121\; \mu(3\nearrow l)\varrho\\
&\sim \lambda'(l\searrow 2)\mu 1(2\nearrow l)\; 1\varrho.
\end{aligned}$$

Since one 4-braid occurs amongst these commutations, C' is adjacent to C. The last case is similar. ∎

2.3.3 LEMMA. Let $\boldsymbol{\lambda}(l\searrow 2)\boldsymbol{\mu}1(2\nearrow l)\boldsymbol{\varrho}$ and $\boldsymbol{\lambda}'(l\searrow 2)\boldsymbol{\mu}'1(2\nearrow l)\boldsymbol{\varrho}'$ be normal representatives of C and C', respectively. Then $C = C'$ if and only if $\boldsymbol{\lambda}\sim\boldsymbol{\lambda}'$, $\boldsymbol{\mu}\sim\boldsymbol{\mu}'$ and $\boldsymbol{\varrho}\sim\boldsymbol{\varrho}'$.

PROOF. Sufficiency is clear.

For the converse, suppose we have commutation-equivalent reduced words $\mathbf{L}(l\searrow 2)\mathbf{M}1(2\nearrow l)\mathbf{R}$ and $\mathbf{L}'(l\searrow 2)\mathbf{M}'1(2\nearrow l)\mathbf{R}'$ satisfying $l\notin \operatorname{supp}(\mathbf{L}), \operatorname{supp}(\mathbf{L}'), \operatorname{supp}(\mathbf{R}), \operatorname{supp}(\mathbf{R}')$ and $1,2\notin\operatorname{supp}(\mathbf{M}),\operatorname{supp}(\mathbf{M}')$. We shall prove that $\mathbf{L}\sim\mathbf{L}'$, $\mathbf{M}\sim\mathbf{M}'$ and $\mathbf{R}\sim\mathbf{R}'$ by induction upon $\ell(\mathbf{R})$.

If $\ell(\mathbf{R}) = 0$ then $\mathbf{L}(l\searrow 2)\mathbf{M}1(2\nearrow l)\sim\mathbf{L}'(l\searrow 2)\mathbf{M}'1(2\nearrow l)\mathbf{R}'$. So, since $l\notin\operatorname{supp}(\mathbf{R}')$, the letter l of $(2\nearrow l)$ in the right-hand word must be able to commute to the rightmost position. This implies $l-1\notin\operatorname{supp}(\mathbf{R}')$. By 1.3.4 we obtain $\mathbf{L}(l\searrow 2)\mathbf{M}1(2\nearrow l-1)\sim\mathbf{L}'(l\searrow 2)\mathbf{M}'1(2\nearrow l-1)\mathbf{R}'$. Similarly, considering the rightmost occurrences of $l-1, l-2,\ldots,2$ in turn, we also obtain $l-2, l-3,\ldots,1\notin\operatorname{supp}(\mathbf{R}')$. Therefore $\mathbf{R}'=\emptyset=\mathbf{R}$. It follows that $\mathbf{L}(l\searrow 2)\mathbf{M}\sim\mathbf{L}'(l\searrow 2)\mathbf{M}'$. Now, since $l\notin\operatorname{supp}(\mathbf{L}),\operatorname{supp}(\mathbf{L}')$ and $1,2\notin\operatorname{supp}(\mathbf{M}),\operatorname{supp}(\mathbf{M}')$, we have exactly the same situation as in the proof of 2.2.3, except with different notation. Therefore $\mathbf{L}\sim\mathbf{L}'$ and $\mathbf{M}\sim\mathbf{M}'$, as required.

Now suppose that $\ell(\mathbf{R})>0$. Write $\mathbf{R}=\mathbf{R}_1 i$, so that $\mathbf{L}(l\searrow 2)\mathbf{M}1(2\nearrow l)\mathbf{R}_1 i\sim\mathbf{L}'(l\searrow 2)\mathbf{M}'1(2\nearrow l)\mathbf{R}'$. So, i commutes to the rightmost position in the right-hand word. Since $i\neq l$, we therefore have $i\in\operatorname{supp}(\mathbf{R}')$, hence $\mathbf{R}'\sim\mathbf{R}_1' i$ for some $\mathbf{R}_1'$. By 1.3.4 we obtain $\mathbf{L}(l\searrow 2)\mathbf{M}1(2\nearrow l)\mathbf{R}_1\sim\mathbf{L}'(l\searrow 2)\mathbf{M}'1(2\nearrow l)\mathbf{R}_1'$. By induction, $\mathbf{L}\sim\mathbf{L}'$, $\mathbf{M}\sim\mathbf{M}'$ and $\mathbf{R}_1\sim\mathbf{R}_1'$, hence $\mathbf{R}\sim\mathbf{R}'$. ∎

2.3.4 LEMMA. Let $\boldsymbol{\lambda}(l\searrow 2)\boldsymbol{\mu}1(2\nearrow l)\boldsymbol{\varrho}$ be a normal representative. Consider the corresponding total order on $\Phi^+(B_l)$.

(a) The root corresponding to the letter 1 adjacent to $\boldsymbol{\mu}$ is $\alpha_1+\alpha_2+\cdots+\alpha_l$.

(b) The roots corresponding to the subwords $(l\searrow 2)$ and $(2\nearrow l)$ are precisely the remaining positive roots with summand α_l, namely

$$\alpha_l,\ \alpha_{l-1}+\alpha_l,\ldots,\alpha_2+\cdots+\alpha_l$$

and

$$2\alpha_1+\alpha_2+\cdots+\alpha_l,\ 2(\alpha_1+\alpha_2)+\alpha_3+\cdots+\alpha_l,\ldots,2(\alpha_1+\cdots+\alpha_{l-1})+\alpha_l.$$

PROOF. This is a simple calculation, using the restrictions on the supports of $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\varrho}$. ∎

We can now prove the converse to 2.3.2

2.3.5 Proposition. Let $C \in \mathcal{C}(B_l)$ have a normal representative $\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho$ and suppose that C' is adjacent to C. Then C' contains a normal representative of precisely one of the nine types listed in 2.3.2.

Proof. Any representative of C' can be obtained from $\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho$ by commutations, a 3- or 4-braid, and further commutations. We track the letters and their corresponding positive roots during the first sequence of commutations, noting whether they come from λ, $(l \searrow 2)$, μ, $1(2 \nearrow l)$ or ϱ.

First consider the case where a 3-braid is applied to iji, and let α, $\alpha+\beta$, β be the corresponding roots; see 1.3.6 (b). Either none or exactly two of these roots may have summand α_l, by inspection of $\Phi^+(B_l)$. So by 2.3.4, either none or exactly two of the letters of iji come from $(l \searrow 2)$ and $(2 \nearrow l)$, since $i, j \neq 1$.

If none of the letters of iji come from $(l \searrow 2)$ and $(2 \nearrow l)$ then they must all come from either λ, μ or ϱ, and hence C' has a representative of type 'left', 'middle' or 'right', respectively.

Suppose two letters of iji come from $(l \searrow 2)$ and $(2 \nearrow l)$. As $i, j \neq 1$, these two letters must come from *either* $(l \searrow 2)$ *or* $(2 \nearrow l)$, otherwise they would be unable to commute into adjacent positions.

Suppose both letters come from $(l \searrow 2)$. If it is the *first* two letters of iji then $j = i-1$ and the third letter, i, comes from μ. So $\mu \sim i\mu'$, for some μ', otherwise all three letters would be unable to commute into adjacent positions. Therefore C' has a representative of type 'middle to left', as $i \neq 1$. If the *last* two letters of iji come from $(l \searrow 2)$ then $j = i+1$ and the first letter, i, comes from λ. Therefore $\lambda \sim \lambda' i$ and C' has a representative of type 'left to middle', as $i \neq 1$.

Symmetrically, if all of iji comes from $(2 \nearrow l)$, we obtain cases 'middle to right' and 'right to middle'.

Now consider the case where a 4-braid is applied, to a subword 2121, say. Let α, $\alpha+\beta$, $\alpha+2\beta$, β be the corresponding roots. Note that none of the letters of 2121 may come from μ.

Clearly, α_l is not a summand of β, otherwise $\alpha+2\beta$ would not be a root.

If α does not have summand α_l either, then the letters of 2121 must all come from λ and ϱ, by 2.3.4. Clearly, 2121 cannot be a *mixture* of letters from these two subwords, so C' must have a representative of type 'left' or 'right'.

Now suppose α has summand α_l, so that $\alpha+\beta$ and $\alpha+2\beta$ also have summand α_l. By 2.3.4, precisely the first three letters of 2121 come from $(l \searrow 2)$ and $1(2 \nearrow l)$; by inspection, the first letter comes from $(l \searrow 2)$, the middle two from $1(2 \nearrow l)$, and the last letter, 1, comes from ϱ. So we have $\varrho \sim 1\varrho'$ and $3 \notin \mathrm{supp}(\mu)$, otherwise the letters of 2121 would be unable to commute into adjacent positions. Therefore C' has a representative of type 'right to left'.

Symmetrically, if a 4-braid is applied to 1212, we obtain case 'left to right'.

Finally, C' cannot have normal representatives of *more* then one of the nine types listed in 2.3.2, as can easily be checked, using 2.3.3. ∎

2.3.6 Proposition. Every commutation class in $\mathcal{C}(B_l)$ contains a normal representative.

Proof. As in the proof of 2.2.6, it suffices to exhibit just one normal representative. If $w_0^{(l)}$ is the longest element of $\mathcal{W}(B_l)$ then we have

$$w_0^{(l)} = w_0^{(l-1)}(s_l s_{l-1} \ldots s_2)s_1(s_2 \ldots s_{l-1}s_l) \text{ and } \ell(w_0^{(l)}) = \ell(w_0^{(l-1)}) + 2l - 1$$

for $l > 2$. As $1212 \in \mathcal{R}(B_2)$, the word

$$1212\,(3 \searrow 2)1(2 \nearrow 3) \ldots (l \searrow 2)1(2 \nearrow l)$$

belongs to $\mathcal{R}(B_l)$ and is visibly a normal representative. ∎

We can now show that our definition of normal representative is slightly stronger than necessary.

2.3.7 Lemma. Any word in $\mathcal{R}(B_l)$ of the form $\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho$ is a normal representative.

Proof. By 2.3.6 there is some normal representative $\lambda'(l \searrow 2)\mu' 1(2 \nearrow l)\varrho'$ such that

$$\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho \sim \lambda'(l \searrow 2)\mu' 1(2 \nearrow l)\varrho'.$$

For a contradiction, suppose there is some letter i in μ with $i \in \{1,2\}$. Consider the following (ordered) list of letters appearing in the left-hand word above:

$$\text{the } l \text{ in } (l \searrow 2), \text{ the } 2 \text{ in } (l \searrow 2), \text{ the } i \text{ in } \mu, \text{ the } 1 \text{ in } 1(2 \nearrow l), \text{ the } 2 \text{ in } (2 \nearrow l), \text{ the } l \text{ in } (2 \nearrow l).$$

As we apply commutations, we track these letters. It is clear that no matter which commutations are applied, each entry in the list must lie to the left to the following entry.

Our letter i in μ must correspond to some letter in the normal representative. First, note that i cannot be any letter in μ', as $1, 2 \notin \mathrm{supp}(\mu')$.

Suppose i corresponds to some letter in $\lambda'(l \searrow 2)$ of the normal representative. Then the second entry in the list must be a letter in λ' (because it lies to the left of i). Hence, the first entry, a letter l, must also lie in λ', which is absurd, as $l \notin \mathrm{supp}(\lambda')$.

Similarly, i cannot correspond to any letter in $(2 \nearrow l)\varrho'$.

So, i must be the letter 1 which is adjacent to μ'. Therefore the fourth entry, also a letter 1, must lie in ϱ'. But this implies that the last entry, an l, lies in ϱ', which is absurd.

We have shown that $1, 2 \notin \mathrm{supp}(\mu)$. We can therefore apply $\ell(\mu)$ 'middle to left' moves to the word $\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho$, (which can certainly be performed without requiring $l \notin \mathrm{supp}(\lambda), \mathrm{supp}(\varrho)$), giving $\lambda\, \mu^-\, (l \searrow 2)1(2 \nearrow l)\varrho$. Now apply the promotion operator (see 2.1) $\ell(\varrho)$ times, noting that the opposition involution is trivial, giving

$$\varrho\, \lambda\, \mu^-\, (l \searrow 2)1(2 \nearrow l) \in \mathcal{R}(B_l).$$

By the calculation in the proof of 2.3.6, this implies that

$$\varrho\, \lambda\, \mu^- \in \mathcal{R}(B_{l-1}),$$

so we certainly have $l \notin \mathrm{supp}(\lambda), \mathrm{supp}(\varrho)$. ∎

Parts of the following technical Lemma will be required here and in subsequent chapters. Refer to 2.1 for the notation.

2.3.8 LEMMA.

(a) All reduced words for the same element of $\mathcal{W}(B_l)$ have the same multiplicity of the letter 1.

(b) For all $\mathbf{i} \in \mathcal{R}(B_l)$ we have $\mathrm{mul}_1(\mathbf{i}) = l$; the l positive roots corresponding to the l occurrences of the letter 1 are

$$\alpha_1,\ \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_l,$$

in some order.

PROOF.

(a) The only braids involving the letter 1 are commutations and 4-braids, both of which leave the multiplicity of 1 unchanged.

(b) The multiplicity of 1 in the word exhibited in the proof of 2.2.6 equal l, so by part (a), the same is true of any $\mathbf{i} \in \mathcal{R}(B_l)$. Now, since α_1 is a short root, it follows that all roots corresponding to the letter 1 are also short; the l listed roots are the only short roots in $\Phi^+(B_l)$. ∎

We now describe an algorithm for constructing all of the commutation classes in $\mathcal{C}(B_l)$ from those of $\mathcal{C}(B_{l-1})$.

Let $C \in \mathcal{C}(B_{l-1})$ and consider its partial order graph. Let I_1 be an ideal. Let I_2 be disjoint from I_1, such that $I_1 \dot{\cup} I_2$ is also an ideal and I_2 contains no vertex corresponding to a letter 1. Thus, the partial order graph of C is partitioned into three parts.

Let $\mathbf{i}_1$, $\mathbf{i}_2$ be linear extensions of I_1, I_2, respectively, so that

$$C = [\mathbf{i}_3\, \mathbf{i}_2\, \mathbf{i}_1],$$

for some $\mathbf{i}_3$. We have $1 \notin \mathrm{supp}(\mathbf{i}_2)$.

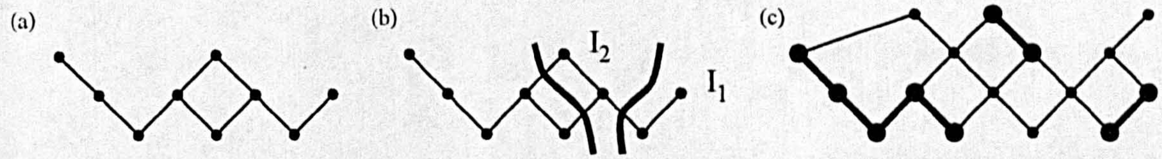
Since $\mathbf{i}_3\ \mathbf{i}_2\ \mathbf{i}_1\ (l \searrow 2)1(2 \nearrow l) \in \mathcal{R}(B_l)$, we may apply 'left to middle' followed by 'middle to right' moves for each letter of $\mathbf{i}_1$ different from 1, and a 'left to right' move for each letter 1 of $\mathbf{i}_1$, giving rise to the word $\mathbf{i}_3\ \mathbf{i}_2\ (l \searrow 2)1(2 \nearrow l)\ \mathbf{i}_1$. Since $1 \notin \mathrm{supp}(\mathbf{i}_2)$, we can apply 'left to middle' moves, giving $\mathbf{i}_3\ (l \searrow 2)\ \mathbf{i}_2^+\ 1(2 \nearrow l)\ \mathbf{i}_1$. We may therefore define

$$C_{I_2,I_1} := [\mathbf{i}_3\ (l \searrow 2)\ \mathbf{i}_2^+\ 1(2 \nearrow l)\ \mathbf{i}_1] \in \mathcal{C}(B_l).$$

Since all linear extensions of I_1 and I_2 are commutation-equivalent to one another, C_{I_2,I_1} is well defined.

For example, if $C \in \mathcal{C}(B_3)$ is shown in Figure 2.3.9 (a), and if I_2 and I_1 are as indicated in 2.3.9 (b), then C_{I_2,I_1} is shown in 2.3.9 (c). Note that, if drawn in, the horizontal lines would be numbered from bottom to top, as usual.

2.3.9 FIGURE. Constructing C_{I_2,I_1}.



In terms of words, $C = [321213212]$ and $C_{I_2,I_1} = [32121\,432\,43\,1234\,12]$.

2.3.10 PROPOSITION. The mapping

$$\begin{aligned} \{(C, I_2, I_1) \mid C \in \mathcal{C}(B_{l-1});\ I_1, I_1\dot{\cup} I_2 \text{ are ideals of } C \text{ and } 1 \notin I_2\} &\rightarrow \mathcal{C}(B_l) \\ (C, I_2, I_1) &\mapsto C_{I_2,I_1} \end{aligned}$$

is bijective.

PROOF. For surjectivity, take any $[\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}] \in \mathcal{C}(B_l)$, set $C := [\boldsymbol{\lambda}\ \boldsymbol{\mu}^-\ \boldsymbol{\varrho}] \in \mathcal{C}(B_{l-1})$ and let I_2, I_1 be the subgraphs of the partial order graph of C formed by $\boldsymbol{\mu}^-$ and $\boldsymbol{\varrho}$, respectively. Note that $1 \notin I_2$ because $2 \notin \mathrm{supp}(\boldsymbol{\mu})$. Clearly (C, I_2, I_1) maps to $[\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}]$.

For injectivity, suppose, with the obvious notation, that $C_{I_2,I_1} = C'_{I'_2,I'_1}$. Then we have

$$[\mathbf{i}_3\ (l \searrow 2)\ \mathbf{i}_2^+\ 1(2 \nearrow l)\ \mathbf{i}_1] = [\mathbf{i}'_3\ (l \searrow 2)\ \mathbf{i}'^+_2\ 1(2 \nearrow l)\ \mathbf{i}'_1].$$

By 2.3.3 we obtain $\mathbf{i}_3 \sim \mathbf{i}'_3$, $\mathbf{i}_2 \sim \mathbf{i}'_2$ and $\mathbf{i}_1 \sim \mathbf{i}'_1$. So $C = C'$, $I_2 = I'_2$ and $I_1 = I'_1$. ∎

We now introduce a map analogous to the one in 2.2.9

2.3.11 PROPOSITION. Define

$$\begin{aligned} \delta : \mathcal{C}(B_l) &\rightarrow \mathcal{C}(B_{l-1}) \\ [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}] &\mapsto [\boldsymbol{\lambda}\ \boldsymbol{\mu}^-\ \boldsymbol{\varrho}]. \end{aligned}$$

(a) The map δ is a surjective graph morphism. Each fibre is connected, and any two vertices in the same fibre are related to one another by 'left to middle', 'middle to right', 'left to right' and their opposite moves.

(b) The δ-fibre of $C \in \mathcal{C}(B_{l-1})$ is

$$\{C_{I_2,I_1} \mid I_1, I_2\dot{\cup} I_1 \text{ are ideals of } C \text{ and } 1 \notin I_2\}.$$

This is a poset with unique top and bottom by setting

$$C_{I_2,I_1} \leq C_{I'_2,I'_1} \text{ if } I_1 \subseteq I'_1 \text{ and } I_2\dot{\cup} I_1 \subseteq I'_2\dot{\cup} I'_1.$$

(c) Each δ-fibre is a ranked poset, with rank function

$$\mathrm{rank}\,(C_{I_2,I_1}) := 2|I_1| + |I_2| - \mathrm{mul}_1(I_1)$$

and highest rank $(l-1)(2l-3)$. (Here, $\mathrm{mul}_1(I_1)$ stands for the number of vertices labelled 1 in I_1.) Adjacent vertices of $\mathcal{C}(B_l)$ related to one another by a 'left', 'middle' or 'right' move have the same rank (in their respective fibres).

PROOF. Note that δ is well defined, by 2.3.3.

(a) If $C = [\mathbf{i}] \in \mathcal{C}(B_{l-1})$ then $[\mathbf{i}\,(l \searrow 2)1(2 \nearrow l)]$ maps to C, so δ is onto. Using 2.3.2, it is easy to check that δ is a morphism of graphs. Now consider a general vertex $[\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}] \in \delta^{-1}(C)$. By definition, $\boldsymbol{\lambda}\,\boldsymbol{\mu}^-\,\boldsymbol{\varrho} \sim \mathbf{i}$, so by applying an appropriate sequence of 'left to middle', 'middle to right' and 'left to right' moves (similar to the sequence of moves used to obtain the definition of C_{I_2,I_1} from C, above), we obtain a path in $\delta^{-1}(C)$ joining $[\mathbf{i}\,(l \searrow 2)1(2 \nearrow l)]$ to $[\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}]$. So $\delta^{-1}(C)$ is connected.

(b) We have, in the notation of part (a),

$$\begin{aligned}
\delta^{-1}(C) &= \{[\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}] \mid C = [\boldsymbol{\lambda}\,\boldsymbol{\mu}^-\,\boldsymbol{\varrho}]\} \\
&= \{C_{I_2,I_1} \mid C = [\boldsymbol{\lambda}\,\boldsymbol{\mu}^-\,\boldsymbol{\varrho}] \text{ and } I_2, I_1 \text{ are formed by } \boldsymbol{\mu}^-, \boldsymbol{\varrho}\} \\
&= \{C_{I_2,I_1} \mid I_1, I_2\dot{\cup}I_1 \text{ are ideals of } C \text{ and } 1 \notin I_2\}.
\end{aligned}$$

To verify the poset structure of this fibre, it is easiest to identify C_{I_2,I_1} with the ordered pair (I_2, I_1) and proceed formally. The details are trivial. The bottom element is $C_{\emptyset,\emptyset}$ and the top element is $C_{\emptyset,C}$.

(c) Write $C_{I_2,I_1} = [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}]$, so that $|I_1| = \ell(\boldsymbol{\varrho})$, $|I_2| = \ell(\boldsymbol{\mu})$ and

$$\mathrm{rank}\,(C_{I_2,I_1}) = 2\ell(\boldsymbol{\varrho}) + \ell(\boldsymbol{\mu}) - \mathrm{mul}_1(\boldsymbol{\varrho}).$$

Now consider a vertex $C_{I_2',I_1'} = [\boldsymbol{\lambda}'(l \searrow 2)\boldsymbol{\mu}'1(2 \nearrow l)\boldsymbol{\varrho}']$ in the δ-fibre, which is one step higher in the partial order. If $C_{I_2',I_1'}$ is obtained from C_{I_2,I_1} by a 'left to middle' move then $\ell(\boldsymbol{\mu}') = \ell(\boldsymbol{\mu})+1$ and $\boldsymbol{\varrho}' \sim \boldsymbol{\varrho}$. Therefore $\mathrm{rank}\,(C_{I_2',I_1'}) = \mathrm{rank}\,(C_{I_2,I_1}) + 1$.

If $C_{I_2',I_1'}$ is obtained from C_{I_2,I_1} by a 'middle to right' move then $\ell(\boldsymbol{\mu}') = \ell(\boldsymbol{\mu}) - 1$, $\mathrm{mul}_1(\boldsymbol{\varrho}') = \mathrm{mul}_1(\boldsymbol{\varrho})$ (because $2 \notin \mathrm{supp}(\boldsymbol{\mu})$) and $\ell(\boldsymbol{\varrho}') = \ell(\boldsymbol{\varrho}) + 1$. Therefore $\mathrm{rank}\,(C_{I_2',I_1'}) = \mathrm{rank}\,(C_{I_2,I_1}) + 1$.

If now $C_{I_2',I_1'}$ is obtained from C_{I_2,I_1} by a 'left to right' move then $\ell(\boldsymbol{\mu}') = \ell(\boldsymbol{\mu})$, $\ell(\boldsymbol{\varrho}') = \ell(\boldsymbol{\varrho}) + 1$ and $\mathrm{mul}_1(\boldsymbol{\varrho}') = \mathrm{mul}_1(\boldsymbol{\varrho})+1$. Thus, $\mathrm{rank}\,(C_{I_2',I_1'}) = \mathrm{rank}\,(C_{I_2,I_1})+1$. Thus, we have a well defined rank function.

The top rank is $\mathrm{rank}\,(C_{\emptyset,C}) = 2|C|+0-\mathrm{mul}_1(C)$. Since C has $(l-1)^2$ vertices and $\mathrm{mul}_1(C) = l-1$ by 2.3.8 (b), we obtain $(l-1)(2l-3)$. The last assertion is clear – simply notice that braids applied to $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\varrho}$ affect neither the lengths of these words nor $\mathrm{mul}_1(\boldsymbol{\varrho})$. ∎

Let $\delta : \mathcal{C}(B_4) \to \mathcal{C}(B_3)$. The δ-fibre of the commutation class C in Figure 2.3.9 (a) is shown in Figure 2.3.12 (a). The poset structure of this fibre is shown in 2.3.12 (b); note that it is ranked with top rank $15 = (4-1)(2.4-3)$, agreeing with 2.3.11 (c).

2.4 Normal Representatives in Type D_l

Let $C \in \mathcal{C}(D_l)$. Any representative of C of the form $\boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}$, where $\{a, b\} = \{1, 2\}$ and also $l \notin \mathrm{supp}(\boldsymbol{\lambda}), \mathrm{supp}(\boldsymbol{\varrho})$ and $1, 2 \notin \mathrm{supp}(\boldsymbol{\mu})$ is called a **normal representative** of C, of **type 12** or **type 21** according as $(a, b) = (1, 2)$ or $(a, b) = (2, 1)$, respectively.

2.4.1 NOTES.

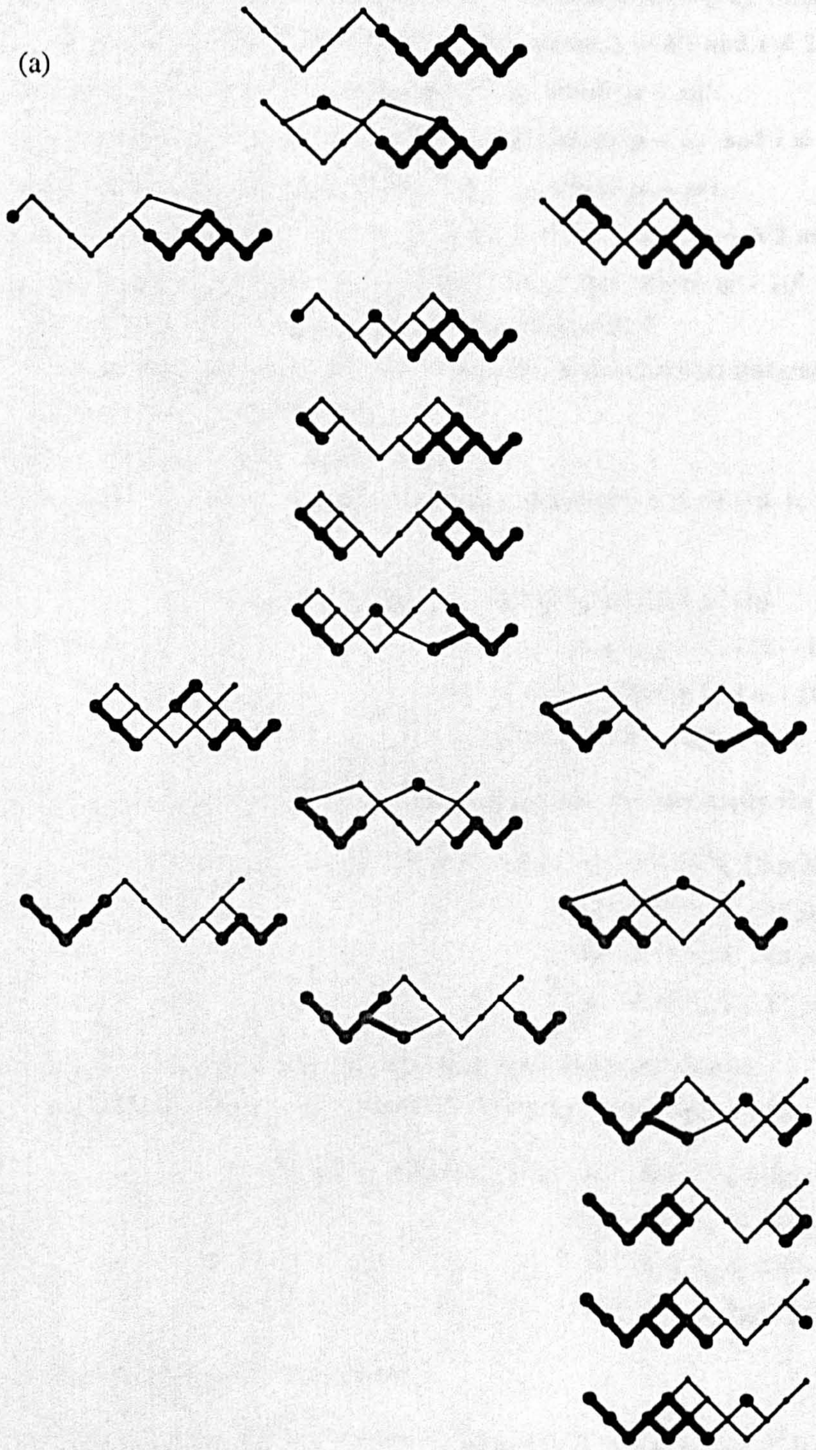
(a) In 2.4.7 we will show that *any* word in $\mathcal{R}(D_l)$ of the form $\boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}$, where $\{a, b\} = \{1, 2\}$, is a normal representative.

(b) Reversing a normal representative of type 12 is a normal representative of type 21 (of a different commutation class, in general).

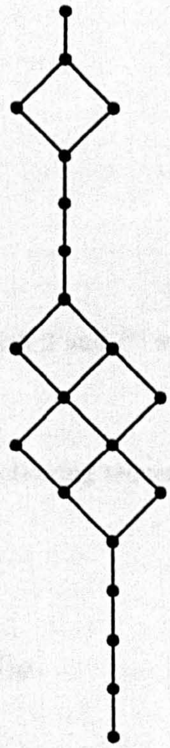
(c) A commutation class may contain normal representatives of both type 12 and type 21.

2.3.12 FIGURE. The δ-fibre of C of Figure 2.3.9 (a).

(a)



(b)



2.4.2 PROPOSITION. Let $C \in \mathcal{C}(D_l)$ have a normal representative $\boldsymbol{\lambda}(l \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho}$. Each of the following is a normal representative of some C' which is adjacent to C.

'left' $\boldsymbol{\lambda}^{\#}(l \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho}$, where $\boldsymbol{\lambda}^{\#}$ is obtained from $\boldsymbol{\lambda}$ by commutations and a 3-braid.

'middle' $\boldsymbol{\lambda}(l \searrow 3)1\boldsymbol{\mu}^{\#}2(3 \nearrow l)\boldsymbol{\varrho}$, where $\boldsymbol{\mu}^{\#}$ is obtained from $\boldsymbol{\mu}$ by commutations and a 3-braid.

'right' $\boldsymbol{\lambda}(l \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho}^{\#}$, where $\boldsymbol{\varrho}^{\#}$ is obtained from $\boldsymbol{\varrho}$ by commutations and a 3-braid.

'left to middle' $\boldsymbol{\lambda}'(l \searrow 3)1\, i^{+}\, \boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho}$, where $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}'i$ and $i \neq 2$.

'middle to left' $\boldsymbol{\lambda}\, i^{--}\, (l \searrow 3)1\boldsymbol{\mu}'2(3 \nearrow l)\boldsymbol{\varrho}$, where $\boldsymbol{\mu} \sim i\boldsymbol{\mu}'$.

'right to middle' $\boldsymbol{\lambda}(l \searrow 3)1\boldsymbol{\mu}\, i^{+}\, 2(3 \nearrow l)\boldsymbol{\varrho}'$, where $\boldsymbol{\varrho} \sim i\boldsymbol{\varrho}'$ and $i \neq 1$.

'middle to right' $\boldsymbol{\lambda}(l \searrow 3)1\boldsymbol{\mu}'2(3 \nearrow l)\, i^{-}\, \boldsymbol{\varrho}$, where $\boldsymbol{\mu} \sim \boldsymbol{\mu}'i$.

'special left to middle' $\boldsymbol{\lambda}'(l \searrow 3)2\,3\,\boldsymbol{\mu}1(3 \nearrow l)\boldsymbol{\varrho}$, where $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}'2$ and $3 \notin \mathrm{supp}(\boldsymbol{\mu})$.

'special right to middle' $\boldsymbol{\lambda}(l \searrow 3)2\boldsymbol{\mu}\,3\,1(3 \nearrow l)\boldsymbol{\varrho}'$, where $\boldsymbol{\varrho} \sim 1\boldsymbol{\varrho}'$ and $3 \notin \mathrm{supp}(\boldsymbol{\mu})$.

(The last two are normal representatives of type 21.)

If C has a normal representative of type 21, a symmetrical statement holds; swap 1 with 2 and i^{-} with i^{--} in the above representatives.

PROOF. This is clear for the first three cases.

Consider case 'left to middle', where $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}'i$ and $i \neq 2$. If $i \neq 1$, we may apply the following sequence of braids:

$$\begin{aligned}
\boldsymbol{\lambda}(l \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho} &\sim \boldsymbol{\lambda}'i\,(l \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho} \\
&\sim \boldsymbol{\lambda}'(l \searrow i+2)\; i\; i+1\; i\; (i-1 \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho} \\
&\xrightarrow{3} \boldsymbol{\lambda}'(l \searrow i+2)\; i+1\; i\; i+1\; (i-1 \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho} \\
&\sim \boldsymbol{\lambda}'(l \searrow 3)1\; i^{+}\; \boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho},
\end{aligned}$$

so C' is adjacent to C. If $i = 1$, the remaining case, we may apply these braids:

$$\begin{aligned}
\boldsymbol{\lambda}(l \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho} &\sim \boldsymbol{\lambda}'1\,(l \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho} \\
&\sim \boldsymbol{\lambda}'(l \searrow 4)\;131\;\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho} \\
&\xrightarrow{3} \boldsymbol{\lambda}'(l \searrow 4)\;313\;\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho} \\
&\sim \boldsymbol{\lambda}'(l \searrow 3)1\;1^{+}\;\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho},
\end{aligned}$$

as required, noting that $1^{+} = 3$. The next three cases are similar.

Consider case 'special left to middle'. We may apply these braids:

$$\begin{aligned}
\boldsymbol{\lambda}(l \searrow 3)1\boldsymbol{\mu}2(3 \nearrow l)\boldsymbol{\varrho} &\sim \boldsymbol{\lambda}'2\,(l \searrow 3)2\boldsymbol{\mu}1(3 \nearrow l)\boldsymbol{\varrho} \\
&\sim \boldsymbol{\lambda}'(l \searrow 4)\;232\;\boldsymbol{\mu}1(3 \nearrow l)\boldsymbol{\varrho} \\
&\xrightarrow{3} \boldsymbol{\lambda}'(l \searrow 4)\;323\;\boldsymbol{\mu}1(3 \nearrow l)\boldsymbol{\varrho} \\
&\sim \boldsymbol{\lambda}'(l \searrow 3)2\;3\;\boldsymbol{\mu}1(3 \nearrow l)\boldsymbol{\varrho},
\end{aligned}$$

as required. The last case is similar. ∎

2.4.3 LEMMA. Let $\boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}$ and $\boldsymbol{\lambda}'(l \searrow 3)a'\boldsymbol{\mu}'b'(3 \nearrow l)\boldsymbol{\varrho}'$ be normal representatives of C and C', respectively. Then $C = C'$ if and only if $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}'$, $\boldsymbol{\mu} \sim \boldsymbol{\mu}'$, $\boldsymbol{\varrho} \sim \boldsymbol{\varrho}'$, and *either*

$$(a, b) = (a', b') \text{ or } (a, b) = (b', a') \text{ and } 3 \notin \mathrm{supp}(\boldsymbol{\mu}) = \mathrm{supp}(\boldsymbol{\mu}').$$

PROOF. Sufficiency is clear, noting that if $\boldsymbol{\mu} \sim \boldsymbol{\mu}'$ and $3 \notin \mathrm{supp}(\boldsymbol{\mu})$ then $a\boldsymbol{\mu}b \sim b\boldsymbol{\mu}'a$.

Conversely, suppose that $\mathbf{L}(l \searrow 3)a\mathbf{M}b(3 \nearrow l)\mathbf{R}$ and $\mathbf{L}'(l \searrow 3)a'\mathbf{M}'b'(3 \nearrow l)\mathbf{R}'$ are commutation-equivalent, satisfying $l \notin \mathrm{supp}(\mathbf{L}), \mathrm{supp}(\mathbf{L}'), \mathrm{supp}(\mathbf{R}), \mathrm{supp}(\mathbf{R}')$ and $1, 2 \notin \mathrm{supp}(\mathbf{M}), \mathrm{supp}(\mathbf{M}')$. We shall prove that $\mathbf{L} \sim \mathbf{L}'$, $\mathbf{M} \sim \mathbf{M}'$, $\mathbf{R} \sim \mathbf{R}'$ and *either*

$$(a, b) = (a', b') \text{ or } (a, b) = (b', a') \text{ and } 3 \notin \operatorname{supp}(\mathbf{M}) = \operatorname{supp}(\mathbf{M}')$$

by induction upon $\ell(\mathbf{R})$.

If $\ell(\mathbf{R}) = 0$ then $\mathbf{L}(l \searrow 3)a\mathbf{M}b(3 \nearrow l) \sim \mathbf{L}'(l \searrow 3)a'\mathbf{M}'b'(3 \nearrow l)\mathbf{R}'$. Since the rightmost l commutes to the end, and as $l \notin \operatorname{supp}(\mathbf{R}')$, we have $l-1 \notin \operatorname{supp}(\mathbf{R}')$. By 1.3.4 we obtain $\mathbf{L}(l \searrow 3)a\mathbf{M}b(3 \nearrow l-1) \sim \mathbf{L}'(l \searrow 3)a'\mathbf{M}'b'(3 \nearrow l-1)\mathbf{R}'$. Similarly, considering the rightmost $l-1, l-2, \ldots, 3$ in turn, we also obtain $l-2, \ldots, 2, 1 \notin \operatorname{supp}(\mathbf{R}')$. Therefore $\mathbf{R}' = \emptyset = \mathbf{R}$, giving

$$\mathbf{L}(l \searrow 3)a\mathbf{M}b \sim \mathbf{L}'(l \searrow 3)a'\mathbf{M}'b'. \qquad (*)$$

If $(a, b) = (a', b')$ then $\mathbf{L}(l \searrow 3)a\mathbf{M} \sim \mathbf{L}'(l \searrow 3)a\mathbf{M}'$; induction upon $\ell(\mathbf{M})$, using an argument similar to the one in the proof of 2.2.3 now yields $\mathbf{L} \sim \mathbf{L}'$, $\mathbf{M} \sim \mathbf{M}'$, as required.

If instead $(a, b) = (b', a')$ then inspection of $(*)$ shows that the rightmost a must commute with $\mathbf{M}$ (as $a \notin \operatorname{supp}(\mathbf{M})$), hence $3 \notin \operatorname{supp}(\mathbf{M})$. Similarly, $3 \notin \operatorname{supp}(\mathbf{M}')$. By 1.3.4 we therefore obtain $\mathbf{L}(l \searrow 3)\mathbf{M} \sim \mathbf{L}'(l \searrow 3)\mathbf{M}'$. Once again, induction upon $\ell(\mathbf{M})$ yields $\mathbf{L} \sim \mathbf{L}'$, $\mathbf{M} \sim \mathbf{M}'$. This completes the case $\ell(\mathbf{R}) = 0$.

When $\ell(\mathbf{R}) > 0$, we proceed just as in the proofs of 2.2.3 and 2.3.3. ∎

2.4.4 LEMMA. Let $\lambda(l \searrow 3)a\mu b(3 \nearrow l)\varrho$ be a normal representative. Consider the corresponding total order on $\Phi^+(D_l)$. The positive roots corresponding to the subwords $(l \searrow 3)a$ and $b(3 \nearrow l)$ are precisely those with summand α_l.

PROOF. This is a simple calculation, using the restrictions on the supports of λ, μ and ϱ. ∎

We can now prove the converse to 2.4.2.

2.4.5 PROPOSITION. Let $C \in \mathcal{C}(D_l)$ have a normal representative $\lambda(l \searrow 3)1\mu 2(3 \nearrow l)\varrho$, and suppose that C' is adjacent to C. Then C' has a normal representative of precisely one of the nine types listed in 2.4.2.

If C has a normal representative of type 21, a symmetrical statement holds; swap 1 with 2 and i^- with i^{--} in each of the listed representatives in 2.4.2.

PROOF. Any representative of C' can be obtained from $\lambda(l \searrow 3)1\mu 2(3 \nearrow l)\varrho$ by commutations, a 3-braid and further commutations. We track the letters and their corresponding positive roots during the first sequence of commutations, noting whether they come from λ, $(l \searrow 3)1$, μ, $2(3 \nearrow l)$ or ϱ.

Let iji be the word to which the 3-braid is applied, and let α, $\alpha + \beta$, β be the corresponding roots. Either none or exactly two of these roots may have summand α_l, so by 2.4.4, either none or exactly two of the letters of iji come from $(l \searrow 3)1$ and $2(3 \nearrow l)$.

If none of the letters of iji come from $(l \searrow 3)1$ and $2(3 \nearrow l)$ then they must all come from either λ, μ or ϱ, and hence C' has a representative of type 'left', 'middle' or 'right', respectively.

Now suppose exactly two letters of iji come from $(l \searrow 3)1$ and $2(3 \nearrow l)$; since $\alpha + \beta$ must have summand α_l, these must be the *first* or *last* two — say the first two, ij, for definiteness. There are now two cases to consider — $\operatorname{supp}(\mu)$ either does or does not contain 3.

If $3 \in \operatorname{supp}(\mu)$ then ij must come from *either* $(l \searrow 3)1$ *or* $2(3 \nearrow l)$, otherwise they could not commute into adjacent positions. For the same reason, the third letter, i, comes from μ or ϱ, and we have commutation-equivalences $\mu \sim i\mu'$ or $\varrho \sim i\varrho'$. Thus, C' has a representative of type 'middle to left' or 'right to middle', respectively.

If instead $3 \notin \operatorname{supp}(\mu)$ then we have the further possibilities that either i (the first letter of iji) is the 3 in $(l \searrow 3)1$ and j is the 2 in $2(3 \nearrow l)$, or, symmetrically, i is the 1 in $(l \searrow 3)1$ and j is the 3 in $2(3 \nearrow l)$. (Note that in this situation, i and j *can* commute into adjacent positions.) However, if $i = 3$, $j = 2$ then the third letter, i, must be the 3 of $2(3 \nearrow l)$, which contradicts the fact that not all of iji comes from $(l \searrow 3)1$ and $2(3 \nearrow l)$. The second case, where $i = 1$, $j = 3$, *is* possible, because the third letter, i, must come from ϱ, and we have a commutation-equivalence $\varrho \sim i\varrho'$, otherwise the three letters iji would be unable to commute into adjacent positions. Therefore C' has a representative of type 'special right to middle'.

If it is the *last* two letters of iji which come form $(l \searrow 3)1$ and $2(3 \nearrow l)$ then we obtain the possibilities 'left to middle', 'middle to right' and 'special left to middle' for C'.

Finally, using 2.4.3, it is easy to check that C' cannot have representatives of *more* than one of the types listed in 2.4.2. ∎

2.4.6 PROPOSITION. Every commutation class in $\mathcal{C}(D_l)$ contains a normal representative.

PROOF. As in the proof of 2.2.6, it suffices to exhibit just one normal representative. If $w_0^{(l)}$ is the longest element of $\mathcal{W}(D_l)$ then we have

$$w_0^{(l)} = w_0^{(l-1)}(s_l s_{l-1} \dots s_3)s_1 s_2(s_3 \dots s_{l-1} s_l) \text{ and } \ell(w_0^{(l)}) = \ell(w_0^{(l-1)}) + 2(l-1)$$

for $l > 2$. As $12 \in \mathcal{R}(D_2)$, the word

$$123123 \dots (l \searrow 3)12(3 \nearrow l)$$

belongs to $\mathcal{R}(D_l)$ and is visibly a normal representative. ∎

We can now show that our definition of normal representative is slightly stronger than necessary.

2.4.7 LEMMA. Any word in $\mathcal{R}(D_l)$ of the form $\lambda(l \searrow 3)a\mu b(3 \nearrow l)\varrho$, where $\{a,b\} = \{1,2\}$ is a normal representative.

PROOF. Here, we will take $(a,b) = (1,2)$; the other possibility is symmetrical. By 2.4.6 there is some normal representative $\lambda'(l \searrow 3)a'\mu'b'(3 \nearrow l)\varrho'$ such that

$$\lambda(l \searrow 3)1\mu 2(3 \nearrow l)\varrho \sim \lambda'(l \searrow 3)a'\mu'b'(3 \nearrow l)\varrho'.$$

For a contradiction, suppose there is some letter i in μ with $i \in \{1,2\}$. If $l = 3$ then we easily obtain a contradiction, strongly constrained by the fact that $\ell(w_0) = 6$ in type D_3. So now let $l > 3$ and consider the following (ordered) list of letters appearing in the left-hand word above:

the l in $(l \searrow 3)$, the 3 in $(l \searrow 3)$, the i in μ.

As we apply commutations, we track these letters. It is clear that no matter which commutations are applied, each entry in the list must lie to the left to the following entry.

Our letter i in μ must correspond to some letter in the normal representative. Note that i cannot be any letter in μ', as $1, 2 \notin \mathrm{supp}(\mu')$.

Suppose i corresponds to some letter in $\lambda'(l \searrow 3)a'$ of the normal representative. Then the second entry in the list must be a letter in λ' (because it lies to the left of i). Hence, the first entry, a letter l, lies in λ' also, which contradicts $l \notin \mathrm{supp}(\lambda')$. Similarly, i cannot correspond to any letter in $b'(3 \nearrow l)\varrho'$.

We have so far shown that $1, 2 \notin \mathrm{supp}(\mu)$. We can therefore apply 'middle to left' moves to the word $\lambda(l \searrow 3)1\mu 2(3 \nearrow l)\varrho$, (which can be performed without requiring $l \notin \mathrm{supp}(\lambda), \mathrm{supp}(\varrho)$), giving rise to $\lambda\, \mu^{--}\, (l \searrow 3)12(3 \nearrow l)\varrho$. Now apply the promotion operator to obtain

$$\overline{\varrho}\, \lambda\, \mu^{--}\, (l \searrow 3)12(3 \nearrow l) \in \mathcal{R}(D_l).$$

By the calculation in the proof of 2.4.6, this implies that

$$\overline{\varrho}\, \lambda\, \mu^{--} \in \mathcal{R}(D_{l-1}),$$

so we have $l \notin \mathrm{supp}(\lambda)$ and $l \notin \mathrm{supp}(\overline{\varrho})$, hence $l \notin \mathrm{supp}(\varrho)$. ∎

We now describe an algorithm for constructing all the commutation classes in $\mathcal{C}(D_l)$ from those of $\mathcal{C}(D_{l-1})$.

Let $C \in \mathcal{C}(D_{l-1})$ and consider its partial order graph. Let I_1 be an ideal. Let I_2 be disjoint from I_1 such that $I_1 \dot\cup I_2$ is also an ideal and I_2 does not contain vertices corresponding to *both* a letter 1 *and* a letter 2.

Let $\mathbf{i}_1$, $\mathbf{i}_2$ be linear extensions of I_1, I_2, respectively, so that

$$C = [\mathbf{i}_3\, \mathbf{i}_2\, \mathbf{i}_1],$$

for some $\mathbf{i}_3$. We have $1 \notin \mathrm{supp}(\mathbf{i}_2)$ or $2 \notin \mathrm{supp}(\mathbf{i}_2)$ (or both).

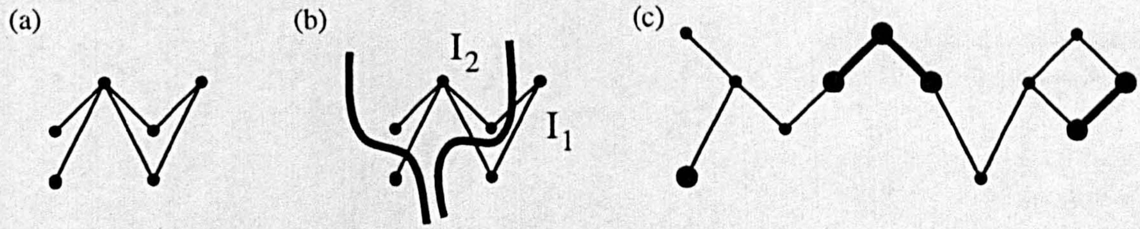
Since $\mathbf{i}_3\ \mathbf{i}_2\ \mathbf{i}_1\ (l \searrow 3)12(3 \nearrow l) \in \mathcal{R}(D_l)$, we may apply a 'left to middle' followed by a 'middle to right' move for each letter of $\mathbf{i}_1$. It is easily checked that these moves change each letter 1 into a 2, and vice-versa. Therefore, we obtain $[\mathbf{i}_3\ \mathbf{i}_2\ (l \searrow 3)12(3 \nearrow l)\ \mathbf{i}_1^{\mathrm{aut}}]$; see 2.1 for the notation. Now we apply 'left to middle' moves for each letter of $\mathbf{i}_2$, giving $[\mathbf{i}_1\ (l \searrow 3)a\ \mathbf{i}_2^{+}\ b(3 \nearrow l)\ \mathbf{i}_1^{\mathrm{aut}}]$, where $(a,b) = (1,2)$ if $1 \in \mathrm{supp}(\mathbf{i}_2)$ and $(a,b) = (2,1)$ if $2 \in \mathrm{supp}(\mathbf{i}_2)$. If *neither* 1 nor 2 occur in $\mathbf{i}_2$ then it does not matter which of the two possibilities we choose for (a,b), because we then have $3 \notin \mathrm{supp}(\mathbf{i}_2^{+})$. We may therefore define

$$C_{I_2,I_1} := \begin{cases} [\mathbf{i}_3\ (l \searrow 3)1\ \mathbf{i}_2^{+}\ 2(3 \nearrow l)\ \mathbf{i}_1^{\mathrm{aut}}] & \text{if } 2 \notin I_2, \\ [\mathbf{i}_3\ (l \searrow 3)2\ \mathbf{i}_2^{+}\ 1(3 \nearrow l)\ \mathbf{i}_1^{\mathrm{aut}}] & \text{if } 1 \notin I_2. \end{cases}$$

Since all linear extensions of I_1, I_2 are commutation-equivalent to one another, C_{I_2,I_1} is well defined.

For example, if $C \in \mathcal{C}(D_3)$ is as shown in Figure 2.4.8 (a), and if I_2 and I_1 are as indicated in 2.4.8 (b), then C_{I_2,I_1} is shown in 2.4.8 (c). Once again, the horizontal lines in these partial order graphs are omitted, but would have been numbered 1, 2, 3,..., from bottom to top.

2.4.8 FIGURE. Constructing C_{I_2,I_1}.



In terms of words, $C = [123213]$ and $C_{I_2,I_1} = [1\,432\,343\,134\,23]$.

2.4.9 PROPOSITION. The mapping

$$\begin{aligned} \{(C, I_2, I_1) \mid C \in \mathcal{C}(D_{l-1}),\ I_1, I_1 \dot{\cup} I_2 \text{ are ideals of } C \text{ and not both } 1,2 \in I_2\} &\longrightarrow \mathcal{C}(D_l) \\ (C, I_2, I_1) &\longmapsto C_{I_2,I_1} \end{aligned}$$

is bijective.

PROOF. For surjectivity, take any $[\boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}] \in \mathcal{C}(D_l)$. Set $C := [\boldsymbol{\lambda}\ \boldsymbol{\mu}^{--}\ \boldsymbol{\varrho}^{\mathrm{aut}}]$ if $(a,b) = (1,2)$ and $C := [\boldsymbol{\lambda}\ \boldsymbol{\mu}^{-}\ \boldsymbol{\varrho}^{\mathrm{aut}}]$ if $(a,b) = (2,1)$. Let I_1 be the ideal formed by $\boldsymbol{\varrho}^{\mathrm{aut}}$. Let I_2 be the subgraph formed by $\boldsymbol{\mu}^{--}$ or $\boldsymbol{\mu}^{-}$, respectively. It is clear that not both 1 and 2 are vertices of I_2, and that (C, I_2, I_1) maps to $[\boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}]$.

For injectivity, suppose, with the obvious notation, that $C_{I_2,I_1} = C'_{I'_2,I'_1}$. Then we have

$$[\mathbf{i}_3\ (l \searrow 3)a\ \mathbf{i}_2^{+}\ b(3 \nearrow l)\ \mathbf{i}_1^{\mathrm{aut}}] = [\mathbf{i}'_3\ (l \searrow 3)a'\ \mathbf{i}'^{+}_2\ b'(3 \nearrow l)\ \mathbf{i}'^{\mathrm{aut}}_1],$$

where $\{a,b\} = \{a',b'\} = \{1,2\}$. Using 2.4.3, we obtain $\mathbf{i}_3 \sim \mathbf{i}'_3$, $\mathbf{i}_1 \sim \mathbf{i}'_1$ and $\mathbf{i}_2^{+} \sim \mathbf{i}'^{+}_2$. It remains to deduce that $\mathbf{i}_2 \sim \mathbf{i}'_2$, for then $C = C'$, $I_1 = I'_1$ and $I_2 = I'_2$. There are two cases.

If $\mathrm{supp}(\mathbf{i}_2)$ contains neither 1 nor 2 then $3 \notin \mathrm{supp}(\mathbf{i}_2^{+})$, hence $3 \notin \mathrm{supp}(\mathbf{i}'^{+}_2)$, which implies $\mathrm{supp}(\mathbf{i}'_2)$ contains neither 1 nor 2, also. It follows at once that $\mathbf{i}_2 \sim \mathbf{i}'_2$, as required.

If instead $1 \in \mathrm{supp}(\mathbf{i}_2)$, say, then we have $(a,b) = (1,2)$, by definition of C_{I_2,I_1}. As $3 \in \mathrm{supp}(\mathbf{i}_2^{+}), \mathrm{supp}(\mathbf{i}'^{+}_2)$, 2.4.3 implies that $(a',b') = (a,b) = (1,2)$. So, by definition of $C_{I'_2,I'_1}$, we must have $1 \in \mathrm{supp}(\mathbf{i}'_2)$ also. So, as $\mathbf{i}_2$ and $\mathbf{i}'_2$ both contain some letter 1 (and hence no letter 2), it follows that $\mathbf{i}_2 \sim \mathbf{i}'_2$, as required. A similar argument applies if $2 \in \mathrm{supp}(\mathbf{i}_2)$. ∎

We now introduce a map analogous to the ones in 2.2.9 and 2.3.11

2.4.10 PROPOSITION. Define

$$\begin{aligned} \delta : \mathcal{C}(D_l) &\longrightarrow \mathcal{C}(D_{l-1}) \\ [\boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}] &\longmapsto \begin{cases} [\boldsymbol{\lambda}\boldsymbol{\mu}^{--}\boldsymbol{\varrho}^{\mathrm{aut}}] & \text{if } (a,b) = (1,2), \\ [\boldsymbol{\lambda}\boldsymbol{\mu}^{-}\boldsymbol{\varrho}^{\mathrm{aut}}] & \text{if } (a,b) = (2,1). \end{cases} \end{aligned}$$

(a) The map δ is a surjective graph morphism. Each fibre is connected, and any two vertices in the same fibre are related to one another by 'left to middle', 'middle to right', 'special left to middle' and their opposite moves.

(b) The δ-fibre of $C \in \mathcal{C}(D_{l-1})$ is

$$\{C_{I_2,I_1} \mid I_1, I_2\dot{\cup}I_1 \text{ are ideals of } C \text{ and not both } 1,2 \in I_2\}.$$

This is a poset with unique top and bottom by setting

$$C_{I_2,I_1} \leq C_{I_2',I_1'} \text{ if } I_1 \subseteq I_1' \text{ and } I_2\dot{\cup}I_1 \subseteq I_2'\dot{\cup}I_1'.$$

(c) Each δ-fibre is a ranked poset, with rank function

$$\operatorname{rank}(C_{I_2,I_1}) := 2|I_1| + |I_2|,$$

and highest rank $2(l-1)(l-2)$. Adjacent vertices of $\mathcal{C}(D_l)$ related to one another by a 'left', 'middle' or 'right' move have the same rank (in their respective fibres).

Proof. To see that δ is well defined, first note that $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\varrho}$ are all defined up to commutations, by 2.4.3. Secondly, if a commutation class in $\mathcal{C}(D_l)$ has representatives of both type 12 and type 21, then, by 2.4.3 again, we have $3 \notin \boldsymbol{\mu}$, so that $\boldsymbol{\mu}^- = \boldsymbol{\mu}^{--}$. Thus, δ is well defined.

(a) If $C = [\mathbf{i}] \in \mathcal{C}(D_{l-1})$ then $[\mathbf{i}\,(l \searrow 3)12(3 \nearrow l)] \mapsto C$, so δ is onto. Using 2.4.2, it is easy to check that δ is a morphism of graphs. Now consider a general vertex $[\boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}] \in \delta^{-1}(C)$. Starting with this vertex, first apply $\ell(\boldsymbol{\mu})$ 'middle to left' moves. Next, for each letter of $\boldsymbol{\varrho}$, apply a 'right to middle' followed by a 'middle to left' move. We obtain a path lying in $\delta^{-1}(C)$, ending with $[\boldsymbol{\lambda}\,\boldsymbol{\mu}^{--}\,\boldsymbol{\varrho}^{\mathrm{aut}}\,(l \searrow 3)12(3 \nearrow l)]$ if $(a,b) = (1,2)$ or $[\boldsymbol{\lambda}\,\boldsymbol{\mu}^{-}\,\boldsymbol{\varrho}^{\mathrm{aut}}\,(l \searrow 3)12(3 \nearrow l)]$ if $(a,b) = (2,1)$. In either case, this equals $[\mathbf{i}\,(l \searrow 3)12(3 \nearrow l)]$. So $\delta^{-1}(C)$ is connected.

(b) We have, in the notation of part (a),

$$\begin{aligned}
\delta^{-1}(C) &= \left\{[\boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}] \mid C = \begin{cases} [\boldsymbol{\lambda}\,\boldsymbol{\mu}^{--}\,\boldsymbol{\varrho}^{\mathrm{aut}}] & \text{if } a=1 \\ [\boldsymbol{\lambda}\,\boldsymbol{\mu}^{-}\,\boldsymbol{\varrho}^{\mathrm{aut}}] & \text{if } a=2 \end{cases}\right\} \\
&= \left\{C_{I_2,I_1} \mid C = \begin{cases} [\boldsymbol{\lambda}\,\boldsymbol{\mu}^{--}\,\boldsymbol{\varrho}^{\mathrm{aut}}] & \text{if } a=1, \text{ and } I_2, I_1 \text{ are formed by } \boldsymbol{\mu}^{--}, \boldsymbol{\varrho} \\ [\boldsymbol{\lambda}\,\boldsymbol{\mu}^{-}\,\boldsymbol{\varrho}^{\mathrm{aut}}] & \text{if } a=2, \text{ and } I_2, I_1 \text{ are formed by } \boldsymbol{\mu}^{-}, \boldsymbol{\varrho} \end{cases}\right\} \\
&= \{C_{I_2,I_1} \mid I_1, I_2\dot{\cup}I_1 \text{ are ideals of } C \text{ and not both } 1,2 \in I_2\}.
\end{aligned}$$

It is trivial to check that the given prescription does indeed give a partial order on the δ-fibres; the bottom element is $C_{\emptyset,\emptyset}$ and the top element is $C_{\emptyset,C}$.

(c) Write $C_{I_2,I_1} = [\boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}]$, so that $|I_1| = \ell(\boldsymbol{\varrho})$, $|I_2| = \ell(\boldsymbol{\mu})$ and

$$\operatorname{rank}(C_{I_2,I_1}) = 2\ell(\boldsymbol{\varrho}) + \ell(\boldsymbol{\mu}).$$

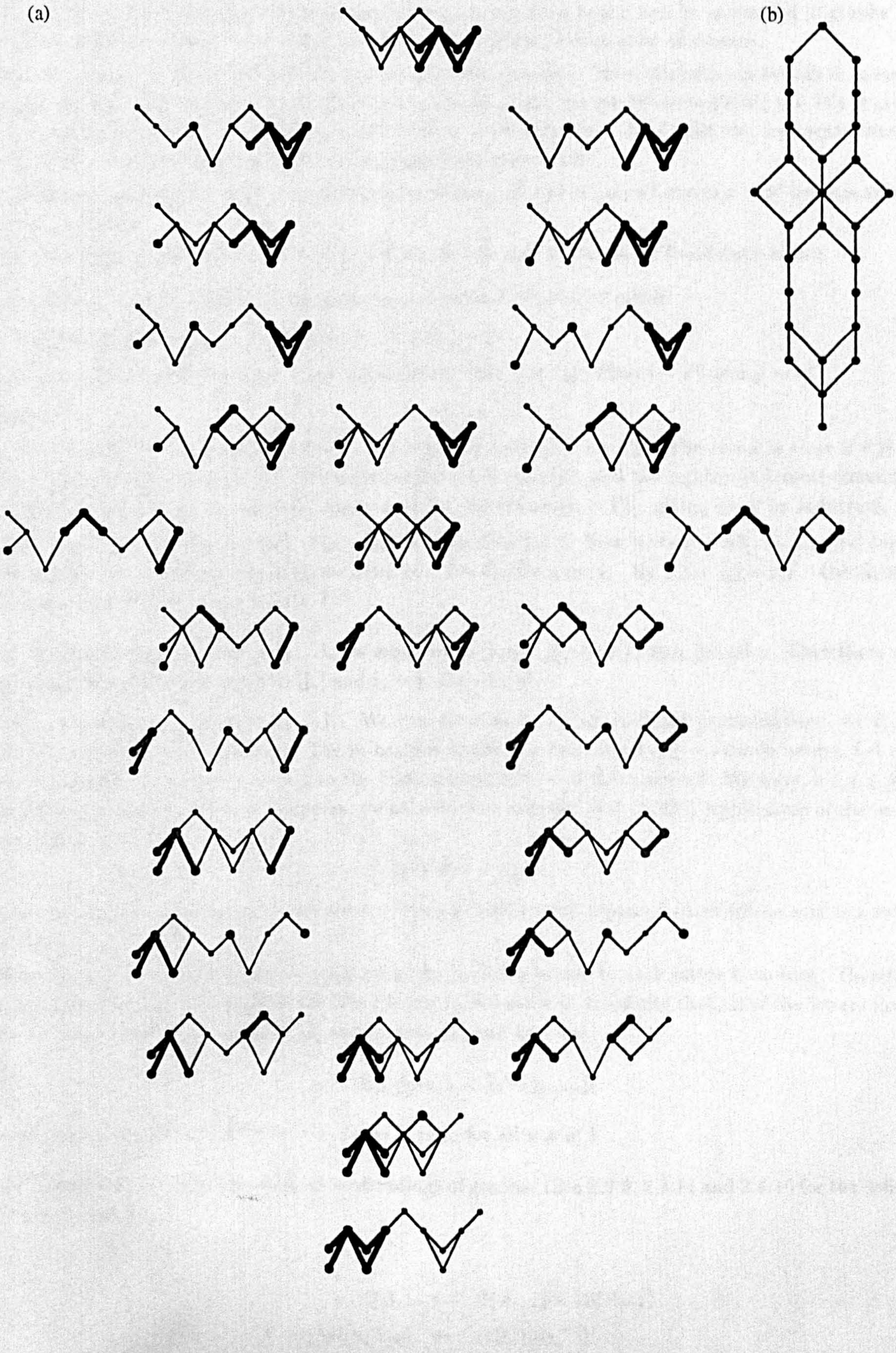
Now consider a vertex $C_{I_2',I_1'} = [\boldsymbol{\lambda}'(l \searrow 3)a'\boldsymbol{\mu}'b'(3 \nearrow l)\boldsymbol{\varrho}']$ in the same fibre, which is one step higher in the partial order. If $C_{I_2',I_1'}$ is obtained from C_{I_2,I_1} by a 'left to middle' or 'special left to middle' move then $\ell(\boldsymbol{\mu}') = \ell(\boldsymbol{\mu}) + 1$ and $\ell(\boldsymbol{\varrho}') = \ell(\boldsymbol{\varrho})$, giving $\operatorname{rank}(C_{I_2',I_1'}) = \operatorname{rank}(C_{I_2,I_1}) + 1$.

If $C_{I_2',I_1'}$ is obtained from C_{I_2,I_1} by a 'middle to right' move then $\ell(\boldsymbol{\mu}') = \ell(\boldsymbol{\mu}) - 1$ and $\ell(\boldsymbol{\varrho}') = \ell(\boldsymbol{\varrho}) + 1$, giving $\operatorname{rank}(C_{I_2',I_1'}) = \operatorname{rank}(C_{I_2,I_1}) + 1$. Therefore, the rank function is well defined.

The top rank is $\operatorname{rank}(C_{\emptyset,C}) = 2|C| + 0 = 2(l-1)(l-2)$. The last part is trivial. ∎

Let $\delta : \mathcal{C}(D_4) \rightarrow \mathcal{C}(D_3)$. The δ-fibre of the commutation class C in Figure 2.4.8 (a) is shown in Figure 2.4.11 (a). The poset structure of this fibre is shown in 2.4.11 (b); note that it is ranked with top rank $12 = 2(4-1)(4-2)$, agreeing with 2.4.10 (c).

2.4.11 FIGURE. The δ-fibre of C of Figure 2.4.8 (a).



An **embedding** of graphs is an injective mapping of vertices which is a graph isomorphism onto its image.

As a simple application of normal representatives, we will show how $\mathcal{C}$ can be embedded in graphs built from Coxeter groups. These embeddings are not used elsewhere, but may be of interest.

We will view $\mathcal{W}$ as the graph underlying its weak order diagram. Thus, the edges in $\mathcal{W}$ join w to $s_i w$ for all simple reflections s_i and all $w \in \mathcal{W}$. If H is a subgroup of $\mathcal{W}$, the set of cosets $\{Hw \mid w \in \mathcal{W}\}$ is given a natural graph structure by deeming two distinct cosets to be adjacent if there exist two representatives, one in each coset, which are adjacent in $\mathcal{W}$. This graph is denoted $\mathcal{W}/H$.

The edges in a **product** $X \times Y$ of graphs join vertices (x, y) and (x', y') whenever $x = x'$ and y is adjacent to y', or vice-versa.

We shall require the following two observations. Recall that $\equiv$ stands for braid-equivalence.

2.5.1 LEMMA. Let $\mathbf{ij} \sim \mathbf{i'j'}$ be a commutation-equivalence of reduced words.

(a) If $\mathbf{i} \equiv \mathbf{i'}$ (or, equivalently, $\mathbf{j} \equiv \mathbf{j'}$) then $\mathbf{i} \sim \mathbf{i'}$ and $\mathbf{j} \sim \mathbf{j'}$.

(b) Suppose that $\mathbf{i} \equiv \mathbf{i'}k$ for some k (or, equivalently, that $\mathbf{j'} \equiv k\mathbf{j}$). Then $\mathbf{i} \sim \mathbf{i'}k$ and $\mathbf{j'} \sim k\mathbf{j}$.

PROOF.

(a) By 1.3.4, it is enough to show $\mathbf{i} \sim \mathbf{i'}$. We argue by induction on $\ell(\mathbf{j})$. The result is clear if $\ell(\mathbf{j}) = 0$. Now let $\ell(\mathbf{j}) > 0$ and write $\mathbf{j} = \mathbf{j}_1 k$. We therefore have $k \in \operatorname{supp}(\mathbf{j'})$, and the rightmost k must commute to the end of $\mathbf{i'j'}$, so $\mathbf{j'} \sim \mathbf{j}'_1 k$ for some $\mathbf{j}'_1$. Applying 1.3.4, we obtain $\mathbf{ij}_1 \sim \mathbf{i'j}'_1$, giving $\mathbf{i} \sim \mathbf{i'}$ by induction.

(b) We argue by induction on $\ell(\mathbf{i'})$. The result is clear if $\ell(\mathbf{i'}) = 0$. Now write $\mathbf{i'} = h\mathbf{i}'_1$, so that $h \in \operatorname{supp}(\mathbf{i})$. As h commutes to the beginning of $\mathbf{ij}$, we must have $\mathbf{i} \sim h\mathbf{i}_1$ for some $\mathbf{i}_1$. By 1.3.4, $\mathbf{i}_1\mathbf{j} \sim \mathbf{i}'_1\mathbf{j'}$. Also, $\mathbf{i}_1 \equiv \mathbf{i}'_1 k$, so by induction, $\mathbf{j'} \sim k\mathbf{j}$, hence $\mathbf{i} \sim \mathbf{i'}k$. ∎

2.5.2 LEMMA. Suppose that $[\mathbf{i}_1\mathbf{i}_2 \ldots \mathbf{i}_n]$ is adjacent to $[\mathbf{j}_1\mathbf{j}_2 \ldots \mathbf{j}_n]$ and $\mathbf{i}_v \equiv \mathbf{j}_v$ for all v. Then there exists some u such that $[\mathbf{i}_u]$ is adjacent to $[\mathbf{j}_u]$ and $\mathbf{i}_v \sim \mathbf{j}_v$ for all $v \neq u$.

PROOF. Set $\mathbf{i} := \mathbf{i}_n \ldots \mathbf{i}_1$, $\mathbf{j} := \mathbf{j}_n \ldots \mathbf{j}_1$. We can go from $\mathbf{i}$ to $\mathbf{j}$ by applying commutations, an m-braid ($m \neq 2$), and further commutations. The m-braid is applied to some word $iji\cdots$ with m letters. Let α and β be the positive roots corresponding to the first and last letters of this subword. We have $\alpha + \beta \in \Phi^+$ by 1.3.6, parts (b) and (c). (For our purposes, we only need to consider $m \in \{3, 4\}$.) Application of the m-braid swaps α and β, hence

$$\beta <_{\mathbf{i}} \alpha \text{ and } \alpha <_{\mathbf{j}} \beta,$$

because any further commutations needed to obtain $\mathbf{j}$ cannot swap a pair of roots whose sum is a root, by 1.3.6 (a).

Since $\mathbf{i}_v \equiv \mathbf{j}_v$ for all v, we may obtain $\mathbf{j}$ from $\mathbf{i}$ by applying braids to each factor $\mathbf{i}_v$ in turn. Therefore, α and β must correspond to letters in the *same* factor $\mathbf{i}_u$, for some u. It follows that all of the letters involved in the m-braid come from $\mathbf{i}_u$. So $[\mathbf{i}_u]$ is adjacent to $[\mathbf{j}_u]$ and we have

$$\mathbf{i}_n \ldots \mathbf{i}_{u+1}\mathbf{j}_u\mathbf{i}_{u-1} \ldots \mathbf{i}_1 \sim \mathbf{j}_n \ldots \mathbf{j}_1.$$

Now applying 2.5.1 (a) $n-1$ times, we obtain $\mathbf{i}_v \sim \mathbf{j}_v$ for all $v \neq u$. ∎

2.5.3 PROPOSITION. The following are embeddings of graphs. (See 2.2.9, 2.3.11 and 2.4.10 for the definition of δ in each case.)

(a)

$$\begin{aligned} \theta : \mathcal{C}(A_l) &\hookrightarrow \mathcal{C}(A_{l-1}) \times \mathcal{W}(A_{l-1}) \\ C = [\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}] &\mapsto (\delta(C), s_{\boldsymbol{\lambda}}{}^{-1}) \end{aligned}$$

(b)

$$\begin{aligned}\theta : \mathcal{C}(B_l) &\hookrightarrow \mathcal{C}(B_{l-1}) \times \mathcal{W}(B_{l-1}) \times \mathcal{W}(B_{l-1})/\langle s_1\rangle \\ C = [\lambda(l \searrow 2)\mu 1(2 \nearrow l)\varrho] &\mapsto (\delta(C), s_\lambda{}^{-1}, \langle s_1\rangle s_\varrho)\end{aligned}$$

(c)

$$\begin{aligned}\theta : \mathcal{C}(D_l) &\hookrightarrow \mathcal{C}(D_{l-1}) \times \mathcal{W}(D_{l-1}) \times \mathcal{W}(D_{l-1}) \\ C = [\lambda(l \searrow 3)a\mu b(3 \nearrow l)\varrho] &\mapsto (\delta(C), s_\lambda{}^{-1}, s_\varrho)\end{aligned}$$

PROOF. The maps are clearly well defined. First we check injectivity.

(a) Let $C' = [\lambda'(l \searrow 1)\varrho']$ be such that $\theta(C) = \theta(C')$. Then $\lambda\varrho^- \sim \lambda'\varrho'^-$ and $\lambda \equiv \lambda'$. By 2.5.1 (a) we obtain $\lambda \sim \lambda'$ and $\varrho^- \sim \varrho'^-$, hence $C = C'$.

(b) Let $C' = [\lambda'(l \searrow 2)\mu' 1(2 \nearrow l)\varrho']$ satisfy $\theta(C) = \theta(C')$. Then $\lambda\mu^-\varrho \sim \lambda'\mu'^-\varrho'$ and $\lambda \equiv \lambda'$, hence $\lambda \sim \lambda'$ by 2.5.1 (a). Now, $s_{\varrho'}s_\varrho{}^{-1} \in \langle s_1\rangle$ and $s_{\varrho'}s_\varrho{}^{-1} = s_{\mu'^-}{}^{-1}s_{\mu^-}$, hence, as $2 \notin \mathrm{supp}(\mu'), \mathrm{supp}(\mu)$, it follows that $s_{\varrho'}s_\varrho{}^{-1} = \mathrm{id}$. Now using 2.5.1 (a) again, we have $\varrho \sim \varrho'$, hence $\mu \sim \mu'$. Thus $C = C'$.

(c) Let $C' = [\lambda'(l \searrow 3)a'\mu' b'(3 \nearrow l)\varrho']$ satisfy $\theta(C) = \theta(C')$. Suppose $(a, b) = (1, 2)$, so that $\delta(C) = [\lambda\mu^{--}\varrho^{\mathrm{aut}}]$. If also $(a', b') = (1, 2)$ then $\delta(C') = [\lambda'\mu'^{--}\varrho'^{\mathrm{aut}}]$, otherwise $\delta(C') = [\lambda'\mu'^-\varrho'^{\mathrm{aut}}]$. Applying 2.5.1 twice, we obtain $\lambda \sim \lambda'$, $\varrho^{\mathrm{aut}} \sim \varrho'^{\mathrm{aut}}$ and either $\mu^{--} \sim \mu'^{--}$ or $\mu^{--} \sim \mu'^-$, respectively. The former clearly implies $\mu \sim \mu'$, as required. The latter implies $3 \notin \mathrm{supp}(\mu), \mathrm{supp}(\mu')$, hence $\mu \sim \mu'$. Therefore $C = C'$. The case $(a, b) = (2, 1)$ is similar.

Using 2.2.5, 2.3.5 and 2.4.5, it is easy to check that each mapping is a morphism of graphs. The reason for using $\langle s_1\rangle$-cosets in type B_l is so that only *one* coordinate changes under 'left to right' moves (see 2.3.2).

Conversely, we must now check that if $\theta(C)$ is adjacent to $\theta(C')$ then C is adjacent to C'. We use the above notation for C and C'.

(a)(i) First suppose that $\delta(C) = \delta(C')$ and $s_\lambda{}^{-1}$ is adjacent to $s'_\lambda{}^{-1}$. Then $\lambda\varrho^- \sim \lambda'\varrho'^-$ and $s_\lambda{}^{-1} = s_i s_{\lambda'}{}^{-1}$ for some s_i. Without loss of generality, assume $\ell(\lambda) > \ell(\lambda')$, so that $\lambda \equiv \lambda' i$. By 2.5.1 (a) we obtain $\lambda \sim \lambda' i$ and $\varrho' \sim i^+\varrho$. It follows that C' can be obtained from C by a single 'left to right' move: see 2.2.1.

(a)(ii) For the other possibility, suppose that $\delta(C)$ is adjacent to $\delta(C')$ and $s_\lambda{}^{-1} = s_{\lambda'}{}^{-1}$. So $[\lambda\varrho^-]$ is adjacent to $[\lambda'\varrho'^-]$, $\lambda \equiv \lambda'$ and $\varrho^- \equiv \varrho'^-$. By 2.5.2, either $[\lambda]$ is adjacent to $[\lambda']$ and $\varrho \sim \varrho'$ or vice-versa. Therefore C' may be obtained from C by a 'left' or 'right' move.

(b)(i) (Throughout case (b) we will define $w := s_{\varrho'}s_\varrho{}^{-1}$.)

Suppose that $\delta(C) = \delta(C')$, $\lambda \equiv \lambda'$ and $\langle s_1\rangle s_\varrho$ is adjacent to $\langle s_1\rangle s_{\varrho'}$. As $\lambda\mu^-\varrho \sim \lambda'\mu'^-\varrho'$, 2.5.1 (a) implies that $\lambda \sim \lambda'$ and $\mu'^-\varrho' \sim \mu^-\varrho$.

Clearly, no reduced word for w can contain the letter 1 (as $2 \notin \mathrm{supp}(\mu), \mathrm{supp}(\mu')$). Using this, we will show that $s_\varrho = s_i s_{\varrho'}$ for some $i \neq 1$. Of course, this is certainly true if s_ϱ and $s_{\varrho'}$ are our adjacent representatives of $\langle s_1\rangle s_\varrho$ and $\langle s_1\rangle s_{\varrho'}$, but there are three other possibilities to consider.

If s_ϱ and $s_1 s_{\varrho'}$ are our adjacent representatives then $s_\varrho = s_i s_1 s_{\varrho'}$ for some $i \neq 1$. Thus, $1i$ is a reduced word for w, which is absurd.

Similarly, $s_1 s_\varrho$ is not adjacent to $s_{\varrho'}$.

If $s_1 s_\varrho$ and $s_1 s_{\varrho'}$ are our adjacent representatives then $s_1 s_\varrho = s_i s_1 s_{\varrho'}$ for some $i \neq 1$. If $i = 2$ then 121 is a reduced word for w, which is absurd. Thus, $i \geq 3$, hence s_i commutes with s_1, giving $s_\varrho = s_i s_{\varrho'}$, as claimed. This completes the four possibilities.

To recap, we have shown that $s_\varrho = s_i s_{\varrho'}$ for some $i \neq 1$.

Without loss of generality, assume $\ell(\varrho) > \ell(\varrho')$, so that $\varrho \equiv i\varrho'$. By 2.5.1 (b) we obtain $\varrho \sim i\varrho'$ and $\mu' \sim \mu i^+$. Recalling $\lambda \sim \lambda'$, it is clear that C' may be obtained from C by a 'right to middle' move (as $i \neq 1$); see 2.3.2.

(b)(ii) Now suppose that $\delta(C) = \delta(C')$, $\langle s_1\rangle s_\varrho = \langle s_1\rangle s_{\varrho'}$ and s_λ^{-1} is adjacent to $s_{\lambda'}^{-1}$. Without loss of generality, let $\ell(\boldsymbol{\lambda}) > \ell(\boldsymbol{\lambda}')$, so that $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}' i$ for some i; as $\boldsymbol{\lambda\mu}^- \boldsymbol{\varrho} \sim \boldsymbol{\lambda}'\boldsymbol{\mu}'^- \boldsymbol{\varrho}'$, this implies $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}' i$ and $\boldsymbol{\mu}'^- \boldsymbol{\varrho}' \sim i\boldsymbol{\mu}^- \boldsymbol{\varrho}$, by 2.5.1 (b). We consider separately the cases $w = \mathrm{id}$ and $w = s_1$.

If $w = \mathrm{id}$ then 2.5.1 (a) implies $\boldsymbol{\varrho} \sim \boldsymbol{\varrho}'$, hence $i^+\boldsymbol{\mu} \sim \boldsymbol{\mu}'$. As $2 \notin \mathrm{supp}(\boldsymbol{\mu}')$ we have $i \neq 1$, so C' may be obtained from C by a 'left to middle' move; see 2.3.2.

If $w = s_1$ then $s_i s_{\boldsymbol{\mu}^-} = s_{\boldsymbol{\mu}'^-} s_1$. Clearly, $i\boldsymbol{\mu}^-$ is reduced (since $\boldsymbol{\lambda}' i\boldsymbol{\mu}^-$ is reduced), and $\boldsymbol{\mu}'^- 1$ is reduced (since $1 \notin \mathrm{supp}(\boldsymbol{\mu}'^-)$). Therefore $i\boldsymbol{\mu}^- \equiv \boldsymbol{\mu}'^- 1$. It follows that $i = 1$, and, being the *only* letter 1 in these two words, it is clear that 1 commutes with $\boldsymbol{\mu}^-$ and $\boldsymbol{\mu}'^-$. In particular, $3 \notin \mathrm{supp}(\boldsymbol{\mu}), \mathrm{supp}(\boldsymbol{\mu}')$.

Now recall (putting $i = 1$) that $\boldsymbol{\mu}'^- \boldsymbol{\varrho}' \sim 1\boldsymbol{\mu}^- \boldsymbol{\varrho}$. Therefore $\boldsymbol{\mu}'^- \boldsymbol{\varrho}' \sim \boldsymbol{\mu}^- 1\boldsymbol{\varrho}$; using this and our assumption $s_{\boldsymbol{\varrho}'} = s_1 s_{\boldsymbol{\varrho}}$ (that is, $w = s_1$), we obtain $\boldsymbol{\varrho}' \sim 1\boldsymbol{\varrho}$ and $\boldsymbol{\mu} \sim \boldsymbol{\mu}'$ by 2.5.1 (a). Recalling that $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}' 1$, we see that C' may be obtained from C by a 'left to right' move.

(b)(iii) Suppose that $\delta(C)$ is adjacent to $\delta(C')$, $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}'$ and $\langle s_1 \rangle s_{\boldsymbol{\varrho}} = \langle s_1 \rangle s_{\boldsymbol{\varrho}'}$. Since $\boldsymbol{\lambda\mu}^- \boldsymbol{\varrho} \equiv \boldsymbol{\lambda}'\boldsymbol{\mu}'^- \boldsymbol{\varrho}'$ we have $w = s_{\boldsymbol{\mu}'^-}^{-1} s_{\boldsymbol{\mu}^-}$, hence $w \neq s_1$. So, $w = \mathrm{id}$, giving $\boldsymbol{\varrho} \equiv \boldsymbol{\varrho}'$ and $\boldsymbol{\mu}^- \equiv \boldsymbol{\mu}'^-$. It follows at once from 2.5.2 that C and C' are related by a 'left', 'middle' or 'right' move.

(c)(i) (Throughout case (c) we will consider only $(a, b) = (1, 2)$, the other case being similar.)

First suppose that $\delta(C) = \delta(C')$, $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}'$ and $s_{\boldsymbol{\varrho}}$ is adjacent to $s_{\boldsymbol{\varrho}'}$. Without loss of generality, let $\ell(\boldsymbol{\varrho}) > \ell(\boldsymbol{\varrho}')$, so that $\boldsymbol{\varrho} \equiv i\boldsymbol{\varrho}'$ for some i. It follows from 2.5.1 (a), applied twice, that $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}'$ and $\boldsymbol{\varrho} \sim i\boldsymbol{\varrho}'$.

If $(a', b') = (1, 2)$ we obtain $\boldsymbol{\mu}^{--} i^{\mathrm{aut}} \sim \boldsymbol{\mu}'^{--}$ using 1.3.4. Thus, $i^{\mathrm{aut}} \neq 2$, that is, $i \neq 1$, hence C' may be obtained from C by a 'right to middle' move; see 2.4.2.

If $(a', b') = (2, 1)$ then $\boldsymbol{\mu}^{--} i^{\mathrm{aut}} \sim \boldsymbol{\mu}'^-$. It follows that $3 \notin \mathrm{supp}(\boldsymbol{\mu})$. So, if $i \neq 1$ then C' may be obtained from C by a 'right to middle' move, and if $i = 1$ then they are related by a 'special right to middle' move.

(c)(ii) The only other essentially different case is when $\delta(C)$ is adjacent to $\delta(C')$ and $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}'$, $\boldsymbol{\varrho} \equiv \boldsymbol{\varrho}'$. We claim that $\boldsymbol{\mu} \equiv \boldsymbol{\mu}'$, for then 2.5.2 shows that C and C' are related by a 'left', 'middle' or 'right' move.

If $(a', b') = (1, 2)$ it follows easily that $\boldsymbol{\mu}^- \equiv \boldsymbol{\mu}'^-$, hence $\boldsymbol{\mu} \equiv \boldsymbol{\mu}'$, as required.

If $(a', b') = (2, 1)$ then $\boldsymbol{\mu}^{--} \equiv \boldsymbol{\mu}'^-$. Thus, $3 \notin \mathrm{supp}(\boldsymbol{\mu}), \mathrm{supp}(\boldsymbol{\mu}')$, which indeed implies $\boldsymbol{\mu} \equiv \boldsymbol{\mu}'$, as required. ▎

2.5.4 NOTES.

(a) The mapping for type B_l looks a little unnatural. In fact, the graphs $\mathcal{W}(B_l)/\langle s_1 \rangle$ and $\mathcal{W}(D_l)$ are isomorphic. (Mapping $w \in \mathcal{W}(D_l)$ to the coset $\langle s_1 \rangle w$ provides a bijection of vertices, and it is easy to verify that this map and its inverse preserve edges.) So, we have an embedding

$$\theta : \mathcal{C}(B_l) \hookrightarrow \mathcal{C}(B_{l-1}) \times \mathcal{W}(B_{l-1}) \times \mathcal{W}(D_{l-1}).$$

(b) If X and Y are posets then so is the **direct product** $X \times Y$ by setting $(x, y) \leq (x', y')$ if and only if $x \leq x'$ and $y \leq y'$. Also, if X and Y are ranked then so is $X \times Y$, with $\mathrm{rank}\,(x, y) = \mathrm{rank}\, x + \mathrm{rank}\, y$. Thus, for each type, we can consider θ as a map into a ranked poset. This clearly induces a ranked poset structure on the domain. Furthermore, θ is a 'tight' mapping in the sense that the ranks of these two posets agree. For example, considering θ in type B_l, we have

$$\mathrm{rank}\, \mathcal{C}(B_l) = \mathrm{rank}\, \mathcal{C}(B_{l-1}) + \mathrm{rank}\, \mathcal{W}(B_{l-1}) + \mathrm{rank}\, \mathcal{W}(D_{l-1}),$$

as can be verified using the formulae in 1.4.9.

(c) By iterating the embeddings of 2.5.3 we obtain an embedding of $\mathcal{C}$ into a direct product of Coxeter groups.

3. Quiver-Compatible Reduced Words

3.1 Equivalence of Quiver-Compatible Words

This chapter has two purposes. The first is to show how normal representatives can be exploited to obtain a numerical characterisation of quiver-compatible longest words. The second is to obtain explicit expressions for quiver-compatible words, which will be used in our subsequent study of root components.

Let Δ_l stand for any labelled connected Coxeter graph with l vertices. A **quiver** is an orientation of Δ_l; if i and j are adjacent vertices of Δ_l, we write $i \to j$ or $i \leftarrow j$. Let $\mathcal{Q}$ denote the set of quivers with underlying Coxeter graph Δ_l. If confusion is possible, we write $\mathcal{Q}(A_6)$, for example, to indicate the underlying graph.

A vertex i of a quiver Q is a **sink** (of Q) if $i \leftarrow j$ for all vertices j adjacent to i.

Let $i(Q)$ denote the quiver obtained from Q by reversing the orientation of all edges adjacent to i.

If $\mathbf{i} := i_r \dots i_1$, let $\mathbf{i}(Q)$ denote the quiver $i_r(\cdots(i_1(Q))\cdots)$.

Following an idea going back at least as far the 1973 paper [BGP], we say $\mathbf{i} := i_r \dots i_1$ is **compatible** with a quiver Q if i_1 is a sink of Q and i_k is a sink of $i_{k-1}\dots i_1(Q)$ for all $k > 1$. (Note that we do not require $\mathbf{i}$ to be a *reduced* word.)

For example, the word $132312 \in \mathcal{R}(A_3)$ is compatible with the quiver $1 \to 2 \leftarrow 3$.

Recall that the promotion operator ∂ on $\mathcal{R}$ sends $i_N \dots i_1$ to $\overline{i_1} i_N \dots i_2$. In this section we will prove that all the quiver-compatible words in $\mathcal{R}$ are equivalent to one another under promotion and commutations. This result may be known, but we have not seen it in the literature. In any case, our proof is very elementary.

3.1.1 LEMMA. Every reduced longest word is compatible with at most one quiver.

PROOF. Suppose that $\mathbf{i} \in \mathcal{R}$ is compatible with quivers Q and Q'. Choose any edge e of the Coxeter graph Δ_l and let j_1, j_2 be its endpoints. Writing $\mathbf{i} := i_N \dots i_1$, let k be the least suffix satisfying $i_k \in \{j_1, j_2\}$. (We are tacitly using the well known fact that the support of any maximal-length word is the whole set $\{1, \dots, l\}$.)

Without loss of generality, let $i_k = j_1$. Therefore j_1 is a sink of both $i_{k-1}\dots i_1(Q)$ and $i_{k-1}\dots i_1(Q')$; in particular, $j_2 \to j_1$ in both these quivers. So $j_2 \to j_1$ in both Q and Q', since, by minimality of k, none of the letters $i_{k-1}, \dots, i_1$ lie in $\{j_1, j_2\}$. Since e was chosen arbitrarily, we have $Q = Q'$. ∎

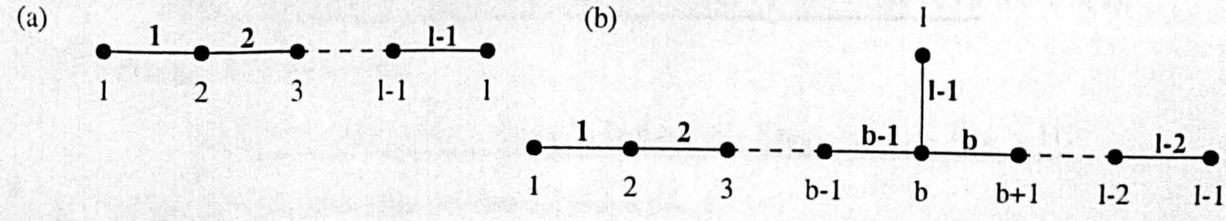
3.1.2 LEMMA. Let $\mathbf{i}$ be reduced and compatible with a quiver Q. If $\mathbf{j} \sim \mathbf{i}$ then $\mathbf{j}$ is also compatible with Q and $\mathbf{j}(Q) = \mathbf{i}(Q)$.

PROOF. We may assume that $\mathbf{i}$ and $\mathbf{j}$ differ by a single commutation — the result is then clear. ∎

We now construct a total order on the set of all quivers with underlying graph Δ_l, where $l > 1$. First, number the vertices and edges of Δ_l as shown figure 3.1.3 (a) or (b), depending upon whether there is a branch point (types D_l, E_6, E_7 and E_8) or not.

We will use this numbering of vertices only for this section, in order to streamline certain arguments.

3.1.3 FIGURE. Constructing a total order on $\mathcal{Q}(\Delta_l)$.



Let $Q \in \mathcal{Q}(\Delta_l)$ and let i, j be adjacent vertices. If $i < j$, the edge orientation $i \leftarrow j$ corresponds to the digit 0 and $i \to j$ corresponds to the digit 1. In this way, Q is uniquely determined by some $(l-1)$-tuple of binary digits, following the ordering of the edges above, which can be viewed as an integer written in base 2. The ordinary total ordering on the integers therefore induces a total order on $\mathcal{Q}(\Delta_l)$. For example, the quiver in $\mathcal{Q}(A_6)$ given by

$$1 \to 2 \leftarrow 3 \leftarrow 4 \to 5 \to 6$$

has binary representation 10011.

Let $\perp_l$ denote the bottom quiver in the total ordering of $\mathcal{Q}(\Delta_l)$.

3.1.4 LEMMA. If $Q \in \mathcal{Q}(\Delta_l)$ is a quiver different from $\perp_l$ then there exists some $j \neq 1$ which is a sink of Q. For any such j we have $j(Q) < Q$ in the total order.

PROOF. First we prove that Q has a sink other than 1, by induction on $l \geq 2$. If $l = 2$ then $Q = 1 \rightarrow 2$, so we can take $j = 2$.

Observe that vertex l of Q lies on a unique edge, e, say. If the arrow on e points towards l then we may take $j := l$ as our sink. Otherwise, let Q' be the subquiver obtained from Q by deleting vertex l and edge e. Clearly, $Q' \neq \perp_{l-1}$ since $Q \neq \perp_l$. By induction, Q' has a sink $j \neq 1$. If j does not lie on e then j is clearly a sink of Q. If j lies on e then, since the arrow on e is pointing towards j, we see that j is a sink of Q in this case too.

We will now show that $j(Q) < Q$. As $j \neq 1$, inspection of Figure 3.1.3 shows that there is a *unique* adjacent vertex j' with $j' < j$. Since j is a sink of Q, we have $j' \rightarrow j$ in Q, corresponding to '1' and $j' \leftarrow j$ in $j(Q)$, corresponding to '0'. Now, the numbering of edges in 3.1.3 has the property that for any other vertex k adjacent to j, edge $\{j, j'\}$ is numbered *lower* than edge $\{j, k\}$. It follows that $j(Q) < Q$. ▮

3.1.5 LEMMA. For all Q and Q' in $\mathcal{Q}(\Delta_l)$, there is some (not necessarily reduced) word ω which is compatible with Q and satisfies $Q' = \omega(Q)$.

PROOF. First we show that there exists some word $\mathbf{j}$ which is compatible with Q and satisfies $\mathbf{j}(Q) = \perp_l$. Then we shall exhibit a word $\mathbf{i}$ which is compatible with $\perp_l$ and satisfies $Q' = \mathbf{i}(\perp_l)$. Setting $\omega := \mathbf{ij}$, we have our desired result.

To prove existence of a suitable $\mathbf{j}$, we argue by induction on the total order on $\mathcal{Q}(\Delta_l)$. If $Q = \perp_l$ we may take $\mathbf{j}$ to be the empty word. Now suppose that $Q \neq \perp_l$; by 3.1.4 there exists some sink $j \neq 1$ of Q with $j(Q) < Q$ in the total order. So by induction there exists some word $\mathbf{j}_1$ which is compatible with $j(Q)$ and satisfies $\mathbf{j}_1(j(Q)) = \perp_l$. Put $\mathbf{j} := \mathbf{j}_1 j$.

Now we shall exhibit a suitable $\mathbf{i}$. Suppose first that Δ_l does not have a branch point. Let Q' have orientation given by $q_i \rightarrow q_i + 1$ for precisely those vertices q_i satisfying $1 \leq q_1 < \cdots < q_k \leq l - 1$. It is easily checked that the following word works:

$$\mathbf{i} := (q_1 \searrow 1)(q_2 \searrow 1) \ldots (q_k \searrow 1).$$

Now suppose that Δ_l has a branch at vertex b. Inspection of 3.1.3 (b) shows that we may take $b = 2$ in type D_l and $b = 3$ in type E_l ($l = 6, 7, 8$). Note that b is adjacent to l; ignoring this edge $\{b, l\}$ temporarily, let Q' have orientation given by $q_i \rightarrow q_i + 1$ for precisely those vertices q_i satisfying $1 \leq q_1 < \cdots < q_k \leq l - 2$. Let $q_1, \ldots, q_{h-1}$ be precisely those q_i which are strictly less than b.

If Q' contains $b \rightarrow l$ we may set

$$\mathbf{i} := \underbrace{(q_1 \searrow 1)(q_2 \searrow 1) \ldots (q_{h-1} \searrow 1)}\ \underbrace{(q_h \searrow 1) l\ (q_{h+1} \searrow 1) l\ \ldots\ (q_k \searrow 1) l}\ (l - 1 \searrow 1),$$

and if Q' contains $b \leftarrow l$ we may set

$$\mathbf{i} := \underbrace{(q_1 \searrow 1)(q_2 \searrow 1)\ \ldots\ (q_{h-1} \searrow 1)}\ \underbrace{l(q_h \searrow 1)\ l(q_{h+1} \searrow 1)\ \ldots\ l(q_k \searrow 1)}.$$

Again, it is easy to check that $\mathbf{i}$ has the claimed properties. ▮

3.1.6 LEMMA. Let $\mathbf{i} := i_r \ldots i_1$ be any reduced word which is compatible with $Q \in \mathcal{Q}$. Suppose j is a sink of Q and that $j \in \operatorname{supp}(\mathbf{i})$. Then there exists some k such that $\mathbf{i} \sim i_r \ldots \widehat{i_k} \ldots i_1 j$. Informally, 'sinks commute to the right'.

PROOF. Consider the least suffix k for which $j = i_k$. If $k = 1$ then there is nothing to prove, so suppose $k > 1$. We claim that for all $h < k$, vertex i_h is not adjacent to i_k. (From this, the result follows at once.)

Suppose not, and consider the least h for which i_h *is* adjacent to i_k. By quiver-compatibility, i_h is a sink of $i_{h-1} \dots i_1(Q)$. Now, Q contains $i_h \rightarrow j = i_k$, and by minimality of k and h, we see that $\{i_h, j\}$ is disjoint from $\{i_{h-1}, \dots, i_2, i_1\}$. Therefore $i_{h-1} \dots i_1(Q)$ *also* contains $i_h \rightarrow j$, contradicting the fact that i_h is a sink of this quiver. ∎

3.1.7 LEMMA. Suppose that $\mathbf{i}$ and $\mathbf{j}$ are reduced words for some element of $\mathcal{W}(\Delta_l)$, which are compatible with the same quiver $Q \in \mathcal{Q}(\Delta_l)$. Then $\mathbf{i} \sim \mathbf{j}$.

PROOF. Write $\mathbf{i} := i_r \dots i_1$ and $\mathbf{j} := j_r \dots j_1$. We argue by induction on r; the result is clear for $r \leq 1$.

We have $j_1 \in \mathrm{supp}(\mathbf{i})$. Since $\mathbf{i}$ is compatible with Q and j_1 is a sink of Q, 3.1.6 shows there is some k such that $\mathbf{i} \sim \mathbf{i}' j_1$, where $\mathbf{i}' := i_r \dots \widehat{i_k} \dots i_1$. Setting $\mathbf{j}' := j_r \dots j_2$, it is clear that $\mathbf{i}'$ and $\mathbf{j}'$ are reduced words for the same element, and which are compatible with $i_1(Q)$. By induction we have $\mathbf{i}' \sim \mathbf{j}'$, hence $\mathbf{i} \sim \mathbf{j}$. ∎

With a fixed labelling of the Coxeter graph Δ_l, for each permutation π of $\{1, \dots, l\}$, the product $s_{\pi(1)} \dots s_{\pi(l)}$ of the l simple reflections is a **Coxeter element**. It is well known that the Coxeter elements form a single conjugacy class and hence have the same order $\hbar = \hbar(\Delta_l)$, called the **Coxeter number**. We have the well known identity $\hbar = 2N/l$ (see [Humphreys, p.79]). The Coxeter numbers corresponding to the root systems are as follows.

Root system	A_l	B_l	D_l	E_6	E_7	E_8	F_4	H_3	H_4	I_2^m
Coxeter number	$l+1$	$2l$	$2(l-1)$	12	18	30	12	10	30	m

The following well known result may be found in [Bourbaki, p.139, Ex.2].

3.1.8 PROPOSITION. The Coxeter graph Δ_l, being bipartite, determines a unique partition of the vertex set $\{1, \dots, l\}$ into two subsets, say $\{e_1, \dots, e_r\}$ and $\{f_1, \dots, f_s\}$, such that no two of the e_i are joined by an edge, and similarly for the f_i. Set $\mathbf{e} := e_1 \dots e_r$ and $\mathbf{f} := f_1 \dots f_s$ (with respect to any numbering) and define

$$\mathbf{i}_0 := \begin{cases} (\mathbf{ef})^{\hbar/2} & \text{if } \hbar \text{ is even,} \\ \mathbf{f}(\mathbf{ef})^{(\hbar-1)/2} & \text{if } \hbar \text{ is odd.} \end{cases}$$

(Where $\mathbf{i}^n$ means $\mathbf{ii}\dots$ with n factors.) Then $\mathbf{i}_0$ is a reduced word for w_0. ∎

For $Q \in \mathcal{Q}(\Delta_l)$, we define $\overline{Q}$ to be the quiver induced by the opposition involution, that is, $i \leftarrow j$ in Q if and only if $\bar{\imath} \leftarrow \bar{\jmath}$ in $\overline{Q}$. Clearly, j is a sink of Q if and only if $\bar{\jmath}$ is a sink of $\overline{Q}$.

3.1.9 LEMMA. For all $Q \in \mathcal{Q}$ there is some $\mathbf{i} \in \mathcal{R}$ which is compatible with Q. Furthermore, any such $\mathbf{i}$ satisfies $\mathbf{i}(Q) = \overline{Q}$.

PROOF. We begin with a special case. In the notation of 3.1.8, let Q_0 be the quiver which has sinks at precisely the vertices $f_1, \dots, f_s$. It is easily checked that $\mathbf{i}_0$ is compatible with Q_0 and satisfies $\mathbf{i}_0(Q_0) = \overline{Q_0}$.

By 3.1.5, it suffices to show the following: if Q is any quiver, if $\mathbf{i} \in \mathcal{R}$ is a compatible word satisfying $\mathbf{i}(Q) = \overline{Q}$, and if j is any sink of Q, then we can find some $\mathbf{j} \in \mathcal{R}$ which is compatible with $j(Q)$ and sends $j(Q)$ to $\overline{j(Q)}$. The special case above shows that this process can begin. Put $\mathbf{i} := i_N \dots i_1$. By 3.1.6 we have $\mathbf{i} \sim \mathbf{i}' := i_N \dots \widehat{i_k} \dots i_1 j$, for some k. By 3.1.2, $\mathbf{i}'$ is compatible with Q and $\mathbf{i}'(Q) = \overline{Q}$. Therefore, since $\bar{\jmath}$ is a sink of $\overline{Q}$, it is clear that $\mathbf{j} := \bar{\jmath} i_N \dots \widehat{i_k} \dots i_1$ is compatible with $j(Q)$, sending $j(Q)$ to $\bar{\jmath}\overline{Q} = \overline{j(Q)}$. Also, $\mathbf{j} = \partial(\mathbf{i}') \in \mathcal{R}$, so $\mathbf{j}$ has the required properties. Thus. every quiver has some compatible word.

Let $\mathbf{i} \in \mathcal{R}$ be compatible with Q such that $\mathbf{i}(Q) = \overline{Q}$. By 3.1.7, all longest words compatible with Q are commutation-equivalent to $\mathbf{i}$, so they send Q to $\overline{Q}$ too. ∎

3.1.10 NOTE. We shall require the following easy observation in Theorem 6.2.17. Let $\mathbf{i} \in \mathcal{R}$ be compatible with Q, so that $\mathbf{i}(Q) = \overline{Q}$ from the previous result. Define $-\overline{Q}$ to be the quiver obtained from $\overline{Q}$ by reversing the orientation of every arrow. Then the reversed word, $\mathbf{i}^{\mathrm{rev}}$, is compatible with $-\overline{Q}$. Thus, in the context of type B_l, where the opposition involution is trivial, $\mathbf{i}^{\mathrm{rev}}$ is compatible with $-Q$.

Putting together 3.1.9, 3.1.7, 3.1.2 and 3.1.1, we obtain the following.

3.1.11 Proposition. The longest words compatible with a quiver Q constitute precisely one commutation class, and none of these words are compatible with any other quiver.

This commutation class will be denoted $[Q]$. ∎

3.1.12 Lemma. If $\mathbf{i} := i_N \ldots i_1 \in \mathcal{R}$ is compatible with some quiver Q then $\partial(\mathbf{i})$ is compatible with $i_1(Q)$.

Proof. Setting $Q' := i_1(Q)$, clearly $\mathbf{i}' := i_N \ldots i_2$ is compatible with Q'. It remains to check that $\overline{i_1}$ is a sink of $\mathbf{i}'(Q')$. This, however, is clear because i_1 is a sink of Q and $\mathbf{i}'(Q') = \overline{Q}$, by 3.1.9. ∎

We introduce a relation on $\mathcal{R}$ whereby two words are related if one may be obtained from the other by some sequence of commutations and applications of the promotion operator. This is an equivalence relation.

3.1.13 Proposition. The quiver-compatible words in $\mathcal{R}$ constitute exactly one equivalence class under commutation and promotion.

Proof. We know that quiver-compatibility is preserved by commutations and promotion (see 3.1.2, 3.1.12).

We must now prove that any two quiver-compatible longest words are equivalent (under commutation and promotion) to one another. It suffices to show that every quiver-compatible word in $\mathcal{R}$ is equivalent to a word compatible with $\perp_l$ (since, by 3.1.7, all the longest words compatible with the same quiver are commutation-equivalent, and also noting that commutation is a symmetric relation and the inverse of ∂ equals ∂^{2N-1}).

Let $\mathbf{i} \in \mathcal{R}$ be compatible with $Q \in \mathcal{Q}$. We argue by induction with respect to the total order on $\mathcal{Q}$. If $Q = \perp_l$, we have nothing to prove. Otherwise, by 3.1.4, there is a sink $j \neq 1$, giving $j(Q) < Q$. By 3.1.6, there exists some k such that $\mathbf{i} \sim \mathbf{i}' := i_N \ldots \widehat{i_k} \ldots i_1 j$. By 3.1.12, $\partial(\mathbf{i}')$ is compatible with $j(Q)$, so by induction, $\partial(\mathbf{i}')$ is equivalent under commutation and promotion to a word compatible with $\perp_l$, whence the same is true of $\mathbf{i}$, as required. ∎

3.2 A Characterisation of Quiver-Compatibility

In this section we will establish, for types A_l, B_l and D_l, a necessary and sufficient condition for a maximal-length reduced word to be quiver-compatible, in terms of letter multiplicities. We shall require the following three observations, whose proofs make use of the existence of normal representatives. (Recall that $\mathrm{mul}_t(\mathbf{i})$ stands for the multiplicity of the letter t in $\mathbf{i}$.)

3.2.1 Lemma. Let $\mathbf{i}$ be a reduced word for some element of $\mathcal{W}(A_l)$.

(a) We have $\mathrm{mul}_1(\mathbf{i}) \leq l$ and $\mathrm{mul}_l(\mathbf{i}) \leq l$.

(b) If $\mathrm{mul}_1(\mathbf{i}) = l$ then $\mathbf{i} \in \mathcal{R}(A_l)$ and

$$\mathbf{i} \sim 1(2 \searrow 1)(3 \searrow 1) \ldots (l \searrow 1).$$

If $\mathrm{mul}_l(\mathbf{i}) = l$ then $\mathbf{i} \in \mathcal{R}(A_l)$ and

$$\mathbf{i} \sim l(l-1 \nearrow l)(l-2 \nearrow l) \ldots (1 \nearrow l).$$

Proof. We will prove both parts by induction on l. The statements are clear if $l = 1$.

(a) Without loss of generality, assume that $\mathbf{i} \in \mathcal{R}(A_l)$. We may write $\mathbf{i} \sim \boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$, where $l \notin \mathrm{supp}(\boldsymbol{\lambda})$ and $1 \notin \mathrm{supp}(\boldsymbol{\varrho})$. By induction $\mathrm{mul}_1(\boldsymbol{\lambda}) \leq l-1$ and $\mathrm{mul}_l(\boldsymbol{\varrho}) \leq l-1$ (because $\mathrm{supp}(\boldsymbol{\varrho}) \subseteq \{2, \ldots, l\}$, which forms a subgraph of A_l isomorphic to A_{l-1}). The desired inequalities follow at once.

(b) Extend $\mathbf{i}$ to a longest word $\mathbf{ij} \in \mathcal{R}(A_l)$. Supposing $\mathrm{mul}_1(\mathbf{i}) = l$, we have $\mathrm{mul}_1(\mathbf{ij}) = l$, by part (a). That is, $1 \notin \mathrm{supp}(\mathbf{j})$.

Using normal representatives, write $\mathbf{ij} \sim \boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$. We have $\mathrm{mul}_1(\mathbf{ij}) = 1 + \mathrm{mul}_1(\boldsymbol{\lambda})$, which implies $\mathrm{mul}_1(\boldsymbol{\lambda}) = l-1$. So by induction we have

$$\boldsymbol{\lambda} \sim 1(2 \searrow 1)(3 \searrow 1) \ldots (l-1 \searrow 1).$$

Comparing word lengths of $\mathbf{ij}$ and $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$, we have $\ell(\boldsymbol{\varrho}) = 0$, hence

$$\mathbf{ij} \sim 1(2 \searrow 1)(3 \searrow 1)\ldots(l-1 \searrow 1)(l \searrow 1),$$

which is easily checked to be compatible with the quiver

$$Q := 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow l\ .$$

It remains only to show that $\ell(\mathbf{j}) = 0$. Supposing otherwise, let j_1 be the rightmost letter of $\mathbf{j}$. Then j_1 must be a sink of Q, hence $j_1 = 1$. This is our required contradiction because we know that $1 \notin \mathrm{supp}(\mathbf{j})$.

The result for the situation $\mathrm{mul}_l(\mathbf{i}) = l$ can be deduced by applying the graph automorphism $i \mapsto l+1-i$ of A_l. ∎

3.2.2 Note. The previous Lemma, and the following two, need to be interpreted carefully. In expressions such as $\mathrm{mul}_l(\mathbf{i}) \le l$ and $\mathrm{mul}_1(\mathbf{i}) = l$, the numbers appearing in the left hand sides are merely *labels* of certain vertices of the Coxeter graph with our standard labelling. These numbers can therefore change, if, as in the next Lemma, we wish to consider a subgraph of type A_{l-2} contained in B_l. The number l appearing in the right hand side is the number of vertices in the Coxeter graph, and is therefore independent of the labelling.

3.2.3 Lemma. If $\mathbf{i}$ is a reduced word for some element of $\mathcal{W}(B_l)$ then $\mathrm{mul}_l(\mathbf{i}) \le l$.

Proof. Without loss of generality, assume that $\mathbf{i} \in \mathcal{R}(B_l)$. We have $\mathbf{i} \sim \boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}$ where $l \notin \mathrm{supp}(\boldsymbol{\lambda}), \mathrm{supp}(\boldsymbol{\varrho})$ and $1, 2 \notin \mathrm{supp}(\boldsymbol{\mu})$. So $\mathrm{mul}_l(\mathbf{i}) = 2 + \mathrm{mul}_l(\boldsymbol{\mu})$. Since $\mathrm{supp}(\boldsymbol{\mu}) \subseteq \{3, \ldots, l\}$, a subgraph of type A_{l-2}, we have $\mathrm{mul}_l(\boldsymbol{\mu}) \le l-2$ by 3.2.1 (a). The required inequality follows at once. ∎

3.2.4 Lemma.

(a) If $\mathbf{i}$ is a reduced word for some element of $\mathcal{W}(D_l)$ then $\mathrm{mul}_l(\mathbf{i}) \le l$ and $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) \le 2(l-1)$.

(b) If $\mathbf{i} \in \mathcal{R}(D_l)$ satisfies $\mathrm{mul}_l(\mathbf{i}) = l$ then $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) = l$.

Proof.

(a) Again, we may assume that $\mathbf{i} \in \mathcal{R}(D_l)$. We have $\mathbf{i} \sim \boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\boldsymbol{\varrho}$, where $l \notin \mathrm{supp}(\boldsymbol{\lambda}), \mathrm{supp}(\boldsymbol{\varrho})$, $1, 2 \notin \mathrm{supp}(\boldsymbol{\mu})$ and $\{a, b\} = \{1, 2\}$. So $\mathrm{mul}_l(\mathbf{i}) = 2 + \mathrm{mul}_l(\boldsymbol{\mu})$. Since $\mathrm{supp}(\boldsymbol{\mu}) \subseteq \{3, \ldots, l\}$, a subgraph of D_l isomorphic to A_{l-2}, we have $\mathrm{mul}_l(\boldsymbol{\mu}) \le l-2$ by 3.2.1 (a). Therefore we have the first inequality, $\mathrm{mul}_l(\mathbf{i}) \le l$.

We prove the second inequality by induction on l. First, note that since $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i})$ is clearly unchanged if we apply the promotion operator to $\mathbf{i}$, we may assume that $\ell(\boldsymbol{\varrho}) = 0$ in the above notation. That is, assume $\mathbf{i} = \boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)$.

If $l = 3$ then $\mathbf{i} \sim \boldsymbol{\lambda}3a\boldsymbol{\mu}b3$. Since $\mathrm{supp}(\boldsymbol{\lambda}) \subseteq \{1, 2\}$, and as 1, 2 commute with one another, we have $\mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda}) \le 2$. Therefore $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) \le 4$, as required.

Now let $l > 3$. We have $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) = 2 + \mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda})$. By induction we have $\mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda}) \le 2(l-2)$, so the desired inequality is clear.

(b) We argue by induction on l. If $l = 3$ then $\ell(\mathbf{i}) = 6$, so that $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) = 6 - \mathrm{mul}_3(\mathbf{i}) = 3$, as required.

Suppose now that $l > 3$. As above, we may consider a normal representative and apply promotion, that is, assume

$$\mathbf{i} = \boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l).$$

Set $\mathbf{j} := \boldsymbol{\lambda}\boldsymbol{\mu}^{--}$ if $a = 1$ or $\mathbf{j} := \boldsymbol{\lambda}\boldsymbol{\mu}^{-}$ if $a = 2$. We have $\mathbf{j} \in \mathcal{R}(D_{l-1})$. (That is, $\mathbf{j}$ is a representative of $\delta([\mathbf{i}])$; see 2.4.10.)

We will now show that $\mathrm{mul}_{l-1}(\mathbf{j}) = l-1$. Applying ∂ to $\mathbf{i}$ brings a letter l to its beginning, so that the leftmost two occurrences of l are separated only by $\boldsymbol{\lambda}$. Consequently, we must have $l-1 \in \mathrm{supp}(\boldsymbol{\lambda})$. Therefore $\mathrm{mul}_{l-1}(\mathbf{j}) \ge 1 + \mathrm{mul}_l(\boldsymbol{\mu})$. Moreover, from our hypothesis that $\mathrm{mul}_l(\mathbf{i}) = l$, we have

$$\mathrm{mul}_l(\boldsymbol{\mu}) = l-2, \tag{$*$}$$

hence $\mathrm{mul}_{l-1}(\mathbf{j}) \geq l-1$. Recalling that $\mathbf{j} \in \mathcal{R}(D_{l-1})$, we have equality, by part (a) of this Lemma.

Having established that $\mathrm{mul}_{l-1}(\mathbf{j}) = l-1$, we have by induction $\mathrm{mul}_1(\mathbf{j}) + \mathrm{mul}_2(\mathbf{j}) = l-1$, that is,

$$\mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda}) + \mathrm{mul}_3(\boldsymbol{\mu}) = l-1,$$

by definition of $\mathbf{j}$. Now, $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i})$ equals $\mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda}) + 2$, which, in view of the above equation, equals $l + 1 - \mathrm{mul}_3(\boldsymbol{\mu})$. It therefore remains to show that $\mathrm{mul}_3(\boldsymbol{\mu}) = 1$.

Since $\mathrm{supp}(\boldsymbol{\mu}) \subseteq \{3, \ldots, l\}$, it follows from $(*)$ and 3.2.1 (b) that $\boldsymbol{\mu} \sim l(l-1 \nearrow l) \ldots (3 \nearrow l)$, in particular, $\mathrm{mul}_3(\boldsymbol{\mu}) = 1$, as we wished to show. ∎

3.2.5 THEOREM. Let Δ_l be a Coxeter graph of type A_l ($l \geq 2$), B_l ($l \geq 2$) or D_l ($l \geq 4$) and let $T \subseteq \Delta_l$ denote the set of all terminal vertices. (Those vertices joined to at most one other vertex.) Then for all $\mathbf{i} \in \mathcal{R}(\Delta_l)$ we have

$$\frac{1}{N} \sum_{t \in T} \mathrm{mul}_t(\mathbf{i}) \leq \frac{|T|}{l}.$$

That is, the average number of letters in $\mathbf{i}$ corresponding to terminal vertices is bounded above by the average number of terminal vertices.

Explicitly, we have the following inequalities.

Type A_l	$\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) \leq l+1$
Type B_l	$\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) \leq 2l$
Type D_l	$\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) \leq 3(l-1)$

Furthermore, we have equality if and only if $\mathbf{i}$ is compatible with some quiver in $\mathcal{Q}(\Delta_l)$.

PROOF. First we establish the inequalities.

Type A_l. Write $\mathbf{i} \sim \boldsymbol{\lambda}(l \searrow 1)\varrho$. Now, since $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i})$ is invariant under promotion and commutations, we may apply promotion $\ell(\varrho)$ times and instead write $\mathbf{i} = \boldsymbol{\lambda}(l \searrow 1)$. By 3.2.1 (a) we have $\mathrm{mul}_1(\boldsymbol{\lambda}) \leq l-1$ because $\mathrm{supp}(\boldsymbol{\lambda}) \subseteq \{1, \ldots, l-1\}$. Therefore $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) = \mathrm{mul}_1(\boldsymbol{\lambda}) + 2 \leq l+1$, as claimed.

Type B_l. Recall from 2.3.8 that $\mathrm{mul}_1(\mathbf{i}) = l$ for all $\mathbf{i} \in \mathcal{R}(B_l)$. We also have $\mathrm{mul}_l(\mathbf{i}) \leq l$ by 3.2.3, so $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) \leq 2l$.

Type D_l. Write $\mathbf{i} \sim \boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)\varrho$. Since both $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i})$ and $\mathrm{mul}_l(\mathbf{i})$ are invariant under promotion and commutations, we may promote $\ell(\varrho)$ times and assume that $\mathbf{i} = \boldsymbol{\lambda}(l \searrow 3)a\boldsymbol{\mu}b(3 \nearrow l)$. By 3.2.4 (a) we know that $\mathrm{mul}_l(\mathbf{i}) \leq l$. There are two cases to consider.

(a) If we have equality, $\mathrm{mul}_l(\mathbf{i}) = l$, then $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) \leq l$ by 3.2.4 (b). Therefore $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) \leq 2l$, which is no greater than $3(l-1)$ when $l \geq 3$, as required.

(b) If the inequality is strict, that is, $\mathrm{mul}_l(\mathbf{i}) < l$, then

$$\begin{aligned}\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) &\leq \mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) + l - 1 \\ &= 2 + \mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda}) + l - 1,\end{aligned}$$

since $1, 2 \notin \mathrm{supp}(\boldsymbol{\mu})$ and $\{a, b\} = \{1, 2\}$. Now, by 3.2.4 (a) we have $\mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda}) \leq 2(l-2)$ (as $s_{\boldsymbol{\lambda}} \in \mathcal{W}(D_{l-1})$). Therefore we obtain

$$\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) \leq 2 + 2(l-2) + l - 1 = 3(l-1),$$

as required.

The next part of the proof is to show that equality holds in the above inequalities if and only if $\mathbf{i}$ is quiver-compatible. First we will establish the easier implication, which in fact holds for *all* types, not just A_l, B_l and D_l, essentially because normal representatives are not used.

Suppose some $\mathbf{i} \in \mathcal{R}$ is quiver-compatible. By 3.1.13, $\mathbf{i}$ is equivalent under commutations and promotion to the special word $\mathbf{i}_0$ of 3.1.8 because, as observed in the proof of 3.1.9, $\mathbf{i}_0$ is itself quiver-compatible.

Since the set T of terminal vertices is permuted by the opposition involution (this may be easily checked for *all* types of Coxeter graph) and since commutations certainly do not affect letter-multiplicities, it follows that

$$\sum_{t\in T} \mathrm{mul}_t(\mathbf{i}) = \sum_{t\in T} \mathrm{mul}_t(\mathbf{i}_0).$$

So, recalling the well known identity $N/l = \hbar/2$, we must prove that

$$\sum_{t\in T} \mathrm{mul}_t(\mathbf{i}_0) = \frac{\hbar}{2}|T|,$$

which will certainly follow if we can show that

$$\sum_{t\in O} \mathrm{mul}_t(\mathbf{i}_0) = \frac{\hbar}{2}|O| \qquad (*)$$

for any orbit O of the opposition involution (because T is a union of such orbits). Refer now to the definition of $\mathbf{i}_0$ in 3.1.8. There are two cases, depending on the parity of the Coxeter number.

If $\hbar$ is even then $\sum_{t\in O} \mathrm{mul}_t(\mathbf{i}_0)$ equals $\frac{\hbar}{2}\sum_{t\in O} \mathrm{mul}_t(\mathbf{ef})$, which equals $\frac{\hbar}{2}|O|$, as desired, since $\mathbf{ef}$ consists of each of the letters $1,\ldots,l$ exactly once.

If $\hbar$ is odd then we need only consider the root systems of type A_l with l even and $I_2(m)$ with m odd. In the case of $I_2(m)$, we must have $O = \{1,2\}$ and the desired equality is trivially true. Finally, consider type A_l with l even. Clearly, $O = \{j, l+1-j\}$ for some j and this orbit has two elements. We have

$$\mathrm{mul}_j(\mathbf{i}_0) + \mathrm{mul}_{l+1-j}(\mathbf{i}_0) = \mathrm{mul}_j(\mathbf{f}(\mathbf{ef})^{(\hbar-1)/2}) + \mathrm{mul}_{l+1-j}(\mathbf{f}(\mathbf{ef})^{(\hbar-1)/2}),$$

which equals $\hbar$, using the fact that, since l is even, $\mathbf{f}$ contains either j or $l+1-j$, but not both. We have therefore verified $(*)$, as required.

We turn now to the converse, that if we have the equality

$$\frac{1}{N}\sum_{t\in T} \mathrm{mul}_t(\mathbf{i}) = \frac{|T|}{l}$$

for some $\mathbf{i} \in \mathcal{R}$ then $\mathbf{i}$ is quiver-compatible. We provide proofs for types A_l, B_l and D_l in turn.

Type A_l. We have some $\mathbf{i} \in \mathcal{R}(A_l)$ which is known to satisfy $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) = l+1$ and we wish to prove that $\mathbf{i}$ is compatible with some quiver in $\mathcal{Q}(A_l)$. Firstly, write $\mathbf{i} \sim \boldsymbol{\lambda}(l,1)\varrho$. As noted when establishing the inequality for type A_l in the first part of this proof, we may apply commutations and promotion without affecting the hypothesis $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) = l+1$. Note also that, by 3.1.13, we will not be in danger of corrupting whether $\mathbf{i}$ is quiver-compatible or not. So, applying the promotion operator $\ell(\varrho)$ times, let us assume that $\mathbf{i} = \boldsymbol{\lambda}(l \searrow 1)$.

Now, $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) = \mathrm{mul}_1(\boldsymbol{\lambda}) + 2$, so $\mathrm{mul}_1(\boldsymbol{\lambda}) = l-1$. This implies that

$$\boldsymbol{\lambda} \sim 1(2\searrow 1)(3\searrow 1)\ldots(l-1\searrow 1)$$

by 3.2.1 (b), since $l \notin \mathrm{supp}(\boldsymbol{\lambda})$. We therefore have

$$\mathbf{i} \sim 1(2\searrow 1)(3\searrow 1)\ldots(l-1\searrow 1)\,(l\searrow 1),$$

which is easily verified to be compatible with the quiver

$$1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow l-1 \leftarrow l\,.$$

This completes the proof for type A_l.

Type B_l. We have some $\mathbf{i} \in \mathcal{R}(B_l)$ satisfying $\mathrm{mul}_l(\mathbf{i}) = l$ (since, by 2.3.8, we automatically have $\mathrm{mul}_1(\mathbf{i}) = l$). We argue by induction on l. If $l = 2$ then $\mathbf{i}$ is either 1212 or 2121; these expressions are compatible with the quivers $1 \rightarrow 2$ and $1 \leftarrow 2$, respectively.

Now suppose $l \geq 3$. Using normal representatives and application of the promotion operator, we may assume that $\mathbf{i} = \boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)$, where $l \notin \mathrm{supp}(\boldsymbol{\lambda})$ and $1, 2 \notin \mathrm{supp}(\boldsymbol{\mu})$. Therefore $\mathrm{mul}_l(\mathbf{i}) = 2 + \mathrm{mul}_l(\boldsymbol{\mu})$, hence $\mathrm{mul}_l(\boldsymbol{\mu}) = l - 2$. Now since $\mathrm{supp}(\boldsymbol{\mu}) \subseteq \{3, \ldots, l\}$, forming a subgraph of B_l isomorphic to A_{l-2}, it follows from 3.2.1 (b) that

$$\boldsymbol{\mu} \sim l(l-1 \nearrow l)(l-2 \nearrow l)\ldots(3 \nearrow l).$$

In order to use our induction hypothesis, set $\mathbf{j} := \boldsymbol{\lambda}\boldsymbol{\mu}^- \in \mathcal{R}(B_{l-1})$ (which is a representative of $\delta([\mathbf{i}])$; see 2.3.11). We have $\mathrm{mul}_{l-1}(\mathbf{j}) = \mathrm{mul}_{l-1}(\boldsymbol{\lambda}) + l - 2$, using our above expression for $\boldsymbol{\mu}$. Observe that $\partial(\mathbf{i}) = l\,\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2, l-1)$, which implies that $l - 1 \in \mathrm{supp}(\boldsymbol{\lambda})$, for otherwise the two leftmost occurrences of the letter l in this expression would be able to commute into adjacent positions, contradicting $\mathbf{i}$ being a reduced word. Thus, $\mathrm{mul}_{l-1}(\mathbf{j}) \geq l - 1$, whence equality holds here, by 3.2.4. Let us record here the following simple consequence of $\mathrm{mul}_{l-1}(\mathbf{j}) = l - 1$, namely

$$\mathrm{mul}_{l-1}(\boldsymbol{\lambda}) = 1. \qquad (\dagger)$$

By induction, $\mathbf{j}$ is compatible with some quiver in $\mathcal{Q}(B_{l-1})$. Recall that $\mathbf{j} = \boldsymbol{\lambda}\boldsymbol{\mu}^-$. Since $\boldsymbol{\mu}^-$ is commutation-equivalent to $l - 1(l-2 \nearrow l-1)\ldots(2 \nearrow l-1)$, it is clear, considering merely this last factor $(2 \nearrow l-1)$, that $\mathbf{j}$ must in fact be compatible with the quiver

$$Q_0 := 1 \to 2 \to 3 \to \cdots \to l-2 \to l-1.$$

So, $\boldsymbol{\lambda}$ is compatible with the quiver $Q_1 := l - 1(l-2 \nearrow l-1)\ldots(2 \nearrow l-1)(Q_0)$, namely,

$$Q_1 = 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow l-2 \leftarrow l-1$$

after a brief calculation.

Now, $\mathbf{i}$ is commutation-equivalent to $\boldsymbol{\lambda}\boldsymbol{\omega}$, where $\boldsymbol{\omega} := (l \searrow 1)\underbrace{l(l-1 \nearrow l)(l-2 \nearrow l)\ldots(2 \nearrow l)}$. It is straightforward to check that $\boldsymbol{\omega}$ is compatible with the quiver Q defined by

$$Q := 1 \to 2 \to 3 \to \cdots \to l-2 \to l-1 \to l$$

and that the resulting quiver is given by

$$\boldsymbol{\omega}(Q) = 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow l-2 \leftarrow l-1 \leftarrow l.$$

We claim that $\boldsymbol{\lambda}$ is compatible with $\boldsymbol{\omega}(Q)$, for then $\mathbf{i}$ will be compatible with Q.

Note that Q_1 is the subquiver of $\boldsymbol{\omega}(Q)$ obtained by deleting the last edge $l - 1 \leftarrow l$. Therefore, since $\boldsymbol{\lambda}$ is compatible with Q_1, a moment's thought shows that $\boldsymbol{\lambda}$ is also compatible with $\boldsymbol{\omega}(Q)$, *provided* $\boldsymbol{\lambda}$ does not contain more than one occurrence of the letter $l - 1$. However, we recorded in $(\dagger)$ that $\mathrm{mul}_{l-1}(\boldsymbol{\lambda}) = 1$, so $\boldsymbol{\lambda}$ is indeed compatible with $\boldsymbol{\omega}(Q)$. This completes the proof for type B_l.

Type D_l. We have some $\mathbf{i} \in \mathcal{R}(D_l)$ satisfying $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) = 3(l-1)$, where $l \geq 4$. We argue by induction on l, but defer the proof of the result for the case $l = 4$ until a little later. By commutations and promotion, we may assume $\mathbf{i} = \boldsymbol{\lambda}(l, 3)a\boldsymbol{\mu}b(3, l)$, where $l \notin \mathrm{supp}(\boldsymbol{\lambda})$, $1, 2 \notin \mathrm{supp}(\boldsymbol{\mu})$ and $\{a, b\} = \{1, 2\}$.

Considering $\delta([\mathbf{i}])$ (see 2.4.10), we obtain some word $\mathbf{j} \in \mathcal{R}(D_{l-1})$, where $\mathbf{j}$ is defined to be $\boldsymbol{\lambda}\boldsymbol{\mu}^{--}$ or $\boldsymbol{\lambda}\boldsymbol{\mu}^-$, depending upon whether $a = 1$ or $a = 2$, respectively. From our hypothesis on $\mathbf{i}$ we have

$$\mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda}) + 4 + \mathrm{mul}_l(\boldsymbol{\mu}) = 3(l-1). \qquad (1)$$

By 3.2.4 (a) we have $\mathrm{mul}_1(\mathbf{j}) + \mathrm{mul}_2(\mathbf{j}) \leq 2(l-2)$, that is

$$\mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda}) + \mathrm{mul}_3(\boldsymbol{\mu}) \leq 2(l-2). \qquad (2)$$

Eliminating $\mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda})$ from (1) and (2), we deduce that

$$\mathrm{mul}_l(\boldsymbol{\mu}) \geq l - 3 + \mathrm{mul}_3(\boldsymbol{\mu}). \qquad (3)$$

Now, $\mathrm{mul}_l(\mathbf{i}) = 2 + \mathrm{mul}_l(\boldsymbol{\mu})$, so in view of (3), we have

$$\mathrm{mul}_l(\mathbf{i}) \geq l - 1 + \mathrm{mul}_3(\boldsymbol{\mu}). \tag{4}$$

Again by 3.2.4 (a) we have $\mathrm{mul}_l(\mathbf{i}) \leq l$, so inspection of (4) shows that $\mathrm{mul}_l(\mathbf{i})$ is either $l-1$ or l. Suppose, for a contradiction, that $\mathrm{mul}_l(\mathbf{i}) = l$. Then by 3.2.4 (b) we have $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) = l$, hence $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) = 2l$, which contradicts our initial hypothesis that $\mathrm{mul}_1(\mathbf{i}) + \mathrm{mul}_2(\mathbf{i}) + \mathrm{mul}_l(\mathbf{i}) = 3(l-1)$, since $l \geq 4$. Thus, $\mathrm{mul}_l(\mathbf{i}) = l - 1$, which implies that $3 \notin \mathrm{supp}(\boldsymbol{\mu})$, by (4). In other words, $\mathrm{supp}(\boldsymbol{\mu}) \subseteq \{4, \dots, l\}$, a subgraph of D_l isomorphic to A_{l-3}.

Now, by (3) we have $\mathrm{mul}_l(\boldsymbol{\mu}) \geq l - 3$, so 3.2.1 (a) implies we have the equality $\mathrm{mul}_l(\boldsymbol{\mu}) = l - 3$, whence

$$\boldsymbol{\mu} \sim l(l-1 \nearrow l) \dots (4 \nearrow l),$$

by 3.2.1 (b). And using the above expression, (1) now says that $\mathrm{mul}_1(\boldsymbol{\lambda}) + \mathrm{mul}_2(\boldsymbol{\lambda}) = 2l - 4$.

Now consider again the definition of $\mathbf{j}$, in order to make an inductive step. First recall that $\boldsymbol{\mu}^- = \boldsymbol{\mu}^{--}$ because $3 \notin \mathrm{supp}(\boldsymbol{\mu})$ and $\mathrm{mul}_{l-1}(\boldsymbol{\mu}^-) = l - 3$. It follows that

$$\mathrm{mul}_1(\mathbf{j}) + \mathrm{mul}_2(\mathbf{j}) + \mathrm{mul}_{l-1}(\mathbf{j}) = (2l-4) + \mathrm{mul}_{l-1}(\boldsymbol{\lambda}) + (l-3) = 3(l-2) + \mathrm{mul}_{l-1}(\boldsymbol{\lambda}) - 1. \tag{5}$$

We claim that $\mathrm{mul}_{l-1}(\boldsymbol{\lambda}) = 1$, for then we will be able to apply the inductive hypothesis to $\mathbf{j}$. Certainly, we must have $l - 1 \in \mathrm{supp}(\boldsymbol{\lambda})$ by considering the leftmost two occurrences of the letter l in $\partial(\mathbf{i})$. For a contradiction, suppose that $\mathrm{mul}_{l-1}(\boldsymbol{\lambda}) \geq 2$. We know already know that $\mathrm{mul}_{l-1}(\mathbf{j}) = \mathrm{mul}_{l-1}(\boldsymbol{\lambda}) + l - 3$, hence $\mathrm{mul}_{l-1}(\mathbf{j}) \geq l - 1$, giving the equality $\mathrm{mul}_{l-1}(\mathbf{j}) = l - 1$ by Lemma 3.2.4 (a). So, part (c) of the same Lemma tells us that $\mathrm{mul}_1(\mathbf{j}) + \mathrm{mul}_2(\mathbf{j}) = l - 1$, hence $\mathrm{mul}_1(\mathbf{j}) + \mathrm{mul}_2(\mathbf{j}) + \mathrm{mul}_{l-1}(\mathbf{j}) = 2(l-1)$ contradicting equation (5) (because $l \geq 4$).

We have therefore shown that

$$\mathrm{mul}_{l-1}(\boldsymbol{\lambda}) = 1, \tag{6}$$

which implies that $\mathrm{mul}_{l-1}(\mathbf{j}) = l - 2$ and

$$\mathrm{mul}_1(\mathbf{j}) + \mathrm{mul}_2(\mathbf{j}) + \mathrm{mul}_{l-1}(\mathbf{j}) = 3(l-2).$$

If $l > 4$ then we may use the induction hypothesis to conclude that $\mathbf{j}$ is compatible with some quiver in $\mathcal{Q}(D_{l-1})$. If $l = 4$ we argue as follows. We have some $\mathbf{j} \in \mathcal{R}(D_3)$ satisfying $\mathrm{mul}_{l-1}(\mathbf{j}) = l - 2$, that is, $\mathrm{mul}_3(\mathbf{j}) = 2$. Since $\ell(\mathbf{j}) = 6$ we therefore have $\mathrm{mul}_1(\mathbf{j}) + \mathrm{mul}_2(\mathbf{j}) = 6 - 2 = 4$. By the characterisation of quiver-compatibility for type A_3, noting that $D_3 = A_3$, except for the labelling of vertices, we see that $\mathbf{j}$ is quiver-compatible. Thus, we have verified that the inductive process begins at $l = 4$.

Recall our expression for $\boldsymbol{\mu}$, above, and that $\mathbf{j} = \boldsymbol{\lambda}\boldsymbol{\mu}^-$. We have shown that $\mathbf{j}$, which is commutation-equivalent to $\boldsymbol{\lambda}\, l-1(l-2 \nearrow l-1) \dots (3 \nearrow l-1)$, is compatible with some quiver $Q_0 \in \mathcal{Q}(D_{l-1})$. It follows from consideration of this final factor $(3 \nearrow l-1)$ that Q_0 must be given by

$$Q_0 = \begin{array}{ccccccccccc} 1 & & & & & & & & & & \\ & \searrow & & & & & & & & & \\ & & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & \cdots & \rightarrow & l-1\,. \\ & \nearrow & & & & & & & & & \\ 2 & & & & & & & & & & \end{array}$$

So $\boldsymbol{\lambda}$ must be compatible with the quiver Q_1 defined by $Q_1 := l-1(l-2 \nearrow l-1) \dots (3 \nearrow l-1)(Q_0)$, pictured below.

$$Q_1 = \begin{array}{ccccccccccc} 1 & & & & & & & & & & \\ & \nwarrow & & & & & & & & & \\ & & 3 & \leftarrow & 4 & \leftarrow & 5 & \leftarrow & \cdots & \leftarrow & l-1\,. \\ & \swarrow & & & & & & & & & \\ 2 & & & & & & & & & & \end{array}$$

Now, a short calculation involving commutations shows that $\mathbf{i} \sim \boldsymbol{\lambda}\omega$, where ω is defined to be the word $\omega := (l \searrow 1)\,\underbrace{l(l-1 \nearrow l) \dots (3 \nearrow l)}$.

We claim that $\mathbf{i}$ is compatible with the quiver Q, below.

$$Q = \begin{array}{ccccccccccccc} 1 & & & & & & & & & & & & \\ & \searrow & & & & & & & & & & & \\ & & 3 & \rightarrow & 4 & \rightarrow & 5 & \rightarrow & \cdots & \rightarrow & l-1 & \rightarrow & l\,. \\ & \nearrow & & & & & & & & & & & \\ 2 & & & & & & & & & & & & \end{array}$$

It is straightforward to check that $\boldsymbol{\omega}$ is compatible with Q and that $\boldsymbol{\omega}(Q)$ is the union of the quiver Q_1 with the edge $l-1\leftarrow l$. Since $\boldsymbol{\lambda}$ is compatible with Q_1, it follows easily that $\boldsymbol{\lambda}$ is also compatible with $\boldsymbol{\omega}(Q)$, *provided* $\mathrm{mul}_{l-1}(\boldsymbol{\lambda}) \leq 1$. Fortunately, by (6) we know that $\mathrm{mul}_{l-1}(\boldsymbol{\lambda}) = 1$. This suffices to prove that $\mathbf{i}$ is compatible with Q, completing the proof for type D_l. ∎

3.2.6 NOTES.

(a) The above proof shows that if $\mathbf{i} \in \mathcal{R}$ is quiver-compatible for *any* type of Coxeter graph then $\frac{1}{N}\sum_{t\in T}\mathrm{mul}_t(\mathbf{i}) = \frac{|T|}{l}$.

(b) The Theorem has also been verified by hand calculation for type H_3 (which has 44 commutation classes). Namely, $\mathrm{mul}_1(\mathbf{i})+\mathrm{mul}_3(\mathbf{i}) \leq 10$ for all $\mathbf{i} \in \mathcal{R}(H_3)$, with equality being achieved only by the quiver-compatible words. It seems reasonable to conjecture that the Theorem is true for *all* types, but naturally, a uniform proof would be preferred.

3.3 Explicit Quiver-Compatible Words

In our subsequent study of root components we shall work in a very explicit way with reduced words. The quiver-compatible words are the simplest and they form a good starting point for understanding certain properties of the whole set $\mathcal{R}$.

We first determine explicitly those words compatible with quivers whose underlying Coxeter graph is A_l. It is convenient to say that an edge with orientation $i\leftarrow i+1$ is a **left arrow** and one with orientation $i\rightarrow i+1$ a **right arrow.**

3.3.1 LEMMA. Let $Q \in \mathcal{Q}(A_l)$ be a quiver with L left arrows and R right arrows, (so that $L+R = l-1$). Define integers $\{x_i\}$, $\{y_j\}$ which satisfy

(a) $l > x_1 > x_2 > \cdots > x_L \geq 1$,

(b) $1 < y_1 < y_2 < \cdots < y_R \leq l$,

(c) $\{l-x_1,\ldots,l-x_L\}\dot\cup\{y_1-1,\ldots,y_R-1\} = \{1,\ldots,l-1\}$,

so that the orientation of Q is given by

$$l-x_i\leftarrow l-x_i+1 \text{ for } 1\leq i\leq L,$$

and

$$y_j-1\rightarrow y_j \text{ for } 1\leq j\leq R.$$

Then the word defined by $\mathbf{i} := \boldsymbol{\lambda}(l\searrow 1)\boldsymbol{\varrho}$, where

$$\begin{aligned} \boldsymbol{\lambda} &:= (x_L\searrow 1)\ldots(x_1\searrow 1), \\ \boldsymbol{\varrho} &:= (l\searrow y_1)\ldots(l\searrow y_R) \end{aligned}$$

belongs to $\mathcal{R}(A_l)$ and is compatible with Q.

PROOF. It is a straightforward task to check that $\mathbf{i}$ is compatible with Q. To prove that $\mathbf{i}\in\mathcal{R}(A_l)$ we shall argue by induction on l. This is clearly true for $l=1$, so now let $l>1$.

Because the sequences $l-x_i$ and y_j-1 are increasing, either $l-x_1 = 1$ or $y_1-1 = 1$. Let us assume $y_1-1=1$; the details for the other possibility are similar. For brevity, set $\boldsymbol{\varrho}_1 := (l\searrow y_2)\ldots(l\searrow y_R)$, omitting the first factor $(l\searrow y_1)$, so that we have $\mathbf{i} = \boldsymbol{\lambda}\,(l\searrow 1)\,(l\searrow 2)\,\boldsymbol{\varrho}_1$. Since every letter to the right of $(l\searrow 1)$ in this expression is different from 1 we may apply, in effect, a 'right to left' move to each letter (see 2.2.1). (Note that this does *not* require us assuming that $\mathbf{i}$ is reduced.) We obtain a word of the form $\mathbf{i}'\,(l \searrow 1)$, where

$$\mathbf{i}' := \boldsymbol{\lambda}\,(l-1 \searrow 1)\,\varrho_1^-.$$

If we set $L' := L$, $R' := R-1$, $x_i' := x_i$ for $1 \le i \le L'$ and $y_j' := y_j - 1$ for $1 \le j \le R'$ then the induction hypothesis applies to the primed letters, giving $\mathbf{i}' \in \mathcal{R}(A_{l-1})$. It follows easily that $\mathbf{i} \in \mathcal{R}(A_l)$, as required. ▮

For example, let $Q = 1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \rightarrow 5 \rightarrow 6$. Then $L = 2$, $R = 3$ and $l = 6$. Considering the left arrows, we have $6 - x_1 = 2$ and $6 - x_2 = 3$. For the right arrows, $y_1 = 2$, $y_2 = 5$ and $y_3 = 6$. Thus $[Q] = [321\ 4321\ 654321\ 65432\ 65\ 6]$.

3.3.2 Lemma. Let $Q \in \mathcal{Q}(B_l)$ have R right arrows $y_i - 1 \rightarrow y_i$ where $1 < y_1 < \cdots < y_R \le l$ and all other edges oriented with left arrows. Then $\mathbf{i} := \boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\varrho$, where

$$\begin{aligned}
\boldsymbol{\lambda} &:= (y_1 - 1 \searrow 1)\ldots(y_R - 1 \searrow 1),\\
\boldsymbol{\mu} &:= (l \searrow 3)\ldots(l \searrow l-1)l,\\
\varrho &:= \underbrace{1(2 \searrow 1)\ldots(l-R-1 \searrow 1)}\ \underbrace{(l-R \searrow y_1)(l-R+1 \searrow y_2)\ldots(l-1 \searrow y_R)}
\end{aligned}$$

belongs to $\mathcal{R}(B_l)$ and is compatible with Q.

Proof. Again, it is easy to check that $\mathbf{i}$ is compatible with Q. To prove that $\mathbf{i} \in \mathcal{R}(B_l)$ we will apply commutations and promotion, proceeding formally, until we reach a word which is known to belong to $\mathcal{R}(B_l)$. This will be enough, since we can simply reverse the steps.

Recall that the opposition involution is trivial in type B_l. First apply inverse promotion $\ell(\boldsymbol{\lambda})$ times, giving

$$(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\,\varrho\,\boldsymbol{\lambda},$$

which, after a little work, is commutation-equivalent to the expression

$$\underbrace{(l \searrow 1)(l \searrow 2)\ldots(l \searrow l-1)l}\ \underbrace{1(2 \searrow 1)\ldots(l-R-1 \searrow 1)}\ \underbrace{(l-R \searrow 1)\ldots(l-1 \searrow 1)}.$$

This is clearly commutation-equivalent to

$$(l \searrow 1)(l \searrow 1)\ldots(l \searrow 1),$$

with l factors. A simple calculation involving a concrete realisation of $\mathcal{W}(B_l)$ as signed permutations, for example (see section 1.2), shows that this last expression belongs to $\mathcal{R}(B_l)$. ▮

For example, if $Q = 1 \rightarrow 2 \leftarrow 3 \leftarrow 4 \rightarrow 5 \rightarrow 6$ (the same quiver as above), we again have $R = 3$ and $y_1 = 2$, $y_2 = 5$, $y_3 = 6$. So this time, $[Q] = [1\ 4321\ 54321\ 65432\ 6543\ 654\ 65\ 6\ 123456\ 1\ 21\ 32]$.

4. General Properties of Root Components

4.1 Introduction

Let β be a positive root. For any $\mathbf{i} = i_N \dots i_1 \in \mathcal{R}$ we know that

$$\beta = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$$

for some uniquely determined k. We say that the **β-value** of $\mathbf{i}$ equals $i_k \in \mathbf{N}$.

It is natural to ask which braids leave the β-value unchanged. Suppose that $\mathbf{i}'$ is obtained from $\mathbf{i}$ by applying a single braid. If the braid is applied entirely within either of the subwords indicated below by braces then trivially, the β-values of $\mathbf{i}$ and $\mathbf{i}'$ agree.

$$\mathbf{i} = \underbrace{i_N \dots i_{k+1}}\ i_k\ \underbrace{i_{k-1} \dots i_1}.$$

If the braid involves the letter i_k then the β-values may differ. Inspection of 1.3.6 shows that the β-value of $\mathbf{i}$ still equals the β-value of $\mathbf{i}'$ if the braid is a commutation or a 4-braid (generally, a braid involving an *even* number of letters), but that the β-values differ if the braid is a 3-braid (generally, a braid involving an *odd* number of letters). In particular we may define the **β-value** of a commutation class C to be the β-value of any of its representatives.

We now introduce some equivalence relations on $\mathcal{R}$ and $\mathcal{C}$. If $\mathbf{i}, \mathbf{i}' \in \mathcal{R}$ are related by some sequence of braids with the property that every word in the sequence has the same β-value, we say that $\mathbf{i}$ and $\mathbf{i}'$ are **β-equivalent** to one another. Analogously, if $C, C' \in \mathcal{C}$ are joined by a connected path in $\mathcal{C}$ such that every commutation class in the path has the same β-value, we say that C and C' are **β-equivalent** to one another, also. These relations are clearly equivalence relations.

The equivalence classes in $\mathcal{C}$ under β-equivalence are called **β-components.** For $C \in \mathcal{C}$, let C_β denote the β-component of C. (So if $\mathbf{i} \in \mathcal{R}$, the β-component of $[\mathbf{i}]$ will be denoted $[\mathbf{i}]_\beta$.)

Each β-component is a connected subgraph of $\mathcal{C}$. Let $\mathcal{C}/\beta$ denote the natural quotient graph of β-components.

Consider type A_3 with $\beta := \alpha_1 + \alpha_2 + \alpha_3$. In Figure 4.1.1 (a) the letters and vertices (of the partial order graphs) corresponding to β are underlined and circled, respectively. The β-components have loops drawn around them and the natural quotient graph $\mathcal{C}(A_3)/(\alpha_1 + \alpha_2 + \alpha_3)$ is shown in (b); each vertex is labelled with its β-value.

The remainder of this thesis is devoted to calculating $\mathcal{C}/\beta$ for the classical types. Along the way we will show how root components are related to root vectors, Coxeter groups, quivers and graph automorphisms.

4.2 Root Components and Root Vectors

In this section we will briefly define root vectors for Hecke algebras and quantized enveloping algebras, following [Bédard]. Then we shall summarise some of Bédard's results and use them to show that root vectors for α_0 are in natural one-to-one correspondence with α_0-components. (Recall that α_0 is the highest root occurring in irreducible crystallographic root systems; see 1.1.6.)

Throughout this section we are considering only the root systems of type A_l, D_l and E_6, E_7, E_8. Let (a_{ij}) be the associated Cartan matrix, so that $a_{ii} = 2$, $a_{ij} = -1$ if i is joined to j and $a_{ij} = 0$ otherwise.

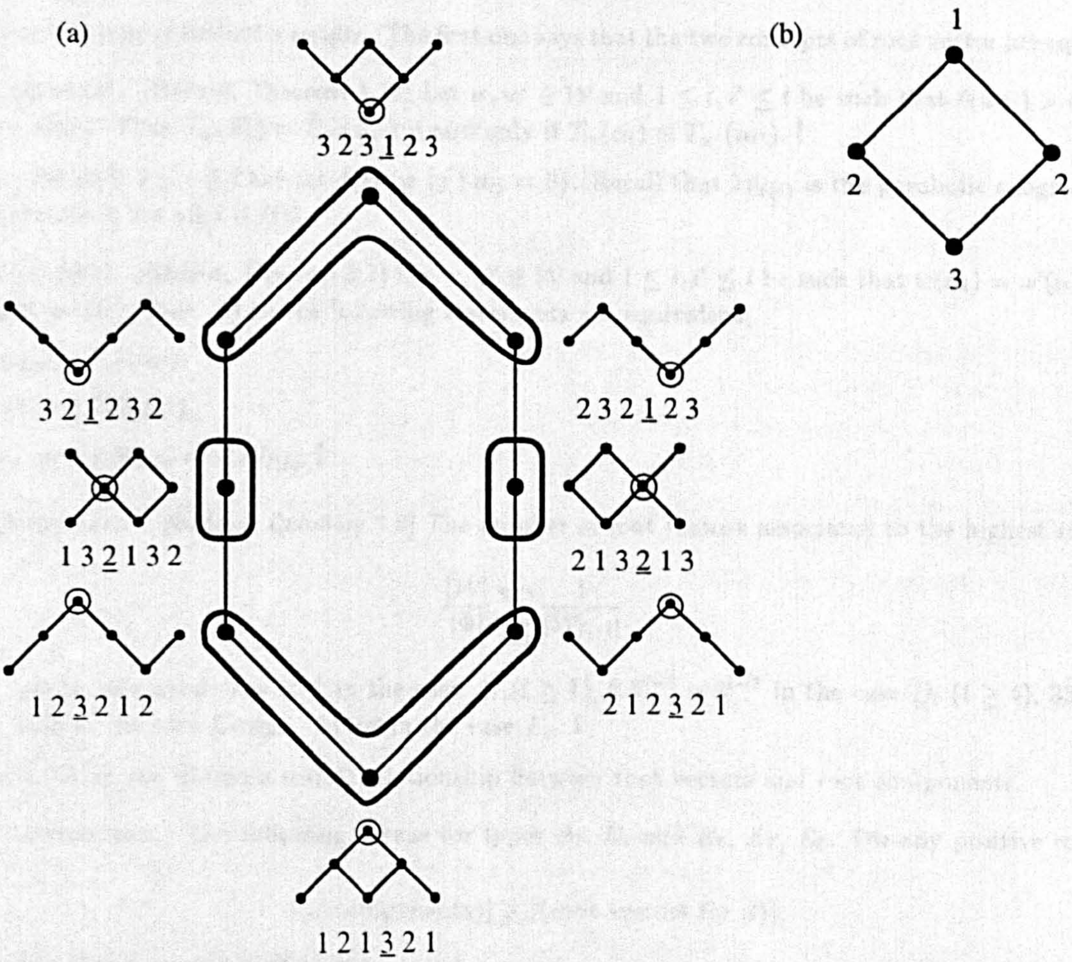
4.2.1 Root Vectors in Hecke Algebras. Let H be the Hecke algebra over $A := \mathbf{Z}[v, v^{-1}]$ (where v is an indeterminate) associated to $\mathcal{W}$. Recall that H is a free A-module with basis elements T_w for $w \in \mathcal{W}$. Multiplication is defined by $T_w T_{w'} := T_{ww'}$ if $\ell(ww') = \ell(w) + \ell(w')$ and $(T_{s_i} + 1)(T_{s_i} - v^2) = 0$ for $1 \le i \le l$.

Let Q be the free abelian group on $\{\alpha_1, \dots, \alpha_l\}$ and set $M := Q \otimes A$, a free A-module with basis elements $\alpha_i \otimes 1$ for $1 \le i \le l$. Abbreviate $\alpha_i \otimes 1$ to α_i.

The **reflection representation** of H is the unique H-module structure on M satisfying

$$T_{s_i}(\alpha_j) = \begin{cases} -\alpha_j & \text{if } i = j, \\ v^2\alpha_j - a_{ij} v \alpha_i & \text{if } i \ne j. \end{cases}$$

4.1.1 FIGURE. (a) The $(\alpha_1+\alpha_2+\alpha_3)$-components of $\mathcal{C}(A_3)$. (b) Corresponding quotient graph.



If $\alpha, \alpha' \in M$, write $\alpha \equiv \alpha'$ if $\alpha' = v^m\alpha$ for some integer m. This is an equivalence relation, with the equivalence class of α being denoted $[\alpha]$. (The notation overlaps with our usage, but will not be used outside of this section.) A class of the form $[T_w(\alpha_i)]$ where $w \in \mathcal{W}$ and $\ell(ws_i) > \ell(w)$ is called a **(positive) root vector** for the root $w(\alpha_i) \in \Phi^+$.

4.2.2 ROOT VECTORS IN QUANTIZED ENVELOPING ALGEBRAS. The quantized enveloping algebra U corresponding to the root system is the associative algebra over $\mathbb{Q}(v)$ with generators E_i, F_i, K_i, K_i^{-1} for $1 \le i \le l$ and relations

(r1) $K_iK_i^{-1} = K_i^{-1}K_i = 1$, $K_iK_j = K_jK_i$;

(r2) $K_iE_j = v^{a_{ij}}E_jK_i$, $K_iF_j = v^{-a_{ij}}F_jK_i$;

(r3) $E_iF_j = F_jE_i$ if $i \neq j$, $E_iF_i - F_iE_i = (K_i - K_i^{-1})/(v - v^{-1})$ if $i = j$;

(r4) $E_iE_j = E_jE_i$ if $a_{ij} = 0$, $E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0$ if $a_{ij} = -1$;

(r5) $F_iF_j = F_jF_i$ if $a_{ij} = 0$; $F_i^2F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0$ if $a_{ij} = -1$.

The algebra automorphism $\tilde{T}_i$ of U is defined as follows:

(a1) $E_i \mapsto -K_i^{-1}F_i$, $F_i \mapsto -E_iK_i$, $K_i \mapsto K_i^{-1}$;

(a2) $E_j \mapsto E_j$, $F_j \mapsto F_j$, $K_j \mapsto K_j$ if $a_{ij} = 0$;

(a3) $E_j \mapsto E_jE_i - v^{-1}E_iE_j$, $F_j \mapsto F_iF_j - vF_jF_i$, $K_j \mapsto K_jK_i$ if $a_{ij} = -1$.

These automorphisms obey the braid relations, that is, $\tilde{T}_i\tilde{T}_j = \tilde{T}_j\tilde{T}_i$ if $a_{ij} = 0$ and $\tilde{T}_i\tilde{T}_j\tilde{T}_i = \tilde{T}_j\tilde{T}_i\tilde{T}_j$ if $a_{ij} = -1$. Thus, if $w \in \mathcal{W}$ and $i_r \dots i_1$ is any reduced word for w then we can define the automorphism $\tilde{T}_w := \tilde{T}_{i_r} \dots \tilde{T}_{i_1}$, being independent of the particular word chosen.

Given any $w \in \mathcal{W}$ and simple root α_i such that $w(\alpha_i) \in \Phi^+$, we define $\tilde{T}_w(E_i)$ to be the **(positive) root vector** for $w(\alpha_i)$.

Now we list some of Bédard's results. The first one says that the two concepts of root vector are equivalent.

4.2.3 Theorem. [Bédard, Theorem 1.18] Let $w, w' \in \mathcal{W}$ and $1 \leq i, i' \leq l$ be such that $\ell(ws_i) > \ell(w)$ and $\ell(w's_{i'}) > \ell(w')$. Then $\tilde{T}_w(E_i) = \tilde{T}_{w'}(E_{i'})$ if and only if $T_w(\alpha_i) \equiv T_{w'}(\alpha_{i'})$. ∎

Define for each $1 \leq i \leq l$ the set $I(i) := \{j \mid a_{ij} = 0\}$. Recall that $\mathcal{W}_{I(i)}$ is the parabolic subgroup of $\mathcal{W}$ with generators s_j for all $j \in I(i)$.

4.2.4 Theorem. [Bédard, Theorem 3.7] Let $w, w' \in \mathcal{W}$ and $1 \leq i, i' \leq l$ be such that $w(\alpha_i) = w'(\alpha_{i'}) = \alpha_0$, the highest positive root. Then the following statements are equivalent:

(a) $T_w(\alpha_i) \equiv T_{w'}(\alpha_{i'})$,

(b) $\tilde{T}_w(E_i) = \tilde{T}_{w'}(E_{i'})$,

(c) $i = i'$ and $w\mathcal{W}_{I(i)} = w'\mathcal{W}_{I(i)}$. ∎

4.2.5 Corollary. [Bédard, Corollary 3.8] The number of root vectors associated to the highest root α_0 is

$$\frac{|\mathcal{W}|}{|\Phi|} \sum_{i=1}^{l} \frac{1}{|\mathcal{W}_{I(i)}|}.$$

In other words, this number is 2^{l-1} in the case A_l $(l \geq 1)$, $2.3^{l-2} - 2^{l-2}$ in the case D_l $(l \geq 4)$, 332 in the case E_6, 3756 in the case E_7 and 131168 in the case E_8. ∎

Using 4.2.4 we can obtain a simple relationship between root vectors and root components.

4.2.6 Proposition. The following is true for types A_l, D_l and E_6, E_7, E_8. For any positive root β we have

$$|\{\beta\text{-components}\}| \geq |\{\text{root vectors for } \beta\}|,$$

with equality if $\beta = \alpha_0$, the highest root.

Proof. We will construct a surjective map

$$\theta : \{\beta\text{-components}\} \twoheadrightarrow \{\text{root vectors for } \beta\}$$

as follows. Take any longest word $\mathbf{i} = i_N \ldots i_1$ and write $\beta = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$ for some uniquely determined k. We associate to $\mathbf{i}$ the root vector $\tilde{T}_{s_{i_1} \ldots s_{i_{k-1}}}(E_{i_k})$ for β. (It does not matter which definition of root vector we use, by 4.2.3.) If $\mathbf{i}'$ differs from $\mathbf{i}$ by a commutation, we obtain the same root vector for β, since $\tilde{T}_{s_i}$ fixes E_j whenever $ij \sim ji$, by (a2), above. If now $\mathbf{i}'$ differs from $\mathbf{i}$ by a 3-braid, subject to their β-letters being equal then the 3-braid clearly cannot involve the letter i_k, so we obtain the same root vector for β. Thus, θ is well defined.

To establish surjectivity, let $\tilde{T}_w(E_i)$ be any root vector for β. So $\beta = w(\alpha_i)$ and $\ell(ws_i) > \ell(w)$. Let $\boldsymbol{\omega}_1$ be any reduced word for w and $\boldsymbol{\omega}_2$ be any reduced word for w_0ws_i, so that $\boldsymbol{\omega}_2\ i\ \boldsymbol{\omega}_1^{\text{rev}}$ is a reduced longest word. Then θ sends the β-component containing this word to $\tilde{T}_w(E_i)$. We have established the inequality stated in this Proposition.

Let $\beta = \alpha_0$. We wish to show that θ is injective. Let $\mathbf{i}$ and $\mathbf{i}'$ be reduced longest words and suppose that θ maps their α_0-components respectively to $\tilde{T}_w(E_i)$ and $\tilde{T}_{w'}(E_{i'})$, where $\alpha_0 = w(\alpha_i) = w'(\alpha_{i'})$. Suppose that these root vectors are equal. Let u be the unique minimal-length representative of the coset $w\mathcal{W}_{I(i)}$ and let $v \in \mathcal{W}_{I(i)}$ be such that $w = uv$ with $\ell(w) = \ell(u) + \ell(v)$ (see 1.1.3). Clearly, $\tilde{T}_w(E_i) = \tilde{T}_u(E_i)$ and $\alpha_0 = u(\alpha_i)$. Similarly, we obtain $\tilde{T}_{w'}(E_{i'}) = \tilde{T}_{u'}(E_{i'})$ and $\alpha_0 = u'(\alpha_{i'})$, where u' is the minimal-length element of $w'\mathcal{W}_{I(i')}$. Again, we have some $v' \in \mathcal{W}_{I(i')}$ such that $w' = u'v'$ with $\ell(w') = \ell(u') + \ell(v')$. By hypothesis, $\tilde{T}_u(E_i) = \tilde{T}_{u'}(E_{i'})$ and $\alpha_0 = u(\alpha_i) = u'(\alpha_{i'})$. By 4.2.4 we obtain $i = i'$ and the equality of cosets $u\mathcal{W}_{I(i)} = u'\mathcal{W}_{I(i)}$. Since u and u' are *both* the unique minimal-length element of this coset, we have $u = u'$, so that $w = uv$ and $w' = uv'$. By definition of θ it is clear that $\mathbf{i}$ ends with a subword which is braid-equivalent to w^{-1}; applying braids to this subword produces words which are α_0-equivalent to $\mathbf{i}$. Thus, $\mathbf{i}$ is α_0-equivalent to some word of the form

$$\text{(reduced word for } w_0 w s_i) \; i \; \text{(reduced word for } v^{-1}) \;\; \text{(reduced word for } u^{-1}).$$

Similarly, $\mathbf{i}'$ is α_0-equivalent to some word of the form

$$\text{(reduced word for } w_0 w' s_i) \; i \; \text{(reduced word for } v'^{-1}) \;\; \text{(reduced word for } u^{-1}).$$

Since i commutes with all reduced words for v^{-1} and v'^{-1} it follows at once that $\mathbf{i}$ is α_0-equivalent to $\mathbf{i}'$. Thus, θ is injective in addition to being surjective. ▮

4.2.7 NOTE. In [Bédard, Corollary 3.9] a formula for the number of root vectors for a general root $\beta \in \Phi^+(D_l)$ is given. When $l = 4$ and $\beta = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ this number equals 8. However, by a hand calculation (or using the theory in chapters 6 and 7), the number of β-components equals 14. Thus, the notions of 'β-component' and 'root vector for β' are in general distinct.

As a consequence of 4.2.6, the numbers listed in 4.2.5 are also the number of α_0-components. We have arrived at these numbers independently (for types A_l and D_l) by different methods, which involve explicit manipulations of reduced words. Further, in the context of root components it is desirable to calculate the *graph structure* of $\mathcal{C}/\beta$; our methods provide a way of doing this.

4.3 Equivalence of Root Components

In this section we will prove that the quotient graphs of root components $\mathcal{C}/\beta$ and $\mathcal{C}/w(\beta)$ are isomorphic to one another for all $\beta \in \Phi^+$ and all $w \in \mathcal{W}$ for which $w(\beta)$ is a positive root. Our argument is valid for types A_l, B_l, D_l, E_l ($l = 6, 7, 8$) and F_4. In particular, $\mathcal{C}/\beta$ depends only upon the *length* of β, by virtue of 1.1.7. This contrasts with the situation for root vectors in type D_l, as we saw at the end of the previous section.

Throughout this section, β is a fixed positive root and s_j is a fixed simple reflection satisfying $s_j(\beta) \in \Phi^+$.

Our first step is to construct a map $\theta : \mathcal{R} \to \mathcal{C}/s_j(\beta)$, for which we need the following easy observation.

4.3.1 LEMMA. Take any $i_N \ldots i_k \ldots i_1 \in \mathcal{R}$ and suppose that $\beta = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$. Exactly one of these two possibilities holds:

(a) $\bar{j}\, i_N \ldots i_{k+1}$ is nonreduced, in which case $i_N \ldots i_{k+1} \equiv \bar{j}\omega$ for some reduced word ω;

(b) $i_{k-1} \ldots i_1\, j$ is nonreduced, in which case $i_{k-1} \ldots i_1 \equiv \sigma j$ for some reduced word σ.

PROOF. If $i_{k-1} \ldots i_1\, j$ is *reduced* then so is $i_k i_{k-1} \ldots i_1 j$ (because $s_j(\beta) \in \Phi^+$) so there exists some ω such that $\omega i_k \ldots i_1 j \in \mathcal{R}$. Promoting, we obtain $\bar{j}\omega i_k \ldots i_1 \in \mathcal{R}$, and comparing this with our original expression $i_N \ldots i_k \ldots i_1 \in \mathcal{R}$ we have $\bar{j}\omega \equiv i_N \ldots i_{k+1}$, by 1.3.3. Hence, $\bar{j}\, i_N \ldots i_{k+1}$ is nonreduced.

We have shown that (a) and (b) are exhaustive. If *both* (a) and (b) hold then we obtain some $\bar{j}\omega i_k \sigma j \in \mathcal{R}$, which is absurd since promotion gives a word containing $\bar{j}$ twice in succession. ▮

For brevity, set $\gamma := s_j(\beta)$. Define a map

$$\begin{aligned} \theta : \mathcal{R} &\to \mathcal{C}/\gamma \\ i_N \ldots i_k \ldots i_1 &\mapsto \begin{cases} [\omega\, i_k i_{k-1} \ldots i_1\, j]_\gamma & \text{if Lemma 4.3.1 (a) holds,} \\ [\bar{j}\, i_N \ldots i_{k+1} i_k\, \sigma]_\gamma & \text{if Lemma 4.3.1 (b) holds,} \end{cases} \end{aligned}$$

where, as usual, we write $\beta = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$.

We need to check that this definition makes sense. First note that the two words displayed above really do belong to $\mathcal{R}$, using the braid-equivalences given in 4.3.1 and then applying promotion ∂ or ∂^{-1}. Secondly, γ corresponds to the letter i_k in both of these words; this is clear for the first word, and Note 1.3.8 should make it clear for the second word. Consequently θ is indeed well defined, despite ω and σ only being determined up to braid-equivalence.

The rest of this section is organised as follows. Our first aim is to show that θ induces a well defined map $\Theta : \mathcal{C}/\beta \rightarrow \mathcal{C}/\gamma$. Secondly, we will exhibit an inverse of Θ. Thirdly, we will show that Θ and Θ^{-1} are morphisms of graphs, and hence that Θ is a graph isomorphism. It will then be a simple matter to deduce that we have a graph isomorphism $\mathcal{C}/\beta \cong \mathcal{C}/w(\beta)$ whenever $w \in \mathcal{W}$ is such that $w(\beta) \in \Phi^+$, which is our desired result.

In order to keep track of β (or indeed any other root) we will introduce some extra notation whereby the particular letter in a word which corresponds to β is marked with the suffix β, as in i_β.

4.3.2 LEMMA. If $\mathbf{i}$ and $\mathbf{i}' \in \mathcal{R}$ are β-equivalent to one another then $\theta(\mathbf{i}) = \theta(\mathbf{i}')$.

PROOF. Write $\mathbf{i} = i_n \dots i_k \dots i_1$ and suppose that β corresponds to the letter i_k. Without loss of generality, $\mathbf{i}'$ is obtained from $\mathbf{i}$ by a single braid. If this braid does not involve the letter i_k then we trivially have $\theta(\mathbf{i}) = \theta(\mathbf{i}')$. Now suppose that i_k is involved; the braid must then be of even length. We will consider the cases when the braid is a commutation or a 4-braid in turn.

$\boxed{\text{The commutation case.}}$

Suppose the commutation swaps a pair of letters x, y so that we have

$$\mathbf{i} = \mathbf{a}\, x\, y_\beta\, \mathbf{b} \text{ and } \mathbf{i}' = \mathbf{a}\, y_\beta\, x\, \mathbf{b}$$

for suitable $\mathbf{a}$ and $\mathbf{b}$. Without loss of generality we are assuming that β corresponds to the y rather than the x in these words.

Considering the definition of $\theta(\mathbf{i})$, there are two cases to consider: either $\mathbf{b} \equiv \boldsymbol{\sigma} j$ for some $\boldsymbol{\sigma}$, or $\mathbf{a}x \equiv \bar{\jmath}\boldsymbol{\omega}$ for some $\boldsymbol{\omega}$.

Case I: $\mathbf{b} \equiv \boldsymbol{\sigma} j$. Here we have $\theta(\mathbf{i}) = [\bar{\jmath}\mathbf{a}xy_\gamma\boldsymbol{\sigma}]_\gamma$ and $\theta(\mathbf{i}') = [\bar{\jmath}\mathbf{a}y_\gamma x\boldsymbol{\sigma}]_\gamma$ so that, as $xy \sim yx$, we trivially have $\theta(\mathbf{i}) = \theta(\mathbf{i}')$, as required.

Case II: $\mathbf{a}x \equiv \bar{\jmath}\boldsymbol{\omega}$. We have

$$\theta(\mathbf{i}) = [\boldsymbol{\omega} y_\gamma \mathbf{b} j]_\gamma. \tag{1}$$

We must consider two Subcases arising from the definition of $\theta(\mathbf{i}')$: either $\mathbf{a} \equiv \bar{\jmath}\boldsymbol{\omega}'$ for some $\boldsymbol{\omega}'$, or $x\mathbf{b} \equiv \boldsymbol{\sigma}' j$ for some $\boldsymbol{\sigma}'$.

Subcase II(i): $\mathbf{a} \equiv \bar{\jmath}\boldsymbol{\omega}'$, so that, by definition

$$\begin{aligned} \theta(\mathbf{i}') &= [\boldsymbol{\omega}' y_\gamma x \mathbf{b} j]_\gamma \\ &= [\boldsymbol{\omega}' x y_\gamma \mathbf{b} j]_\gamma \text{ since } yx \sim xy, \end{aligned}$$

which equals $\theta(\mathbf{i})$, because by comparison with (1) we have $\boldsymbol{\omega}' x \equiv \boldsymbol{\omega}$, and these braids do not involve the letter corresponding to γ.

Subcase II(ii): $x\mathbf{b} \equiv \boldsymbol{\sigma}' j$, so that $\theta(\mathbf{i}') = [\bar{\jmath}\mathbf{a}y_\gamma\boldsymbol{\sigma}']_\gamma$.

Now, the following type of argument will be often used in this proof. From our Subcase hypothesis we easily obtain $s_{\mathbf{b}} s_j = s_x s_{\boldsymbol{\sigma}'}$. Also, $\mathbf{b}j$ clearly has the same number of letters as $x\boldsymbol{\sigma}'$. So, since $\mathbf{b}j$ is reduced (by inspection of (1)) we have the braid-equivalence $\mathbf{b}j \equiv x\boldsymbol{\sigma}'$ of reduced words.

So, by (1) we have

$$\begin{aligned} \theta(\mathbf{i}) &= [\boldsymbol{\omega} y_\gamma x \boldsymbol{\sigma}']_\gamma \text{ using } \mathbf{b}j \equiv x\boldsymbol{\sigma}' \\ &= [\boldsymbol{\omega} x y_\gamma \boldsymbol{\sigma}']_\gamma \text{ by applying a commutation,} \end{aligned}$$

which equals $\theta(\mathbf{i}')$ because the braid-equivalence $\boldsymbol{\omega}x \equiv \bar{\jmath}\mathbf{a}$ is visibly forced. This completes the case where $\mathbf{i}$ and $\mathbf{i}'$ are related by a commutation.

$\boxed{\text{The 4-braid case.}}$

The argument for this case is considerably longer and a little less trivial than the previous case. We have

$$\mathbf{i} = \mathbf{a}xyxy\mathbf{b} \text{ and } \mathbf{i}' = \mathbf{a}yxyx\mathbf{b}$$

for suitable $\mathbf{a}$ and $\mathbf{b}$, where $xyxy \overset{4}{\leftrightarrow} yxyx$ is the 4-braid. Without loss of generality, let α_x be the short root and α_y the long root, so that

$$s_x(\alpha_y) = 2\alpha_x + \alpha_y \text{ and } s_y(\alpha_x) = \alpha_x + \alpha_y.$$

Also, we may assume that β is a long root for the following reason. In the root system of type B_l there is only one β-component if β is a short root, which is a trivial case. As for type F_4, the simple roots α_1, α_2 can be taken as long roots and α_3, α_4 as short roots. The diagram automorphism of F_4 of order two provides a bijection between the long and short roots, as well as a graph automorphism of $\mathcal{R}(F_4)$. Consequently, if β is a short root and β' is the corresponding long root then $\mathcal{C}/\beta$ is isomorphic to $\mathcal{C}/\beta'$. So, we may assume that β is long.

Consider now the expression for $\mathbf{i}$ above; β corresponds to either the first or second occurrence of y (since α_y rather then α_x is the long root). Accordingly, our proof splits into two main parts.

Part one: $\mathbf{i} = \mathbf{a}xy_\beta xy\mathbf{b}$ and hence $\mathbf{i}' = \mathbf{a}yxy_\beta x\mathbf{b}$. That is, β corresponds to the *first* letter y in the expression for $\mathbf{i}$. Considering the definition of $\theta(\mathbf{i})$ there are two cases to consider: either $\mathbf{a}x \equiv \bar{\jmath}\omega$ for some ω, or $xy\mathbf{b} \equiv \sigma j$ for some σ.

Case I: $\mathbf{a}x \equiv \bar{\jmath}\omega$, so that

$$\theta(\mathbf{i}) = [\omega y_\gamma xy\mathbf{b}j]_\gamma. \tag{3}$$

Considering now $\theta(\mathbf{i}')$, we have either $\mathbf{a}yx \equiv \bar{\jmath}\omega'$ for some ω', or $x\mathbf{b} \equiv \sigma' j$ for some σ'.

Subcase I(i): $\mathbf{a}yx \equiv \bar{\jmath}\omega'$, hence we have

$$\theta(\mathbf{i}') = [\omega' y_\gamma x\mathbf{b}j]_\gamma. \tag{4}$$

The key to establishing $\theta(\mathbf{i}) = \theta(\mathbf{i}')$ is to prove that $\bar{\jmath}\mathbf{a}$ is a *nonreduced* word. We separate out this observation because we will need it again.

Claim A. Assuming only the Subcase hypothesis that $\mathbf{a}yx \equiv \bar{\jmath}\omega'$, the word $\omega' x$ is nonreduced.

Proof of Claim A. We must show that $s_{\omega'}(\alpha_x) \in \Phi^-$. We have easily

$$s_{\omega'}(\alpha_x) = -s_{\bar{\jmath}}\underbrace{s_{\mathbf{a}}s_y(\alpha_x)}_{\in\Phi^+},$$

where the braced expression is a positive root because $\mathbf{a}yx$ is reduced (by inspection of $\mathbf{i}'$). So $\omega' x$ is reduced if and only if $s_{\mathbf{a}}s_y(\alpha_x) = \alpha_{\bar{\jmath}}$; for a contradiction suppose this is so. We then have

$$\mathbf{i}' = \mathbf{a}\, y\, x_{\alpha_j}\, y_\beta\, x\, \mathbf{b},$$

which is absurd since inspection of 1.3.6 (c) shows that no simple root can correspond to a 'middle' letter in a 4-braid configuration. This completes the proof of the Claim.

So, by Claim A and 1.3.2 we have

$$\omega' \equiv \omega'' x \tag{5}$$

for some ω''. Substituting this into $\mathbf{a}yx \equiv \bar{\jmath}\omega'$ we obtain

$$\mathbf{a}y \equiv \bar{\jmath}\omega''. \tag{6}$$

We will now show that $\bar{\jmath}\mathbf{a}$ is nonreduced, which is equivalent to showing that $s_{\mathbf{a}}^{-1}(\alpha_{\bar{\jmath}}) \in \Phi^-$. By our hypothesis in Case I, namely $\mathbf{a}x \equiv \bar{\jmath}\omega$, we have

$$s_{\mathbf{a}}^{-1}(\alpha_{\bar{\jmath}}) = s_x \underbrace{s_{\omega}^{-1}(\alpha_{\bar{\jmath}})}_{\in\Phi^+},$$

where the braced expression is a positive root because $\bar{\jmath}\omega$ is reduced. So $\bar{\jmath}\mathbf{a}$ is reduced if and only if $s_{\omega}^{-1}(\alpha_{\bar{\jmath}}) = \alpha_x$; for a contradiction suppose this is so. Using $\mathbf{a}x \equiv \bar{\jmath}\omega$ again, consider the right hand side of

$$\mathbf{i} \equiv \bar{\jmath}\omega y_\beta xy\mathbf{b}.$$

Of course, the root corresponding to the leftmost letter is α_j, but this can also be calculated as

$$\begin{aligned} s_{\mathbf{b}}^{-1} s_y s_x s_y s_\omega^{-1}(\alpha_{\bar{\jmath}}) &= s_{\mathbf{b}}^{-1} s_y s_x s_y(\alpha_x) \text{ by our supposition that } s_\omega^{-1}(\alpha_{\bar{\jmath}}) = \alpha_x \\ &= s_{\mathbf{b}}^{-1}(\alpha_x). \end{aligned}$$

Thus, $s_{\mathbf{b}}^{-1}(\alpha_x) = \alpha_j$, which certainly implies that $s_j s_{\mathbf{b}}^{-1}(\alpha_x) \in \Phi^-$, hence $x\mathbf{b}j$ is nonreduced, contradicting the expression in (4).

We have now established that $\bar{\jmath}\mathbf{a}$ is nonreduced. This implies that

$$\mathbf{a} \equiv \bar{\jmath}\omega''' \tag{7}$$

for some ω'''. Thus, using (7) and $\mathbf{a}x \equiv \bar{\jmath}\omega$ we obtain

$$\omega \equiv \omega''' x. \tag{8}$$

Similarly, by (6) and (7) we obtain

$$\omega'' \equiv \omega''' y. \tag{9}$$

All the necessary facts are now assembled. We have

$$\begin{aligned} \theta(\mathbf{i}) &= [\omega y_\gamma x y \mathbf{b} j]_\gamma \text{ by (3)} \\ &= [\omega''' x y_\gamma x y \mathbf{b} j]_\gamma \text{ by (8)} \\ &= [\omega''' y x y_\gamma x \mathbf{b} j]_\gamma \text{ by applying a 4-braid} \\ &= [\omega'' x y_\gamma x \mathbf{b} j]_\gamma \text{ by (9)} \\ &= [\omega' y_\gamma x \mathbf{b} j]_\gamma \text{ by (5)} \\ &= \theta(\mathbf{i}'), \end{aligned}$$

by (4), as we wished to show.

Subcase I(ii): $x\mathbf{b} \equiv \sigma' j$, so that

$$\theta(\mathbf{i}') = [\bar{\jmath}\mathbf{a} y x y_\gamma \sigma']_\gamma. \tag{10}$$

By the hypothesis in this Subcase we may obtain $\mathbf{b}j \equiv x\sigma'$, since $\mathbf{b}j$ is reduced, by inspection of $\theta(\mathbf{i})$ in (3). We therefore have

$$\begin{aligned} \theta(\mathbf{i}) &= [\omega y_\gamma x y \mathbf{b} j]_\gamma \text{ by (3)} \\ &= [\omega y_\gamma x y x \sigma']_\gamma \text{ using } \mathbf{b}j \equiv x\sigma' \\ &= [\omega x y x y_\gamma \sigma']_\gamma \text{ applying a 4-braid} \\ &= \theta(\mathbf{i}'), \end{aligned}$$

since, by comparison with the expression in (10), the braid-equivalence $\omega x \equiv \bar{\jmath}\mathbf{a}$ is forced. This completes Case I.

Case II: $xy\mathbf{b} \equiv \sigma j$, so that our new expression for $\theta(\mathbf{i})$ is

$$\theta(\mathbf{i}) = [\bar{\jmath}\mathbf{a} x y_\gamma \sigma]_\gamma. \tag{11}$$

Considering the definition of $\theta(\mathbf{i}')$ we have either $\mathbf{a}yx \equiv \bar{\jmath}\omega'$ for some ω', or $x\mathbf{b} \equiv \sigma' j$ for some σ'.

Subcase II(i): $\mathbf{a}yx \equiv \bar{\jmath}\omega'$, so that

$$\theta(\mathbf{i}') = [\omega' y_\gamma x \mathbf{b} j]_\gamma. \tag{12}$$

First note that Claim A (in Subcase I(i), above) applies, as do equations (5) and (6) which follow from it. In particular, (6), which says $\mathbf{a}y \equiv \bar{\jmath}\omega''$, implies that

$$\bar{\jmath}\mathbf{a} \equiv \omega'' y \tag{13}$$

because inspection of (11) shows $\bar{\jmath}\mathbf{a}$ to be reduced.

Claim B. $x\sigma$ is nonreduced.

Proof of Claim B. It suffices to show that $s_\sigma^{-1}(\alpha_x) \in \Phi^-$. From our Case II hypothesis that $xy\mathbf{b} \equiv \sigma j$ we have

$$s_\sigma^{-1}(\alpha_x) = -s_j \underbrace{s_{\mathbf{b}}^{-1} s_y(\alpha_x)}_{\in\Phi^+},$$

where the braced term lies in Φ^+ because $xy\mathbf{b}$ is reduced, by inspection of our expression for $\mathbf{i}$. So $x\sigma$ is reduced if and only if $s_{\mathbf{b}}^{-1} s_y(\alpha_x) = \alpha_j$; however, if this holds then we have

$$\mathbf{i} = \mathbf{a}\, x\, y_\beta\, x_{\alpha_j}\, y\, \mathbf{b}$$

which contradicts α_j being a simple root (because it lies in the middle of a 4-braid configuration). This contradiction establishes Claim B.

By Claim B we therefore have

$$\sigma \equiv x\sigma'' \tag{14}$$

for some σ''. (We have used σ'' rather than σ' in order to avoid stealing the notation of the Subcase II(ii).) We can now say that

$$\begin{aligned}
\theta(\mathbf{i}) &= [\bar{\jmath}\mathbf{a}xy_\gamma\sigma]_\gamma \text{ by (11)}\\
&= [\omega'' yxy_\gamma\sigma]_\gamma \text{ by (13)}\\
&= [\omega'' yxy_\gamma x\sigma'']_\gamma \text{ by (14)}\\
&= [\omega'' xy_\gamma xy\sigma'']_\gamma \text{ applying a 4-braid}\\
&= [\omega' y_\gamma xy\sigma'']_\gamma \text{ by (5), which } \textit{is} \text{ applicable here}\\
&= \theta(\mathbf{i}'),
\end{aligned}$$

since we have the braid-equivalence $y\sigma'' \equiv \mathbf{b}j$ by comparison with (12). This completes Subcase II(i).

Subcase II(ii): $x\mathbf{b} \equiv \sigma' j$, so that

$$\theta(\mathbf{i}') = [\bar{\jmath}\mathbf{a}yxy_\gamma\sigma']_\gamma. \tag{15}$$

We claim that $x\sigma'$ is nonreduced. We have, using $x\mathbf{b} \equiv \sigma' j$,

$$s_{\sigma'}^{-1}(\alpha_x) = -s_j \underbrace{s_{\mathbf{b}}^{-1}(\alpha_x)}_{\in\Phi^+},$$

so $x\sigma'$ is reduced if and only if $s_{\mathbf{b}}^{-1}(\alpha_x) = \alpha_j$, which, for a contradiction we will suppose holds. We can use our Case II hypothesis $xy\mathbf{b} \equiv \sigma j$ to obtain an expression for $s_{\mathbf{b}}^{-1}$, giving $s_j s_\sigma^{-1} s_x s_y(\alpha_x) = \alpha_j$. This can be re-written as $s_\sigma^{-1} s_y(\alpha_x) = -\alpha_j \in \Phi^-$, whence $xy\sigma$ is *nonreduced*. However, considering our expression $\mathbf{i} = \mathbf{a}xyxy\mathbf{b}$ and substituting $xy\mathbf{b} \equiv \sigma j$ we obtain $\mathbf{a}xy\sigma j \in \mathcal{R}$, in particular, the subword $xy\sigma$ is *reduced*. This is our required contradiction.

We have therefore shown $x\sigma'$ to be nonreduced. Consequently,

$$\sigma' \equiv x\sigma''' \tag{16}$$

for some σ'''. We now have, beginning with $\theta(\mathbf{i}')$ this time, that

$$\begin{aligned}
\theta(\mathbf{i}') &= [\bar{\jmath}\mathbf{a}yxy_\gamma\sigma']_\gamma \text{ by (15)}\\
&= [\bar{\jmath}\mathbf{a}yxy_\gamma x\sigma''']_\gamma \text{ by (16)}\\
&= [\bar{\jmath}\mathbf{a}xy_\gamma xy\sigma''']_\gamma \text{ applying a 4-braid}\\
&= \theta(\mathbf{i}),
\end{aligned}$$

since we necessarily have the braid-equivalence $xy\sigma''' \equiv \sigma$ by comparison with (11). Part one of the proof is now finished.

Part two: $\mathbf{i} = \mathbf{a}xyxy_\beta\mathbf{b}$ and hence $\mathbf{i}' = \mathbf{a}y_\beta xyx\mathbf{b}$. In this new situation, β corresponds to the *second* letter y in the expression for $\mathbf{i}$. Considering the definition of $\theta(\mathbf{i})$ there are two cases to consider: either $\mathbf{b} \equiv \boldsymbol{\sigma} j$ for some $\boldsymbol{\sigma}$, or $\mathbf{a}xyx \equiv \bar{\jmath}\boldsymbol{\omega}$ for some $\boldsymbol{\omega}$.

Case I: $\mathbf{b} \equiv \boldsymbol{\sigma} j$, so that

$$\theta(\mathbf{i}) = [\bar{\jmath}\mathbf{a}xyxy_\gamma\boldsymbol{\sigma}]_\gamma. \tag{17}$$

Considering now $\theta(\mathbf{i}')$, we have either $\mathbf{a} \equiv \bar{\jmath}\boldsymbol{\omega}'$ for some $\boldsymbol{\omega}'$, or $xyx\mathbf{b} \equiv \boldsymbol{\sigma}' j$ for some $\boldsymbol{\sigma}'$.

Subcase I(i): $\mathbf{a} \equiv \bar{\jmath}\boldsymbol{\omega}'$. Together with the Case I hypothesis that $\mathbf{b} \equiv \boldsymbol{\sigma} j$ we may obtain $\mathbf{i} \equiv \bar{\jmath}\boldsymbol{\omega}' xyxy\boldsymbol{\sigma} j$, which is nonreduced (as promoting will give a word containing $\bar{\jmath}\bar{\jmath}$). So, this Subcase cannot arise.

Subcase I(ii): $xyx\mathbf{b} \equiv \boldsymbol{\sigma}' j$, so that

$$\theta(\mathbf{i}') = [\bar{\jmath}\mathbf{a}y_\gamma\boldsymbol{\sigma}']_\gamma. \tag{18}$$

We have our desired result at once, for by (17) we have $\theta(\mathbf{i}) = [\bar{\jmath}\mathbf{a}y_\gamma xyx\boldsymbol{\sigma}]_\gamma$ upon application of a 4-braid, which equals $\theta(\mathbf{i}')$ by comparison with (18). (The braid-equivalence $xyx\boldsymbol{\sigma} \equiv \boldsymbol{\sigma}'$ is forced.)

Case II: $\mathbf{a}xyx \equiv \bar{\jmath}\boldsymbol{\omega}$, so that

$$\theta(\mathbf{i}) = [\boldsymbol{\omega} y_\gamma \mathbf{b} j]_\gamma. \tag{19}$$

Considering $\theta(\mathbf{i}')$, we have either $\mathbf{a} \equiv \bar{\jmath}\boldsymbol{\omega}'$ for some $\boldsymbol{\omega}'$, or $xyx\mathbf{b} \equiv \boldsymbol{\sigma}' j$ for some $\boldsymbol{\sigma}'$.

Subcase II(i): $\mathbf{a} \equiv \bar{\jmath}\boldsymbol{\omega}'$, so that

$$\begin{aligned} \theta(\mathbf{i}') &= [\boldsymbol{\omega}' y_\gamma xyx\mathbf{b} j]_\gamma \text{ by definition} \\ &= [\boldsymbol{\omega}' xyxy_\gamma \mathbf{b} j]_\gamma \text{ applying a 4-braid} \\ &= \theta(\mathbf{i}), \end{aligned}$$

by comparison with (19), noting that $\boldsymbol{\omega}' xyx \equiv \boldsymbol{\omega}$ is forced.

Subcase II(ii): $xyx\mathbf{b} \equiv \boldsymbol{\sigma}' j$, so that

$$\theta(\mathbf{i}') = [\bar{\jmath}\mathbf{a}y_\gamma\boldsymbol{\sigma}']_\gamma. \tag{20}$$

Claim C. $\bar{\jmath}\mathbf{a}xy$ is nonreduced.

Proof of Claim C. We know from the form of $\mathbf{i}$ that $\mathbf{a}xy$ is reduced, so we must show that $s_y s_x s_{\mathbf{a}}^{-1}(\alpha_{\bar{\jmath}}) \in \Phi^-$. We can use our Case II hypothesis $\mathbf{a}xyx \equiv \bar{\jmath}\boldsymbol{\omega}$ to obtain an expression for $s_{\mathbf{a}}^{-1}$, giving

$$s_y s_x s_{\mathbf{a}}^{-1}(\alpha_{\bar{\jmath}}) = -s_x \underbrace{s_{\boldsymbol{\omega}}^{-1}(\alpha_{\bar{\jmath}})}_{\in\Phi^+}$$

after routine simplification. Note that the braced term is a positive root because $\bar{\jmath}\boldsymbol{\omega}$ is reduced. Therefore $\bar{\jmath}\mathbf{a}xy$ is reduced if and only if $s_{\boldsymbol{\omega}}^{-1}(\alpha_{\bar{\jmath}}) = \alpha_x$. For a contradiction, suppose this is so. Substituting $\mathbf{a}xyx \equiv \bar{\jmath}\boldsymbol{\omega}$ into our expression for $\mathbf{i}$, we obtain the braid-equivalence

$$\mathbf{i} \equiv \bar{\jmath}\boldsymbol{\omega} y_\beta \mathbf{b}.$$

The root corresponding to the leftmost letter in this expression is clearly α_j, which also equals

$$s_{\mathbf{b}}^{-1} s_y s_{\boldsymbol{\omega}}^{-1}(\alpha_{\bar{\jmath}}) = s_{\mathbf{b}}^{-1} s_y(\alpha_x)$$

by our assumption that $s_{\boldsymbol{\omega}}^{-1}(\alpha_{\bar{\jmath}}) = \alpha_x$. So, we have $\alpha_j = s_{\mathbf{b}}^{-1} s_y(\alpha_x)$, which clearly implies that

$$\mathbf{i} = \mathbf{a}\, x\, y\, x_{\alpha_j}\, y_\beta\, \mathbf{b}.$$

Once again, this contradicts α_j being a simple root; Claim C is established. As a consequence, we have

$$\mathbf{a}xy \equiv \bar{\jmath}\boldsymbol{\omega}' \tag{21}$$

for some $\boldsymbol{\omega}'$; substituting this into our Case II hypothesis $\mathbf{a}xyx \equiv \bar{\jmath}\boldsymbol{\omega}$ we obtain

$$\boldsymbol{\omega}' x \equiv \boldsymbol{\omega}. \tag{22}$$

Claim D. $\bar{\jmath}ax$ is nonreduced.

Proof of Claim D. Inspection of $\mathbf{i}$ shows $\mathbf{a}x$ to be reduced, so we must prove that $s_x s_{\mathbf{a}}^{-1}(\alpha_{\bar{\jmath}}) \in \Phi^-$. This time we use (21) to obtain an expression for $s_{\mathbf{a}}^{-1}$, giving

$$s_x s_{\mathbf{a}}^{-1}(\alpha_{\bar{\jmath}}) = -s_y \underbrace{s_{\omega'}^{-1}(\alpha_{\bar{\jmath}})}_{\in\Phi^+},$$

where the braced term lies in Φ^+ because $\bar{\jmath}\omega'$ is reduced. So, $\bar{\jmath}ax$ is reduced if and only if $s_{\omega'}^{-1}(\alpha_{\bar{\jmath}}) = \alpha_y$; for a contradiction, suppose this is true.

Consider again the root corresponding to the leftmost letter of $\mathbf{i} \equiv \bar{\jmath}\omega y\mathbf{b}$, as we did in the proof of Claim C. Equating the two ways of calculating this root we have

$$\begin{aligned} \alpha_j &= s_{\mathbf{b}}^{-1} s_y s_{\omega}^{-1}(\alpha_{\bar{\jmath}}) \\ &= s_{\mathbf{b}}^{-1} s_y s_x s_{\omega'}^{-1}(\alpha_{\bar{\jmath}}) \text{ using (22)} \\ &= s_{\mathbf{b}}^{-1} s_y s_x(\alpha_y) \text{ by our supposition that } s_{\omega'}^{-1}(\alpha_{\bar{\jmath}}) = \alpha_y \\ &= s_{\mathbf{b}}^{-1} s_x(\alpha_y), \end{aligned}$$

because s_y fixes $s_x(\alpha_y) = 2\alpha_x + \alpha_y$. It follows at once that

$$\mathbf{i}' = \mathbf{a}\, y_\beta\, x\, y_{\alpha_j}\, x\, \mathbf{b},$$

which, as usual, contradicts α_j being a simple root. This proves Claim D.

By Claim D we have

$$\mathbf{a}x \equiv \bar{\jmath}\omega'' \tag{23}$$

for some ω'', and substituting this braid-equivalence into (21) we obtain

$$\omega'' j \equiv \omega'. \tag{24}$$

Another consequence of (23) is that

$$\bar{\jmath}\mathbf{a} \equiv \omega'' x, \tag{25}$$

because $\bar{\jmath}\mathbf{a}$ is reduced (by inspection of (20)).

We turn now to proving that $yx\mathbf{b}j$ and $x\mathbf{b}j$ are nonreduced. First, if we look at our Case II and Subcase II(ii) hypotheses side-by-side,

$$\mathbf{a}xyx \equiv \bar{\jmath}\omega \text{ and } xyx\mathbf{b} \equiv \sigma' j,$$

we see that these equations have a symmetrical form to one another. Now, Claims C and D used the first equation to prove that $\bar{\jmath}\mathbf{a}xy$ and $\bar{\jmath}\mathbf{a}x$ are nonreduced. Symmetrically, we can use the second equation to prove that $yx\mathbf{b}j$ and $x\mathbf{b}j$ are nonreduced. We will not include the details because they are entirely similar to the proofs of Claims C and D. In fact, for our purposes, we need use only the fact that $x\mathbf{b}j$ is nonreduced. So, since $x\mathbf{b}$ *is* reduced (by inspection of $\mathbf{i}'$) we therefore have $x\mathbf{b} \equiv \sigma'' j$ for some σ''. From this we can say that

$$\mathbf{b}j \equiv x\sigma'' \tag{30}$$

because inspection of (19) shows $\mathbf{b}j$ to be reduced.

Thus,

$$\begin{aligned} \theta(\mathbf{i}) &= [\omega y_\gamma \mathbf{b} j]_\gamma \text{ from (19)} \\ &= [\omega' x y_\gamma \mathbf{b} j]_\gamma \text{ by (22)} \\ &= [\omega'' y x y_\gamma \mathbf{b} j]_\gamma \text{ by (24)} \\ &= [\omega'' y x y_\gamma x \sigma'']_\gamma \text{ by (30)} \\ &= [\omega'' x y_\gamma x y \sigma'']_\gamma \text{ applying a 4-braid} \\ &= [\bar{\jmath}\mathbf{a} y_\gamma x y \sigma'']_\gamma \text{ by (25)} \\ &= \theta(\mathbf{i}'), \end{aligned}$$

by comparison with (20), since the braid-equivalence $xy\sigma'' \equiv \sigma'$ is forced. The proof is finished. ∎

By the result just proven we can define a map

$$\Theta : \mathcal{C}/\beta \rightarrow \mathcal{C}/\gamma$$

which sends the β-component containing any $\mathbf{i} \in \mathcal{R}$ to $\theta(\mathbf{i})$. Note that, since $s_j(\gamma) = \beta$, we can define a map

$$\Psi : \mathcal{C}/\gamma \rightarrow \mathcal{C}/\beta$$

in exactly the same way as Θ, except with the rôles of β and γ reversed.

4.3.3 LEMMA. The maps Θ and Ψ are mutual inverses.

PROOF. For reasons of symmetry we will only prove that $\Psi \circ \Theta$ is the identity map on $\mathcal{C}/\beta$.

Take any word $\mathbf{i} := i_n \dots i_k \dots i_1$ in $\mathcal{R}$ and suppose that β corresponds to the letter i_k, that is, $\beta = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$. We must show that $\Psi \circ \Theta$ fixes the β-component $[\mathbf{i}]_\beta$.

By 4.3.1, either $i_N \dots i_{k+1} \equiv \bar{\jmath}\omega$ for some ω, or $i_{k-1} \dots i_1 \equiv \sigma j$ for some σ. Let us suppose the former holds. We then have $\Theta([\mathbf{i}]_\beta) := [\omega\,(i_k)_\gamma\, i_{k-1} \dots i_1\, j]_\gamma$, where γ corresponds to the letter i_k. Now since $i_{k-1} \dots i_1 j$ is obviously braid-equivalent to a word ending with j, by definition Ψ sends the above γ-component to $[\bar{\jmath}\omega\,(i_k)_\beta\, i_{k-1} \dots i_1]_\beta$, which equals $[\mathbf{i}]_\beta$. Thus, $\Psi \circ \Theta$ does indeed fix $[\mathbf{i}]_\beta$.

The other case, when $i_{k-1} \dots i_1 \equiv \sigma j$, is equally trivial. ∎

4.3.4 LEMMA. The map $\Theta : \mathcal{C}/\beta \rightarrow \mathcal{C}/\gamma$ is a morphism of graphs.

PROOF. A pair of adjacent β-components must contain a pair of adjacent words, so suppose that $\mathbf{i}$ and $\mathbf{i}' \in \mathcal{R}$ differ by a single braid and belong to different β-components. Since this braid must be of odd length, and since we are not concerned with types H_3 and H_4, this braid is a 3-braid. We must show that $\theta(\mathbf{i})$ and $\theta(\mathbf{i}')$ belong to adjacent γ-components; (they cannot belong to the *same* γ-component because Θ is bijective).

Write $\mathbf{i} = \mathbf{a}xyx\mathbf{b}$ and $\mathbf{i}' = \mathbf{a}yxy\mathbf{b}$, where $xyx \overset{3}{\leftrightarrow} yxy$ is a 3-braid. Certainly, β must be involved in this 3-braid; β can correspond to any of the three letters of xyx in $\mathbf{i}$. However, we need only consider two essentially different cases: β corresponds either to the letter y in the middle, or the *first* x. Accordingly, the proof has two main parts.

Part one: $\mathbf{i} = \mathbf{a}\,x\,y_\beta\,x\,\mathbf{b}$ and $\mathbf{i}' = \mathbf{a}\,y\,x_\beta\,y\,\mathbf{b}$. That is, β corresponds to the middle letters of the 3-braid configurations.

Considering the definition of $\theta(\mathbf{i})$, either $\mathbf{a}x \equiv \bar{\jmath}\omega$ for some ω, or $x\mathbf{b} \equiv \sigma j$ for some σ. Here we will consider only $\mathbf{a}x \equiv \bar{\jmath}\omega$ because the other case is symmetrical. So, we have

$$\theta(\mathbf{i}) = [\omega\, y_\gamma\, x\,\mathbf{b}\, j]_\gamma. \tag{1}$$

Considering now $\theta(\mathbf{i}')$, either $\mathbf{a}y \equiv \bar{\jmath}\omega'$ for some ω', or $y\mathbf{b} \equiv \sigma' j$ for some σ'.

Subcase (i): $\mathbf{a}y \equiv \bar{\jmath}\omega'$, so that

$$\theta(\mathbf{i}') = [\omega'\, x_\gamma\, y\,\mathbf{b}\, j]_\gamma. \tag{2}$$

We claim that ωx is nonreduced; we have

$$\begin{aligned} s_\omega(\alpha_x) &= s_{\bar{\jmath}} s_{\mathbf{a}} s_x(\alpha_x) \text{ using } \mathbf{a}x \equiv \bar{\jmath}\omega \\ &= -s_{\omega'} s_y(\alpha_x) \text{ using } \mathbf{a}y \equiv \bar{\jmath}\omega' \\ &= -s_{\omega'} s_x(\alpha_y) \text{ because } s_x(\alpha_y) = s_y(\alpha_x) = \alpha_x + \alpha_y. \end{aligned}$$

Now, $s_{\omega'} s_x(\alpha_y)$ is a positive root because upon substituting $\mathbf{a}y \equiv \bar{\jmath}\omega'$ into our expression $\mathbf{a}yxy\mathbf{b}$ for $\mathbf{i}'$, we see that $\omega' xy$ is reduced. Thus, $s_\omega(\alpha_x)$ is a negative root, showing ωx to be nonreduced, as claimed.

We therefore have $\omega \equiv \omega'' x$ for some ω''. So,

$$\begin{aligned} \theta(\mathbf{i}) &= [\omega y_\gamma x \mathbf{b} j]_\gamma \text{ by (1)} \\ &= [\omega'' x y_\gamma x \mathbf{b} j]_\gamma \text{ using } \omega \equiv \omega'' x \\ &\leftrightarrow [\omega'' y x_\gamma y \mathbf{b} j]_\gamma \text{ by applying a 3-braid involving } \gamma, \end{aligned}$$

where we have introduced the shorthand notation $\leftrightarrow$ to mean 'is adjacent to' (in the graph $\mathcal{C}/\gamma$). The 3-braid changes the γ-value from y to x. The last expression equals $\theta(\mathbf{i}')$, for by comparison with (2), the braid-equivalence $\omega''y \equiv \omega'$ is forced. Thus, $\theta(\mathbf{i})$ is adjacent to $\theta(\mathbf{i}')$ in $\mathcal{C}/\gamma$, as we wished to show.

Subcase (ii): $y\mathbf{b} \equiv \sigma' j$, so that

$$\theta(\mathbf{i}') = [\overline{\jmath}\,\mathbf{a}\,y\,x_\gamma\,\sigma']_\gamma. \tag{3}$$

From $y\mathbf{b} \equiv \sigma' j$ we can obtain $\mathbf{b}j \equiv y\sigma'$ because inspection of (1) shows that $\mathbf{b}j$ is reduced. Therefore

$$\begin{aligned} \theta(\mathbf{i}) &= [\omega y_\gamma x \mathbf{b} j]_\gamma \text{ by (1)} \\ &= [\omega y_\gamma x y \sigma']_\gamma \text{ using } \mathbf{j} \equiv y\sigma' \\ &\leftrightarrow [\omega x y x_\gamma \sigma']_\gamma \text{ applying a 3-braid,} \end{aligned}$$

for comparison with (3) shows that we necessarily have $\omega x \equiv \overline{\jmath}\mathbf{a}$. Thus $\theta(\mathbf{i})$ is adjacent to $\theta(\mathbf{i}')$, completing the first part of the proof.

Part two: $\mathbf{i} = \mathbf{a}\,x_\beta\,y\,x\,\mathbf{b}$ and $\mathbf{i}' = \mathbf{a}\,y\,x\,y_\beta\,\mathbf{b}$. That is, β corresponds to the first x in the 3-braid configuration in $\mathbf{i}$.

Considering the definition of $\theta(\mathbf{i})$, either $\mathbf{a} \equiv \overline{\jmath}\omega$ for some ω, or $yx\mathbf{b} \equiv \sigma j$ for some σ.

Case I: $\mathbf{a} \equiv \overline{\jmath}\omega$. We then have $\theta(\mathbf{i}) = [\omega\,x_\gamma\,y\,x\,\mathbf{b}\,j]_\gamma$ and $\theta(\mathbf{i}') = [\omega\,y\,x\,y_\gamma\,\mathbf{b}\,j]$, which are visibly adjacent γ-components.

Case II: $yx\mathbf{b} \equiv \sigma j$, so that

$$\theta(\mathbf{i}) = [\overline{\jmath}\,\mathbf{a}\,x_\gamma\,\sigma]_\gamma. \tag{4}$$

Considering now $\theta(\mathbf{i}')$, either $\mathbf{b} \equiv \sigma' j$ for some σ', or $\mathbf{a}yx \equiv \overline{\jmath}\omega'$ for some ω'.

Subcase II(i): $\mathbf{b} \equiv \sigma' j$. We then have, by definition,

$$\begin{aligned} \theta(\mathbf{i}') &= [\overline{\jmath}\mathbf{a}yxy_\gamma\sigma']_\gamma \\ &\leftrightarrow [\overline{\jmath}\mathbf{a}x_\gamma yx\sigma']_\gamma \text{ by applying a 3-braid} \\ &= \theta(\mathbf{i}), \end{aligned}$$

since (4) shows that the braid-equivalence $yx\sigma' \equiv \sigma$ is forced. We come now to the final Subcase.

Subcase II(ii): $\mathbf{a}yx \equiv \overline{\jmath}\omega'$, so that

$$\theta(\mathbf{i}') = [\omega' y_\gamma \mathbf{b} j]_\gamma. \tag{5}$$

We first claim that $y\sigma$ is nonreduced. Using our Case II hypothesis that $yx\mathbf{b} \equiv \sigma j$ to obtain an expression for s_σ^{-1}, we have

$$s_\sigma^{-1}(\alpha_y) = -s_j \underbrace{s_{\mathbf{b}}^{-1} s_x(\alpha_y)}_{\in\Phi^+},$$

where the braced term lies in Φ^+ because $yx\mathbf{b}$ is reduced (by inspection of $\mathbf{i}$). So, $y\sigma$ is reduced if and only if $s_{\mathbf{b}}^{-1} s_x(\alpha_y) = \alpha_j$. For a contradiction, suppose this is so. Then we have

$$\mathbf{i} = \mathbf{a}\,x_\beta\,y_{\alpha_j}\,x\,\mathbf{b},$$

which is absurd, because no simple root can correspond to the middle letter in a 3-braid configuration. This proves that $y\sigma$ is nonreduced and hence that

$$\sigma \equiv y\sigma'' \tag{6}$$

for some σ''.

Next, we claim that $\omega' x$ is nonreduced. This is proved in an entirely symmetrical way to the proof just above, this time making use of the Subcase II(ii) hypothesis to compute $s_{\omega'}(\alpha_x)$. We omit the details. We therefore have, by analogy with (6), the braid-equivalence

$$\omega' \equiv \omega'' x \tag{7}$$

for some ω''. Substituting this into the Subcase II(ii) hypothesis that $\mathbf{a}yx \equiv \bar{\jmath}\omega'$ yields $\mathbf{a}y \equiv \bar{\jmath}\omega''$, from which we obtain

$$\bar{\jmath}\mathbf{a} \equiv \omega'' y, \tag{8}$$

because inspection of (4) shows that $\bar{\jmath}\mathbf{a}$ is reduced. Putting all the pieces together gives

$$\begin{aligned}
\theta(\mathbf{i}) &= [\bar{\jmath}\mathbf{a}x_\gamma\sigma]_\gamma \text{ by (4)}\\
&= [\bar{\jmath}\mathbf{a}x_\gamma y\sigma'']_\gamma \text{ by (6)}\\
&= [\omega'' yx_\gamma y\sigma'']_\gamma \text{ by (8)}\\
&\leftrightarrow [\omega'' xy_\gamma x\sigma'']_\gamma \text{ applying a 3-braid}\\
&= [\omega' y_\gamma x\sigma'']_\gamma \text{ by (7)}\\
&= \theta(\mathbf{i}'),
\end{aligned}$$

for by comparison with (5), the braid-equivalence $x\sigma'' \equiv \mathbf{b}j$ is forced. This completes the proof. ∎

Before we can prove the main result of this chapter we need one more observation, which is sure to be well known, but we provide a proof for completeness.

4.3.5 LEMMA. Let β and γ be any two positive roots which lie in the same $\mathcal{W}$-orbit, that is, $\gamma = w(\beta)$ for some $w \in \mathcal{W}$. Then there is a sequence $(\alpha^0, \alpha^1, \ldots, \alpha^n)$ of positive roots such that $\alpha^0 = \beta$, $\alpha^n = \gamma$, and α^i is obtained from α^{i-1} by the application of some simple reflection for $i = 1, 2, \ldots, n$.

PROOF. This can be achieved if we allow each α^i to be a positive *or* negative root, simply by considering any reduced word for w. Choose such a sequence $(\alpha^0, \ldots, \alpha^n)$ for the least possible n. We claim that such a minimum-length sequence necessarily lies in Φ^+. For a contradiction, suppose that some interval $(\alpha^i, \alpha^{i+1}, \ldots, \alpha^{i+r})$ of this sequence consists of negative roots only. We will further assume that i is minimal and, with this i fixed, that r is maximal, so that α^{i-1} and α^{i+r+1} are positive roots.

Since the only positive root sent to some negative root by a simple reflection is a simple root, it follows that α^{i-1} and α^{i+r+1} are simple roots and that $\alpha^{i-1} = -\alpha^i$, $\alpha^{i+r+1} = -\alpha^{i+r}$. If now we delete α^{i-1} and α^{i+r+1} from our original sequence and change the signs of the intermediate negative roots $(\alpha^i, \ldots, \alpha^{i+r})$, we visibly obtain a *shorter* valid sequence from β to γ, contradicting the minimality of n. ∎

We will now summarise our calculations in the following result, which holds for types A_l, B_l, D_l, E_l ($l = 6, 7, 8$) and F_4.

Note that we can view $\mathcal{C}/\beta$ as a **labelled** graph, where each vertex is labelled with the corresponding β-value.

4.3.6 THEOREM. Let β and γ be two positive roots which lie in the same $\mathcal{W}$-orbit. Then there is a label-preserving isomorphism $\mathcal{C}/\beta \cong \mathcal{C}/\gamma$ of quotient graphs of root components.

PROOF. By 4.3.5 we may assume that $\gamma = s_j(\beta)$ for some s_j. By 4.3.3, Θ is bijective. By 4.3.4, Θ is a morphism, and by symmetry so is Θ^{-1}. Thus, Θ is a graph isomorphism, which is visibly label-preserving from the way that θ is defined. ∎

The remainder of this thesis is concerned with root components in types A_l, B_l and D_l. Because of Theorem 4.3.6 it is enough to understand the root components for just one root in each $\mathcal{W}$-orbit.

The root systems of type A_l and D_l have only one orbit. Type B_l has two orbits – one consisting of long roots, the other, short roots. But as remarked before, if β is a *short* root then the β-value of any $\mathbf{i} \in \mathcal{R}(B_l)$ equals 1. Therefore, we need not concern ourselves any further with this trivial case.

The highest root α_0 is a long root, and as might be expected, the α_0-components have favourable properties. Chapters 5 and 6 will be mainly concerned with understanding the α_0-components in types A_l and B_l. In Chapter 7 we will deduce information about the α_0-components in type D_l from those of type B_l.

5. Root Components in Type A_l

5.1 Introduction

In this chapter we will calculate the graph of β-components $\mathcal{C}(A_l)/\beta$ for all $\beta \in \Phi^+(A_l)$ in two different ways.

The first way is by induction on the rank, l. The main part of the inductive step is to calculate the graph structure whenever β does not have α_1 as a summand. To then deduce the structure for general β, we have a choice of two methods. One method is simply to invoke Theorem 4.3.6. The other method is to define a certain graph automorphism, φ, of $\mathcal{C}(A_l)$, which induces a bijection, φ_0, of $\Phi^+(A_l)$ in a natural way. It happens that φ provides a bijection between the β-components and the $\varphi_0(\beta)$-components. This allows the inductive step to be completed.

The second method of proof considers only α_0-components, where $\alpha_0 = \alpha_1 + \cdots + \alpha_l$. We show, by explicit manipulations of reduced words, that each α_0-component contains a unique quiver-compatible commutation class. This tells us that there are exactly 2^{l-1} α_0-components. A key step in the method is to show how the α_0-components induce a partition of the Coxeter group $\mathcal{W}(A_{l-1})$ of rank $l-1$. This enables us to calculate the *edges* of $\mathcal{C}(A_l)/\alpha_0$.

The following two sections collect some observations on graph automorphisms of $\mathcal{C}(A_l)$, in preparation for the inductive proof, which is completed in section 5.4. The remainder of the chapter concerns the second proof.

5.2 Graph Automorphisms of $\mathcal{C}$

From now on we will usually suppress the A_l in our notation. Let $C \in \mathcal{C}$ have normal representative $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$. Applying the promotion operator ∂ to this word $\ell(\boldsymbol{\varrho})$ times, we obtain $(\mathbf{l{+}1}-\boldsymbol{\varrho})\ \boldsymbol{\lambda}(l \searrow 1)$, where we have temporarily employed the notation $\mathbf{l+1}-\boldsymbol{\varrho}$ to mean the word obtained from $\boldsymbol{\varrho}$ by replacing each letter i by $l+1-i$. Next, apply the opposition involution of A_{l-1} to the word $(\mathbf{l{+}1}-\boldsymbol{\varrho})\ \boldsymbol{\lambda}$ in $\mathcal{R}(A_{l-1})$, giving rise to $(\boldsymbol{\varrho}-\mathbf{1})\ (\mathbf{l}-\boldsymbol{\lambda})\ (l \searrow 1)$ in $\mathcal{R}(A_l)$. Now apply ∂^{-1} to this word $\ell(\boldsymbol{\varrho})$ times, giving $\mathbf{l}-\boldsymbol{\lambda}\ (l \searrow 1)\ \mathbf{l{+}2}-\boldsymbol{\varrho}$. Finally, apply the opposition involution of A_l to obtain $\boldsymbol{\lambda}+\mathbf{1}\ (1 \nearrow l)\ \boldsymbol{\varrho}-\mathbf{1}$, which we traditionally write as $\boldsymbol{\lambda}^+(1 \nearrow l)\boldsymbol{\varrho}^-$. Accordingly, we define a map as follows.

$$\begin{aligned} \varphi : \mathcal{C} &\rightarrow \mathcal{C} \\ [\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}] &\mapsto [\boldsymbol{\lambda}^+(1 \nearrow l)\boldsymbol{\varrho}^-]. \end{aligned}$$

Clearly, φ is well defined since $\boldsymbol{\lambda}$ and $\boldsymbol{\varrho}$ are determined up to commutations. Any commutation class may be expressed as $[\mathbf{a}(1 \nearrow l)\mathbf{b}]$, where $1 \notin \operatorname{supp}(\mathbf{a})$, $l \notin \operatorname{supp}(\mathbf{b})$ and $\mathbf{a}$, $\mathbf{b}$ are well defined up to commutations. Thus φ is bijective, its inverse being $[\mathbf{a}(1 \nearrow l)\mathbf{b}] \mapsto [\mathbf{a}^-(l \searrow 1)\mathbf{b}^+]$. It is easily checked that φ sends adjacent vertices to adjacent vertices, and that the same is true of φ^{-1}. Hence, φ is an automorphism of $\mathcal{C}$.

Recall from 1.4.7 the involution $C \mapsto C^*$ of $\mathcal{C}$ which is the composite of the opposition involution and reversal. We will show that the automorphism φ^{l+1} is this involution. The proof requires several steps.

Firstly, for $1 \le m \le l+1$, we define the subset Φ^+_m of Φ^+ by

$$\Phi^+_m := \underbrace{\{\alpha_1 + \cdots + \alpha_{m-1}, \alpha_2 + \cdots + \alpha_{m-1}, \ldots, \alpha_{m-1}\}}_{\text{empty if } m=1} \cup \underbrace{\{\alpha_m, \alpha_m + \alpha_{m+1}, \ldots, \alpha_m + \cdots + \alpha_l\}}_{\text{empty if } m=l+1}.$$

5.2.1 Lemma. The partial order on Φ^+ induced by any $C \in \mathcal{C}$ induces a total order on each Φ^+_m.

Proof. We must show that any pair of commutation-equivalent words in $\mathcal{R}$ induce the same total order on each Φ^+_m. Consider any pair of reduced longest words which differ by a single commutation, say

$$\mathbf{a}ij\mathbf{b} \text{ and } \mathbf{a}ji\mathbf{b},$$

where $|i-j| > 1$. Set $w := s_{\mathbf{b}}^{-1}$. The corresponding total orders on Φ^+ differ from one another only in that $w(\alpha_j)$ and $w(\alpha_i)$ are transposed. These transposed positive roots are

$$\alpha_{w(j)} + \alpha_{w(j)+1} + \cdots + \alpha_{w(j+1)-1} \text{ and } \alpha_{w(i)} + \alpha_{w(i)+1} + \cdots + \alpha_{w(i+1)-1},$$

because $w(\alpha_i) = w(e_i - e_{i+1}) = e_{w(i)} - e_{w(i+1)}$, where we are viewing w as an element of the symmetric group on $\{1, \dots, l+1\}$.

Since $i \neq j$, these roots do not have the same simple root as the first nor last summand. Also, since $|i-j| > 1$, the sum of these two roots is not a root. Consequently, these roots cannot both belong to Φ_m^+, for any m. ∎

5.2.2 LEMMA. Let $C \in \mathcal{C}$ and let α, β be distinct positive roots which are comparable with respect to $\leq_C$, say $\alpha \leq_C \beta$. Then there exist positive roots $\gamma_1, \dots \gamma_r$ such that

$$\alpha = \gamma_1 \leq_C \gamma_2 \leq_C \cdots \leq_C \gamma_r = \beta$$

and such that each pair $\{\gamma_k, \gamma_{k+1}\}$ of consecutive roots is a subset of some Φ_m^+, where m depends on k. That is, the partial order arising from C is completely determined by the induced total orders on the Φ_m^+.

PROOF. We may assume that β covers α with respect to $\leq_C$. We must show that α and β belong to a common Φ_m^+. By 1.4.2 (b), there exists some $\mathbf{i} \in C$ such that β covers α with respect to $<_{\mathbf{i}}$. Thus, we may write $\mathbf{i} = \mathbf{a}ij\mathbf{b}$ in such a way that $\alpha = w(\alpha_j)$ and $\beta = ws_j(\alpha_i)$, where $w := s_{\mathbf{b}}^{-1}$. Now, since α and β are comparable and consecutive with respect to $\leq_C$, we have $|i-j| = 1$. If $j = i+1$ then

$$\alpha = \alpha_{w(i+1)} + \alpha_{w(i+1)+1} + \cdots + \alpha_{w(i+2)-1}$$

and

$$\beta = \alpha_{w(i)} + \alpha_{w(i)+1} + \cdots + \alpha_{w(i+2)-1}.$$

Since α and β visibly have the same rightmost summand, they both belong to $\Phi_{w(i+2)}^+$. The case $j = i-1$ is similar. ∎

Let $\varphi_0 : \Phi^+ \to \Phi^+$ be the map defined by

$$\begin{array}{rcll} \alpha_1 + \alpha_2 + \cdots + \alpha_r & \mapsto & \alpha_r + \alpha_{r+1} + \cdots + \alpha_l & \text{if } r \geq 1, \\ \alpha_t + \alpha_{t+1} + \cdots + \alpha_r & \mapsto & \alpha_{t-1} + \alpha_t + \cdots + \alpha_{r-1} & \text{if } r \geq t > 1. \end{array}$$

For example, if $l = 3$ then φ_0 has the following two orbits.

$$\alpha_1 + \alpha_2 + \alpha_3 \mapsto \alpha_3 \mapsto \alpha_2 \mapsto \alpha_1 \mapsto \alpha_1 + \alpha_2 + \alpha_3$$

$$\alpha_1 + \alpha_2 \mapsto \alpha_2 + \alpha_3 \mapsto \alpha_1 + \alpha_2$$

If $l > 1$ it is clear that φ_0 has order $l+1$; in particular, φ_0 is bijective. Also note that $\varphi_0(\Phi_m^+) = \Phi_{m-1}^+$, taking the subscripts modulo $l+1$. We can now prove the main result of this section.

5.2.3 PROPOSITION. The automorphism φ^{l+1} of $\mathcal{C}$ is the involution $C \mapsto C^*$.

PROOF. Let $C \in \mathcal{C}$. We will investigate how the partial orders on Φ^+ induced by C and $\varphi(C)$ differ from one another.

Write $C = [\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}]$ and $\boldsymbol{\lambda} = \lambda_m \dots \lambda_1$, $\boldsymbol{\varrho} = \varrho_n \dots \varrho_1$. We will show how the letter corresponding to any $\beta \in \Phi^+$ (with respect to $\leq_C$) is related to the letter corresponding to $\varphi_0(\beta)$ (with respect to $\leq_{\varphi(C)}$).

Recall that $\varphi(C) = [\boldsymbol{\lambda}^+(1 \nearrow l)\boldsymbol{\varrho}^-]$. There are three cases to consider depending upon whether, in the expression for C above, β corresponds to a letter in $\boldsymbol{\varrho}$, $\boldsymbol{\lambda}$ or $(l \searrow 1)$.

(1) Suppose that $\beta = s_{\varrho_1} \dots s_{\varrho_{k-1}}(\alpha_{\varrho_k})$, for some k. By 2.2.4, α_1 is not a summand of β. Define

$$\gamma := s_{\varrho_1^-} \dots s_{\varrho_{k-1}^-}(\alpha_{\varrho_k^-}).$$

Clearly, γ may be obtained from β by lowering the suffix of each simple root that occurs in β by 1. By definition of φ_0 we therefore have $\gamma = \varphi_0(\beta)$.

(2) Suppose that $\beta = s_{\boldsymbol{\varrho}}^{-1} s_1 \dots s_l s_{\lambda_1} \dots s_{\lambda_{k-1}}(\alpha_{\lambda_k})$, for some k. Since β corresponds to a letter in $\boldsymbol{\lambda}$, by 2.2.4 α_1 is not a summand. Considering $\varphi(C)$, define the positive root

$$\gamma := s_{\boldsymbol{\varrho}^-}^{-1} s_l \dots s_1 s_{\lambda_1^+} \dots s_{\lambda_{k-1}^+}(\alpha_{\lambda_k^+});$$

we claim that $\gamma = \varphi_0(\beta)$. Well, $(l \searrow 1)\lambda_1^+ \dots \lambda_{k-1}^+$ is braid-equivalent to $\lambda_1 \dots \lambda_{k-1}(l \searrow 1)$, hence

$$\gamma = s_{\varrho^-}^{-1} s_{\lambda_1} \dots s_{\lambda_{k-1}}(\alpha_{\lambda_k}),$$

using the fact that $s_l \dots s_1$ sends $\alpha_{\lambda_k^+}$ to α_{λ_k}. Now, $(\varrho^-)^{\mathrm{rev}}(1 \nearrow l)$ is braid-equivalent to $(1 \nearrow l)\varrho^{\mathrm{rev}}$, hence

$$\gamma = s_l \dots s_1 \, s_{\varrho}^{-1} \, s_1 \dots s_l \, s_{\lambda_1} \dots s_{\lambda_{k-1}}(\alpha_{\lambda_k}),$$

which equals $s_l \dots s_1(\beta)$. Again, this is obtained from β by decreasing the suffix of each simple root summand by 1, and so γ equals $\varphi_0(\beta)$, as claimed.

(3) Suppose now that β corresponds to some letter in the subword $(l \searrow 1)$. Thus, $\beta = s_{\varrho}^{-1} s_1 s_2 \dots s_{k-1}(\alpha_k)$, for some k. This time, 2.2.4 tells us that α_1 *is* a summand of β. Considering $\varphi(C)$, define the positive root

$$\gamma := s_{\varrho^-}^{-1} s_l s_{l-1} \dots s_{k+1}(\alpha_k);$$

we claim that $\gamma = \varphi_0(\beta)$. First, write γ in the form

$$\gamma = -s_{\varrho^-}^{-1}(s_l \dots s_1)s_1 s_2 \dots s_{k-1}(\alpha_k).$$

Recalling that $(\varrho^-)^{\mathrm{rev}}(l \searrow 1)$ is braid-equivalent to $(l \searrow 1)\varrho^{\mathrm{rev}}$, we obtain $\gamma = -s_l \dots s_1(\beta)$. If we now write $\beta = \alpha_1 + \alpha_2 + \dots + \alpha_r$ (for some r) it can be easily checked that γ equals $\varphi_0(\beta)$, as claimed.

We can now establish the relationship between the partial orders induced by C and $\varphi(C)$. Let α, β be any two distinct roots in some Φ_m^+. Since C induces a *total order* on Φ_m^+, we may assume that $\alpha \leq_C \beta$. Since $\varphi_0(\Phi_m^+) = \Phi_{m-1}^+$, certainly $\varphi_0(\alpha)$ and $\varphi_0(\beta)$ are comparable with respect to $\leq_{\varphi(C)}$, too; we wish to know in which order these two roots appear. It suffices to know whether $\varphi_0(\alpha)$ precedes, or is preceded by $\varphi_0(\beta)$ with respect to $<_{\mathbf{j}}$, for any $\mathbf{j} \in \varphi(C)$. This can be easily deduced from our three calculations above, as follows.

The calculations show that $\varphi_0(\alpha) \leq_{\varphi(C)} \varphi_0(\beta)$ whenever not both of α and β have α_1 as a summand. This condition is equivalent to saying that α and β do not belong to Φ_1^+, that is, $m \neq 1$. Otherwise, if α, β *both* have α_1 as a summand, and hence belong to Φ_1^+, then case (2) shows that $\varphi_0(\beta) \leq_{\varphi(C)} \varphi_0(\alpha)$.

To summarise these facts, write (Φ_m^+, C) for the totally ordered set Φ_m^+ with respect to $\leq_C$, say

$$(\Phi_m^+, C) = \beta_{(1)} < \beta_{(2)} < \dots < \beta_{(l)},$$

where the $\beta_{(i)}$ denote the roots in Φ_m^+. Using this concrete expression, we define the two related total orders

$$\varphi_0(\Phi_m^+, C) := \varphi_0(\beta_{(1)}) < \varphi_0(\beta_{(2)}) < \dots < \varphi_0(\beta_{(l)})$$

and

$$-(\Phi_m^+, C) := \beta_{(1)} > \beta_{(2)} > \dots > \beta_{(l)},$$

obtained from the original total order by pointwise application of φ_0 and reversal, respectively.

We have thus shown that

$$(\Phi_{m-1}^+, \varphi(C)) = \varphi_0(\Phi_m^+, C) \quad \text{whenever } m \neq 1,$$

and

$$(\Phi_{l+1}^+, \varphi(C)) = -\varphi_0(\Phi_1^+, C).$$

Using these two equations we have the following easy calculation, valid for any m and any C.

$$\begin{aligned}
(\Phi_m^+, \varphi^{l+1}(C)) &= (\varphi_0)^{l+1-m}(\Phi_{l+1}^+, \varphi^m(C)) \\
&= -(\varphi_0)^{l+2-m}(\Phi_1^+, \varphi^{m-1}(C)) \\
&= -(\varphi_0)^{l+1}(\Phi_m^+, C) \\
&= -(\Phi_m^+, C),
\end{aligned}$$

the last line due to the fact that φ_0 has order $l+1$. Thus $\varphi^{l+1}(C)$ induces the *opposite* total order on each Φ_m^+ to that induced by C.

Consequently, by 5.2.2, we see that $\varphi^{l+1}(C)$ induces the opposite partial order on all of Φ^+ to that of C. However, C^* also has this property (see 1.4.7), so by 1.4.3 we have $\varphi^{l+1}(C) = C^*$, as we wished to show. ∎

5.3 Action on Root Components

As an easy example of how graph automorphisms act on root components, we have the following, which is true generally, not just for type A_l. (Recall that $\bar{\beta}$ is the effect of the opposition involution on Φ^+ and that rev indicates reversal of words.)

5.3.1 Proposition. For all $\beta \in \Phi^+$ and all $C \in \mathcal{C}$ we have

$$\mathrm{rev}(\text{the } \beta\text{-component of } C) = \text{the } \bar{\beta}\text{-component of } C^{\mathrm{rev}}$$

and

$$\text{the } \beta\text{-value of } C = \text{the } \bar{\beta}\text{-value of } C^{\mathrm{rev}}.$$

Proof. This is easily checked, noting that if $i_N \ldots i_1$ is a representative of C and if $\beta = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$ then we have $\bar{\beta} = s_{i_N} \ldots s_{i_{k+1}}(\alpha_{i_k})$. ▎

The following action will be used at the last step of our inductive calculation of $\mathcal{C}/\beta$.

5.3.2 Proposition. For all $\beta \in \Phi^+$ and all $C \in \mathcal{C}$ we have

$$\varphi(\text{the } \beta\text{-component of } C) = \text{the } \varphi_0(\beta)\text{-component of } \varphi(C).$$

Proof. We must show that any two commutation classes C, C' are β-equivalent to one another if and only if $\varphi(C)$ is $\varphi_0(\beta)$-equivalent to $\varphi(C')$.

Forward implication. We may assume that C is adjacent to C'. Write $C = [\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}]$ and $\varphi(C) = [\boldsymbol{\lambda}^+(1 \nearrow l)\boldsymbol{\varrho}^-]$.

Considering C, first suppose that β corresponds to some letter in the subword $(l \searrow 1)$. From part (3) of the proof of 5.2.3 we can surmise that the β-value of C equals the $\varphi_0(\beta)$-value of $\varphi(C)$. A similar equality holds for C'. Thus, since the β-values of C and C' agree (by assumption), it follows that the $\varphi_0(\beta)$-values of $\varphi(C)$ and $\varphi(C')$ agree also. So $\varphi(C)$ is $\varphi_0(\beta)$-equivalent to $\varphi(C')$, as required.

Now suppose that β corresponds to some letter in $\boldsymbol{\varrho}$ in the expression for C. This time, part (1) of the proof of 5.2.3 shows that the β-value of C equals the $\varphi_0(\beta)$-value of $\varphi(C)$ plus 1. A similar equality holds for C', and we argue exactly as above.

Finally, if β corresponds to some letter in $\boldsymbol{\lambda}$ in the expression for C, part (2) of the proof of 5.2.3 shows that we can simply replace 'plus 1' with 'minus 1' in the above.

Backward implication. Suppose that $\varphi(C')$ is $\varphi_0(\beta)$-equivalent to $\varphi(C)$, for some C and C'. Applying the forward implication $2l+1$ times, we deduce that $\varphi^{2l+2}(C')$ is $\varphi_0^{2l+2}(\beta)$-equivalent to $\varphi^{2l+2}(C)$. This simply says that C' is β-equivalent to C, since, by 5.2.3, the order of φ divides $2(l+1)$, and we know that the order of φ_0 is $l+1$. ▎

From the proof of 5.3.2 we separate out the following observation.

5.3.3 Lemma. If $\beta \in \Phi^+$ has α_1 as a summand then

$$\text{the } \beta\text{-value of } C = \text{the } \varphi_0(\beta)\text{-value of } \varphi(C)$$

for all $C \in \mathcal{C}$. ▎

As a consequence of our calculations so far, we can compute the order of φ, although we will not use this result in the sequel.

5.3.4 Proposition. If $l \geq 3$ then the order of φ is $2(l+1)$.

Proof. Let the order of φ be n. By 5.2.3, we know that φ^{l+1} is an automorphism of order 2. It will follow that $n = 2(l+1)$ if we can show that n is divisible by $l+1$. For any $C \in \mathcal{C}$, we have, by 5.3.2,

$$\text{the } \alpha_1\text{-component of } C = \text{the } \varphi_0^n(\alpha_1)\text{-component of } C. \qquad (\dagger)$$

Now, set

$$C := [\underline{1}(2 \searrow \underline{1})(3 \searrow \underline{1}) \dots (l-1 \searrow \underline{1})(\underline{l} \searrow \underline{1})].$$

Put $\gamma := \varphi_0^n(\alpha_1)$, which belongs to $\{\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_l\}$, where $\alpha_0 = \alpha_1 + \dots + \alpha_l$. Clearly, C has α_0-value l and α_i-value 1 whenever $i \neq 0$. (The corresponding letters are underlined in the above expression.)

Consider first the possibility that $\gamma = \alpha_0$. Define

$$C' := [\underbrace{1(2 \searrow 1) \dots (l-2 \searrow 1)}(l-1 \searrow 2)(\underline{l} \searrow 3)\underline{2}12],$$

which is adjacent to C. The α_1-value of C' is 2 (underlined), so that C' is *not* α_1-equivalent to C. However, the α_0-value of C' is l (underlined), so that C' *is* C-equivalent to C. This contradicts the assertion (†).

Now let us suppose that $\gamma = \alpha_k$ for some $k \geq 3$. This time, define

$$C' := [\underbrace{1(2 \searrow 1)(3 \searrow 1) \dots (l-k \searrow 1)}(l+1-k \searrow 2)(l+2-k \searrow 1)\underline{2}\ \underbrace{(l+3-k \searrow 1) \dots (l-1 \searrow 1)(l \searrow \underline{1})}],$$

which is adjacent to C. The α_1-value of C' equals 1 (underlined), and the α_k-value of C' equals 2 (underlined). Thus, C' is α_1-equivalent, but not γ-equivalent to C, contradicting (†).

If $\gamma = \alpha_2$, we set

$$C' := [\underbrace{1(2 \searrow 1)(3 \searrow 1) \dots (l-3 \searrow 1)}(l-2 \searrow 2)(l-1 \searrow 3)\underline{2}12(l \searrow \underline{1})],$$

which is adjacent to C. Now, C' has α_1-value equal to 1 (underlined), and C-value equal to 2 (underlined). (Note that this uses our hypothesis $l \geq 3$.) Thus, C' is α_1-equivalent, but not γ-equivalent to C, contradicting (†).

Therefore, the only remaining possibility is that $\gamma = \alpha_1$. Thus, α_1 is fixed by φ_0^n, which implies that $l+1$ divides n, as required. ∎

In [Elnitsky] a bijection between $\mathcal{C}(A_l)$ and the set of all possible tilings of a regular $2(l+1)$-gon by unit rhombi is established. In this context, φ can be viewed as rotation of tilings.

5.4 An Inductive Calculation of $\mathcal{C}/\beta$

The underlying graph of the poset of all subsets of an n-element set, ordered by inclusion, is called an **n-cube**. We will prove that $\mathcal{C}/\beta$ is isomorphic to an $(l-1)$-cube by induction on l (for all β), but first we will separate out four technical observations.

5.4.1 Lemma. Let $\mathbf{i}, \mathbf{j} \in \mathcal{R}$ and let $\beta, \gamma \in \Phi^+$ be such that $\beta + \gamma \in \Phi^+$. If $\beta <_{\mathbf{i}} \gamma$ and $\gamma <_{\mathbf{j}} \beta$ then $\mathbf{i}$ is *not* β-equivalent to $\mathbf{j}$.

Proof. Towards a contradiction, suppose that $\mathbf{i}$ *is* $\mathbf{i}$-equivalent to $\mathbf{j}$, and consider some connected path joining $\mathbf{i}$ to $\mathbf{j}$ on which β-values are constant. Now, commutations leave all root-values unchanged, as well as the relative positions of β and γ, since their sum is a root. Therefore, β must swap positions with γ at a 3-braid. However, we certainly know that when β is involved in a braid of odd length, the β-values change. This contradicts our choice of path. ∎

5.4.2 Lemma. If $\boldsymbol{\lambda}(l \searrow 1)$ and $(l \searrow 1)\boldsymbol{\varrho}$ belong to $\mathcal{R}$ then they lie in distinct α-components for all $\alpha \in \Phi^+$.

Proof. For suitable $r \leq t$ we have $\alpha \in \{\beta, \gamma\}$, where $\beta := \alpha_1 + \dots + \alpha_r$ and $\gamma := \alpha_{r+1} + \dots + \alpha_t$. We have $\beta < \gamma$ with respect to the total order induced by $\boldsymbol{\lambda}(l \searrow 1)$, because β has summand α_1 and so corresponds to a letter of $(l \searrow 1)$ (see 2.2.4).

Similarly, $\gamma < \beta$ with respect to $(l \searrow 1)\boldsymbol{\sigma}$. Since $\beta + \gamma \in \Phi^+$, we may apply 5.4.1, which is symmetrical in β and γ, to conclude that these longest words lie in distinct β- and γ-components. ∎

Define maps τ and $\hat{\tau}$ as follows.

$$\begin{aligned} \mathcal{C}(A_{l-1}) &\rightarrow \mathcal{C} \\ \tau : [\mathbf{i}] &\mapsto [\mathbf{i}(l \searrow 1)] \\ \hat{\tau} : [\mathbf{i}] &\mapsto [(l \searrow 1)\mathbf{i}^+] \end{aligned}$$

These maps are injective, by 2.2.3. It is simple to check that they are in fact graph *isomorphisms* onto their images.

Recall from 2.2.9 the map $\delta : \mathcal{C}(A_l) \rightarrow \mathcal{C}(A_{l-1})$ given by $[\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}] \mapsto [\boldsymbol{\lambda}\boldsymbol{\varrho}^-]$.

5.4.3 LEMMA. Let $C \in \mathcal{C}$ and suppose that $\beta \in \Phi^+$ does not have α_1 as a summand. Then C is β-equivalent to either $(\tau \circ \delta)(C)$ or $(\hat{\tau} \circ \delta)(C)$, but not both.

PROOF. Write $C = [\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}]$, so that $(\tau \circ \delta)(C) = [\boldsymbol{\lambda}\boldsymbol{\varrho}^-(l \searrow 1)]$ and $(\hat{\tau} \circ \delta)(C) = [(l \searrow 1)\boldsymbol{\lambda}^+\boldsymbol{\varrho}]$. By hypothesis, β must correspond to some letter of $\boldsymbol{\lambda}$ or $\boldsymbol{\varrho}$ within C. If the former, then C is clearly β-equivalent to $(\tau \circ \delta)(C)$, by moving applying $\ell(\boldsymbol{\varrho})$ 'right to left' moves. If the latter, then C is β-equivalent to $(\hat{\tau} \circ \delta)(C)$ by applying $\ell(\boldsymbol{\lambda})$ 'left to right' moves. Finally, if C is β-equivalent to *both* expressions then $\boldsymbol{\lambda}\boldsymbol{\varrho}^-(l \searrow 1)$ is β-equivalent to $(l \searrow 1)\boldsymbol{\lambda}^+\boldsymbol{\varrho}$, which contradicts 5.4.2. ∎

5.4.4 LEMMA. Let $\beta \in \Phi^+$ be such that α_1 is not a summand. Let $C, C' \in \operatorname{Im}\tau$. If C is β-equivalent to C' within $\mathcal{C}$, then C is moreover β-equivalent to C' within $\operatorname{Im}\tau$. The same is true with $\hat{\tau}$ in place of τ.

PROOF. We will only give the details for the τ map, because a similar argument works for $\hat{\tau}$.

By hypothesis, there exists some connected path $\pi \subseteq \mathcal{C}$ joining C to C' on which β-values are constant. Consider the path $(\tau \circ \delta)(\pi) \subseteq \operatorname{Im}\tau$; this is connected because τ and δ are morphisms, and it coincides with π at C and C'. (Use the fact that $C, C' \in \operatorname{Im}\tau$.) It remains to show that adjacent vertices on π map to vertices under $\tau \circ \delta$ which have the same β-value. (Refer to 2.2.1 for the notation.)

If our adjacent vertices on π are of the form $[\boldsymbol{\lambda}'i(l \searrow 1)\boldsymbol{\varrho}]$ and $[\boldsymbol{\lambda}'(l \searrow 1)i^+\boldsymbol{\varrho}]$ then these both map to $[\boldsymbol{\lambda}\boldsymbol{\varrho}^-(l \searrow 1)]$, so there is nothing to prove. The same applies to adjacent pairs such as $[\boldsymbol{\lambda}(l \searrow 1)i\boldsymbol{\varrho}']$, $[\boldsymbol{\lambda}i^-(l \searrow 1)\boldsymbol{\varrho}']$.

Now suppose that our adjacent vertices on π have the form $[\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}]$ and $[\boldsymbol{\lambda}^\#(l \searrow 1)\boldsymbol{\varrho}]$, where the hash denotes the application of a 3-braid. These map to $[\boldsymbol{\lambda}\boldsymbol{\varrho}^-(l \searrow 1)]$ and $[\boldsymbol{\lambda}^\#\boldsymbol{\varrho}^-(l \searrow 1)]$, respectively. Now, β must correspond to a letter of $\boldsymbol{\lambda}$ or $\boldsymbol{\varrho}$, because α_1 is not a summand.

If β corresponds to a letter in $\boldsymbol{\lambda}$ then the 3-braid applied to $\boldsymbol{\lambda}$ cannot involve this letter (otherwise $[\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}]$ and $[\boldsymbol{\lambda}^\#(l \searrow 1)\boldsymbol{\varrho}]$ would have different β-values). It follows at once that $[\boldsymbol{\lambda}\boldsymbol{\varrho}^-(l \searrow 1)]$ and $[\boldsymbol{\lambda}^\#\boldsymbol{\varrho}^-(l \searrow 1)]$ do indeed have the same β-value.

On the other hand, if β corresponds to some letter in $\boldsymbol{\varrho}$ then, writing $\boldsymbol{\varrho} = \varrho_m \cdots \varrho_1$, we have $\beta = s_{\varrho_1} \cdots s_{\varrho_{k-1}}(\alpha_{\varrho_k})$ for some k. Now, decreasing all the subscripts on the right hand side by one, and premultiplying by $s_1 \ldots s_l$ also gives β. This implies that $[\boldsymbol{\lambda}\boldsymbol{\varrho}^-(l \searrow 1)]$ and $[\boldsymbol{\lambda}^\#\boldsymbol{\varrho}^-(l \searrow 1)]$ have the same β-value as one another, as required.

A similar argument to the one above works for adjacent pairs of the form $[\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}]$, $[\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}^\#]$, completing the proof. ∎

We come now to our main result.

5.4.5 THEOREM. For all $\beta \in \Phi^+$ the quotient graph $\mathcal{C}/\beta$ of β-components is an $(l-1)$-cube.

PROOF. The proof is by induction on l, the result being true for $l = 1$. Let $l > 1$ and first suppose that α_1 is not a summand of β. Writing β as a sum of the α_k, define β^- by decreasing each suffix by one. We wish to know how the β-components of the subgraphs $\operatorname{Im}\tau$ and $\operatorname{Im}\hat{\tau}$ of $\mathcal{C}(A_l)$ relate to the β^--components of $\mathcal{C}(A_{l-1})$.

We claim that for all $C \in \mathcal{C}(A_{l-1})$, the β-value of $\tau(C)$ equals the β^--value of C, and that the β-value of $\hat{\tau}(C)$ equals the β^--value of C plus 1.

To verify this, let $\mathbf{i} := i_N \ldots i_1 \in C$ and let $\beta^- = s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k})$. So, the β^--value of C is i_k. Since $\tau(C) = [\mathbf{i}(l \searrow 1)]$ and $(s_1 \ldots s_l)s_{i_1} \ldots s_{i_{k-1}}(\alpha_{i_k}) = s_1 \ldots s_l(\beta^-) = \beta$, the β-value of $\tau(C)$ equals i_k too. The second part of the claim is trivial.

Therefore, since the β^--values of the vertices of $\mathcal{C}(A_{l-1})$ differ from the β-values of their images under τ and $\hat{\tau}$ by *constants*, it follows from the induction hypothesis that $(\operatorname{Im}\tau)/\beta$ and $(\operatorname{Im}\hat{\tau})/\beta$ are $(l-2)$-cubes.

Now, 5.4.4 says that distinct β-components within $\operatorname{Im}\tau$ are contained in distinct β-components of $\mathcal{C}$, and similarly for $\hat{\tau}$. Also, by 5.4.3, each vertex of $\mathcal{C}$ lies in the same β-component as some vertex of $\operatorname{Im}\tau$ or $\operatorname{Im}\hat{\tau}$, but not both, by 5.4.2. Thus, we have proved that $\mathcal{C}$ has exactly 2^{l-1} β-components.

Now we shall determine the edges in $\mathcal{C}/\beta$. Consider the map

$$\begin{aligned}\theta : (\operatorname{Im}\tau)/\beta \quad &\rightarrow \quad (\operatorname{Im}\hat{\tau})/\beta \\ \text{the } \beta\text{-component of } \tau(C) \text{ in } \operatorname{Im}\tau \quad &\mapsto \quad \text{the } \beta\text{-component of } \hat{\tau}(C) \text{ in } \operatorname{Im}\hat{\tau},\end{aligned}$$

where C is any vertex of $\mathcal{C}(A_{l-1})$. We claim that θ is a graph isomorphism. To check that θ is well defined, suppose the τ-images of some $[\mathbf{i}], [\mathbf{i}'] \in \mathcal{C}(A_{l-1})$ belong to the same β-component within $\operatorname{Im}\tau$. Then there is a sequence of words joining $\mathbf{i}(l \searrow 1)$ to $\mathbf{i}'(l \searrow 1)$ with the same β-value, such that the braids only take place within the subword $\mathbf{i}$ (because the sequence lies in $\operatorname{Im}\tau$). Thus, $\mathbf{i}$ is braid-equivalent to $\mathbf{i}'$. It now follows easily that $(l \searrow 1)\mathbf{i}^+$ is β-equivalent to $(l \searrow 1)(\mathbf{i}')^+$ within $\operatorname{Im}\hat{\tau}$, as required. A symmetrical argument shows θ to be injective. Thus, θ is bijective, being a mapping between equinumerous sets. To prove that θ is a morphism of graphs, consider a pair of words $\mathbf{i}, \mathbf{i}' \in \mathcal{R}(A_{l-1})$ such that $[\mathbf{i}(l \searrow 1)]$ and $[\mathbf{i}'(l \searrow 1)]$ lie in adjacent β-components within $\operatorname{Im}\tau$. We may assume that $\mathbf{i}$ differs from $\mathbf{i}'$ by a single 3-braid. Clearly now, $[(l \searrow 1)\mathbf{i}^+]$ is adjacent to $[(l \searrow 1)(\mathbf{i}')^+]$ and these have distinct β-values, so they lie in adjacent β-components within $\operatorname{Im}\hat{\tau}$, as required. Symmetrically, θ^{-1} is also a morphism. Thus, θ is an isomorphism.

Now, every β-component within $\operatorname{Im}\tau$ or $\operatorname{Im}\hat{\tau}$ is a subgraph of a larger β-component in $\mathcal{C}(A_l)$. Now consider some particular β-component in $\operatorname{Im}\tau$, and consider also its image under θ, which is a β-component in $\operatorname{Im}\hat{\tau}$. We claim that the corresponding larger β-components in $\mathcal{C}(A_l)$ are adjacent to one another. In other words, for any $C \in \mathcal{C}(A_{l-1})$ we claim that $\tau(C)$ and $\hat{\tau}(C)$ belong to adjacent β-components of $\mathcal{C}(A_l)$.

To prove this, first observe that $\tau(C)$ lies in the same δ-fibre as $\hat{\tau}(C)$. Let F be this fibre, and let C' be any vertex of F. By 5.4.3, C' is β-equivalent to exactly one of $\tau(C)$ or $\hat{\tau}(C)$, using that fact that $\delta \circ \tau$ and $\delta \circ \hat{\tau}$ are both the identity map on $\mathcal{C}(A_{l-1})$. Thus, the β-components of $\tau(C)$ and $\hat{\tau}(C)$ are adjacent, as claimed. (The point is that F, which is known to be connected by 2.2.9, is partitioned into two nonempty connected subgraphs by the β-components, and must therefore have an edge which straddles the partition.)

So far, we have identified the following structure of $\mathcal{C}/\beta$.

- Two disjoint $(l-2)$-cubes coming from $(\operatorname{Im}\tau)/\beta$ and $(\operatorname{Im}\hat{\tau})/\beta$.
- Edges joining the vertices on these two cubes which correspond to one another via θ.

Since θ is an *isomorphism* of $(l-2)$-cubes, it follows that the $(l-1)$-cube is a subgraph of $\mathcal{C}/\beta$. Thus, having already verified that $\mathcal{C}/\beta$ has exactly 2^{l-1} vertices, it remains only to rule out the possibility that any other edges exist.

So, consider any pair of adjacent vertices $C, C' \in \mathcal{C}$ which lie in adjacent β-components. By 5.4.3, C is β-equivalent to either $(\tau \circ \delta)(C)$ or $(\hat{\tau} \circ \delta)(C)$; suppose the former, the other case being similar. By 5.4.3 again, there are two possibilities for C'.

(a) Suppose that C' is β-equivalent to $(\tau \circ \delta)(C')$. Thus, C and C' are both β-equivalent to vertices in $\operatorname{Im}\tau$, so the edge $\{C, C'\}$ is one of the known edges joining two β-components in $\operatorname{Im}\tau$.

(b) Suppose that C' is β-equivalent to $(\hat{\tau} \circ \delta)(C')$. We claim that $\delta(C) = \delta(C')$. First, write $C = [\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}]$ and $C' = [\boldsymbol{\lambda}'(l \searrow 1)\boldsymbol{\varrho}']$. For an eventual contradiction, suppose that $\delta(C) \neq \delta(C')$. Referring to 2.2.1, we see that C and C' must therefore be related by a 'left' or 'right' move; in particular we have the braid-equivalences $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}'$ and $\boldsymbol{\varrho} \equiv \boldsymbol{\varrho}'$. Next, let $\alpha \in \Phi^+$ be the unique root such with summand α_1 such that $\alpha + \beta \in \Phi^+$. We claim that $\alpha \leq_C \beta$ if and only if $\alpha \leq_{C'} \beta$. To see this, note that β corresponds to some letter in $\boldsymbol{\lambda}$ if and only if β corresponds to some letter in $\boldsymbol{\lambda}'$. So, since α must come from $(l \searrow 1)$, having summand α_1, the relative positions of α and β are the same in C and C', as claimed. Consequently, the possibilities $\alpha \leq_{C'} \beta$ and $\beta \leq_C \alpha$ are exhaustive and mutually exclusive — we consider them in turn.

(b)(i) Suppose that $\alpha \leq_{C'} \beta$. Recall that $(\hat{\tau} \circ \delta)(C') = [(l \searrow 1)\boldsymbol{\lambda}'^{+}\varrho']$. Now, β must correspond to some letter of $\boldsymbol{\lambda}'^{+}\varrho$, since α_1 is not a summand. Thus $\beta \leq \alpha$ with respect to the partial order induced by *this* commutation class. So, the relative positions of α and β with respect to C' and $(\hat{\tau} \circ \delta)(C')$ are reversed. So, by 5.4.1, C' is *not* β-equivalent to $(\hat{\tau} \circ \delta)(C')$. This contradicts our case (b) hypothesis.

(b)(ii) Suppose that $\beta \leq_C \alpha$. Recall that $(\tau \circ \delta)(C) = [\boldsymbol{\lambda}\varrho^{-}(l \searrow 1)]$. We have $\alpha \leq \beta$ with respect to this commutation class, so the relative positions of α and β with respect to C and $(\tau \circ \delta)(C)$ are reversed. Using 5.4.1 again, C cannot be β-equivalent to $(\tau \circ \delta)(C)$, which plainly contradicts our initial assumption.

This overall contradiction establishes our claim that $\delta(C) = \delta(C')$; call this common vertex C_0. Our hypotheses at this point now say that C is β-equivalent to $\tau(C_0)$ and that C' is β-equivalent to $\hat{\tau}(C_0)$. Consequently, the edge $\{C, C'\}$ joins the β-components of $\tau(C_0)$ and $\hat{\tau}(C_0)$, and all such edges have already been described, using the map θ.

Thus, we have described *all* the edges of $\mathcal{C}/\beta$, as well as the vertices. That is, $\mathcal{C}/\beta$ is an $(l-1)$-cube whenever α_1 is not a summand of β.

To complete the induction step, we need to establish the structure of $\mathcal{C}/\beta$ for general β. Recall from 5.3.2 that the graph automorphism φ of $\mathcal{C}$ sends each β-component to a certain $\varphi_0(\beta)$-component, for all β. Now, since the φ_0-iterates of the set of all positive roots without summand α_1 clearly exhaust Φ^+, it follows that $\mathcal{C}/\beta$ is an $(l-1)$-cube for *all* β. ∎

5.4.6 COROLLARY. Let $\beta \in \Phi^+$. There is a unique vertex of $\mathcal{C}/\beta$ with associated β-value 1. Viewing the $(l-1)$-cube $\mathcal{C}/\beta$ as a ranked poset with this vertex at the bottom, the β-value corresponding to any vertex equals one more than its rank.

PROOF. We argue by induction on $l > 1$, being true for $l = 1$. First suppose that α_1 is not a summand of β. Under the graph embedding τ, we have seen that all β-values remain unchanged, so by induction, there is a unique vertex of $(\operatorname{Im}\tau)/\beta$ with β-value 1, and the resulting poset has the required rank-labelling by β-values. Under $\hat{\tau}$, however, all β-values increase by 1, so $(\operatorname{Im}\hat{\tau})/\beta$ is the same as $(\operatorname{Im}\tau)/\beta$, except that all vertices are labelled 1 higher.

Now, recall that $(\operatorname{Im}\tau)/\beta$ and $(\operatorname{Im}\hat{\tau})/\beta$ are embedded as disjoint $(l-2)$-cubes in $\mathcal{C}/\beta$. We have seen that the vertices in these two cubes which correspond to one another under θ are adjacent in $\mathcal{C}/\beta$, and so have β-values which differ by exactly 1. So, the pair of lowest-labelled vertices in $(\operatorname{Im}\tau)/\beta$ and $(\operatorname{Im}\hat{\tau})/\beta$ are joined in $\mathcal{C}/\beta$. Since θ is an isomorphism, it follows that each vertex of $(\operatorname{Im}\tau)/\beta$ is joined to some vertex of $(\operatorname{Im}\hat{\tau})/\beta$ which is labelled 1 higher. Clearly, this endows $\mathcal{C}/\beta$ with the labelling described in this Corollary.

To deduce that $\mathcal{C}/\beta$ has the correct labelling whenever α_l is not a summand of β, we apply the reversal isomorphism of $\mathcal{C}$: since α_1 is not a summand of $\bar{\beta}$, and since we know that $\mathcal{C}/\bar{\beta}$ has the correct labelling, it follows at once from 5.3.1 that $\mathcal{C}/\beta$ has the correct labelling.

It remains to deal with the case when β has summands α_1 and α_l. This implies that $\beta = \alpha_1 + \cdots + \alpha_l$. We know already that $\mathcal{C}/\alpha_1$ has the correct labelling. Now, applying 5.3.3 to the root α_1, noting that $\varphi_0(\alpha_1) = \alpha_1 + \cdots + \alpha_l$, it follows that $\mathcal{C}/\beta$ has the correct labelling. ∎

5.5 Highest-Root Components

Let $\alpha_0 = \alpha_1 + \cdots + \alpha_l$ be the highest root. In this section we will prove that each α_0-component of $\mathcal{C}(A_l)$ contains exactly one quiver-compatible commutation class.

5.5.1 LEMMA. Define the map

$$\begin{aligned} \varepsilon : \mathcal{C}(A_l) &\rightarrow \mathcal{W}(A_{l-1}) \\ [\boldsymbol{\lambda}(l \searrow 1)\varrho] &\mapsto s_{\varrho^-}. \end{aligned}$$

(a) The map ε is a surjective morphism of graphs. Each ε-fibre is connected, and any two vertices in the same fibre are related to one another by 'left' and 'right' moves (see 2.2.1).

(b) Each α_0-component of $\mathcal{C}(A_l)$ is a union of ε-fibres.

PROOF. Since ϱ is determined up to commutations, ε is certainly well defined, noting that $\text{supp}(\varrho^-) \subseteq \{1, 2, \ldots, l-1\}$. As usual, we are viewing $\mathcal{W}$ as the graph underlying the weak order poset diagram.

(a) Let $w \in \mathcal{W}(A_{l-1})$ and let $\mathbf{i}$ be reduced for w. Extend to a longest word $\mathbf{ji} \in \mathcal{R}(A_{l-1})$. We have $[\mathbf{ji}(l \searrow 1)] \in \mathcal{C}(A_l)$, and applying $\ell(\mathbf{i})$ 'left to right' moves gives $[\mathbf{j}(l \searrow 1)\mathbf{i}^+]$, which maps to w under ε, proving surjectivity.

Using the characterisation of adjacent vertices (2.2.1) it is easy to check that ε is a morphism of graphs. If now $[\boldsymbol{\lambda}(l \searrow 1)\varrho]$ and $[\boldsymbol{\lambda}'(l \searrow 1)\varrho']$ are any two vertices in the same ε-fibre then we have the braid-equivalences $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}'$ and $\varrho \equiv \varrho'$, so the other claims follow at once.

(b) Since α_0, having summand α_1, corresponds to some letter of the subword $(l \searrow 1)$ of any normal representative, braids applied to $\boldsymbol{\lambda}$ or ϱ do not change the α_0-value. Thus, by part (a), each ε-fibre is wholly contained in some α_0-component. ∎

5.5.2 PROPOSITION. If C and C' belong to the same α_0-component of $\mathcal{C}(A_l)$, with α_0-value k, then

$$\varepsilon(C')\varepsilon(C)^{-1} \in \mathcal{W}(A_{\{1,2,\ldots,k-2\}}) \times \mathcal{W}(A_{\{k+1,\ldots,l-2,l-1\}}),$$

an (internal) direct product of parabolic subgroups with the indicated generating sets.

PROOF. Write $C = [\boldsymbol{\lambda}(l \searrow 1)\varrho]$ and without loss of generality assume that C' is adjacent to C. If C' is related to C by a 'left' or 'right' move then of course $\varepsilon(C) = \varepsilon(C')$. If C' is related to C by a 'left to right' move then $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}'i$ for some i, and $C' = [\boldsymbol{\lambda}'(l \searrow 1)i^+\varrho]$. Since the α_0-value of C' is also k, it follows that s_{i^+} must fix $\alpha_1 + \cdots + \alpha_k$, hence $i^+ \neq 1, k, k+1$ or rather $i \neq k-1, k$. Thus, $\varepsilon(C')\varepsilon(C)^{-1}$, which equals s_i, belongs to the subgroup of $\mathcal{W}(A_{l-1})$ with generating set $\{1, \ldots, l-1\} \setminus \{k-1, k\}$, as required. A similar calculation applies for 'right to left' moves. ∎

Now we shall introduce the quivers. Recall from 3.3.1 that the unique commutation class $[Q]$ of longest words compatible with a quiver $Q \in \mathcal{Q}(A_l)$ has an expression of the form

$$[Q] = [\underbrace{(x_L \searrow 1)\ldots(x_1 \searrow 1)(l \searrow 1)}\underbrace{(l \searrow y_1)\ldots(l \searrow y_R)}],$$

for certain $l > x_1 > x_2 > \cdots > x_L \geq 1$ and $1 < y_1 < y_2 < \cdots < y_R \leq l$. Here, L is the number of left arrows in Q and $R = l - 1 - L$ the number of right arrows. In what follows, the above expression for $[Q]$ will always be used. When we need to introduce another quiver, Q', an analogous expression for $[Q']$ will be assumed, replacing L, R, x_i and y_j by L', R', x_i' and y_j'. We record here the following easy consequences of the above inequalities.

5.5.3 LEMMA. We have $x_i \geq L + 1 - i$ for $1 \leq i \leq L$ and $y_j \leq L + 1 + j$ for $1 \leq j \leq R$. ∎

5.5.4 LEMMA. The α_0-value of $[Q]$ is $L + 1$.

PROOF. Let γ be the positive root

$$\gamma = s_{(y_R \nearrow l)} \cdots s_{(y_1 \nearrow l)} s_1 \cdots s_L(\alpha_{L+1});$$

we wish to show that $\gamma = \alpha_0$. We clearly have

$$\gamma = s_{(y_R \nearrow l)} \cdots s_{(y_1 \nearrow l)}(\alpha_1 + \cdots + \alpha_{L+1}).$$

Since $(y_1 \nearrow l)$ contains $L + 2$ but not 1 (by 5.5.3), we obtain

$$\gamma = s_{(y_R \nearrow l)} \cdots s_{(y_2 \nearrow l)}(\alpha_1 + \cdots + \alpha_{L+1} + \alpha_{L+2}).$$

Next we use the fact that $(y_2 \nearrow l)$ contains $L + 3$ but not 1, giving

$$\gamma = s_{(y_R \nearrow l)} \cdots s_{(y_3 \nearrow l)}(\alpha_1 + \cdots + \alpha_{L+1} + \alpha_{L+2} + \alpha_{L+3}).$$

Continuing in this way, we obtain $\gamma = \alpha_0$. ∎

5.5.5 Lemma. If the quiver-compatible commutation classes $[Q]$ and $[Q']$ belong to the same α_0-component of $\mathcal{C}(A_l)$ then $Q = Q'$.

Proof. By 5.5.4 we have $L = L'$, hence $R = R'$, and also by 5.5.2 that the element $w := \varepsilon([Q'])\varepsilon([Q])^{-1}$ belongs to the direct product of symmetric groups $\mathrm{Sym}(\{1,2,\ldots,L\}) \times \mathrm{Sym}(\{L+2,\ldots,l-1,l\})$. For a contradiction, suppose that $Q \neq Q'$. Since the numbers $\{y_j\}_{j=1}^{R}$ completely determine Q, we must therefore have $y_p \neq y'_p$ for some p, which we take to be *maximal*. Thus, the following is a (certainly nonreduced) word representing w:

$$\underbrace{(l-1\searrow y'_1-1)\ldots(l-1\searrow y'_p-1)}\,\underbrace{(y_p-1\nearrow l-1)\ldots(y_1-1\nearrow l-1)}. \qquad (*)$$

Without loss of generality, assume that $y_p > y'_p$. Set $\eta := l+1-p$. We have $\eta \in \{L+2,\ldots,l-1,l\}$ (because $1 \leq p \leq R$), and we will obtain a contradiction by showing that $w(\eta) \in \{1,2,\ldots,L\}$. (Recall that s_i is the transposition $(i\ ,\ i+1)$.)

We will first consider the rightmost $p-1$ 'factors' of $(*)$, namely $(y_{p-1}-1\nearrow l-1)\ldots(y_1-1\nearrow l-1)$. Of course, if $p=1$, this word is null.

If $p \geq 2$ then $y_1 - 1 < l+1-p \leq l-1$. (Use $y_1 \leq L+2$ from 5.5.3, $p \leq R$ and $L+R = l-1$ for the left inequality.) It follows that

$$s_{(y_1-1\nearrow l-1)} \text{ sends } \eta := l+1-p \text{ to } l+2-p.$$

If $p \geq 3$ then $y_2 - 1 < l+2-p \leq l-1$. Thus

$$s_{(y_2-1\nearrow l-1)} \text{ sends } l+2-p \text{ to } l+3-p.$$

We may continue in this way until

$$s_{(y_{p-1}-1\nearrow l-1)} \text{ sends } \eta+p-2 \text{ to } \eta+p-1 = l.$$

Now,

$$s_{(y_p-1\nearrow l-1)} \text{ sends } l \text{ to } y_p - 1.$$

Next we calculate the effect of the left part of $(*)$ on $y_p - 1$.
We have $l-1 \geq y_p - 1 > y'_p - 1$ (by our assumption that $y_p > y'_p$). Thus

$$s_{(l-1\searrow y'_p-1)} \text{ sends } y_p-1 \text{ to } y_p-2.$$

Now, $l-1 \geq y_p - 2 > y'_{p-1} - 1$ (since $y_p > y'_p > y'_{p-1}$). Thus

$$s_{(l-1\searrow y'_{p-1}-1)} \text{ sends } y_p-2 \text{ to } y_p-3.$$

We may continue in this way until

$$s_{(l-1\searrow y'_1-1)} \text{ sends } y_p-p \text{ to } y_p-p-1.$$

Thus, $w(\eta) = y_p-p-1$. By 5.5.3 we have $y_p \leq L+1+p$, hence $w(\eta) \in \{1,\ldots,L\}$, our required contradiction. So Q=Q'. ∎

So far, we know that any α_0-component contains at *most* one quiver-compatible commutation class. To establish a bijection, we will use an argument which involves counting the elements of $\mathcal{W}(A_{l-1})$ in two different ways.

Consider any quiver $Q \in \mathcal{Q}(A_l)$ with L left and R right arrows. Define the subgroups

$$\mathcal{W}_L := \mathcal{W}(A_{\{1,2,\ldots,L-1\}}) \text{ and } \mathcal{W}_R := \mathcal{W}(A_{\{l+2-R,\ldots,l-1,l\}}),$$

of orders $L!$ and $R!$, respectively. We will shortly define an injective mapping from $\mathcal{W}_L \times \mathcal{W}_R$ to $\mathcal{W}(A_{l-1})$.

In our standard expression for the commutation class $[Q]$, the subword $(x_L \searrow 1)\ldots(x_1 \searrow 1)$ is easily seen to be commutation-equivalent to

$$\underbrace{(x_L \searrow 1)(x_{L-1} \searrow 2)(x_{L-2} \searrow 3)\ldots(x_1 \searrow L)}_{\mathbf{i}_Q}\ \underbrace{1(2 \searrow 1)\ldots(L-1 \searrow 1)}_{\mathbf{i}}$$

and that $(l \searrow y_1)\ldots(l \searrow y_R)$ is commutation-equivalent to

$$\underbrace{(l \searrow l+2-R)\ldots(l \searrow l-1)l}_{\mathbf{j}}\ \underbrace{(l+1-R \searrow y_1)\ldots(l-2 \searrow y_{R-2})(l-1 \searrow y_{R-1})(l \searrow y_R)}_{\mathbf{j}_Q}.$$

Let $\mathbf{i}_Q$ and $\mathbf{j}_Q$ be the indicated subwords, which clearly depend only upon Q. Also note that $\mathbf{i}$ and $\mathbf{j}$ are reduced words for the longest elements of $\mathcal{W}_L$ and $\mathcal{W}_R$, respectively. (This is easily checked by calculating their lengths.) Take any pair of elements $(w_L, w_R) \in \mathcal{W}_L \times \mathcal{W}_R$. Let ω_L, ω_R be any reduced words for w_L, w_R, respectively. Thus, we have braid-equivalences

$$\mathbf{i} \equiv \sigma_L \omega_L \text{ and } \mathbf{j} \equiv \omega_R \sigma_R$$

for certain reduced words σ_L and σ_R. So, by comparison with our expression $[\mathbf{i}_Q\, \mathbf{i}\, (l \searrow 1)\, \mathbf{j}\, \mathbf{j}_Q]$ for $[Q]$, we obtain

$$[\mathbf{i}_Q\, \sigma_L\, \omega_L\, (l \searrow 1)\, \omega_R\, \sigma_R\, \mathbf{j}_Q],$$

which is clearly α_0-equivalent to $[Q]$ (with α_0-value $L+1$). Now, since $L, L+1 \notin \operatorname{supp}(\omega_L)$, it is clear from the proof of 5.5.2 that we may apply a 'left to right' move for each letter of ω_L whilst staying in the same α_0-component. Thus,

$$[\mathbf{i}_Q\, \sigma_L\, (l \searrow 1)\, \omega_L^+\, \omega_R\, \sigma_R\, \mathbf{j}_Q]$$

is α_0-equivalent to $[Q]$. We can commute ω_L^+ and ω_R with one another because $\operatorname{supp}(\omega_L^+) \subseteq \{2, 3, \ldots, L\}$ and $\operatorname{supp}(\omega_R) \subseteq \{L+3, \ldots, l-1, l\}$, giving

$$[\mathbf{i}_Q\, \sigma_L\, (l \searrow 1)\, \omega_R\, \omega_L^+\, \sigma_R\, \mathbf{j}_Q].$$

Since $L+1, L+2 \notin \operatorname{supp}(\omega_R)$ we may apply a 'right to left' move for each letter of ω_R whilst staying in the same α_0-component. Thus

$$[\mathbf{i}_Q\, \sigma_L\, \omega_R^-\, (l \searrow 1)\, \omega_L^+\, \sigma_R\, \mathbf{j}_Q] \tag{†}$$

is α_0-equivalent to $[Q]$.

Now define the map

$$\begin{aligned} f_Q : \mathcal{W}_L \times \mathcal{W}_R &\to \mathcal{W}(A_{l-1}) \\ (w_L, w_R) &\mapsto s_{\omega_L}\, s_{\sigma_R^-}\, s_{\mathbf{j}_Q^-}. \end{aligned}$$

Note that f_Q is well defined because $\mathbf{j}_Q^-$ depends only upon Q, and ω_L, σ_R^- are determined up to braid-equivalence.

5.5.6 Lemma. The image of f_Q is contained in the ε-image of the α_0-component of $[Q]$.

Proof. Simply observe that $f_Q(w_L, w_R)$ equals the ε-image of the commutation class exhibited in (†), above, which we know to be α_0-equivalent to $[Q]$. ∎

Later we will be able to show that equality holds in the previous Lemma.

5.5.7 Lemma. The map f_Q is injective.

Proof. If, with the obvious notation, $f_Q(w_L, w_R) = f_Q(w'_L, w'_R)$ then

$$s_{\omega_L}\, s_{\sigma_R^-}\, s_{\mathbf{j}_Q^-} = s_{\omega'_L}\, s_{\sigma'^-_R}\, s_{\mathbf{j}_Q^-},$$

hence the element

$$(w'_L)^{-1} w_L = s_{\sigma'^-_R} (s_{\sigma^-_R})^{-1}$$

belongs to the trivial group $\mathcal{W}(A_{\{1,\ldots,L-1\}}) \cap \mathcal{W}(A_{\{L+2,\ldots,l-1\}})$. So $w_L = w'_L$ and $w_R = w'_R$. ∎

The following is an immediate consequence of 5.5.6 and 5.5.7.

5.5.8 LEMMA. The ε-image of the α_0-component of $[Q]$ contains at least $L!R!$ elements. ∎

This next observation will be required in section 5.6. The proof is an immediate consequence of the definition of f_Q, noting that we have renamed $s_{\mathbf{j}^-}$ as w_Q.

5.5.9 LEMMA. We have

$$\operatorname{Im} f_Q = \left(\mathcal{W}(A_{\{1,2,\ldots,L-1\}}) \times \mathcal{W}(A_{\{L+2,\ldots,l-2,l-1\}})\right) w_Q,$$

where $w_Q := s_{(l-R\searrow y_1-1)} \cdots s_{(l-2\searrow y_{R-1}-1)} s_{(l-1\searrow y_R-1)}$. (View the direct product as internal.) Thus, $\operatorname{Im} f_Q$ is a certain coset of a parabolic subgroup of $\mathcal{W}(A_{l-1})$. ∎

Incidentally, although not used later, we note that $\varepsilon([Q])$ may be described either as $f_Q(\mathrm{id}, \mathrm{id})$ or $(\mathrm{id}, w_0)w_Q$, in the notation of the above Lemma.

5.5.10 PROPOSITION. We have the disjoint union

$$\mathcal{W}(A_{l-1}) = \dot{\bigcup}_{Q \in \mathcal{Q}(A_l)} \operatorname{Im} f_Q.$$

PROOF. To show that the union is disjoint, suppose that some intersection $\operatorname{Im} f_Q \cap \operatorname{Im} f_{Q'}$ is nonempty, containing an element w. Applying 5.5.6 twice, we can write $w = \varepsilon(C)$ and $w = \varepsilon(C')$ where C and C' are α_0-equivalent to $[Q]$ and $[Q']$, respectively. Since C and C' belong to the same ε-fibre, they also belong to the same α_0-component, by 5.5.1 (b). Thus $[Q]$ is α_0-equivalent to $[Q']$, which implies that $Q = Q'$ by 5.5.5, as required.

We establish the equality of sets by a simple counting argument. Of course, the right side is contained in the left side. Recall from 5.5.7 that $|\operatorname{Im} f_Q| = L!(l-1-L)!$, where we have written $R = l-1-L$. The number of quivers in $\mathcal{Q}(A_l)$ with exactly L left arrows is $\binom{l-1}{L}$. Therefore

$$\begin{aligned} \left|\dot{\bigcup}_{Q \in \mathcal{Q}(A_l)} \operatorname{Im} f_Q\right| &= \sum_{L=0}^{l-1} \binom{l-1}{L} L!(l-1-L)! \\ &= l!, \end{aligned}$$

which is the order of $\mathcal{W}(A_{l-1})$. ∎

5.5.11 THEOREM. Each α_0-component of $\mathcal{C}(A_l)$ contains precisely one quiver-compatible commutation class. Thus, the number of α_0-components is 2^{l-1}.

PROOF. Take any $C \in \mathcal{C}(A_l)$. By 5.5.10 there is some quiver Q such that $\varepsilon(C) \in \operatorname{Im} f_Q$. By 5.5.6 we have $\varepsilon(C) = \varepsilon(C')$ for some C' in the same α_0-component as $[Q]$. Hence C is also α_0-equivalent to $[Q]$ (because C lies in the same ε-fibre as C'). Finally, 5.5.5 says that C is not α_0-equivalent to any other quiver-compatible commutation class. ∎

5.5.12 PROPOSITION. The ε-image of the α_0-component of $[Q]$ is precisely $\operatorname{Im} f_Q$. Thus, $\mathcal{W}(A_{l-1})$ is partitioned by these ε-images (using 5.5.10).

PROOF. Inclusion one way has been proved in 5.5.6. For the other inclusion, consider any element of the form $\varepsilon(C)$ where C is α_0-equivalent to $[Q]$. We must prove that $\varepsilon(C) \in \operatorname{Im} f_Q$. Well, by 5.5.10 we have $\varepsilon(C) \in \operatorname{Im} f_{Q'}$ for some Q'. And by 5.5.6, this time applied to $f_{Q'}$, we have $\varepsilon(C) = \varepsilon(C')$ for some C' in the same α_0-component as $[Q']$. So $[Q]$ is α_0-equivalent to $[Q']$, giving $Q = Q'$, so that $\varepsilon(C) \in \operatorname{Im} f_Q$, as required. ∎

5.6 The Hypercube Poset

In this section we complete our calculation of $\mathcal{C}(A_l)/\alpha_0$ by determining the edges. We begin with a general observation which we will use to obtain more refined representatives of commutation classes in $\mathcal{C}(A_l)$ than the normal representatives.

If we have a commutation-equivalence of reduced words $\mathbf{i} \sim \mathbf{i}'\mathbf{i}''$ then $\mathbf{i}'$ is called an **initial part** of $\mathbf{i}$. (In general, $\mathbf{i}$ will have many initial parts.)

5.6.1 LEMMA. Any initial part of a reduced word $\mathbf{i} := \mathbf{i}_1\mathbf{i}_2\ldots\mathbf{i}_n$ is commutation-equivalent to some product of initial parts $\mathbf{i}'_1\mathbf{i}'_2\ldots\mathbf{i}'_n$, where $\mathbf{i}'_r$ is an initial part of $\mathbf{i}_r$.

PROOF. This is intuitively obvious, but we can give a rigorous proof if we argue by induction on the length of the initial part of $\mathbf{i}$. Clearly, we may also assume that $n = 2$. Let $\mathbf{i}_1\mathbf{i}_2 \sim \mathbf{i}'\mathbf{i}''$. If the initial part $\mathbf{i}'$ has length zero then the result is true. Now let k be the first letter of $\mathbf{i}'$, so that $\mathbf{i}' = k\mathbf{j}'$ for some $\mathbf{j}'$. There are two cases, depending upon whether k can be traced back to a letter of $\mathbf{i}_1$ or $\mathbf{i}_2$.

If k comes from $\mathbf{i}_1$ then k commutes to the beginning of $\mathbf{i}_1$, so that $\mathbf{i}_1 \sim k\mathbf{j}_1$ for some $\mathbf{j}_1$. It follows that $\mathbf{j}_1\mathbf{i}_2 \sim \mathbf{j}'\mathbf{i}''$, so by induction,

$$\mathbf{j}' \sim (\text{an initial part of } \mathbf{j}_1)\ (\text{an initial part of } \mathbf{i}_2).$$

Now, since k followed by an initial part of $\mathbf{j}_1$ is an initial part of $\mathbf{i}_1$, it follows that $\mathbf{i}'$ has the required form.

If instead k comes from $\mathbf{i}_2$ then clearly k must commute with $\mathbf{i}_1$, and also commute with the letters to its left in $\mathbf{i}_2$, so that $\mathbf{i}_2 \sim k\mathbf{j}_2$ for some $\mathbf{j}_2$. It follows that $\mathbf{i}_1\mathbf{j}_2 \sim \mathbf{j}'\mathbf{i}''$, so by induction,

$$\mathbf{j}' \sim (\text{an initial part of } \mathbf{i}_1)\ (\text{an initial part of } \mathbf{j}_2).$$

Since k certainly commutes with any initial part of $\mathbf{i}_1$, and since k followed by an initial part of $\mathbf{j}_2$ is an initial part of $\mathbf{i}_2$, it follows that $\mathbf{i}'$ has the required form. ∎

5.6.2 PROPOSITION. Every commutation class $C \in \mathcal{C}(A_l)$ has a representative of the form

$$\mathbf{a}(1 \nearrow k-1)\mathbf{b}(l \searrow 1)\mathbf{c}(k+1 \nearrow l)\mathbf{d},$$

where k is the α_0-value of C and $\mathrm{supp}(\mathbf{b}) \subseteq \{1, 2, \ldots, k-2\}$, $\mathrm{supp}(\mathbf{c}) \subseteq \{k+2, \ldots, l-1, l\}$ and $1, l \notin \mathrm{supp}(\mathbf{a}), \mathrm{supp}(\mathbf{d})$.

PROOF. Let $\boldsymbol{\lambda}(l \searrow 1)\boldsymbol{\varrho}$ be a normal representative for C. We have $\delta(C) = [\boldsymbol{\lambda}\boldsymbol{\varrho}^-] \in \mathcal{C}(A_{l-1})$. By considering a normal representative for the reverse of this commutation class, we have

$$\boldsymbol{\lambda}\boldsymbol{\varrho}^- \sim \mathbf{x}(1 \nearrow l-1)\mathbf{y}$$

for some $\mathbf{x}$, $\mathbf{y}$ which are determined up to commutations, satisfying $1 \notin \mathrm{supp}(\mathbf{x})$ and $l-1 \notin \mathrm{supp}(\mathbf{y})$.

Since $\boldsymbol{\lambda}$ is an initial part of $\mathbf{x}(1 \nearrow l-1)\mathbf{y}$, by 5.6.1 we have commutation-equivalences $\mathbf{x} \sim \mathbf{x}'\mathbf{x}''$, $\mathbf{y} \sim \mathbf{y}'\mathbf{y}''$ and $(1 \nearrow l-1) = (1 \nearrow k-1)(k \nearrow l-1)$ such that $\boldsymbol{\lambda} \sim \mathbf{x}'(1 \nearrow k-1)\mathbf{y}'$.

Now, $\mathbf{x}(1 \nearrow l-1)\mathbf{y}$ is commutation-equivalent to $\mathbf{x}'\mathbf{x}''(1 \nearrow k-1)(k \nearrow l-1)\mathbf{y}'\mathbf{y}''$, so in order that $\mathbf{x}'(1 \nearrow k-1)\mathbf{y}'$ may commute to the far left of this expression, $\mathbf{x}''$ commutes with $(1 \nearrow k-1)$ and $\mathbf{y}'$ commutes with $(k \nearrow l-1)$. These conditions are equivalent to $\mathrm{supp}(\mathbf{x}'') \subseteq \{k+1, \ldots, l-1\}$ and $\mathrm{supp}(\mathbf{y}') \subseteq \{1, \ldots, k-2\}$. In particular, $\mathbf{x}''$ and $\mathbf{y}'$ commute with one another. Putting these facts together we obtain

$$\boldsymbol{\lambda}\boldsymbol{\varrho}^- \sim \underbrace{\mathbf{x}'(1 \nearrow k-1)\mathbf{y}'}_{\sim\boldsymbol{\lambda}}\ \underbrace{\mathbf{x}''(k \nearrow l-1)\mathbf{y}''}_{\sim\boldsymbol{\varrho}^-}.$$

Thus $\boldsymbol{\varrho} \sim (\mathbf{x}'')^+(k+1 \nearrow l)(\mathbf{y}'')^+$ and we have

$$C = [\mathbf{x}'(1 \nearrow k-1)\,\mathbf{y}'\,(l \searrow 1)\,(\mathbf{x}'')^+\,(k+1 \nearrow l)\,(\mathbf{y}'')^+].$$

Define $\mathbf{a} := \mathbf{x}'$, $\mathbf{b} := \mathbf{y}'$, $\mathbf{c} := (\mathbf{x}'')^+$ and $\mathbf{d} := (\mathbf{y}'')^+$. The restrictions on the supports of these words have already been established.

Finally, it is straightforward to check that the positive root corresponding to the letter k of the subword $(l \searrow 1)$, above, corresponds to α_0, making use of what we know about the supports of $\mathbf{c}$ and $\mathbf{d}$. ∎

5.6.3 NOTE. In the above Proposition, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are in fact all well defined up to commutations. We do not need to know this for our purposes, but a proof can be obtained if we first use the result of 2.2.3 to say that $\mathbf{a}(1 \nearrow k-1)\mathbf{b}$ and $\mathbf{c}(k+1 \nearrow l)\mathbf{d}$ are well defined up to commutations, and then imitate the proof of 2.2.3 for both of these words.

For the remainder of this section let $\widehat{C} \in \mathcal{C}(A_{l-1})$ be a fixed commutation class. By considering a normal representative for the reverse of $\widehat{C}$ we may write

$$\widehat{C} = [\boldsymbol{\varrho}(1 \nearrow l-1)\boldsymbol{\lambda}], \tag{1}$$

where $\operatorname{supp}(\boldsymbol{\varrho}) \subseteq \{2, \ldots, l-1\}$, $\operatorname{supp}(\boldsymbol{\lambda}) \subseteq \{1, \ldots, l-2\}$ and $\boldsymbol{\lambda}$, $\boldsymbol{\varrho}$ are well defined up to commutations.

5.6.4 DEFINITION. For $R = 0, 1, \ldots, l-1$, let Q_R be the quiver in $\mathcal{Q}(A_l)$ with the R right arrows

$$s_{\boldsymbol{\lambda}}^{-1}(l-i) \rightarrow 1 + s_{\boldsymbol{\lambda}}^{-1}(l-i) \text{ for } i = 1, 2, \ldots, R, \tag{2}$$

and $L := l - 1 - R$ left arrows on the remaining edges.

5.6.5 NOTE. We have suppressed the dependence upon $\boldsymbol{\lambda}$ in our notation because $\widehat{C}$ is considered to be fixed in advance.

The following key Proposition makes use of the mappings $\delta : \mathcal{C}(A_l) \rightarrow \mathcal{C}(A_{l-1})$ and $\varepsilon : \mathcal{C}(A_l) \rightarrow \mathcal{W}(A_{l-1})$ (see 2.2.9 and 5.5.1).

5.6.6 PROPOSITION. The δ-fibre of $\widehat{C}$ intersects precisely those α_0-components of $\mathcal{C}(A_l)$ which contain $[Q_0], [Q_1], \ldots, [Q_{l-1}]$.

PROOF. Let C be any vertex of the δ-fibre of $\widehat{C}$, and let k be the α_0-value of C. The main task lies in showing that C is α_0-equivalent to $[Q_{l-k}]$. Assuming the truth of this, we can deduce that the fibre intersects the α_0-component of each $[Q_i]$. For every fibre contains some vertex of the form $[\boldsymbol{\omega}(l \searrow 1)]$ (with α_0-value l) and also the vertex $[(l \searrow 1)\boldsymbol{\omega}^+]$ (with α_0-value 1). Since the fibre is connected, and since α_0-values can change by at most 1 as we pass between adjacent vertices, it follows that the fibre contains vertices with α_0-values from 1 to l, and thus intersects the α_0-components of $[Q_{l-1}]$ to $[Q_0]$ respectively, completing the proof.

Now, in the notation of 5.6.2, write

$$C = [\mathbf{a}(1 \nearrow k-1)\mathbf{b}(l \searrow 1)\mathbf{c}(k+1 \nearrow l)\mathbf{d}]. \tag{3}$$

Since $\widehat{C} = \delta(C)$ we have by (1) that

$$\boldsymbol{\varrho}(1 \nearrow l-1)\boldsymbol{\lambda} \sim \mathbf{a}(1 \nearrow k-1)\mathbf{b}\mathbf{c}^-(k \nearrow l-1)\mathbf{d}^-.$$

By 5.6.2 we see that $\mathbf{c}^-$ can commute to the left of $(1 \nearrow k-1)\mathbf{b}$, and then $\mathbf{b}$ can commute to the right of $(k \nearrow l-1)$, whence

$$\boldsymbol{\varrho}(1 \nearrow l-1)\boldsymbol{\lambda} \sim \mathbf{a}\mathbf{c}^-(1 \nearrow k-1)(k \nearrow l-1)\mathbf{b}\mathbf{d}^-.$$

By a symmetrical version of 2.2.3 we therefore have

$$\boldsymbol{\varrho} \sim \mathbf{a}\mathbf{c}^- \text{ and } \boldsymbol{\lambda} \sim \mathbf{b}\mathbf{d}^-. \tag{4}$$

These equations will be used shortly. Returning to expression (3) for C, since $k, k+1 \notin \operatorname{supp}(\mathbf{c})$, the proof of 5.5.2 shows that C is α_0-equivalent to $[\mathbf{a}(1 \nearrow k-1)\mathbf{b}\mathbf{c}^-(l \searrow 1)(k+1 \nearrow l)\mathbf{d}]$, which equals $[\mathbf{a}\mathbf{c}^-(1 \nearrow k-1)\mathbf{b}(l \searrow 1)(k+1 \nearrow l)\mathbf{d}]$, upon commuting $\mathbf{c}^-$ past $(1 \nearrow k-1)\mathbf{b}$. Similarly, since $k-1$ and k

do not belong to supp($\mathbf{b}$), this commutation class is α_0-equivalent to $[\mathbf{ac}^-(1 \nearrow k-1)(l \searrow 1)\mathbf{b}^+(k+1 \nearrow l)\mathbf{d}]$, which equals $[\mathbf{ac}^-(1 \nearrow k-1)(l \searrow 1)(k+1 \nearrow l)\mathbf{b}^+\mathbf{d}]$, upon commuting $\mathbf{b}$ past $(k+1 \nearrow l)$. So by (4), we have shown that

$$C \text{ is } \alpha_0\text{-equivalent to } D := [\varrho(1 \nearrow k-1)(l \searrow 1)(k+1 \nearrow l)\boldsymbol{\lambda}^+], \tag{5}$$

where we have introduced a new letter, D. By 5.5.12, D (and hence C) is α_0-equivalent to $[Q_{l-k}]$ if and only if $\varepsilon(D)$ belongs to $\operatorname{Im} f_{Q_{l-k}}$. Well,

$$\varepsilon(D) = s_{(k \nearrow l-1)} s_{\boldsymbol{\lambda}}$$

and by 5.5.9 we have

$$\operatorname{Im} f_{Q_{l-k}} = (\mathcal{W}(A_{\{1,2,\ldots,k-2\}}) \times \mathcal{W}(A_{\{k+1,k+2,\ldots,l-1\}}))\ w_{Q_{l-k}}$$

where

$$w_{Q_{l-k}} = s_{(k \searrow y_1-1)} s_{(k+1 \searrow y_2-1)} \cdots s_{(l-1 \searrow y_{l-k}-1)} \tag{6}$$

and the numbers $y_1 < y_2 < \cdots < y_{l-k}$ are a rearrangement of the vertices $1 + s_{\boldsymbol{\lambda}}^{-1}(l-i)$, $1 \le i \le l-k$, corresponding to the right arrows of Q_{l-k}. So by the last three equations, we must show that the element

$$w := w_{Q_{l-k}}\, s_{\boldsymbol{\lambda}}^{-1}\, s_{(l-1 \searrow k)} \tag{7}$$

belongs to $\operatorname{Sym}(\{1,\ldots,k-1\}) \times \operatorname{Sym}(\{k+1,\ldots,l\})$. It is enough to verify that w fixes k and permutes $\{k+1,\ldots,l\}$.

First we show that w fixes k. Referring to (7), note that $s_{(l-1 \searrow k)}$ sends k to l, which is unchanged by $s_{\boldsymbol{\lambda}}^{-1}$ because $\operatorname{supp}(\boldsymbol{\lambda}) \subseteq \{1,\ldots,l-2\}$. Now referring to (6), the $l-k$ factors successively decrease the image of l by one, ending with image k, as required.

Now we show that w permutes $\{k+1,\ldots,l\}$. Take any element η in this set. Note that $s_{(l-1 \searrow k)}$ sends η to $\eta-1$. Because $k \le \eta - 1 \le l-1$, we must have $s_{\boldsymbol{\lambda}}^{-1}(\eta-1) = y_j - 1$ for some j (depending on η), by construction of Q_{l-k} and by definition of $y_1,\ldots,y_{l-k}$. Note that $1 \le j \le l-k$. We must calculate the image of $y_j - 1$ under $w_{Q_{l-k}}$. Since y_j is strictly less than $y_{j+1}, y_{j+2},\ldots,y_{l-k}$, all but the first j factors of expression (6) leave $y_j - 1$ fixed. Now, the j^{th} factor, $s_{(k+j-1 \searrow y_j-1)}$, sends $y_j - 1$ to $k+j$, which is visibly fixed by the remaining factors. Thus $w(\eta) = k+j$. Now using $1 \le j \le l-k$, we have $k+1 \le w(\eta) \le l$, completing the proof. ∎

Define a partial order on $\mathcal{Q}(A_l)$ as follows. Let Q' cover Q in the partial order if Q' is obtained from Q by changing just one left arrow into a right arrow. The resulting poset is order-isomorphic to a hypercube with 2^{l-1} vertices. We next describe a well known one-to-one correspondence between the $(l-1)!$ maximal chains in this hypercube $\mathcal{Q}(A_l)$ and the $(l-1)!$ permutations of $\{1,\ldots,l-1\}$.

5.6.7 Lemma. Let the edge of the Coxeter graph A_l with vertices e and $e+1$ be **edge number** e. Consider any maximal chain in $\mathcal{Q}(A_l)$. Starting from the bottom quiver, all of whose arrows point to the left, let the $(l-1)$-tuple $(e_{(1)}, e_{(2)},\ldots,e_{(l-1)})$ be the numbers of the edges whose orientations change from $\leftarrow$ to $\rightarrow$ as we proceed to the top quiver, one step at a time. We associate to our maximal chain the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & l-1 \\ e_{(1)} & e_{(2)} & \cdots & e_{(l-1)} \end{pmatrix}.$$

Each element of $\mathcal{W}(A_{l-2})$ occurs in this manner exactly once. ∎

We now define the following variant of ε, namely

$$\begin{aligned} \widehat{\varepsilon} : \mathcal{C}(A_{l-1}) &\to \mathcal{W}(A_{l-2}) \\ \widehat{C} := [\varrho(1 \nearrow l-1)\boldsymbol{\lambda}] &\mapsto s_{\boldsymbol{\lambda}}^{-1}\, w_0^{(l-2)}, \end{aligned}$$

where $w_0^{(l-2)}$ is the longest element of $\mathcal{W}(A_{l-2})$. Like ε, the map $\widehat{\varepsilon}$ is surjective.

In the following Proposition, we consider the natural map $\mathcal{C}(A_l) \to \mathcal{Q}(A_l)$ which sends a commutation class C to that (unique) quiver Q for which $[Q]$ is α_0-equivalent to C.

5.6.8 Proposition. The image of the δ-fibre of $\widehat{C}$ under the map $\mathcal{C}(A_l) \to \mathcal{Q}(A_l)$ is a maximal chain in the hypercube poset $\mathcal{Q}(A_l)$. This maximal chain is the permutation $\widehat{\varepsilon}(\widehat{C})$, via the identification in 5.6.7.

Proof. In the notation of 5.6.6, the image of the δ-fibre of $\widehat{C}$ is $\{Q_0, Q_1, \ldots, Q_{l-1}\}$. By construction, for $R = 1, 2, \ldots, l-1$, the quiver Q_R differs from Q_{R-1} only in that edge number $s_\lambda^{-1}(l-R)$ is a right arrow. Thus, the image is a maximal chain, which visibly corresponds to the permutation

$$\widehat{\varepsilon}(\widehat{C}) = \begin{pmatrix} 1 & 2 & \cdots & l-1 \\ s_\lambda^{-1}(l-1) & s_\lambda^{-1}(l-2) & \cdots & s_\lambda^{-1}(1) \end{pmatrix},$$

as claimed. ∎

We will pause before stating the main result in order to give an example of the previous observation.

5.6.9 Example. In Figure 5.6.10 (a) we show the α_0-components of $\mathcal{C}(A_3)$ and the bijection with the quiver-compatible commutation classes. Now, $\delta : \mathcal{C}(A_3) \to \mathcal{C}(A_2)$ has two fibres. The δ-fibre of [121] is the heavily-drawn subgraph of $\mathcal{C}(A_3)$ on the left, and the δ-fibre of [212] is the heavily-drawn subgraph on the right.

Passing to the quotient graph $\mathcal{C}(A_3)/\alpha_0$ in (b), the δ-fibre of [121] visibly corresponds to the sequence of quivers

$$1 \leftarrow 2 \leftarrow 3 \ , \ 1 \rightarrow 2 \leftarrow 3 \ , \ 1 \rightarrow 2 \rightarrow 3,$$

and the δ-fibre of [212] corresponds to

$$1 \leftarrow 2 \leftarrow 3 \ , \ 1 \leftarrow 2 \rightarrow 3 \ , \ 1 \rightarrow 2 \rightarrow 3.$$

Both of these sequences are indeed maximal chains of $\mathcal{Q}(A_3)$, which is order-isomorphic to a 2-cube (a square). In the first sequence, the pair of edge numbers which change from $\leftarrow$ to $\rightarrow$ is $(1, 2)$. Thus we obtain the permutation $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, which is the identity. We can check that this is consistent with Proposition 5.6.8 by calculating $\widehat{\varepsilon}([121])$, which equals $s_1^{-1} w_0^{(1)}$, the identity element, as expected.

In the second sequence of quivers, the pair of edge numbers is $(2, 1)$, giving rise to the permutation $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, which equals s_1. Again we check our answer: $\widehat{\varepsilon}([212]) = \mathrm{id}^{-1} w_0^{(1)} = s_1$.

5.6.11 Theorem. The graph $\mathcal{C}(A_l)/\alpha_0$ is an $(l-1)$-cube and each α_0-component contains a unique quiver-compatible commutation class.

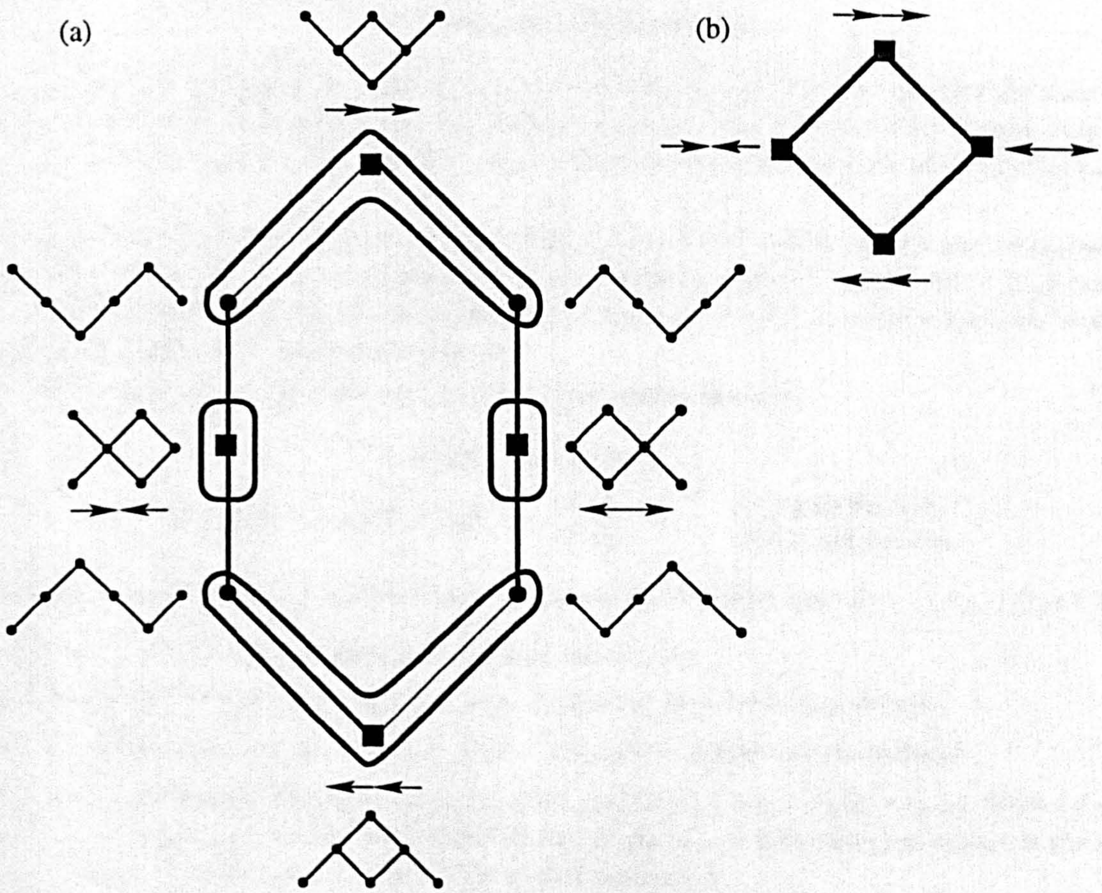
Proof. The second assertion is Theorem 5.5.11. We must show that two quivers Q and Q' are adjacent in the hypercube $\mathcal{Q}(A_l)$ if and only if the α_0-components of $[Q]$ and $[Q']$ are adjacent.

First suppose that the α_0-components of $[Q]$ and $[Q']$ are adjacent. Then we can find a pair of adjacent representatives C and C' in $\mathcal{C}(A_l)$. Inspection of 2.2.9 and 5.5.1 shows that any pair of adjacent vertices of $\mathcal{C}(A_l)$ belong either to the same δ-fibre or ε-fibre; if the latter then they also belong to the same α_0-component, by 5.5.1 (b). Thus, C and C' belong to the δ-fibre of some $\widehat{C}$. (We will use our usual notation for $\widehat{C}$.) Now by the first clause of 5.6.8, Q is necessarily adjacent to Q'.

Conversely, suppose that Q is adjacent to Q' in $\mathcal{Q}(A_l)$. Then the edge $\{Q, Q'\}$ is part of some maximal chain in this hypercube, which we shall identify with an element of $\mathrm{Sym}(\{1, \ldots, l-1\})$. Since $\widehat{\varepsilon}$ is surjective, this element equals $\widehat{\varepsilon}(\widehat{C})$ for some $\widehat{C} \in \mathcal{C}(A_{l-1})$. By the second clause of 5.6.8, the image (under $\mathcal{C}(A_l) \to \mathcal{Q}(A_l)$) of the δ-fibre of $\widehat{C}$ is our maximal chain of $\mathcal{Q}(A_l)$. By definition therefore, we can find some C, C' in this δ-fibre such that C is α_0-equivalent to $[Q]$ and C' is α_0-equivalent to $[Q']$. We may assume that Q has one more left arrow than Q', so that the α_0-value of $[Q]$ is k and that of $[Q']$ is $k-1$, for some k.

Now, one way to finish the proof is to use (5) from the proof of 5.6.6: we may say that

5.6.10 FIGURE. Interpreting δ-fibres as permutations.



$$C \text{ is } \alpha_0\text{-equivalent to } D := [\varrho(1 \nearrow k-1)(l \searrow 1)(k+1 \nearrow l)\boldsymbol{\lambda}^+]$$

and

$$C' \text{ is } \alpha_0\text{-equivalent to } D' := [\varrho(1 \nearrow k-2)(l \searrow 1)(k \nearrow l)\boldsymbol{\lambda}^+],$$

and then simply note that D is adjacent to D'.

A different, more geometrical way to finish is to recall that the δ-fibre of $\widehat{C}$, and hence its image under $\mathcal{C}(A_l) \rightarrow \mathcal{C}(A_l)/\alpha_0$, is *connected*. Since the α_0-values of adjacent vertices can differ by at most 1, it follows, in the notation of 5.6.6, that the α_0-components of $[Q_{i-1}]$ and $[Q_i]$ are adjacent for $i = 1, \ldots, l-1$. In particular, the α_0-components of $[Q]$ and $[Q']$ are adjacent. ∎

6. Root Components in Type B_l

6.1 Properties of Handedness

In this chapter we calculate the graph of α_0-components for type B_l, broadly following the same method of the last two sections of chapter 5. In this section we will go some way towards understanding the α_0-components with α_0-value 2. These will be seen to form the boundary of a partition of $\mathcal{C}(B_l)$ into two halves.

The highest root in $\Phi^+(B_l)$ is $\alpha_0 := 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l$. Let $C \in \mathcal{C}(B_l)$ have a normal representative $\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}$. By 2.3.4, α_0 corresponds to some letter in either $(l \searrow 2)$ or $(2 \nearrow l)$; if the former then C is **lefthanded**, otherwise C is **righthanded.** Clearly, $C \mapsto C^{\mathrm{rev}}$ provides a bijection between the lefthanded and righthanded commutation classes.

Let $w_0^{(l-1)}$ denote the longest element of $\mathcal{W}(B_{l-1})$ and define the map

$$\begin{array}{rcl} \varepsilon : \mathcal{C}(B_l) & \longrightarrow & \mathcal{W}(B_{l-1}) \\ C := [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}] & \mapsto & \begin{cases} s_{\boldsymbol{\lambda}}^{-1} w_0^{(l-1)} & \text{if } C \text{ is lefthanded,} \\ s_{\boldsymbol{\varrho}} & \text{if } C \text{ is righthanded.} \end{cases} \end{array}$$

Consider the action of $\mathcal{W}(B_{l-1})$ as signed permutations of the standard basis $\{e_1, \ldots, e_{l-1}\}$ (see 1.2.2).

6.1.1 LEMMA. Let $C = [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}]$ have α_0-value k.

(a) If C is lefthanded then $\varepsilon(C)$ sends e_{l-1} to e_{k-1}; if $k = 2$ then $\boldsymbol{\lambda}1$ is nonreduced.

(b) If C is righthanded then $\varepsilon(C)$ sends e_{l-1} to $-e_{k-1}$; if $k = 2$ then $1\boldsymbol{\varrho}$ is nonreduced.

PROOF. If C is lefthanded then $\alpha_0 = s_{\boldsymbol{\lambda}} s_{(l \searrow k+1)}(\alpha_k)$, that is, $e_{l-1} + e_l = s_{\boldsymbol{\lambda}}(e_l - e_{k-1})$. Since $l \notin \mathrm{supp}(\boldsymbol{\lambda})$ we have $e_{l-1} = -s_{\boldsymbol{\lambda}}(e_{k-1})$ and the first claim follows. If also $k = 2$ then $s_{\boldsymbol{\lambda}}(e_1) = s_{\boldsymbol{\lambda}}(\alpha_1)$ is the negative root $-e_{l-1}$, hence $\boldsymbol{\lambda}1$ is nonreduced. Part (b) is proved similarly. ∎

6.1.2 LEMMA.

(a) The map ε is surjective and each fibre is connected, of one handedness.

(b) Each α_0-component of $\mathcal{C}(B_l)$ is a union of ε-fibres.

PROOF. Let $w \in \mathcal{W}(B_{l-1})$ and consider any $\mathbf{ji} \in \mathcal{R}(B_{l-1})$ for which $w = s_{\mathbf{i}}$. Starting from the commutation class $[\mathbf{ji}(l \searrow 2)1(2 \nearrow l)] \in \mathcal{C}(B_l)$, apply either a 'left to middle', 'middle to right' sequence, or a 'left to right' move for each letter of $\mathbf{i}$, giving $[\mathbf{j}(l \searrow 2)1(2 \nearrow l)\mathbf{i}]$, which maps to w, irrespective of handedness. Thus ε is surjective.

Suppose that $C := [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}]$ and $C' := [\boldsymbol{\lambda}'(l \searrow 2)\boldsymbol{\mu}'1(2 \nearrow l)\boldsymbol{\varrho}']$ lie in the same ε-fibre. By 6.1.1, the handedness of a commutation class is determined by its ε-image. If C and C' are both lefthanded, say, then $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}'$ and hence $\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho} \equiv \boldsymbol{\mu}'1(2 \nearrow l)\boldsymbol{\varrho}'$. These braid-equivalences do not change the handedness, the ε-image nor the α_0-value. The same is true if C is righthanded. ∎

Our immediate aim is to find an analogue of 5.5.2 for type B_l. We shall first summarise how the handedness and α_0-values of adjacent commutation classes differ. Let $C = [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}]$ be lefthanded with α_0-value k. Let C' be adjacent to C. Clearly, if C' is obtained from C by a 'left', 'middle' or 'right' move then C' is also lefthanded, has the same α_0-value, and $\varepsilon(C')\varepsilon(C)^{-1} = \mathrm{id}$. The other six possibilities for C' pair off into three essentially different types, as listed below. The entries are all easily verified.

6.1.3

Type of move	C'	Lefthanded?	α_0-value $= k$?	$\varepsilon(C')\varepsilon(C)^{-1}$
'middle to right'	$[\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}'1(2 \nearrow l)i^-\boldsymbol{\varrho}]$, $\boldsymbol{\mu} \sim \boldsymbol{\mu}'i$	yes	yes	id
'middle to left'	$[\boldsymbol{\lambda}i^-(l \searrow 2)\boldsymbol{\mu}'1(2 \nearrow l)\boldsymbol{\varrho}]$, $\boldsymbol{\mu} \sim i\boldsymbol{\mu}'$	yes	iff $i \neq k, k+1$	s_{i^-}
'left to right'	$[\boldsymbol{\lambda}'(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)1\boldsymbol{\varrho}]$, $\boldsymbol{\lambda} \sim \boldsymbol{\lambda}'1$	iff $k \neq 2$	yes	s_1 (if $k \neq 2$)

A similar table can be constructed, starting from the assumption that C is *righthanded*. The following observation is now clear.

6.1.4 LEMMA. Adjacent vertices in $\mathcal{C}(B_l)$ have opposite handedness if and only if they are related by a 'left to right' or 'right to left' move and have common α_0-value 2. ∎

6.1.5 LEMMA. Let C and C' be lefthanded vertices of $\mathcal{C}(B_l)$ which belong to the same α_0-component, with α_0-value 2. Then these vertices are moreover joined by a path of *lefthanded* vertices within their α_0-component. (Similarly for righthanded vertices.)

PROOF. Consider a path $(C, C_1, \ldots, C_n, C')$ joining C to C' within their α_0-component. If this path consists of lefthanded vertices only, we are done. Otherwise, without loss of generality, suppose that *all* the C_i are righthanded. Write

$$C_i := [\boldsymbol{\lambda}_i(l \searrow 2)\boldsymbol{\mu}_i 1(2 \nearrow l)\boldsymbol{\varrho}_i]$$

and apply 'middle to left' moves, yielding $[\boldsymbol{\lambda}_i\boldsymbol{\mu}_i^-(l \searrow 2)1(2 \nearrow l)\boldsymbol{\varrho}_i]$, which is also righthanded with α_0-value 2. By 6.1.1 (b) we have $\boldsymbol{\varrho}_i \equiv 1\boldsymbol{\varrho}'_i$ for some $\boldsymbol{\varrho}'_i$, whence by 6.1.4, the commutation class defined by

$$C'_i := [\boldsymbol{\lambda}_i\boldsymbol{\mu}_i^- 1(l \searrow 2)1(2 \nearrow l)\boldsymbol{\varrho}'_i] \tag{$*$}$$

is *lefthanded*. (It does not matter that $\boldsymbol{\varrho}'_i$ is determined only up to braid-equivalence, since braids applied to this subword yield only lefthanded vertices in the same α_0-component.)

Our first claim is that each C'_i is α_0-equivalent to C'_{i+1} via a path of lefthanded vertices. By inspection of $(*)$, this is certainly true if $\boldsymbol{\varrho}_i \equiv \boldsymbol{\varrho}_{i+1}$, for then $\boldsymbol{\varrho}'_i \equiv \boldsymbol{\varrho}'_{i+1}$ and $\boldsymbol{\lambda}_i\boldsymbol{\mu}_i^- 1 \equiv \boldsymbol{\lambda}_{i+1}\boldsymbol{\mu}_{i+1}^- 1$. Also, 6.1.1 implies that C_i cannot be joined to C_{i+1} by a 'left to right' or 'right to left' move, for these vertices are both lefthanded. So the only remaining possibility is that C_{i+1} is obtained from C_i by a 'middle to right' move. (The 'right to middle' case is similar.) Let us write $C_i = [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}' j 1(2 \nearrow l)\boldsymbol{\varrho}]$ and $C_{i+1} = [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}' 1(2 \nearrow l) j^- \boldsymbol{\varrho}]$. Since the α_0-values are both 2, we have $j \neq 2, 3$. Writing $\boldsymbol{\varrho} \equiv 1\boldsymbol{\varrho}'$, we have

$$C'_i = [\boldsymbol{\lambda}\boldsymbol{\mu}'^- j^- 1(l \searrow 2)1(2 \nearrow l)\boldsymbol{\varrho}'] \text{ and } C'_{i+1} = [\boldsymbol{\lambda}\boldsymbol{\mu}'^- 1(l \searrow 2)1(2 \nearrow l) j^- \boldsymbol{\varrho}'],$$

where, in the second expression we have used the fact that $j^-\boldsymbol{\varrho} \equiv 1 j^- \boldsymbol{\varrho}'$, as j^- commutes with 1. So, starting with C'_{i+1}, we may apply a 'right to middle', 'middle to left' sequence to obtain C'_i, and the intermediate steps are lefthanded with α_0-value 2, completing the claim.

So far we know that C'_1 is α_0-equivalent to C'_n via a path of lefthanded vertices. It remains to show that there are similar paths from C to C'_1 and from C'_n to C'. It is enough consider only the first task — the other is similar. Since C_1 has opposite handedness to C, it is obtained by a 'left to right' or 'right to left' move, by 6.1.4. So, setting $C := [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}]$ we have in particular that $3 \notin \mathrm{supp}(\boldsymbol{\mu})$. Now, $\boldsymbol{\lambda}1$ is *nonreduced* by 6.1.1 (a). So, if 1 could commute to the beginning of $\boldsymbol{\varrho}$, applying a 'right to left' move would yield a nonreduced word, which is absurd. Hence C_1 must be obtained from C by a 'left to right' move. So, writing $\boldsymbol{\lambda} \equiv \boldsymbol{\lambda}'1$, we obtain

$$C = [\boldsymbol{\lambda}'1(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}] \text{ and } C'_1 = [\boldsymbol{\lambda}'\boldsymbol{\mu}^- 1(l \searrow 2)1(2 \nearrow l)\boldsymbol{\varrho}].$$

Clearly C'_1 is obtainable from C by 'middle to left' moves. Since $2, 3 \notin \mathrm{supp}(\boldsymbol{\mu})$, the intermediate vertices have α_0-value 2, and are obviously lefthanded. ∎

The following is an analogue of Proposition 5.5.2.

6.1.6 PROPOSITION. If C and C' have the same handedness and belong to the same α_0-component of $\mathcal{C}(B_l)$, with α_0-value k, then

$$\varepsilon(C')\varepsilon(C)^{-1} \in \mathcal{W}(B_{\{1,2,\ldots,k-2\}}) \times \mathcal{W}(B_{\{k+1,\ldots,l-2,l-1\}}).$$

PROOF. For definiteness take C, C' to be lefthanded. We know that these vertices are joined by a path of lefthanded vertices within their α_0-component: if $k \neq 2$ this follows from 6.1.4, and if $k = 2$ we need to invoke 6.1.5. So without loss of generality we may assume that C is *adjacent* to C'. The result now follows by inspection of the last column of the table in 6.1.3. ∎

The following will be used in section 6.3. (Refer to 2.3.11 for the definition of δ.)

6.1.7 LEMMA. The lefthanded and righthanded vertices in each δ-fibre form connected subgraphs.

PROOF. Fix some $[\mathbf{i}] \in \mathcal{C}(B_{l-1})$ and set $C_0 := [\mathbf{i}(l \searrow 2)1(2 \nearrow l)]$, which is lefthanded and lies in the δ-fibre of $[\mathbf{i}]$. Now let C be *any* lefthanded vertex in this δ-fibre and write $C = [\boldsymbol{\lambda}(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}]$. We will construct a path of lefthanded vertices, lying in this δ-fibre, from C to C_0 as follows.

First apply 'middle to left' moves to give a lefthanded path from C to $C_1 := [\boldsymbol{\lambda\mu}^-(l \searrow 2)1(2 \nearrow l)\boldsymbol{\varrho}]$. Let $\boldsymbol{\varrho} = i\boldsymbol{\varrho}'$. If $i \neq 1$ we can apply a 'right to middle', 'middle to left' (lefthanded) sequence. Or, if $i = 1$ and the α_0-value of C_1 is different from 2, we can instead apply a 'right to left' move; in either of these two cases we obtain the lefthanded vertex $[\boldsymbol{\lambda\mu}^- i(l \searrow 2)1(2 \nearrow l)\boldsymbol{\varrho}']$. However, if C_1 has α_0-value 2 then $\boldsymbol{\lambda\mu}^-1$ is nonreduced (by 6.1.1) so that necessarily $i \neq 1$, otherwise a 'left to right' move would give a nonreduced word.

Iterating the above for each letter of $\boldsymbol{\varrho}$ in turn, we obtain our desired path from C to C_0. A symmetrical argument applies for righthanded vertices. ∎

6.1.8 COROLLARY. The subgraphs of lefthanded and righthanded vertices in $\mathcal{C}(B_l)$ are connected.

PROOF. Every δ-fibre contains a vertex of the form $[\mathbf{i}(l \searrow 2)1(2 \nearrow l)]$, and any two such vertices are clearly joined by a lefthanded path. The righthanded case is symmetrical. ∎

Now we will introduce some explicit representatives for the α_0-components with α_0-value 2. These α_0-components 'glue together' the lefthanded and righthanded halves of $\mathcal{C}(B_l)$.

Let $Q \in \mathcal{Q}(B_l)$ be a quiver with R right arrows. In 3.3.2 we obtained an expression of the form

$$[Q] = [\boldsymbol{\lambda}_Q(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\boldsymbol{\varrho}_Q]$$

in which

$$\begin{aligned}
\boldsymbol{\lambda}_Q &:= (y_1 - 1 \searrow 1)\dots(y_R - 1 \searrow 1),\\
\boldsymbol{\mu} &:= (l \searrow 3)\dots(l \searrow l-1)l,\\
\boldsymbol{\varrho}_Q &:= \underbrace{1(2 \searrow 1)\dots(l-R-1 \searrow 1)}\ \underbrace{(l-R \searrow y_1)(l-R+1 \searrow y_2)\dots(l-1 \searrow y_R)}.
\end{aligned}$$

(Note that $\boldsymbol{\mu}$ is independent of Q.) The right arrows are $y_i - 1 \to y_i$ for certain $1 < y_1 < \dots < y_R \leq l$.

6.1.9 PROPOSITION. Let Q, Q' be quivers in $\mathcal{Q}(B_l)$ which are identically oriented except that Q contains $l-1 \to l$ and Q' contains $l-1 \leftarrow l$. Then $[Q]$ is lefthanded, $[Q']$ is righthanded, and $[Q]$ is α_0-equivalent to $[Q']$, with common α_0-value 2.

PROOF. In the expression for $[Q]$ we will have $y_R = l$, so that the root corresponding to the letter 2 in $(l \searrow 2)$ is

$$s_{(y_1-1\searrow 1)}s_{(y_2-1\searrow 1)}\cdots s_{(y_{R-1}-1\searrow 1)}\underbrace{s_{(l-1\searrow 1)}.s_l\dots s_3(\alpha_2)}.$$

The braced term evaluates to α_0, which is fixed by the remainder of the expression because no s_{l-1} occurs there. Thus, $[Q]$ is lefthanded, with α_0-value 2.

To show that $[Q]$ is α_0-equivalent to $[Q']$, first observe that $\boldsymbol{\mu} \sim l(l-1 \nearrow l)\dots(3 \nearrow l)$, so that applying 'middle to right' moves to the last factor $(3 \nearrow l)$, we see that $[Q]$ is α_0-equivalent to

$$[\boldsymbol{\lambda}_Q(l \searrow 2)\ \underbrace{l(l-1 \nearrow l)\dots(4 \nearrow l)}\ 1(2 \nearrow l)(2 \nearrow l-1)\boldsymbol{\varrho}_Q].$$

Note that $\boldsymbol{\lambda}_Q = \boldsymbol{\lambda}_{Q'}(l-1 \searrow 1)$. By 6.1.4, applying now a 'left to right' move (noting that no letter 3 occurs in the braced subword, above) gives a *righthanded* vertex; applying 'left to middle' moves to $(l-1 \searrow 2)$

therefore keeps us in the same α_0-component. Thus $[Q]$ is α_0-equivalent to the righthanded vertex

$$[\boldsymbol{\lambda}_{Q'}(l \searrow 2)(l \searrow 3)\,\underbrace{l(l-1 \nearrow l)\ldots(4 \nearrow l)}\,1(2 \nearrow l)(1 \nearrow l-1)\varrho_Q].$$

Now, $(l \searrow 3)$ followed by the braced subword is easily commutation-equivalent to $\boldsymbol{\mu}$. Thus, applying braids to $(1 \nearrow l-1)\varrho_Q$ is enough to obtain $[Q']$. So $[Q]$ is α_0-equivalent to the righthanded vertex $[Q']$. ∎

(Note that, unfortunately, the handedness of $[Q]$ is *opposite* to the orientation of the last edge of Q.)

6.2 Highest-Root Components

In this section we will enumerate the α_0-components by exhibiting explicit representatives.

Let $Q \in \mathcal{Q}(B_l)$ contain $l-1 \to l$, so that $y_R = l$ in our standard expression $[\boldsymbol{\lambda}_Q(l \searrow 2)\boldsymbol{\mu}1(2 \nearrow l)\varrho_Q]$ for $[Q]$. Fix some $2 \le k \le l$; we will consider those α_0-components with α_0-value k.

Consider any $(k-2)$-tuple $\mathbf{u} = (u_1, u_2, \ldots, u_{k-2})$ such that $3 \le u_1 < u_2 < \cdots < u_{k-2} \le l$.

Recall that $\boldsymbol{\mu} = (l \searrow 3)\ldots(l \searrow l-1)l$; we have the braid-equivalence

$$\boldsymbol{\mu} \equiv \underbrace{(u_1 \searrow 3)\ldots(u_{k-2} \searrow 3)}\,\underbrace{l(l-1 \nearrow l)\ldots(k+1 \nearrow l)}\,\boldsymbol{\mu}_{\mathbf{u}}, \qquad (*)$$

where $\boldsymbol{\mu}_{\mathbf{u}}$ is defined as follows:

$$\boldsymbol{\mu}_{\mathbf{u}} := (k \nearrow l+2-u_1)(k-1 \nearrow l+2-u_2)\ldots(3 \nearrow l+2-u_{k-2}).$$

(To see this, first observe that $\boldsymbol{\mu} \in \mathcal{R}(A_{\{3,4,\ldots,l\}})$. Promoting the first braced subword in $(*)$ gives rise to $l(l-1 \nearrow l)\ldots(3 \nearrow l)$ after commutations, which itself is commutation-equivalent to $\boldsymbol{\mu}$.) Substitute the right side of $(*)$ into the standard expression for $[Q]$, and perform 'middle to left' moves for all the letters in the first braced subword. This leads to the following key definition.

6.2.1 DEFINITION. Define $C(Q, \mathbf{u})$ to be the commutation class

$$C(Q,\mathbf{u}) := [\boldsymbol{\lambda}_Q\,\underbrace{(u_1-1 \searrow 2)\ldots(u_{k-2}-1 \searrow 2)}\,(l \searrow 2)\,\underbrace{l(l-1 \nearrow l)\ldots(k+1 \nearrow l)}\,\boldsymbol{\mu}_{\mathbf{u}}\,1(2 \nearrow l)\,\varrho_Q].$$

6.2.2 LEMMA. The commutation class $C(Q, \mathbf{u})$ is lefthanded and has α_0-value k.

PROOF. Note that $s_{(u_1-1\searrow 2)}\ldots s_{(u_{k-2}-1\searrow 2)}$ sends $\alpha_k + \cdots + \alpha_l$ to $\alpha_2 + \cdots + \alpha_l$ (because each $u_i \ge i+2$), which we already know is sent to α_0 by $s_{\boldsymbol{\lambda}_Q}$, from the proof of 6.1.9. ∎

6.2.3 LEMMA. If $C(Q,\mathbf{u}) = C(Q',\mathbf{u}')$ then $Q = Q'$ and $\mathbf{u} = \mathbf{u}'$. So the total number of commutation classes $C(Q,\mathbf{u})$ is $\sum_{k=2}^{l} 2^{l-2}\binom{l-2}{k-2}$.

PROOF. With the obvious notation we have $\varrho_Q \sim \varrho_{Q'}$ (by 2.3.3). Equating multiplicities of the letter 1 gives $l-R-1 = l-R'-1$, that is, $R = R'$. Thus

$$(l-R \searrow y_1)(l-R+1 \searrow y_2)\ldots(l-1 \searrow y_R) \sim (l-R \searrow y_1')(l-R+1 \searrow y_2')\ldots(l-1 \searrow y_R').$$

Comparing lowest letters we have $y_1 = y_1'$. So we can delete $(l-R \searrow y_1)$ from both sides and iterate. We obtain $y_j = y_j'$ for all j, hence $Q = Q'$. Therefore $\boldsymbol{\lambda}_Q = \boldsymbol{\lambda}_{Q'}$ and we obtain from 2.3.3 again that

$$(u_1-1 \searrow 2)\ldots(u_{k-2}-1 \searrow 2) \sim (u_1'-1 \searrow 2)\ldots(u_{k-2}'-1 \searrow 2),$$

where we have used the fact that $k = k'$, in view of the previous Lemma. Now, comparing highest letters, we obtain $u_{k-2} = u_{k-2}', \ldots, u_1 = u_1'$ in turn, whence $\mathbf{u} = \mathbf{u}'$. The formula is easily obtained. ∎

6.2.4 DEFINITION. Let $Q \in \mathcal{Q}(B_l)$ contain $l-1 \to l$ and suppose further that $R \ge k-1$. Let $\mathbf{u} = (u_1, \ldots, u_{k-2})$ satisfy $3 \le u_1 < u_2 < \cdots < u_{k-2} \le R+1$. Define the symbol $[Q, \mathbf{u}]$ to be the commutation class $C(Q, \mathbf{u})$.

6.2.5 NOTES.

(a) In the above Definition, observe that $\mathbf{u}$ depends upon R (and hence upon Q), and that R in turn depends upon k.

(b) When $k = 2$ we have $\mathbf{u} = ()$ and $[Q, ()]$ is simply the quiver-compatible commutation class $[Q]$.

(c) The $[Q, \mathbf{u}]$ will turn out to be a complete, irredundant set of representatives of those α_0-components which intersect the lefthanded half of $\mathcal{C}(B_l)$.

(d) Understand that $C(Q, \mathbf{u})$ is defined for *any* $\mathbf{u}$ with increasing entries between 3 and l, but $[Q, \mathbf{u}]$ is only defined when Q and $\mathbf{u}$ obey the indicated restrictions.

6.2.6 LEMMA. The total number of distinct $[Q, \mathbf{u}]$ is $\sum_{k=2}^{l} 2^{l-k} \binom{l-2}{k-2}$, which equals 3^{l-2}.

PROOF. For fixed k we must choose at least $r \geq k-2$ right arrows on the subgraph B_{l-1}, and choose $k-2$ distinct integers between 3 and $r+2$ inclusive. So, we must compute

$$\sum_{k=2}^{l} \sum_{r=k-2}^{l-2} \binom{l-2}{r} \binom{r}{k-2}.$$

Since $\binom{l-2}{r}\binom{r}{k-2}$ equals $\binom{l-2}{k-2}\binom{l-k}{r-k+2}$, this sum is now routinely evaluated. ∎

6.2.7 PROPOSITION. If $[Q, \mathbf{u}]$ and $[Q', \mathbf{u}']$ belong to the same α_0-component then $Q = Q'$ and $\mathbf{u} = \mathbf{u}'$.

PROOF. Let k be their common α_0-value, so that $\mathbf{u}$ and $\mathbf{u}'$ are both $(k-2)$-tuples. By 6.1.6, the element $w := \varepsilon([Q, \mathbf{u}])\varepsilon([Q', \mathbf{u}'])^{-1}$ represented by the following (nonreduced) word

$$\underbrace{(2 \nearrow u_{k-2}^-) \dots (2 \nearrow u_1^-)}\, \underbrace{(1 \nearrow y_R^-) \dots (1 \nearrow y_1^-)} \cdot \underbrace{(y_1'^- \searrow 1) \dots (y_{R'}'^- \searrow 1)}\, \underbrace{(u_1'^- \searrow 2) \dots (u_{k-2}'^- \searrow 2)} \quad (*)$$

belongs to $\mathcal{W}(B_{\{1,\dots,k-2\}}) \times \mathcal{W}(B_{\{k+1,\dots,l-1\}})$. (We have written i^- instead of $i-1$ to save space.) The first factor is the group of signed permutations of $\{1, 2, \dots, k-2\}$ and the second factor is the symmetric group on $\{k, k+1, \dots, l-1\}$. Recall that $s_1 = (1\ ,\ -1)$ and $s_i = (i-1\ ,\ i)$ for $i > 1$.

Step 1: Suppose that $y_{R-j} \neq y'_{R'-j}$ for some $1 \leq j < \min(R, R')$. (Of course, $j \neq 0$ because $y_R = y'_{R'} = l$.) Take j to be minimal. Without loss of generality, let $y_{R-j} > y'_{R'-j}$. Set $\eta := y_{R-j} + j - 1$. We have $R' + 1 \leq \eta \leq l - 1$. (For the first inequality use $y_{R-j} > y'_{R'-j} \geq 1 + R' - j$. For the second inequality use $y_{R-j} \leq l - j$.) In particular, $k \leq \eta \leq l-1$ because $R' \geq k-1$, by construction of $[Q', \mathbf{u}']$.

We will obtain a contradiction by showing that $w(\eta)$ is *signed*.

In $(*)$ above, the fourth braced subword, when interpreted as a product of simple reflections, fixes η. This is because $\eta \geq R' + 1 > u_i' - 1$ for each i, remembering that $3 \leq u_1' < \dots < u_{k-2}' \leq R' + 1$. The next term, $(y'_{R'} - 1 \searrow 1) = (l - 1 \searrow 1)$, sends η to $\eta - 1$. The next $j-1$ factors can be written as

$$(y_{R-j+1} - 1 \searrow 1) \dots (y_{R-1} - 1 \searrow 1),$$

by minimality of j. Now, for $i = 1, 2, \dots, j-1$ we have $2 \leq \eta - i \leq y_{R-i} - 1$. (For the first inequality, $\eta - i \geq \eta - j + 1 = y_{R-j} \geq 2$. For the second inequality, note that $y_{R-i} \geq y_{R-i-(j-i)} + (j - i)$ and use the definition of η.) It follows at once that these $j-1$ factors each decrease the image of $\eta - 1$ by 1, ending with $\eta - j$, which is $y_{R-j} - 1$. This is visibly fixed by the remaining factors of the third braced subword of $(*)$ because of our hypothesis that $y_{R-j} > y'_{R'-j}$.

Now consider the second braced subword of $(*)$. Clearly, $y_{R-j} - 1$ is fixed by the rightmost $R - j - 1$ factors. The next factor, $(1 \nearrow y_{R-j} - 1)$, sends $y_{R-j} - 1$ to -1. The remaining factors send -1 to -2, -3, and so on, until $(1 \nearrow y_R - 1) = (1 \nearrow l - 1)$ sends $-j$ to $-(j+1)$. Finally, since the first braced subword of $(*)$ contains no letter 1, the image of $-(j+1)$ remains signed. That is, $w(\eta)$ is signed, which is our required contradiction.

Step 2: We now know that $y_{R-j} = y'_{R'-j}$ for all $1 \leq j < \min(R, R')$. Without loss of generality, let $R \geq R'$, so that

$$y_R = y'_{R'},\ y_{R-1} = y'_{R'-1}, \dots, y_{R-R'+1} = y'_1. \qquad (\dagger)$$

We wish to prove that $R = R'$. Supposing otherwise, we have $R > R'$. This time, set $\eta := y_{R-R'} + R' - 1$. We again have $k \leq \eta \leq l - 1$. In fact, the argument in Step 1 is valid if we replace j by R' throughout — we only have to be aware that $y'_{R'-j} = y'_0$ is not defined. We find that $w(\eta)$ is signed, and this contradiction shows that $R = R'$. Now (†) says that $y_i = y'_i$ for all i, whence $Q = Q'$.

Step 3: It remains to prove that $\mathbf{u} = \mathbf{u}'$. If not, then $u_j \neq u'_j$ for some j, which we take to be minimal. Considering $(*)$, w can now be represented by the following word:

$$\underbrace{(2 \nearrow u_{k-2} - 1)\ldots(2 \nearrow u_j - 1)}\,\underbrace{(u'_j - 1 \searrow 2)\ldots(u'_{k-2} - 1 \searrow 2)}.$$

Without loss of generality, let $u'_j > u_j$. Set $\eta := u_j + k - 2 - j$. We have $k \leq \eta \leq l - 1$. (For the left inequality use $u_j \geq j + 2$. For the right inequality use $u_j < u'_j$ and fact that, starting from $u'_{k-2} \leq l$, we may easily obtain $u'_j \leq l + j - k + 2$.) We will obtain a contradiction by showing that $1 \leq w(\eta) \leq k - 2$.

Now, for $i = 0, 1, \ldots, k - 2 - j$ we have $2 \leq \eta - i \leq u'_{k-2-i} - 1$. (For the left inequality, $\eta - i \geq k - i \geq 2 + j \geq 3$. For the right inequality use $u_j < u'_j$ and the easily obtainable $u'_j \leq u'_{j+(k-2-j-i)} - (k - 2 - j - i)$.) It follows that each of the factors in the second braced subword, above, decreases the image of η by 1, ending with $\eta - (k - 1 - j)$ which equals $u_j - 1$.

Next, $(2 \nearrow u_j - 1)$ sends $u_j - 1$ to 1. The remaining $k - j - 2$ factors each increase the image by 1, hence $w(\eta) = k - j - 1$. Clearly, $1 \leq w(\eta) \leq k - 2$, our required contradiction. Thus, $\mathbf{u} = \mathbf{u}'$. ∎

Our next aim is to obtain an analogue of the map f_Q of section 5.5. Define the subgroups

$$\mathcal{W}_1 := \mathcal{W}(B_{\{2,3,\ldots,k-2\}}) \text{ and } \mathcal{W}_2 := \mathcal{W}(B_{\{k+2,k+3,\ldots,l\}}),$$

of orders $(k-2)!$ and $(l-k)!$ respectively. We will shortly define an injection from $\mathcal{W}_1 \times \mathcal{W}_2$ to $\mathcal{W}(B_{l-1})$.

Consider our standard expression for $C(Q, \mathbf{u})$. We have the commutation-equivalence

$$(u_1 - 1 \searrow 2)\ldots(u_{k-2} - 1 \searrow 2) \sim \boldsymbol{\lambda}_{\mathbf{u}}\, 2(3 \searrow 2)\ldots(k - 2 \searrow 2),$$

where $\boldsymbol{\lambda}_{\mathbf{u}}$ is defined as follows:

$$\boldsymbol{\lambda}_{\mathbf{u}} := (u_1 - 1 \searrow 2)(u_2 - 1 \searrow 3)\ldots(u_{k-2} - 1 \searrow k - 1).$$

Thus,

$$C(Q, \mathbf{u}) = [\boldsymbol{\lambda}_Q \boldsymbol{\lambda}_{\mathbf{u}}\, \underbrace{2(3 \searrow 2)\ldots(k - 2 \searrow 2)}_{\mathbf{i}_1}\, \underbrace{(l \searrow 2)\, l(l - 1 \nearrow l)\ldots(k + 2 \nearrow l)}_{\mathbf{i}_2}.(k + 1 \nearrow l)\, \boldsymbol{\mu}_{\mathbf{u}} 1(2 \nearrow l)\boldsymbol{\varrho}_Q].$$

Note that $\mathbf{i}_1$ and $\mathbf{i}_2$, as defined above, are respectively longest words for $\mathcal{W}_1$ and $\mathcal{W}_2$. Take any (w_1, w_2) in $\mathcal{W}_1 \times \mathcal{W}_2$. Let $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}_2$ be reduced words for w_1, w_2 respectively, and extend to maximal-length words, so that

$$\mathbf{i}_1 \equiv \boldsymbol{\omega}_1 \boldsymbol{\sigma}_1 \text{ and } \mathbf{i}_2 \equiv \boldsymbol{\omega}_2 \boldsymbol{\sigma}_2$$

for some $\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$. So $C(Q, \mathbf{u})$ is certainly α_0-equivalent to the lefthanded commutation class

$$[\boldsymbol{\lambda}_Q \boldsymbol{\lambda}_{\mathbf{u}} \boldsymbol{\omega}_1 \boldsymbol{\sigma}_1 (l \searrow 2) \boldsymbol{\omega}_2 \boldsymbol{\sigma}_2 (k + 1 \nearrow l) \boldsymbol{\mu}_{\mathbf{u}} 1(2 \nearrow l)\boldsymbol{\varrho}_Q].$$

Since $\operatorname{supp}(\boldsymbol{\sigma}_1) \subseteq \{2, \ldots, k - 2\}$ and $\operatorname{supp}(\boldsymbol{\omega}_2) \subseteq \{k + 2, \ldots, l\}$ we may apply 'left to middle' moves to $\boldsymbol{\sigma}_1$, commute $\boldsymbol{\sigma}_1^+$ past $\boldsymbol{\omega}_2$, and then apply 'middle to left' moves to each letter of $\boldsymbol{\omega}_2$, whilst staying in the same α_0-component and remaining lefthanded. Thus, $C(Q, \mathbf{u})$ is α_0-equivalent to the lefthanded commutation class

$$[\boldsymbol{\lambda}_Q \boldsymbol{\lambda}_{\mathbf{u}} \boldsymbol{\omega}_1 \boldsymbol{\omega}_2^- (l \searrow 2) \boldsymbol{\sigma}_1^+ \boldsymbol{\sigma}_2 (k + 1 \nearrow l) \boldsymbol{\mu}_{\mathbf{u}} 1(2 \nearrow l)\boldsymbol{\varrho}_Q]. \tag{$**$}$$

6.2.8 Definition. Let $f_{Q,\mathbf{u}}$ be the map

$$\begin{aligned} f_{Q,\mathbf{u}} : \mathcal{W}_1 \times \mathcal{W}_2 &\longrightarrow \mathcal{W}(B_{l-1}) \\ (w_1, w_2) &\longmapsto (s_{\boldsymbol{\lambda}_Q} s_{\boldsymbol{\lambda}_{\mathbf{u}}} s_{\boldsymbol{\omega}_1} s_{\boldsymbol{\omega}_2^-})^{-1} w_0^{(l-1)}. \end{aligned}$$

This is well defined because ω_1, ω_2 are determined up to braid-equivalence.

6.2.9 LEMMA. The image of $f_{Q,\mathbf{u}}$ is contained in the ε-image of the intersection of the α_0-component containing $C(Q,\mathbf{u})$ with the lefthanded subgraph of $C(B_l)$.

PROOF. Simply note that $f_{Q,\mathbf{u}}(w_1,w_2)$ equals the ε-image of the commutation class (**), above, which we know to be lefthanded and α_0-equivalent to $C(Q,\mathbf{u})$. ∎

6.2.10 LEMMA. The map $f_{Q,\mathbf{u}}$ is injective.

PROOF. If $f_{Q,\mathbf{u}}(w_1,w_2) = f_{Q,\mathbf{u}}(w_1',w_2')$ then $w_1 s_{\omega_2^-} = w_1' s_{\omega_2'^-}$, hence $w_1^{-1}w_1'$ belongs to the trivial group $\mathcal{W}(B_{\{2,\ldots,k-2\}}) \cap \mathcal{W}(B_{\{k+1,\ldots,l-1\}})$. Thus $w_1 = w_1'$ and $w_2 = w_2'$. ∎

By the previous two observations we obtain the following.

6.2.11 LEMMA. The ε-image of the intersection of the α_0-component containing $C(Q,\mathbf{u})$ with the lefthanded subgraph of $C(B_l)$ has at least $(k-2)!(l-k)!$ elements. ∎

The next Lemma is an immediate consequence of the definition of $f_{Q,\mathbf{u}}$.

6.2.12 LEMMA. We have

$$\operatorname{Im} f_{Q,\mathbf{u}} = \left(\mathcal{W}(B_{\{2,3,\ldots,k-2\}}) \times \mathcal{W}(B_{\{k+1,k+2,\ldots,l-1\}})\right)\, s_{\lambda_{\mathbf{u}}}^{-1}\, s_{\lambda_Q}^{-1}\, w_0^{(l-1)},$$

which is thus a certain coset of a parabolic subgroup. ∎

The next result is an analogue of Proposition 5.5.10. Define

$$\mathcal{W}(B_{l-1})^+ := \{w \in \mathcal{W}(B_{l-1}) \mid w(e_{l-1}) \text{ is unsigned}\},$$

a subset of cardinality $\frac{1}{2}|\mathcal{W}(B_{l-1})| = 2^{l-2}(l-1)!$. Since ε is surjective, it follows from 6.1.1 that

$$\mathcal{W}(B_{l-1})^+ = \{\varepsilon(C) \mid C \text{ is a lefthanded member of } \mathcal{C}(B_l)\}.$$

6.2.13 PROPOSITION. We have the disjoint union

$$\mathcal{W}(B_{l-1})^+ = \bigcup_{k=2}^{l}\bigcup_{Q}\bigcup_{\mathbf{u}} \operatorname{Im} f_{Q,\mathbf{u}}.$$

The union is taken over all quivers $Q \in \mathcal{Q}(B_l)$ containing $l-1 \to l$ and, for fixed k, all $\mathbf{u} = (u_1,\ldots,u_{k-2})$ such that $3 \le u_1 < \cdots < u_{k-2} \le l$.

PROOF. By 6.2.9, the right side is contained in the left side. We know that $|\operatorname{Im} f_{Q,\mathbf{u}}| = (k-2)!(l-k)!$ whenever $\mathbf{u}$ is a $(k-2)$-tuple, so once disjointness is proven, the right side of the desired identity will have cardinality $\sum_{k=2}^{l} 2^{l-2}\binom{l-2}{k-2}(k-2)!(l-k)!$, which equals $2^{l-2}(l-1)!$, the cardinality of the left side. So, it remains to establish disjointness.

Introduce notation for $f_{Q',\mathbf{u}'}$ in the obvious way, and suppose that $\operatorname{Im} f_{Q,\mathbf{u}}$ intersects $\operatorname{Im} f_{Q',\mathbf{u}'}$ in some element w. We will prove that $k=k'$, $R=R'$, $Q=Q'$ and $\mathbf{u}=\mathbf{u}'$ in four steps.

Step 1: By 6.2.9 applied twice, w lies in the ε-images of certain α_0-components with α_0-values k and k'. Therefore $k=k'$ because each ε-fibre is wholly contained in some α_0-component; see 6.1.2 (b).

Step 2: Set $I := \{2,3,\ldots,l-1\} \setminus \{k-1,k\}$. By 6.2.12, $\operatorname{Im} f_{Q,\mathbf{u}}$ and $\operatorname{Im} f_{Q',\mathbf{u}'}$ are certain $\mathcal{W}(B_I)$-cosets, which by hypothesis are not disjoint, so they must coincide. Inverting, we obtain

$$s_{\lambda_Q} s_{\lambda_{\mathbf{u}}} \mathcal{W}(B_I) = s_{\lambda_{Q'}} s_{\lambda_{\mathbf{u}'}} \mathcal{W}(B_I).$$

An arbitrary element of the left side can be written as $s_{\lambda_Q} s_{\lambda_{\mathbf{u}}} s_{\omega_1} s_{\omega_2'}$, in the notation used for the definition of $f_{Q,\mathbf{u}}$. We saw there that $\lambda_Q\lambda_{\mathbf{u}}\omega_1\omega_2'$ is a *reduced* word. Therefore $s_{\lambda_Q}s_{\lambda_{\mathbf{u}}}$ is the unique minimum-length element this coset. Similarly, so is $s_{\lambda_{Q'}}s_{\lambda_{\mathbf{u}'}}$. So we have the braid-equivalence

$$\lambda_Q\lambda_{\mathbf{u}} \equiv \lambda_{Q'}\lambda_{\mathbf{u}'}.$$

By 2.3.8 (a) we may equate the multiplicities of the letter 1 in these two words, which yields $R = R'$ at once.

Step 3: To prove that $Q = Q'$, we must establish $y_j = y_j'$ for all $1 \le j \le R-1$. For a contradiction suppose that $y_j \ne y_j'$ for some j, which we take to be minimal. In view of the previous equation we may define

$$w := s_{\lambda_Q}^{-1} s_{\lambda_{Q'}} = s_{\lambda_{\mathbf{u}}} s_{\lambda_{\mathbf{u}'}}^{-1}. \tag{1}$$

Considering the first equality, w is represented by the following (nonreduced) word:

$$\underbrace{(1 \nearrow y_R - 1)(1 \nearrow y_{R-1} - 1)\dots(1 \nearrow y_j - 1)}\ \underbrace{(y_j' - 1 \searrow 1)\dots(y_{R-1}' - 1 \searrow 1)(y_R' - 1 \searrow 1)}. \tag{2}$$

We may assume that $y_j' > y_j$. Set $\eta := y_j + R - j$. For $i = 0, 1, \dots, R-j$ we have $2 \le \eta - i \le y_{R-i}' - 1$. (For the left inequality, $\eta - i \ge \eta - (R-j) = y_j \ge 2$. For the right inequality use $y_j < y_j'$ and the easily obtainable inequality $y_j' \le y_{j+(R-i-j)}' - (R-i-j)$.) It follows that each of the $R+1-j$ factors in the second part of (2) decreases the image of η by 1, ending with $\eta - (R+1-j) = y_j - 1$.

Next, $(1 \nearrow y_j - 1)$ sends $y_j - 1$ to -1. The remaining factors send -1 to -2, and so on, so that $w(\eta) = -(R+1-j)$, which is *signed*. But now recall the second equality in (1); since $1 \notin \operatorname{supp}(\lambda_{\mathbf{u}}), \operatorname{supp}(\lambda_{\mathbf{u}'})$ it follows that $w(\eta)$ must be *unsigned*. This contradiction establishes that $Q = Q'$.

Step 4: From (1) we now have $\lambda_{\mathbf{u}} \equiv \lambda_{\mathbf{u}'}$, that is

$$(u_1 - 1 \searrow 2)(u_2 - 1 \searrow 3)\dots(u_{k-2} - 1 \searrow k-1) \equiv (u_1' - 1 \searrow 2)(u_2' - 1 \searrow 3)\dots(u_{k-2}' - 1 \searrow k-1).$$

The support of a reduced word is unchanged by braids; equating the largest members of $\operatorname{supp}(\lambda_{\mathbf{u}})$ and $\operatorname{supp}(\lambda_{\mathbf{u}'})$ implies that $u_{k-2} = u_{k-2}'$. Now delete $(u_{k-2} - 1 \searrow k-1)$ from both sides of the braid-equivalence and iterate. This gives $u_i = u_i'$ for all i, whence $\mathbf{u} = \mathbf{u}'$. ∎

We come now to a key technical Lemma which involves some intricate manipulations with words.

6.2.14 LEMMA. Consider a fixed $[Q, \mathbf{u}]$ with α_0-value k, so that in the usual notation,

$$y_1 < y_2 < \dots < y_{R-1}$$

and

$$1 \le R + 2 - u_{k-2} < \dots < R + 2 - u_2 < R + 2 - u_1 \le R - 1.$$

Visibly, the numbers $R + 2 - u_i$ occur as subscripts of $k-2$ terms of the list $y_1, y_2, \dots, y_{R-1}$. Select any orientation of the corresponding $k-2$ edges which are oriented as right arrows $\to y_{R+2-u_i}$ in Q (there are 2^{k-2} choices), and let Q' be the resulting quiver. Then $[Q, \mathbf{u}]$ is α_0-equivalent to $C(Q', \mathbf{u}')$, for some $\mathbf{u}'$.

PROOF. We first consider a special case. Consider a fixed u_i and set

$$h := R + 2 - u_i. \tag{1}$$

Let Q' denote the quiver obtained from Q by reversing the orientation of $y_h - 1 \to y_h$. Now,

$$[Q, \mathbf{u}] = [\underbrace{(y_1 - 1 \searrow 1)\dots(y_R - 1 \searrow 1)}_{\lambda_Q}\ \underbrace{(u_1 - 1 \searrow 2)\dots(u_{k-2} - 1 \searrow 2)}(l \searrow 2) * 1(2 \nearrow l)*], \tag{2}$$

where the precise subwords marked $*$ are unimportant for our purposes. Clearly, $\lambda_{Q'}$ is obtained from λ_Q by deleting the subword $(y_h - 1 \searrow 1)$.

The deletion details: First observe that $(u_1 - 1 \searrow 2)\dots(u_{k-2} - 1 \searrow 2)$ is braid-equivalent to

$$(u_i - 1 \searrow 2)\ \underbrace{(u_1 \searrow 3)\dots(u_{i-1} \searrow 3)}\ \underbrace{(u_{i+1} - 1 \searrow 2)\dots(u_{k-2} - 1 \searrow 2)}. \tag{3}$$

Note that $(u_i - 1 \searrow 2)$ equals $(R + 1 - h \searrow 2)$, by (1). We now assert the following braid-equivalence:

$$\lambda_Q (R + 1 - h \searrow 2) \equiv \lambda_{Q'} (y_h + R - h - 1 \searrow 1). \tag{4}$$

Proof of (4). Consider the left side of (4). For $j = 1, 2, \ldots, R+1-(h+2)$ we have $2 \leq R+2-h-j \leq y_{r+1-j}-1$. (The left inequality is trivial; the 2 can even be replaced by 3. For the right inequality, use $h \geq 1$ and $R+1-j < y_{R+1-j}$.) It follows that, using braids, the letter $R+1-h$ can 'hop' leftwards over each of the factors $(y_R - 1 \searrow 1)$, $(y_{R-1} - 1 \searrow 1), \ldots, (y_{h+2} - 1 \searrow 1)$ in turn, decreasing by one each time, ending up equal to 2. Now perform these braids:

$$\begin{aligned}
(y_h - 1 \searrow 1)(y_{h+1} - 1 \searrow 1).2 \quad &\sim \quad (y_h - 1 \searrow 2)(y_{h+1} - 1 \searrow 3)\, 1212 \\
&\overset{4}{\leftrightarrow} \quad (y_h - 1 \searrow 2)(y_{h+1} - 1 \searrow 3)\, 2121 \\
&= \quad (y_h - 1 \searrow 2)(y_{h+1} - 1 \searrow 1)\, 21 \\
&\equiv \quad (y_{h+1} - 1 \searrow 1)(y_h \searrow 3)\, 21 \\
&= \quad (y_{h+1} - 1 \searrow 1)(y_h \searrow 1).
\end{aligned}$$

So, we have managed to move the factor $(y_h - 1 \searrow 1)$ to the right of its neighbouring factor. The only anomaly is that this factor has grown by an extra letter at the front, becoming $(y_h \searrow 1)$.

Now iterate the whole process until all of the letters from $(R+1-h \searrow 2)$ in (4) have been used up — that is $R-h$ iterations in total. Consequently, our original factor $(y_h - 1 \searrow 1)$ in $\boldsymbol{\lambda}$ moves $R-h$ places to the right, and increases in length by $R-h$ too, becoming $(y_h + R - h - 1 \searrow 1)$, establishing (4).

So by (2), (3) and (4), $[Q, \mathbf{u}]$ is certainly α_0-equivalent to

$$[\boldsymbol{\lambda}_{Q'}(y_h + R - h - 1 \searrow 1)\, \underbrace{(u_1 \searrow 3)\ldots(u_{i-1} \searrow 3)}\, \underbrace{(u_{i+1} - 1 \searrow 2)\ldots(u_{k-2} - 1 \searrow 2)}(l \searrow 2)1(2 \nearrow l)*], \quad (5)$$

noting that we have also taken the opportunity to apply 'middle to right' moves to remove the first * in (2).

Calculation of u′: Having obtained $\boldsymbol{\lambda}_{Q'}$ as a subword, we must now show that (5) is α_0-equivalent to some $C(Q', \mathbf{u}')$. Firstly,

$$(y_h + R - h - 1 \searrow 1)\, \underbrace{(u_1 \searrow 3)\ldots(u_{i-1} \searrow 3)} \equiv \underbrace{(u_1 - 1 \searrow 2)\ldots(u_{i-1} - 1 \searrow 2)}\, (y_h + R - h - 1 \searrow 1) \quad (6)$$

because $y_h + R - h - 1 \geq R = h + u_i - 2$ by (1), which is $\geq u_i - 1 \geq u_{i-1}$. So, we are led to consider the word σ, defined by

$$\sigma := (y_h + R - h - 1 \searrow 1)\, \underbrace{(u_{i+1} - 1 \searrow 2)\ldots(u_{k-2} - 1 \searrow 2)},$$

which motivates us to define m to be the largest subscript $\geq i$ for which $u_m - 1 \leq y_h + R - h - 1$. (Certainly m exists, because this inequality holds with $m = i$, using (1).) We claim that

$$\sigma \equiv \underbrace{(u_{i+1} - 2 \searrow i+1)\ldots(u_m - 2 \searrow m)}(y_h + R - h - 1 \searrow m+1)\, \underbrace{(u^-_{m+1} \searrow m+2)\ldots(u^-_{k-2} \searrow k-1)}\, \omega, \quad (7)$$

where ω is defined to be

$$\omega := (m \searrow 1)\, \underbrace{(i+1 \searrow 2)\ldots(k-2 \searrow 2)}.$$

Proof of (7). Starting from the definition of σ, write

$$(u_{i+1} - 1 \searrow 2)\ldots(u_{k-2} - 1 \searrow 2) \sim \underbrace{(u_{i+1} - 1 \searrow i+2)\ldots(u_{k-2} - 1 \searrow k-1)}\, \underbrace{(i+1 \searrow 2)\ldots(k-2 \searrow 2)}. \quad \text{(7a)}$$

Now, because $y_h + R - h - 1 \geq u_m - 1$ we easily see that

$$(y_h + R - h - 1 \searrow 1)\, \underbrace{(u_{i+1} - 1 \searrow i+2)\ldots(u_m - 1 \searrow m+1)}$$

is braid-equivalent to

$$\underbrace{(u_{i+1} - 2 \searrow i+1)\ldots(u_m - 2 \searrow m)}(y_h + R - h - 1 \searrow 1). \quad \text{(7b)}$$

Of course, $u_m - 1$, and hence $y_h + R - h - 1$ is $\geq m+1$, so we may split the last factor of (7b) into $(y_h + R - h - 1 \searrow m+1)(m \searrow 1)$. Finally, since $(m \searrow 1)$ commutes with $(u_{m+1}^- \searrow 1)\dots(u_{k-2}^- \searrow k-1)$, equation (7) follows by substituting (7a) and (7b) into the defining expression for $\boldsymbol{\sigma}$.

Insert (6) followed by (7) into (5), to obtain a new (lefthanded) expression which is α_0-equivalent to $[Q, \mathbf{u}]$. Since $\mathrm{supp}(\boldsymbol{\omega}) \subseteq \{1, 2, \dots, k-2\}$, we may apply either a 'left to right' move, or a 'left to middle', 'middle to right' sequence for each of its letters, whilst staying in the same α_0-component. Also note that we have $1 \in \mathrm{supp}(\boldsymbol{\omega})$ only if $m \geq 1$, that is, only if $k > 2$. So, by 6.1.4, applying a 'left to right' move involving this letter 1 will not change the handedness. To summarise, we have shown that $[Q, \mathbf{u}]$ is α_0-equivalent to a lefthanded commutation class of the form

$$[\boldsymbol{\lambda}_{Q'} \underbrace{(u_1^- \searrow 2)\dots(u_{i-1}^- \searrow 2)} \underbrace{(u_{i+1} - 2 \searrow i+1)\dots(u_m - 2 \searrow m)}$$

$$(y_h + R - h - 1 \searrow m+1) \underbrace{(u_{m+1}^- \searrow m+2)\dots u_{k-2}^- \searrow k-1)}(l \searrow 2)1(2 \nearrow l)*]. \tag{8}$$

This suggests that we should define

$$\mathbf{u}' := (\underbrace{u_1, \dots, u_{i-1}}_{\text{unchanged}}, \underbrace{u_{i+1} - 1, \dots, u_m - 1}_{\text{empty if } m = i}, y_h + R - h, \underbrace{u_{m+1}, \dots, u_{k-2}}_{\text{empty if } m = k-2}). \tag{9}$$

By definition of m, this is an *increasing* $(k-2)$-tuple; let the entries be denoted $u_1' < u_2' < \dots < u_{k-2}'$ for brevity.

We claim that $C(Q', \mathbf{u}')$ is α_0-equivalent to (8). Well, in the standard expression for $C(Q', \mathbf{u}')$, the associated subword

$$\boldsymbol{\lambda}_{\mathbf{u}'} := (u_1' - 1 \searrow 2)\dots(u_{k-2}' - 1 \searrow 2)$$

is easily seen to be commutation-equivalent to

$$\underbrace{(u_1^- \searrow 2)\dots(u_{i-1}^- \searrow 2)}\underbrace{(u_i' - 1 \searrow i+1)\dots(u_{k-2}' - 1 \searrow k-1)}\underbrace{(i \searrow 2)\dots(k-2 \searrow 2)}_{\boldsymbol{\omega}'},$$

where $\boldsymbol{\omega}'$ is defined as indicated. Since $\mathrm{supp}(\boldsymbol{\omega}') \subseteq \{2, \dots, k-2\}$, we may, starting with $C(Q', \mathbf{u}')$, apply $\ell(\boldsymbol{\omega}')$ 'left to middle' moves and so obtain a commutation class which is visibly α_0-equivalent to the expression in (8). Thus, $[Q, \mathbf{u}]$ is α_0-equivalent to $C(Q', \mathbf{u}')$.

The general case: The proof is finished by iterating the previous procedure of deleting some $(y_h - 1 \searrow 1)$ from our current expression and then calculating a new $\mathbf{u}'$. In detail, define the increasing sequence of numbers

$$h_1 := R + 2 - u_{k-2}, \ h_2 := R + 2 - u_{k-3}, \dots, h_{k-2} := R + 2 - u_1, \tag{10}$$

so that Q by construction contains at least these $k-2$ right arrows:

$$\rightarrow y_{h_1}, \ \rightarrow y_{h_2}, \dots, \ \rightarrow y_{h_{k-2}}.$$

(This in noted in the statement of the Lemma.) Now Q' is any one of the possible 2^{k-2} quivers obtainable from Q by reversing some subset of these right arrows. In what order should we reverse our subset of arrows? Well, from (10) we see that *low* values of h_j correspond to *high* values of the u_i and hence to high values of i. And recall from (9) that the first $i-1$ entries of our new $\mathbf{u}'$ remain unchanged. Thus, we should reverse the *leftmost* of our subset of arrows *first*, that is, the arrow corresponding to the lowest h_j. Thus, $[Q, \mathbf{u}]$ is α_0-equivalent to $C(Q', \mathbf{u}')$ for some $\mathbf{u}'$, which in general will depend upon Q and Q' in a complicated way. ∎

6.2.15 COROLLARY. Each $C(Q', \mathbf{u}')$ lies in the same α_0-component as some $[Q, \mathbf{u}]$.

PROOF. We will use a simple counting argument. Fix k. Consider a particular $[Q, \mathbf{u}]$ with α_0-value k. By Lemma 6.2.14, $[Q, \mathbf{u}]$ lies in the same α_0-component as (at least) 2^{k-2} distinct commutation classes of the form $C(Q', \mathbf{u}')$; distinctness is proven in 6.2.3.

By 6.2.6 there are $2^{l-k}\binom{l-2}{k-2}$ possibilities for $[Q, \mathbf{u}]$. Further, by 6.2.7, we are in no danger of duplicating any of the above $C(Q', \mathbf{u}')$ as Q and $\mathbf{u}$ vary. So, we have counted

$$2^{l-k}\binom{l-2}{k-2}.2^{k-2}$$

commutation classes $C(Q', \mathbf{u}')$ which lie in the same α_0-component as some $[Q, \mathbf{u}]$. Summing over k, Lemma 6.2.3 shows that we have in fact counted *all* of the $C(Q', \mathbf{u}')$, hence the result. ▎

Intuitively, the 2^{k-2} commutation classes $C(Q', \mathbf{u}')$ which belong to the same α_0-component as some $[Q, \mathbf{u}]$ (with α_0-value k) may be viewed as the vertices of a $(k-2)$-dimensional hypercube, $[Q, \mathbf{u}]$ merely being a particular vertex.

We now have enough information to start counting the α_0-components.

6.2.16 PROPOSITION. Each α_0-component of $\mathcal{C}(B_l)$ which contains some lefthanded commutation class contains precisely one commutation class of the form $[Q, \mathbf{u}]$. So, the total number of such α_0-components is 3^{l-2}.

PROOF. Take any lefthanded $C \in \mathcal{C}(B_l)$. By 6.2.13 we have $\varepsilon(C) \in \operatorname{Im} f_{Q',\mathbf{u}'}$ for some Q' and $\mathbf{u}'$. So, by 6.2.9 we have $\varepsilon(C) = \varepsilon(C')$ for some lefthanded C' which is α_0-equivalent to $C(Q', \mathbf{u}')$. By 6.1.2 (b) it follows that C is α_0-equivalent to $C(Q', \mathbf{u}')$, whence by 6.2.15, C is α_0-equivalent to some $[Q, \mathbf{u}]$. Finally, 6.2.7 says that C cannot be α_0-equivalent to any other commutation class of this form. (The number 3^{l-2} was calculated in 6.2.6.) ▎

6.2.17 THEOREM. Consider $\mathcal{C}(B_l)$.

(a) An α_0-component contains both lefthanded and righthanded vertices if and only if it has α_0-value 2.

(b) Each α_0-component with α_0-value 2 contains exactly two quiver-compatible vertices $[Q]$ and $[Q']$, where Q differs from Q' only in the orientation of the last edge $\{l-1, l\}$.

(c) The total number of α_0-components is $2.3^{l-2} - 2^{l-2}$.

PROOF.

(a) If some α_0-component contains lefthanded and righthanded vertices then, being a connected subgraph, we can find an *adjacent* such pair, whence by 6.1.4, the α_0-value is 2.

Conversely, consider any $C \in \mathcal{C}(B_l)$ with α_0-value 2. It is easy to see that C^{rev} also has α_0-value 2, and that either C or C^{rev} is lefthanded.

Suppose first that C is lefthanded. Then 6.2.16 says that C is α_0-equivalent to some $[Q, \mathbf{u}]$, which by Note 6.2.5 (b) is just $[Q]$. By 6.1.9, Q necessarily contains $l-1 \longrightarrow l$, and defining Q' to be the quiver obtained by reversing this arrow, 6.1.9 also implies that C is α_0-equivalent to $[Q']$, which is righthanded.

Now suppose that C^{rev} is lefthanded. Arguing as above, the α_0-component of C^{rev} contains some lefthanded $[Q]$ and righthanded $[Q']$. Thus, the α_0-component of C contains the righthanded $[Q]^{\text{rev}}$ and lefthanded $[Q']^{\text{rev}}$. This is enough to prove the result, but recall from 3.1.10 that $[Q]^{\text{rev}} = [-Q]$, where $-Q$ is the result of reversing every arrow in Q. Similarly, $[Q']^{\text{rev}} = [-Q']$.

(b) We saw in the proof of part (a) that any α_0-component with α_0-value 2 contains *at least* two vertices of the form $[Q]$ and $[Q']$, as described in this Lemma. If this α_0-component contains a *third* quiver-compatible vertex $[Q'']$ then without loss of generality (applying reversal of words if necessary), Q'' and Q both contain $l-1 \longrightarrow l$. Now, remembering that $[Q''] = [Q'', ()]$ and $[Q] = [Q, ()]$, we obtain $Q = Q''$ from 6.2.7.

(c) By a symmetrical version of Proposition 6.2.16 (applying reversal), the number of α_0-components containing some *righthanded* vertex is also 3^{l-2}. However, from (a) and (b), the number of α_0-components containing both lefthanded and righthanded vertices is exactly 2^{l-2}, half the number of quivers. The result follows at once. ▎

6.3 The Face Poset of a Hypercube

In this section we determine the edges of $\mathcal{C}(B_l)/\alpha_0$. As usual, we shall consider the lefthanded half first. Fix an arbitrary commutation class

$$\widehat{C} := [\boldsymbol{\lambda}(l-1 \searrow 2)\boldsymbol{\mu}1(2 \nearrow l-1)\boldsymbol{\varrho}] \in \mathcal{C}(B_{l-1})$$

throughout this section. Recall from 2.3.11 the map $\delta : \mathcal{C}(B_l) \twoheadrightarrow \mathcal{C}(B_{l-1})$.

6.3.1 LEMMA. Let C be any vertex in the δ-fibre of $\widehat{C}$, such that C is lefthanded and has α_0-value k. Then C is α_0-equivalent to the following lefthanded vertex of the same fibre:

$$[\boldsymbol{\lambda}(l-1 \searrow 2)\boldsymbol{\mu}1(2 \nearrow k-1)\,(l \searrow 2)1(2 \nearrow l)\,(k \nearrow l-1)\boldsymbol{\varrho}].$$

PROOF. Let $\mathbf{a}(l \searrow 2)\mathbf{b}1(2 \nearrow l)\mathbf{c}$ be a normal representative of C. Considering $\delta(C)$ we have

$$\mathbf{a}\mathbf{b}^{-}\mathbf{c} \sim \boldsymbol{\lambda}(l-1 \searrow 2)\boldsymbol{\mu}1(2 \nearrow l-1)\boldsymbol{\varrho}. \tag{1}$$

Consider the letter 1 adjacent to $\boldsymbol{\mu}$, above; the corresponding root is $\alpha_1 + \cdots + \alpha_l$ by 2.3.4 (b), which equals e_{l-1} in terms of the standard basis (see 1.2.2). Since C is lefthanded, 6.1.1 implies that $s_{\mathbf{a}}^{-1}$ sends e_{l-1} to a *negative* root. It follows that e_{l-1} is the root corresponding to some letter of $\mathbf{a}$ (in the normal representative of C). Put differently, as commutations are applied to the right side of (1), this letter 1 ends up as a letter in $\mathbf{a}$. Hence, by inspection, the whole subword $\boldsymbol{\lambda}(l-1 \searrow 2)$ lying to the left of this letter 1 also commutes into $\mathbf{a}$. Consequently we have $\mathbf{a} \sim \boldsymbol{\lambda}(l-1 \searrow 2)1\mathbf{a}_1$ for some $\mathbf{a}_1$, which, using (1) implies that

$$\mathbf{a}_1\mathbf{b}^{-}\mathbf{c} \sim \boldsymbol{\mu}(2 \nearrow l-1)\boldsymbol{\varrho}. \tag{2}$$

Now, $\mathbf{a}_1$ is clearly an initial part of the right side, so by 5.6.1 we have some commutation-equivalence

$$\mathbf{a}_1 \sim \boldsymbol{\mu}'(2 \nearrow k-1)\boldsymbol{\varrho}', \tag{3}$$

where $\boldsymbol{\mu} \sim \boldsymbol{\mu}'\boldsymbol{\mu}''$ and $\boldsymbol{\varrho} \sim \boldsymbol{\varrho}'\boldsymbol{\varrho}''$, and for some k (which will turn out to be the α_0-value of C).

Comparing (2) and (3) we see that $\boldsymbol{\mu}'(2 \nearrow k-1)\boldsymbol{\varrho}'$ is an initial part of

$$\boldsymbol{\mu}'\boldsymbol{\mu}''(2 \nearrow k-1)(k \nearrow l-1)\,\boldsymbol{\varrho}'\boldsymbol{\varrho}''.$$

Visibly, $\boldsymbol{\mu}''$ must be able to commute past $(2 \nearrow k-1)$, and $\boldsymbol{\varrho}'$ must be able to commute past $(k \nearrow l-1)$. Thus,

$$\operatorname{supp}(\boldsymbol{\mu}'') \subseteq \{k+1, \ldots, l-1\} \text{ and } \operatorname{supp}(\boldsymbol{\varrho}') \subseteq \{1, \ldots, k-2\}.$$

(Of course, $\boldsymbol{\varrho}'$ now automatically commutes with $\boldsymbol{\mu}''$.) Eliminating $\mathbf{a}_1$ between (2) and (3) yields

$$\mathbf{b}^{-}\mathbf{c} \sim \boldsymbol{\mu}''(k \nearrow l-1)\boldsymbol{\varrho}''. \tag{4}$$

Substitute (3) into the normal representative for C to obtain

$$C = [\boldsymbol{\lambda}(l-1 \searrow 2)1\,\boldsymbol{\mu}'(2 \nearrow k-1)\boldsymbol{\varrho}'\,(l \searrow 2)\mathbf{b}1(2 \nearrow l)\mathbf{c}]. \tag{5}$$

To verify that k is the α_0-value of C, calculate the positive root

$$s_{\boldsymbol{\lambda}}s_{(l-1\searrow 2)}s_1\, s_{\boldsymbol{\mu}'}s_{(2\nearrow k-1)}s_{\boldsymbol{\varrho}'}\, s_{(l\searrow k+1)}(\alpha_k).$$

This equals α_0, using the restrictions on the supports of $\boldsymbol{\varrho}'$, $\boldsymbol{\mu}'$ and $\boldsymbol{\lambda}$. Thus, k is the α_0-value of C.

Now, apply $\ell(\mathbf{b})$ 'middle to right' moves to (5) and then substitute for $\mathbf{b}^{-}\mathbf{c}$ from (4), so that C is α_0-equivalent to

$$[\boldsymbol{\lambda}(l-1 \searrow 2)1\,\boldsymbol{\mu}'(2 \nearrow k-1)\boldsymbol{\varrho}'\,(l \searrow 2)1(2 \nearrow l)\,\boldsymbol{\mu}''(k \nearrow l-1)\boldsymbol{\varrho}''].$$

As $1 \notin \mathrm{supp}(\mu'')$ we may perform $\ell(\mu'')$ 'right to middle' moves, and then, since $\mathrm{supp}((\mu'')^+) \subseteq \{k+2,\ldots,l\}$, we may perform the same number of 'middle to left' moves whilst staying in the same α_0-component. So C is α_0-equivalent to the (lefthanded) vertex

$$[\lambda(l-1 \searrow 2)1\,\mu'(2 \nearrow k-1)\varrho'\,\mu''\,(l \searrow 2)1(2 \nearrow l)\,(k \nearrow l-1)\varrho''],$$

which equals

$$[\lambda(l-1 \searrow 2)1\mu(2 \nearrow k-1)\varrho'\,(l \searrow 2)1(2 \nearrow l)\,(k \nearrow l-1)\varrho'']$$

because μ'' commutes with $(2 \nearrow k-1)\varrho'$ and $\mu \sim \mu'\mu''$.

Recall that $\mathrm{supp}(\varrho') \subseteq \{1,\ldots,k-2\}$. If $k=2$ then ϱ' is the null word, so suppose that $k \neq 2$. If the rightmost letter of ϱ' is not 1, we can apply a 'left to middle', 'middle to right' sequence, otherwise simply apply a 'left to right' move. These moves leave unchanged the α_0-value. Nor do they change the handedness, by 6.1.1. So, iterating the above for each letter of ϱ' we see that C is α_0-equivalent to the lefthanded vertex

$$[\lambda(l-1 \searrow 2)1\mu(2 \nearrow k-1)\,(l \searrow 2)1(2 \nearrow l)\,\varrho'(k \nearrow l-1)\varrho''].$$

Now commute ϱ' past $(k \nearrow l-1)$ and write $\varrho \sim \varrho'\varrho''$ to obtain

$$[\lambda(l-1 \searrow 2)1\mu(2 \nearrow k-1)\,(l \searrow 2)1(2 \nearrow l)\,(k \nearrow l-1)\varrho],$$

which clearly lies in the δ-fibre of $\widehat{C}$. ∎

We want to determine which α_0-components intersect the δ-fibre of $\widehat{C}$. Recall that ϱ is involved in the definition of $\widehat{C}$. As in type A_l, the edge of the Coxeter graph B_l with vertices $i, i+1$ is **edge number** i.

6.3.2 Definition. Let the data s_ϱ and k $(2 \leq k \leq l)$ be given. We will construct a commutation class of the form $[Q,\mathbf{u}]$ with α_0-value k, as follows.

- Set $\pi := s_\varrho^{-1} = \begin{pmatrix} 1 & 2 & \cdots & l-2 \\ \pi(1) & \pi(2) & \cdots & \pi(l-2) \end{pmatrix} \in \mathcal{W}(B_{l-2})$.
- Let $\pi_1 < \cdots < \pi_{k-2}$ be the unsigned numbers $|\pi(1)|,\ldots,|\pi(k-2)|$ arranged in increasing order.
- Orient edge numbers $\pi_1,\ldots,\pi_{k-2}$ with right arrows.

And for $i = k-1,\ldots,l-2$, orient edge $|\pi(i)|$ with a $\begin{cases} \text{right arrow} & \text{if } \pi(i) > 0, \\ \text{left arrow} & \text{if } \pi(i) < 0. \end{cases}$

- Orient the last edge with a right arrow: $l-1 \rightarrow l$.
- Let Q_k denote the resulting quiver, with R right arrows $\rightarrow y_1,\ldots, \rightarrow y_R$, where $1 < y_1 < \cdots < y_R = l$.
- Implicitly define $u_1,\ldots,u_{k-2}$ via $\pi_{k-1-j} = y_{R+2-u_j} - 1$ for $j = 1,\ldots,k-2$. Set $\mathbf{u}_k := (u_1,\ldots,u_{k-2})$.
- We thus obtain the commutation class $[Q_k,\mathbf{u}_k]$, with α_0-value k.

6.3.3 Note. A moment's thought shows that the symbol $[Q_k,\mathbf{u}_k]$ makes sense, that is, $R \geq k-1$ and $\mathbf{u}_k$ is an increasing sequence with entries between 3 and $R+1$ inclusive. We have suppressed the dependence upon ϱ in our notation because $\widehat{C}$ is considered to be fixed in advance.

To illustrate the definition, suppose that $l=10$, $k=5$ and $s_\varrho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ -3 & 4 & -6 & -7 & -1 & 8 & 2 & 5 \end{pmatrix}$. Then $\pi := s_\varrho^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ -5 & 7 & -1 & 2 & 8 & -3 & -4 & 6 \end{pmatrix}$. We have $\pi_1 = 1$, $\pi_2 = 5$ and $\pi_3 = 7$, so these edges are right arrows. Also, edges 2, 8, 6 are right arrows and edges 3, 4 are left arrows. Edge 9 is the final right arrow. Thus $Q_5 = 1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10$.

The $R=7$ right arrows $\rightarrow y_j$ give us $y_1 = 2$, $y_2 = 3$, $y_3 = 6$, $y_4 = 7$, $y_5 = 8$, $y_6 = 9$ and $y_7 = 10$. Now,

$$\begin{aligned} \pi_3 = y_{9-u_1} - 1 \quad &\text{implies} \quad y_{9-u_1} = 8, \quad \text{hence} \quad u_1 = 4, \\ \pi_2 = y_{9-u_2} - 1 \quad &\text{implies} \quad y_{9-u_2} = 6, \quad \text{hence} \quad u_2 = 6, \\ \pi_1 = y_{9-u_3} - 1 \quad &\text{implies} \quad y_{9-u_3} = 2, \quad \text{hence} \quad u_3 = 8, \end{aligned}$$

so that $\mathbf{u}_5 = (u_1,u_2,u_3) = (4,6,8)$.

The following is a key observation.

6.3.4 Proposition. The lefthanded subgraph of the δ-fibre of $\widehat{C}$ intersects precisely those α_0-components containing $[Q_2, \mathbf{u}_2], \ldots, [Q_l, \mathbf{u}_l]$.

Proof. Let C be any lefthanded vertex of the δ-fibre of $\widehat{C}$ and let k be the α_0-value of C. The main task will be to show that C is α_0-equivalent to $[Q_k, \mathbf{u}_k]$. Assuming the truth of this, we can deduce that the fibre intersects the α_0-component of each $[Q_i, \mathbf{u}_i]$. For every fibre contains some *lefthanded* vertex of the form $[\omega(l \searrow 2)1(2 \nearrow l)]$ (with α_0-value l) and also the *righthanded* vertex $[(l \searrow 2)1(2 \nearrow l)\omega]$. Since the fibre is *connected*, it follows from 6.1.4 that it must in particular contain some lefthanded vertex with α_0-value 2. So, since the subgraph of lefthanded vertices in the fibre is also connected (by 6.1.7), and since α_0-values can change by at most 1 as we pass between adjacent vertices, it follows that the fibre contains lefthanded vertices with α_0-values from 2 to l, and thus intersects the α_0-components of $[Q_2, \mathbf{u}_2]$ to $[Q_l, \mathbf{u}_l]$ respectively, completing the proof.

So now let us return to our lefthanded C with α_0-value k. By 6.3.1, C is α_0-equivalent to the lefthanded vertex

$$D := [\boldsymbol{\lambda}(l-1 \searrow 2)\boldsymbol{\mu}1(2 \nearrow k-1)(l \searrow 2)1(2 \nearrow l)\,\boldsymbol{\varrho}].$$

For brevity, write $Q := Q_k$ and $\mathbf{u} := \mathbf{u}_k$ for the rest of the proof. Now, *if it happens that* $\varepsilon(D)$ belongs to $\operatorname{Im} f_{Q,\mathbf{u}}$ then we have $\varepsilon(D) = \varepsilon(C')$ for some C' in the same α_0-component as $C(Q, \mathbf{u}) = [Q, \mathbf{u}]$, by 6.2.9. Hence, D is α_0-equivalent to $[Q, \mathbf{u}]$ by 6.1.2 (b), which is what we want to show.

Recall from 6.2.12 that

$$\operatorname{Im} f_{Q,\mathbf{u}} = (\mathcal{W}(B_{\{2,3,\ldots,k-2\}}) \times \mathcal{W}(B_{\{k+1,k+2,\ldots,l-1\}}))\; s_{\boldsymbol{\lambda}_{\mathbf{u}}}^{-1}\, s_{\boldsymbol{\lambda}_Q}^{-1}\, w_0^{(l-1)},$$

where $\boldsymbol{\lambda}_Q$ and $\boldsymbol{\lambda}_{\mathbf{u}}$ (which have nothing to do with $\boldsymbol{\lambda}$ in the expression for $\widehat{C}$) were defined as follows:

$$\begin{aligned}\boldsymbol{\lambda}_Q &:= (y_1 - 1 \searrow 1)(y_2 - 1 \searrow 1)\ldots(y_R - 1 \searrow 1),\\ \boldsymbol{\lambda}_{\mathbf{u}} &:= (u_1 - 1 \searrow 2)(u_2 - 1 \searrow 3)\ldots(u_{k-2} - 1 \searrow k-1).\end{aligned}$$

We also have the restrictions $y_R = l$, $R \geq k-1$ and $3 \leq u_1 < \ldots < u_{k-2} \leq R+1$.

A brief calculation shows that $\varepsilon(D) = s_{(k \nearrow l-1)} s_{\boldsymbol{\varrho}}$. (Reassuringly, this depends only upon $\boldsymbol{\varrho}$ and k.) So, we wish to show that the element w defined as

$$w := s_{(k \nearrow l-1)} s_{\boldsymbol{\varrho}}\, w_0^{(l-1)} s_{\boldsymbol{\lambda}_Q} s_{\boldsymbol{\lambda}_{\mathbf{u}}}$$

belongs to the direct product of symmetric groups $\operatorname{Sym}(\{1, 2, \ldots, k-2\}) \times \operatorname{Sym}(\{k, k+1, \ldots, l-1\})$. We will do this in three steps.

Step 1: w fixes $k-1$.

Firstly, $s_{\boldsymbol{\lambda}_{\mathbf{u}}}$ sends $k-1$ to 1. Then $s_{\boldsymbol{\lambda}_Q}$ sends 1 to $-(y_R - 1) = -(l-1)$, which becomes $l-1$ under $w_0^{(l-1)}$. Next, $s_{\boldsymbol{\varrho}}$ fixes $l-1$ because $\operatorname{supp}(\boldsymbol{\varrho}) \subseteq \{1, \ldots, l-2\}$. Finally $s_{(k \nearrow l-1)}$ sends $l-1$ to $k-1$, as required.

Step 2: w permutes $\{1, 2, \ldots, k-2\}$.

Let $1 \leq \eta \leq k-2$. First, $s_{\boldsymbol{\lambda}_{\mathbf{u}}}$ sends η to $u_\eta - 1$. (Look at $\eta = k-2$ first to see how this works.) Now consider $s_{\boldsymbol{\lambda}_Q}$.

Since $y_R - 1 \geq R \geq u_\eta - 1 \geq 2$ it follows that $s_{(y_R - 1 \searrow 1)}$ sends $u_\eta - 1$ to $u_\eta - 2$.

Next, $y_{R-1} - 1 \geq R - 1 \geq u_\eta - 2 \geq 2$, so $s_{(y_{R-1} - 1 \searrow 1)}$ sends $u_\eta - 2$ to $u_\eta - 3$.

We may continue in this fashion until $s_{(y_{R+3-u_\eta} - 1 \searrow 1)}$ sends 2 to 1. The next factor, $s_{(y_{R+2-u_\eta} - 1 \searrow 1)}$, sends 1 to $-(y_{R+2-u_\eta} - 1)$, whose sign is then removed by $w_0^{(l-1)}$.

Now recall from Definition 6.3.2 that $s_{\boldsymbol{\varrho}}$ equals π^{-1}, which necessarily sends $y_{R+2-u_\eta} - 1 =: \pi_{k-1-\eta}$ to some element of $\{1, 2, \ldots, k-2\}$, as required.

Step 3: w permutes $\{k, k+1, \ldots, l-1\}$.

In fact, it is easier to consider w^{-1}. Since $w^{-1} \in \mathcal{W}(B_{l-1})$, it follows from the previous two steps that w^{-1} permutes $\{k, k+1, \ldots, l-1\}$, *but possibly introducing sign changes.* Let $k \leq \eta \leq l-1$; we know that $k \leq |w^{-1}(\eta)| \leq l-1$ but we must show that $w^{-1}(\eta) > 0$. So, consider

$$w^{-1} = s_{\lambda_{\mathbf{u}}}^{-1} s_{\lambda_Q}^{-1} w_0^{(l-1)} \pi \, s_{(l-1 \searrow k)},$$

where we have substituted π for s_{ϱ}^{-1}. Firstly, $s_{(l-1\searrow k)}$ sends η to $\eta - 1$; since this belongs to $\{k-1, \ldots, l-2\}$ there are two cases to consider.

Case A: $\pi(\eta-1) > 0$. Then Q must contain the right arrow $\rightarrow \pi(\eta-1)+1$, that is, $\pi(\eta-1)+1 = y_j$ for some $1 \leq j \leq R-1$. So, the image under $w_0^{(l-1)}$ is $-(y_j - 1)$. Consider now $s_{\lambda_Q}^{-1}$, represented by the word

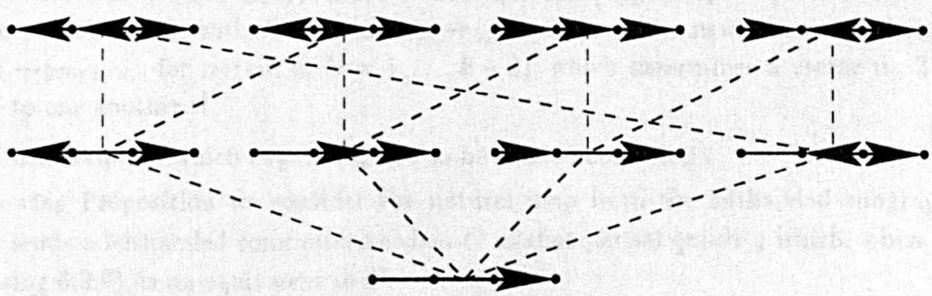
$$(1 \nearrow y_R - 1) \ldots (1 \nearrow y_2 - 1)(1 \nearrow y_1 - 1). \tag{$*$}$$

The rightmost $j-1$ factors leave $-(y_j - 1)$ fixed, but $s_{(1 \nearrow y_j - 1)}$ sends it to 1 (unsigned). The remaining factors successively increase the image by one, resulting in an *unsigned* number. Finally, because $1 \notin \operatorname{supp}(\lambda_{\mathbf{u}})$, the image must remain unsigned under $s_{\lambda_{\mathbf{u}}}^{-1}$, as required.

Case B: $\pi(\eta-1) < 0$. This time Q contains the left arrow $\leftarrow |\pi(\eta-1)|+1$, so for *no* j is $|\pi(\eta-1)|+1$ equal to y_j. Now, the image under $w_0^{(l-1)}$ is the unsigned number $\xi := -\pi(\eta-1)$. Considering ($*$), let m be the least suffix for which $y_m - 1 \geq \xi$. But equality has already been ruled out, so $y_m - 1 > \xi$. Thus, the rightmost $m-1$ factors of ($*$) fix ξ, and $s_{(1 \nearrow y_m - 1)}$ sends ξ to $\xi + 1$. Similarly, we may show that $\xi + 1 < y_{m+1} - 1$ and hence $s_{(1 \nearrow y_{m+1} - 1)}$ sends $\xi + 1$ to $\xi + 2$. Continuing in this way, the image under $s_{\lambda_Q}^{-1}$ is unsigned, which again remains unsigned under $s_{\lambda_{\mathbf{u}}}^{-1}$, as required. ∎

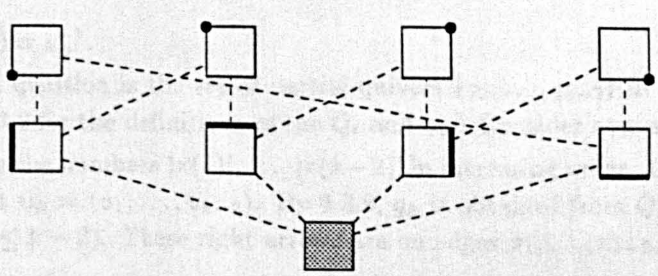
A **partial quiver** is an orientation of the Coxeter graph in which edges can remain unoriented; for example $q := 1 \rightarrow 2 - 3 \leftarrow 4$ has an unoriented edge. Define $\mathcal{Q}(B_l)^+$ to be the set of all partial quivers containing the right arrow $l-1 \rightarrow l$, a set of cardinality 3^{l-2}. Define a partial order on $\mathcal{Q}(B_l)^+$ as follows. Let $q, q' \in \mathcal{Q}(B_l)^+$. Set $q \leq q'$ if every oriented edge in q is also oriented in q', and these orientations agree.

6.3.5 FIGURE. The partial quiver poset $\mathcal{Q}(B_4)^+$.



In fact $\mathcal{Q}(B_l)^+$ is the **face poset** of an $(l-2)$-dimensional hypercube. Consider the n-cube $[-1,1]^n$. The d-dimensional faces are the d-cubes with vertices $(*, \ldots, *, \pm 1, *, \ldots, *, \pm 1, \ldots)$, where each $*$ is a fixed 1 or -1, and ± 1 occurs d times. The face poset is defined by containment of faces with the n-cube itself at the bottom. See Figure 6.3.6 for the $n = 2$ case.

6.3.6 FIGURE. The face poset of a 2-cube.



To see the correspondence, select any $(k-2)$-dimensional face of the $(l-2)$-cube and consider the coordinates of all the vertices in the face. If the i^{th} coordinate remains constant (1 or -1), orient edge i with a right arrow if it is 1, otherwise a left arrow. The $k-2$ coordinates which change correspond to unoriented edges. Finally, the last edge is oriented as a right arrow. It is easy to check that this prescription gives an order-isomorphism between the face poset and $\mathcal{Q}(B_l)^+$. We find it notationally easier to work with partial quivers rather than faces of a hypercube.

The number of maximal chains in $\mathcal{Q}(B_l)^+$ is $2^{l-2}(l-2)!$, the order of $\mathcal{W}(B_{l-2})$. (After all, the Coxeter group of type B_l is the **hyperoctahedral group**.) We shall now describe a bijection between these objects, which is surely known (especially if formulated in terms of faces of a hypercube rather than partial quivers).

6.3.7 Lemma. Consider any maximal chain in $\mathcal{Q}(B_l)^+$. Let Q be the quiver at the top of the chain. Let $e_{(1)}, e_{(2)}, \ldots, e_{(l-2)}$ be the numbers of the edges which successively become *unoriented* as we proceed to the bottom of the chain, one step at a time. We associate to our maximal chain that signed permutation of $\{1,2,\ldots,l-2\}$ which maps i to $\begin{cases} +e_{(i)} & \text{if edge } e_{(i)} \text{ of } Q \text{ is a right arrow,} \\ -e_{(i)} & \text{if edge } e_{(i)} \text{ of } Q \text{ is a left arrow.} \end{cases}$ Each element of $\mathcal{W}(B_{l-2})$ occurs in this manner exactly once. ∎

We now define the following variant of ε, namely

$$\begin{aligned} \widehat{\varepsilon} : \mathcal{C}(B_{l-1}) &\rightarrow \mathcal{W}(B_{l-2}) \\ \widehat{C} := [\lambda(l-1 \searrow 2)\mu 1(2 \nearrow l-1)\varrho] &\mapsto s_{\varrho}^{-1}. \end{aligned}$$

6.3.8 Lemma. The map $\widehat{\varepsilon}$ is surjective.

Proof. Let $w \in \mathcal{W}(B_{l-2})$ and consider any $\mathbf{ij} \in \mathcal{R}(B_{l-1})$ for which $s_{\mathbf{i}} = w^{-1}$. Starting from the word $(l-1 \searrow 2)1(2 \nearrow l-1)\mathbf{ij}$, promote $\ell(\mathbf{j})$ times and so consider $[\mathbf{j}(l-1 \searrow 2)1(2 \nearrow l-1)\mathbf{i}]$, which maps to w. ∎

6.3.9 Lemma. There is a bijection between the partial quivers in $\mathcal{Q}(B_l)^+$ and the symbols $[Q, \mathbf{u}]$.

Proof. Given $[Q,\mathbf{u}]$, the quiver Q determines R right arrows $\rightarrow y_j$ where $1 < y_1 < \cdots < y_R = l$, and $\mathbf{u}$ determines integers $3 \le u_1 < \cdots < u_{k-2} \le R+1$ for some k. Simply replace the right arrows $\rightarrow y_{R+2-u_i}$ $(i = 1, \ldots, k-2)$ by unoriented edges to give a partial quiver $q \in \mathcal{Q}(B_l)^+$.

In the other direction, let $q \in \mathcal{Q}(B_l)^+$ have $k-2$ unoriented edges, say. Replace every unoriented edge by a right arrow to obtain Q, with R right arrows $\rightarrow y_j$ as before. The newly-introduced right arrows will have the form $\rightarrow y_{R+2-u_i}$, for certain u_i $(i = 1, \ldots, k-2)$, which determines a vector $\mathbf{u}$. These maps are clearly inverse to one another. ∎

(Note that $\mathbf{u}$ determines which edges in Q are to be made *unoriented*.)

In the following Proposition we consider the natural map from the lefthanded subgraph of $\mathcal{C}(B_l)$ to $\mathcal{Q}(B_l)^+$ which sends a lefthanded commutation class C to that partial quiver q which, when identified with some $[Q, \mathbf{u}]$ (using 6.3.9), is α_0-equivalent to C.

6.3.10 Proposition. The image of the lefthanded subgraph of the δ-fibre of $\widehat{C}$ under the map

$$\{\text{All lefthanded } C \in \mathcal{C}(B_l)\} \rightarrow \mathcal{Q}(B_l)^+$$

is a maximal chain in the poset $\mathcal{Q}(B_l)^+$. This maximal chain is the signed permutation $\widehat{\varepsilon}(\widehat{C})$, via the identification in 6.3.7.

Proof. Define $\pi := \widehat{\varepsilon}(\widehat{C}) = s_{\varrho}^{-1}$.

By 6.3.4, the image in question is the set of partial quivers $\{q_2, \ldots, q_{k-2}\}$ in which each q_i is identified with $[Q_i, \mathbf{u}_i]$. (Refer to 6.3.2 for the definitions of the Q_i and $\mathbf{u}_i$.) Consider now a fixed $k \in \{2, 3, \ldots, l-1\}$.

Let $\pi_1 < \cdots < \pi_{k-2}$ be the numbers $|\pi(1)|, \ldots, |\pi(k-2)|$ in increasing order. Let Q_k have R right arrows $\rightarrow y_1, \ldots, \rightarrow y_R = l$ and let $\mathbf{u}_k = (u_1, \ldots, u_{k-2})$. By 6.3.9, q_k is obtained from Q_k by unorienting the right arrows $\rightarrow y_{R+2-u_i}$ $(1 \le i \le k-2)$. These right arrows are on edges $\pi_1, \ldots, \pi_{k-2}$, by the definition of $\mathbf{u}_k$ in 6.3.2.

Now let $\pi'_1 < \cdots < \pi'_{k-1}$ be the numbers $|\pi(1)|, \ldots, |\pi(k-2)|, |\pi(k-1)|$ in increasing order. In analogy to the previous paragraph, q_{k+1} is obtained from Q_{k+1} by unorienting the right arrows on edges $\pi'_1, \ldots, \pi'_{k-1}$.

The main task is to determine exactly how q_{k+1} differs from q_k. There are two cases.

Case 1: $\pi(k-1) > 0$.

By definition, edge $|\pi(k-1)|$ is a right arrow in both Q_k and Q_{k+1}, and these quivers clearly agree on all the other edges too, whence $Q_{k+1} = Q_k$. So q_{k+1} is obtained from $Q_{k+1} = Q_k$ by unorienting the right arrows on edges $\pi_1, \dots, \pi_{k-2}$ and also on edge $|\pi(k-1)|$. Clearly therefore,

$$q_{k+1} \text{ is obtained from } q_k \text{ by unorienting the \textit{right} arrow on edge } |\pi(k-1)|. \tag{1}$$

Case 2: $\pi(k-1) < 0$.

By definition, edge $|\pi(k-1)|$ is a left arrow in Q_k but a right arrow in Q_{k+1}, and these quivers agree on all the other edges. Therefore, since q_k is obtained from Q_k by unorienting only certain *right* arrows, and since edge $|\pi(k-1)|$ in Q_k is not a right arrow, it follows that

$$\text{edge } |\pi(k-1)| \text{ in } q_k \text{ is a left arrow.} \tag{$*$}$$

Now, q_{k+1} is obtained from Q_{k+1} by unorienting the right arrows on edges $\pi_1, \dots, \pi_{k-2}$ and also on edge $|\pi(k-1)|$. So, remembering $(*)$,

$$q_{k+1} \text{ is obtained from } q_k \text{ by unorienting the \textit{left} arrow on edge } |\pi(k-1)|. \tag{2}$$

A little thought shows that our maximal chain is indeed identified with π: in the notation of 6.3.7, just set $Q := q_2$ and $e_{(k-1)} := |\pi(k-1)|$. If edge $e_{(k-1)}$ of Q is a right arrow then $\pi(k-1) > 0$, hence $+e_{(k-1)} = \pi(k-1)$ by (1). On the other hand, if this edge is a left arrow then $\pi(k-1) < 0$, hence $-e_{(k-1)}$ again equals $\pi(k-1)$, by (2), as required. ∎

6.3.11 THEOREM. Those α_0-components which intersect the lefthanded half of $\mathcal{C}(B_l)$ form a connected subgraph of $\mathcal{C}(B_l)/\alpha_0$, isomorphic to the graph underlying the poset $\mathcal{Q}(B_l)^+$. Furthermore, each such α_0-component contains a unique $[Q, \mathbf{u}]$.

PROOF. The second clause is Proposition 6.2.16. Consider now any $[Q, \mathbf{u}]$, $[Q', \mathbf{u}']$ and let q, q' respectively be their corresponding partial quivers. We must show that q is adjacent to q' in $\mathcal{Q}(B_l)^+$ if and only if the α_0-components of $[Q, \mathbf{u}]$ and $[Q', \mathbf{u}']$ are adjacent.

First suppose that the α_0-components of $[Q, \mathbf{u}]$ and $[Q', \mathbf{u}']$ are adjacent. Then we can find a pair of adjacent representatives C and C' in $\mathcal{C}(B_l)$. We claim that C and C' are lefthanded; by 6.1.4 this is certainly true if their α_0-values are different from 2. Otherwise, suppose that C has α_0-value 2. Then C' has α_0-value 3, and as such is lefthanded. Now, if C is righthanded then by 6.1.4 again, C and C' both have α_0-value 2 — a contradiction. This proves that C and C' are lefthanded.

We claim that $\delta(C) = \delta(C')$; for otherwise C and C' would be related to one another by a 'left', 'middle' or 'right' move, hence $\varepsilon(C) = \varepsilon(C')$ and we reach a contradiction by 6.1.2 (b). So, suppose that C and C' lie in the δ-fibre of some $\widehat{C}$ (with the usual expression we have been using all along). Therefore, by the first clause of 6.3.10 it is clear that q is adjacent to q'.

Conversely, let q be adjacent to q' in $\mathcal{Q}(B_l)^+$. Then the edge $\{q, q'\}$ is part of some maximal chain in this face poset, which we shall identify with an element of $\mathcal{W}(B_{l-2})$ (using 6.3.7); since $\widehat{\varepsilon}$ is surjective, this element equals $\varepsilon(\widehat{C})$ for some $\widehat{C}$. The second clause of 6.3.10 says that the image under the natural map $\{\text{All lefthanded } C \in \mathcal{C}(B_l)\} \longrightarrow \mathcal{Q}(B_l)^+$ is our maximal chain. So by the *definition* of this mapping, there exist some lefthanded C, C' in the δ-fibre of $\widehat{C}$ such that C is α_0-equivalent to $[Q, \mathbf{u}]$ and C' is α_0-equivalent to $[Q', \mathbf{u}']$.

Next, using the hypothesis that q is adjacent to q', we may assume that q has one more unoriented edge than q'. So the α_0-values of $[Q, \mathbf{u}]$ and $[Q', \mathbf{u}']$ are respectively k and $k-1$, for some k. (The α_0-value is 2 plus the number of unoriented edges.) Note that $k \neq 2$. By 6.3.1, C is α_0-equivalent to the lefthanded vertex

$$D := [\lambda(l-1 \searrow 2)\mu 1(2 \nearrow k-1)\,(l \searrow 2)1(2 \nearrow l)\,(k \nearrow l-1)\varrho]$$

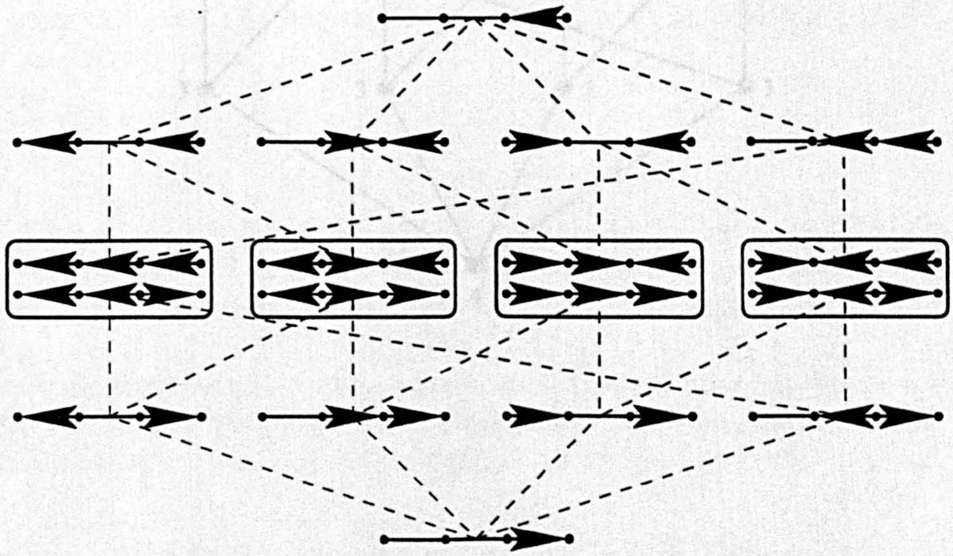
and C' is α_0-equivalent to the lefthanded vertex

$$D' := [\lambda(l-1 \searrow 2)\mu 1(2 \nearrow k-2)\,(l \searrow 2)1(2 \nearrow l)\,(k-1 \nearrow l-1)\varrho].$$

Visibly, we can apply a 'left to middle' move to D (since $k \neq 2$) and obtain a vertex which is α_0-equivalent to D' (just apply a 'middle to right' move). So C, C' and hence $[Q, \mathbf{u}]$, $[Q', \mathbf{u}']$ belong to adjacent α_0-components. ▮

Penultimate to the main result, we shall describe a symmetrical poset obtained from $\mathcal{Q}(B_l)^+$ by 'reflecting about the axis of quivers'. Formally, let $\mathcal{Q}(B_l)^-$ be the set of all partial quivers with the left arrow $l-1 \leftarrow l$, and partially order it as follows. Let q, $q' \in \mathcal{Q}(B_l)^-$ and let q^+, q'^+ be the respective members of $\mathcal{Q}(B_l)^+$ obtained by changing $l-1 \leftarrow l$ to $l-1 \rightarrow l$. Define $q \leq q'$ if and only if $q'^+ \leq q^+$. Finally, identify the pairs of quivers whose orientations differ only in the last edge. Let $\mathcal{Q}(B_l)^0$ denote the resulting poset.

6.3.12 FIGURE. The poset $\mathcal{Q}(B_4)^0$.



6.3.13 THEOREM. The graph of α_0-components $\mathcal{C}(B_l)/\alpha_0$ is isomorphic to the graph underlying $\mathcal{Q}(B_l)^0$.

PROOF. Recall from 6.3.11 that the subgraph of $\mathcal{C}(B_l)/\alpha_0$ consisting of those α_0-components intersecting the lefthanded half of $\mathcal{C}(B_l)$ is the graph of $\mathcal{Q}(B_l)^+$. Symmetrically, the subgraph of $\mathcal{C}(B_l)/\alpha_0$ consisting of righthanded-intersecting α_0-components is the graph of $\mathcal{Q}(B_l)^-$; of course, these graphs are isomorphic, despite the different notation. Certainly therefore, every vertex or edge in $\mathcal{Q}(B_l)^0$ corresponds to some vertex or edge in $\mathcal{C}(B_l)/\alpha_0$, respectively.

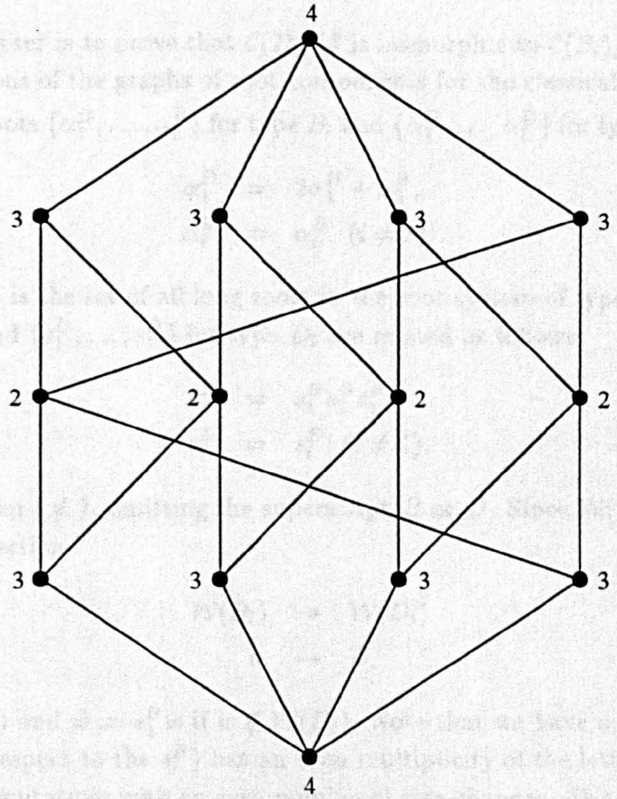
Conversely, we must check that every vertex or edge in $\mathcal{C}(B_l)/\alpha_0$ corresponds to some (unique) vertex or edge in the graph of $\mathcal{Q}(B_l)^0$.

If an α_0-component contains only lefthanded or only righthanded commutation classes then it corresponds to a vertex in *either* $\mathcal{Q}(B_l)^+$ *or* $\mathcal{Q}(B_l)^-$, and hence to a unique vertex in $\mathcal{Q}(B_l)^0$. Otherwise, by 6.2.17 this α_0-component contains some unique pair $[Q]$, $[Q']$ of opposite handedness to one another, and such that Q, Q' differ in the orientation of the last edge only. So, although this α_0-component corresponds to a vertex in both $\mathcal{Q}(B_l)^+$ and $\mathcal{Q}(B_l)^-$, it gives rise to a *unique* vertex in $\mathcal{Q}(B_l)^0$, by construction.

Now consider any pair of adjacent commutation class C, C' which belong to different α_0-components. By 6.1.4, C and C' necessarily have the *same* handedness. Thus, the edge in $\mathcal{C}(B_l)/\alpha_0$ induced by the edge $\{C, C'\} \subseteq \mathcal{C}(B_l)$ corresponds to an edge in *either* $\mathcal{Q}(B_l)^+$ *or* $\mathcal{Q}(B_l)^-$, and hence corresponds to a unique edge in $\mathcal{Q}(B_l)^0$. ▮

Finally, here is the graph $\mathcal{C}(B_4)/\alpha_0$, labelled with α_0-values.

6.3.14 FIGURE. The labelled graph of α_0-components in type B_4.



7. Root Components in Type D_l

7.1 Relationship with Type B_l

The purpose of this chapter is to prove that $\mathcal{C}(D_l)/\beta$ is isomorphic to $\mathcal{C}(B_l)/\beta$ for all $\beta \in \Phi^+(D_l)$. This will complete our calculations of the graphs of root components for the classical types.

Recall that the simple roots $\{\alpha_1^B, \ldots, \alpha_l^B\}$ for type B_l and $\{\alpha_1^D, \ldots, \alpha_l^D\}$ for type D_l are related as follows:

$$\begin{aligned} \alpha_1^D &= 2\alpha_1^B + \alpha_2^B, \\ \alpha_i^D &= \alpha_i^B \quad (i \neq 1). \end{aligned}$$

The root system of type D_l is the set of all long roots in the root system of type B_l. The simple reflections $\{s_1^B, \ldots, s_l^B\}$ for type B_l and $\{s_1^D, \ldots, s_l^D\}$ for type D_l are related as follows:

$$\begin{aligned} s_1^D &= s_1^B s_2^B s_1^B, \\ s_i^D &= s_i^B \quad (i \neq 1). \end{aligned}$$

We will write α_i and s_i when $i \neq 1$, omitting the superscript B or D. Since $\mathcal{W}(B_l) = \mathcal{W}(D_l) \dot{\cup} s_1^B \mathcal{W}(D_l)$ we can define the natural projection

$$\begin{aligned} \mathcal{W}(B_l) &\twoheadrightarrow \mathcal{W}(D_l) \\ w &\mapsto \widehat{w}, \end{aligned}$$

where $\widehat{w} := w$ if $w \in \mathcal{W}(D_l)$ and $\widehat{w} := s_1^B w$ if $w \notin \mathcal{W}(D_l)$. Note that we have $w \in \mathcal{W}(D_l)$ if and only if any reduced word for w (with respect to the s_i^B) has an *even* multiplicity of the letter 1; this is because $\mathcal{W}(D_l)$ consists of those signed permutations with an even number of sign changes. The longest elements are related via $\widehat{w_0^B} = w_0^D$.

Consider now the total order on $\Phi^+(B_l)$ induced by some $\mathbf{i} \in \mathcal{R}(B_l)$, and restrict to a total order on the long roots, $\Phi^+(D_l)$. I am grateful to Roger Carter for suggesting that this new total order might be induced by some $\widehat{\mathbf{i}} \in \mathcal{R}(D_l)$. This is indeed true, and the resulting map $\mathcal{R}(B_l) \to \mathcal{R}(D_l)$ is captured in the following simple algorithm.

7.1.1 Proposition. Let $\mathbf{i}$ be a reduced word (with respect to the s_i^B) for some $w \in \mathcal{W}(B_l)$. Immediately below each letter k of $\mathbf{i}$, write:

$$\begin{cases} k & \text{if } k \geq 3, \\ 2 & \text{if } k = 2 \text{ and the multiplicity of 1 in } \mathbf{i} \text{ to the right of } k \text{ is even}, \\ 1 & \text{if } k = 2 \text{ and the multiplicity of 1 in } \mathbf{i} \text{ to the right of } k \text{ is odd}, \\ \emptyset & \text{if } k = 1. \end{cases}$$

(Recall that $\emptyset$ is the null word.) The resulting word, denoted $\widehat{\mathbf{i}}$, is reduced (with respect to the s_i^D) for $\widehat{w}$.

Proof. The result is true if $\ell(\mathbf{i}) = 0$. Let $\mathbf{i}$ be reduced for w and assume inductively that $\widehat{\mathbf{i}}$ is reduced for $\widehat{w}$. Suppose now that $k\mathbf{i}$ is reduced. If $k = 1$ then $\widehat{k\mathbf{i}}$ equals $\widehat{\mathbf{i}}$, which is already reduced for $\widehat{w} = \widehat{s_1^B w}$, as required. If $k \neq 1$ then since $k\mathbf{i}$ is reduced, we have

$$w^{-1}(\alpha_k) \in \Phi^+(D_l), \tag{$*$}$$

because α_k is a long root.

If now $k \geq 3$ then $\widehat{k\mathbf{i}}$ equals $k\widehat{\mathbf{i}}$, so to establish that the latter word is reduced, we must show that $\widehat{w}^{-1}(\alpha_k^D)$ belongs to $\Phi^+(D_l)$. Well, $\widehat{w}$ equals either w or $s_1^B w$, and since s_1^B fixes $\alpha_k^B = \alpha_k^D$ we have our result by $(*)$. Thus $k\widehat{\mathbf{i}}$ is a reduced word for $s_k\widehat{w}$, which equals $\widehat{s_k w}$, since s_k commutes with s_1^B.

Now suppose that $k = 2$ and $\widehat{w} = w$ (that is, $\mathrm{mul}_1(\mathbf{i})$ is *even*). We have $\widehat{k\mathbf{i}} = 2\widehat{\mathbf{i}}$ and $\widehat{w}^{-1}(\alpha_2) \in \Phi^+(D_l)$ by $(*)$. So $2\widehat{\mathbf{i}}$ is reduced for $s_2\widehat{w} = s_2 w = \widehat{s_2 w}$.

Finally, suppose that $k = 2$ and $\widehat{w} = s_1^B w$ (that is, $\mathrm{mul}_1(\mathbf{i})$ is *odd*). We have $\widehat{k\mathbf{i}} = 1\widehat{\mathbf{i}}$ and $\widehat{w}^{-1}(\alpha_1^D) = w^{-1}s_1^B(2\alpha_1^B + \alpha_2)$, which equals $w^{-1}(\alpha_2)$ and so lies in $\Phi^+(D_l)$ by $(*)$. So $1\widehat{\mathbf{i}}$ is reduced for the element $s_1^D \widehat{w} = (s_1^B s_2 s_1^B) s_1^B w = \widehat{s_2 w}$. ∎

Here is an example of the previous Proposition.

$$\begin{array}{rccccccccccl} \mathbf{i} = & 1 & 2 & 3 & 2 & 1 & 2 & 3 & 1 & 2 & & \in \mathcal{R}(B_3) \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ \widehat{\mathbf{i}} = & & 2 & 3 & 2 & & 1 & 3 & & 2 & & \in \mathcal{R}(D_3) \end{array}$$

7.1.2 Proposition. Let $\beta \in \Phi^+(D_l)$.

(a) The map $\mathcal{R}(B_l) \to \mathcal{R}(D_l)$ given by $\mathbf{i} \mapsto \widehat{\mathbf{i}}$ is a morphism of graphs.

(b) If $\mathbf{i}$ is β-equivalent to $\mathbf{i}'$ in $\mathcal{R}(B_l)$ then $\widehat{\mathbf{i}}$ is β-equivalent to $\widehat{\mathbf{i}'}$.

Proof.

(a) The following is easily checked using 7.1.1. A 4-braid $1212 \overset{4}{\leftrightarrow} 2121$ induces a commutation $12 \sim 21$; a 3-braid $232 \overset{3}{\leftrightarrow} 323$ induces a 3-braid $a3a \overset{3}{\leftrightarrow} 3a3$ for some $a \in \{1,2\}$; and any commutation involving a letter 1 in $\mathcal{R}(B_l)$ gives the same word in $\mathcal{R}(D_l)$. The other types of braids in $\mathcal{R}(B_l)$ are exactly mirrored in $\mathcal{R}(D_l)$.

(b) Let $\mathbf{i}'$ differ from $\mathbf{i}$ by a single braid. If this braid does not involve the letter in $\mathbf{i}$ which corresponds to β then trivially $\widehat{\mathbf{i}}$ is β-equivalent to $\widehat{\mathbf{i}'}$. Otherwise, this braid must be a commutation or a 4-braid, both of which induce only commutations in $\mathcal{R}(D_l)$, hence the result. ∎

By virtue of the above Proposition we have a well defined map

$$\Theta : \mathcal{C}(B_l)/\beta \to \mathcal{C}(D_l)/\beta,$$

which sends the β-component containing $\mathbf{i} \in \mathcal{R}(B_l)$ to the β-component of $\widehat{\mathbf{i}}$. Further, Θ is certainly a morphism, by 7.1.2 (a).

We turn now to finding a candidate for the inverse of Θ. As usual, let β be a fixed long root. Take any $\mathbf{i} \in \mathcal{R}(D_l)$ and write

$$\mathbf{i} = \mathbf{a}\, k_\beta\, \mathbf{b},$$

where, as in chapter 4, the subscript indicates which letter corresponds to β. Thus, setting $w := s_{\mathbf{b}}^D$ we have $\beta = w^{-1}(\alpha_k^D)$. Certainly we have $w \in \mathcal{W}(B_l)$, so we can define $\boldsymbol{\omega}$ to be any reduced word with respect to the s_i^B for the element w (if $k \neq 1$) or the element $s_1^B w$ (if $k = 1$). So, if $k \neq 1$ then $(s_{\boldsymbol{\omega}}^B)^{-1}(\alpha_k) = w^{-1}(\alpha_k) = \beta$ and if $k = 1$ then $(s_{\boldsymbol{\omega}}^B)^{-1}(\alpha_2) = w^{-1}s_1^B(\alpha_2) = w^{-1}(\alpha_1^D) = \beta$. These calculations show that $k\boldsymbol{\omega}$ is reduced if $k \neq 1$, and $2\boldsymbol{\omega}$ is reduced if $k = 1$. Now extend the appropriate word to any *maximal-length* reduced word, yielding $\boldsymbol{\sigma} k_\beta \boldsymbol{\omega}$ or $\boldsymbol{\sigma} 2_\beta \boldsymbol{\omega}$, for some $\boldsymbol{\sigma}$. Although $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$ are only determined up to braid-equivalence, our maximal-length word belongs to an unambiguous β-component of $\mathcal{C}(B_l)$; accordingly we define the map

$$\begin{aligned} \psi : \mathcal{R}(D_l) &\to \mathcal{C}(B_l)/\beta \\ \mathbf{a}\, k_\beta\, \mathbf{b} &\mapsto \begin{cases} [\boldsymbol{\sigma}\, k_\beta\, \boldsymbol{\omega}]_\beta & \text{if } k \neq 1;\ (\boldsymbol{\omega} \text{ is reduced for } s_{\mathbf{b}}^D), \\ [\boldsymbol{\sigma}\, 2_\beta\, \boldsymbol{\omega}]_\beta & \text{if } k = 1;\ (\boldsymbol{\omega} \text{ is a reduced word for } s_1^B s_{\mathbf{b}}^D). \end{cases} \end{aligned}$$

(Recall that C_β stands for the β-component containing the commutation class C.)

7.1.3 Proposition. Let $\beta \in \Phi^+(D_l)$.

(a) If $\mathbf{i}$ is β-equivalent to $\mathbf{i}'$ in $\mathcal{R}(D_l)$ then $\psi(\mathbf{i}) = \psi(\mathbf{i}')$.

(b) The map $\psi : \mathcal{R}(D_l) \to \mathcal{C}(B_l)/\beta$ is a morphism of graphs.

Proof.

(a) Let $\mathbf{i}'$ be obtained from $\mathbf{i}$ by a single braid. If this braid does not involve the letter corresponding to β then clearly $\psi(\mathbf{i}) = \psi(\mathbf{i}')$. Otherwise, this braid must be a commutation, in order for the β-values of $\mathbf{i}$ and $\mathbf{i}'$ to agree.

$\boxed{\text{The commutation case.}}$

Write $\mathbf{i} = \mathbf{a}y_\beta x\mathbf{b}$ and $\mathbf{i}' = \mathbf{a}xy_\beta\mathbf{b}$, where x commutes with y in D_l. We must prove that $\psi(\mathbf{i}) = \psi(\mathbf{i}')$.

Case I: $y \neq 1$. By definition we have

$$\psi(\mathbf{i}) = [\sigma y_\beta \omega]_\beta \text{ and } \psi(\mathbf{i}') = [\sigma' y_\beta \omega']_\beta, \tag{1}$$

where ω is reduced for $s_x^D s_{\mathbf{b}}^D$ and ω' is reduced for $s_{\mathbf{b}}^D$.

Subcase I(i): $x \neq 1$. Since $s_x^B = s_x^D$ and $\alpha_x^B = \alpha_x^D$ we have $(s_\omega^B)^{-1}(\alpha_x^B) = -(s_{\mathbf{b}}^D)^{-1}(\alpha_x^D)$, which is a negative root because $x\mathbf{b}$ is reduced in type D_l. Thus $x\omega$ is nonreduced in type B_l, which implies that $\omega \equiv x\omega'$. So from (1),

$$\begin{aligned}
\psi(\mathbf{i}) &= [\sigma y_\beta x \omega']_\beta \\
&= [\sigma x y_\beta \omega']_\beta \text{ applying a commutation} \\
&= \psi(\mathbf{i}'),
\end{aligned}$$

as required, since $\sigma x \equiv \sigma'$ is forced by comparison with the expression for $\psi(\mathbf{i}')$ in (1).

Subcase I(ii): $x = 1$. This time ω is reduced for $s_1^D s_{\mathbf{b}}^D = s_1^B s_2 s_1^B s_{\mathbf{b}}^D$. Eliminating $s_{\mathbf{b}}^D$ from the definitions of s_ω^B and $s_{\omega'}^B$ we have

$$s_\omega^B = s_1^B s_2 s_1^B s_{\omega'}^B. \tag{2}$$

There are now three further possibilities to consider. Observe that if $1\omega'$ is reduced then so is $21\omega'$, for $(s_{\omega'}^B)^{-1} s_1^B(\alpha_2)$ is the positive root $(s_{\mathbf{b}}^D)^{-1}(\alpha_1^D)$ (by inspection of $\mathbf{i}$). Also note that $y \neq 3$, since y commutes with $x = 1$ in D_l.

Possibility A: $121\omega'$ is reduced. By (2) we then have $\omega \equiv 121\omega'$. So, if $y \geq 4$ then

$$\begin{aligned}
\psi(\mathbf{i}) &= [\sigma y_\beta 121\omega']_\beta \text{ using } \omega \equiv 121\omega' \\
&= [\sigma 121 y_\beta \omega']_\beta \text{ as } y \text{ commutes with 1 and 2} \\
&= \psi(\mathbf{i}').
\end{aligned}$$

And if $y = 2$ then

$$\begin{aligned}
\psi(\mathbf{i}) &= [\sigma 2_\beta 121\omega']_\beta \\
&= [\sigma 1212_\beta \omega']_\beta \text{ applying a 4-braid} \\
&= \psi(\mathbf{i}').
\end{aligned}$$

Possibility B: $121\omega'$ is nonreduced, but $21\omega'$ is reduced. By these hypotheses we have $21\omega' \equiv 1\omega''$ for some ω''. By (2) we obtain $\omega \equiv \omega''$, hence

$$21\omega' \equiv 1\omega. \tag{3}$$

Also by our hypotheses in Possibility B, we have $(s_{\omega'}^B)^{-1}(\alpha_1^B + \alpha_2) \in \Phi^-$. So, since $(s_{\omega'}^B)^{-1}(\alpha_1^B)$ is a *positive* root, it follows that $(s_{\omega'}^B)^{-1}(\alpha_2)$ is a *negative* root, whence $2\omega'$ is nonreduced and $\omega' \equiv 2\omega_1$ for some ω_1. Now, $1\omega_1$ is nonreduced because $(s_{\omega_1}^B)^{-1}(\alpha_1^B)$ is the negative root $(s_{\omega'}^B)^{-1}(\alpha_1^B + \alpha_2)$, whence $\omega_1 \equiv 1\omega_2$ for some ω_2. So,

$$\omega' \equiv 21\omega_2. \tag{4}$$

Eliminating ω' between (3) and (4), and applying a 4-braid $1212 \xleftrightarrow{4} 2121$ gives

$$\omega \equiv 212\omega_2. \tag{5}$$

As $y\omega$ is reduced (see (1)), we deduce from (5) that $y \neq 2$. Thus $y \geq 4$, which commutes with 1 and 2, so that $\psi(\mathbf{i}) = [\sigma 212 y_\beta \omega_2]_\beta$ (by (5)) and $\psi(\mathbf{i}') = [\sigma' 21 y_\beta \omega_2]_\beta$ (by (4)). Therefore $\psi(\mathbf{i}) = \psi(\mathbf{i}')$.

Possibility C: $1\omega'$ is nonreduced. This implies that

$$\omega' \equiv 1\omega'' \tag{6}$$

for some ω''. From (2) we therefore have

$$s_\omega^B = s_1^B s_2 s_{\omega''}^B. \tag{7}$$

A routine calculation shows that $(s_{\omega''}^B)^{-1}(\alpha_2)$ is the positive root $(s_{\mathbf{b}}^D)^{-1}(\alpha_1^D)$, hence $2\omega''$ is reduced. In fact, $12\omega''$ is reduced, because the root $(s_{\omega''}^B)^{-1}s_2(\alpha_1^B)$ is clearly the sum of two *positive* roots, upon writing $s_2(\alpha_1^B)$ as $\alpha_1^B + \alpha_2$. By (7) we therefore have

$$\omega \equiv 12\omega''. \tag{8}$$

So, when $y \geq 4$ we easily obtain $\psi(\mathbf{i}) = [\sigma 12 y_\beta \omega'']_\beta$ by (8) and $\psi(\mathbf{i}') = [\sigma' 1 y_\beta \omega'']_\beta$ by (6). Thus $\psi(\mathbf{i}) = \psi(\mathbf{i}')$. If $y = 2$ then we must first verify that $\sigma 1$ is nonreduced: from (7) and the expression for $\psi(\mathbf{i})$, we calculate $s_\sigma^B(\alpha_1^B)$ to be the negative root $-(s_{\omega''}^B)^{-1}(\alpha_1^B)$. So, $\sigma \equiv \sigma'' 1$ for some σ'', hence

$$\begin{aligned}
\psi(\mathbf{i}) &= [\sigma'' 12_\beta \omega]_\beta \\
&= [\sigma'' 12_\beta 12\omega'']_\beta \text{ by (8)} \\
&= [\sigma'' 212_\beta 1\omega'']_\beta \text{ applying a 4-braid} \\
&= [\sigma'' 212_\beta \omega']_\beta \text{ by (6)} \\
&= \psi(\mathbf{i}').
\end{aligned}$$

Case II: $y = 1$. So $\mathbf{i} = \mathbf{a}1_\beta x\mathbf{b}$ and $\mathbf{i}' = \mathbf{a}x1_\beta\mathbf{b}$, hence

$$\psi(\mathbf{i}) = [\sigma 2_\beta \omega]_\beta \text{ and } \psi(\mathbf{i}') = [\sigma' 2_\beta \omega']_\beta, \tag{9}$$

where ω is reduced for $s_1^B s_x^D s_{\mathbf{b}}^D$ and ω' is reduced for $s_1^B s_{\mathbf{b}}^D$. As x commutes with 1 in D_l, we have $x \geq 4$ or $x = 2$.

Subcase II(i): $x \geq 4$. We have $s_\omega^B = s_x s_{\omega'}^B$. Now, $(s_{\omega'}^B)^{-1}(\alpha_x)$ equals the positive root $(s_{\mathbf{b}}^D)^{-1}(\alpha_x)$, since s_1^B fixes α_x. So $x\omega'$ is reduced and we have $\omega \equiv x\omega'$. Thus

$$\begin{aligned}
\psi(\mathbf{i}) &= [\sigma 2_\beta x\omega']_\beta \\
&= [\sigma x 2_\beta \omega']_\beta \text{ since } x \text{ commutes with 2 in } B_l \\
&= \psi(\mathbf{i}') \text{ by comparison with (9).}
\end{aligned}$$

Subcase II(ii): $x = 2$. We now have

$$s_\omega^B = s_1^B s_2 s_1^B s_{\omega'}^B. \tag{10}$$

Of course, this is identical to equation (2), and in fact most of the steps in Subcase I(ii) go through with little change, so we will be a little more brief here. First note that if $1\omega'$ is reduced then so is $21\omega'$. Thus there are again three main possibilities to consider.

Possibility A: $121\omega'$ is reduced. By (10) we have $\omega \equiv 121\omega'$, hence

$$\begin{aligned}
\psi(\mathbf{i}) &= [\sigma 2_\beta 121\omega']_\beta \\
&= [\sigma 1212_\beta \omega']_\beta \text{ applying a 4-braid} \\
&= \psi(\mathbf{i}') \text{ by (9).}
\end{aligned}$$

Possibility B: $121\omega'$ is nonreduced, but $21\omega'$ is reduced. Exactly as in Possibility B for Subcase I(ii) we obtain some braid-equivalence $\omega' \equiv 21\omega_2$ (this is equation (4)). But this contradicts the expression for $\psi(\mathbf{i}')$ in (9) being reduced.

Possibility C: $1\omega'$ is nonreduced. As in (6) and (7) we have $\omega' \equiv 1\omega''$ for some ω'', and $s_\omega^B = s_1^B s_2 s_{\omega''}^B$. It is again true that $(s_{\omega''}^B)^{-1}(\alpha_2)$ is a positive root, but this time because it equals $(s_{\mathbf{b}}^D)^{-1}(\alpha_2)$ (noting that $2\mathbf{b}$ is a subword of $\mathbf{i}$). The remainder of the argument applies verbatim, except for the sentence dealing with $y \geq 4$, which is inapplicable here. This completes part (a) of the Proposition.

(b) $\boxed{\text{The 3-braid case.}}$ We now suppose that the letter in **i** which corresponds to β is involved in a 3-braid, so that [**i**] and [**i'**] belong to adjacent β-components of $\mathcal{C}(D_l)/\beta$; we must prove that $\psi(\mathbf{i})$ is adjacent to $\psi(\mathbf{i}')$ in $\mathcal{C}(B_l)/\beta$.

Write $\mathbf{i} = \mathbf{a}xyx\mathbf{b}$ and $\mathbf{i}' = \mathbf{a}yxy\mathbf{b}$, where $xyx \overset{3}{\leftrightarrow} yxy$ is a 3-braid. The only essentially different cases occur when β corresponds to either the middle y or the *first* x in **i**, say. Accordingly, the proof splits into two main parts.

Part one: $\mathbf{i} = \mathbf{a}\,x\,y_\beta\,x\,\mathbf{b}$ and $\mathbf{i}' = \mathbf{a}\,y\,x_\beta\,y\,\mathbf{b}$, that is, β corresponds to the *middle* letters of the 3-braid configurations.

Case I: $y \neq 1$. So $\psi(\mathbf{i}) = [\sigma y_\beta \omega]_\beta$ where ω is reduced for $s_x^D s_{\mathbf{b}}^D$.

Subcase I(i): $x \neq 1$. So $\psi(\mathbf{i}') = [\sigma' x_\beta \omega']_\beta$ where ω' is reduced for $s_y^D s_{\mathbf{b}}^D$. We have

$$s_\omega^B = s_x s_y s_{\omega'}^B. \tag{11}$$

Now, $y\omega'$ is nonreduced because $s_{\omega'}^{-1}(\alpha_y)$ is the negative root $-(s_{\mathbf{b}}^D)^{-1}(\alpha_y)$, whence

$$\omega' \equiv y\omega'' \tag{12}$$

for some ω''. By (11) and (12) we obtain

$$s_\omega^B = s_x s_{\omega''}^B. \tag{13}$$

Next, $x\omega''$ is reduced because $(s_{\omega''}^B)^{-1}(\alpha_x)$ equals the positive root $(s_{\mathbf{b}}^D)^{-1}(\alpha_x)$. So by (13) we conclude that

$$\omega \equiv x\omega''. \tag{14}$$

Since our hypotheses are symmetrical at this point, we have symmetrical versions of (12) and (14), namely, $\sigma' \equiv \sigma'' y$ and $\sigma \equiv \sigma'' x$, for some σ''. Therefore

$$\begin{aligned} \psi(\mathbf{i}) &= [\sigma'' x y_\beta x \omega'']_\beta \text{ substituting for } \sigma \text{ and } \omega \\ &\leftrightarrow [\sigma'' y x_\beta y \omega'']_\beta \text{ applying a 3-braid} \\ &= [\sigma' x_\beta \omega']_\beta \text{ substituting } \sigma' \text{ and } \omega' \\ &= \psi(\mathbf{i}'), \end{aligned}$$

where, as in chapter 4, we have used the symbol $\leftrightarrow$ to indicate adjacency in $\mathcal{C}(B_l)/\beta$. Thus, $\psi(\mathbf{i})$ is indeed adjacent to $\psi(\mathbf{i}')$.

Subcase I(ii): $x = 1$. So $y = 3$ because y is joined to x by an edge in D_l. We now have

$$\psi(\mathbf{i}) = [\sigma 3_\beta \omega]_\beta \text{ and } \psi(\mathbf{i}') = [\sigma' 2_\beta \omega']_\beta, \tag{15}$$

where ω is reduced for $s_1^D s_{\mathbf{b}}^D$ and ω' is reduced for $s_1^B s_3 s_{\mathbf{b}}^D$, so that

$$s_\omega^B = s_1^B s_2 s_3 s_{\omega'}^B. \tag{16}$$

Now, $3\omega'$ is nonreduced because $(s_{\omega'}^B)^{-1}(\alpha_3)$ equals the negative root $-(s_{\mathbf{b}}^D)^{-1}(\alpha_3)$. So

$$\omega' \equiv 3\omega'' \tag{17}$$

for some ω'', whence by (16) we have

$$s_\omega^B = s_1^B s_2 s_{\omega''}^B. \tag{18}$$

Now, $2\omega''$ is reduced because $(s_{\omega''}^B)^{-1}(\alpha_2)$ is the positive root $(s_{\mathbf{b}}^D)^{-1}(\alpha_1^D)$ (using (18)). So, since both $2\omega''$ and $3\omega''$ are reduced, it follows that $323\omega''$ is also reduced, that is, $32\omega'$ is reduced, by (17). From this and inspection of the expression for $\psi(\mathbf{i}')$ in (15), it follows that $\sigma' \equiv \sigma'' 3$ for some σ''. So $\psi(\mathbf{i}') = [\sigma'' 32_\beta 3\omega'']_\beta$, which is adjacent to the β-component $[\sigma'' 23_\beta 2\omega'']_\beta$; it remains to prove that this equals $\psi(\mathbf{i})$.

Possibility A: $12\omega''$ is reduced. Then we have $\omega \equiv 12\omega''$ by (18), hence $\psi(\mathbf{i}) = [\sigma 3_\beta 12\omega'']_\beta = [\sigma 13_\beta 2\omega'']_\beta$, which visibly matches the expression just above, so that $\psi(\mathbf{i})$ is adjacent to $\psi(\mathbf{i}')$.

Possibility B: $12\omega''$ is nonreduced. Since $2\omega''$ *is* reduced, we have from (18) that $2\omega'' \equiv 1\omega$, which, substituted into the expression just above, yields $[\sigma''23_\beta 1\omega]_\beta$, which equals $[\sigma''213_\beta\omega]_\beta = \psi(\mathbf{i})$, as required.

Case II: $y = 1$. This is the same as Subcase I(ii) above, with the rôles of x and y reversed.

Part two: $\mathbf{i} = \mathbf{a}\, x_\beta\, y\, x\, \mathbf{b}$ and $\mathbf{i}' = \mathbf{a}\, y\, x\, y_\beta\, \mathbf{b}$, that is, β now corresponds to the *first* x in the 3-braid configuration in $\mathbf{i}$.

Case I: $x \neq 1$. So $\psi(\mathbf{i}) = [\sigma x_\beta \omega]_\beta$, where ω is reduced for $s_y^D s_x s_{\mathbf{b}}^D$.

Subcase I(i): $y \neq 1$. So $\psi(\mathbf{i}') = [\sigma' y_\beta \omega']_\beta$, where ω' is reduced for $s_{\mathbf{b}}^D$. We have

$$s_\omega^B = s_y s_x s_{\omega'}^B. \tag{19}$$

Now, $x\omega'$ is reduced because $(s_{\omega'}^B)^{-1}(\alpha_x)$ is the positive root $(s_{\mathbf{b}}^D)^{-1}(\alpha_x)$. So, since both $x\omega'$ and $y\omega'$ are reduced (see the expression for $\psi(\mathbf{i}')$), it follows that $yx\omega'$ is also reduced, whence by (19) we have $\omega \equiv yx\omega'$. Thus

$$\begin{aligned} \psi(\mathbf{i}) &= [\sigma x_\beta y x \omega']_\beta \\ &\leftrightarrow [\sigma y x y_\beta \omega']_\beta \text{ applying a 3-braid} \\ &= \psi(\mathbf{i}'), \end{aligned}$$

so that $\psi(\mathbf{i})$ is adjacent to $\psi(\mathbf{i}')$.

Subcase I(ii): $y = 1$. We have $x = 3$ and

$$\psi(\mathbf{i}) = [\sigma 3_\beta \omega]_\beta \text{ and } \psi(\mathbf{i}') = [\sigma' 2_\beta \omega']_\beta, \tag{20}$$

where ω is reduced for $s_1^D s_3 s_{\mathbf{b}}^D$ and ω' is reduced for $s_1^B s_{\mathbf{b}}^D$. We have

$$s_\omega^B = s_1^B s_2 s_3 s_{\omega'}. \tag{21}$$

Now, $3\omega'$ is reduced because $(s_{\omega'}^B)^{-1}(\alpha_3)$ is the positive root $(s_{\mathbf{b}}^D)^{-1}(\alpha_3)$. So because both $3\omega'$ and $2\omega'$ are reduced (see the expression for $\psi(\mathbf{i}')$), it follows that $23\omega'$ is reduced.

Possibility A: $123\omega'$ is reduced. Then by (21) we have $\omega \equiv 123\omega'$, so that

$$\begin{aligned} \psi(\mathbf{i}) &= [\sigma 3_\beta 123\omega']_\beta \\ &= [\sigma 13_\beta 23\omega']_\beta \text{ applying a commutation} \\ &\leftrightarrow [\sigma 1232_\beta \omega']_\beta \text{ applying a 3-braid} \\ &= \psi(\mathbf{i}'). \end{aligned}$$

Possibility B: $123\omega'$ is nonreduced. Since we know that $23\omega'$ *is* reduced, we obtain

$$23\omega' \equiv 1\omega \tag{22}$$

by (21). Now, 31ω is reduced because $(s_\omega^B)^{-1} s_1^B(\alpha_3)$ is the positive root $(s_\omega^B)^{-1}(\alpha_3)$ (see the expression for $\psi(\mathbf{i})$ in (20)). Therefore $323\omega'$, and hence $232\omega'$ is reduced. Comparing this last expression with $\psi(\mathbf{i}')$, it follows that $\sigma' \equiv \sigma''23$ for some σ''. So we have

$$\begin{aligned} \psi(\mathbf{i}') &= [\sigma''232_\beta \omega']_\beta \\ &\leftrightarrow [\sigma''3_\beta 23\omega']_\beta \text{ applying a 3-braid} \\ &= [\sigma''3_\beta 1\omega]_\beta \text{ by (22)} \\ &= [\sigma''13_\beta \omega]_\beta \text{ applying a commutation} \\ &= \psi(\mathbf{i}). \end{aligned}$$

Case II: $x = 1$, so that $y = 3$. This situation is symmetrical to Subcase I(ii). ∎

By virtue of the previous Proposition we have a well defined graph morphism

$$\Psi : \mathcal{C}(D_l)/\beta \rightarrow \mathcal{C}(B_l)/\beta,$$

which sends the β-component containing any $\mathbf{i} \in \mathcal{R}(D_l)$ to $\psi(\mathbf{i})$.

7.1.4 Theorem. For all $\beta \in \Phi^+(D_l)$ we have a graph isomorphism $\mathcal{C}(B_l)/\beta \cong \mathcal{C}(D_l)/\beta$.

Proof. We will prove that the morphisms Θ and Ψ are mutual inverses. First we will show that $\Psi \circ \Theta$ is the identity on $\mathcal{C}(B_l)/\beta$. Take any $\mathbf{i} \in \mathcal{R}(B_l)$ and write

$$\mathbf{i} = \mathbf{x}\, k_\beta\, \mathbf{y}, \tag{1}$$

so that the β-value of $\mathbf{i}$ is k, noting that $k \neq 1$ because β is long. By definition, Θ sends the β-component containing $\mathbf{i}$ to the β-component of $\widehat{\mathbf{i}}$, in the notation of 7.1.1. We consider the cases $k \geq 3$ and $k = 2$ in turn.

Case I: $k \geq 3$, so that $\widehat{\mathbf{i}}$ has the form $\widehat{\mathbf{i}} = \mathbf{a} k_\beta \mathbf{b}$. Clearly, $\mathbf{b} = \widehat{\mathbf{y}}$. Now, Ψ sends the β-component of $\widehat{\mathbf{i}}$ to the β-component

$$\psi(\widehat{\mathbf{i}}) := [\sigma k_\beta \omega]_\beta, \tag{2}$$

where ω is reduced (with respect to the s_i^B) for $s_{\mathbf{b}}^D$.

Subcase I(i): $s_{\mathbf{y}}^B \in \mathcal{W}(D_l)$. Then, since $\mathbf{b} = \widehat{\mathbf{y}}$ we have $s_{\mathbf{b}}^D = s_{\mathbf{y}}^B$ and hence $\omega \equiv \mathbf{y}$. Insert this into (2) to obtain $\psi(\widehat{\mathbf{i}}) = [\sigma k_\beta \mathbf{y}]_\beta$, which equals $[\mathbf{i}]_\beta$ by comparison (1), as required.

Subcase I(ii): $s_{\mathbf{y}}^B \notin \mathcal{W}(D_l)$. This time we have $s_{\mathbf{b}}^D = s_1^B s_{\mathbf{y}}^B$, so that

$$s_\omega^B = s_1^B s_{\mathbf{y}}^B. \tag{3}$$

Now if $1\mathbf{y}$ is *reduced* then (3) implies that $\omega \equiv 1\mathbf{y}$, hence $\psi(\widehat{\mathbf{i}}) = [\sigma k_\beta 1 \mathbf{y}]_\beta$, which equals $[\sigma 1 k_\beta \mathbf{y}]_\beta$ because $k \geq 3$, and this last expression is $[\mathbf{i}]_\beta$.

If $1\mathbf{y}$ is *nonreduced* then we obtain $\mathbf{y} \equiv 1\omega$ from (3). Thus $[\mathbf{i}]_\beta = [\mathbf{x} k_\beta 1 \omega]_\beta$, which equals $\psi(\widehat{\mathbf{i}})$.

Case II: $k = 2$.

Subcase II(i): $s_{\mathbf{y}}^B \in \mathcal{W}(D_l)$. Then $\widehat{\mathbf{i}} = \mathbf{a} 2_\beta \mathbf{b}$, where $\mathbf{b} = \widehat{\mathbf{y}}$, so that $s_{\mathbf{b}}^D = s_{\mathbf{y}}^B$. We have $\psi(\widehat{\mathbf{i}}) = [\sigma 2_\beta \omega]_\beta$, where ω is reduced for $s_{\mathbf{b}}^D$. Thus, $\omega \equiv \mathbf{y}$, giving $\psi(\widehat{\mathbf{i}}) = [\sigma 2_\beta \mathbf{y}]_\beta = [\mathbf{i}]_\beta$.

Subcase II(ii): $s_{\mathbf{y}}^B \notin \mathcal{W}(D_l)$. Then $\widehat{\mathbf{i}} = \mathbf{a} 1_\beta \mathbf{b}$, where $\mathbf{b} = \widehat{\mathbf{y}}$, so that $s_{\mathbf{b}}^D = s_1^B s_{\mathbf{y}}^B$. We have $\psi(\widehat{\mathbf{i}}) = [\sigma 2_\beta \omega]_\beta$, where ω is reduced for $s_1^B s_{\mathbf{b}}^D = s_{\mathbf{y}}^B$. So we have $\omega \equiv \mathbf{y}$ and $\psi(\widehat{\mathbf{i}}) = [\mathbf{i}]_\beta$, just as before.

It remains to verify that $\Theta \circ \Psi$ is the identity map on $\mathcal{C}(D_l)/\beta$. Take any $\mathbf{i} \in \mathcal{R}(D_l)$ and write

$$\mathbf{i} = \mathbf{a} k_\beta \mathbf{b}, \tag{4}$$

so that the β-value of $\mathbf{i}$ is k. We consider the cases $k \geq 2$ and $k = 1$ in turn.

Case I: $k \geq 2$. Then $\psi(\mathbf{i}) = [\sigma k_\beta \omega]_\beta$, where ω is reduced for $s_{\mathbf{b}}^D$. So, since s_ω^B trivially belongs to $\mathcal{W}(D_l)$, by definition Θ sends this β-component to some $[\mathbf{x} k_\beta \mathbf{y}]_\beta$, where $\mathbf{y} = \widehat{\omega}$. Thus $s_{\mathbf{y}}^D = s_\omega^B = s_{\mathbf{b}}^D$, which implies $\mathbf{y} \equiv \mathbf{b}$, so that $[\mathbf{x} k_\beta \mathbf{y}]_\beta = [\mathbf{x} k_\beta \mathbf{b}]_\beta$, which equals $[\mathbf{i}]_\beta$ by comparison with (4), as required.

Case II: $k = 1$. Then $\psi(\mathbf{i}) = [\sigma 2_\beta \omega]_\beta$, where ω is reduced for $s_1^B s_{\mathbf{b}}^D$. Since $s_\omega^B \notin \mathcal{W}(D_l)$, by definition Θ sends this β-component to some $[\mathbf{x} 1_\beta \mathbf{y}]_\beta$ in which $\mathbf{y} = \widehat{\omega}$. So $s_{\mathbf{y}}^D = s_1^B s_\omega^B$, which equals $s_{\mathbf{b}}^D$. Hence $\mathbf{y} \equiv \mathbf{b}$ and we have $[\mathbf{x} 1_\beta \mathbf{y}]_\beta = [\mathbf{x} 1_\beta \mathbf{b}]_\beta = [\mathbf{i}]_\beta$, completing the proof. ∎

7.2 Labelling of Components

The mapping of β-components $\Theta : \mathcal{C}(B_l)/\beta \rightarrow \mathcal{C}(D_l)/\beta$ does not quite preserve β-values. Inspection of the map $\mathbf{i} \mapsto \widehat{\mathbf{i}}$ of 7.1.1 shows that β-values different from 2 are preserved. However, a β-component of $\mathcal{C}(B_l)$

with β-value 2 may get mapped to a β-component of $\mathcal{C}(D_l)$ with β-value 2 or 1. By Theorem 4.3.6 it is enough to consider the α_0-components.

Recall from Theorem 6.2.17 (b) that each α_0-component of $\mathcal{C}(B_l)$ with α_0-value 2 contains a lefthanded quiver-compatible vertex $[Q]$, which necessarily contains the right arrow $l-1 \rightarrow l$.

7.2.1 PROPOSITION. The α_0-component of a lefthanded quiver-compatible vertex $[Q] \in \mathcal{C}(B_l)$ maps under Θ to an α_0-component of $\mathcal{C}(D_l)$ with α_0-value 2 or 1 according as L, the number of left arrows in Q, is odd or even, respectively.

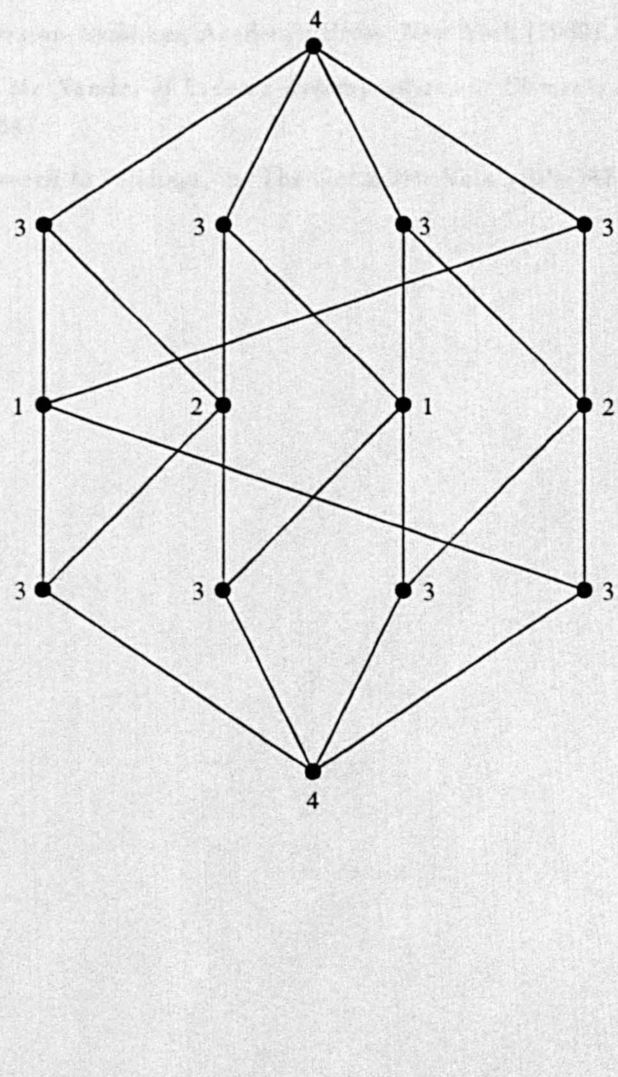
PROOF. If R is the number of right arrows in Q then from 3.3.2 we have an expression of the form

$$[Q] = [(y_1 - 1 \searrow 1)\dots(y_R - 1 \searrow 1)\,(l \searrow \underline{2})\mu 1(2 \nearrow l)\varrho],$$

where we have underlined the letter corresponding to α_0. There are R occurrences of 1 to the left of this letter, hence $l - R = L + 1$ to the right, using 2.3.8 (b). The result follows immediately from 7.1.1. ▮

From the graph $\mathcal{Q}(B_l)^0$ of partial quivers (see Figure 6.3.12 for the case $l = 4$), it is now easy to deduce the labelling by α_0-values of $\mathcal{C}(D_l)/\alpha_0$: simply consider the (fully-oriented) quivers containing $l-1 \rightarrow l$ and count how may left arrows they contain, modulo 2. Here is an example.

7.2.2 FIGURE. The labelled graph of α_0-components in type D_4.



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