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Revised July 2014
Analytic Methods In Combinatorics

by

Jan Volec

Thesis

Submitted to the University of Warwick

and Université Paris Diderot – Paris 7

for the degree of

Doctor of Philosophy

Mathematics Institute and UFR d’Informatique

September 2014


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Acknowledgments

My big thanks go to Dan and Jean-Sébastien for their supervising and without any doubts an exceptional care during my whole studies. I also want to thank my collaborators – Jozsi Balogh, Roman Glebov, Ping Hu, Dan Král’, Bernard Lidický, Oleg Pikhurko and Balázs Udvari. Working together on the joint projects was always full of enjoyable moments.

My acknowledgments go to all the three institutions I was based at during my doctoral studies. I spent the first year at Computer Science Institute of Charles University in Prague, where I was also supported by the Grant Agency of Charles University (GAUK 601812). In the following two years, my studies were split between the University of Warwick and LORIA in Nancy. I am very grateful to both of the institutions for their great and inspiring atmosphere. My stays in France were also supported by a grant from French government.

Thanks to my friends Rafael Barbosa, Zdeněk Dvořák, Florent Foucaud, Andrzej Grzesik, Tereza Klímošová, Matjaž Krnc, Martin Kupec, Anita Liebenau, Lukáš Mach, Guillem Perarnau, Will Perkins, Florian Pfender, Jakub Sliačan, Robert Šámal and Vojta Tůma for both mathematical and non-mathematical discussions. Last but not least, I want to thank my family for their support.
Declarations

This thesis consists of four chapters.

a) Chapter 1 contains an introduction to flag algebras, which is a framework developed by Razborov [59] for problems in extremal combinatorics. Most of the content of this chapter follows the exposition in [59].

b) The results from Chapter 2 were obtained together with Roman Glebov and Dan Král’. The corresponding paper [34] is submitted for publication. An extended abstract was accepted to the proceedings of Erdős Centennial conference [35] and to the proceedings of Eurocomb 2013 conference [36].

c) The results from Chapter 3 were obtained together with Jozef Balogh, Ping Hu, Bernard Lidický, Oleg Pikhurko, and Balázs Udvari. The paper [8] contains the results from Section 3.2 and it has been accepted for publication in Combinatorics, Probability and Computing. The results from Section 3.2 are also a part of the PhD thesis of Ping Hu. The paper [9] with the results from Section 3.5 and Section 3.6 will be submitted for publication.

d) The results from Chapter 4 were obtained together with Roman Glebov and Dan Král’. The corresponding paper [33] is submitted for publication.

With the exception described under the point (c), none of the results appeared in any other thesis.

All the collaborators have agreed with the inclusion of our joint work into this thesis.
Abstract

In the thesis, we apply the methods from the recently emerged theory of limits of discrete structures to problems in extremal combinatorics. The main tool we use is the framework of flag algebras developed by Razborov.

We determine the minimum threshold $d$ that guarantees a 3-uniform hypergraph to contain four vertices which span at least three edges, if every linear-size subhypergraph of the hypergraph has density more than $d$. We prove that the threshold value $d$ is equal to $1/4$. The extremal configuration corresponds to the set of cyclically oriented triangles in a random orientation of a complete graph. This answers a question raised by Erdős.

We also use the flag algebra framework to answer two questions from the extremal theory of permutations. We show that the minimum density of monotone subsequences of length five in any permutation is asymptotically equal to $1/256$, and that the minimum density of monotone subsequences of length six is asymptotically equal to $1/3125$. Furthermore, we characterize the set of (sufficiently large) extremal configurations for these two problems. Both the values and the characterizations of extremal configurations were conjectured by Myers.

Flag algebras are also closely related to the theory of dense graph limits, where the main objects of study are convergent sequences of graphs. Such a sequence can be assigned an analytic object called a graphon. In this thesis, we focus on finitely forcible graphons. Those are graphons determined by finitely many subgraph densities. We construct a finitely forcible graphon such that the topological space of its typical vertices is not compact. In our construction, the space even fails to be locally compact. This disproves a conjecture of Lovász and Szegedy.
Résumé

Dans cette thèse, nous appliquons à des problèmes de combinatoire extrême les méthodes de la théorie des limites de structures discrètes, qui a été récemment développée. L’outil principal utilisé est celui des algèbres de drapeaux, développé par Razborov.

Nous déterminons le seuil minimum, $d$, garantissant que tout hypergraphe 3-uniforme contient quatre sommets induisant au moins trois arêtes, si tout sous-hypergraphe d’ordre linéaire en l’ordre de l’hypergraphe a une densité strictement plus grande que $d$. Nous prouvons que cette valeur seuil $d$ est égale à $1/4$. La configuration extrême correspond à un ensemble de triangles orientés de façon cyclique dans une orientation aléatoire d’un graphe complet. Ceci répond à une question posée par Erdős.

Nous utilisons également la théorie des algèbres de drapeaux pour répondre à deux questions de théorie extrême des permutations. Nous montrons que la densité minimum d’une sous-suite monotone de longueur cinq dans toute permutation est asymptotiquement égale à $1/256$, et que la densité minimum d’une sous-suite monotone de longueur six est asymptotiquement égale à $1/3125$. Par ailleurs, nous caractérisons l’ensemble des configurations extrêmes (suffisamment grandes) pour ces deux problèmes. Les deux valeurs ainsi que les caractérisations de configurations extrêmes avaient été conjecturées par Myers.

Les algèbres de drapeaux sont également étroitement liées à la théorie des limites de graphes denses, dont les objets d’étude principaux sont les suites de graphes convergentes. On peut associer à une telle suite un objet analytique appelé graphon. Nous nous intéressons aux graphons forçables de façon finie. Il s’agit des graphons déterminés par un ensemble fini de densités de sous-graphes. Nous construisons un graphon forçable de façon finie tel que l’espace topologique de ses sommets typiques n’est pas compact. Dans notre construction, cet espace n’est même pas localement compact. Ceci réfute une conjecture de Lovász et Szegedy.
Notation and preliminaries

We follow the basic graph theory notation from the book of Bondy and Murty [10]. For a graph $G$, we denote the set of vertices of $G$ by $V(G)$ and the set of edges of $G$ by $E(G) \subseteq \binom{V(G)}{2}$. Analogously, for every integer $r$ and every $r$-uniform hypergraph $H$, or an $r$-graph for short, we again let $V(H)$ to be the set of vertices of $H$ and $E(H) \subseteq \binom{V(H)}{r}$ to be the set of (hyper)edges of $H$. Note that the definition of a 2-graph coincide with the one for a graph. We call the cardinality of $V(H)$ the order of $H$ and denote it by $v(H)$, and the cardinality of $E(H)$ the size of $H$ and denote it by $e(H)$.

For a vertex $v$ in an graph $H$, we define $N_H(v) := \{ u \in V(H) : uv \in E(H) \}$ to be the neighborhood of $v$, and $N_H[v] := N_H(v) \cup \{ v \}$ to be the closed neighborhood of $v$. If the graph $H$ is clear from the context, we omit the $H$ from the subscript and write only $N(v)$ or $N[v]$. We refer to the size of $N(v)$ as to the degree of $v$. If $H$ is a 3-graph and $u$ and $v$ are two of its vertices, we define the co-degree of $u$ and $v$ to be the size of the set $N_H(u,v) := \{ w \in V(H) : uvw \in E(H) \}$.

For an $r$-graph $H$, we denote by $\overline{H}$ the complement of $H$, i.e., the $r$-graph with the vertex-set $V(H)$ and the edge-set $\binom{V(H)}{r} \setminus E(H)$. For a subset of vertices $S \subseteq V(H)$, we denote by $H[S]$ the induced subhypergraph (or simply subgraph if $r = 2$), i.e., the $r$-graph with the vertex-set $S$ and the edge-set $\{ e \in E(H) : e \subseteq S \}$.

An independent set of an $r$-graph $H$ is a subset of $I \subseteq V(H)$ such that $e(H[I]) = 0$. The chromatic number of $H$, which we denote by $\chi(H)$, is the smallest integer $k$ such that $V(H)$ can be partitioned into $k$ independent sets.

We say that an $r$-graph $H$ is linear if every two edges intersect in at most one vertex. Note that every 2-graph, i.e., every graph, is linear.

One of the basic questions in extremal combinatorics is to determine the maximum possible number of edges in an $n$-vertex $r$-graph that does not contain a copy of some fixed $r$-graph $F$. For an $r$-graph $F$, we define the extremal number of $F$ as

$$\text{ex}(n, F) := \max\{ e(H) : H \text{ is an } r\text{-graph on } n \text{ vertices with no copy of } F \},$$
and the Turán density of $F$, which is denoted by $\pi(F)$, as $\lim_{n \to \infty} \frac{\text{ex}(n,F)}{\binom{n}{r}}$. Note that for a fixed $r$-graph $F$, the function $\frac{\text{ex}(n,F)}{\binom{n}{r}}$ is non-increasing, the hence limit always exists.

For two graphs $G$ and $H$, we define the composition $G \circ H$ (which is sometimes called the lexicographical product of $G$ and $H$) to be the graph on the vertex set $V(G) \times V(H)$ in which a vertex $(u,v)$ is adjacent to a vertex $(u',v')$ if and only if either $uu' \in E(G)$, or $u = u'$ and $vv' \in E(H)$. In other words, we replace each vertex of $G$ by a copy of $H$, and linking these copies by complete bipartite graphs according to the edges of $G$. This notion has a naturally generalizes to $r$-graphs. The composition-closure of a family $\mathcal{F}$ of $r$-graphs is the smallest family of $r$-graphs $\mathcal{F}'$ that contains $\mathcal{F}$ as a subfamily and which satisfies $G \circ H \in \mathcal{F}'$ for every $G, H \in \mathcal{F}'$.

For an $r$-graph $H$ and an integer $\ell$, the $\ell$-th blow-up of $H$ is the $r$-graph on $\ell \cdot v(H)$ vertices which is constructed from $H$ by replacing each vertex $v$ of $H$ with an independent set of $\ell$ vertices $I_v$, and each edge of $H$ with a complete $r$-partite $r$-graph, where each part has size $\ell$. Analogously, for an $r$-graph $H$ and an integer $k$, the $k$-th iterated blow-up of $H$ is the $r$-graph on the vertex-set $V(H)^k$ isomorphic to $\underbrace{H \circ H \circ \cdots \circ H}_{k\text{-times}}$. In other words, we first take an $\ell$-th blow-up of $H$ for $\ell$ being $v(H)^{k-1}$, and then we place a copy of the $(k - 1)$-th iterated blow-up of $H$ inside $I_v$ for every vertex $v \in V(H)$. 
Chapter 1

Flag Algebras

The main tool used in the thesis is the framework of flag algebras. It was introduced by Razborov [59] as a general tool to approach problems from extremal combinatorics. The work of Razborov was inspired by the theory of dense graph limits, which is discussed in Section 4.1.

The flag algebra method have been successfully applied to various problems in extremal combinatorics. To name some of the applications, they were used for attacking the Caccetta-Häggkvist conjecture [43, 63], Turán-type problems in graphs [60, 54, 57, 64, 19, 38, 40, 58, 70, 42] 3-graphs [61, 56, 5, 29, 28, 34] and hypercubes [4, 7], extremal problems in a colored setting [41, 44, 6, 18], or in geometry [45]. More details on these applications can be found in a recent survey of Razborov [62].

In this chapter, we follow the approach of Razborov [59] and introduce the framework of flag algebras for the graphs. Exactly the same scheme can also be used to setup flag algebras for the oriented graphs, the \(\ell\)-uniform hypergraphs (\(\ell\)-graphs), the permutations, and many others. In fact, Razborov introduced in [59] the framework for an arbitrary universal first-order logic theory without constants or function symbols. We decided to present in this section the flag algebra setup for the particular instance of graphs rather than in the general setting, since it might be easier to understand the ideas of the framework in this way.

1.1 Flag algebra setting for graphs

The central notions we are going to introduce are an algebra \(A\) and algebras \(A^\sigma\), where \(\sigma\) is a fixed graph with a fixed labelling of its vertex set. In order to precisely describe the algebras \(A\) and \(A^\sigma\) on formal linear combinations of graphs, we first need to introduce some additional notation. Let \(F\) be the set of all finite non-
Figure 1.1: Two examples of linear combinations used in generating $K$.

isomorphic graphs. Next, for every $\ell \in \mathbb{N}$, let $F_\ell \subset F$ be the set of all graphs of order $\ell$. For convenience, we fix an arbitrary ordering on the elements the set $F_\ell$ for every $\ell \in \mathbb{N}$, i.e., we always assume that $F_\ell = \{F_1, F_2, \ldots, F_{|F_\ell|}\}$.

For $H \in F_\ell$ and $H' \in F_{\ell'}$, we define $p(H, H')$ to be the probability that a randomly chosen subset of $\ell$ vertices in $H'$ induces a subgraph isomorphic to $H$. Note that $p(H, H') = 0$ if $\ell' < \ell$. Let $\mathbb{RF}$ be the set of all formal linear combinations of elements of $F$ with real coefficients. Furthermore, let $K$ be the linear subspace of $\mathbb{RF}$ generated by all the linear combinations of the form

$$H - \sum_{H' \in F_{v(H)+1}} p(H, H') \cdot H'.$$

Two examples of such linear combinations are depicted in Figure 1.1. Finally, we set $\mathcal{A}$ to be the space $\mathbb{RF}$ factored by $K$, and the element corresponding to $K$ in $\mathcal{A}$ to be the zero element of $\mathcal{A}$.

The space $\mathcal{A}$ comes with a natural definition of an addition and a multiplication by a real number. We now introduce the notion of a product of two elements from $\mathcal{A}$. We start with the definition for the elements of $F$. For $H_1, H_2 \in F$, and $H \in F_{v(H_1)+v(H_2)}$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H)$ of size $v(H_1)$ and its complement induce in $H$ subgraphs isomorphic to $H_1$ and $H_2$, respectively. We set

$$H_1 \times H_2 := \sum_{H \in F_{v(H_1)+v(H_2)}} p(H_1, H_2; H) \cdot H.$$

See Figure 1.2 for an example of a product with $H_1 = \bullet$ and $H_2 = \blacksquare$. The multiplication on $F$ has a unique linear extension to $\mathbb{RF}$, which yields a well-defined multiplication also in the factor algebra $\mathcal{A}$. A formal proof of this is given in [59, Lemma 2.4]. Observe that the one-vertex graph $\bullet \in F$ is, modulo $K$, the neutral element of the product in $\mathcal{A}$.

Let us now move to the definition of the algebra $\mathcal{A}^\sigma$, where $\sigma$ is a fixed finite graph with a fixed labelling of its vertices. The labelled graph $\sigma$ is usually called
A type. We follow the same lines as in the definition of $\mathcal{A}$. Let $\mathcal{F}^\sigma$ be the set of all finite graphs $H$ with a fixed embedding of $\sigma$, i.e., an injective mapping $\theta$ from $V(\sigma)$ to $V(H)$ such that $\theta$ is an isomorphism between $\sigma$ and $H[\text{Im}(\theta)]$. The elements of $\mathcal{F}^\sigma$ are usually called $\sigma$-flags, the subgraph induced by $\text{Im}(\theta)$ is called the root of a $\sigma$-flag, and the vertices $\text{Im}(\theta)$ are called the rooted or the labelled vertices. The vertices that are not rooted are called the non-rooted or the non-labelled vertices. For every $\ell \in \mathbb{N}$, we define $\mathcal{F}_\ell^\sigma \subset \mathcal{F}^\sigma$ to be the set of all $\ell$-vertex $\sigma$-flags from $\mathcal{F}^\sigma$. Also, for each type $\sigma$ and each integer $\ell$, we fix an arbitrary ordering on the elements of the set $\mathcal{F}_\ell^\sigma$.

In the analogy to the case of $\mathcal{A}$, for two $\sigma$-flags $H \in \mathcal{F}^\sigma$ and $H' \in \mathcal{F}^\sigma$ with the embeddings of $\sigma$ given by $\theta$ and $\theta'$, respectively, we set $p(H, H')$ to be the probability that a randomly chosen subset of $v(H) - v(\sigma)$ vertices in $V(H') \setminus \theta'(V(\sigma))$ together with $\theta'(V(\sigma))$ induces a subgraph that is isomorphic to $H$ through an isomorphism $f$ that preserves the embedding of $\sigma$. In other words, the isomorphism $f$ has to satisfy $f(\theta') = \theta$. Let $\mathbb{R}\mathcal{F}^\sigma$ be the set of all formal linear combinations of elements of $\mathcal{F}^\sigma$ with real coefficients, and let $\mathcal{K}^\sigma$ be the linear subspace of $\mathbb{R}\mathcal{F}^\sigma$ generated by all the linear combinations of the form

$$H - \sum_{H' \in \mathcal{F}_\ell^\sigma(H)+1} p(H, H') \cdot H'.$$

See Figure 1.3 for two examples of such linear combinations in the case $\sigma$ is the one-vertex type. We define $\mathcal{A}^\sigma$ to be $\mathbb{R}\mathcal{F}^\sigma$ factored by $\mathcal{K}^\sigma$ and, analogously to the case for the algebra $\mathcal{A}$, we let the element corresponding to $\mathcal{K}^\sigma$ to be the zero element of $\mathcal{A}^\sigma$. 

---

Figure 1.2: An example of a product in the algebra $\mathcal{A}$.

Figure 1.3: Two examples of linear combinations used in generating $\mathcal{K}^\sigma$, where $\sigma$ is the one-vertex type.
Figure 1.4: Two examples of a product in the algebra $A^\sigma$, where $\sigma$ is the one-vertex type.

We now define the product of two elements from $F^\sigma$. Let $H_1, H_2 \in F^\sigma$ and $H \in F^\sigma_{v(H_1) + v(H_2) - v(\sigma)}$ be $\sigma$-flags, and $\theta$ be the fixed embedding of $\sigma$ in $H$. Similarly to the definition of the multiplication for $H$ and $1$, we define $p(H_1, H_2; H)$ to be the probability that a randomly chosen subset of $V(H) \setminus \theta(V(\sigma))$ of size $v(H_1) - v(\sigma)$ and its complement in $V(H) \setminus \theta(V(\sigma))$ of size $v(H_2) - v(\sigma)$, extend $\theta(V(\sigma))$ in $H$ to subgraphs isomorphic to $H_1$ and $H_2$, respectively. Again, by isomorphic we mean that there is an isomorphism that preserves the fixed embedding of $\sigma$. We set

$$H_1 \times H_2 := \sum_{H \in F^\sigma_{v(H_1) + v(H_2) - v(\sigma)}} p(H_1, H_2; H) \cdot H.$$ 

Two examples of a product in $A^\sigma$ for $\sigma$ being the one-vertex type are depicted in Figure 1.4. The definition of the product for the elements of $F^\sigma$ naturally extends to $A^\sigma$. It follows that the unique $\sigma$-flag of size $v(\sigma)$ represents, modulo $\mathcal{K}^\sigma$, the neutral element of the product in $A^\sigma$.

Now consider an infinite sequence $(G_n)_{n \in \mathbb{N}}$ of graphs with increasing orders. We say that the sequence $(G_n)_{n \in \mathbb{N}}$ is convergent if the probabilities $p(H, G_n)$ converge for every $H \in F$. A standard compactness argument (e.g., using Tychonoff’s theorem [73]) yields that every such infinite sequence has a convergent subsequence.

Fix a convergent sequence $(G_n)_{n \in \mathbb{N}}$ of graphs with increasing orders. For every $H \in F$, we set $\phi(H) = \lim_{n \to \infty} p(H, G_n)$, and we then linearly extend $\phi$ to $A$. We usually refer to the mapping $\phi$ as to the limit of the sequence. The obtained mapping $\phi$ is a homomorphism from $A$ to $\mathbb{R}$, see [59, Theorem 3.3a]. Moreover, for every $H \in F$, it holds $\phi(H) \geq 0$. Let $\text{Hom}^+(A, \mathbb{R})$ be the set of all such homomorphisms, i.e., the set of all homomorphisms $\psi$ from the algebra $A$ to $\mathbb{R}$ such that $\psi(H) \geq 0$ for every $H \in F$. It is interesting to see that this set is exactly the set of all the limits of convergent sequences of graphs [59, Theorem 3.3b].

Let $(G_n)_{n \in \mathbb{N}}$ be a convergent sequence of graphs and $\phi \in \text{Hom}^+(A, \mathbb{R})$ its limit. For a type $\sigma$ and an embedding $\theta$ of $\sigma$ in $G_n$, we define $G_n^\sigma$ to be the graph rooted on the copy of $\sigma$ that corresponds to $\theta$. For every $n \in \mathbb{N}$ and $H^\sigma \in F^\sigma$, we define $p_n^\sigma(H^\sigma) = p(H^\sigma, G_n^\sigma)$. Picking $\theta$ at random gives rise to a probability distribution $P_n^\sigma$ on mappings from $A^\sigma$ to $\mathbb{R}$, for every $n \in \mathbb{N}$. Since $p(H, G_n)$ con-
verges (as $n$ tends to infinity) for every $H \in \mathcal{F}$, the sequence of these probability distributions on mappings from $A^\sigma$ to $\mathbb{R}$ weakly converges to a Borel probability measure on $\text{Hom}^+(A^\sigma, \mathbb{R})$, see [59, Theorems 3.12 and 3.13]. We denote the limit probability distribution by $P^\sigma$. In fact, for any $\sigma$ such that $\phi(\sigma) > 0$, the homomorphism $\phi$ itself fully determines the probability distribution $P^\sigma$ [59, Theorem 3.5]. Furthermore, any mapping $\phi^\sigma$ from the support of the distribution $P^\sigma$ is in fact a homomorphism from $A^\sigma$ to $\mathbb{R}$ such that $\phi^\sigma(H^\sigma) \geq 0$ for all $H^\sigma \in \mathcal{F}^\sigma$ [59, Proof of Theorem 3.5].

The last notion we introduce is the averaging (or downward) operator $[\cdot]_\sigma : A^\sigma \to A$. It is the linear operator defined on the elements of $H \in \mathcal{F}^\sigma$ by

$$[H]_\sigma := p_H^\sigma \cdot H^\emptyset,$$

where $H^\emptyset$ is the (unlabelled) graph from $\mathcal{F}$ corresponding to $H$ after unlabelling all its vertices, and $p_H^\sigma$ is the probability that a random injective mapping from $V(\sigma)$ to $V(H^\emptyset)$ is an embedding of $\sigma$ in $H^\emptyset$ yielding a $\sigma$-flag isomorphic to $H$. See Figure 1.5 for three examples of applying the averaging operator $[\cdot]_\sigma$, where $\sigma$ is the one-vertex type.

The key relation between $\phi$ and $\phi^\sigma$ is the following

$$\forall H^\sigma \in A^\sigma, \quad \phi([H^\sigma]_\sigma) = \phi([\sigma]_\sigma) \cdot \int \phi^\sigma(H^\sigma), \quad (1.1)$$

where the integration is with respect to the probability measure given by the random distribution $P^\sigma$ on $\phi^\sigma$. Note that

$$\phi([\sigma]_\sigma) = \frac{|\text{Aut}(\sigma^\emptyset)|}{v(\sigma^\emptyset)!} \cdot \phi(\sigma^\emptyset).$$

Vaguely speaking, the relation 1.1 corresponds to the conditional probability formula $P[A \cap B] = P[B] \cdot P[A \mid B]$, where $B$ is the event that a random injective mapping $\theta$ is an embedding of $\sigma$, and $A$ is the event that a random subset of $v(H) - v(\sigma)$ vertices extends $\theta$ to the $\sigma$-flag $H^\sigma$. A formal proof is given in [59, Lemma 3.11].
The relation (1.1) implies that if $\phi(\sigma) \geq 0$ almost surely for some $A^\sigma \in \mathcal{A}^\sigma$, then $\phi(\|A^\sigma\|_\sigma) \geq 0$. In particular, for every homomorphism $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ and every linear combination $A^\sigma \in \mathcal{A}^\sigma$ it holds
\[
\phi\left(\left\lfloor (A^\sigma)^2 \right\rfloor_\sigma\right) \geq 0.
\] (1.2)

We note that a stronger variant of (1.2) can be proven using Cauchy-Schwarz’s inequality. Specifically, [59, Theorem 3.14] states that
\[
\forall \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}), \forall A^\sigma, B^\sigma \in \mathcal{A}^\sigma, \phi\left(\left\lfloor (A^\sigma)^2 \right\rfloor_\sigma \times \left\lfloor (B^\sigma)^2 \right\rfloor_\sigma\right) \geq \phi\left(\|A^\sigma \times B^\sigma\|_\sigma\right)^2.
\]

Let $\sigma$ be a type, $A^\sigma \in \mathcal{A}^\sigma$, and $m$ the minimum integer such that
\[
A^\sigma = \sum_{F^\sigma \in \mathcal{F}^\sigma_m} \alpha_{F^\sigma} \cdot F_i^\sigma.
\]

We say that $\ell := 2m - v(\sigma)$ is the order of the expression $\left\lfloor (A^\sigma)^2 \right\rfloor_\sigma$. It follows that
\[
\left\lfloor (A^\sigma)^2 \right\rfloor_\sigma = \sum_{F \in \mathcal{F}_\ell} \alpha_F \cdot F_i.
\]

Since the operator $\left\lfloor \cdot \right\rfloor_\sigma$ is linear, it immediately follows that for every $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$, $A_1^\sigma, A_2^\sigma, \ldots, A_K^\sigma \in \mathcal{A}^\sigma$, and non-negative reals $\alpha_1, \alpha_2, \ldots, \alpha_K$ we have
\[
\phi\left(\left\lfloor \alpha_1 \cdot (A_1^\sigma)^2 + \alpha_2 \cdot (A_2^\sigma)^2 + \cdots + \alpha_K \cdot (A_K^\sigma)^2 \right\rfloor_\sigma\right) \geq 0.
\]

Hence for every finite set $S \subseteq \mathcal{F}^\sigma$ and every real symmetric positive semidefinite matrix $M$ of size $|S| \times |S|$ it holds that
\[
\phi\left(\left\lfloor x_S^T M x_S \right\rfloor_\sigma\right) \geq 0,
\]
where $x_S$ is the $|S|$-dimensional vector from whose $i$-th coordinate is equal to the $i$-th element of $S$. Note that we used the fact that every real symmetric positive semidefinite matrix $M$ of size $s \times s$ can be written as a sum of squares. In other words, there exists an integer $d \leq s$ such that
\[
M = \sum_{j=1}^{d} \lambda_j \cdot v_j \times (v_j)^T,
\]
where vectors $v_i$, for $i \in [d]$, form an orthonormal eigen-basis of $M$ and $\lambda_i$ are the corresponding (always non-negative) eigenvalues. On the other hand, for every set
of \( d' \) non-negative reals \( \lambda_j' \) and vectors \( v_j' \in \mathbb{R}^s \), the matrix
\[
M' := \sum_{j \in [d']} \lambda_j' \cdot v_j' \times (v_j')^T
\]
is a symmetric positive semidefinite matrix of size \( s \times s \).

### 1.2 Semidefinite method

The heart of the semidefinite method are inequalities of the form (1.2). These inequalities turned out to be very useful while dealing with problems from extremal combinatorics. Before moving to the description of the method itself, let us illustrate this on a small example. Consider the following inequality, where \( \sigma \) is the one-vertex type:
\[
\left[ 3 \times \left( \begin{array}{c}
1 \\
1
\end{array} \right)^2 \right]_{\sigma} \geq 0 . \tag{1.3}
\]
The definition of the product on \( \mathcal{A}_\sigma \) implies that the left-hand side of (1.3) is equal to
\[
\left[ 3 \times \left( \begin{array}{c}
\bullet + \bullet \\
1 + 1
\end{array} \right) - 6 \times \left( \begin{array}{c}
\frac{1}{2} \bullet + \frac{1}{2} \bullet \\
1 + 1
\end{array} \right) + 3 \times \left( \begin{array}{c}
\bullet + \bullet \\
1 + 1
\end{array} \right) \right]_{\sigma},
\]
and an application of the averaging operator yields that
\[
\left[ 3 \times \left( \begin{array}{c}
1 \cdot
1
\end{array} \right)^2 \right]_{\sigma} = 3 \times \bullet + 3 \times \bullet - \bullet - \bullet .
\]
Therefore, every \( \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \) satisfies
\[
\phi \left( 3 \times \bullet + 3 \times \bullet - \bullet - \bullet \right) \geq 0,
\]
and hence also
\[
\phi \left( 4 \times \bullet + 4 \times \bullet \right) \geq \phi \left( \bullet + \bullet + \bullet + \bullet \right).
Since the right-hand side of the last inequality is equal to one, we conclude that
\[
\phi \left( \begin{array}{c}
\cdot \\
\cdot \\
\end{array} \right) + \left( \begin{array}{c}
\cdot \\
\end{array} \right) \geq \frac{1}{4}
\tag{1.4}
\]
for every \( \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \). This is a well-known inequality due to Goodman [37]. Note that the inequality (1.4) is best possible. This can be seen, for example, by considering the limit of the sequence of Erdős-Renyi random graphs \( G_{n,1/2} \) with increasing orders (the sequence is convergent with probability 1). Another example where the inequality (1.4) is tight is the sequence of complete balanced bipartite graphs with increasing orders (it is straightforward to check that this sequence is convergent).

Now consider a general linear combination \( A \in \mathcal{A} \). One of the fundamental problems in extremal combinatorics is to determine the smallest value of \( \phi(A) \) over all \( \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \). The semidefinite method is a tool from the flag algebra framework that systematically searches for inequalities of the form (1.2), like the inequality (1.3) in the case when \( A = \left( \begin{array}{c}
\cdot \\
\cdot \\
\end{array} \right) + \left( \begin{array}{c}
\end{array} \right) \), in order to find a lower bound on
\[
\min_{\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})} \phi(A). 
\tag{1.5}
\]
Note that since \( \text{Hom}^+(\mathcal{A}, \mathbb{R}) \) is compact, such a minimum exists for every \( A \in \mathcal{A} \).

The semidefinite method works as follows. First, fix an upper bound \( \ell \) on the order of flags in all the terms of linear inequalities we are going to consider, including also the terms of the objective function \( A \). Without loss of generality, \( A = \sum_{F \in \mathcal{F}_\ell} \alpha_F \cdot F \). Next, fix an arbitrary list of types \( \sigma_1, \ldots, \sigma_K \) of order at most \( \ell \). Recall our aim is to find a lower bound on (1.5). The semidefinite method finds a way how to express \( A \) in the algebra \( \mathcal{A} \) as follows:
\[
A = \left( \sum_{k \in [K]} \sum_{j \in [J_k]} b_{k,j} \cdot \left( \left( A_j^{\sigma_k} \right)^2 \right)_{\sigma_k} \right) + \left( \sum_{F \in \mathcal{F}_\ell} \beta_F \cdot F \right) + \left( \frac{c}{\sum_{F \in \mathcal{F}_\ell} F} \right), 
\tag{1.6}
\]
where
- \( J_1, \ldots, J_K \) are non-negative integers,
- \( A_j^{\sigma_1} \in \mathcal{A}^{\sigma_1} \) so that the order of \( \left( A_j^{\sigma_1} \right)^2 \) is at most \( \ell \) for every \( j \in [J_1] \),
\[
\begin{align*}
  &\bullet A^{\sigma_k}_{j} \in A^{\sigma_k} \text{ so that the order of } \left( A^{\sigma_k}_{j} \right) \leq \ell \text{ for every } j \in [J_K], \\
  &\bullet b^k_j \geq 0 \text{ for every } k \in [K] \text{ and } j \in [J_k], \\
  &\bullet \beta_F \geq 0 \text{ for every } F \in \mathcal{F}_\ell, \text{ and} \\
  &\bullet c \in \mathbb{R}. 
\end{align*}
\]

Since \( \phi(R) \geq 0, \phi(S) \geq 0, \) and \( \phi(T) = c \) for all \( \phi \in \text{Hom}^+(A, \mathbb{R}) \), we conclude that \( \phi(A) \geq c. \) Note that \( R \) is a positive linear combination of inequalities (1.2) of order at most \( \ell \), hence \( R = \sum_{F \in \mathcal{F}_\ell} r_F \cdot F \) for some choice of reals \( r_F. \)

For a fixed choice of the parameters \( \ell \) and \( \sigma_1, \ldots, \sigma_K \), finding such an expression of \( A \) can be formulated as a semidfinite program. Note that all the expressions \( A, R, S \) and \( T \) can be written as linear combinations of the elements from \( \mathcal{F}_\ell \), i.e., they can be viewed as vectors in \( \mathbb{R}^{||\mathcal{F}_\ell||} \). Furthermore, the bound obtained by the semidefinite method is “best possible” in the following sense. Let \( c_0 \) be the obtained bound. For every expression of \( A \) as a linear combination of the form (1.6), the coefficient \( c \) in this combination is at most \( c_0 \). Note that there might be (and often there are) different combinations that yield the bound \( c_0 \).

Let us now describe the corresponding semidefinite program in more detail. Fix one of the types \( \sigma_k \in \{\sigma_1, \ldots, \sigma_K\} \). Since the semidefinite method uses inequalities \( \left( (A^{\sigma_k})^2 \right)_{\sigma_k} \) of order at most \( \ell \), it follows that

\[
A^{\sigma_k} = \sum_{F_i \in \mathcal{F}_{m(k)}} \alpha_i \cdot F_i
\]

for some integer \( m(k) \) such that \( m(k) \leq \frac{\ell + v(\sigma_k)}{2}. \) Without loss of generality, \( m(k) = \left\lfloor \frac{\ell + v(\sigma_k)}{2} \right\rfloor. \) Therefore,

\[
R = \sum_{k \in [K]} \sum_{j \in [J_k]} b^k_j \cdot \left( \left( \sum_{F_i \in \mathcal{F}_{m(k)}} \alpha_{k,j,i} \cdot F_i \right)_{\sigma_k} \right)^2 = \sum_{k \in [K]} \left\| x_{\sigma_k}^T M_{\sigma_k} x_{\sigma_k} \right\|_{\sigma_k}, \quad (1.7)
\]

where

\[
\bullet \text{ each vector } x_{\sigma_k} \text{ is the } |\mathcal{F}_{m(k)}| \text{-dimensional vector whose } i\text{-th coordinate is equal to the } i\text{-th element of } \mathcal{F}_{m(k)}; \text{ and}
\]
each matrix $M_{\sigma_k}$ is equal to

$$\sum_{j \in [J_k]} b_j^k \cdot \left( \alpha_{k,j,1}, \alpha_{k,j,2}, \ldots, \alpha_{k,j,\lfloor \sigma_k m(k) \rfloor} \right) \times \left( \alpha_{k,j,1}, \alpha_{k,j,2}, \ldots, \alpha_{k,j,\lfloor \sigma_k m(k) \rfloor} \right)^T.$$

Note that the matrices $M_{\sigma_1}, \ldots, M_{\sigma_K}$ are symmetric positive semidefinite matrices with real entries.

Now recall that $R = \sum_{F \in F_\ell} r_F \cdot F$. The equation (1.7) implies that all the coefficients $r_F$ depends only on the entries of the matrices $M_{\sigma_1}, \ldots, M_{\sigma_K}$. For a given set of matrices $M_{\sigma_1}, \ldots, M_{\sigma_K}$, we write $r_F(M_{\sigma_1}, \ldots, M_{\sigma_K})$ to denote the coefficient in front of $F$ in $R$. Using this notation, the semidefinite program for the objective value $A = \sum_{F \in F_\ell} \alpha_F \cdot F$ can be written as

$$\begin{align*}
\text{maximize} & \quad c \\
\text{subject to} & \quad \alpha_F \geq c + r_F(M_{\sigma_1}, \ldots, M_{\sigma_K}) \quad \forall F \in F_\ell, \\
& \quad M_{\sigma_1} \succeq 0, \\
& \quad \vdots \\
& \quad M_{\sigma_K} \succeq 0,
\end{align*}$$

where the constraints $M_{\sigma_k} \succeq 0$, for $k \in [K]$, denote that the matrices $M_{\sigma_k}$ are positive semidefinite.

Let us now focus on the dual program of the semidefinite program (1.8). We start with introducing some additional notation. For a homomorphism $\phi \in \text{Hom}^+(A, \mathbb{R})$ and an integer $\ell$, the local density $\ell$-profile of $\phi$ is the vector

$$\phi|_\ell := (\phi(F_1), \phi(F_2), \ldots, \phi(F_{|F_\ell|})).$$

We denote the $i$-th coordinate of $\phi|_\ell$ by $\phi_i|_\ell(F_i)$. Furthermore, for $A = \sum_{F \in F_\ell} \alpha_F \cdot F$, where $\alpha_F$ are arbitrary fixed reals, we define

$$\phi_i|_\ell(A) := \sum_{F \in F_\ell} \alpha_F \cdot \phi_i|_\ell(F).$$

With a slight abuse of notation, we use this notion also for an arbitrary vector $z \in \mathbb{R}^{|F_\ell|}$, i.e., we write $z(F_i)$ for the $i$-th coordinate of $z$, and we use $z(A)$ to denote the value of $\sum_{F \in F_\ell} \alpha_F \cdot z(F)$. 

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Let \( P_{\mathcal{F}_\ell} := \{ \phi|_{\ell} : \phi \in \text{Hom}^+(A, \mathbb{R}) \} \) be the set of all local density \( \ell \)-profiles. Note that \( P_{\mathcal{F}_\ell} \subseteq [0,1]|_{\mathcal{F}_\ell} \). For a combination \( A = \sum_{F \in \mathcal{F}_\ell} \alpha_F \cdot F \), it follows from the definitions that the value of (1.5) is equal to the minimum value of \( \phi|_{\ell}(A) \), where the minimum is taken over all \( \phi|_{\ell} \in P_{\mathcal{F}_\ell} \).

Now fix \( K \) types \( \sigma_1, \ldots, \sigma_K \) of order at most \( \ell \). Let \( \mathcal{S}_{\mathcal{F}_\ell} \) be the set of vectors \( z \in \mathbb{R}^{|\mathcal{F}_\ell|} \) that satisfy

- all the linear inequalities of the form \( z \left( \left[ (A^{\sigma_k})^2 \right]_{\sigma_k} \right) \geq 0 \), where \( k \in [K] \), \( A^{\sigma_k} \in A^{\sigma_k} \), and the order of the expression \( \left[ (A^{\sigma_k})^2 \right]_{\sigma_k} \) is at most \( \ell \),
- the non-negative inequalities \( z(F) \geq 0 \) for every \( F \in \mathcal{F}_\ell \), and
- the equation \( z \left( \sum_{F \in \mathcal{F}_\ell} F \right) = 1 \).

It immediately follows that \( P_{\mathcal{F}_\ell} \subseteq \mathcal{S}_{\mathcal{F}_\ell} \). It also follows that the set \( \mathcal{S}_{\mathcal{F}_\ell} \) is a convex set.

Recall that the aim of the semidefinite method is to find the minimum of \( \phi(A) \), where \( \phi \in \text{Hom}^+(A, \mathbb{R}) \), which is the same as the minimum of \( \phi|_{\ell}(A) \) for \( \phi|_{\ell} \in P_{\mathcal{F}_\ell} \). The duality of semidefinite programing (see, e.g., [39, Theorem 4.1.1]) implies that the dual of (1.8) is the following semidefinite program:

\[
\begin{align*}
\text{minimize} & \quad z(A), \\
\text{subject to} & \quad z(A) \geq c,
\end{align*}
\]

Therefore, if the semidefinite method finds a proof that for every \( \phi \in \text{Hom}^+(A, \mathbb{R}) \) it holds that \( \phi(A) \geq c \), where \( c \in \mathbb{R} \), it also holds that \( z(A) \geq c \) for any \( z \in \mathcal{S}_{\mathcal{F}_\ell} \).

In particular, if \( \phi, \psi \in \text{Hom}^+(A, \mathbb{R}) \) and \( \lambda \in (0,1) \), then

\[
\lambda \cdot \phi|_{\ell}(A) + (1 - \lambda) \cdot \psi|_{\ell}(A) \geq c.
\]

Note that the \( |\mathcal{F}_\ell| \)-dimensional vector \( \lambda \cdot \phi|_{\ell} + (1 - \lambda) \cdot \psi|_{\ell} \) is usually not a local density \( \ell \)-profile of any convergent sequence of graphs.
Chapter 2

Hypergraphs with positive lower density

One of the most well-known results in extremal graph theory is Turán’s theorem [72], which determines the largest possible number of edges in an $n$-vertex graph without a complete subgraph of a given size. Erdős, Simonovits, and Stone [23, 24] generalized Turán’s theorem by showing that the extremal number of a fixed graph $F$ is asymptotically determined by its chromatic number. Specifically, for every graph $F$ with at least one edge,

$$
ex(n, F) = \left( \frac{\chi(F) - 2}{\chi(F) - 1} + o(1) \right) \binom{n}{2}.\]

These extremal questions are dual to determining the minimum number of edges $m(F)$ that guarantees that an $n$-vertex graph $G$ with at least $m(n, F)$ edges contains a copy of $F$, since $m(n, F) = \text{ex}(n, F) + 1$.

For problems studied in this chapter, we impose a stronger density assumption on a (hyper)graph, where we wish to find a copy of some fixed (hyper)graph $F$. Instead of only assuming that the whole graph has a sufficiently large density, we assume that also every sufficiently large subset of vertices, say of size at least $\delta n$, has large relative density. Formally, we define the $\delta$-linear density of a graph $G$ to be the smallest density induced by a $\delta$-fraction of the set of vertices, i.e.,

$$d(G, \delta) := \min \left\{ \frac{e(G[A])}{\binom{|A|}{2}} : A \subseteq V(G), |A| \geq \delta \cdot v(G) \right\}.\]

Note that for any graph $G$, the function $d(G, \delta)$ is a non-decreasing function of $\delta$ taking values in $[0, 1]$.

However, requiring a positive $\delta$-linear density immediately forces large graphs to contain every given graph as a subgraph.
Observation 2.1. For every \( \varepsilon > 0 \) and a fixed graph \( F \), there exist \( \delta > 0 \) and \( n_0 \in \mathbb{N} \) such that every graph \( G \) on at least \( n_0 \) vertices with \( d(G, \delta) \geq \varepsilon \) contains \( F \) as a subgraph.

A proof of this observation follows, e.g., from [65, Theorem 1].

The notion of the \( \delta \)-linear density has a natural generalization to \( r \)-graphs. For an \( r \)-graph \( H \), we define

\[
d(H, \delta) := \min \left\{ \frac{e(H_n[A])}{\binom{|A|}{r}} : A \subseteq V(H), |A| \geq \delta \cdot v(H) \right\}.
\]

Since in this chapter we mostly deal with sequences of 3-graphs, it is natural to define the lower density of an increasing sequence of 3-graphs \( (H_n)_{n \in \mathbb{N}} \) to be the smallest \( \delta \)-linear density and denote it by \( \lambda ((H_n)_{n \in \mathbb{N}}) \). Formally,

\[
\lambda ((H_n)_{n \in \mathbb{N}}) := \lim_{\delta \to 0^+} \left\{ \liminf_{n \to \infty} d(H_n, \delta) \right\}.
\]

It is natural to ask for what \( r \)-graphs \( F \) we can generalize Observation 2.1, i.e., what is the set of \( r \)-graphs \( F \) such that that every sufficiently large \( r \)-graph \( H \) with a positive \( \delta \)-linear density contains a copy of \( F \). Frankl and Rödl [30] showed that for every \( r \)-graph \( F \) from the composition-closure of the set of all \( r \)-partite \( r \)-graphs, there exists a positive constant \( \delta \) such that every large \( r \)-graph \( H \) with a positive \( \delta \)-linear density contains a copy of \( F \). Until now, these are the only \( r \)-graphs \( F \) with this property we know.

2.1 Random tournaments construction

In order to simplify the notation in this chapter, we denote by \( K_4^{(3)} \) the complete 3-graph on 4 vertices, and by \( K_4^{-} \) the 3-graph on 4 vertices with 3 edges; see Figure 2.1. Erdős and Sós [26, Problem 5] asked whether a sufficiently large 3-graph \( H \) with a positive \( \delta \)-linear density contains a copy of \( K_4^{(3)} \), or at least a copy of \( K_4^{-} \). However, Füredi observed that the following construction of Erdős and Hajnal [22] gives a negative answer to the above question in a very strong sense.

Construction 2.2. Consider a random tournament \( T_n \) on \( n \) vertices. Let \( H_n \) be the 3-graph on the same vertex set consisting of exactly those triples that span a cyclically oriented triangle in \( T_n \).

For the exact reference to the observation, we refer the reader to [30], where Frankl and Rödl cite a personal communication with Füredi in 1983. Additional
One can check that in every 3-graph obtained in this way, any four vertices span at most two edges, i.e., for every $n \in \mathbb{N}$ the 3-graph $H_n$ does not contain $K_4^-$. It remains to show that for every $\delta > 0$, the $\delta$-linear density of a typical $H_n$ tends to $1/4$ as $n$ goes to infinity. In fact, we prove that even $(1/\log n)$-linear density almost surely tends to $1/4$.

We start with the following standard concentration lemma.

**Lemma 2.3.** Let $p \in [0,1]$, $r \in \mathbb{N}$ such that $r \geq 2$, and $n \in \mathbb{N}$. In an $n$-vertex $r$-graph $H$, we associate with every edge $e \in E(H)$ a random event $\mathcal{E}(e)$ such that

- $\mathbb{P}[\mathcal{E}(e)] = p$ for every $e \in E(H)$, and
- The events $\mathcal{E}(e)$ and $\mathcal{E}(e')$ are independent whenever $|e \cap e'| \leq 1$.

Then the probability that the number of occurred events on some large vertex set is far from its expectation tends to zero super-exponentially fast in $n$. More precisely,
there exists a positive constant $c$ such that

$$
\Pr \left[ \exists A \subseteq V(H), |A| \geq \frac{n}{\log n} : \left| \{e \in E(H[A]) : \mathcal{E}(e)\} - p|E(H[A])| \right| > \frac{c^2}{\log n} \right] = e^{-c^2 n^2}.
$$

**Proof.** Fix an arbitrary set $A \subseteq V(H)$ with $|A| \geq n/\log n$. We claim that

$$
\Pr \left[ \left| \{e \in E(H[A]) : \mathcal{E}(e)\} - p|E(H[A])| \right| > \frac{c^2}{\log n} \right] = e^{-c'^2 n^2} \tag{2.4}
$$

for some $c' > 0$. The statement of the lemma then follows easily by a union bound over all subsets of $V(H)$ of size at least $n/\log n$, so it remains to prove the claim.

Consider an edge-coloring of $H[A]$ with $\binom{|A|-2}{r-2}$ colors, such that every color class is a linear $r$-graph, i.e., any two edges of the same color intersect in at most one vertex. Since every fixed edge intersects in at least two vertices less than $\binom{|A|-2}{r-2}$ other edges, such a coloring can be found in a greedy way. Since

$$
\binom{r}{2} \binom{|A|-2}{r-2} > \frac{|A|(|A|-1)}{r^4 \cdot \log n},
$$

we infer that the event inside the probability formula in (2.4) implies that in at least one of the color classes, say $C$, the size of $\{e \in C : \mathcal{E}(e)\}$ deviates from its expectation by at least $|A|(|A|-1)/(r^4 \cdot \log n)$.

Fix a color class $C$. First, observe that $|C| \leq \frac{|A|}{2} / \frac{r}{2} = \frac{|A|(|A|-1)}{r(r-1)}$. The events $\mathcal{E}(e)$ and $\mathcal{E}(e')$ are independent for every $e, e' \in C$, hence by Chernoff’s inequality (see, e.g., [2, Corollary A.1.7]) we obtain

$$
\Pr \left[ \left| \{e \in C : \mathcal{E}(e)\} - p|C| \right| > \frac{|A|(|A|-1)}{r^4 \cdot \log n} \right] < 2 \cdot \exp \left( \frac{-2|A|^2(|A|-1)^2}{|C| \cdot r^8 \cdot \log^2 n} \right),
$$

which is equal to $e^{-c'^2 n^2}$ for $c'$ sufficiently small (recall that $|A| \geq n/\log n$ and $|C| < |A|(|A|-1)$). The claim now follows by a union bound over all color classes. \qed

We are ready to show that the lower density in Construction 2.2 is equal to $1/4$.

**Observation 2.5.** Consider a random sequence $(H_n)_{n \in \mathbb{N}}$ from Construction 2.2. With probability 1, the lower density of $(H_n)_{n \in \mathbb{N}}$ is equal to $1/4$.

**Proof.** Recall that $|V(H_n)| = n$. Let $H$ be the complete 3-graph on $n$ vertices, and $\mathcal{E}(e)$, for $e = uvw$, be the event that the three arcs on the set $\{u, v, w\}$ in
the underlying tournament of $H_n$ form an oriented cycle. Lemma 2.3 yields that with probability $1 - e^{-\frac{n^2}{4\log n}}$, for all subsets $A \subseteq V(H_n)$ of size at least $n/\log n$, the number of edges in $H_n[A]$ is between $(1/4 - 1/\log n)(\binom{|A|}{3})$ and $(1/4 + 1/\log n)(\binom{|A|}{3})$. Therefore, by Borel-Cantelli Lemma (see, e.g., [2, Lemma 8.6.1]), with probability one there exists $n_0 \in \mathbb{N}$ such that the property above is true for every $n \geq n_0$. But this immediately implies that the lower density of $(H_n)_{n \in \mathbb{N}}$ is $1/4$. \hfill \Box

The main result of this chapter is that the Construction 2.2 is essentially the best possible. This answers a question of Erdős from [21], which is based on the original problem of Erdős and Sós, positively.

2.2 Flag algebra setting

The proof of the main theorem is based on the semidefinite method described in Section 1.2. Through the whole chapter, we will use $\mathcal{F}$ to denote the set of all non-isomorphic finite $K_4^-$-free 3-graphs, and $\mathcal{F}_k$ to denote the subset of $\mathcal{F}$ containing all $k$-vertex $K_4^-$-free 3-graphs. Next, for a fixed $K_4^-$-free 3-graph $\sigma$ with a given labelling of its vertices, we define $\mathcal{F}_\sigma$ to be the set all finite $K_4^-$-free 3-graphs with a fixed embedding of $\sigma$, and $\mathcal{F}_k^\sigma$ to be the appropriate subsets. Analogously to the graph setting presented in Section 1.1, we construct an algebra $A$, algebras $A_\sigma$, where $\sigma$ is a fixed labelled $K_4^-$-free 3-graph, and averaging operators $\Psi_\sigma: A_\sigma \rightarrow A$. Finally, we denote by $\text{Hom}^+(A, \mathbb{R})$ the set of all algebra homomorphisms $\psi$ from $A$ to $\mathbb{R}$ such that $\psi(F) \geq 0$ for every $F \in \mathcal{F}$.

A sequence of $K_4^-$-free 3-graphs $(H_n)_{n \in \mathbb{N}}$ of increasing order is convergent if the limit $\lim_{n \to \infty} p(F, H_n)$ exists for every $F \in \mathcal{F}$. As in the graph case, there is a one-to-one correspondence between the set $\text{Hom}^+(A, \mathbb{R})$ and the set of all vectors in $[0, 1]^\mathcal{F}$ that represents the limit probabilities of convergent sequences.

We derive now some additional inequalities that are valid for any convergent sequence of 3-graphs $(H_n)_{n \in \mathbb{N}}$ with lower density at least $1/4$.

Let us first informally explain the main idea behind the inequalities. Consider an arbitrary type $\sigma$ that has a positive density in the limit, and let $F$ be a $\sigma$-flag with exactly one non-rooted vertex. Furthermore, assume that $n$ is sufficiently large. Now fix a copy $S$ of $\sigma$ in $H_n$ and consider the set $U(S)$ of all vertices $u \in V(H_n)$ that extends $S$ to a copy of $F$. The following two outcomes can happen:

- The size of $U(S)$ is small, i.e., $o(n)$. But then, for this choice of $S$, both the probability that three random points belongs to $U(S)$, and the probability that three random points belongs to $U(S)$ and span an edge, are $o(1)$.
The size of $U(S)$ is large, i.e., $\Omega(n)$. But then, by the assumption on the lower density, for this choice of $S$, the probability that three random points belong to $U(S)$ and span an edge is asymptotically at least a quarter of the probability that three random points belongs to $U(S)$.

Analysis of these two outcomes is given in the following lemma.

**Lemma 2.6.** Let $(H_n)_{n \in \mathbb{N}}$ be a convergent sequence of 3-graphs with lower density at least $1/4$. Let $\phi$ be its limit, $\sigma$ a type and $F \in \mathcal{F}_r^{\sigma}$ a $\sigma$-flag. In addition, let

$$\kappa := F^3 = \sum_{G \in \mathcal{F}_r^{\sigma}+(3)} \alpha_G \cdot G \quad \text{and} \quad \kappa^+ := \sum_{G \in \mathcal{F}_r^{\sigma}+(3)} \alpha^+_G \cdot G,$$

where

$$\alpha^+_G := \begin{cases} \alpha_G & \text{if the three non-labelled vertices of } G \text{ span an edge, and} \\ 0 & \text{otherwise.} \end{cases}$$

It holds that

$$\phi \left( \| 4\kappa^+ - \kappa \|_{\sigma} \right) \geq 0. \quad (2.7)$$

Note that the values of the coefficients $\alpha_G$ are uniquely determined by the choice of $F$. Also note that $\alpha_G \in \{0, 1\}$ for every $G \in \mathcal{F}_n^{\sigma}+(\sigma)+3$.

**Proof.** First observe that if $\phi(\sigma^\emptyset) = 0$, then all the terms in the left-hand side of (2.7) are equal to zero. For the rest of the proof, we assume $\phi(\sigma^\emptyset) > 0$.

Suppose for the contrary that (2.7) is not true, i.e., $\phi(\| 4\kappa^+ - \kappa \|_{\sigma}) = -\varepsilon_r$ for some $\varepsilon_r > 0$. Let $q_n$ be the values of the expression $\| 4\kappa^+ - \kappa \|_{\sigma}$ evaluated on the densities of $H_n$. Since the sequence $(H_n)_{n \in \mathbb{N}}$ is convergent, there exists $n_0 \in \mathbb{N}$ such that $q_{n_0} \leq -\varepsilon_r/2$. Moreover, since $(H_n)$ has lower density at least 1/4, we may assume that $d(H_{n_0}, \beta_\varepsilon) \geq 1/4$ for $\beta_\varepsilon := \sqrt{\varepsilon_r}/13$. It follows that one can get $\beta_\varepsilon \varepsilon_r$ for free.

Fix an embedding $\theta$ of $\sigma$ in $H_{n_0}$. Let $q_{n_0}^\theta$ be the value of the expression $(4\kappa^+ - \kappa)$ evaluated on the rooted densities of $H_{n_0}^\theta$. If there are less than $\beta_\varepsilon n$ vertices that extend $\theta$ to $F$, then $q_{n_0}^\theta$ is at least $-6(\beta_\varepsilon)^3 = -o(1)$. On the other hand, if at least $\beta_\varepsilon n$ vertices extend $\theta$ to $F$, then the density of the subhypergraph induced by those vertices is at least 1/4 and hence $4\kappa^+ \geq \kappa - o(1)$. In other words, $q_{n_0}^\theta \geq -o(1)$.

Recall that $p^\sigma_{\sigma^\emptyset}$ denotes the probability that a random labelling of $V(\sigma^\emptyset)$ with labels $\{1, \ldots, v(\sigma^\emptyset)\}$ produces the type $\sigma$, and note that $p^\sigma_{\sigma^\emptyset} \cdot p(\sigma^\emptyset, H_{n_0}) \leq 1$. 19
The definition of the conditional probability yields that
\[ q_{n_0} = p_{\theta}^* \cdot p(\theta, H_{n_0}) \cdot E[q_{n_0}] \geq -6(\beta_\varepsilon)^3 - o(1), \]
where the average is taken over all possible embeddings \( \theta \) of \( \sigma \) in \( H_{n_0} \). However, the choice of \( \beta_\varepsilon \) implies that \( q_{n_0} \geq -6\varepsilon/13 - o(1) \) which contradicts the fact that \( q_{n_0} \leq -\varepsilon_r/2 \).

We are now ready to present the two inequalities we are going to use in the next step. Let \( \sigma \) be the unique type on 2 vertices; from now on, we also write “2” to denote this type. We denote the 3-vertex \( \sigma \)-flag with no edges by \( F_0 \), and the 3-vertex \( \sigma \)-flag with one edge by \( F_1 \). The first inequality is an instance of (2.7) for \( \sigma = 2 \) and \( F = F_1 \). Let \( \kappa_1 \) be the corresponding \( \kappa \) and let \( \kappa_1^+ \) be the corresponding \( \kappa^+ \). The first line in Figure 2.3 shows the flag \( F_1 \) and the corresponding linear combinations \( \kappa_1 \) and \( \kappa_1^+ \). Since all \( \sigma \)-flags in \( \mathcal{F}^\sigma \) are \( K_4^- \)-free, it follows that \( \kappa_1 = L_1 + L_1^+ \) and \( \kappa_1^+ = L_1^+ \). The \( \sigma \)-flags \( L_1 \) and \( L_1^+ \) are depicted in Figure 2.4.

Analogously, let \( \kappa_2 \) be \( \kappa \) and let \( \kappa_2^+ \) be \( \kappa^+ \) from (2.7) for \( F = F_0 \) (again, \( \sigma \) is equal to 2). The second line in Figure 2.3 shows the flag \( F_0 \), and the combinations \( \kappa_2 \) and \( \kappa_2^+ \). It holds that
\[ \kappa_2 = \sum_{i=2}^{14} L_i + \sum_{i=2}^{6} L_i^+ \quad \text{and} \quad \kappa_2^+ = \sum_{i=2}^{6} L_i^+, \]
where the \( \sigma \)-flags \( L_2, \ldots, L_{14} \) and \( L_2^+, \ldots, L_6^+ \) are again depicted in Figure 2.4.

Lemma 2.6 yields that
\[ \phi([4\kappa_1^+ - \kappa_1]) \geq 0, \quad (2.8) \]
and
\[ \phi([4\kappa_2^+ - \kappa_2]) \geq 0. \quad (2.9) \]

### 2.3 3-graphs with 4 vertices spanning at most 2 edges

In this section, we present the main result of this chapter. Specifically, we prove the following theorem.

**Theorem 2.10.** For every \( \varepsilon_{\text{THM}} > 0 \) there exist \( \delta_{\text{THM}} > 0 \) and \( n_{\text{THM}} \in \mathbb{N} \) such that every 3-graph \( H \) on at least \( n_{\text{THM}} \) vertices with \( d(H, \delta_{\text{THM}}) \geq 1/4 + \varepsilon_{\text{THM}} \) contains a copy of \( K_4^- \).
Figure 2.3: The expressions used in the inequalities (2.8) and (2.9). A solid line
denotes an edge, a dashed line a non-edge, and finally we sum over all the possible
choices edge/non-edge for the triples where there is neither a dashed nor a solid line.

Recall that $F_7$ is the set of all $K_4^-$-free 3-graphs of size 7. It holds that
$|F_7| = 8157$. Let $\text{EXT}_7$ be the set of all 3-graphs from $F_7$ that can possibly appear
as induced subhypergraphs of the 3-graphs from Construction 2.2. It holds that
$|\text{EXT}_7| = 247$. We start the proof of Theorem 2.10 with the following lemma.

**Lemma 2.11.** There exist rational numbers $(\alpha_G)_{G \in F_7}$, where $\alpha_G = 0$
for $G \in \text{EXT}_7$, and $\alpha_G < 0$ otherwise, such that the following holds. If $\phi$
is an element of $\text{Hom}^+(\mathcal{A}, \mathbb{R})$ that satisfies (2.8) and (2.9), then

$$
\phi \left( \sum_{G \in F_7} \alpha_G \cdot G \right) \geq 0.
$$

**Proof.** Using the method from Section 1.2, an instance of the SDP was used to
find positive rationals $\gamma_1$ and $\gamma_2$ and 8 symmetric positive semidefinite matrices
$M_1, M_2, \ldots, M_8$ with rational entries such that the following holds:

$$
\phi \left( \gamma_1 \cdot \|4\kappa_1^+ - \kappa_1\|_2 + \gamma_2 \cdot \|4\kappa_2^+ - \kappa_2\|_2 + \sum_{i \in [8]} \|x_i^T M_i x_i\|_{\sigma_i} \right) = \phi \left( \sum_{G \in F_7} \alpha_G \cdot G \right),
$$

where
Figure 2.4: The $\sigma$-flags from the inequalities (2.8) and (2.9).
• the type $\sigma_1$ is the type with one vertex,
• the type $\sigma_2$ is the type with three vertices and no edge,
• the type $\sigma_3$ is the type with three vertices and one edge,
• the types $\sigma_4, \ldots, \sigma_8$ are the five specific types on five vertices given in Figure 2.5,
• the vector $x_1 \in (\mathbb{R} F_{\sigma_1}^4)|_{F_{\sigma_1}^4}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R} F_{\sigma_1}^4$,
• for $i = 2, 3$, the vector $x_i \in (\mathbb{R} F_{\sigma_i}^5)|_{F_{\sigma_i}^5}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R} F_{\sigma_i}^5$,
• for $i = 4, 5, \ldots, 8$, each vector $x_i \in (\mathbb{R} F_{\sigma_i}^6)|_{F_{\sigma_i}^6}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R} F_{\sigma_i}^6$, and
• $\alpha_G = 0$ if $G \in \text{EXT}_7$, and $\alpha_G < 0$ otherwise.

The left-hand side of the inequality above is non-negative by (1.2), (2.8), and (2.9).

The proof of the previous lemma was found with a computer assistance. We used semidefinite programming libraries CSDP [11] and SDPA [74] to find an approximate solution of the corresponding semidefinite program. The approximate solution was then turned into an exact one by a careful rounding. For the rounding part, we used a mathematical software Sage [71]. Also note that the sizes $a_i$ of the sets $F_{\sigma_1}^4, F_{\sigma_2}^5, F_{\sigma_3}^5, F_{\sigma_4}^5, F_{\sigma_5}^6, F_{\sigma_6}^6, F_{\sigma_7}^6$, and $F_{\sigma_8}^6$, which coincide with the order of the matrices $M_i$, are 5, 95, 47, 191, 135, 95, 101, and 148, respectively.

The numerical values of $\gamma_1, \gamma_2$, and the entries of the matrices $M_1, \ldots, M_8$ can be downloaded from \url{http://honza.ucw.cz/phd/}. Each matrix $M_i$ is not stored directly but as an appropriate number of vectors $v^j_i \in \mathbb{Q}^{a_i}$ and positive rationals $w^j_i$ such that

$$M_i = \sum_{j=1}^{r_i} w^j_i \cdot v^j_i \times (v^j_i)^T,$$

where $r_1 = 3, r_2 = 45, r_3 = 21, r_4 = 63, r_5 = 60, r_6 = 43, r_7 = 31$, and $r_8 = 25$.

In order to make an independent verification of our computations easier, we created a sage script called “lemma_2.13-verify.sage”, which is also available on the web page mentioned above.

Lemma 2.11 immediately yields the following.
Corollary 2.12. Let $F$ be a 3-graph of size at most 7 such that its induced density in the 3-graphs from Construction 2.2 is zero. If $\phi \in \text{Hom}^+(A, \mathbb{R})$ satisfies (2.8) and (2.9), then $\phi(F) = 0$.

We are now ready to present the proof of Theorem 2.10.

Proof of Theorem 2.10. Suppose to the contrary that there exists $d > 1/4$ and a sequence $(H_n)_{n \in \mathbb{N}}$ of $K_4^-$-free 3-graphs of increasing orders with lower density $d$. In other words
\[ \lambda((H_n)_{n \in \mathbb{N}}) = d > 1/4. \]
Let us assume, without loss of generality, that $(H_n)_{n \in \mathbb{N}}$ converges and $\phi_C$ is the limit; We aim to show that the edge density of $\phi_C$ is zero, contradicting the fact that it must be at least the lower density of the sequence.

Let $B$ be the 3-graph depicted in Figure 2.6, which we call the butterfly 3-graph. Since the 3-graphs from Construction 2.2 are $B$-free, Corollary 2.12 implies that $\phi(B) = 0$ whenever a homomorphism $\phi \in \text{Hom}^+(A, \mathbb{R})$ satisfies (2.8) and (2.9).

In particular, for the limit $\phi$ of any convergent sequence of $K_4^-$-free 3-graphs with lower density at least $1/4$, we have $\phi(B) = 0$.

Instead of applying the claim directly to $\phi_C$, we first construct from $(H_n)_{n \in \mathbb{N}}$ a new sequence $(H'_n)_{n \in \mathbb{N}}$ such that $\lambda((H'_n)_{n \in \mathbb{N}}) = 1/4$. The new sequence is obtained by a random sparsification of $(H_n)_{n \in \mathbb{N}}$, i.e., removing each edge of each $H_n$ at random with the appropriately chosen probability. This is formulated in the

Figure 2.5: The types $\sigma_1$ through $\sigma_8$. A solid line denotes an edge, a triple without a solid line spans a non-edge.
following observation, which is an immediate corollary of Lemma 2.3.

**Observation 2.13.** Let \((H_n)_{n \in \mathbb{N}}\) be a sequence of 3-uniform \(K_4\)-free 3-graphs with 
\[ d := \lambda \left( (H_n)_{n \in \mathbb{N}} \right) > 1/4. \]
Furthermore, for every \(n \in \mathbb{N}\), let \(H'_n\) be a random subhypergraph of \(H_n\) obtained by removing every hyperedge of \(H_n\) independently with probability \(1 - \frac{1}{4d}\). Then
\[ \mathbb{P} \left[ \lambda \left( (H'_n)_{n \in \mathbb{N}} \right) = 1/4 \right] = 1. \]

Let \((H'_n)_{n \in \mathbb{N}}\) be a sequence of 3-graphs obtained from \((H_n)_{n \in \mathbb{N}}\) by a random sparsification with probability \(1 - \frac{1}{4d}\). All the following holds with probability one. Let \(\phi'_C\) be its limit (the sequence \((H'_n)_{n \in \mathbb{N}}\) must be convergent). Lemma 2.6 implies that \(\phi'_C\) satisfies both (2.8) and (2.9). Hence, by Corollary 2.12, the induced density of \(B\) in \(\phi'_C\) is equal to zero. But this implies that \(\phi_C(F) = 0\) for any \(F \in \mathcal{F}\) that contains \(B\) as a non-induced subhypergraph. This holds because otherwise there would be some \(F \in \mathcal{F}\) that contains a non-induced copy of \(B\) and \(\phi_C(F) > 0\). However, a positive proportion of the induced copies of \(F\) from \(\phi_C\) will be then turned, by a random sparsification, to induced copies of \(B\) in \(\phi'_C\). We conclude that the \textit{non-induced} density of the butterfly 3-graph \(B\) in \(\phi_C\) is equal to zero. In particular, any non-negative combination of the elements of \(\mathcal{F}_5\) that contain the butterfly as a non-induced subhypergraph must be zero in \(\phi_C\).

We are now ready to conclude that the edge density of \(\phi_C\) is zero. Let \(\rho \in \mathcal{F}_3\) be the 3-graph on three (non-rooted) vertices that span an edge. Let \(\sigma\) be the one-vertex type and \(\rho_1 \in \mathcal{F}_3^\sigma\) the flag corresponding to \(\rho\) with exactly one rooted vertex. Observe that all the elements of \(\mathcal{F}_3^\sigma\) with positive coefficients in the expression \((\rho_1)^2\) contain \(B\) as a subhypergraph. So, \(\phi_C \left( [\rho_1]^2 \right)_\sigma = 0\), which implies that
\[ \phi_C (\rho)^2 = \phi_C (\rho_1^2\sigma)^2 \leq \phi_C (\rho_1^2\sigma) = 0. \]
Theorem 2.10 together with the 3-uniform version of the Hypergraph Removal Lemma (see, e.g., [67, Theorem 3] for the general $r$-uniform version of the lemma) immediately implies that if the $\delta$-linear density is more than $1/4$, we do not have only one but actually many copies of $K_4^-$. Let us first precisely state the version of the removal lemma for 3-graphs, which actually follows already from the results in [31].

**Lemma 2.14** (Hypergraph Removal Lemma, 3-uniform case). For every $\varepsilon_{RL} > 0$ and fixed $k$-vertex 3-graph $F$, there exists $\gamma_{RL} > 0$ such that every 3-graph $H$ on $n$ vertices with less than $\gamma_{RL} \cdot \binom{n}{k}$ copies of $F$ can be made $F$-free by removing less than $\varepsilon_{RL} \cdot \binom{n}{3}$ edges.

We are now ready to derive the counting version of Theorem 2.10.

**Corollary 2.15.** For every $\varepsilon_{CNT} > 0$ there exist $\gamma_{CNT} > 0$, $\delta_{CNT} > 0$ and $n_{CNT} \in \mathbb{N}$ such that every 3-graph $H$ on at least $n_{CNT}$ vertices with $d(H, \delta_{CNT}) \geq 1/4 + \varepsilon_{CNT}$ contains $\gamma_{CNT} \cdot \binom{|S|}{4}$ copies of $K_4^-$. 

**Proof.** Suppose for a contradiction there is $\varepsilon_0 > 0$ such that for every $\gamma > 0$, $\delta > 0$ and $n \in \mathbb{N}$ we can find a 3-graph $H(n, \delta, \gamma)$ on at least $n$ vertices, with $\delta$-linear density at least $1/4 + \varepsilon_0$, and with less than $\gamma \cdot \binom{n}{3}$ copies of $K_4^-$. Fix such $\varepsilon_0$ and for every $\gamma, \delta$ and $n$, fix one such 3-graph $H(n, \delta, \gamma)$.

Let $n_{THM}$ and $\delta_{THM}$ be the constants from Theorem 2.10 applied for $\varepsilon_{THM} := \varepsilon_0/2$. Fix an integer $k$ so that $k \log k > n_{THM}$ and $1/k < \delta_{THM}$. Next, let $\gamma_{RL} := \gamma_{RL}(k)$ be the constant given by the Hypergraph Removal Lemma for $K_4^-$ applied for $\varepsilon_{RL} := \varepsilon_0/(3k^3)$. Finally, let

$$G := H(k \cdot \log k, 1/k, \gamma_{RL}(k)).$$

This means that every $S \subseteq V(G)$ of size at least $v(G)/k \geq \log k$ contains at least $(1/4 + \varepsilon_0)(\frac{|S|}{3})$ edges. Moreover, $G$ can be transformed to a $K_4^-$-free 3-graph $G'$ by removing less than $\frac{\varepsilon_0}{3k^3} \cdot \binom{v(G')}{3}$ edges. Consequently, every $S \subseteq V(G')$ of size at least $v(G')/k$ contains at least $(1/4 + 2\varepsilon_0/3)(\frac{|S|}{3}) - O(|S|^2)$ edges. Since $k$ is large, it follows that

$$\left( \frac{1}{4} + \frac{2\varepsilon_0}{3} \right) \cdot \left( \frac{|S|}{3} \right) - O(|S|^2) > \left( \frac{1}{4} + \frac{\varepsilon_0}{2} \right) \cdot \left( \frac{|S|}{3} \right).$$

Therefore, $d(G', 1/k) > 1/4 + \varepsilon_0/2$ and Theorem 2.10 imply that $G'$ contains a copy of $K_4^-$, a contradiction. \qed

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Also, exploiting the fact that the semidefinite method seeks for a vector minimizing a given linear function over a convex superset of the set Hom\(^+(\mathcal{A},\mathbb{R})\), we can replace the \(d(H,\delta) \geq 1/4 + \varepsilon\) condition by another that controls only the densities inside subsets that are common neighborhood of two vertices in \(H\) and inside subsets that are common neighborhoods in the complement of \(H\).

**Theorem 2.16.** For every \(\varepsilon > 0\) there exist \(\delta > 0\) and \(n_0 \in \mathbb{N}\) such that the following is true. Every 3-graph \(H\) on \(n \geq n_0\) vertices such that for every \(\{u,v\} \in (V(H))_2\) satisfies the following two conditions:

- if \(|N_H(u,v)| \geq \delta n\), then \(H[N_H(u,v)]\) has density at least \((1/4 + \varepsilon)\), and
- if \(|\overline{N_H}(u,v)| \geq \delta n\), then \(H[\overline{N_H}(u,v)]\) has density at least \((1/4 + \varepsilon)\),

contains \(K_4^-\) as a subhypergraph.

**Proof.** Suppose for a contradiction the theorem is false. Hence there exist a positive \(\varepsilon_0\) and a convergent sequence of \(K_4^-\)-free 3-graphs \((H_n)_{n \in \mathbb{N}}\) of increasing orders such that each \(H_n\) satisfies the two conditions from the statement of this theorem. We denote the limit of this sequence by \(\phi_C\).

First, we claim that the edge density of \(\phi_C\) is positive (in fact, it is strictly more than \(1/32\)). Fix two arbitrary vertices \(u\) and \(v\) in \(H_n\). Since either \(uvw \in E(H_n)\), or \(uvw \in E(\overline{H_n})\) for every \(w \in V(H_n) \setminus \{u,v\}\), it follows that there is a set \(S \subseteq V(H_n)\) with at least \(v(H_n)/2 - 1\) vertices such that \(H_n[S]\) has edge density at least \((1/4 + \varepsilon_0)\). But this means that \(H_n\) has edge density at least \(1/32 + \varepsilon_0/8 - o(1)\).

Let us now analyze some further properties of \(\phi_C\). Let \(\sigma = 2\) be the two-vertex type. Recall that \(F_0\) is the 3-vertex \(\sigma\)-flag with no edges, and \(F_1\) is the edge with two rooted vertices. If \(P[\phi_C^\sigma(F_1) = 0] = 0\), then, by averaging over all pairs of the vertices, we conclude that the edge density of \(\phi_C\) is zero. However, we know the edge density is more than \(1/32\). Analogously, if \(P[\phi_C^\sigma(F_0) = 0] = 1\), then the edge density of \(\phi_C\) must be one. However, since \(\phi_C(K_4^-) = \phi_C(K_4^{(3)}) = 0\), the edge density of \(\phi_C\) is trivially at most \(1/2\), a contradiction. Therefore, there exists a positive \(\gamma\) such that

\[P[\phi_C^\sigma(F_1) \geq \gamma] \geq \gamma,\]

and similarly

\[P[\phi_C^\sigma(F_0) \geq \gamma] \geq \gamma.\]

Recall the inequalities (2.8) and (2.9). The reasoning from the previous paragraph yields that a positive proportion of pairs \(\{u,v\}\) have a large co-degree of \(\{u,v\}\) in the underlying sequence of \(\phi_C\). Analogously, a positive proportion of
pairs \{u, v\} have large co-degree in the complement. Since the density on every such subset is at least \(1/4 + \varepsilon_0 > 1/4\), the inequalities (2.8) and (2.9) have some slack for \(\phi = \phi_C\). More precisely, there exists a positive \(\zeta\) such that

\[
\phi_C \left( \left\| k^+_1 - k_1 \right\|_2 \right) \geq \zeta,
\]

(2.17)

and

\[
\phi_C \left( \left\| k^+_2 - k_2 \right\|_2 \right) \geq \zeta.
\]

(2.18)

Now recall the semidefinite program (1.8) and its dual (1.9). In the proof of Lemma 2.11, we used the semidefinite method to find two positive rationals \(\gamma_1\) and \(\gamma_2\) and an expression

\[
R = \sum_{i=1}^{8} \sum_{j=1}^{r_i} b_{ij} \cdot \left\| \left( A_{ij}^{\sigma_1} \right)^2 \right\|_{\sigma_1},
\]

where \(b_{ij} \geq 0\), \(A_{ij}^{\sigma_1} \in A^{\sigma_1}\), and the order of the expression \(\left\| \left( A_{ij}^{\sigma_1} \right)^2 \right\|_{\sigma_1}\) is at most 7 for every \(i \in [8]\) and \(j \in [r_i]\), such that the following is true:

\[
\gamma_1 \cdot \left\| 4k^+_1 - k_1 \right\|_2 + \gamma_2 \cdot \left\| 4k^+_2 - k_2 \right\|_2 + R = \sum_{G \in \mathcal{F}_7} \alpha_G \cdot G,
\]

where \(\alpha_G = 0\) for every \(G \in \text{EXT}_7\), and \(\alpha_G < 0\) for every \(G \in \mathcal{F}_7 \setminus \text{EXT}_7\). It holds that \(\phi(R) \geq 0\) for every \(\phi \in \text{Hom}^+(A, \mathbb{R})\). Furthermore, the reasoning from Section 1.2 yields that the bound \(z(R) \geq 0\) holds also for every vector \(z \in \mathbb{R}^{\left| \mathcal{F}_7 \right|}\) that satisfies the following:

- all the linear inequalities of the form \(z \left( \left\| (A^{\sigma_1})^2 \right\|_{\sigma_1} \right) \geq 0\), where \(A^{\sigma_1} \in A^{\sigma_1}\) and the order of the expression \(\left\| (A^{\sigma_1})^2 \right\|_{\sigma_1}\) is at most 7,

\[
\vdots
\]

- all the linear inequalities of the form \(z \left( \left\| (A^{\sigma_8})^2 \right\|_{\sigma_8} \right) \geq 0\), where \(A^{\sigma_8} \in A^{\sigma_8}\) and the order of the expression \(\left\| (A^{\sigma_8})^2 \right\|_{\sigma_8}\) is at most 7,

- the non-negative inequalities \(z(F) \geq 0\) for every \(F \in \mathcal{F}_7\), and

- the equation \(z \left( \sum_{F \in \mathcal{F}_7} F \right) = 1\).

The types \(\sigma_1, \ldots, \sigma_8\) are the same types as in Lemma 2.11, i.e., the types depicted in Figure 2.5. We denote the set of all such vectors \(z \in \mathbb{R}^{\left| \mathcal{F}_7 \right|}\) by \(S_{\mathcal{F}_7}\).
Finally, Recall the 3-graph $B$ depicted in Figure 2.6. We slightly abuse the notation and use $B$ also to denote the linear combination of the elements of $F_7$ that is equal to $B$ in $A$. Our aim is to use the homomorphism $\phi_C$ for constructing a vector $z \in S_{F_7}$ that satisfies the inequalities (2.8), (2.9) and $z(B) > 0$. This is an immediate contradiction, because on one hand

$$z\left(\gamma_1 \cdot \|4\kappa_1 - \kappa_1\|_2 + \gamma_2 \cdot \|4\kappa_2 - \kappa_2\|_2 + R\right) \geq 0,$$

but on the other hand every 3-graph $G \in \text{EXT}_7$ does not contain $B$ as an induced subhypergraph, and hence

$$z\left(\sum_{G \in F_7} \alpha_G \cdot G\right) < 0.$$

Let us emphasize that in the argument we do not need that the vector $z \in S_{F_7}$ to be a 7-local density profile of some convergent sequence of 3-graphs.

Let $\phi_b \in \text{Hom}^+(A, \mathbb{R})$ be the limit corresponding to the sequence of 3-graphs $(B_k)_{k \in \mathbb{N}}$, where $B_k$ is the $k$-th balanced blow-up of the butterfly 3-graph $B$. Every 3-graph $B_k$ is $K^-_4$-free, and the density of $B$ tends to $24/625$, i.e., $\phi_b(B) = 24/625$. For every $\xi \in [0,1]$, let $\phi_\xi := \xi \cdot \phi_C + (1 - \xi) \cdot \phi_b$. Since the set $S_{F_7}$ is convex, it follows that $(\phi_\xi)_{\xi} \in S_{F_7}$ for every $\xi \in [0,1]$. However, since $\phi_1 = \phi_C$ satisfies even the inequalities (2.17) and (2.18), there exists a point $\xi_0 \in (0,1)$ such that

$$\phi_{\xi_0}\left(\|4\kappa_1^+ - \kappa_1\|_2\right) \geq \zeta / 2 \geq 0,$$

and

$$\phi_{\xi_0}\left(\|4\kappa_2^+ - \kappa_2\|_2\right) \geq \zeta / 2 \geq 0.$$

However, $\phi_{\xi_0}(B) = (1 - \xi_0) \cdot \phi_b(B) > 0$, a contradiction.

\[ \square \]

### 2.4 Uniqueness of the tournament construction

In the last section of this chapter, we show that the limit of any convergent sequence of $K^-_4$-free 3-graphs with lower density $1/4$ has to be equal to the limit of the sequence of 3-graphs from Construction 2.2. In order to do so, we use the argument of Falgas-Rarvy, Pikhurko and Vaughan [27], who proved the following closely related result.

**Theorem 2.19** (Falgas-Rarvy, Pikhurko and Vaughan). *For every $\varepsilon > 0$ there exists*
\( n_0 \in \mathbb{N} \) such that every 3-graph \( H \) on \( n \geq n_0 \) vertices with the minimum co-degree at least \( (1/4 + \varepsilon) \cdot n \) contains a copy of \( K^-_4 \).

The relation between Theorems 2.10 and 2.19 is that the 3-graphs from Construction 2.2 serve as extremal configurations for the corresponding two problems. The entire section is devoted to a proof of the following theorem.

**Theorem 2.20.** Let \( (H_n)_{n \in \mathbb{N}} \) be a convergent sequence of \( K^-_4 \)-free 3-graphs so that \( \lambda((H_n)) = 1/4 \), and let \( \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \) be its limit. It holds that \( \phi \) is equal to the homomorphism \( \psi \), which is the limit of the sequence of 3-graphs from Construction 2.2 with increasing orders.

**Proof.** Without loss of generality, \( v(H_n) = n \). Recall that the set \( \text{EXT}_7 \) is the set of all 7-vertex induced subhypergraphs of the 3-graphs from Construction 2.2. Corollary 2.12 and the induced version of the Hypergraph Removal Lemma [66, Theorem 6] implies that there exists a sequence of 3-graphs \( (H'_n)_{n \in \mathbb{N}} \) such that each \( H'_n \) can be obtained from \( H_n \) by adding or removing \( o(n^3) \) edges, and every 7-vertex induced subhypergraph of \( H'_n \) is isomorphic to one of the 3-graphs from \( \text{EXT}_7 \). Note that \( (H'_n)_{n \in \mathbb{N}} \) is a convergent sequence of 3-graphs. Also note that the density of any fixed 3-graph \( G \) in \( (H'_n)_{n \in \mathbb{N}} \) tends to \( \phi(G) \). Since the application of the Hypergraph Removal Lemma changed only \( o(n^3) \) edges, it holds that \( \lambda((H'_n)) = 1/4 \). Therefore, it is enough to show that the limit of \( (H'_n)_{n \in \mathbb{N}} \) is equal to \( \psi \).

The induced subhypergraph property satisfied by the 3-graphs \( H'_n \) yields the following important fact: for every induced subhypergraph \( Z \) of any \( H'_n \) on at most 7 vertices, there exists a tournament \( T_Z \) such that the edges of \( Z \) are in one-to-one correspondence to the cyclically oriented triangles in \( T_Z \).

We now setup a variant of the flag algebra framework for convergent sequences of 3-graphs that satisfy this property on 7-vertex induced subhypergraphs. Let \( \mathcal{G} \) be the set of all 3-graphs such that each of their 7-vertex induced subhypergraphs is isomorphic to one of the 3-graphs from \( \text{EXT}_7 \). Also, for any type \( \sigma \) with the underlying 3-graph from \( \mathcal{G} \), let \( \mathcal{G}^\sigma \) be the corresponding set of all the \( \sigma \)-flags with this property. Finally, let \( \mathcal{A}' \) be the corresponding flag algebra defined by \( \mathcal{G} \). It follows that there exists a homomorphism \( \phi' \in \text{Hom}^+(\mathcal{A}', \mathbb{R}) \) which represents the limit of the sequence \( (H'_n)_{n \in \mathbb{N}} \).

Let \( \sigma = 2 \) be the 2-vertex type and let \( F_0 \) and \( F_1 \) be the \( \sigma \)-flags on 3 vertices with zero edges and one edge, respectively. Since \( H'_n \) was obtained from \( H_n \) by adding or removing \( o(n^3) \) edges, the homomorphism \( \phi' \) satisfies the following two inequalities, which are direct analogues of (2.8) and (2.9) in the algebra \( \mathcal{A}' \):

\[
\phi' (\|4t_1^+ - t_1\|_2) \geq 0 \quad \text{and} \quad \phi' (\|4t_2^+ - t_2\|_2) \geq 0,
\]
where

\[ \nu_1 := (F_1)^3 = \sum_{G \in \mathcal{G}_5^0} \alpha_G \cdot G, \quad \nu_1^+ := \sum_{G \in \mathcal{G}_5^0} \alpha_G^+ \cdot G, \]

\[ \nu_2 := (F_0)^3 = \sum_{G \in \mathcal{G}_5^0} \beta_G \cdot G, \quad \text{and} \quad \nu_2^+ := \sum_{G \in \mathcal{G}_5^0} \beta_G^+ \cdot G. \]

The values of the coefficients \( \alpha_G \) and \( \beta_G \) are uniquely determined,

\[ \alpha_G^+ := \begin{cases} \alpha_G & \text{if the three non-labelled vertices of } G \text{ span an edge,} \\
0 & \text{otherwise,} \end{cases} \]

and

\[ \beta_G^+ := \begin{cases} \beta_G & \text{if the three non-labelled vertices of } G \text{ span an edge,} \\
0 & \text{otherwise.} \end{cases} \]

Note that all the values of \( \alpha_G, \alpha_G^+, \beta_G \) and \( \beta_G^+ \) are either zero or one for every \( G \in \mathcal{G}_5^0 \).

Let \( \rho \in \mathcal{G} \) be the 3-graph on 3 vertices with one edge. Using the semidefinite method, we find four positive definite matrices \( M_1', \ldots, M_4' \) such that the following inequality in the algebra \( \mathcal{A}' \) holds:

\[ \rho = \sum_{F \in \mathcal{G}_6} \alpha_F \cdot G \leq \frac{1}{4} \cdot \sum_{F' \in \mathcal{G}_6} F - \sum_{i \in \{1, 2\}} \gamma_i' \cdot \left[ 4 \nu_i^+ - \nu_i \right]_2 - \sum_{i \in \{3, 4\}} \left[ (x_i I_i)^T \cdot M_i' \cdot (I_i x_i) \right]_{\sigma_i}, \]

(2.21)

where

- \( \alpha_G = e(G)/20 \) for every \( G \in \mathcal{G}_6 \),
- \( \gamma_1' = 148803/16384 \) and \( \gamma_2' = 70943/16384 \),
- the type \( \sigma_1 = 2 \) is the two-vertex type,
- the type \( \sigma_2 \) is the type with four vertices \( a, b, c, d \) and no edge,
- the type \( \sigma_3 \) is the type with four vertices \( a, b, c, d \) and the edge \( abc \),
- the type \( \sigma_4 \) is the type with four vertices \( a, b, c, d \) and the edges \( abc \) and \( abd \),
- the vector \( x_1 \in (\mathbb{R}^{\mathcal{G}_4^0})^{|\mathcal{G}_4^0|} \) is the vector whose \( j \)-th coordinate is equal to the \( j \)-th element of the canonical base of \( \mathbb{R}^{\mathcal{G}_4^0} \),
- for \( i = 2, 3, 4 \), the vector \( x_i \in (\mathbb{R}^{\mathcal{G}_5^0})^{|\mathcal{G}_5^0|} \) is the vector whose \( j \)-th coordinate is equal to the \( j \)-th element of the canonical base of \( \mathbb{R}^{\mathcal{G}_5^0} \), and
the matrices $I_1, \ldots, I_4$ are some specific matrices of sizes $8 \times 7$, $38 \times 26$, $17 \times 15$ and $17 \times 15$, respectively. The entries of the matrices $I_i$ can be found in Appendix A.1.

Similarly to the case of Lemma 2.11, the numerical values of the entries of the matrices $M'_1, \ldots, M'_4$ are available online and they can be downloaded from http://honza.ucw.cz/phd/. A sage script called “theorem_2_22-verify.sage”, which is also available on the web page, can be used to verify the computations.

The equation (2.21) implies that the edge density of $\phi'$ is at most $1/4$. On the other hand, since $\lambda((H'_n)) = 1/4$, the edge density of $\phi'$ must be at least $1/4$. We conclude that $\phi'(\rho) = 1/4$. Therefore, all the inequalities that were used in (2.21) must be in fact equations for the homomorphism $\phi'$. In particular, for every $i \in [4]$ it holds that

$$\phi' \left( \left\langle (x_i I_i)^T \cdot M'_i \cdot (I_i x_i) \right\rangle_{\sigma_i} \right) = 0.$$  

Since the matrices $M'_i$ are positive definite, we conclude that for a random homomorphism $\phi_{\sigma_i}'$ drawn the probability distribution $P^{\sigma_i}$ given by $\phi'$, the vector $\phi_{\sigma_i}'(x_i I_i)$ has all the coordinates equal to zero with probability one. By $\phi_{\sigma_i}'(x_i I_i)$ we mean the random vector $x'_i \in \mathbb{R}^{J_i}$ whose $j$-th coordinate is equal to $\phi_{\sigma_i}$ applied to the $j$-th coordinate of $x_i I_i$, where the values of $J_1, \ldots, J_4$ are equal to 7, 26, 15, and 15, respectively.

Therefore, asymptotically almost surely each coordinate of $\phi_{\sigma_i}'(x_i I_i)$ is equal to $o(1)$, where $\phi_{\sigma_i}'$ is drawn from the probability distribution $P^{\sigma_i}$ (recall that the probability distributions $P^{\sigma_i}_n$ arise from picking a copy of $\sigma_i$ in $H'_n$ at random, and the sequence $(P^{\sigma_i}_n)$ weakly converges to the distribution $P^{\sigma_i}$).

The next step of our proof is to deduce some structural properties of the 3-graphs $H'_n$ from the fact that the vectors $\phi_{\sigma_i}'(x_i I_i)$ have for most of the choices of the quadruple $a, b, c$ and $d$ all its coordinates equal to $o(1)$. Before doing so, we need to introduce some additional notation. We say that a pair of 2-sets of vertices $(\{a, b\}, \{c, d\})$ is tightly connected in $H'_n$ if $H'_n$ contains (at least) one of the following 12 configurations:

- a vertex $e$ and the edges $abc$, $ace$, $cde$
- a vertex $e$ and the edges $abd$, $ade$, $cde$
- a vertex $e$ and the edges $abe$, $ade$, $cde$
- a vertex $e$ and the edges $abe$, $ace$, $cde$
- the edges $abc$, $bce$, $cde$
- the edges $abc$, $bcd$
- the edges $abd$, $bce$, $cde$
- the edges $abd$, $ace$, $cde$
- a vertex $e$ and the edges $abe$, $ace$, $cde$
- a vertex $e$ and the edges $abe$, $bce$, $cde$
- a vertex $e$ and the edges $abc$, $bce$, $cde$
- a vertex $e$ and the edges $abc$, $bcd$
- a vertex $e$ and the edges $abe$, $ace$, $cde$
- a vertex $e$ and the edges $abe$, $bce$, $cde$

Note that we do allow the subhypergraph induced by the vertices $a, b, c, d$ and $e$
to contain also some other edges. Also note that the definition is symmetric, i.e.,

\( (\{a, b\}, \{c, d\}) \) is tightly connected if and only if \( (\{c, d\}, \{a, b\}) \) is. In other words, the pair \( (\{a, b\}, \{c, d\}) \) is tightly connected if there exists a tight path from \( \{a, b\} \) to \( \{c, d\} \) on at most 5 vertices. Our aim is to show that most of the pairs \( (\{a, b\}, \{c, d\}) \) in \( H' \) are tightly connected.

**Claim 1.** For every \( H' \), the number of pairs \( (\{a, b\}, \{c, d\}) \) that are not tightly-connected is \( o(n^4) \).

Fix one pair \( (\{a, b\}, \{c, d\}) \) that is not tightly-connected and let \( \sigma \) be the subhypergraph induced by \( a, b, c, d \) in \( H' \) labelled with \( a, b, c \) and \( d \). Since there is no tight path of length four between the pairs, we may assume by the symmetry that \( \sigma = \sigma_2, \sigma_3 \) or \( \sigma_4 \).

First, let us consider the case when \( \sigma = \sigma_4 \). We know that for all but \( o(n^4) \) copies of \( \sigma_4 \) it holds that \( \phi_{n, 4}^\sigma(x_4I_4) \) is an almost zero-vector. It holds that there are 6 (out of 17) \( \sigma_4 \)-flags of size 5 that contain a tight path from \( \{a, b\} \) to \( \{c, d\} \). For brevity, we let the last 6 coordinates of \( x_4 \) to be their corresponding densities. Also, we assume that \( G_5^\sigma = \{ G_1^\sigma, \ldots, G_{17}^\sigma \} \) so that the \( j \)-th coordinate of \( x_i \) is equal to the density of \( G_j^\sigma \). Taking the sum of the 13th and the 15th column of \( I_4 \), we conclude that asymptotically almost surely

\[
\phi_{n, 4}^\sigma \left( \sum_{G^\sigma \in G_5^\sigma} \alpha_{G^\sigma} \cdot G^\sigma \right) = o(1),
\]

where \( \alpha_{G^\sigma} < 0 \) for \( G^\sigma \in \{ G_1^\sigma, \ldots, G_{11}^\sigma \} \), i.e., for those \( G^\sigma \)s that do not contain a tight path from \( \{a, b\} \) to \( \{c, d\} \), and, additionally also for \( G_{12}^\sigma, G_{15}^\sigma, G_{17}^\sigma \). In the other cases, i.e., for \( G^\sigma \in \{ G_{13}^\sigma, G_{14}^\sigma, G_{16}^\sigma \} \), the value of \( \alpha_{G^\sigma} \) is positive. See Appendix A.1 for the entries of the matrix \( I_4 \) as well as the vector corresponding to the sum of the 13th and the 15th column of \( I_4 \), we conclude that asymptotically almost surely

\[
\phi_{n, 4}^\sigma \left( \sum_{G^\sigma \in G_5^\sigma} \alpha_{G^\sigma} \cdot G^\sigma \right) = o(1),
\]

for \( \alpha_{G^\sigma} < 0 \) for \( G^\sigma \in \{ G_1^\sigma, \ldots, G_{11}^\sigma \} \cup \{ G_3^\sigma, \ldots, G_{13}^\sigma \} \), and, additionally also for \( G_{12}^\sigma, G_{15}^\sigma, G_{17}^\sigma \). In the other cases, i.e., for \( G^\sigma \in \{ G_{13}^\sigma, G_{14}^\sigma, G_{16}^\sigma \} \), the value of \( \alpha_{G^\sigma} \) is positive. See Appendix A.1 for the entries of the matrix \( I_4 \) as well as the vector corresponding to the sum of the 13th and the 15th column of \( I_4 \), we conclude that asymptotically almost surely

\[
\phi_{n, 3}^\sigma \left( \sum_{G^\sigma \in G_3^\sigma} \alpha_{G^\sigma} \cdot G^\sigma \right) = o(1),
\]

asymptotically almost surely. This time, \( \alpha_{G^\sigma} \) is negative for \( G^\sigma \in \{ G_1^\sigma, \ldots, G_{11}^\sigma \} \cup \{ G_3^\sigma, \ldots, G_{13}^\sigma \} \), and, additionally also for \( G_{12}^\sigma, G_{15}^\sigma, G_{17}^\sigma \).
\{G_{12}^{\sigma_3}, G_{13}^{\sigma_3}, G_{15}^{\sigma_3}, G_{17}^{\sigma_3}\}$, and positive otherwise. Note that $\mathcal{F}_5^{\sigma_3}$ also has size 17 and again, there are 6 $\sigma_3$-flags of size 5 that contain the desired tight path. As in the case $\sigma = \sigma_4$, we used $G_{12}^{\sigma_3}, \ldots, G_{17}^{\sigma_3}$ to denote these $\sigma_3$-flags. See Appendix A.1 for the additional details.

Finally, if $\sigma = \sigma_2$, then there are exactly 38 $\sigma_2$-flags of size 5. Also in this case, six of them contain a tight path from $\{a, b\}$ to $\{c, d\}$. We denote these $\sigma_2$-flags by $G_{33}^{\sigma_2}, \ldots, G_{38}^{\sigma_2}$. A particular linear combination of 8 columns of $I_2$ shows that asymptotically almost surely

$$
\phi_n^{\sigma_2} \left( \sum_{G^{\sigma_2} \in G_n^{\sigma_2}} \alpha_{G^{\sigma_2}} \cdot G^{\sigma_2} \right) = o(1),
$$

where $\alpha_{G^{\sigma_2}} > 0$ if and only if $G \in \{G_{33}^{\sigma_2}, \ldots, G_{38}^{\sigma_2}\}$. Note that this linear combination was found with an assistance of a software for solving linear programs called QSopt.ex [3]. Analogously to the previous two cases, the exact computation is given in Appendix A.1. This finishes the proof of Claim 1.

Our final goal is to establish the following claim:

**Claim 2.** For every $H_n'$, there exists an oriented graph $O_n$ on the same set of vertices such that its arc density is at least $1 - o(1)$, and if three vertices $u, v, w$ in $O_n$ span a transitive triangle, then uvw is a non-edge in $H_n'$.

Before proving this claim, let us look at how it implies the theorem. A simple application of Cauchy-Schwarz’s inequality shows that the density of cyclically oriented triangles in any oriented graph is at most $1/4 + o(1)$. Furthermore, the equality holds if and only if the oriented graph is an almost balanced almost tournament, i.e., for all but $o(n)$ vertices $v$ both the out-degree and the in-degree of $v$ are equal to $n/2 \pm o(n)$. Since every three vertices that span an edge in $H_n'$ either correspond to a cyclically oriented triangle in $O_n$, or contain at least one of the $o(n^2)$ pairs that do not span an arc in $O_n$, we conclude that the density of cyclically oriented triangles in $O_n$ is at least $1/4 - o(1)$ (hence it must be equal to $1/4 \pm o(1)$). Therefore, $O_n$ is an almost balanced almost tournament. Moreover, since $\lambda((H_n')) = 1/4$, the same reasoning can be also applied to every subgraph of $O_n$ of size $n/2$.

Now since every $(n/2)$-vertex subgraph of $O_n$ is an almost balanced almost tournament, a classical result of Chung and Graham [16] on quasi-random tournaments yields that the sequence $(O_n)_{n \in \mathbb{N}}$ must be (after adding $o(n^2)$ arcs to each $O_n$) a sequence of quasi-random tournaments (see the property P6 in [16, Theorem 1]). This immediately concludes the theorem, since the density of any subtournament in

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a sequence of quasi-random tournaments is the same as the corresponding density in the sequence of the tournaments that are truly random [16].

It only remains to prove Claim 2. For every $n \in \mathbb{N}$, we use the tightly-connected property proved in Claim 1 to define an orientation for almost every pair of vertices $u,v \in V(H'_n)$. For two vertices $u$ and $v$, we write $u \to v$ to denote the fact that we put to $O_n$ an arc from $u$ to $v$. Fix two vertices $a,b \in V(H'_n)$ such that for all but $o(n^2)$ subsets $\{c,d\} \subseteq V(H'_n)$ the pair $\{(a,b),\{c,d\}\}$ is tightly-connected. Claim 1 implies that there exists such a choice of $a$ and $b$. First, we orient $a \to b$. Next for any $\{c,d\}$ that forms a tightly-connected pair with $\{(a,b)\}$, we use a tight path $P$ on at most 5 vertices to define the orientation of $\{c,d\}$ as follows. Consider the induced subhypergraph $H'_n[V(P)]$. There is always a unique way how to orient the pairs of the vertices of $H'_n[V(P)]$ that are contained in some of the edges of $P$ so that $a \to b$, and every edge of $P$ corresponds to a cyclically oriented triangle in this orientation. In particular, there is a unique orientation of $\{c,d\}$, which is the way how we orient $\{c,d\}$ in $O_n$.

Let us first show that the orientation of almost all $\{c,d\} \subseteq V(H'_n)$ is well-defined. For any two tight paths $P_1,P_2$ from $\{a,b\}$ to $\{c,d\}$ on at most 5 vertices, consider the induced subhypergraph $H'_n[V(P_1) \cup V(P_2)]$. It follows that the subhypergraph has at most 6 vertices. Recall that for every induced subhypergraph $Z$ of $H'_n$ on at most 7 vertices, there must exist an underlying tournament $T_Z$ on $V(Z)$ such that the edges of $Z$ are in one-to-one correspondence with the cyclically oriented triangles in $T_Z$. Therefore, both paths $P_1$ and $P_2$ must define the same orientation of $\{c,d\}$ in $O_n$.

The last part of Claim 2 we need to establish is that if three vertices $u,v,w$ span a transitive triangle in $O_n$, then $uvw$ is a non-edge in $H'_n$. The argument is very similar to the one used in the last paragraph. Without loss of generality, the arcs on $\{u,v\}$ and $\{u,w\}$ in $O_n$ are $u \to v$ and $u \to w$. Let $P_{uv}$ and $P_{uw}$ be tight paths on at most 5 vertices from $\{a,b\}$ to $\{u,v\}$ and $\{u,w\}$, respectively. Since the induced subhypergraph $Z := H'_n[V(P_{uv} \cup V(P_{uw})]$ has at most 7 vertices, there is an underlying tournament $T_Z$ on $V(Z)$ so that the edges of $Z$ correspond to the cyclically oriented triangles in $T_Z$. Without loss of generality, $a \to b$ in $T_Z$. But then also $u \to v$ and $u \to w$ in $T_Z$ and hence $uvw$ cannot be an edge of $Z$. \qed
Chapter 3

Monotone subsequences in permutations

The problem we study here is inspired by a well-known result of Erdős and Szekeres [25] which states that every permutation on $\{1, \ldots, n\}$, where $n \geq (k - 1)^2 + 1$, contains a monotone subsequence of length $k$. When $n$ is much larger than $k^2$, one expects that the number of monotone subsequences of length $k$ is much more than just one. In fact, a simple double-counting argument implies that the result of Erdős and Szekeres guarantees that there are at least $\binom{n}{k}/\binom{(k-1)^2+1}{k}$ such subsequences. A natural question to ask is what is the minimum number of monotone subsequences of length $k$ inside a permutation of length $n$.

According to Myers [53], this problem was first posed by Atkinson, Albert and Holton. In this chapter, we use $F_k(\tau)$ to denote the number of monotone subsequences of length $k$ in a permutation $\tau$. Note that $F_k(\tau) = F_k(\tau^{-1})$, where $\tau^{-1}$ is the reverse permutation of $\tau$. The minimum of $F_k(\tau)$ over all permutations $\tau \in S_n$ is denoted by $F_k(n)$. For brevity, we also define $f_k(\tau)$ to denote the density of monotone subsequences of length $k$ in $\tau$, i.e., $f_k(\tau) = F_k(\tau)/\binom{n}{k}$.

Myers [53] described a permutation $\tau_k(n) \in S_n$ which gives an upper bound on $F_k(n)$ of the form $(k - 1)^{1-k} \cdot \binom{n}{k} + O(n^{k-1})$. It consists of $k - 1$ increasing sequences $A_1, A_2, \ldots, A_{k-1}$ whose sizes differ by at most one, and, every monotone subsequence of length $k$ is entirely contained in $A_i$ for some $i \in [k - 1]$. In other words, with $a_j = \lfloor jn/k \rfloor$, an example of such a permutation is

$$
\tau_k(n) = (\begin{array}{ccccccc}
a_{k-2} + 1, & a_{k-2} + 2, & \ldots, & n - 1, & n, \\
a_{k-3} + 1, & a_{k-3} + 2, & \ldots, & a_{k-2} - 1, & a_{k-2}, \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1, & 2, & \ldots, & a_1 - 1, & a_1
\end{array})
$$

See Figure 3 for $\tau_4(15)$. 
Let $r \equiv n \pmod{k-1}$, where $0 \leq r < k-1$. It follows that value of $F_k(\tau_k(n))$ is equal to
\[
\frac{\binom{n}{k} - \binom{r}{k}}{(k-1)^{k-1}} = O\left(n^{k-1}\right).
\]
Myers [53] proved that $F_3(n) = F_3(\tau_3(n))$ and he described all permutations $\tau \in S_n$ where $F_3(\tau) = F_3(n)$. He conjectured that the upper bound given by $F_k(\tau_k(n))$ actually holds for every $k \in \mathbb{N}$.

**Conjecture 3.1** (Myers [53]). Let $n$ and $k$ be positive integers. In any permutation $\tau \in S_n$ there are at least $F_k(\tau_k(n))$ monotone subsequences of length $k$.

The main result of this chapter is to show that the conjecture is true for $k = 5$ and $k = 6$, when $n$ is sufficiently large. In fact, for these values of $k$ and $n$, we also determine a full description of the set of extremal permutations $\tau \in S_n$, i.e., $\tau \in S_n$ such that $F_k(\tau) = F_k(n) = F_k(\tau_k(n))$.

As we already mentioned, Myers showed the conjecture is true for $k = 3$, which is actually a consequence of Goodman’s formula. In [8], the conjecture was verified for $k = 4$ and $n$ sufficiently large. In Section 3.2, we present a slightly different proof of this result as a “warm-up” for the next two cases. Very recently, Samotij and Sudakov [68] confirmed the conjecture if $n \leq k^2 + ck^{3/2}/\log k$ for some absolute positive constant $c$, provided $k$ is sufficiently large.

Subject to the additional constraint that all the monotone subsequences of length $k$ are either only increasing or only decreasing (and also $n \geq (k-1)(2k-3)$), Myers proved that every such a permutation contains at least the conjectured number of monotone subsequences of length $k$. He also gave the list $\mathcal{W}_k(n)$ of all such permutations $\tau \in S_n$ that satisfy the additional constraint and have $F_k(\tau) = F_k(\tau_k(n))$. Every permutation from $\mathcal{W}_k(n)$ can be decomposed into $k$ disjoint monotone subse-

Figure 3.1: Permutation $\tau_4(15)$, its permutation graph $T_3(15)$, the reverse of $\tau_4(15)$, and its permutation graph $T_3(15)$. 

Let $r \equiv n \pmod{k-1}$, where $0 \leq r < k-1$. It follows that value of $F_k(\tau_k(n))$ is equal to
\[
\frac{\binom{n}{k} - \binom{r}{k}}{(k-1)^{k-1}} = O\left(n^{k-1}\right).
\]
Myers [53] proved that $F_3(n) = F_3(\tau_3(n))$ and he described all permutations $\tau \in S_n$ where $F_3(\tau) = F_3(n)$. He conjectured that the upper bound given by $F_k(\tau_k(n))$ actually holds for every $k \in \mathbb{N}$.

**Conjecture 3.1** (Myers [53]). Let $n$ and $k$ be positive integers. In any permutation $\tau \in S_n$ there are at least $F_k(\tau_k(n))$ monotone subsequences of length $k$.

The main result of this chapter is to show that the conjecture is true for $k = 5$ and $k = 6$, when $n$ is sufficiently large. In fact, for these values of $k$ and $n$, we also determine a full description of the set of extremal permutations $\tau \in S_n$, i.e., $\tau \in S_n$ such that $F_k(\tau) = F_k(n) = F_k(\tau_k(n))$.

As we already mentioned, Myers showed the conjecture is true for $k = 3$, which is actually a consequence of Goodman’s formula. In [8], the conjecture was verified for $k = 4$ and $n$ sufficiently large. In Section 3.2, we present a slightly different proof of this result as a “warm-up” for the next two cases. Very recently, Samotij and Sudakov [68] confirmed the conjecture if $n \leq k^2 + ck^{3/2}/\log k$ for some absolute positive constant $c$, provided $k$ is sufficiently large.

Subject to the additional constraint that all the monotone subsequences of length $k$ are either only increasing or only decreasing (and also $n \geq (k-1)(2k-3)$), Myers proved that every such a permutation contains at least the conjectured number of monotone subsequences of length $k$. He also gave the list $\mathcal{W}_k(n)$ of all such permutations $\tau \in S_n$ that satisfy the additional constraint and have $F_k(\tau) = F_k(\tau_k(n))$. Every permutation from $\mathcal{W}_k(n)$ can be decomposed into $k$ disjoint monotone subse-
quences $A_1, \ldots, A_{k-1}$ that are either all increasing or all decreasing, and their sizes differ by at most one. Moreover, every monotone subsequence of length $k$ is a subsequence of $A_i$ for some $i \in [k-1]$. These permutations look very similar to $\tau_k(n)$ or $\tau_k^{-1}(n)$ and the only parts where they can vary are at the first / last $k-1$ elements of each block. It turns out that $|\mathcal{W}_k(n)|$ has size $2 \cdot (\frac{k-1}{n \mod (k-1)}) \cdot C(k-1)^{2k-4}$, where $C(k)$ is the $k$th Catalan number. The precise description of the sets $\mathcal{W}_k(n)$ can be found in [53].

**Theorem 3.2** ([53, Theorem 9]). Let $n \geq (k-1)(2k-3)$. If a permutation $\tau \in S_n$ contains no increasing subsequence of length $k$, then $F_k(\tau) \geq F_k(\tau_k(n))$.

Furthermore, there are exactly $(\frac{k-1}{n \mod (k-1)}) \cdot C(k-1)^{2k-4}$ permutations $\tau$ such that $F_k(\tau) = F_k(\tau_k(n))$. We denote the set of all such extremal permutations by $\mathcal{W}_k^{-}(n)$.

Symmetrically, if a permutation $\tau \in S_n$ contains no decreasing subsequence of length $k$, then $F_k(\tau) \geq F_k(\tau_k(n))$. Furthermore, there are exactly $(\frac{k-1}{n \mod (k-1)}) \cdot C(k-1)^{2k-4}$ permutations $\tau$ such that $F_k(\tau) = F_k(\tau_k(n))$. We denote the set of all such extremal permutations by $\mathcal{W}_k^{+}(n)$.

Note that it immediately follows that $\mathcal{W}_k(n) = \mathcal{W}_k^{+}(n) \cup \mathcal{W}_k^{-}(n)$.

### 3.1 Flag algebra setting

Instead of working directly with permutations, we formulate the problem in the language of permutation graphs and tweak the flag algebra framework for this particular setting. The permutation graph of a permutation $\pi \in S_n$ is a graph $G_\pi$ with the vertex set $[n]$ and two vertices $i, j \in [n]$ form an edge if and only if the pair $\{i, j\}$ is an inversion in $\pi$. For a permutation graph $G$ and an integer $k$, we define $F_k(G)$ to denote the number of induced subgraphs of size $k$ that are either complete, or empty, and we let $f_k(G) := F_k(G)/\binom{n}{k}$.

For an integer $k$, we let $K_k$ to be the $k$-vertex complete graph. Since our problem is symmetric under taking the graph complement, we apply flag algebras in the complement-blind setting, i.e., we do not distinguish between a graph $G$ and its complement. Formally, we say that two graphs $G_1$ and $G_2$ are blindly isomorphic, if $G_1$ is isomorphic either to $G_2$, or to the complement of $G_2$. In particular, for a permutation graph $G$, the value of $F_k(G)$ is exactly the number of $k$-vertex subgraphs blindly isomorphic to $K_k$.

Let $\mathcal{F}$ be the set of all finite permutation graphs up to a blind isomorphism, and let $\mathcal{F}_\ell \subseteq \mathcal{F}$ be the set of permutation graphs of order exactly $\ell$. Note that $\mathcal{F}_3 = 2, \mathcal{F}_4 = 6, \mathcal{F}_5 = 17, \mathcal{F}_6 = 71, \mathcal{F}_7 = 388$ and $\mathcal{F}_8 = 2852$. 

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Finally, we want to exploit the fact that the vertex set of a permutation graph comes with a natural linear order, which gives an additional restriction on possibilities how to extend a fixed small subgraph in a permutation graph to a larger one. Therefore, we do a more careful setting of flag algebras compared to the one in Chapter 1. Let us describe the differences in more detail.

For the first step, we follow the standard approach and construct an algebra $\mathcal{A}$, which is the set $\mathbb{RF}$ factored by the space of all linear combinations of the form $H - \sum_{H' \in F_{n}(H)+1} p(H, H') \cdot H'$. The space $\mathcal{A}$ has naturally defined linear operations of an addition, and we follow Chapter 1 in order to define also a multiplication on the elements of $\mathcal{A}$.

Our next task is to define algebras $\mathcal{A}_\sigma$, where $\sigma$ is a permutation graph with a fixed labelling of its vertex set. Again, the labelled graph $\sigma$ is called a type, any supergraph of $\sigma$ that preserves the labelling of the vertices of $\sigma$ is called a $\sigma$-flag, and the labelled part of a $\sigma$-flag is called the root. At this point, we are going to differ from the Chapter 1 so before proceeding further, let us demonstrate our approach on a simple example.

Suppose $abc$ is a triangle in an $n$-vertex permutation graph $G_\pi$ such that $a < b < c$. That means, all three pairs $\{a, b\}$, $\{b, c\}$ and $\{a, c\}$ are inversions in the permutation $\pi$. Now observe that for every point $d \in [n] \setminus \{a, b, c\}$, if $\{b, d\}$ is an inversion in $\pi$, then $\{a, d\}$ or $\{c, d\}$ has to be an inversion as well. Translated back to the graph language, there is no vertex $d$ in $G_\pi$ that would extend the triangle $abc$ to a 4-vertex graph with the edge set $\{ab, bc, ac, bd\}$, see Figure 3.1. Note that the other two options of adding a vertex of degree 1 to $abc$, i.e., the edge sets $\{ab, bc, ac, ad\}$ and $\{ab, bc, ac, cd\}$, can be realized. On the other hand, if we consider a triangle $abc$ with $b < a < c$, there might be a vertex $d$ such that the 4-vertex subgraph induced by $\{a, b, c, d\}$ has the edge set $\{ab, bc, ac, bd\}$. The non-realizable extension for this case is $\{ab, bc, ac, ad\}$. Our plan is to construct the algebra $\mathcal{A}_\sigma$, for $\sigma$ being a triangle, in such a way that it will be generated by a set of $\sigma$-flags $\mathcal{F}_\sigma$ that in
particular contains only two $\sigma$-flags with 4 vertices and 4 edges.

Let us now move to the general definition of $A_{\sigma}$. Fix $\sigma$ a type and let $\tau_1, \tau_2, \ldots, \tau_t \in S_{v(\sigma)}$ be all the permutations such that their permutation graph is blindly isomorphic to $\sigma$. We denote the set \{\tau_1, \ldots, \tau_t\} by $T(\sigma)$. For each $\tau \in T(\sigma)$, we choose a bijection $b_\tau$ from the points of $\tau$ to the vertices of $\sigma$ that maps the permutation graph of $\tau$ to $\sigma$ with respect to blind isomorphism. For example, if $\sigma$ is a triangle $abc$, we have $\tau_1 = 321$ and $\tau_2 = 123$. If we set $b_{\tau_1} := b_{\tau_2} := abc$, by which we mean the function that maps 1 to $a$, 2 to $b$ and 3 to $c$, then we infer that in both cases we cannot realize any $\sigma$-flag that contains a vertex adjacent with $b$ but not adjacent with $a$ and $c$.

In general, let $B(\tau)$ be the set of all blind isomorphisms $b : V(G_\tau) \to V(\sigma)$. Note that $B(\tau) \cong \text{Aut}(\sigma)$. For every $\tau \in T(\sigma)$ and $b \in B(\tau)$, we define $F^{\tau,b}$ as the set of all finite $\sigma$-flags that satisfy the following:

a) they can be realized as permutation graphs with the root $\sigma$,

b) the vertices of the root induce $\tau$ in at least one of the choices of an underlying permutation, and

c) the bijection between the root and the points of $\tau$ is equal to $b$.

We set $F^\sigma := \bigcup_{\tau \in T(\sigma)} F^{\tau,b}$. Note that if $\sigma$ has a non-trivial automorphism group, we had some freedom how to choose the bijections $(b_\tau)_{\tau \in T(\sigma)}$. For each $\sigma$, the particular choice of $(b_\tau)$ was made (with a computer assistance) in order to make the sets $F^\sigma$ as minimal as possible. Next, for any blind isomorphism $b : V(G_\tau) \to V(\sigma)$, we define an injective mapping $z_{\tau,b}$ from $F^{\tau,b}$ to $F^\sigma$ as the relabelling function of the vertices of the root of $H \in F^{\tau,b}$ using $b^{-1} \circ b_\tau$.

Now, we again follow the scheme described in Chapter 1 and construct the algebras $A^\sigma$ by factoring the space $\mathbb{R}F^\sigma$ with all linear combinations of the form $H - \sum_{H' \in F^\sigma_{v(H)+1}} p(H, H') \cdot H'$, and equip $A^\sigma$ with a multiplication. Following the same lines, we also construct algebras $A^{\tau,b}$ for every $\tau \in T(\sigma)$ and $b \in B(\tau)$.

The algebras $A^\sigma$ and $A^{\tau,b}$ are closely related. Recall the injective mapping $z_{\tau,b} : F^{\tau,b} \to F^\sigma$. We slightly abuse the notation and will use $z_{\tau,b}$ both for the mapping from $F^{\tau,b}$ to $F^\sigma$ and its unique extension to a linear operator from $A^{\tau,b}$ to $A^\sigma$. The operator $z_{\tau,b}$ has the following property for every $H_1, H_2 \in F^{\tau,b}$:

$$H_1^\sigma \times H_2^\sigma = z_{\tau,b}(H_1 \times H_2) + \sum_{H^\sigma \in F^\sigma \setminus z_{\tau,b}(F^{\tau,b})} p(H_1^\sigma, H_2^\sigma; H^\sigma) \cdot H^\sigma, \quad (3.3)$$
where \( H_1^\sigma = \tau_{\sigma,b}(H_1), H_2^\sigma = \tau_{\sigma,b}(H_2) \) and \( \ell = v(H_1) + v(H_2) - v(\sigma) \). We also define the “inverse” operator \( \tau_{\sigma,b}^{-1} \) from \( \mathcal{A}^\sigma \) to \( \mathcal{A}^{\tau,b} \) as the unique linear extension of the following map of the basis of \( \mathcal{A}^\sigma \):

- \( \tau_{\sigma,b}^{-1}(H^\sigma) \) for \( H^\sigma \in \tau_{\sigma,b}(\mathcal{F}^{\tau,b}) \), and
- \( 0 \) for \( H^\sigma \in \mathcal{F}^\sigma \setminus \tau_{\sigma,b}(\mathcal{F}^{\tau,b}) \).

Now consider an infinite sequence \( (\pi_n)_{n \in \mathbb{N}} \) of permutations of increasing size. We say that the sequence is convergent if for every fixed permutation \( \tau \), the probabilities that a random \( |\tau| \) points in \( \pi_n \) induce the subpermutation equal to \( \tau \) has a limit as \( n \) tends to infinity. We denote this limit probability by \( q(\tau,(\pi_n)) \). In particular, for every convergent sequence of permutations \( (\pi_n)_{n \in \mathbb{N}} \), the probabilities \( p(H,G_{\pi_n}) \) have a limit for every \( H \in \mathcal{F} \). As in Chapter 1, we know that every infinite sequence of permutations has a convergent subsequence. Fix a convergent increasing sequence \( (\pi_n)_{n \in \mathbb{N}} \) of permutations. For every \( H \in \mathcal{F} \), we set \( \phi(H) := \lim_{n \to \infty} p(H,G_{\pi_n}) \) and linearly extend \( \phi \) to \( \mathcal{A} \). The obtained mapping \( \phi \) is a homomorphism from \( \mathcal{A} \) to \( \mathbb{R} \) such that \( \phi(H) \geq 0 \) for every \( H \in \mathcal{F} \). Let \( \text{Hom}^+ (\mathcal{A},\mathbb{R}) \) be the set of all homomorphisms \( \psi \) from the algebra \( \mathcal{A} \) to \( \mathbb{R} \) such that \( \psi(H) \geq 0 \) for every \( H \in \mathcal{F} \).

Fix a convergent sequence of permutations \( (\pi_n)_{n \in \mathbb{N}} \) and let \( \phi \in \text{Hom}^+ (\mathcal{A},\mathbb{R}) \) be the corresponding homomorphism from \( \mathcal{A} \) to \( \mathbb{R} \). Fix a type \( \sigma \). Recall that \( T(\sigma) \) is the set of all permutations \( \tau \in S_{n(\sigma)} \) with the permutation graph \( G_{\tau} \) blindly isomorphic to \( \sigma \), and \( B(\tau) \) is the set of bijections between the vertices of \( \sigma \) and the points of \( \tau \) that preserve the blind isomorphism between \( \sigma \) and \( G_{\tau} \). For an embedding \( \theta \) of a permutation \( \tau \in T(\sigma) \) in \( \pi_n \), we define \( G_{\pi_n}^\theta \) to be the permutation graph of \( \pi_n \) rooted on the copy of \( \sigma \) that corresponds to \( \theta \). Furthermore, let \( b \in B(\tau) \) be the bijection between the points of \( \theta \) and the vertices of the copy of \( \sigma \). For every \( n \in \mathbb{N} \) and \( H^{\tau,b} \in \mathcal{F}^{\tau,b} \), we define \( p_n^\theta(H^{\tau,b}) = p(H^{\tau,b},G_{\pi_n}^\theta) \). Picking such \( \theta \) with a fixed bijection \( b \) in \( \pi_n \) at random gives rise to a probability distribution \( \mathbf{P}_n^{\tau,b} \) on mappings from \( \mathcal{A}^{\tau,b} \) to \( \mathbb{R} \), and, via the operator \( \tau_{\sigma,b} \), also on mappings from \( \mathcal{A}^\sigma \) to \( \mathbb{R} \).

Since the sequence \( (\pi_n)_{n \in \mathbb{N}} \) is convergent, the sequence of probability distributions \( \mathbf{P}_n^{\tau,b} \) on mappings from \( \mathcal{A}^{\tau,b} \) to \( \mathbb{R} \) also converge. As in Chapter 1, this follows from [59, Theorems 3.12 and 3.13]. We denote the limit probability distribution by \( \mathbf{P}^{\tau,b} \). Furthermore, the relation (3.3) implies that for the labelled permutation graph \( \sigma \) of the permutation \( \tau \), any mapping \( \phi^{\tau,b}(\tau_{\sigma,b}(\cdot)) \), where \( \phi^{\tau,b} \) is taken from the support of the distribution \( \mathbf{P}^{\tau,b} \), is a homomorphism from \( \mathcal{A}^\sigma \) to \( \mathbb{R} \) such that \( \phi^\sigma(H^\sigma) \geq 0 \) for all \( H^\sigma \in \mathcal{F}^\sigma \) [59, Proof of Theorem 3.5].
The remaining bit we need to introduce is the averaging operator $\mathbb{[\cdot]}_\sigma : A^\sigma \to A$. Again, it is an analogue of the averaging operator from Chapter 1. The operator is the linear extension of a mapping defined on the elements of $H^\sigma \in F^\sigma$ by

$$\mathbb{[H^\sigma]}_\sigma := p_H^0 \cdot H^0,$$

where $H^0$ is the (unlabelled) permutation graph from $F$ corresponding to $H^\sigma$, and $p_H^0$ is probability that a random injective mapping from $V(\sigma)$ to $V(H^0)$ is an embedding of $\sigma$ in $H^0$ yielding a $\sigma$-flag blindly isomorphic to $H^\sigma$. However, the corresponding relation between the homomorphism $\phi$ and the homomorphisms $\phi_{\tau,b}$, where $\tau \in T(\sigma)$ and $b \in B(\tau)$, is slightly different from the analogue in Chapter 1. Specifically,

$$\forall H^\sigma \in A^\sigma, \quad \phi(\mathbb{[H^\sigma]}_\sigma) = \phi([\sigma]_\sigma) \cdot \sum_{\tau \in T(\sigma)} \sum_{b \in B(\tau)} \sum_{n} q(\tau, (\pi_n)) \cdot \phi(\sigma^0) [B(\tau)] \cdot \int \phi_{\tau,b} (z_{\tau,b}(A^\sigma)) dP_{\tau,b}.$$

Therefore, if for some fixed $A^\sigma \in A^\sigma$ and every $\tau \in T(\sigma), b \in B(\tau)$ we have

$$\phi_{\tau,b} (z_{\tau,b}(A^\sigma)) \geq 0$$

almost surely, then $\phi(\mathbb{[A^\sigma]}_\sigma) \geq 0$. In particular, the analogue of the inequality (1.2), i.e.,

$$\phi \left( \mathbb{[\mathbb{[A^\sigma]}]}_\sigma \right) \geq 0,$$

holds for every $A^\sigma \in A^\sigma$.

### 3.2 Monotone subsequences of length four

In this section, we reprove the main result of [8]. Our proof essentially follows the same lines as in [8]. There are the following two differences between the proofs:

- We use the flag algebra setup presented in the previous section, which is a slightly different than the one used in [8].
- Our flag algebra proof provides a somewhat better control on the substructures appearing in (almost) extremal configurations, which helps to simplify our stability arguments presented in Section 3.3.

In Section 3.5 and Section 3.6, we use the same approach to resolve also the question of minimizing monotone subsequences of length 5 and 6.
Recall we use the flag algebra framework in the complement-blind way. In particular, $\phi(K_4)$ is the sum of the density of $K_4$ and the density of the complement of $K_4$. Let $\text{EXT}_7^4$ be the set of all 7-vertex subgraphs up to a blind isomorphism that have a positive density in $T_3(n)$, or in $\overline{T_3(n)}$. It follows that $\text{EXT}_7^4 = \{H_1, H_2, \ldots, H_8\}$. The set $\text{EXT}_7^4$ is depicted in Figure 3.3. Next, let

$$\mathcal{E}_7^4 := \mathcal{F}_7 \setminus \text{EXT}_7^4.$$

We are now ready to present the main theorem of this section.

**Theorem 3.4.** There exists a positive rational $\alpha$ such that the following is true. If $(\pi)_n \in \mathbb{N}$ is a convergent sequence of permutations and $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ is its limit, then

$$\phi \left( K_4 - \alpha \cdot \sum_{H \in \mathcal{E}_7^4} H \right) \geq \frac{1}{27}.$$

Theorem 3.4 immediately implies the following two corollaries.

**Corollary 3.5.** For every positive $\varepsilon_{\text{ASYM}}$ there exists $n_{\text{ASYM}} \in \mathbb{N}$ such that the following is true. If $G$ is a permutation graph on $n \geq n_{\text{ASYM}}$ vertices, then $f_4(G) > (1/27 - \varepsilon)$.

**Corollary 3.6.** For every positive $\varepsilon_{\text{CONF}}$ there exist a positive $\delta_{\text{CONF}}$ and $n_{\text{CONF}} \in \mathbb{N}$ such that the following is true. If $G$ is a permutation graph on $n \geq n_{\text{CONF}}$ vertices that satisfies $f_4(G) \leq (1/27 + \delta_{\text{CONF}})$, then $G$ contains at most $\varepsilon_{\text{CONF}} \cdot \binom{n}{4}$ induced copies of $F$ and at most $\varepsilon_{\text{CONF}} \cdot \binom{7}{2}$ induced copies of $\overline{F}$, where $F \in \mathcal{E}_7^4$.

**Proof of Theorem 3.4.** We use the flag algebra framework presented in Section 3.1 in the following way. Let $\sigma_1$ is the 3-vertex type with no edges and $\sigma_2$ is the 3-vertex type with the edge $bc$. The types $\sigma_1$ and $\sigma_2$ are depicted in Figure 3.4.

Next, the set $T(\sigma_1)$ contains two permutations $\tau_{1,1} := 123$ and $\tau_{1,2} := 321$. We set $b_{\tau_{1,1}} := b_{\tau_{1,2}} := abc$. It follows that $|\mathcal{F}_5^{\sigma_1}| = 54$ and $|\mathcal{F}_7^{\sigma_1}| = 5388$.

Analogously, the set $T(\sigma_2)$ contains the remaining four permutations from $S_3$. Specifically, $T(\sigma_2)$ contains the permutations $\tau_{2,1} = 132$, $\tau_{2,2} = 231$, $\tau_{2,3} = 231$, and $\tau_{2,4} = 312$. We set $b_{\tau_{2,1}} := b_{\tau_{2,2}} := abc$ and $b_{\tau_{2,3}} := b_{\tau_{2,4}} := bca$. It follows that $|\mathcal{F}_5^{\sigma_2}| = 71$ and $|\mathcal{F}_7^{\sigma_2}| = 9055$. 43
Using the method from Section 1.2, an instance of the SDP program was used to find two symmetric positive semidefinite matrices $M_1$ and $M_2$ with rational entries such that for every $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ we have

$$
\phi \left( [x_1^T M_1 x_1]_{\sigma_1} + [x_2^T M_2 x_2]_{\sigma_2} \right) \leq \phi \left( K_4 - \frac{1}{27} - \alpha \cdot \sum_{H \in \mathcal{E}_5} H \right),
$$

where

- the vector $x_1 \in (\mathbb{R}^{F_{\sigma_1}^5})^{\mathcal{F}_{\sigma_1}^5}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R}^{\mathcal{F}_{\sigma_1}^5}$,
- the vector $x_2 \in (\mathbb{R}^{\mathcal{F}_{\sigma_2}^5})^{\mathcal{F}_{\sigma_2}^5}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R}^{\mathcal{F}_{\sigma_2}^5}$, and
- $\alpha = 2629661/9106944 \sim 0.28875$.

The left-hand side of the inequality above is non-negative by (1.2). \hfill \square

As in Chapter 2, the proof was found with an assistance of computer programs CSDP [11], SDPA [74] and SAGE [71]. We also used a software package nauty [52] for graph isomorphism tests performed while the appropriate sets of flags were being generated.

The numerical values of the entries of $M_1$ and $M_2$ can be downloaded from the web page \url{http://honza.ucw.cz/phd/}. In fact, the $54 \times 54$ matrix $M_1$ is not stored directly, but as a pair of matrices $(D_1, M_1)$ such that $N_1$ is a $52 \times 52$ positive definite matrix and $M_1 = D_1 \cdot N_1 \cdot (D_1)^T$. Analogously, the $71 \times 71$ matrix $M_2$ is stored as a pair of matrices $(D_2, M_2)$ such that $N_2$ is a $70 \times 70$ positive definite matrix and $M_2 = D_2 \cdot N_2 \cdot (D_2)^T$. 

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Similarly to the flag algebra computations presented in Chapter 2, we created a sage script called “thm.3.4-verify.sage”, which can be used for an independent verification of the computations in Theorem 3.4. The script is also available on the web page mentioned above.

3.3 Stability of almost extremal configurations

In this section, we show that the following two properties that are satisfied by (at this moment only conjectured) extremal graphs – Turán graph and its complement – force every almost extremal graph to be close in the edit distance to one of the conjectured extremal graphs.

We say that a graph $G$ satisfies Property $A(k)$, if for every $k$-vertex subgraph $H$ of $G$ that is either complete or empty, and every vertex $v \in V(G') \setminus V(H)$, we have either $V(H) \subseteq N(v)$, or $V(H) \cap N(v) = \emptyset$. See Figure 3.5 for an illustration of Property A(5). Next, we say that a graph $G$ satisfies Property B, if $G$ does not contain a copy of the graph $E_4$ and $\overline{E}_4$, where $E_4$ is the paw graph, i.e., the 4-vertex graph containing a triangle and one pendant edge. The graph $E_4$ is depicted in Figure 3.6.

![Figure 3.5: The possible extensions of $H$ in Property A(5).](image)

![Figure 3.6: The graph $E_4$ from Property B.](image)

We start with the following lemma that helps us to describe the global structure of almost extremal configurations from the two (local) properties described in the previous paragraph.
Lemma 3.7. For a fixed integer \( k \geq 3 \), let \( G \) be a graph and \( H \) a \( k \)-vertex subgraph of \( G \) that is a clique or an independent set. If \( G \) satisfies Property \( A(k) \) and Property \( B \), then \( G \) is either a disjoint union of at most \( k - 1 \) cliques, or the complement, i.e, a complete \( r \)-partite graph for some \( r < k \).

Proof. Without loss of generality, the graph \( H \) is a clique on \( k \) vertices. Let \( h_1, h_2, \ldots, h_k \) be the vertices of \( H \). Property \( A(k) \) implies that every vertex \( u \in N(h_1) \setminus V(H) \) extends \( H \) to a clique on \( k + 1 \) vertices. Therefore, \( N[h_i] = N[h_j] \) for every \( 1 \leq i, j \leq k \). Furthermore, any two neighbors \( u, v \in N(h_1) \) are connected by an edge, otherwise the graph induced by \( (V(H) \setminus \{h_k\}) \cup \{u, v\} \) violates Property \( A(k) \). Therefore, the vertices \( N[h_1] \) form a clique and every vertex \( z \in V(G) \setminus N[h_1] \) has its neighborhood \( N(z) \) disjoint from \( N[h_1] \).

If \( N[h_1] = V(G) \), we are done. Otherwise, we claim that the graph \( G[V(G) \setminus N[h_1]] \) is a disjoint union of cliques. Indeed, suppose there exist distinct vertices \( x, y, z \in V(G) \setminus N[h_1] \) such that \( \{x, y\} \subseteq N(z) \) and \( xy \notin E(G) \). However, the subgraph \( \{h_1, x, y, z\} \) is isomorphic to the graph \( E_4 \), a contradiction.

So the whole graph \( G \) is a disjoint collection of cliques and we only need to show that the number of components must be less than \( k \). Suppose for a contradiction the number of components of \( G \) is at least \( k \). However, any independent set of size \( k \) containing \( h_1 \) is extended by the vertex \( h_2 \) to a subgraph that violates Property \( A(k) \). \( \square \)

Our next tool is the Infinite Removal Lemma of Alon and Shapira [1].

Lemma 3.8 (Infinite Removal Lemma [1]). For any (possibly infinite) family of graphs \( \mathcal{H} \) and \( \varepsilon_{RL} > 0 \), there exists \( \delta_{RL} > 0 \) such that if a graph \( G \) on \( n \) vertices contains at most \( \delta_{RL} \cdot \binom{n}{\text{w}(H)} \) induced copies of \( H \) for every graph \( H \in \mathcal{H} \), then it is possible to make \( G \) induced \( \mathcal{H} \)-free by adding and/or deleting at most \( \varepsilon_{RL} \cdot \binom{n^2}{2} \) edges.

We are ready to present the stability result for the function \( F_4(G) \).

Theorem 3.9. For every \( \varepsilon_{STAB} > 0 \) there exist \( \delta_{STAB} > 0 \) and \( n_{STAB} \) such that the following is true. If \( G \) is a permutation graph on \( n_{STAB} \geq n_0 \) vertices with \( f_4(G) \leq \frac{1}{27} + \delta_{STAB} \), then \( G \) is isomorphic to either \( T_3(n) \) or \( \overline{T_3(n)} \) after adding and/or deleting at most \( \varepsilon_{STAB} \cdot \binom{n}{2} \) edges.

Note that we do not optimize the proof for the best possible values of \( \delta_{STAB} \) and \( n_{STAB} \) but rather try to keep our computations as simple as possible.

Proof. Without loss of generality, \( \varepsilon_{STAB} < 1/2 \). Fix \( C \) a sufficiently large constant \( (C \geq 25 \times 3^3 = 675 \) will suffice).
Let $\mathcal{X}$ be the set of all non-permutation graphs, i.e.,

$$\mathcal{X} := \{G : G \text{ is a graph such that } G \notin \mathcal{F} \text{ and } \overline{G} \notin \mathcal{F}\},$$

and $\delta_{RL}$ the value from Infinite Removal Lemma applied for $\varepsilon_{RL} := (\varepsilon_{STAB})^2 / C$ and the family $\mathcal{X} \cup \mathcal{E}^4$. Next, let $\delta_{CONF}$ and $n_{CONF}$ be the values from Corollary 3.6 applied for $\varepsilon_{CONF} := \varepsilon_{RL}$. Finally, let $n_{ASYM}$ be the value from Corollary 3.5 for $\varepsilon_{ASYM} := (\varepsilon_{STAB})^2 / C$. We set $n_{STAB} := \max\{C/\varepsilon_{STAB}, n_{CONF}, 2 \cdot n_{ASYM}\}.$

Let $G$ be a given graph on $n$ vertices satisfying the assumptions of the theorem. Corollary 3.6 and Infinite Removal Lemma imply that we can change at most $\varepsilon_{RL} \cdot \binom{n}{2} < 1/5 \cdot \varepsilon_{STAB} \cdot \binom{n}{2}$ edges and obtain a permutation graph $G'$ that does not contain neither a copy of a graph $E$, nor a copy of a graph $\overline{E}$, where $E \in \mathcal{E}^4$. In particular, $G'$ satisfies Property A(4) and Property B. Furthermore, $f_4(G') \leq 1/27 + \delta_{STAB} + \varepsilon_{RL}$. Lemma 3.7 implies we can partition $G'$ into at most 3 parts such that either every part is a clique and there are no edges between the parts, or every part is an independent set and all pairs of vertices from different parts are connected by an edge.

Without loss of generality, $G'$ is a disjoint union of at most 3 cliques. We claim that every clique has size at most $(1/3 + \varepsilon_{STAB}/5) \cdot n$. This immediately concludes the theorem since the number of cliques must be 3, and in order to balance the sizes of the cliques we need to add/remove edges incident to at most $2/5 \cdot \varepsilon_{STAB} \cdot n$ vertices, which means changing at most $4/5 \cdot \varepsilon_{STAB} \cdot \binom{n}{2}$ edges. Therefore, it is enough to show the claim.

Let $\gamma := \varepsilon_{STAB}/5$. Suppose for a contradiction $G'$ contains a clique of size more than $(1/3 + \gamma) \cdot n$. Let $H_0 := G'$. For each $i \in [\gamma \cdot n/3]$, let $v_i$ be an arbitrary vertex from a maximum clique inside $H_{i-1}$, and let $H_i := H_{i-1} - v_i$. Let $Z := \{v_1, v_2, \ldots, v_{\gamma \cdot n/3}\}$. It follows that every vertex $v \in Z$ is contained in at least $(1/27 + 2\gamma/3) \cdot n$ copies of $K_4$ that are disjoint from $Z \setminus \{v\}$. Since $n_{STAB} > 12/\gamma = 60/\varepsilon_{STAB}$, we have

$$\left(\frac{(1/3 + 2 \cdot \gamma/3) \cdot n}{3}\right) > \left(\frac{1}{3} + \frac{\gamma}{2}\right)^3 \cdot \frac{n^3}{6} > \left(\frac{1}{27} + \frac{\gamma}{6}\right) \cdot \frac{n^3}{6}.$$ 

Furthermore,

$$f_4(H_{i+1}) \leq f_4(H_i) \leq f_4(G') \leq \frac{1}{27} + 2 \cdot \frac{(\varepsilon_{STAB})^2}{C} = \frac{1}{27} + 2 \cdot \frac{\gamma^2}{27}.$$
Our aim is to show that $f_4(H_i) - f_4(H_{i+1}) > \gamma/(3 \cdot n)$. Indeed, we have

\[
\begin{align*}
f_4(H_i) - f_4(H_{i+1}) &> \left(\frac{1}{27} + \frac{\gamma}{6}\right) \cdot n^3 \cdot \frac{n^3}{6} + f_4(H_{i+1}) \cdot \left(\frac{v(H_i)^{-1}}{v(H_i)}\right) - f_4(H_{i+1}) \cdot \left(\frac{v(H_i)}{4}\right) \\
&> \left(\frac{1}{27} + \frac{\gamma}{6}\right) \cdot n^3 \cdot \frac{n^3}{6} - f_4(H_{i+1}) \cdot \left(\frac{v(H_i)^{-1}}{v(H_i)}\right) > 4 \cdot \left(\frac{1}{27} + \frac{\gamma}{6} - f_4(H_{i+1})\right) \\
&> \frac{4}{n} \cdot \left(\frac{\gamma}{6} - \frac{2\gamma^2}{27}\right) > \frac{\gamma}{n} \cdot \left(\frac{\gamma}{6} - \frac{2\gamma}{27}\right) \\
&> \frac{\gamma}{3n}.
\end{align*}
\]

Since $n - \gamma \cdot n/3 \geq n/2 \geq n_{ASYM}$, Corollary 3.5 implies that $f_4(H_{\gamma \cdot n/3}) \geq 1/27 - (\varepsilon_{STAB})^2/C$. However,

\[
\frac{1}{27} + 2 \cdot \frac{(\varepsilon_{STAB})^2}{C} \geq f_4(G') = f_4(H_0) > f_4(H_{\gamma \cdot n/3}) + \frac{\gamma^2}{27} \geq \frac{1}{27} - \frac{(\varepsilon_{STAB})^2}{C} + (\varepsilon_{STAB})^2 \cdot \frac{25 \times 9}{27},
\]

which contradicts the choice of $C$.

\section*{3.4 Extremal configurations}

For a fixed integer $k$, we say a permutation $\tau \in S_n$ is $k$-extremal if $F_k(\tau) = F_k(n)$. Analogously, we say an $n$-vertex permutation graph $G$ is $k$-extremal if $F_k(G) = F_k(n)$. In this section, we present a method for obtaining the exact description of $k$-extremal permutations for $k \geq 4$. Using the stability result from the previous section and the asymptotic result given by Theorem 3.4, we apply this method to the case $k = 4$. The same analysis will be then used for the cases $k = 5$ and $k = 6$ in Section 3.5 and Section 3.6, respectively.

We start with the following definition. We call $u \in V(G)$ a clone of $v \in V(G)$ if $N(u) = N(v)$. In particular, $uv$ is not an edge of $G$. The next simple proposition shows that if we add a clone of a vertex $x$ in a permutation graph $G$, then the new graph $G'$ is still a permutation graph.

\textbf{Proposition 3.10.} Let $G$ be a permutation graph of order $n$. If we add a clone $x'$ of some $x \in V(G)$ to form a new graph $G'$ of order $n + 1$, i.e., $N_{G'}(x') = N_{G}(x)$, then $G'$ is still a permutation graph.

\textbf{Proof.} The graph $G$ comes from some permutation $\tau \in S_n$. Let $k$ be the number in $[n]$ that corresponds to $x$, then we can construct a new permutation $\tau' \in S_{n+1}$ as
follows:
\[
\tau'(i) = \begin{cases} 
\tau(i) & \text{if } i \leq k \text{ and } \tau(i) \leq \tau(k) \\
\tau(i) + 1 & \text{if } i \leq k \text{ and } \tau(i) > \tau(k) \\
\tau(k) + 1 & \text{if } i = k + 1 \\
\tau(i - 1) & \text{if } i > k \text{ and } \tau(i - 1) < \tau(k) \\
\tau(i - 1) + 1 & \text{if } i > k \text{ and } \tau(i - 1) \geq \tau(k). 
\end{cases}
\]

The permutation graph of \(\tau'\) is \(G'\) with \(k+1\) corresponding to the new vertex \(x'\).

For an integer \(k\), a graph \(G\) and a vertex \(u \in V(G)\), let \(I^u_k(G)\) be the number of independent sets of size \(k\) that contain \(u\). Analogously, let \(J^u_k(G)\) be the number of cliques of size \(k\) that contain \(u\). We let \(F^u_k(G):= I^u_k(G) + J^u_k(G)\) and \(f^u_k(G) := F^u_k(G)/(n-1)\). An immediate corollary of the previous proposition is that in a \(k\)-extremal graph \(G\), every two vertices contributes to \(F_k(G)\) by roughly the same amount.

**Corollary 3.11.** Fix an integer \(k\) and let \(G\) be a \(k\)-extremal graph of order \(n\). For every two vertices \(u\) and \(v\), we have \(|F^u_k(G) - F^v_k(G)| \leq \frac{n-2}{k-2}\). Therefore,

\[
f_k(G) - \frac{k}{n} < F^u_k(G) < f_k(G) + \frac{k}{n}
\]

for every \(u \in V(G)\).

**Proof.** Without loss of generality, \(F^u_k(G) \geq F^v_k(G)\). Let \(G'\) be the \(n\)-vertex permutation graph obtained from \(G\) by removing \(u\) and adding a clone of \(v\). It follows that

\[
0 \leq F(G') - F(G) \leq F^u_k(G) - F^u_k(G) + \binom{n-2}{k-2}.
\]

Let \(E_7\) be the 7-vertex graph obtained by gluing three paths \(x - y_i - z_i, \ i = 1, 2, 3\), at the common vertex \(x\), see Figure 3.7. We continue our exposition by observing that there is no permutation \(\tau \in S_7\) such that the permutation graph of \(\tau\) would be isomorphic to \(E_7\).

**Observation 3.12.** The graph \(E_7\) is not a permutation graph.

**Proof.** Suppose there is a permutation \(\tau D \in S_7\) such that its permutation graph is isomorphic to \(E_7\). Without loss of generality, \(y_1 < y_2 < y_3\). Since the only neighbor of \(z_2\) is \(y_2\), it follows that \(y_1 < z_2 < y_3\) and \(\tau(y_1) < \tau(z_2) < \tau(y_3)\). However, the points \(\{y_1, y_2, y_3\}\) form an independent set in \(E_7\) and they are all connected to the point \(x\), hence either \(x > y_i\) and \(\tau(x) < \tau(y_i)\) for every \(i \in [3]\), or \(x < y_i\).
and \( \tau(x) > \tau(y_i) \) for every \( i \in [3] \). In the first case, we conclude that \( x > z_2 \) and \( \tau(x) < \tau(z_2) \). In the latter case \( x < z_2 \) and \( \tau(x) > \tau(z_2) \), a contradiction.

Alternatively, it is possible to check all the permutation graphs on 7 vertices, i.e., the set \( \mathcal{F}_7 \), and conclude that it does not contain neither \( E_7 \), nor the complement of \( E_7 \).

**Observation 3.13.** Fix an integer \( k \geq 4 \). The minimum value of the polynomial \( p(x, y) = x^{k-1} + y^{k-1} + (k-1)! \cdot (1-x) \cdot (1-y) \) on \([0,1]^2\) is equal to 1, and \( p(x_0, y_0) = 1 \) if and only if \( \{x_0, y_0\} = \{0,1\} \). Furthermore, if the value of \( p(x_1, y_1) \) is close to 1, then \( \{x_1, y_1\} \) is close \( \{0,1\} \).

**Proof.** If \( x \in \{0,1\} \) is fixed, then \( p(x, y) \) is clearly minimized when \( y = 1 - x \), and symmetrically for \( y \in \{0,1\} \). Now suppose there are \( x_0, y_0 \in (0,1) \) such that \( \frac{\partial p}{\partial x}(x_0, y_0) = 0 \) and \( \frac{\partial p}{\partial y}(x_0, y_0) = 0 \). It follows that

\[
(k-1) \cdot \left( x^{k-2} - y^{k-2} \right) = (x - y) \cdot (k-1)!,
\]

and hence

\[
\sum_{i=0}^{k-3} x^i \cdot y^{k-2-i} = (k-2)!. 
\]

However, since both \( x \) and \( y \) are less than one and \( k \geq 4 \), the sum on the left-hand side is equal to \( (k-2) \) summands, each of them less than one; a contradiction.

Since \( p(x, y) \) is continuous, the second part of the statement follows from the compactness of \([0,1]^2\).}

The main result of this section is the following theorem.
**Theorem 3.14.** For every integer \( k \geq 4 \) there exist a positive \( \varepsilon \) and an integer \( n_0 \) such that the following is true. If \( G \) is a \( k \)-extremal permutation graph on \( n \geq n_0 \) vertices that can be transformed to \( T_{k-1}(n) \) or \( \overline{T_{k-1}(n)} \) by adding and/or removing at most \( \varepsilon \binom{n}{2} \) edges, then the vertices of \( G \) can be partitioned into either \( k-1 \) independent sets, or \( k-1 \) cliques.

Note that combining Theorems 3.9 and 3.14 yields that every extremal permutation \( \tau \in S_n \), where \( n \) is sufficiently large, does not contain either any increasing subsequence of length 4, or any decreasing subsequence of length 4. Therefore, using Theorem 3.2 we conclude that \( \tau \in \mathcal{W}_4(n) \).

**Proof of Theorem 3.14.** Let \( \gamma := 1/(100 \cdot k^k) \) be an auxiliary constant and fix the choice of \( \varepsilon \) small enough so that any solution \((x,y)\) of the polynomial

\[
p(x, y) := x^{k-1} + y^{k-1} + (k-1)! \cdot (1-x) \cdot (1-y)
\]

that has value at most \( 1 + 20k^{2k} \cdot \varepsilon^{1/6} \) satisfy either \( x \in [0, \gamma] \) and \( y \in [1-\gamma, 1] \), or \( x \in [1-\gamma, 1] \) and \( y \in [0, \gamma] \). Such a choice of \( \varepsilon > 0 \) exists by Observation 3.13. Furthermore, we also assume that \( \varepsilon < 1/k^{100k} \). Finally, let \( n_0 := k^2/\varepsilon \).

Without loss of generality, we can modify \( \varepsilon \binom{n}{2} \) pairs in \( \binom{V(G)}{2} \) to obtain \( \overline{T_{k-1}(n)} \). Our aim is to show that \( G \) can be partitioned into \( k-1 \) cliques. Since \( G \) is close to \( \overline{T_{k-1}(n)} \), it follows that

\[
\frac{1}{(k-1)^{k-1}} \cdot \frac{k^2}{2} \cdot \varepsilon < f_k(G) < \frac{1}{(k-1)^{k-1}} + \frac{k^2}{2} \cdot \varepsilon,
\]

and, by Corollary 3.11,

\[
\frac{1}{(k-1)^{k-1}} - k^2 \cdot \varepsilon < f_k^v(G) < \frac{1}{(k-1)^{k-1}} + k^2 \cdot \varepsilon \tag{3.15}
\]

for every \( v \in V(G) \).

Let \( C \) be a graph on \( n \) vertices that is a union of \( k-1 \) disjoint cliques \( A_1, A_2, \ldots, A_{k-1} \) such that the size of the symmetric difference \( E(C) \Delta E(G) \) is as small as possible. In particular, \( |E(C) \Delta E(G)| \leq \varepsilon \binom{n}{2} \). Furthermore, every clique \( A_i \), where \( i \in [k-1] \), has to have size between \( \left( \frac{1}{k-1} - 2\varepsilon \right) n \) and \( \left( \frac{1}{k-1} + 2\varepsilon \right) n \).

Fix a vertex \( v \in V(G) \). We claim that there are at most \( \gamma \cdot n \) edges \( e \) in \( E(C) \Delta E(G) \) such that \( v \) is one of the endpoints of \( e \). We call such edges \( v \)-wrong.

Consider the partition of the set \( V(G) \) into parts \( P_1, P_2, \ldots, P_{k-1} \) according to the cliques \( A_1, A_2, \ldots, A_{k-1} \) of \( C \), and let \( d^v_i := \frac{|e(v, P_i)|}{|P_i|} \). Furthermore, let \( \gamma' := 10 \cdot k^{k+1} \cdot \varepsilon^{1/6} \). By our choice of \( \varepsilon \), it follows that \( \gamma > \gamma' \). First, suppose for
a contradiction there exist 3 distinct numbers \(i, i', i'' \in [k - 1]\) such that for every \(j \in \{i, i', i''\}\), we have \(d_{ij} < 1 - \frac{\gamma'}{2}\). That means there are at least

\[
\left(\frac{\gamma'}{2}\right)^6 \cdot \left(\frac{1}{k - 1} - 2\varepsilon\right)^6 \cdot n^6 \geq \left(\frac{\gamma'}{2}\right)^6 \cdot \left(\frac{1}{(k - 1)k - 1} - 12\varepsilon\right) \cdot n^6 \geq \varepsilon \cdot n^6
\]

choices of \(y_i, z_i \in P_i\), \(y_{i'}, z_{i'} \in P_{i'}\), and \(y_{i''}, z_{i''} \in P_{i''}\) such that \(v y_j \in E(G)\) and \(v z_j \notin E(G)\) for every \(j \in \{i, i', i''\}\). However, at most \(\varepsilon \cdot \left(\frac{n}{k - 1} - 2\varepsilon\right) < \varepsilon \cdot n^6\) such choices of \(y_i, y_{i'}, y_{i''}, z_i, z_{i'}, z_{i''}\) can contain an edge from \(E(C)\Delta E(G)\). Therefore, \(G\) contains a copy of \(E_7\), which contradicts Observation 3.12.

Without loss of generality, let \(d_i^v \in [0, 1] \setminus \left(\frac{\gamma'}{2}, 1 - \frac{\gamma'}{2}\right)\) for every \(i \in [k - 3]\). If there is an \(i \in [k - 3]\) such that \(d_i^v \geq 1 - \frac{\gamma'}{2}\), then \(d_i^v < \gamma'\) for every \(j \in [k - 1] \setminus \{i\}\). Indeed, otherwise \(f_k^v(G)\) is at least

\[
(1 - \gamma' / 2 + \gamma') \cdot \left(\frac{1}{(k - 1)k - 1} - 2k \cdot \varepsilon\right) - \varepsilon \geq \frac{1}{(k - 1)k - 1} + k \cdot \varepsilon,
\]

which contradicts (3.15). Therefore, we conclude that \(v \in P_i\) (otherwise moving \(v\) to \(A_i\) would decrease \(|E(G)\Delta E(C)|\)), and hence the number of \(v\)-wrong edges is at most \(\gamma' \cdot n < \gamma \cdot n\).

Now suppose \(d_i^v \leq \gamma' / 2\) for every \(i \in [k - 3]\). It follows that \(f_k^v(G)\) is at least

\[
(k - 1)! \cdot \left(\frac{1}{(k - 1)k - 1} - 2k \cdot \varepsilon\right) \cdot \left(1 - \frac{\gamma'}{2}\right)^{k - 3} \cdot (1 - d_{i-2}^v) \cdot (1 - d_{i-1}^v) + \left(\frac{1}{(k - 1)k - 1} - 2k \cdot \varepsilon\right) \cdot (d_{i-2}^v)^{k - 1} + \left(\frac{1}{(k - 1)k - 1} - 2k \cdot \varepsilon\right) \cdot (d_{i-1}^v)^{k - 1} - \varepsilon,
\]

where the first summand corresponds to the independent sets of size \(k\) that contain \(v\) and one other vertex from each \(P_i\), the next two summands correspond to the cliques of size \(k\) with all the other \(k - 1\) vertices inside the part \(P_{k-2}\) or \(P_{k-1}\), and finally the summand \(-\varepsilon\) comes from an upper bound for the \((k-1)\)-sets that contain at least one of the edges from \(E(G)\Delta E(C)\). It follows that (3.16) is at least

\[
p(d_{i-2}^v \cdot d_{i-1}^v) \frac{(k - 1)^{k - 1} - 7 \cdot k! \cdot \varepsilon - (k - 3) \cdot (k - 1)!) \cdot \gamma'}{2 \cdot (k - 1)^{k - 1}} > p(d_{i-2}^v \cdot d_{i-1}^v) \frac{(k - 1)^{k - 1} - 7 \cdot k! \cdot \varepsilon + \gamma'}{(k - 1)^{k - 1}}.
\]

However, \(f_k^v(G) - \frac{1}{(k - 1)^{k - 1}} < 2k^2 \cdot \varepsilon\), so our choice of the parameters imply that either \(d_{i-2}^v \in [0, \gamma]\) and \(d_{i-1}^v \in [1 - \gamma, 1]\), or \(d_{i-2}^v \in [1 - \gamma, 1]\) and \(d_{i-1}^v \in [0, \gamma]\). In the first case, \(v\) must be in \(P_{k-1}\) (otherwise moving \(v\) to \(A_{k-1}\) would decrease \(|E(G)\Delta E(C)|\)), symmetrically in the other case \(v \in P_{k-2}\). Furthermore, the number
of \( v \)-wrong edges is at most \( \gamma \cdot n \).

Suppose there are two vertices \( u \in P_i \) and \( v \in P_i \) for some \( i \in [k - 1] \) so that \( uv \notin E(G) \). The number of independent sets of size \( k \) in \( G \) that contain both \( u \) and \( v \) is then at least

\[
(1 - 2\gamma) \cdot \left(1 - 2\varepsilon \right) \cdot \frac{n}{k - 1} - \varepsilon \cdot \left( \frac{n}{k - 2} \right) > \left( \frac{1}{(k - 1)^{k-2}} - 5k \cdot \gamma \right) \cdot n^{k-2}. \tag{3.17}
\]

On the other hand, the number of cliques of size \( k \) in \( G + uv \) that contain both \( u \) and \( v \) is at most

\[
\left( \frac{1+2\varepsilon}{k - 1} \right) + \left( \gamma \cdot \frac{n}{k - 2} \right) < \left( \frac{1 + 2k \cdot \varepsilon}{(k - 1)^{k-2} \cdot (k - 2)!} + \gamma \right) \cdot n^{k-2}. \tag{3.18}
\]

By the choice of \( \gamma \), the value of (3.17) - (3.18) is positive. But that means \( F_k(G + uv) < F_k(G) \), a contradiction. We conclude that for every \( i \in [k - 1] \), the set of vertices \( P_i \) form a clique in \( G \). \( \square \)

### 3.5 Monotone subsequences of length five

In this section, we follow the approach presented in Sections 3.2, 3.3 and 3.4 in order to fully characterize all sufficiently large 5-extremal permutations.

Let \( \text{EXT}_7^5 \) be the set of all 7-vertex subgraphs up to a blind isomorphism that have a positive density in the conjectured extremal construction. It follows that \( \text{EXT}_7^5 = \{ H_1, H_2, \ldots, H_{11} \} = \text{EXT}_7^4 \cup \{ H_9, H_{10}, H_{11} \} \). The set \( \text{EXT}_7^5 \) is depicted in Figure 3.8. Let \( \mathcal{E}_7^5 := \mathcal{F}_7 \setminus \text{EXT}_7^5 \). The main theorem of this section is the following.

**Theorem 3.19.** There exists a positive rational \( \alpha \) such that the following is true. If \( (\pi)_{n \in \mathbb{N}} \) is a convergent sequence of permutations and \( \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \) is its limit,
then
\[
\phi \left( K_5 - \alpha \cdot \sum_{H \in \mathcal{E}^2} H \right) \geq \frac{1}{256}.
\]

**Proof.** As in the proof of Theorem 3.4, we let \( \sigma_1 \) to be the 3-vertex type with no edges and \( \sigma_2 \) the 3-vertex type with the edge \( bc \). Additionally, let \( \sigma_3 \) and \( \sigma_4 \) be two specific types of order 5 given in Figure 3.9.

Again, we set \( b_{\tau_{1,1}} := b_{\tau_{1,2}} := abc \), \( b_{\tau_{2,1}} := b_{\tau_{2,4}} := abc \), and \( b_{\tau_{2,2}} := b_{\tau_{2,3}} := bca \). The set \( T(\sigma_1) \) contains 16 permutations \( \tau_{3,j} \). For each \( j \in [16] \), we define \( b_{\tau_{3,j}} \) in the following way:

- for \( \tau_{3,1} = 13542 \), set \( b_{\tau_{3,1}} := abcde \), \( \phi(\sigma) = 34251 \), set \( b_{\tau_{3,9}} := cdbea \),
- for \( \tau_{3,2} = 14352 \), set \( b_{\tau_{3,2}} := acdbe \), \( \phi(\sigma) = 41325 \), set \( b_{\tau_{3,10}} := ebda \),
- for \( \tau_{3,3} = 15243 \), set \( b_{\tau_{3,3}} := aebdc \), \( \phi(\sigma) = 42135 \), set \( b_{\tau_{3,11}} := edcba \),
- for \( \tau_{3,4} = 15324 \), set \( b_{\tau_{3,4}} := aedcb \), \( \phi(\sigma) = 42351 \), set \( b_{\tau_{3,12}} := bcdea \),
- for \( \tau_{3,5} = 24315 \), set \( b_{\tau_{3,5}} := bcdea \), \( \phi(\sigma) = 51342 \), set \( b_{\tau_{3,13}} := aedcb \),
- for \( \tau_{3,6} = 24531 \), set \( b_{\tau_{3,6}} := edcba \), \( \phi(\sigma) = 51423 \), set \( b_{\tau_{3,14}} := aebdc \),
- for \( \tau_{3,7} = 25341 \), set \( b_{\tau_{3,7}} := ebda \), \( \phi(\sigma) = 52314 \), set \( b_{\tau_{3,15}} := acde \),
- for \( \tau_{3,8} = 32415 \), set \( b_{\tau_{3,8}} := cdbea \), \( \phi(\sigma) = 53124 \), set \( b_{\tau_{3,16}} := abcde \).

It holds that \( |F_6^\sigma| = 26 \) and \( |F_7^\sigma| = 574 \). Similarly, the set \( T(\sigma_4) \) has size 12. For each \( j \in [12] \), we set \( b_{\tau_{4,j}} \) as follows:

- for \( \tau_{4,1} = 14532 \), set \( b_{\tau_{4,1}} := abcde \), \( \phi(\sigma) = 34215 \), set \( b_{\tau_{4,7}} := cbdea \),
- for \( \tau_{4,2} = 15432 \), set \( b_{\tau_{4,2}} := adbc \), \( \phi(\sigma) = 42315 \), set \( b_{\tau_{4,8}} := dcbea \),
- for \( \tau_{4,3} = 15423 \), set \( b_{\tau_{4,3}} := aebd \), \( \phi(\sigma) = 43125 \), set \( b_{\tau_{4,9}} := edcab \),
- for \( \tau_{4,4} = 23541 \), set \( b_{\tau_{4,4}} := edcba \), \( \phi(\sigma) = 51243 \), set \( b_{\tau_{4,10}} := aeidb \),
- for \( \tau_{4,5} = 24351 \), set \( b_{\tau_{4,5}} := dcbea \), \( \phi(\sigma) = 51324 \), set \( b_{\tau_{4,11}} := abdce \),
- for \( \tau_{4,6} = 32451 \), set \( b_{\tau_{4,6}} := cbdea \), \( \phi(\sigma) = 52134 \), set \( b_{\tau_{4,12}} := abcde \).

In this case, \( |F_6^\sigma| = 28 \) and \( |F_7^\sigma| = 624 \).

Based on the semidefinite method presented in Section 1.2, an instance of the SDP was used to find 4 symmetric positive semidefinite matrices \( M_1, M_2, M_3 \) and \( M_4 \) with rational entries such that for every \( \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \) we have

\[
\phi \left( \frac{4}{\phi} \sum_{i=1}^{4} [x_i^T M_i x_i] \sigma_i \right) \leq \phi \left( K_5 - \frac{1}{256} \alpha \cdot \sum_{H \in \mathcal{E}^2} H \right),
\]

where
The types $\sigma_1$ through $\sigma_4$ used in the proof of Theorem 3.19.

- the vector $x_1 \in (\mathbb{R}F_{5}^{\sigma_1})|_{F_{5}^{\sigma_1}}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R}F_{5}^{\sigma_1}$,
- the vector $x_2 \in (\mathbb{R}F_{5}^{\sigma_2})|_{F_{5}^{\sigma_2}}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R}F_{5}^{\sigma_2}$,
- the vector $x_3 \in (\mathbb{R}F_{6}^{\sigma_3})|_{F_{6}^{\sigma_3}}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R}F_{6}^{\sigma_3}$,
- the vector $x_4 \in (\mathbb{R}F_{6}^{\sigma_4})|_{F_{6}^{\sigma_4}}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R}F_{6}^{\sigma_4}$, and
- $\alpha = 22961176619/6306641510400 \sim 0.00364$.

The left-hand side of the inequality above is non-negative by (1.2). 

As in the case of Theorem 3.4, the numerical values of the entries of the matrices $M_1, M_2, M_3$ and $M_4$ can be downloaded from the web page http://honza.ucw.cz/phd/. We also created a sage script called “thm_3.19-verify.sage”, which can be used to verify the computations.

The methods presented in Section 3.3 can be straightforwardly adopted also to the case of monotone subsequences of length 5. Specifically, they yield the following stability result, which is an analogue of Theorem 3.9.

**Theorem 3.20.** For every $\varepsilon_{\text{STAB}} > 0$ there exist $\delta_{\text{STAB}} > 0$ and $n_{\text{STAB}}$ such that the following is true. If $G$ is a permutation graph on $n_{\text{STAB}} \geq n_0$ vertices with $f_5(G) \leq \frac{1}{256} + \delta_{\text{STAB}}$, then $G$ is isomorphic to either $T_4(n)$ or $\overline{T_4(n)}$ after adding and/or deleting at most $\varepsilon_{\text{STAB}} \cdot \binom{n}{2}$ edges.
The reasoning is essentially the same as for Theorem 3.9. For the sake of completeness, we give a proof in Appendix B.1.

The stability result together with Theorem 3.14 and Theorem 3.2 gives a complete characterization of 5-extremal permutations.

**Theorem 3.21.** There exists an integer $n_0$ such that for every permutation $\tau \in S_n$, where $n \geq n_0$, we have $F_5(\tau) \geq F_5(\tau_5(n))$. Furthermore, if $F_5(\tau) = F_5(\tau_5(n))$, then $\tau \in \mathcal{W}_5(n)$.

### 3.6 Monotone subsequences of length six

In the last section of this chapter, we use our methods to characterize also all large 6-extremal permutations. Analogously to the previous cases, we let $\text{EXT}^6_8$ to be the set of all non-blindly-isomorphic 8-vertex graphs that have a positive density in the conjectured extremal example. It holds that $|\text{EXT}^6_8| = 18$ and the graphs $H'_1, \ldots, H'_{18}$ are depicted in Figure 3.10. We also define $\mathcal{E}^6_8$ to be the complement of the set $\text{EXT}^6_8$, i.e., $\mathcal{E}^6_8 := \mathcal{F}_8 \setminus \text{EXT}^6_8$.

We start with the main theorem of this section.

**Theorem 3.22.** There exists a positive rational $\alpha$ such that the following is true. If $(\pi)_{n \in \mathbb{N}}$ is a convergent sequence of permutations and $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ is its limit,
then

\[ \phi \left( K_6 - \alpha \cdot \sum_{H \in \mathcal{E}_6^G} H \right) \geq \frac{1}{3125}. \]

**Proof.** Let \( \sigma_1, \ldots, \sigma_{31} \) be the types depicted in Figure 3.11. Note that the type \( \sigma_1 \) is the only type (up to a blind isomorphism) of order 2, the types \( \sigma_2, \ldots, \sigma_7 \) are all 6 the types of order 4, and the remaining types \( \sigma_8, \ldots, \sigma_{31} \) are 24 types of order 6. Note we do not use all the possible types of order 6 (there are 71 non-isomorphic types of order 6 in total) and the particular choice was suggested by a computer.

Going through the lists of all the permutations of sizes 2, 4, and 6 and their corresponding permutation graphs yields that the sets \( T(\sigma_1), \ldots, T(\sigma_{31}) \) have the following sizes: 2, 2, 6, 8, 4, 2, 2, 2, 8, 8, 8, 8, 8, 8, 8, 8, 8, 4, 8, 4, 8, 4, 8, 4, 8, 4, 8, 8, 8, 8, and 2, respectively. Note that

\[ \sum_{i=2}^{7} |T(\sigma_i)| = |S_4| = 24 \quad \text{and} \quad \sum_{i=8}^{31} |T(\sigma_i)| = 160. \]

The second sum is less than \(|S_6| = 720\), since we do not use all the types of size six.

Now we need to choose the bijections \( b_{\tau_{i,j}} \), where \( i \in [31] \) and \( \tau_{i,j} \in T(\sigma_i) \). The choice of \( b_{\tau_{i,j}} \) has been made with a computer assistance and they bijections are given in Appendix B.3. It follows that

- the set \( \mathcal{F}_{S_5}^{\sigma_1} \) has size 119, the set \( \mathcal{F}_{S_8}^{\sigma_1} \) has size 99440,
- the sets \( \mathcal{F}_{S_6}^{\sigma_i} \), where \( i \in \{2, \ldots, 7\} \), have sizes 122, 243, 191, 170, 191, and 220, respectively,
- the sets \( \mathcal{F}_{S_8}^{\sigma_i} \) have sizes 22361, 66186, 44698, 39286, 44698, and 51540, respectively,
- the sets \( \mathcal{F}_{S_7}^{\sigma_i} \), where \( i \in \{8, \ldots, 31\} \), have sizes 22, 29, 34, 34, 32, 31, 32, 34, 33, 34, 33, 35, 34, 34, 32, 35, 35, 35, 34, 34, and 35, respectively,
- the sets \( \mathcal{F}_{S_8}^{\sigma_i} \) have sizes 436, 719, 904, 904, 822, 791, 791, 837, 904, 871, 871, 904, 871, 938, 904, 904, 837, 938, 938, 938, 938, 938, 938, 938, 938, 938, respectively.

We use the semidefinite method to find 31 symmetric positive semidefinite matrices \( M_1, \ldots, M_{31} \) such that the following is true. If \( \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R}) \), then

\[ \phi \left( \sum_{i=1}^{31} [x_i^T M_i x_i]_{\sigma_i} \right) \leq \phi \left( K_6 - \frac{1}{3125} - \alpha \cdot \sum_{H \in \mathcal{E}_6^G} H \right), \]

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where

- the vector $x_1 \in (\mathbb{R}^{F_{\sigma 1}^{5}})^{|F_{\sigma 1}^{5}|}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R}^{F_{\sigma 1}^{5}}$,
- for $i \in \{2, \ldots, 7\}$, the vector $x_i \in (\mathbb{R}^{F_{\sigma i}^{6}})^{|F_{\sigma i}^{6}|}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R}^{F_{\sigma i}^{6}}$,
- for $i \in \{8, \ldots, 31\}$, the vector $x_i \in (\mathbb{R}^{F_{\sigma i}^{7}})^{|F_{\sigma i}^{7}|}$ is the vector whose $j$-th coordinate is equal to the $j$-th element of the canonical base of $\mathbb{R}^{F_{\sigma i}^{7}}$, and
- $\alpha = 13271039154489483541126912165592130242585059628042945811885064599181621701235552471267309993094496421736848347313984284148255111212206507583585697425682443380032858294928639014516332583745304406174382777990388793236389 / 20686661332608331930207369662236348167139641759068592728375337467555700205174729933734826748826074449891537835054260913260013612982750348884354508780363273210218336138890262647318509494337222405421715841786157274759168000000 \sim 6 \cdot 10^{-5}$. 

The numerical values of the entries of the matrices $M_1, \ldots, M_{31}$ can be downloaded from the web page http://honza.ucw.cz/phd/. A sage script called “thm_3.22-verify.sage”, which is also available on the web page, can be used to verify our computations.

Analogously to the case of monotone subsequences of length 4 and 5, the methods from Section 3.3 yield the following stability result.

**Theorem 3.23.** For every $\varepsilon_{\text{STAB}} > 0$ there exist $\delta_{\text{STAB}} > 0$ and $n_{\text{STAB}}$ such that the following is true. If $G$ is a permutation graph on $n_{\text{STAB}} \geq n_0$ vertices with $f_6(G) \leq \frac{1}{375} + \delta_{\text{STAB}}$, then $G$ is isomorphic to either $T_3(n)$ or $\overline{T_3(n)}$ after adding and/or deleting at most $\varepsilon_{\text{STAB}} \cdot \binom{n}{2}$ edges.

The proof of Theorem 3.23 is given in Appendix B.2.

As in the case of 5-extremal permutations, the stability result together with Theorem 3.14 and Theorem 3.2 immediately gives a complete characterization of sufficiently large 6-extremal permutations.

**Theorem 3.24.** There exists an integer $n_0$ such that for every permutation $\tau \in S_n$, where $n \geq n_0$, we have $F_6(\tau) \geq F_6(\tau_6(n))$. Furthermore, if $F_6(\tau) = F_6(\tau_6(n))$, then $\tau \in \mathcal{W}_6(n)$. 

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Figure 3.11: The types $\sigma_1$ through $\sigma_{31}$ used in the proof of Theorem 3.22.
Chapter 4

Finitely forcible graphons

Recently, a theory of limits of combinatorial structures emerged and attracted substantial attention. In this chapter, we address limits of dense graphs, which is the most studied case. Its study was initiated in a series of papers by Borgs, Chayes, Lovász, Sós, Szegedy and Vesztergombi [12, 13, 14, 47, 49]. Dense graph limits are also closely related to the framework of flag algebras, which we discussed in Chapter 1.

As in Section 1.1, our object of study are convergent sequences of graphs, i.e., sequences, where the induced densities of every fixed graph as a subgraph in the graphs from the sequence converge. In Section 1.1, such a convergent sequence of graphs was assigned an algebra homomorphism from the set of all graphs to reals. The homomorphism naturally represents the limit subgraph densities. A convergent sequence of graphs can also be associated with an analytic object (graphon), which is a symmetric measurable function from the unit square $[0, 1]^2$ to $[0, 1]$. Note that a graphon also contains the information about the limit subgraph densities in the sequence.

In this chapter, we are concerned with finitely forcible graphons, i.e., those graphons that are uniquely determined (up to a natural notion of equivalence) by finitely many subgraph densities. Such graphons are related to uniqueness of extremal configurations in extremal graph theory as well as to other problems.

4.1 Dense graph limits

In this section, we introduce basic notions from the theory of dense graph limits. We follow a recent monograph on the topic by Lovász [46].
4.1.1 Graphons

Recall from Section 1.1 that $p(H, G_i)$ denotes the probability that random subset of $V(G_i)$ of size $v(H)$ induces a copy of $H$. Also recall a sequence of graphs $(G_i)_{i \in \mathbb{N}}$ is convergent if the subgraph density of every graph in $G_i$ converges, i.e., the limit $\lim_{i \to \infty} p(H, G_i)$ exists for every graph $H \in \mathcal{F}$. As we discussed in Chapter 1, such a sequence naturally defines a limit object – a homomorphism $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$. In the whole chapter, we study a different (and more analytical) representation of the limit object of a convergent sequence of graphs.

A graphon $W$ is a symmetric measurable function from $[0, 1]^2$ to $[0, 1]$. Here, symmetric stands for the property that $W(x, y) = W(y, x)$ for every $x, y \in [0, 1]$. Given a graphon $W$, we define a $W$-random graph of order $k$ in the following way. First, we sample $k$ random points $v_1, v_2, \ldots, v_k \in [0, 1]$ uniformly and independently, and then we join the $i$-th and the $j$-th vertex by an edge with probability $W(v_i, v_j)$ (again independently of the other edges and on the sampling of the vertices $v_1, v_2, \ldots, v_k$). Since the points of $[0, 1]$ play the role of vertices, we refer to them as to the vertices of $W$.

The density $p(H, W)$ of a graph $H$ in a graphon $W$ is equal to the probability that a $W$-random graph of order $v(H)$ is isomorphic to $H$. Clearly, the following holds:

$$p(H, W) = \frac{v(H)!}{|\text{Aut}(H)|} \int_{[0, 1]^{v(H)}} \prod_{ij \in E(H)} W(v_i, v_j) \prod_{ij \notin E(H)} (1 - W(v_i, v_j)) \, d\lambda_{v(H)},$$

where $\text{Aut}(H)$ is the automorphism group of $H$ and $\lambda_{v(H)}$ is the uniform Borel measure on $[0, 1]^{v(H)}$.

One of the key results of the theory of dense graph limits asserts that for every convergent sequence $(G_i)_{i \in \mathbb{N}}$ of graphs with the limit $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$, there exists a graphon $W$ such that

$$p(H, W) = \phi(H) = \lim_{i \to \infty} p(H, G_i)$$

for every graph $H \in \mathcal{F}$. For a proof, see, e.g., [46, Theorem 11.21]. Conversely, if $W$ is a graphon, then the sequence of $W$-random graphs with increasing orders converges with probability one and its limit is $W$ [46, Corollary 11.15].

An example of a graphon corresponding to the sequence of complete balanced bipartite graphs $K_{n,n}$ is depicted in Figure 4.1. Through of this chapter, we use the following convention when drawing graphons. The point $(0, 0)$ is always in the top-left corner of the square $[0, 1]^2$, black points represent the value one, gray points,
Figure 4.1: A complete balanced bipartite graph $K_{n,n}$, its adjacency matrix, and a
graphon representing the sequence $(K_{n,n})_{n \in \mathbb{N}}$.

Figure 4.2: Two graphons weakly isomorphic to each other. They both represent
the limit of a sequence of complete balanced tripartite graphs.

depending on their shade, represent values between zero and one, and white points
represent the value zero.

Two graphons $W_1$ and $W_2$ are weakly isomorphic if $p(H, W_1) = p(H, W_2)$ for
every graph $H$. If $\varphi : [0, 1] \to [0, 1]$ is a measure preserving map, then the graphon
$W^\varphi(x, y) := W(\varphi(x), \varphi(y))$ is always weakly isomorphic to $W$. The opposite is
true in the following sense [15]: if two graphons $W_1$ and $W_2$ are weakly isomorphic,
then there exist measure preserving maps $\varphi_1 : [0, 1] \to [0, 1]$ and $\varphi_2 : [0, 1] \to [0, 1]$
such that $W_1^{\varphi_1} = W_2^{\varphi_2}$ almost everywhere. An example of two different graphons,
both representing the limit of the sequence of complete balanced tripartite graphs
of increasing order, is depicted in Figure 4.2.

### 4.1.2 Finite forcibility

A graphon $W$ is finitely forcible if there exist finitely many graphs $H_1, \ldots, H_k$ such
that every graphon $W'$ satisfying $p(H_i, W) = p(H_i, W')$ for every $i \in [k]$ is weakly
isomorphic to $W$. Such graphons are related to uniqueness of extremal configu-
rations in extremal graph theory as well as to other problems. For example, the
classical result of Chung, Graham and Wilson [17] asserting that a large graph is
pseudorandom if and only if the non-induced densities of $K_2$ and $C_4$ are the same
as in the Erdős-Rényi random graph $G_{n,1/2}$ can be cast in the language of graphons as follows: the graphon identically equal to $1/2$ is uniquely determined by the non-induced densities of $K_2$ and $C_4$. In other words, it is finitely forcible. Another example that can be cast in the language of finite forcibility is the asymptotic version of the theorem of Turán [72]: there exists a unique graphon with edge density $\frac{r-1}{r}$ and zero density of $K_{r+1}$, namely the graphon corresponding to the sequence of Turán graphs $T_r(n)$.

A systematic study of finitely forcible graphons, which was started by Lovász and Szegedy in [50], was motivated by a possibility of a better understanding of extremal configurations for problems in extremal graph theory. For every finitely forcible graphon $W$, there exists an extremal graph theory problem such that $W$ is its (unique) solution. Perhaps the most important open problem in the area of finite forcibility is whether there exists also a certain converse of this statement. Specifically, Lovász and Szegedy asked the following question:

**Conjecture 4.1** ([50, Conjecture 7]). Let $k$ be an integer, let $F_1, \ldots, F_k$ be $k$ fixed graphs and let $a_1, \ldots, a_k$ be $k$ fixed reals. If a finite set of constraints of the form $p(F_i, W) = a_i$ is satisfied by some graphon, then it is satisfied by a finitely forcible graphon.

If an extremal problem has a unique solution then clearly the graphon corresponding to the solution is a finitely forcible graphon. However, the conjecture for a general extremal graph theory problem remains open.

Let us now describe some other known examples of finitely forcible graphons. A *stepfunction* is a graphon $W$ such that its vertex-set $[0,1]$ can be partitioned into finitely many measurable sets (also called parts) $A_1, A_2, \ldots, A_k$ in such a way that for every $i, j \in [k]$ if $u, u' \in A_i$ and $v, v' \in A_j$, then $W(u,v) = W(u',v')$. Note that a stepfunction with one part is a graphon of a sequence of Erdős-Rényi graphs $G_{n,p}$ for a fixed $p \in [0,1]$, and vice versa. An example of a stepfunction with four parts is depicted in Figure 4.3. The result of Chung, Graham, and Wilson was generalized by Lovász and Sós [51] who proved that any graphon that is a stepfunction is finitely forcible.

Next, a result of Diaconis, Homes, and Janson [20] asserts that the half graphon $W_\Delta(x, y)$ defined as $W_\Delta(x, y) = 1$ if $x + y \geq 1$, and $W_\Delta = 0$, otherwise, is finitely forcible; see Figure 4.4. This was probably the first example of a finitely forcible graphon that is not a stepfunction. Further examples of finitely forcible graphons were found by Lovász and Szegedy in [50].

When dealing with finitely forcible graphons, we usually give a set of equality constraints that uniquely determines $W$ instead of specifying the finitely many
subgraphs that uniquely determine $W$. A constraint is an equality between two density expressions, where a density expression is recursively defined as follows: a real number or a graph $F \in \mathcal{F}$ are density expressions, and if $D_1$ and $D_2$ are two density expression, then the sum $D_1 + D_2$ and the product $D_1 \times D_2$ are also density expressions. The value of a density expression is the value obtained by substituting for every subgraph $F$ its density in the graphon. Observe that if $W$ is a unique (up to weak isomorphism) graphon that satisfies a finite set $C$ of constraints, then it is finitely forcible. In particular, $W$ is the unique (up to weak isomorphism) graphon with densities of subgraphs appearing in $C$ equal to their densities in $W$. This holds since any graphon with these densities satisfies all constraints in $C$ and thus it must be weakly isomorphic to $W$.

Following [48], each graphon $W$ can be assigned a topological space of so-called typical vertices of $W$. To simplify our notation, if $A \subseteq [0, 1]$ is measurable, we use $|A|$ to denote its measure. For $x \in [0, 1]$, we define the neighborhood function of $x$ as $f^W_x(y) := W(x, y)$. For an open set $A \subseteq L_1[0, 1]$, we write $A^W$ for $\{x \in [0, 1], f^W_x \in A\}$. Let $T(W)$ be the set formed by the functions $f \in L_1[0, 1]$ such that $|U^W| > 0$ for every neighborhood $U$ of $f$. The set $T(W)$ inherits topology from $L_1[0, 1]$. The vertices $x \in [0, 1]$ with $f^W_x \in T(W)$ are called typical vertices of a graphon $W$. Almost every vertex of a graphon is typical [48].
4.1.3 Finite forcibility and non-compactness

If $W$ is a finitely forcible graphon, how complicated can the space of its typical vertices $T(W)$ be? If the structure of the space $T(W)$ would be somewhat “simple”, we might hope that together with a positive answer to Conjecture 4.1 we will be able to establish a general machinery for solving problems in extremal graph theory. Unfortunately, it turned out that there are finitely forcible graphons $W$ where the structure of the space $T(W)$ is rather complicated.

The known examples of finitely forcible graphons led Lovász and Szegedy to make the following two conjectures.

**Conjecture 4.2 ([50, Conjecture 9]).** If $W$ is a finitely forcible graphon, then $T(W)$ is a compact space.

They noted that they could not even prove that $T(W)$ had to be locally compact. The main result of this chapter is a construction of a finitely forcible graphon $W_R$, which we call Rademacher graphon, such that $T(W_R)$ fails to be locally compact. In particular, $T(W)$ is not compact.

**Theorem 4.3.** There exists a finitely forcible graphon $W_R$ such that the topological space $T(W_R)$ is not locally compact.

The other conjecture of Lovász and Szegedy asks whether the dimension of $T(W)$ of a finitely forcible graphon $W$ is always finite.

**Conjecture 4.4 ([50, Conjecture 10]).** If $W$ is a finitely forcible graphon, then $T(W)$ is finite dimensional.

Note that Lovász and Szegedy stated in their paper that they intentionally did not specify the notion of dimension they had in mind. This conjecture has been recently disproved by Glebov, Klímašová and Král’ [32]. Specifically, they constructed a finitely forcible graphon $W$ such that the space $T(W)$ contains a subset $A$ which is homeomorphic to $[0, 1]^\infty$.

4.2 Partitioned graphons

In this section, we introduce the notion of partitioned graphons. Some of the methods presented in this section are analogous to those used by Lovász and Sós in [51], and by Norin in [55]. In particular, they used similar types of arguments to specialize their constraints to parts of graphons they were forcing as we do in this section.
We adapt the notion of rooted densities from the flag algebra framework to graphons in order to extend the notion of density expressions to the rooted case. Recall from Chapter 1 that a type \( \sigma \) is a graph with a fixed labelling of its vertex-set and a \( \sigma \)-flag \( F^\sigma \in \mathcal{F}^\sigma \) is a graph containing a fixed labelled embedding of \( \sigma \). The subgraph induced by the labelled vertices is called the root of \( F^\sigma \) and the labelled vertices are also referred to as the rooted vertices of \( F^\sigma \).

Fix a type \( \sigma \), a \( \sigma \)-flag \( F^\sigma \in \mathcal{F}^\sigma \), and let \( m = v(\sigma) \). Recall \( \sigma^\emptyset \) is the unlabelled graph from \( \mathcal{F} \) that corresponds to \( \sigma \). For a graphon \( W \) with \( p(\sigma^\emptyset, W) > 0 \), we let the auxiliary function \( c_\sigma : [0, 1]^m \to [0, 1] \) denote the probability that an \( m \)-tuple \((x_1, \ldots, x_m) \in [0, 1]^m \) induces a copy of \( \sigma \) in \( W \) respecting the labeling of vertices of \( \sigma \):

\[
c_\sigma(x_1, \ldots, x_m) := \left( \prod_{ij \in E(\sigma)} W(x_i, x_j) \right) \cdot \left( \prod_{ij \notin E(\sigma)} (1 - W(x_i, x_j)) \right).
\]

We next define a probability measure \( \mu_\sigma \) on \([0, 1]^m\). If \( A \subseteq [0, 1]^m \) is a Borel set, then

\[
\mu_\sigma(A) := \frac{\int_A c_\sigma(x_1, \ldots, x_m) \, d\lambda_m}{\int_{[0, 1]^m} c_\sigma(x_1, \ldots, x_m) \, d\lambda_m} = \frac{m!}{|\text{Aut}(\sigma^\emptyset)|} \cdot \frac{\int_A c_\sigma(x_1, \ldots, x_m) \, d\lambda_m}{p(\sigma^\emptyset, W)}.
\]

Note that the probability measure \( \mu_\sigma \) is an analogue of the probability distribution \( P^\sigma \) on \( \text{Hom}^+(\mathcal{A}^\sigma, \mathbb{R}) \) from Section 1.1.

When \( x_1, \ldots, x_m \in [0, 1] \) are fixed, then the density of \( F \) with the rooted vertices being \( x_1, \ldots, x_m \) is the probability that a random sample of the non-roots yields a copy of \( F \) conditioned on the event that the roots induce \( \sigma \). Noticing that an automorphism of a \( \sigma \)-flag has all the rooted vertices as fixed points, we obtain that this is equal to

\[
\frac{(v(F) - m)!}{|\text{Aut}(F)|} \cdot \frac{\int_{[0, 1]^m} \prod_{ij \in E(F), E(\sigma)} W(x_i, x_j) \prod_{ij \in E(F) \cup (V(\sigma))^2} (1 - W(x_i, x_j)) \, d\lambda_{v(F)-m}}{p(\sigma^\emptyset, W)}.
\]

Two flags \( F_1 \) and \( F_2 \) are compatible if the their types are isomorphic, i.e., both \( F_1 \in \mathcal{F}^\sigma \) and \( F_2 \in \mathcal{F}^\sigma \) for some type \( \sigma \). A rooted density expression \( D \) is a density expression such that all flags that appear in it are mutually compatible rooted graphs. Note we will also speak about compatible rooted density expressions to emphasize that the flags in all of them are mutually compatible. First let \( x_1, \ldots, x_m \in [0, 1] \) be fixed. Analogously to the non-rooted case we use the notion
of density of $F \in \mathcal{F}$ from the previous paragraph to determine the value of $D$ with the rooted vertices being $x_1, \ldots, x_m$. For different choices of $x_1, \ldots, x_m$, we obtain different values. Finally, the rooted density of $D$ is then a random variable determined by the choice of the rooted vertices according to the probability measure $\mu_\sigma$.

We now consider a constraint such that both the left and the right hand sides $D$ and $D'$ are compatible rooted density expressions. Let $\sigma$ be the corresponding type of all the flags in $D$ and $D'$. Such a constraint should be interpreted to mean that it holds that $D - D' = 0$ with probability one, where the randomness comes from picking the root according to $\mu_\sigma$. It follows from (1.1) that the expected value of a rooted density expression $D$ with the root $\sigma$ is equal to $\langle D \rangle_\sigma / p(\sigma, W)$, where $\langle D \rangle_\sigma$ is an ordinary density expression (in particular, it does not dependent of $W$). Observe that if $D$ and $D'$ are compatible rooted density expressions, both with the roots $\sigma$, then a graphon satisfies $D = D'$ if and only if it satisfies the (ordinary) constraint $\langle (D - D') \times (D - D') \rangle_\sigma = 0$. This allows us to express constraints involving rooted density expressions as ordinary constraints, hence we will not distinguish between the two types of constraints in what follows.

A degree of a vertex $x \in [0, 1]$ of a graphon $W$ is equal to

$$\int_{[0,1]} W(x, y) \, dy = \int_{[0,1]} f^W_x(y) \, dy.$$  

Note that the degree is well-defined for almost every vertex of $W$. A graphon $W$ is partitioned if there exist $k \in \mathbb{N}$, positive reals $a_1, \ldots, a_k$ with $\sum_i a_i = 1$ and distinct reals $d_1, \ldots, d_k \in [0, 1]$ such that the set of vertices of $W$ with degree $d_i$ has measure $a_i$. An example of a partitioned graphon with two parts is depicted in Figure 4.5. We will often speak about partitioned graphons when having in mind fixed values.

Figure 4.5: A partitioned graphon with two parts $A_1$ and $A_2$. The measure of $A_1$ is $1/3$ and the degree of each of its vertices is $2/3$, and the measure of $A_2$ is $2/3$ and the degree of each of its vertices is $1/3$. 


of \( k, a_1, \ldots, a_k \), and \( d_1, \ldots, d_k \). Being a partitioned graphon can be finitely forced as shown in the next lemma.

**Lemma 4.5.** Let \( k \) be an integer, \( a_1, \ldots, a_k \) positive real numbers summing to one, and \( d_1, \ldots, d_k \) distinct reals between zero and one. There exists a finite set of constraints \( C \) such that any graphon \( W \) satisfying \( C \) also satisfies the following:

The set of vertices of \( W \) with degree \( d_i \) has measure \( a_i \).

In other words, such \( W \) must be a partitioned graphon with parts of sizes \( a_1, \ldots, a_k \) and degrees \( d_1, \ldots, d_k \).

**Proof.** The desired property of being a partitioned graphon with the given choice of parameters is forced by the following set of constraints:

\[
\prod_{i=1}^{k} (e^1 - d_i) = 0, \quad \text{and} \\
\left[ \prod_{i=1, i \neq j}^{k} (e^1 - d_i) \right]_{1} = a_j \prod_{i=1, i \neq j}^{k} (d_j - d_i) \quad \text{for every } j, 1 \leq j \leq k,
\]

where “1” denotes the 1-vertex type and \( e^1 \) is an edge with one rooted and one non-rooted vertex. The first constraint says that the degree of almost every vertex is equal to one of the numbers \( d_1, \ldots, d_k \). For \( j \leq k \), the left hand side of the second constraint before applying the \( [\cdot] \)-operator is non-zero only if the degree of the rooted vertex is \( d_j \), assuming the degree is one of \( d_1, \ldots, d_k \). Hence, the left hand side is equal to

\[
\prod_{i=1, i \neq j}^{k} (d_j - d_i)
\]

in that case. Therefore, the measure of vertices of degree \( d_j \) is forced to be \( a_j \). \( \square \)

Assume that \( W \) is a partitioned graphon. We write \( A_i \) for the set of vertices of degree \( d_i \) for \( i, 1 \leq i \leq k \) and identify \( A_i \) with the interval \([0, a_i]\) (note that the measure of \( A_i \) is \( a_i \)). This will be convenient when defining partitioned graphons. For example, we can use the following when defining a graphon \( W \): \( W(x, y) = 1 \) if \( x \in A_1, y \in A_2 \) and \( x \geq y \).

A graph \( H \) is decorated if its vertices are labelled with parts \( A_1, \ldots, A_k \). The density of a decorated graph \( H \) in a partitioned graphon \( W \) is the probability that randomly chosen \( v(H) \) vertices induce a subgraph isomorphic to \( H \) with its vertices
Figure 4.6: Examples of decorated constraints.

contained in the parts corresponding to the labels. For example, if $H$ is an edge with vertices decorated with parts $A_1$ and $A_2$, then the density of $H$ is the density of edges between the parts $A_1$ and $A_2$, i.e.,

$$p(H, W) = \int_{A_1} \int_{A_2} W(x, y) \, dx \, dy.$$ 

If $W$ is the graphon depicted in Figure 4.5, then $p(H, W) = 4/9$.

Similarly as in the case of non-decorated graphs, we define rooted decorated subgraphs. A constraint that uses (rooted or non-rooted) decorated subgraphs is referred to as decorated. In this chapter, we use the following convention for drawing graphs in density expressions: edges of graphs are always drawn solid, non-edges dashed, and if two vertices are not joined, then the picture represents the sum over both possibilities. If a graph contains some roots, the roots are depicted by square vertices, and the non-root vertices by circles. If there are more roots from the same part of a graphon, then the squares are rotated to distinguish the roots. If a graph is decorated, then the decorations of its vertices are always drawn inside their circles or squares. See Figure 4.6 for the following five examples of decorated expressions:

- the first expression from the left denotes the edge density between the parts $A_1$ and $A_2$,
- the next one denotes the edge density inside the part $A_1$ multiplied by $|A_1|^2$,
- the third one is the non-edge density between the parts $A_1$ and $A_2$,
- the fourth one corresponds to the degree of a fixed vertex from $A_1$ inside the part $A_1$ multiplied by $|A_1|$, and
- the last expression is equal to the degree of a fixed vertex from $A_1$ inside the entire partitioned graphon (assuming the graphon has $k$ parts).

The next lemma shows that decorated constraints are not more powerful than non-decorated ones, and therefore they can be used to show that a graphon is

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Lemma 4.6. Let \( k \) be an integer, \( a_1, \ldots, a_k \) positive real numbers summing to one, and \( d_1, \ldots, d_k \) distinct reals between zero and one. If \( W \) is a partitioned graphon with \( k \) parts formed by vertices of degree \( d_i \) and measure \( a_i \) each, then a decorated (rooted or non-rooted) constraint can be expressed as a non-decorated one. In other words, \( W \) satisfies the decorated constraint if and only if it satisfies the non-decorated constraint.

Proof. By the argument analogous to the non-decorated case, it is enough to show that the density of a non-rooted decorated subgraph can be expressed as a combination of densities of non-decorated subgraphs. Fix a non-rooted decorated subgraph \( H \) with vertices \( v_1, \ldots, v_n \) such that \( v_i \) is labelled with a part \( A_{i_j} \). Let \( H^U \) be the ordinary subgraph corresponding to \( H \) after removing the decorations, \( \sigma \) the type corresponding to \( H^U \) labelled with \( 1, \ldots, n \) (the vertex \( v_i \) has label \( i \)), and \( H_i \) the sum of all \( \sigma \)-flags on \( n + 1 \) vertices where the only non-rooted vertex is always adjacent to \( v_i \) (an example is given in Figure 4.7). We claim that the proportion of copies of \( H^U \) that have their vertices inside the parts according to the decoration of \( H \) is equal to

\[
\int_{\mu, \sigma} \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{k} \frac{H_i - d_j}{d_{i_j} - d_j},
\]

hence the density of \( H \) is equal to

\[
\frac{v(H)!}{|\text{Aut}(H)|} \times \left[ \prod_{i=1}^{n} \prod_{j=1, j \neq i}^{k} \frac{H_i - d_j}{d_{i_j} - d_j} \right]_{\sigma}. \tag{4.7}
\]

Indeed, if the \( n \) rooted vertices are chosen on a copy of \( H^U \) such that the \( i \)-th
rooted vertex is not from \(A_{\ell, i}\), then the second product of the above expression is zero. Otherwise, the second product is one. Hence the value of (4.7) is exactly the probability that randomly chosen \(n\) vertices induce a labelled copy of \(H^U\) such that the \(i\)-th vertex belong to \(A_{\ell, i}\).

Since a decorated constraint can be expressed by a non-decorated one, we will not distinguish between decorated and non-decorated constraints in what follows.

We finish this section with two lemmas that are straightforward corollaries of Lemma 4.6. The first one says that we can finitely force a copy of a finitely forcible graphon inside one of the parts of a partitioned graphon.

**Lemma 4.8.** Let \(W_0\) be a finitely forcible graphon. Then for every choice of \(k \in \mathbb{N}\), positive reals \(a_1, \ldots, a_k\) summing to one, distinct reals \(d_1, \ldots, d_k\) between zero and one, and \(\ell \leq k\), there exists a finite set of constraints \(C\) such that the graphon induced by the \(\ell\)-th part of every graphon \(W\) that is a partitioned graphon with \(k\) parts \(A_1, \ldots, A_k\) of measures \(a_1, \ldots, a_k\) and degrees \(d_1, \ldots, d_k\), respectively, and that satisfies \(C\) is weakly isomorphic to \(W_0\). More precisely, there exist measure preserving maps \(\varphi\) and \(\varphi'\) from \(A_\ell\) to itself such that

\[
W_0\left(\frac{\varphi(x)}{|A_\ell|}, \frac{\varphi(y)}{|A_\ell|}\right) = W\left(\varphi'(x), \varphi'(y)\right)
\]

for almost every \(x, y \in A_\ell\).

**Proof.** Assume that \(W_0\) is forced by some \(m\) constraints of the form \(p(H_i, W) = d_i\), where \(i \in [m]\) and \(H_i \in \mathcal{F}\). The set \(C\) is then formed by constraints of the form

\[
p(H'_i, W) = a_\ell^{\varphi(H_i)} d_i,
\]

where \(H'_i\) is the graph \(H_i\) with all vertices decorated with \(A_\ell\).

The second lemma asserts finite forcibility of pseudorandom bipartite graphs between different parts of a partitioned graphon.

**Lemma 4.9.** For every choice of \(k \in \mathbb{N}\), positive reals \(a_1, \ldots, a_k\) summing to one, distinct reals \(d_1, \ldots, d_k\) between zero and one, \(\ell, \ell' \leq k\), \(\ell \neq \ell'\), and \(p \in [0, 1]\), there exists a finite set of constraints \(C\) such that every graphon \(W\) that is a partitioned graphon with \(k\) parts \(A_1, \ldots, A_k\) of measures \(a_1, \ldots, a_k\) and degrees \(d_1, \ldots, d_k\), respectively, and that satisfies \(C\) also satisfies that \(W(x, y) = p\) for almost every \(x \in A_\ell\) and \(y \in A_{\ell'}\).
Proof. Let $H$ be a rooted edge with the root decorated with $A_\ell$ and the non-root decorated with $A_{\ell'}$, let $H_1$ be a triangle with two roots such that the roots are decorated with $A_\ell$ and the non-root with $A_{\ell'}$, and let $H_2$ be a cherry (a path on three vertices) with two roots on its non-edge such that the roots are decorated with $A_\ell$ and the non-root with $A_{\ell'}$. The set $\mathcal{C}$ is formed by three constraints: $H = p$, $H_1 = p^2$, and $H_2 = p^2$ (also see Figure 4.8). These constraints imply that

$$
\int_{A_{\ell'}} W(x, y) \, dy = a_{\ell'} \cdot p \quad \text{and} \quad \int_{A_{\ell'}} W(x, y) \cdot W(x', y) \, dy = a_{\ell'} \cdot p^2
$$

for almost every $x, x' \in A_\ell$. Following the reasoning given in [50, proof of Lemma 3.3], the second equation implies that

$$
\int_{A_{\ell'}} W^2(x, y) \, dy = a_{\ell'} \cdot p^2
$$

for almost every $x \in A_\ell$. Cauchy-Schwarz’s inequality yields that $W(x, y) = p$ for almost every $x \in A_\ell$ and $y \in A_{\ell'}$.

4.3 Rademacher graphon

In this section, we introduce the graphon $W_R$ which we refer to as Rademacher graphon. The name comes from the fact that the adjacencies between its parts $A$ and $C$ resemble the Rademacher system of functions on the unit interval. Note that such adjacencies also appear in [46, Example 13.30] as an example of a graphon with non-compact space of typical vertices.

The graphon $W_R$ has eight parts and we use $A, A', B, B', B''$, $C, C'$ and $D$ to denote the parts. All the parts except for $C$ have the same measure $a := 1/9$; the measure of $C$ is $2a = 2/9$.

For $x \in [0, 1)$, let us denote by $\langle x \rangle$ the smallest integer $k$ such that $x + 2^{-k} < 1$. The graphon $W_R$ is then defined as follows (also see Figure 4.9). Note that whenever
Figure 4.9: Rademacher graphon $W_R$. 
we prescribe the value of $W_R(x, y)$ for some pair $(x, y) \in [0, 1]^2$, we implicitly assume that we also defined the value $W_R(y, x) := W_R(x, y)$. Let $x$ and $y$ be two vertices of $W_R$. The value $W_R(x, y)$ is equal to one in the following cases:

- $x, y \in A$ and $\langle x/a \rangle \neq \langle y/a \rangle$,
- $x, y \in A'$ and $\langle x/a \rangle \neq \langle y/a \rangle$,
- $x \in A$, $y \in A'$ and $\langle x/a \rangle = \langle y/a \rangle$,
- $x \in A$, $y \in B$ and $x + y \leq a$,
- $x \in A$, $y \in B''$ and $x + y \geq a$,
- $x \in A'$, $y \in B''$ and $y \leq a$,
- $x, y \in C$ and $\left\lfloor \frac{y}{2a} \cdot 2^{\langle x/a \rangle} \right\rfloor$ is even, and
- $x \in A'$, $y \in C'$ and $(1 - 2^{-\langle x/a \rangle} - x/a) \cdot 2^{\langle x/a \rangle} + y/a \leq 1$.

If $x \in A'$, $y \in C$ and $\left\lfloor \frac{y}{2a} \cdot 2^{\langle x/a \rangle} \right\rfloor$ is even, then

$$W_R(x, y) := \left(1 - 2^{-\langle x/a \rangle} - x/a\right) \cdot 2^{\langle x/a \rangle}.$$

For $x, y \in C$ such that $x + y \geq 2a$, the value $W_R(x, y)$ is equal to $3/4$. If $y \in D$, then

$$W_R(x, y) := \begin{cases} 0.2 & \text{if } x \in A' \text{ or } x \in B', \\ 0.4 & \text{if } x \in B'' \text{, and} \\ 0.8 & \text{if } x \in C'. \end{cases}$$

Finally, $W_R(x, y) := 0$ if neither $(x, y)$ nor the symmetric pair fall in any of the described cases.

The degrees of vertices in the eight parts of Rademacher graphon $W_R$ are routine to compute and they are given in Table 4.1.

We finish this section with establishing that Rademacher graphon, assuming its finite forcibility, yields Theorem 4.3.

**Proposition 4.10.** The topological space $T(W_R)$ is not locally compact.

**Proof.** We understand the interval $[0, 1]$ to be partitioned by the intervals $A$, $A'$, $B$,
Let \( g : [0, 1] \to [0, 1] \) be the function defined as follows:

\[
g(x) := \begin{cases} 
1 & \text{if } x \in A' \cup B'' \cup C', \\
0.2 & \text{if } x \in D, \text{ and} \\
0 & \text{otherwise.}
\end{cases}
\]

Further, let \( g_{i, \delta} : [0, 1] \to [0, 1] \) for \( i \in \mathbb{N} \) and \( \delta \in (0, 1) \) be defined as follows:

\[
g_{i, \delta}(x) := \begin{cases} 
1 & \text{if } x \in A \text{ and } x/a = i, \\
1 & \text{if } x \in A' \text{ and } x/a \neq i, \\
1 & \text{if } x \in B' \text{ and } x \leq (1 + \delta) \cdot 2^{-i}, \\
1 & \text{if } x \in B'' \text{ and } x \leq 1 - (1 + \delta) \cdot 2^{-i}, \\
\delta & \text{if } x \in C \text{ and } \lfloor 2^i \cdot x/2a \rfloor \text{ is even,} \\
1 & \text{if } x \in C' \text{ and } x/a \leq 1 - \delta, \\
0.2 & \text{if } x \in D, \text{ and} \\
0 & \text{otherwise.}
\end{cases}
\]

Observe that \( W_R (x, 2/9 - (1 + \delta) \cdot 2^{-i}/9) = g_{i, \delta}(x) \) for every \( i \in \mathbb{N}, \delta \in (0, 1), \) and \( x \in [0, 1] \). It follows that

\[
\| g_{i, \delta} - g \|_1 = \frac{1}{9} \cdot (2 \cdot 2^{-i} + 2 \cdot (1 + \delta) \cdot 2^{-i} + 2 \cdot \delta) = \frac{(4 + 2\delta) \cdot 2^{-i} + 2\delta}{9},
\]

and

\[
\| g_{i, \delta} - g_{i', \delta'} \|_1 \geq \int_C |g_{i, \delta}(x) - g_{i', \delta'}(x)| \, dx \geq \frac{\delta + \delta'}{18} \quad \text{for } i \neq i'.
\]

Hence, since \( g_{i, \delta} \in T(W_R) \) for every \( i \in \mathbb{N} \) and \( \delta \in (0, 1) \), we obtain that \( g \in T(W_R) \).

However, for every \( \varepsilon > 0 \), all the functions \( g_{i, \varepsilon} \) with \( i > \log_2 \varepsilon^{-1} \) are at \( L_1 \)-distance at most \( \varepsilon \) from \( g \) and the \( L_1 \)-distance between any pair of them is at least \( \varepsilon/9 \). We conclude that no neighborhood of \( g \) in \( T(W) \) is compact.

### 4.4 Forcing the graphon

In this section, we prove that Rademacher graphon \( W_R \) defined in the previous section is finitely forcible. We first describe the set of constraints \( C_R \) we use to force a graphon to be weakly isomorphic to \( W_R \). We give names to the different kinds of the constraints to refer to them in our exposition.

- The **partition constraints** forcing the existence of eight parts of sizes as in \( W_R \) and with vertex degrees as in \( W_R \) (the existence of such constraints follows...
from Lemma 4.5),

- the zero constraints setting the edge density inside $B''$ and $D$ to zero as well as setting the edge density between the following pairs of parts to zero: $A$ and $C'$, $A$ and $D$, $A'$ and $B$, $B$ and $B'$, $B$ and $B''$, $B$ and $C$, $B$ and $C'$, $B$ and $D$, $B'$ and $B''$, $B'$ and $C$, $B'$ and $C'$, $B''$ and $C$, $B''$ and $C'$, and $C$ and $D$,

- the triangular constraints forcing the half graphons on $B$, $B'$, $C$, and $C'$ with densities $1$, $1$, $1$, and $3/4$ (see Lemma 4.8 and [50, Corollaries 3.15 and 5.2] for their existence), respectively,

- the pseudorandom constraints forcing the pseudorandom bipartite graph between $D$ and the parts $A'$, $B'$, $B''$, and $C'$ with densities $0.2$, $0.2$, $0.4$, and $0.8$, respectively (see Lemma 4.9 for their existence),

- the monotonicity constraints depicted in Figure 4.10,

- the split constraints depicted in Figure 4.11,

- the infinitary constraints depicted in Figure 4.12, and

- the orthogonality constraints depicted in Figure 4.13.

The existence of the corresponding monotonicity, split, infinitary, and orthogonality constraints as ordinary constraints follows from Lemma 4.6. Also note that the first five monotonicity constraints imply that the graphon has values zero and one almost everywhere between the parts $A$ and $B$, $A$ and $B''$, $A'$ and $B'$, $A'$ and $B''$, and $A'$ and $C'$ (see [50, Lemma 3.3] for further details).

Before we proceed to the proof of the main theorem of this section, let us recall a useful proposition for one-variable measurable functions called Monotone Reordering Theorem (see, e.g., [46, Proposition A.19]).
Figure 4.11: The split constraints.

Figure 4.12: The infinitary constraints.

Figure 4.13: The orthogonality constraints.
In the rest of the proof, we establish that $W$ (the existence of such maps and functions follows from Monotone Reordering Theorem): where (see Figure 4.14).

Proposition 4.11 (Monotone Reordering Theorem). For every measurable function $f : [0, 1] \to [0, \infty)$ there exist a monotone decreasing measurable function $h : [0, 1] \to [0, \infty)$ and a measure preserving map $\phi : [0, 1] \to [0, 1]$ such that $f = \phi \circ h$. The function $h$ is uniquely determined up to a set of measure zero.

We are now ready to show that Rademacher graphon is finitely forcible.

Theorem 4.12. If $W$ is a graphon satisfying all constraints in $C_R$, then there exist measure preserving maps $\varphi, \psi : [0, 1] \to [0, 1]$ such that $W^{\varphi}$ and $W^{\psi}_R$ are equal almost everywhere.

Proof. Since $W$ satisfies the partition constraints contained in $C_R$, Lemma 4.5 yields that the interval $[0, 1]$ can be partitioned into eight parts all but one having measure 1/9 and the remaining one with measure 2/9 such that almost all vertices in the parts have degrees as those in the corresponding parts of $W_R$. In particular, there exists a measure preserving map $\varphi : [0, 1] \to [0, 1]$ such that the subintervals of $[0, 1]$ corresponding to the parts of $W_R$ are mapped to the corresponding parts of $W$. From now on, we use $A, A', B, B', B'', C, C'$, and $D$ for the subintervals of $[0, 1]$ corresponding to the parts.

We next construct a measure preserving map $\psi$ consisting of measure preserving maps on the intervals $A, A', B, B', B'', C$ and $C'$. We choose these maps such that there exist decreasing functions $f_A : A \to [0, 1]$ and $f_{A'} : A' \to [0, 1]$, and increasing functions $f_B : B \to [0, 1]$, $f_{B'} : B' \to [0, 1]$, $f_{B''} : B'' \to [0, 1]$, $f_C : C \to [0, 1]$ and $f_{C'} : C' \to [0, 1]$ such that the following holds almost everywhere (the existence of such maps and functions follows from Monotone Reordering Theorem):

$$\forall x \in A \quad f_A(\psi(x)) = \int_B W^{\varphi}(x, y) \, dy$$  
$$\forall x \in A' \quad f_{A'}(\psi(x)) = \int_{B'} W^{\varphi}(x, y) \, dy$$  
$$\forall x \in B \quad f_B(\psi(x)) = \int_B W^{\varphi}(x, y) \, dy$$  
$$\forall x \in B' \quad f_{B'}(\psi(x)) = \int_{B'} W^{\varphi}(x, y) \, dy$$  
$$\forall x \in B'' \quad f_{B''}(\psi(x)) = \int_A W^{\varphi}(x, y) \, dy$$  
$$\forall x \in C \quad f_C(\psi(x)) = \int_C W^{\varphi}(x, y) \, dy$$  
$$\forall x \in C' \quad f_{C'}(\psi(x)) = \int_{C'} W^{\varphi}(x, y) \, dy$$  

In the rest of the proof, we establish that $W^{\varphi}$ and $W^{\psi}_R$ are equal almost everywhere.

The pseudorandom and zero constraints in $C_R$ imply that $W^{\varphi}$ and $W^{\psi}_R$ agree almost everywhere on $D \times [0, 1]$ and $[0, 1] \times D$. The zero and triangular constraints and the choice of $\psi$ on $B, B', C$, and $C'$ yield the same conclusion for $(B \cup B' \cup B'' \cup C \cup C')^2, A \times B', B' \times A, A \times C, C \times A, A' \times B$, and $B \times A'$ (see Figure 4.14).
Figure 4.14: The forced structure of \( W \) after the first step of the proof of Theorem 4.12. Question marks denote the parts with no forced structure so far.

Let us now introduce some additional notation. If \( x \) is a vertex and \( Y \) is one of the parts, let \( N_Y(x) \) denote the set of \( y \in Y \) such that \( W^\varphi(x, y) > 0 \). If \( x \) and \( y \) belong to the same part, then we write \( x \preceq y \) iff \( \psi(x) \leq \psi(y) \). Observe that the monotonicity constraint (a) from Figure 4.10 and the choice of \( \psi \) implies the existence of a set \( Z \) of measure zero such that \( N_B(x') \setminus N_B(x) \) has measure zero for \( x, x' \in A \setminus Z \) if and only if \( x \preceq x' \). Since the degree of every vertex in \( B \) is \( 1/9 \), this yields that the graphons \( W^\varphi \) and \( W^\psi_R \) agree almost everywhere on \( A \times B \). The same reasoning applies to \( A' \) and \( B' \). Thus, we conclude that the graphons \( W^\varphi \) and \( W^\psi_R \) agree almost everywhere on \( (A \cup A') \times (B \cup B') \) and \( (B \cup B') \times (A \cup A') \).

We now apply the same reasoning using the monotonicity constraint (b) and the split constraints (b) to deduce the existence of a zero measure set \( Z \) such that \( N_{B''}(x) \setminus N_{B''}(x') \) has measure zero if and only if \( x \preceq x' \) for \( x, x' \in A \setminus Z \). The monotonicity constraint also imply that \( W^\varphi \) has only values zero and one almost everywhere on \( A \times B'' \). Since the measure of \( N_B(x) \cup N_{B''}(x) \) is \( 1/9 \) for almost all \( x \in A \) by the split constraint (b), the choice of \( \psi \) on \( B'' \) implies that the graphons \( W^\varphi \) and \( W^\psi_R \) agree almost everywhere on \( A \times B'' \). The degree regularity in \( B'' \), the split constraint (d), and the monotonicity constraint (d), which yields that \( W^\varphi \) has values zero and one almost everywhere on \( A' \times B'' \), yield the agreement almost everywhere on \( A' \times B'' \). Symmetrically, they agree almost everywhere on \( B'' \times (A \cup A') \). The forced structure of \( W \) forced so far is depicted in Figure 4.15.

We now focus on the graphon \( W^\varphi \) on \( A^2 \). Observe first that the measure of
Figure 4.15: The forced structure of $W$ after the second step of the proof of Theorem 4.12. Again, question marks denote the parts with no forced structure so far.

$N_B(x)$ is equal to $\psi(x)$ for almost all $x \in A$. The monotonicity constraints (f) and (h) from Figure 4.10 imply that there exists a set $Z$ of measure zero such that every point $x \in A \setminus Z$ can be associated with a unique open interval $J_x \subseteq A$ such that $W^c(x, x') = 0$ for almost every $x' \in \psi^{-1}(J_x)$, and $W^c(x, x') = 1$ for almost every $x' \in A \setminus \psi^{-1}(J_x)$. The interval $J_x$ can be empty for some choice of $x$. Recall that $|J_x|$ is the measure of the interval $J_x$, and let $\mathcal{J}$ be the set of all intervals $J_x$, $x \in A$, with $|J_x| > 0$. Since the intervals in $\mathcal{J}$ are disjoint, the set $\mathcal{J}$ is equipped with a natural linear order.

Let us now focus on the infinitary constraint (b) from Figure 4.12. Fix three vertices (two from $A$ and one from $B$) as in the figure and let $x$ be the left vertex from $A$. Observe that if $x \in A$ is fixed, then the set of choices of the other two vertices has non-zero measure unless $\psi(x) = \sup J_x$. The left hand side of the constraint is equal to the measure of $J_x$, i.e., $\sup J_x - \inf J_x$. The right hand side is equal to $1/9 - \sup J_x$. We conclude that $\inf J_x = 1/9 - 2|J_x|$. This implies that the set $\mathcal{J}$ is well-ordered and countable.

Let us write $J_k$ for the $k$-th interval contained in $\mathcal{J}$. Furthermore, for $k \geq 1$, define

$$\beta_k = \frac{2(1 - 9 \inf J_{k+1})}{1 - 9 \inf J_k} = \frac{2|J_{k+1}|}{|J_k|},$$

and let $\beta_0$ be equal to $1 - 9 \inf J_1$. Note that by the observations made in the last paragraph and since $\inf J_{k+1} \geq \sup J_k$, we obtain $\beta_k \leq 1$ for every $k \geq 0$. In case
that \( \mathcal{J} \) is finite, we define \( \beta_k = 0 \) for \( k > |\mathcal{J}| \). We can now express the density of non-edges with both end-vertices in \( A \) as

\[
\sum_{j \in \mathcal{J}} |J|^2 = \sum_{k=1}^{\infty} \left( \frac{1}{9} \cdot \frac{1}{2^k} \prod_{k'=0}^{k-1} \beta_{k'} \right)^2.
\]

Since the sum is forced to be 1/243 by the infinitary constraint (a), we get that \( \beta_k = 1 \) for every \( k \). This implies that for every \( k \), \( J_k = \left( \frac{1}{9} \cdot \frac{1}{2^k}, \frac{1}{9} \cdot \frac{2^{k+1}}{9} \right) \). In particular, the graphons \( W^\varphi \) and \( W_R^\psi \) agree almost everywhere on \( A^2 \).

The same reasoning as for \( A^2 \) yields that the graphons \( W^\varphi \) and \( W_R^\psi \) agree almost everywhere on \( A' \). Let \( \mathcal{J}' \) be the corresponding set of intervals for \( A' \) and let \( J_1', J_2', \ldots \) be their ordering. The split constraints (e) and (f) from Figure 4.11 imply that for almost every \( x \in A \) with \( |N_{A'}(x)| > 0 \), there exists \( J' \in \mathcal{J}' \) such that 
\( N_{A'}(x) \Delta \psi^{-1}(J') \) has measure zero and \( W^\varphi(x, y) = 1 \) for almost every \( y \in \psi^{-1}(J') \).

Let \( x \in \psi^{-1}(J_k) \). The split constraint (a) from Figure 4.11 yields that \( |N_{A'}(x)| = \frac{1}{9} \). Consequently, \( N_{A'}(x) \Delta \psi^{-1}(J_k) \) has measure zero for almost every \( x \in \psi^{-1}(J_k) \) and \( W(x, x') = 1 \) for almost every \( x \in \psi^{-1}(J_k) \) and \( x' \in \psi^{-1}(J'_k) \). We conclude that the graphons \( W^\varphi \) and \( W_R^\psi \) agree almost everywhere on \( A \times A' \) and \( A' \times A \).

The orthogonality constraints (a) and (b) from Figure 4.13 yield that there exist measurable subsets \( I_k \subseteq C \) with \( |I_k| = 1/9 \) for every \( k \geq 1 \) such that it holds for almost every \( x \in \psi^{-1}(J_k) \) that \( N_C(x) \) differs from \( I_k \) on a set of measure zero and \( W^\varphi(x, y) = 1 \) for almost every \( y \in I_k \). The construction of \( \psi \) and the split constraint (h) from Figure 4.11 imply that \( |N_A(x)| = 1/9 - \psi(x)/2 \) for almost every \( x \in C \). Since \( \psi^{-1}(J_1) \setminus N_A(x) \) has measure zero for almost every \( x \in I_1 \), we get that \( |J_1| \leq |N_A(x)| \) for almost every \( x \in I_1 \). This implies that \( I_1 \) and \( \psi^{-1}([0, 1/9]) \) differ on a set of measure zero (also see Figure 4.16). Since \( \psi^{-1}(J_2) \setminus N_A(x) \) has measure zero for almost every \( x \in J_2 \) and \( J_1 \cap J_2 \) has measure zero, we get that \( |J_1| + |J_2| \leq |N_A(x)| \) for almost every \( x \in I_1 \cap I_2 \) and that \( |J_2| \leq |N_A(x)| \) for almost every \( x \in J_2 \setminus I_1 \). This implies that \( I_2 \) and \( \psi^{-1}([0, 1/18] \cup [1/9, 1/6]) \) differ on a set of measure zero. Iterating the argument, we obtain that \( I_k \) differs from the preimage with respect to \( \psi \) of the set

\[
\bigcup_{i=1}^{2^{k-1} - 1} \left[ \frac{2i - 2}{9 \cdot 2^{k-1}}, \frac{2i - 1}{9 \cdot 2^{k-1}} \right]
\]

on a set of measure zero for every \( k \in \mathbb{N} \). This yields that the graphons \( W^\varphi \) and \( W_R^\psi \) agree almost everywhere on \( A \times C \).
Figure 4.16: Illustration of the argument used in the proof of Theorem 4.12 to establish that the graphons $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $A \times C$.

The orthogonality constraint (c) from Figure 4.13 implies that $(C \setminus N_C(x)) \cap N_C(x')$ has measure zero for every $k$, almost every $x \in A \setminus \psi^{-1}(J_k)$, and almost every $x' \in \psi^{-1}(J'_k)$. In particular, almost every $x' \in \psi^{-1}(J'_k)$ satisfies that $N_C(x') \setminus I_k$ has measure zero, i.e., $W^\varphi(x', y) = 0$ for almost every $x' \in \psi^{-1}(J'_k)$ and $y \not\in I_k$.

We now interpret the orthogonality constraint (d) from Figure 4.13. Fix an integer $k \geq 1$ and a typical vertex $x' \in \psi^{-1}(J'_k)$. The left term in the product on the left hand side of the constraint is equal to the square of

$$\int_C W^\varphi(x', y) \, dy = \int_{I_k} W^\varphi(x', y) \, dy.$$ 

The right term in the product is equal to the square of $|J'_k| = 2^{-\langle \psi(x')/a \rangle} / 9$. The term on the right hand side is equal to the probability that randomly chosen $x''$ and $y$ satisfy $x'' \in A'$, $y \in B'$, $x'' \in \psi^{-1}(J'_k)$, and $\psi(x') \leq \psi(y) < \psi(x'')$. This is equal to

$$\frac{\left(1 - 2^{-\langle \psi(x')/a \rangle} - \frac{\psi(x')}{a} \right)^2}{2 \cdot 9^2}.$$

We deduce that almost every $x' \in \psi^{-1}(J'_k)$ satisfies

$$\int_{I_k} W^\varphi(x', y) \, dy = \frac{1 - 2^{-\langle \psi(x')/a \rangle} - \frac{\psi(x')}{a}}{9 \cdot 2^{-\langle \psi(x')/a \rangle}}. \quad (4.13)$$

We apply the same reasoning to the orthogonality constraint (e) from Figure 4.13.
and deduce that almost every pair of vertices $x', x'' \in \psi^{-1}(J'_k)$ satisfies
\[
\frac{92}{1} \cdot \left( \int_{I_k} W^\varphi(x', y) W^\varphi(x'', y) \, dy \right)^2 \cdot \left( 2^{-\langle \psi(x')/a \rangle} \right)^4 \cdot \frac{(1 - 2^{-\langle \psi(x')/a \rangle} - \psi(x')/a)^2}{2} \cdot \frac{(1 - 2^{-\langle \psi(x'')/a \rangle} - \psi(x'')/a)^2}{2}.
\]
This implies (similarly as in the proof of Lemma 4.9) that almost every $x' \in \psi^{-1}(J'_k)$ satisfies:
\[
\left( \int_{I_k} W^\varphi(x', y)^2 \, dy \right)^{1/2} = 1 - 2^{-\langle \psi(x'/a) \rangle} - \psi(x')/a \cdot \frac{3 \cdot 2^{-(\psi(x'/a)}}{2}. \tag{4.14}
\]
Using Cauchy-Schwartz Inequality, we deduce from (4.13) and (4.14) (recall that $|I_k| = 1/9$) that the following holds for almost every $x' \in \psi^{-1}(J'_k)$ and $y \in I_k$,
\[
W^\varphi(x', y) = \frac{1 - 2^{-\langle \psi(x'/a) \rangle} - \psi(x')/a}{2^{-(\psi(x'/a)}}.
\]
In other words, $W^\varphi(x', y)$ is constant almost everywhere on $I_k$ for almost every $x' \in \psi^{-1}(J'_k)$ and its value linearly decreases from one to zero almost everywhere inside $\psi^{-1}(J'_k)$. Hence, the graphons $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $A' \times C$ and $C \times A'$ (recall that $W^\varphi(x', y) = 0$ for almost every pair $x' \in \psi^{-1}(J'_k)$ and $y \notin I_k$).

The monotonicity constraint (e) from Figure 4.10 yields that at least one of the sets $N_{C'}(x) \setminus N_{C'}(x')$ or $N_{C'}(x') \setminus N_{C'}(x)$ has measure zero for every $k$ and almost every pair $x, x' \in A'$, and also that the graphon $W^\varphi$ has values zero and one almost everywhere on $A' \times C'$. This together with the regularity on $A'$ and $C'$ imply that the graphons $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $A' \times C'$. We have shown that the graphons $W^\varphi$ and $W^\psi_R$ agree almost everywhere on $(A \cup A') \times (C \cup C')$ and $(C \cup C') \times (A \cup A')$. Since these were the last subsets of their domains that remained to be analyzed, we proved that the graphon $W^\varphi$ is equal to $W^\psi_R$ almost everywhere.

Theorem 4.12 immediately yields the following.

**Corollary 4.15.** The graphon $W_R$ is finitely forcible.
Appendix A

Supplementary computations for Chapter 2

A.1 Matrices $I_1, I_2, I_3$ and $I_4$ from proof of Theorem 2.20

In this appendix, we display the matrices $I_1, I_2, I_3$ and $I_4$ that appear in the proof of Theorem 2.20. For the matrices $I_2, I_3$ and $I_4$, we also present the appropriate linear combinations of their rows that were used in the proof of Claim 1.

We start with the matrix $I_1$. It has size $8 \times 7$ and corresponds to the type $\sigma_1 = 2$.

$$I_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{3}{19} & \frac{37}{19} & 0 & 0 & 0 & 0 & -\frac{1}{19} \\
-\frac{2}{19} & -\frac{3}{19} & \frac{38}{19} & 0 & 0 & 0 & -\frac{1}{19} \\
-\frac{3}{19} & -\frac{1}{19} & -\frac{1}{19} & \frac{39}{19} & 0 & 0 & -\frac{1}{19} \\
-\frac{2}{19} & -\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & \frac{40}{19} & 0 & 0 \\
-\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & \frac{12}{19} & -\frac{1}{19} \\
-\frac{3}{19} & -\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & \frac{12}{19} \\
-\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & -\frac{1}{19} & \frac{12}{19}
\end{pmatrix}$$

The matrix $I_2$ has size $26 \times 38$. It corresponds to the type $\sigma_2$, which is the 4-vertex type with no edges. Unfortunately, the whole matrix is too large to be fully displayed here. Therefore, we display its transpose $I_2^T$ decomposed into 3 block in the following way:

$$I_2^T = \begin{pmatrix}
I_{2A} & I_{2B} & I_{2C}
\end{pmatrix},$$

where $I_{2A}, I_{2B}$ and $I_{2C}$ are the corresponding blocks of sizes $38 \times 10$, $38 \times 7$ and $38 \times 9$, respectively.

Now let us move to the linear combination of the rows of $I_2$ used in the proof Claim 1. Let $r_1, \ldots, r_8$ be the first, 7th, 14th, 19th, 20th, 23rd, 24th and the last row of $I_2$, respectively. In other words, $r_1$ is the first column and $r_2$ is the 7th column of the matrix $I_{2A}$, $r_3$ is the 4th column of $I_{2B}$, and $r_4, \ldots, r_8$ are the second, third,
6th, 7th and 9th column of $I_{2C}$, respectively. Next, let $w_1, \ldots, w_8$ be the following 8 rationals:

\[
\begin{align*}
    w_1 &:= -\frac{17848347258759844}{667105311709433}, & w_2 &:= -\frac{2907703547217221}{100732902068124383}, \\
    w_3 &:= -\frac{182634053866072}{667105311709433}, & w_4 &:= -\frac{20013159574516}{667105311709433}, \\
    w_5 &:= -\frac{360}{20013159574516}, & w_6 &:= -\frac{1334210623418866}{667105311709433}, \\
    w_7 &:= -\frac{352}{20013159574516}, & w_8 &:= -\frac{21227849574516}{667105311709433}.
\end{align*}
\]

Recall that the last six coordinates of the vector $x_2$ are the ones that correspond to six $\sigma_2$-flags on five vertices containing a tight path from $\{a, b\}$ to $\{c, d\}$. The vector $q_2 := \sum_{i \in [8]} w_i \cdot r_i$ is a 38-dimensional vector with rational entries such that the last six coordinates of $q_2$ are positive and all the other ones are negative. Note that there is no linear combination of (at most) 7 rows from $I_2$ with such a property. The vector $q_2$ is also generated and then verified in the script “theorem_2_22-verify.sage”.

The next matrix is $I_3$. It has size $15 \times 17$ and corresponds to the type $\sigma_3$, the 4-vertex type that contains a single edge $abc$. For brevity, we display the transpose $I_3^T$ of $I_3$ instead of $I_3$ itself. Let $r_1$ be the 13th and $r_2$ the 15th row of $I_3$, i.e., the 13th and 15th column of $I_3^T$. The vector $q_3 := 6 \cdot r_2 - 5 \cdot r_1$ has the desired property that its first 11 entries are negative. We write the last six coordinates of the vector $q_3$ in a **bold** font.

Finally, the last matrix is $I_4$. It corresponds to the type $\sigma_4$, which is the 4-vertex type with the edge-set $\{abc, abd\}$. Like the matrix $I_3$, the matrix $I_4$ also has size $15 \times 17$ and again, we display rather its transpose $I_4^T$. The sum of the 13th and the 15th row of $I_4$, i.e., the 13th and the 15th column of $I_4^T$, is equal to the vector $q_4$. One more time, since the last six coordinates of the vector $x_4$ correspond to the $\sigma_4$-flags on five vertices that contain a tight path from $\{a, b\}$ to $\{c, d\}$, we just need to check that that $q_4$ has positive values only on the last six coordinates. Note that the last six coordinates of $q_4$ are again written in a **bold** font.
\[ q_3 = \begin{pmatrix} -66 \\ -1225 \\ 44 \\ -1225 \\ 11 \\ -1225 \\ 11 \\ -1225 \\ 11 \\ -1225 \\ 11 \\ -1225 \\ -22 \\ -1225 \\ 11 \\ -1225 \\ 11 \\ -1225 \\ +136 \\ 25 \\ -1225 \end{pmatrix} \]

\[ q_4 = \begin{pmatrix} -99 \\ -1225 \\ -27 \\ -7350 \\ 13 \\ -3075 \\ 13 \\ -3075 \\ -99 \\ -1225 \\ -13 \\ -3075 \\ -271 \\ -7350 \\ -271 \\ -7350 \end{pmatrix} \]

\[ I_T^2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]
\[
I_5 = \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
I_{2A} = \\
\begin{bmatrix}
-20 & -5 & -3710 & -920 & -4990 & 525 \\
0 & 1291 & 1291 & 1291 & 1291 & 1291 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
\[
I_{2N} =
\]

\[
\begin{pmatrix}
-204315 & -47499 & 1895873 & -3703870 & -1883725 & -10841505 & -149105 \\
-204315 & -292074 & 5392744 & -2588874 & -2958977 & 10241639 & -561085 \\
292074 & 5392744 & -10881876 & 2588874 & 2958977 & -10241639 & 561085 \\
-2588874 & 2588874 & -27468159 & -2588874 & 2958977 & 10241639 & -561085 \\
-2958977 & 2958977 & 2958977 & -10881876 & 2588874 & -2588874 & 561085 \\
10241639 & -10241639 & 10241639 & 10241639 & -27468159 & 2588874 & -2958977 \\
-561085 & 561085 & -561085 & -561085 & -561085 & 2588874 & -2958977 \\
42811395 & 42811395 & 42811395 & 42811395 & 42811395 & 42811395 & -14270465 \\
14270465 & 14270465 & 14270465 & 14270465 & 14270465 & 14270465 & -14270465 \\
42811395 & 42811395 & 42811395 & 42811395 & 42811395 & 42811395 & -14270465 \\
14270465 & 14270465 & 14270465 & 14270465 & 14270465 & 14270465 & -14270465 \\
42811395 & 42811395 & 42811395 & 42811395 & 42811395 & 42811395 & -14270465 \\
14270465 & 14270465 & 14270465 & 14270465 & 14270465 & 14270465 & -14270465 \\
42811395 & 42811395 & 42811395 & 42811395 & 42811395 & 42811395 & -14270465 \\
14270465 & 14270465 & 14270465 & 14270465 & 14270465 & 14270465 & -14270465 \\
\end{pmatrix}
\]
\[
I_{2c} = \begin{pmatrix}
-474 & 100 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 \\
-213170 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 \\
-2682272 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 \\
-58492 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 \\
-6941 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 \\
-124 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 \\
-460 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 \\
-26 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 & 15081675 \\
\end{pmatrix}
\]
Appendix B

Supplementary computations for Chapter 3

B.1 Proof of Theorem 3.20

Up to a few changes in the choice of the parameters, we follow the lines of the proof of Theorem 3.9 presented in Section 3.3. As in Section 3.2, Theorem 3.19 has the following two corollaries.

**Corollary B.1.** For every positive $\varepsilon_{ASYM}$ there exists $n_{ASYM} \in \mathbb{N}$ such that the following is true. If $G$ is a permutation graph on $n \geq n_{ASYM}$ vertices, then $f_5(G) > (1/256 - \varepsilon)$.

**Corollary B.2.** For every positive $\varepsilon_{CONF}$ there exist a positive $\delta_{CONF}$ and $n_{CONF} \in \mathbb{N}$ such that the following is true. If $G$ is a permutation graph on $n \geq n_{CONF}$ vertices that satisfies $f_5(G) \leq (1/256 + \delta_{CONF})$, then $G$ contains at most $\varepsilon_{CONF} \cdot \binom{n}{7}$ induced copies of $F$ and at most $\varepsilon_{CONF} \cdot \binom{n}{7}$ induced copies of $\overline{F}$, where $F \in \mathcal{E}_{57}$.

**Proof of Theorem 3.20.** Let $C := 49 \times 140 = 6860$. Recall $X$ is the set of all non-permutation graphs. We do the analogous set-up of the parameters as in Theorem 3.9. Specifically, let $\delta_{RL}$ be the value from Infinite Removal Lemma applied for $\varepsilon_{RL} := (\varepsilon_{STAB})^2 / C$ and the family $X \cup \mathcal{E}_7$, let $\delta_{CONF}$ and $n_{CONF}$ be the values from Corollary B.2 applied for $\varepsilon_{CONF} := \delta_{RL}$, and let $n_{ASYM}$ be the value from Corollary B.1 for $\varepsilon_{ASYM} := (\varepsilon_{STAB})^2 / C$. We set $\delta_{STAB} := \min\{(\varepsilon_{STAB})^2 / C, \delta_{CONF}\}$ and $n_{STAB} := \max\{C / \varepsilon_{STAB}, n_{CONF}, 2 \cdot n_{ASYM}\}$.

Let $G$ be a graph on $n$ vertices satisfying the assumptions of the theorem. Corollary B.2 and Infinite Removal Lemma implies that we can add or remove less than $\frac{1}{7} \cdot \varepsilon_{STAB} \cdot \binom{n}{7}$ edges in $G$ and obtain a permutation graph $G'$ that satisfies Property A(5) and Property B. Furthermore, $f_5(G') < 1/256 + 2 \cdot \varepsilon_{STAB}^2 / C$. Hence, by Lemma 3.7, we can partition $G'$ into at most 4 parts such that either each part is a clique and there are no edges between the parts, or every edge between the
parts is present and the parts themselves form an independent set. Without loss of
generality, \( G' \) is a disjoint union of at most 4 cliques.

It is enough to show that every clique has size at most \((1/4 + \varepsilon_{\text{STAB}}/7) \cdot n\). Let
\( \gamma := \varepsilon_{\text{STAB}}/7 \) and suppose for a contradiction \( G' \) contains a clique of size more than
\((1/4 + \gamma) \cdot n\). Let \( H_0 := G' \). For each \( i \in [\gamma \cdot n/3] \), let \( v_i \) be an arbitrary vertex from a
maximum clique inside \( H_{i-1} \), and let \( H_i := H_{i-1} - v_i \). Let \( Z := \{v_1, v_2, \ldots, v_{\gamma \cdot n/3}\} \).
It follows that every vertex \( v \in Z \) is contained in at least
\( \left( \frac{1}{256} + 2 \cdot \frac{\gamma}{32} \right) \cdot \frac{n^4}{24} \).

Furthermore,
\[
\frac{1}{256} + 2 \cdot \left( \frac{\varepsilon_{\text{STAB}}}{C} \right) \geq f_5(G') = f_5(H_0) > f_5(H_{\gamma \cdot n/3}) + \frac{\gamma^2}{36} \geq \frac{1}{256} \cdot \frac{(\varepsilon_{\text{STAB}})^2}{C} + \frac{(\varepsilon_{\text{STAB}})^2}{49 \times 36},
\]
a contradiction.

B.2 Proof of Theorem 3.23

Again, we adapt the proof of Theorem 3.9. Analogously to Theorems 3.4 and 3.19,
Theorem 3.22 has the following two corollaries.

Corollary B.3. For every positive \( \varepsilon_{\text{ASYM}} \) there exists \( n_{\text{ASYM}} \in \mathbb{N} \) such that the
following is true. If \( G \) is a permutation graph on \( n \geq n_{ASYM} \) vertices, then \( f_6(G) > \left( \frac{1}{3125} - \varepsilon \right) \).

Corollary B.4. For every positive \( \varepsilon_{CONF} \) there exist a positive \( \delta_{CONF} \) and \( n_{CONF} \in \mathbb{N} \) such that the following is true. If \( G \) is a permutation graph on \( n \geq n_{CONF} \) vertices that satisfies \( f_6(G) \leq \left( \frac{1}{3125} + \delta_{CONF} \right) \), then \( G \) contains at most \( \varepsilon_{CONF} \cdot \left( \frac{n}{8} \right)^5 \) induced copies of \( F \) and at most \( \varepsilon_{CONF} \cdot \left( \frac{n}{8} \right)^5 \) induced copies of \( \overline{F} \), where \( F \in \mathcal{E}^6_8 \).

Proof of Theorem 3.23. We let \( C := 81 \times 900 = 72900 \) this time. Again, \( \mathcal{X} \) is the set of all non-permutation graphs. We do the following set-up of the parameters. Let \( \delta_{RL} \) be the value from Infinite Removal Lemma applied for \( \varepsilon_{RL} := \left( \frac{\varepsilon_{STAB}}{2} \right)^2 \) and the family \( \mathcal{X} \cup \mathcal{E}^6_8 \), let \( \delta_{CONF} \) and \( n_{CONF} \) be the values from Corollary B.4 applied for \( \varepsilon_{CONF} := \delta_{RL} \), and let \( n_{ASYM} \) be the value from Corollary B.3 for \( \varepsilon_{ASYM} := \left( \frac{\varepsilon_{STAB}}{2} \right)^2 \). We set \( \delta_{STAB} := \min \{ \left( \frac{\varepsilon_{STAB}}{2} \right)^2 / C, \delta_{CONF} \} \) and \( n_{STAB} := \max \{ C / \varepsilon_{STAB}, n_{CONF}, 2 \cdot n_{ASYM} \} \).

Let \( G \) be a graph on \( n \) vertices satisfying the assumptions of the theorem. Corollary B.4 and Infinite Removal Lemma implies that \( G \) is in edit distance less than \( \frac{1}{9} \cdot \varepsilon_{STAB} \cdot \left( \frac{n}{2} \right)^5 \) to a permutation graph \( G' \) that satisfies Property A(6) and Property B. Furthermore, \( f_6(G') < \frac{1}{3125} + 2 \cdot \varepsilon_{STAB}^2 / C \). By Lemma 3.7, there is a partition of \( G' \) into at most 5 parts such that either each part is a clique and there are no edges between the parts, or every edge between the parts is present and the parts themselves form an independent set. Without loss of generality, \( G' \) is a disjoint union of at most 5 cliques.

It is enough to show that every clique has size at most \( \left( \frac{1}{5} + \varepsilon_{STAB} / 9 \right) \cdot n \). Let \( \gamma := \varepsilon_{STAB} / 9 \) and suppose for a contradiction \( G' \) contains a clique of size more than \( \left( \frac{1}{5} + \gamma \right) \cdot n \). Again, let \( H_0 := G' \). For each \( i \in [\gamma \cdot n / 3] \), let \( v_i \) be an arbitrary vertex from a maximum clique inside \( H_{i-1} \), and let \( H_i := H_{i-1} - v_i \). Let \( \gamma := \{ v_1, v_2, \ldots, v_{\gamma \cdot n / 3} \} \). It follows that every vertex \( v \in Z \) is contained in at least \( (1/3125 + 2 \cdot \gamma / 3 \cdot n) \) copies of \( K_6 \) that are disjoint from \( Z \setminus \{ v \} \). Since \( n_{STAB} > 24 / \gamma = 216 / \varepsilon_{STAB} \), we have

\[
\left( \frac{(1/5 + 2 \cdot \gamma / 3 \cdot n)}{5} \right) > \left( \frac{1}{5} + \frac{\gamma}{2} \right)^5 \cdot \frac{n^5}{120} > \left( \frac{1}{3125} + \frac{\gamma}{250} \right) \cdot \frac{n^5}{120}.
\]

Furthermore,

\[
f_6(H_{i+1}) \leq f_6(H_i) \leq f_6(G') \leq \frac{1}{3125} + 2 \cdot \frac{(\varepsilon_{STAB})^2}{C} = \frac{1}{3125} + 2 \cdot \frac{\gamma^2}{900}.
\]

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Our aim is to show that $f_6(H_i) - f_6(H_{i+1}) > \gamma/(100 \cdot n)$. Indeed, we have

$$f_6(H_i) - f_6(H_{i+1}) > \left( \frac{1}{3125} + \frac{\gamma}{250} \right) \cdot \frac{n^5}{120} + f_6(H_{i+1}) \cdot \frac{v(H_i - 1)}{\binom{v(H_i)}{6}} - f_6(H_{i+1}) \cdot \frac{v(H_i)}{\binom{v(H_i)}{6}}$$

$$> \left( \frac{1}{3125} + \frac{\gamma}{250} \right) \cdot \frac{n^5}{120} - f_6(H_{i+1}) \cdot \frac{v(H_i - 1)}{\binom{v(H_i)}{6}}$$

$$> \frac{6}{n} \cdot \left( \frac{\gamma}{250} - \frac{\gamma}{450} \right) \geq \frac{\gamma}{100n}.$$  

Corollary B.1 implies that $f_6(H_{\gamma n/3}) \geq 1/3125 - (\varepsilon_{\text{STAB}})^2/C$. Therefore,

$$\frac{1}{3125} + 2 \cdot \frac{\varepsilon_{\text{STAB}}^2}{C} \geq f_6(G') > \frac{1}{3125} - \frac{(\varepsilon_{\text{STAB}})^2}{C} + \frac{(\varepsilon_{\text{STAB}})^2}{81 \times 300},$$

during which $a$ contradiction.

\[\square\]
B.3 The bijections $\tau_{i,j}$ from the proof of Theorem 3.22

- $\tau_{1,1} = 12$: $b_{1,1} = ab$
- $\tau_{1,2} = 21$: $b_{1,2} = ab$
- $\tau_{2,1} = 1234$: $b_{2,1} = abcd$
- $\tau_{2,2} = 4321$: $b_{2,2} = abcd$
- $\tau_{3,1} = 1243$: $b_{3,1} = abcd$
- $\tau_{3,2} = 1324$: $b_{3,2} = abcd$
- $\tau_{3,3} = 2134$: $b_{3,3} = cdab$
- $\tau_{3,4} = 3421$: $b_{3,4} = cdab$
- $\tau_{4,1} = 3241$: $b_{4,1} = cdab$
- $\tau_{4,2} = 4321$: $b_{4,2} = cdab$
- $\tau_{4,3} = 4132$: $b_{4,3} = adbc$
- $\tau_{4,4} = 2143$: $b_{4,4} = adbc$
- $\tau_{4,5} = 3124$: $b_{4,5} = dabc$
- $\tau_{4,6} = 3241$: $b_{4,6} = dabc$
- $\tau_{4,7} = 4312$: $b_{4,7} = dabc$
- $\tau_{5,1} = 1432$: $b_{5,1} = abcd$
- $\tau_{5,2} = 2341$: $b_{5,2} = abcd$
- $\tau_{5,3} = 3214$: $b_{5,3} = abcd$
- $\tau_{5,4} = 4123$: $b_{5,4} = abcd$
- $\tau_{5,5} = 5213$: $b_{5,5} = abcd$
- $\tau_{6,1} = 2431$: $b_{6,1} = abcd$
- $\tau_{6,2} = 3124$: $b_{6,2} = abcd$
- $\tau_{6,3} = 3241$: $b_{6,3} = abcd$
- $\tau_{6,4} = 4312$: $b_{6,4} = abcd$
- $\tau_{6,5} = 5124$: $b_{6,5} = abcd$
- $\tau_{7,1} = 2431$: $b_{7,1} = abcd$
- $\tau_{7,2} = 3142$: $b_{7,2} = abcd$
- $\tau_{7,3} = 4321$: $b_{7,3} = abcd$
- $\tau_{7,4} = 2341$: $b_{7,4} = abcd$
- $\tau_{7,5} = 3142$: $b_{7,5} = abcd$
- $\tau_{7,6} = 4321$: $b_{7,6} = abcd$
- $\tau_{8,1} = 123456$: $b_{8,1} = abcdef$
- $\tau_{8,2} = 456321$: $b_{8,2} = abcdef$
- $\tau_{8,3} = 134562$: $b_{8,3} = abcdef$
- $\tau_{8,4} = 163542$: $b_{8,4} = abcdef$
- $\tau_{8,5} = 243156$: $b_{8,5} = abcdef$
- $\tau_{8,6} = 324156$: $b_{8,6} = abcdef$

- $\tau_{10,1} = 135264$: $b_{10,1} = abedef$
- $\tau_{10,2} = 142635$: $b_{10,2} = abedef$
- $\tau_{10,3} = 241536$: $b_{10,3} = abedef$
- $\tau_{10,4} = 315246$: $b_{10,4} = abedef$
- $\tau_{10,5} = 462531$: $b_{10,5} = abedef$
- $\tau_{10,6} = 536241$: $b_{10,6} = abedef$
- $\tau_{10,7} = 635142$: $b_{10,7} = abedef$
- $\tau_{10,8} = 642513$: $b_{10,8} = abedef$
- $\tau_{11,1} = 136425$: $b_{11,1} = abedef$
- $\tau_{11,2} = 152463$: $b_{11,2} = abedef$
- $\tau_{11,3} = 263415$: $b_{11,3} = abedef$
- $\tau_{11,4} = 436251$: $b_{11,4} = abedef$
- $\tau_{11,5} = 413526$: $b_{11,5} = abedef$
- $\tau_{11,6} = 524631$: $b_{11,6} = abedef$
- $\tau_{11,7} = 625314$: $b_{11,7} = abedef$
- $\tau_{11,8} = 641352$: $b_{11,8} = abedef$
- $\tau_{12,1} = 134652$: $b_{12,1} = abedef$
- $\tau_{12,2} = 146352$: $b_{12,2} = abedef$
- $\tau_{12,3} = 256341$: $b_{12,3} = abedef$
- $\tau_{12,4} = 325416$: $b_{12,4} = abedef$
- $\tau_{12,5} = 452361$: $b_{12,5} = abedef$
- $\tau_{12,6} = 521436$: $b_{12,6} = abedef$
- $\tau_{12,7} = 645213$: $b_{12,7} = abedef$
- $\tau_{12,8} = 634215$: $b_{12,8} = abedef$
- $\tau_{13,1} = 124563$: $b_{13,1} = abedef$
- $\tau_{13,2} = 134563$: $b_{13,2} = abedef$
- $\tau_{13,3} = 234156$: $b_{13,3} = abedef$
- $\tau_{13,4} = 365142$: $b_{13,4} = abedef$
- $\tau_{13,5} = 413526$: $b_{13,5} = abedef$
- $\tau_{13,6} = 452136$: $b_{13,6} = abedef$
- $\tau_{13,7} = 564213$: $b_{13,7} = abedef$
- $\tau_{13,8} = 653214$: $b_{13,8} = abedef$
- $\tau_{14,1} = 135642$: $b_{14,1} = abedef$
- $\tau_{14,2} = 142653$: $b_{14,2} = abedef$
- $\tau_{14,3} = 231564$: $b_{14,3} = abedef$
- $\tau_{14,4} = 321564$: $b_{14,4} = abedef$
Bibliography


