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Iteration of order preserving subhomogeneous maps on a cone

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Abstract

We investigate the iterative behaviour of continuous order preserving subhomogeneous maps $f\colon K \to K$, where K is a polyhedral cone in a finite dimensional vector space. We show that each bounded orbit of f converges to a periodic orbit and, moreover, the period of each periodic point of f is bounded by

$$\beta_N = \max_{q+r+s=N} \frac{N!}{q!r!s!} = \frac{N!}{\left\lfloor \frac{N}{3} \right\rfloor! \left\lfloor \frac{N+1}{3} \right\rfloor! \left\lfloor \frac{N+2}{3} \right\rfloor!} \sim \frac{3^{N+1}\sqrt{3}}{2\pi N},$$

where N is the number of facets of the polyhedral cone. By constructing examples on the standard positive cone in $\mathbb{R}^n$, we show that the upper bound is asymptotically sharp.

These results are an extension of work by Lemmens and Scheutzow concerning periodic orbits in the interior of the standard positive cone in $\mathbb{R}^n$.

1. *Introduction*

Let K be a polyhedral cone in a finite dimensional real vector space X and $f\colon K \to K$ be a continuous map. A basic problem in the theory of discrete dynamical systems is to describe qualitatively the asymptotic behaviour of the orbits $\{f^k(x)\colon k = 0, 1, 2, \ldots\}$ for each initial point $x \in K$, as $k \to \infty$. In this paper we investigate this problem for continuous maps $f\colon K \to K$ that are, in addition, order

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preserving and subhomogeneous. In particular, we prove in Theorem 2·1 that each bounded orbit of f converges to a periodic orbit and that the period of each periodic point of f is bounded by

$$\beta_N = \max_{q+r+s=N} \frac{N!}{q!r!s!} = \frac{N!}{\left\lfloor \frac{N}{3} \right\rfloor! \left\lfloor \frac{N+1}{3} \right\rfloor! \left\lfloor \frac{N+2}{3} \right\rfloor!}, \tag{1·1}$$

where N is the number of facets of the polyhedral cone K. Here $\lfloor a \rfloor$ denotes the greatest integer not exceeding a. As a second result we show in Theorem 2·2 that the upper bound is asymptotically sharp in case the polyhedral cone is the standard positive cone in $\mathbb{R}^n$ given by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant n\}$.

Order preserving subhomogeneous maps have been studied intensively in nonlinear Perron-Frobenius theory. They arise in various fields, such as optimal control and game theory [**1**, **24**, **29**], idempotent analysis [**17**, **23**], the analysis of monotone dynamical systems [**15**, **16**, **18**, **19**, **32**, **33**], and discrete event systems [**4**, **12**, **13**]. In this list we have quoted only a few recent works and we suggest the reader consult [**25**, **26**] for further references. The dynamical behaviour of these maps has been investigated in [**1**, **11**, **15**, **16**, **18**, **19**, **20**, **25**–**28**, **32**, **33**, **35**]; often under the additional assumption that f maps K into its interior, which is denoted by $\operatorname{int}(K)$. In particular, it is known that if $f: \operatorname{int}(K) \to \operatorname{int}(K)$ is an order preserving subhomogeneous map, then f is nonexpansive with respect to Thompson's part metric (see [**8**] and [**25**]). For maps that are nonexpansive with respect to Thompson's part metric, Weller [**35**] has proved that every bounded orbit in the interior of the polyhedral cone K converges to a periodic orbit. Moreover, for the standard positive cone, $\mathbb{R}^n_+$, it has been shown by Martus [**22**] that if $f: \operatorname{int}(\mathbb{R}^n_+) \to \operatorname{int}(\mathbb{R}^n_+)$ is nonexpansive with respect to the part metric, then the periods of periodic points of f are bounded by $n!2^n$. The upper bound of Martus is not sharp. In fact, Nussbaum [**27**, p. 525] has conjectured that 2^n is the optimal upper bound; but at present this conjecture is proved only for $n \leqslant 3$. The case $n = 3$ is proved by Lyons and Nussbaum in [**21**], in which additional evidence supporting the conjecture is also given. The current best general estimate is $\max_k 2^k \binom{n}{k}$ by Lemmens and Scheutzow [**20**]. Other upper bounds have been obtained in [**5**, **27**, **31**]. For order preserving homogeneous maps $f: \operatorname{int}(\mathbb{R}^n_+) \to \operatorname{int}(\mathbb{R}^n_+)$, it was expected that stronger estimates hold for the periods of periodic points. Indeed, Gunawardena and Sparrow conjectured (see [**12**]) that $\binom{n}{\lfloor n/2 \rfloor}$ is the optimal upper bound. A proof of this conjecture was given by Lemmens and Scheutzow in [**20**]. We shall see that the arguments in [**20**] can be refined to show that if $f: \operatorname{int}(K) \to \operatorname{int}(K)$ is an order preserving subhomogeneous map, then the periods of periodic points of f do not exceed $\binom{N}{\lfloor N/2 \rfloor}$, where N is the number of facets of the polyhedral cone K. In connection with these results it is useful to mention that each order preserving subhomogeneous map $f: \operatorname{int}(K) \to K$ is continuous and has a continuous extension $f: K \to K$, which is again order preserving and subhomogeneous (see [**7**, theorem 3·10]).

With these results in mind the following questions are natural. Given a polyhedral cone K and a continuous order preserving subhomogeneous map $f: K \to K$, does every bounded orbit of f converge to a periodic orbit? Does there exist an a priori upper bound for the periods of periodic points in terms of the number of facets of K? If so, what is the optimal upper bound? In this paper we answer these questions.

To conclude the introduction we outline the organisation of the paper. In Section 2 we state the two main results, Theorems 2·1 and 2·2. In Section 3 we collect some

preliminary results. Subsequently, we study in Section 4 periodic points of order preserving subhomogeneous maps on polyhedral cones, whose orbit is contained in a part of the cone. Using a result of Lemmens and Scheutzow [**20**] we give an upper bound for the possible periods of these periodic points. This upper bound is then used in Section 5 to show that the period of any periodic point does not exceed β_N, where β_N is given in (1·1). In Section 6 we prove that each bounded orbit converges to a periodic orbit. Combining this result with the results in Section 5 yields the first main result, Theorem 2·1. In Section 7 we prove the second main result, Theorem 2·2.

2. *Statement of the main results*

Let X be a real topological vector space. A subset K of X is called a *cone* if it is a convex subset of X such that $\lambda K \subset K$ for all $\lambda \geqslant 0$ and $K \bigcap (-K) = \{0\}$. A cone K in X is called a *closed cone* if it is a closed subset of X. If X is a finite dimensional topological vector space, then it is known that X has exactly one Hausdorff vector space topology and it coincides with the standard topology. The main results of this paper concern closed cones in finite dimensional vector spaces. In that case the vector space topology will always be the standard topology. Many preliminary results will however be stated and proved for more general topological vector spaces.

A closed cone K in a finite dimensional vector space X is said to be a *polyhedral cone* if it is the intersection of finitely many closed half spaces, i.e., there exist linear functionals $\varphi_1, \ldots, \varphi_m$ such that $K = \{x \in X : \varphi_i(x) \geqslant 0 \text{ for } 1 \leqslant i \leqslant m\}$. A *face* of a polyhedral cone K is any set of the form $F = K \bigcap \{x \in X : \varphi(x) = 0\}$, where $\varphi : X \to \mathbb{R}$ is a linear functional such that $K \subset \{x \in X : \varphi(x) \geqslant 0\}$. Note that the cone itself is a face. The *dimension* of a face F, denoted $\dim(F)$, is the dimension of its linear span. A face F is called a *facet* if $\dim(F) = \dim(K) - 1$. We remark that if K is a polyhedral cone with N facets, then there exist N linear functionals $\psi_i : X \to \mathbb{R}$, where $1 \leqslant i \leqslant N$, such that

$$K = \{x \in X : \psi_i(x) \geqslant 0 \text{ for } 1 \leqslant i \leqslant N\} \bigcap \operatorname{span}(K) \tag{2·1}$$

and each linear functional ψ_i defines a facet of K (see [**30**, section 8·4]). In this paper the closed cone will often be polyhedral and we reserve the notation ψ_i, where $1 \leqslant i \leqslant N$, to denote the linear functionals that define its facets. A natural example of a polyhedral cone is the *standard positive cone* in $\mathbb{R}^n$ given by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \geqslant 0 \text{ for } 1 \leqslant i \leqslant n\}$, which has n facets.

A cone K in a topological vector space X induces a partial ordering $\leqslant_K$ on X by $x \leqslant_K y$ if $y - x \in K$. We simply write $\leqslant$ if K is obvious from the context. Subsets of X will always inherit the partial ordering of X. If $(S, \leqslant)$ and $(T, \leqslant)$ are two partial ordered sets, then we call a map $f : S \to T$ *order preserving* if $f(x) \leqslant f(y)$ for all $x, y \in S$ with $x \leqslant y$. If X is a vector space with a partial ordering $\leqslant$ and if $f : D \to X$, where $D \subset X$, has the property that $\lambda f(x) \leqslant f(\lambda x)$ for every $x \in D$ and $0 < \lambda < 1$ satisfying $\lambda x \in D$, then f is said to be *subhomogeneous*. If $\lambda f(x) = f(\lambda x)$ for every $x \in D$ and $\lambda \geqslant 0$ satisfying $\lambda x \in D$, then f is said to be *homogeneous*.

If S is a set and $f : S \to S$, then a point $x \in S$ is called a *periodic point* if $f^p(x) = x$ for some integer $p \geqslant 1$; the minimal such $p \geqslant 1$ is said to be the *period* of x under f. The *orbit* of $x \in S$ under f is given by $\mathcal{O}(x; f) = \{f^k(x) : k = 0, 1, 2, \ldots\}$. If x is a periodic point, then $\mathcal{O}(x; f)$ is called a *periodic orbit*.

Equipped with these notions we now state the main results.

Table 1. *The lower and upper bound for* $1 \leqslant N \leqslant 15$

α_N	1, 2, 6, 12, 30, 78, 210, 540, 1660, 4180, 11480, 34510, 90090, 251874, 756252
β_N	1, 2, 6, 12, 30, 90, 210, 560, 1680, 4200, 11550, 34650, 90090, 252252, 756756

Theorem 2·1. *Let K be a polyhedral cone with N facets in a finite dimensional vector space X. If $f\colon K \to K$ is a continuous order preserving subhomogeneous map and the orbit of $x \in K$ is bounded, then there exists a periodic point ξ of f, with period p, such that* $\lim_{k\to\infty} f^{kp}(x) = \xi$ *and* $p \leqslant \beta_N$, *where*

$$\beta_N = \max_{q+r+s=N} \frac{N!}{q!r!s!} = \frac{N!}{\left\lfloor \frac{N}{3} \right\rfloor! \left\lfloor \frac{N+1}{3} \right\rfloor! \left\lfloor \frac{N+2}{3} \right\rfloor!}. \tag{2·2}$$

To show that the upper bound β_N is asymptotically sharp we prove in Section 7 the following theorem.

Theorem 2·2. *For every $1 \leqslant m \leqslant n$, $1 \leqslant p \leqslant \binom{m}{\lfloor m/2 \rfloor}$, and $1 \leqslant q \leqslant \binom{n}{m}$, there exists a continuous order preserving homogeneous map $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ that has a periodic point with period equal to the least common multiple of p and q.*

From Theorem 2·2 it follows that

$$\alpha_N = \max\left\{ \operatorname{lcm}(p,q)\colon 1 \leqslant p \leqslant \binom{m}{\lfloor m/2 \rfloor}, 1 \leqslant q \leqslant \binom{N}{m}, \text{ and } 1 \leqslant m \leqslant N \right\} \tag{2·3}$$

is a lower bound for the maximum period of periodic points of continuous order preserving subhomogeneous map $f\colon \mathbb{R}^N_+ \to \mathbb{R}^N_+$. We show in Section 7 that $\lim_{N\to\infty} \alpha_N / \beta_N = 1$. This implies that the upper bound in Theorem 2·1 is asymptotically sharp in case K is the standard positive cone in $\mathbb{R}^N$. This fact is illustrated in Table 1 below. Moreover, by using Stirling's formula, it can be shown that β_N has the following asymptotics:

$$\beta_N \sim \frac{3^{N+1}\sqrt{3}}{2\pi N}.$$

3. *Preliminary results*

3·1. *Partially ordered sets*

Partially ordered sets occur frequently in this exposition and it is useful to recall several basic concepts concerning them. Let $(S, \leqslant)$ be a partially ordered set. We say that a and b are *comparable* if $a \leqslant b$ or $b \leqslant a$. A subset $\mathcal{A}$ of S is called an *antichain* if no two distinct elements in $\mathcal{A}$ are comparable. A subset $\mathcal{C}$ of S is called a *chain* if every two elements in $\mathcal{C}$ are comparable, and it is said to be a *maximal chain* if there exists no chain $\mathcal{D} \subset S$ that properly contains $\mathcal{C}$. We have the following basic lemma.

Lemma 3·1. *Let $(S, \leqslant)$ be a partially ordered set and let $f\colon S \to S$ be an order preserving map. If $x \in S$ is a periodic point of f, then $\mathcal{O}(x; f)$ is an antichain.*

Proof. Let $x \in S$ be a periodic point of f with period p. Suppose that $y, z \in \mathcal{O}(x; f)$ and $y \leqslant z$. As $\mathcal{O}(x; f)$ is a periodic orbit with period p, there exists $0 \leqslant k < p$ such that $z = f^k(y)$ and hence $y \leqslant f^k(y)$. Since f^k is order preserving, this implies that

$f^k(y) \leqslant f^{2k}(y) \leqslant \cdots \leqslant f^{kp}(y) = y$ and therefore $z \leqslant y$. Thus $y = z$ and hence $\mathcal{O}(x; f)$ is an antichain.

3·2. *Parts and Thompson's part metric*

Let K be a cone in a topological vector space X. For the analysis it is convenient to define an equivalence relation $\sim$ on K by $x \sim y$ if there exist constants $0 < \alpha \leqslant \beta$ such that $\alpha x \leqslant y \leqslant \beta x$. We write $[x]$ to denote the equivalence class of x. The equivalence classes in K are called *parts* (or *constituents*) (see [**3**, **34**]) and we denote the set of all parts of K by $P(K)$. We say that x *dominates* y if there exists $\beta > 0$ such that $y \leqslant \beta x$. We observe that if $x \sim x'$ and $y \sim y'$, then x dominates y if and only if x' dominates y'. This observation allows us to define a partial ordering $\preceq$ on the set of parts, $P(K)$, in the following manner: $P \preceq Q$ if x dominates y for some $x \in Q$ and $y \in P$.

For $x, y \in K$ we define

$$M(y/x; K) = \inf\{\beta > 0 : y \leqslant \beta x\} \tag{3·1}$$

and we put $M(y/x; K) = \infty$ if the set is empty. If K is obvious from the context, we simply write $M(y/x)$. Remark that $M(y/x) < \infty$ if and only if x dominates y. Moreover, if in addition K is closed, then the infimum in (3·1) is attained and in that case $y \leqslant M(y/x)x$. We have the following lemma.

Lemma 3·2. *Let K be a cone in a topological vector space X and let P be a part of K. If $\mathcal{A}$ is an antichain in the partially ordered set $(P, \leqslant)$ and $f: P \to P$ is an order preserving subhomogeneous map, then*

$$M(f(y)/f(x)) \leqslant M(y/x) \quad \textit{for all } x, y \in \mathcal{A}. \tag{3·2}$$

Moreover, if $f(\mathcal{A}) \subset \mathcal{A}$ and each $x \in \mathcal{A}$ is a periodic point of f, then

$$M(f(y)/f(x)) = M(y/x) \quad \textit{for all } x, y \in \mathcal{A}. \tag{3·3}$$

Proof. Clearly the equations (3·2) and (3·3) are true if $x = y$. So, let $x, y \in \mathcal{A}$ with $x \neq y$. As x and y belong to the same part, $M(y/x)$ is finite. Consider $\lambda > 0$ such that $\lambda > M(y/x)$. Then $y \leqslant \lambda x$ and, since $\mathcal{A}$ is an antichain, we have that $\lambda > 1$. Using the fact that f is order preserving and subhomogeneous, we deduce that $\lambda^{-1} f(y) \leqslant f(\lambda^{-1} y) \leqslant f(x)$, so that $M(f(y)/f(x)) \leqslant \lambda$. Since this inequality holds for all $\lambda > M(y/x)$, inequality (3·2) follows.

To prove the second assertion we assume that $f^p(x) = x$ and $f^q(y) = y$. By applying the previous observation iteratively we deduce for each $k \geqslant 1$ that $M(f^k(y)/f^k(x)) \leqslant M(f(y)/f(x)) \leqslant M(y/x)$. Now by taking $k = pq$ we find that $M(y/x) \leqslant M(f(y)/f(x)) \leqslant M(y/x)$, which completes the proof.

Using the function $M(y/x)$ we define a map $d_T: K \times K \to [0, \infty]$ by

$$d_T(x, y) = \log(\max\{M(y/x), M(x/y)\}) \tag{3·4}$$

for all $(x, y) \in K \times K$, with $(x, y) \neq (0, 0)$, and we put $d_T(0, 0) = 0$. The function d_T is called (*Thompson's*) *part metric* [**34**]. It is well known that if K is a closed cone, then d_T is a genuine metric on each part of the cone, but not on the whole cone. Indeed, $d_T(x, y)$ is finite if and only if $x \sim y$. If K is not a closed cone, then in general d_T is a semi-metric on each part. Moreover, if K is a closed cone in a finite dimensional vector space X and P is a part of K, then (P, d_T) is a complete metric space and the

topology coincides with the topology induced by the standard topology on X. More general results concerning the part metric can be found in [**3**, **25**, **26**, **34**].

To conclude this subsection we mention the relation between the part metric and the sup-norm on $\mathbb{R}^n$ given by $\|z\|_\infty = \max_i |z_i|$. Consider the standard positive cone $\mathbb{R}^n_+$ and the part corresponding to the interior of $\mathbb{R}^n_+$. There exists an isometry from the metric space $(\mathrm{int}(\mathbb{R}^n_+), d_T)$ onto the metric space $(\mathbb{R}^n, \|\cdot\|_\infty)$ (cf. [**25**, proposition 1·6]). Indeed, one can use the map $L\colon \mathrm{int}(\mathbb{R}^n_+) \to \mathbb{R}^n$ given by

$$L(x) = (\log x_1, \ldots, \log x_n) \quad \text{for } x = (x_1, \ldots, x_n) \in \mathrm{int}(\mathbb{R}^n_+). \tag{3·5}$$

The inverse of L is, of course, the map $E\colon \mathbb{R}^n \to \mathrm{int}(\mathbb{R}^n_+)$ given by

$$E(x) = (e^{x_1}, \ldots, e^{x_n}) \quad \text{for } x = (x_1, \ldots, x_n) \in \mathbb{R}^n. \tag{3·6}$$

To see that the map L is an isometry it is convenient to first define a map $t\colon \mathbb{R}^n \to \mathbb{R}$ by $t(x) = \max_i x_i$ for $x \in \mathbb{R}^n$, and subsequently to remark that

$$\|x\|_\infty = \max\{t(x), t(-x)\}. \tag{3·7}$$

Now note that if $x, y \in \mathrm{int}(\mathbb{R}^n_+)$, then

$$M(x/y) = \inf\{\beta \geqslant 0 \colon x \leqslant \beta y\} = \max_i (x_i/y_i)$$

and $\log(\max_i (x_i/y_i)) = \max_i (\log x_i - \log y_i) = t(L(x) - L(y))$. Thus,

$$\log M(x/y) = t(L(x) - L(y)) \quad \text{for all } x, y \in \mathrm{int}(\mathbb{R}^n_+), \tag{3·8}$$

so that (3·4) and (3·7) yield

$$d_T(x, y) = \|L(x) - L(y)\|_\infty \quad \text{for all } x, y \in \mathrm{int}(\mathbb{R}^n_+). \tag{3·9}$$

The function $t\colon \mathbb{R}^n \to \mathbb{R}$ above appears naturally in the study of topical functions (see Gunawardena and Keane [**14**]) and also played an important role in [**20**]. It has certain properties of a norm. For instance, $t(x+y) \leqslant t(x) + t(y)$ for all $x, y \in \mathbb{R}^n$; but, $t(x) \neq t(-x)$ in general.

3·3. *Nonexpansiveness and order preserving maps on a cone*

If (C, d) is a metric space, then $f\colon C \to C$ is called *nonexpansive with respect to* d, or, simply *d-nonexpansive* if

$$d(f(x), f(y)) \leqslant d(x, y) \quad \text{for all } x, y \in C. \tag{3·10}$$

The map f is called a *d-isometry* if (3·10) is an equality for all $x, y \in C$. Although d_T is not a proper metric on a cone K (and in general only a semi-metric on each part of K, when K is not closed), we say that $f\colon K \to K$ is *d_T-nonexpansive* if (3·10) holds for d_T. Here the inequality only makes sense if the right-hand side is finite. In the same way we abuse terminology for the function $t : \mathbb{R}^n \to \mathbb{R}$ given by $t(x) = \max_i x_i$. We call a map $f\colon S \to S$, where $S \subset \mathbb{R}^n$, *t-nonexpansive* if $t(f(x) - f(y)) \leqslant t(x - y)$ for all $x, y \in S$. The map f is called a *t-isometry* if $t(f(x) - f(y)) = t(x - y)$ for all $x, y \in S$. We have the following lemma (cf. [**25**, proposition 1·5]), which is similar to results in [**9**].

LEMMA 3·3. *Let K be a closed cone in a topological vector space X. If $f\colon K \to K$ is order preserving, then f is d_T-nonexpansive if and only if f is subhomogeneous.*

Proof. Assume first that f is subhomogeneous. If $x, y \in K$ and

$$\lambda \geqslant \max\{M(y/x), M(x/y)\}, \tag{3·11}$$

then $y \leqslant \lambda x$ and $x \leqslant \lambda y$, so that $x \leqslant \lambda y \leqslant \lambda^2 x$ and therefore $\lambda \geqslant 1$. As f is order preserving and subhomogeneous, we obtain

$$\lambda^{-1} f(y) \leqslant f(\lambda^{-1} y) \leqslant f(x) \quad \text{and} \quad \lambda^{-1} f(x) \leqslant f(\lambda^{-1} x) \leqslant f(y).$$

This implies that $\max\{M(f(y)/f(x)), M(f(x)/f(y))\} \leqslant \lambda$ and hence

$$d_T(f(x), f(y)) \leqslant \log \lambda = d_T(x, y).$$

Now assume that f is nonexpansive with respect to d_T on K. Let $x \in K$ and put $y = \lambda^{-1} x$, where $\lambda \geqslant 1$. Clearly $d_T(x, y) = \log \lambda$ if $x \neq 0$ and $d_T(x, y) \leqslant \log \lambda$ if $x = 0$. As f is nonexpansive with respect to d_T, we have that

$$\log M(f(x)/f(y)) \leqslant d_T(f(x), f(y)) \leqslant d_T(x, y) \leqslant \log \lambda,$$

so that $f(x) \leqslant \lambda f(y)$. This implies that $\lambda^{-1} f(x) \leqslant f(y) = f(\lambda^{-1} x)$ and hence f is subhomogeneous.

Maps that are nonexpansive with respect to the part metric map parts into parts. Indeed, we have the following lemma.

Lemma 3·4. *If K is a cone in a topological vector space X and $f\colon K \to K$ is d_T-nonexpansive, then $f([x]) \subset [f(x)]$ for each $x \in K$.*

Proof. If $y \in f([x])$, then there exists $z \in [x]$ such that $f(z) = y$. Since $z \sim x$ we have that $d_T(x, z)$ is finite. As f is nonexpansive with respect to d_T on $[x]$, we find that $d_T(f(x), y)$ is finite and hence $y \sim f(x)$.

This lemma has the following corollary.

Corollary 3·5. *If K is a cone in a topological vector space X and $f\colon K \to K$ is d_T-nonexpansive, then the map $F\colon P(K) \to P(K)$ given by $F(P) = [f(x)]$ for $x \in P$ is well defined. Moreover, if f is order preserving and K is closed, then F preserves the ordering $\preceq$ on $P(K)$.*

Proof. To see that F is well defined we let P be a part of the cone K. For each $x, y \in P$ we have that $f(x) \sim f(y)$, by Lemma 3·4, and hence $[f(x)] = [f(y)]$. Thus, F is well defined. If $f\colon K \to K$ is an order preserving d_T-nonexpansive map and K is closed, then f is subhomogeneous by Lemma 3·3. Now let $P, Q \in P(K)$ be such that $P \preceq Q$. If $x \in Q$ and $y \in P$, then x dominates y and therefore there exists $\lambda \geqslant 1$ such that $y \leqslant \lambda x$. Since f is order preserving and subhomogeneous, it follows that $\lambda^{-1} f(y) \leqslant f(\lambda^{-1} y) \leqslant f(x)$. Thus $f(x)$ dominates $f(y)$, so that $[f(y)] \preceq [f(x)]$. From this we conclude that $F(P) \preceq F(Q)$, which completes the proof.

4. *Periodic orbits in a part of the polyhedral cone*

In [**20**] Lemmens and Scheutzow proved the following theorem.

Theorem 4·1 ([**20**]). *If $\mathcal{A}$ is a finite antichain in $(\mathbb{R}^n, \leqslant)$, where the partial ordering $\leqslant$ is induced by $\mathbb{R}^n_+$, and on $\mathcal{A}$ a commutative group of t-isometries acts transitively, then $\mathcal{A}$ has at most $\binom{n}{\lfloor n/2 \rfloor}$ elements.*

We use this theorem to derive the following result.

Theorem 4·2. *Let K be a polyhedral cone with nonempty interior in a finite dimensional vector space X. If K has N facets and $f\colon \mathrm{int}(K) \to \mathrm{int}(K)$ is order preserving and subhomogeneous, then the periods of periodic points of f do not exceed $\binom{N}{\lfloor N/2 \rfloor}$.*

Proof. Let ξ be a periodic point of f with period p and let $\mathcal{A} = \mathcal{O}(\xi; f)$. From Lemma 3·1 it follows that $\mathcal{A}$ is an antichain in $(\mathrm{int}(K), \leqslant_K)$. Furthermore Lemma 3·2 implies that

$$M(f(y)/f(x); K) = M(y/x; K) \quad \text{for all } x, y \in \mathcal{A}. \tag{4·1}$$

Define $\Psi\colon X \to \mathbb{R}^N$ by $\Psi(x) = (\psi_1(x), \ldots, \psi_N(x))$ for all $x \in X$. Here $\psi_i\colon X \to \mathbb{R}$, with $1 \leqslant i \leqslant N$, are the linear functionals that define the facets of K. The map Ψ is linear and, by (2·1), $x \in K$ if and only if $\Psi(x) \in \mathbb{R}^N_+$. Hence $\Psi(K) = \Psi(X) \bigcap \mathbb{R}^N_+$ and if $\mathbb{R}^N$ is endowed with the partial ordering induced by $\mathbb{R}^N_+$, we get that $x \leqslant_K \lambda y$ is equivalent to $\Psi(x) \leqslant \lambda \Psi(y)$. It follows that

$$M(y/x; K) = M(\Psi(y)/\Psi(x); \mathbb{R}^n_+) \quad \text{for all } x, y \in K. \tag{4·2}$$

Moreover, Ψ is injective, because $\Psi(x) = 0$ implies that $x \in K$ and $-x \in K$, so that $x = 0$. We also have that $\Psi(\mathrm{int}(K)) \subset \mathrm{int}(\mathbb{R}^N_+)$. Indeed, if $y \in \mathrm{int}(K)$, then for each $z \in X$ there exists $\varepsilon > 0$ such that $y - \varepsilon z \in K$. This implies that $\psi_i(y) \geqslant \varepsilon \psi_i(z)$ for all $1 \leqslant i \leqslant N$. Since ψ_i is nonzero, there exists $z \in X$ such that $\psi_i(z) > 0$. Therefore $\psi_i(y) > 0$ for all $1 \leqslant i \leqslant N$ and hence $\Psi(y) \in \mathrm{int}(\mathbb{R}^N_+)$.

Let Ψ^{-1} be the inverse of Ψ on $\Psi(X)$. Put $\mathcal{A}' = \Psi(\mathcal{A})$ and let $g\colon \mathcal{A}' \to \mathcal{A}'$ be given by $g = \Psi \circ f \circ \Psi^{-1}$. By using (4·1) and (4·2) we find that if $u, v \in \mathcal{A}'$, $u = \Psi(x)$, and $v = \Psi(y)$, then

$$M(v/u; \mathbb{R}^n_+) = M(y/x; K) = M(f(y)/f(x); K) = M(g(u)/g(v); \mathbb{R}^n_+). \tag{4·3}$$

Now put $\mathcal{A}'' = L(\mathcal{A}')$ and define $h\colon \mathcal{A}'' \to \mathcal{A}''$ by $h = L \circ g \circ E$, where the maps L and E are given in (3·5) and (3·6), respectively. The set $\mathcal{A}''$ is well defined, as $\mathcal{A}' \subset \Psi(\mathrm{int}(K)) \subset \mathrm{int}(\mathbb{R}^N_+)$. It follows from (3·8) and (4·3) that

$$t(h(r) - h(s)) = t(r - s) \quad \text{for all } r, s \in \mathcal{A}''.$$

One can verify that $\mathcal{A}''$ is a periodic orbit of h with period p; in fact, $\mathcal{A}'' = \mathcal{O}(L(\Psi(\xi)); h)$. Therefore $G = \{h^k\colon \mathcal{A}'' \to \mathcal{A}'' \mid 0 \leqslant k < p\}$ is a commutative group of t-isometries that acts transitively on $\mathcal{A}''$ and hence Theorem 4·1 implies that $p = |\mathcal{A}''| \leqslant \binom{N}{\lfloor N/2 \rfloor}$.

We shall generalize Theorem 4·2 to the case where f maps a part of the cone into itself; but before we do this we introduce some definitions. Let K be a polyhedral cone with N facets and let $\psi_i\colon X \to \mathbb{R}$, with $1 \leqslant i \leqslant N$, be the linear functionals that define the facets of K. We define for each $x \in K$ a set I_x by

$$I_x = \{i \in \{1, \ldots, N\}\colon \psi_i(x) > 0\}. \tag{4·4}$$

It easy to verify that $I_y \subset I_x$ if and only if x dominates y. Therefore $I_x = I_y$ is equivalent to $x \sim y$. This allows us to make the following definition.

Definition 4·3. If K is a polyhedral cone with N facets in a finite dimensional vector space X, then for each part $P \in P(K)$ we define $I(P) = I_x$, where $x \in P$.

The same observation shows that the map $I: P(K) \to 2^{[N]}$ given by $P \mapsto I(P)$ is injective. Here $2^{[N]}$ denotes the set of all subsets of $\{1, \ldots, N\}$. In particular, this implies that there are at most 2^N parts in K. Moreover, $I(P) \subset I(Q)$ if and only if $P \preceq Q$, and hence $I: (P(K), \preceq) \to (2^{[N]}, \subset)$ and its inverse I^{-1} are both order preserving.

Corollary 4·4. *Let K be a polyhedral cone in a finite dimensional vector space X. If P is a part of K and $f: P \to P$ is order preserving and subhomogeneous, then the periods of periodic points of f do not exceed $\binom{m}{\lfloor m/2 \rfloor}$, where $m = |I(P)|$.*

Proof. Let K be given by $\{x \in X: \psi_i(x) \geqslant 0 \text{ for } 1 \leqslant i \leqslant N\} \bigcap \operatorname{span}(K)$, where each ψ_i is a linear functional that defines a facet of K. Put $J = I(P)$ and let J' denotes its complement. Define

$$Y = \{x \in X: \psi_j(x) = 0 \text{ for all } j \in J'\} \bigcap \operatorname{span}(K).$$

Remark that Y is a linear subspace of X. Now let $C = K \bigcap Y$. We observe that $C = \{y \in Y: \psi_j(y) \geqslant 0 \text{ for all } j \in J\}$ and hence C is a polyhedral cone with at most $|J|$ facets in the vector space Y. Since

$$\begin{aligned} P &= \{x \in X: \psi_j(x) > 0 \text{ for } j \in J \text{ and } \psi_j(x) = 0 \text{ for } j \in J'\} \bigcap \operatorname{span}(K) \\ &= \{y \in Y: \psi_j(y) > 0 \text{ for } j \in J\}, \end{aligned}$$

we have that P is the interior of C in Y. Thus we can apply Theorem 4·2 to conclude that the periods of periodic points of $f: P \to P$ do not exceed $\binom{q}{\lfloor q/2 \rfloor}$, where q is the number of facets of C. Since $q \leqslant |J| = |I(P)| = m$, we find that $\binom{q}{\lfloor q/2 \rfloor} \leqslant \binom{m}{\lfloor m/2 \rfloor}$ and this completes the proof.

Theorem 4·2 and Corollary 4·4 generalize theorem 5·2 in [**20**] by allowing subhomogeneous maps rather than homogeneous maps and allowing general polyhedral cones. To conclude this section we mention one other consequence of Theorem 4·2, which refines another result in [**20**]. It concerns order preserving sup-norm nonexpansive maps. Recall that a map $f: \mathbb{R}^n \to \mathbb{R}^n$ is sup-norm nonexpansive if $\|f(x) - f(y)\|_\infty \leqslant \|x - y\|_\infty$ for all $x, y \in \mathbb{R}^n$.

Theorem 4·5. *If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a sup-norm nonexpansive map and f is order preserving with respect to the ordering induced by $\mathbb{R}^n_+$, then the periods of periodic points of f do not exceed $\binom{n}{\lfloor n/2 \rfloor}$.*

Proof. Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be an order preserving sup-norm nonexpansive map and suppose that $\xi \in \mathbb{R}^n$ is a periodic point of f with period p. Define a map $h: \operatorname{int}(\mathbb{R}^n_+) \to \operatorname{int}(\mathbb{R}^n_+)$ by $h = E \circ f \circ L$, where L and E are respectively given in (3·5) and (3·6). From (3·9) we know that L is an isometric homeomorphism between $(\operatorname{int}(\mathbb{R}^n_+), d_T)$ and $(\mathbb{R}^n, \|\cdot\|_\infty)$ and the inverse isometry is the map E. As f is sup-norm nonexpansive this implies that h is nonexpansive with respect to d_T. Since f is order preserving, the map h also preserves the ordering induced by $\mathbb{R}^n_+$. Therefore it follows from Lemma 3·3 that h is an order preserving subhomogenous map. Clearly $E(\xi)$ is a periodic point of h, with period p, and hence we conclude from Theorem 4·2 that p is at most $\binom{n}{\lfloor n/2 \rfloor}$.

5. *Periods of periodic points in a polyhedral cone*

The main goal of this section is to prove that the periods of all periodic points of order preserving subhomogenous maps on a polyhedral cone with N facets do not exceed β_N, where β_N is given in (2·2). As a start we show the following theorem.

THEOREM 5·1. *Let K be a polyhedral cone with N facets in a finite dimensional vector space X. If $f: K \to K$ is an order preserving subhomogenous map and $x \in K$ is a periodic point of f with period p, then there exist integers q_1 and q_2 such that $p = q_1 q_2$,*

$$1 \leqslant q_1 \leqslant \binom{N}{\max\{m, \lfloor N/2 \rfloor\}}, \quad \textit{and} \quad 1 \leqslant q_2 \leqslant \binom{m}{\lfloor m/2 \rfloor},$$

where $m = \min\{|I_{f^j(x)}|: 0 \leqslant j < p\}$.

Proof. It follows from Lemma 3·3 and Corollary 3·5 that the map $F: P(K) \to P(K)$ given by $F(P) = [f(x)]$ for $x \in P$, is well defined and order preserving. Let $I: P(K) \to 2^{[N]}$ be given as in Definition 4·3. We denote by I^{-1} the inverse of I on $I(P(K))$. Define a map $G: I(P(K)) \to I(P(K))$ by $G = I \circ F \circ I^{-1}$. As F, I, and I^{-1} are all order preserving, the map G preserves the partial ordering $\subset$ on $I(P(K))$.

Let $x \in K$ be a periodic point of f, with period p, and let $m = \min\{|I_{f^j(x)}|: 0 \leqslant j < p\}$. Take $z \in \mathcal{O}(x; f)$ such that $|I_z| = m$ and put $Q = [z]$. We observe that $F^j(Q) = [f^j(z)]$ for all $j \geqslant 0$ and hence $F^p(Q) = Q$. Let k be the period of Q under F. Obviously k divides p and $I(Q)$ is a periodic point of G with period k. Let $\mathcal{A} = \mathcal{O}(I(Q); G)$. Since

$$G^j(I(Q)) = I(F^j(Q)) = I([f^j(z)]) = I_{f^j(z)} \quad \text{for all } j \geqslant 0,$$

we have that $\mathcal{A} = \{I_{f^j(z)}: 0 \leqslant j < k\}$. As G is order preserving, it follows from Lemma 3·1 that $\mathcal{A}$ is an antichain in $(2^{[N]}, \subset)$.

A maximal chain $\mathcal{C}$ in $(2^{[N]}, \subset)$ is a sequence of $N+1$ subsets $A_0, A_1, \ldots, A_N$ of $\{1, \ldots, N\}$ such that $A_0 \subset A_1 \subset \cdots \subset A_N$ and $|A_i| = i$ for $0 \leqslant i \leqslant N$. Hence there are exactly $N!$ maximal chains. If $A \subset \{1, \ldots, N\}$ and $|A| = s$, then there are precisely $s!(N-s)!$ maximal chains $\mathcal{C}$ in $(2^{[N]}, \subset)$ which contain A. As $\mathcal{A}$ is an antichain, each maximal chain $\mathcal{C}$ contains at most one element of $\mathcal{A}$. Since $m = \min\{|I_{f^j(x)}|: 0 \leqslant j < p\}$, we know that $|A| \geqslant m$ for all $A \in \mathcal{A}$.

Now for $m \leqslant s \leqslant N$, let ν_s be the number of elements of $\mathcal{A}$ with cardinality s. As each maximal chain contains at most one element of $\mathcal{A}$ and each $A \in \mathcal{A}$ with cardinality s is contained in $s!(N-s)!$ maximal chains, we find that

$$\sum_{s=m}^{N} \nu_s s!(N-s)! \leqslant N! \quad \text{so that} \quad \sum_{s=m}^{N} \nu_s \binom{N}{s}^{-1} \leqslant 1.$$

Put $M(m) = \max_{m \leqslant s \leqslant N} \binom{N}{s}$. It is well known that $M(m) = \binom{N}{m}$ if $m \geqslant \lfloor N/2 \rfloor$, and $M(m) = \binom{N}{\lfloor N/2 \rfloor}$ if $0 \leqslant m \leqslant \lfloor N/2 \rfloor$. From this it follows that

$$k = |\mathcal{A}| = \sum_{s=m}^{N} \nu_s \leqslant M(m) = \binom{N}{\max\{m, \lfloor N/2 \rfloor\}}. \tag{5·1}$$

For k, z, and Q as above, it follows from Lemma 3·4 that $f^k(Q) \subset [f^k(z)] = Q$. As z is a periodic point of f^k, we can use Corollary 4·4 to see that the period of z under

f^k is less than or equal to $\binom{m}{\lfloor m/2 \rfloor}$. Put $q_1 = k$ and let q_2 be the period of z under f^k. Since k divides the period p of z under f, we get that $p = kq_2 = q_1 q_2$, which completes the proof.

We would like to remark that the arguments to derive inequality (5·1) in the proof of Theorem 5·1 appear in the study of Sperner systems and are known in combinatorics as the LYM technique; see [**6**, p. 10–11].

As a consequence of Theorem 5·1 we find that if K is a polyhedral cone with N facets, then the periods of periodic points of order preserving subhomogeneous maps $f\colon K \to K$ are bounded by

$$\max_{1 \leqslant m \leqslant N} \binom{N}{\max\{m, \lfloor N/2 \rfloor\}} \binom{m}{\lfloor m/2 \rfloor}.$$

To see that this upper bound coincides with β_N, where β_N is given in (2·2), we prove the following equalities.

Lemma 5·2. *For each $n \geqslant 1$ we have that*

$$\max_{1 \leqslant m \leqslant n} \binom{n}{\max\{m, \lfloor n/2 \rfloor\}} \binom{m}{\lfloor m/2 \rfloor} = \max_{q+r+s=n} \frac{n!}{q!r!s!} = \frac{n!}{\lfloor \frac{n}{3} \rfloor! \lfloor \frac{n+1}{3} \rfloor! \lfloor \frac{n+2}{3} \rfloor!}.$$

Proof. We first remark that for $1 \leqslant m \leqslant \lfloor n/2 \rfloor$ we have that

$$\binom{n}{\max\{m, \lfloor n/2 \rfloor\}} \binom{m}{\lfloor m/2 \rfloor} \leqslant \binom{n}{\lfloor n/2 \rfloor} \binom{\lfloor n/2 \rfloor}{\lfloor \lfloor n/2 \rfloor / 2 \rfloor},$$

so that

$$\max_{1 \leqslant m \leqslant n} \binom{n}{\max\{m, \lfloor n/2 \rfloor\}} \binom{m}{\lfloor m/2 \rfloor} = \max_{\lfloor n/2 \rfloor \leqslant m \leqslant n} \binom{n}{m} \binom{m}{\lfloor m/2 \rfloor}. \tag{5·2}$$

Further we have that

$$\binom{n}{m} \binom{m}{\lfloor m/2 \rfloor} = \frac{n!}{q!r!s!}, \tag{5·3}$$

where $q = n - m$, $r = \lfloor m/2 \rfloor$, and $s = m - \lfloor m/2 \rfloor$. This implies that

$$\max_{\lfloor n/2 \rfloor \leqslant m \leqslant n} \binom{n}{m} \binom{m}{\lfloor m/2 \rfloor} \leqslant \max_{q+r+s=n} \frac{n!}{q!r!s!}. \tag{5·4}$$

Let us now consider the right-hand side of (5·4). Assume that the maximum is attained for $0 \leqslant q^* \leqslant r^* \leqslant s^*$. We claim that $s^* \leqslant q^* + 1$. Indeed, suppose by way of contradiction that $s^* > q^* + 1$. Then $q^*!s^*! = q^*!(s^* - 1)!s^* > (q^* + 1)!(s^* - 1)!$, so that

$$\frac{n!}{q^*!r^*!s^*!} < \frac{n!}{(q^* + 1)!r^*!(s^* - 1)!},$$

which contradicts the maximality assumption.

Since $n = q^* + r^* + s^*$ and $q^* \leqslant r^* \leqslant s^* \leqslant q^* + 1$ we have that $3q^* \leqslant n \leqslant 3q^* + 2$ and hence $q^* = \lfloor \frac{n}{3} \rfloor$. Furthermore, $n + 1 = r^* + s^* + q^* + 1$ and $r^* \leqslant s^* \leqslant q^* + 1 \leqslant r^* + 1$, as $q^* \leqslant r^*$. This implies that $3r^* \leqslant n + 1 \leqslant 3r^* + 2$ and hence $r^* = \lfloor \frac{n+1}{3} \rfloor$. Similarly, $n + 2 = s^* + q^* + 1 + r^* + 1$ and $s^* \leqslant q^* + 1 \leqslant r^* + 1 \leqslant s^* + 1$ imply $3s^* \leqslant n + 2 \leqslant 3s^* + 2$, so that $s^* = \lfloor \frac{n+2}{3} \rfloor$. Thus, we find that

$$\max_{q+r+s=n} \frac{n!}{q!r!s!} = \frac{n!}{\lfloor \frac{n}{3} \rfloor! \lfloor \frac{n+1}{3} \rfloor! \lfloor \frac{n+2}{3} \rfloor!}. \tag{5·5}$$

Now put $m = \lfloor \frac{n+1}{3} \rfloor + \lfloor \frac{n+2}{3} \rfloor$ and compute q, r, and s in the right-hand side of (5·3). As $2\lfloor \frac{n+1}{3} \rfloor \leqslant m \leqslant 2\lfloor \frac{n+1}{3} \rfloor + 1$, we find that $r = \lfloor m/2 \rfloor = \lfloor \frac{n+1}{3} \rfloor = r^*$. Moreover, $s = m - \lfloor m/2 \rfloor = \lfloor \frac{n+2}{3} \rfloor = s^*$. Since $n = q + r + s$ we also have that $q = \lfloor \frac{n}{3} \rfloor = q^*$. Further we remark that $m = n - q = n - \lfloor n/3 \rfloor \geqslant 2n/3 \geqslant n/2 \geqslant \lfloor n/2 \rfloor$ so that equation (5·3) implies

$$\max_{\lfloor n/2 \rfloor \leqslant m \leqslant n} \binom{n}{m} \binom{m}{\lfloor m/2 \rfloor} \geqslant \frac{n!}{\lfloor \frac{n}{3} \rfloor ! \lfloor \frac{n+1}{3} \rfloor ! \lfloor \frac{n+2}{3} \rfloor !}.$$

Finally we combine this inequality with (5·2), (5·4), and (5·5) to obtain the desired result.

A combination of Theorem 5·1 and Lemma 5·2 immediately gives the following corollary.

COROLLARY 5·3. *If K is a polyhedral cone with N facets in a finite dimensional vector space X, then the periods of periodic points of order preserving subhomogenous maps $f: K \to K$ do not exceed β_N, where β_N is given in* (2·2).

6. *Asymptotic behaviour of bounded orbits*

In this section we prove Theorem 2·1. To establish this result we need to understand the asymptotic behaviour of bounded orbits. It is therefore natural to study the structure of the ω-limit sets. If D is a metrizable topological space and $f: D \to D$ is a continuous map, then for each $x \in D$ the *ω-limit set* of x under f is given by

$$\omega(x; f) = \{y \in D: f^{k_i}(x) \to y \text{ for some sequence } (k_i) \text{ with } k_i \to \infty\}.$$

It is easy to verify that each ω-limit set is a (possibly empty) closed subset of D and that $f(\omega(x; f)) \subset \omega(x; f)$. Furthermore, if $\mathcal{O}(x; f)$ has a compact closure, then $\omega(x; f)$ is a nonempty compact subset of D and $f(\omega(x; f)) = \omega(x; f)$. The ω-limit sets also enjoy the following elementary property.

LEMMA 6·1. *Let D be a metrizable topological space. If $f: D \to D$ is a continuous map and $x \in D$ is such that $\mathcal{O}(x; f)$ has a compact closure and $\omega(x; f)$ is finite, then there exists a periodic point ξ of f, with period p, such that* $\lim_{k\to\infty} f^{kp}(x) = \xi$ *and* $\omega(x; f) = \mathcal{O}(\xi; f)$.

Proof. Since f is continuous and $\mathcal{O}(x; f)$ has a compact closure, $f(\omega(x; f)) = \omega(x; f)$. As $\omega(x; f)$ is a finite set, this implies that each $y \in \omega(x; f)$ is a periodic point of f. Moreover, as $\omega(x; f)$ is finite and D is a metrizable topological space, there exist pairwise disjoint neighbourhoods U_y for each $y \in \omega(x; f)$. Every $y \in \omega(x; f)$ also has a neighbourhood $V_y \subset U_y$ such that for each $u \in V_y$ we have that $f^q(u) \in U_y$, where q is the period of y under f, because f is continuous.

Let $\mathrm{cl}(\mathcal{O}(x; f))$ denote the closure of $\mathcal{O}(x; f)$ in D. Then there exists $m \geqslant 1$ such that for all $k \geqslant m$ we have that $f^k(x) \in V_y$ for some $y \in \omega(x; f)$. Indeed, if such an integer m does not exist, then there exists a sequence $(k_i)_i$ such that $k_i \to \infty$ and $f^{k_i}(x) \notin V_y$ for all $y \in \omega(x; f)$. But $\mathrm{cl}(\mathcal{O}(x; f))$ is compact, so that $(f^{k_i}(x))_i$ has a convergent subsequence, which has its limit outside $\omega(x; f)$. This is obviously a contradiction.

Now let $m \geqslant 1$ be such an integer. Suppose that $f^m(x) \in V_z$ and let p be the period of z. Then $f^{m+p}(x) \in U_z$; but, as the neighbourhoods U_y are pairwise disjoint and $f^{m+p}(x) \in V_y$ for some $y \in \omega(x; f)$, we find that $f^{m+p}(x) \in V_z$. By iterating the

argument we deduce that $f^{m+kp}(x) \in V_z$ for all $k \geqslant 1$. As z is the only limit point of $(f^k(x))_k$ in V_z, we conclude that $(f^{m+kp}(x))_k$ converges to z. This implies that $(f^{kp}(x))_k$ converges to $f^r(z)$, where $r \equiv -m \bmod p$, because f is continuous. Thus, if we take $\xi = f^r(z)$, then $\omega(x; f) = \mathcal{O}(\xi; f)$ and this completes the proof.

To prove Theorem 2·1 we first show that if $f: K \to K$ is a continuous order preserving subhomogenous map on a polyhedral cone, then the ω-limit sets of points, with a bounded orbit, are finite. A combination of this result with Lemma 6·1, and Corollary 5·3 will yield Theorem 2·1.

We shall use the following result of Nussbaum [**27**, corollary 2].

THEOREM 6·2 ([**27**]). *Let P be a part of a polyhedral cone K in a finite dimensional vector space X and let $m = |I(P)|$. If C is a compact subset of P and $f: C \to C$ is nonexpansive with respect to d_T, then there exists an integer τ_m, which only depends on m, such that $|\omega(x; f)| \leqslant \tau_m$ for every $x \in C$.*

The first ideas for this theorem go back to Weller [**35**, corollary 4·10], who proved a similar assertion, only without the upper bound.

In case K is the standard positive cone $\mathbb{R}^n_+$ and P is the part corresponding to $\operatorname{int}(\mathbb{R}^n_+)$, we know that the map $f: C \to C$, with $C \subset P$, is nonexpansive with respect to d_T if and only if the map $g: C' \to C'$ given by $g = L \circ f \circ E$, where L and E are given in (3·5) and (3·6), is nonexpansive with respect to the sup-norm. Using this observation it is not hard to show that Theorem 6·2 is equivalent to the following assertion: if C is a compact set of $\mathbb{R}^n$ and $g: C \to C$ is sup-norm nonexpansive, then there exists an integer τ_n, which only depends on n, such that $|\omega(x; g)| \leqslant \tau_n$ for all $x \in C$. It has been conjectured by Nussbaum [**27**, p. 525] that the optimal choice for τ_n is 2^n; but at present the conjecture is proved only for $n \leqslant 3$ (see [**21**]). The current best general estimate for τ_n is $\max_k 2^k \binom{n}{k}$ (see [**20**]).

We know by Lemma 3·3 that every order preserving subhomogeneous map is nonexpansive with respect to the part metric. Therefore Theorem 6·2 implies that if $f: K \to K$ is an order preserving subhomogenous map and $\mathcal{O}(x; f) \subset P$ has a compact closure in P, then $\omega(x; f)$ is finite. A difficulty arises when $\mathcal{O}(x; f)$ is an orbit in a part P, but its closure is not contained in P. To overcome this and other difficulties we shall use several technical lemmas.

Let us first recall the following old result of Freudenthal and Hurewicz [**10**].

LEMMA 6·3 ([**10**]). *Let (C, d) be a compact metric space and let D be a nonempty subset of C. If $f: D \to D$ is nonexpansive and f maps D onto itself, then f has a unique continuous extension $F: \operatorname{cl}(D) \to \operatorname{cl}(D)$, where $\operatorname{cl}(D)$ denotes the closure of D, and F is an isometry of $\operatorname{cl}(D)$ onto itself.*

A combination of this lemma with Lemma 6·1 and Theorem 6·2 yields the following corollary.

COROLLARY 6·4. *Let P be a part of a polyhedral cone K in a finite dimensional vector space X. If C is a compact subset of P and $f: C \to C$ is d_T-nonexpansive map that maps C onto itself, then every point $x \in C$ is a periodic point of f.*

Proof. We first remark that as C is a compact subset of a part of the cone, (C, d_T) is a compact metric space. Since f is d_T-nonexpansive and maps C onto itself, it follows from Lemma 6·3 that f is an isometry with respect to d_T.

Now let $x \in C$ and remark that $\omega(x; f)$ is finite by Theorem 6·2. As $\mathcal{O}(x; f) \subset C$, it has a compact closure. Therefore Lemma 6·1 implies that there exists a periodic point $\xi \in C$ of f, with period p, such that $f^{kp}(x)$ converges to ξ, as k goes to infinity. Since f is an isometry with respect to d_T, we find that

$$d_T(f^p(x), x) = d_T\left(f^{(k+1)p}(x), f^{kp}(x)\right) \quad \text{for all } k \geqslant 0.$$

We now observe that the right-hand side of this equality converges to 0, as k goes to infinity, and hence $d_T(f^p(x), x) = 0$. Thus $f^p(x) = x$ and this completes the proof.

The following technical lemma is stated in considerably greater generality than is actually needed here. Recall (see [**7**, p. 41]) that if K is a closed cone in a topological vector space X and $x \in K$, we say that K *satisfies condition G at* x if for every $0 < \lambda < 1$ and every sequence $(x_k)_k$ in K such that $\lim_{k\to\infty} x_k = x$, there exists $k^* \geqslant 1$ such that $\lambda x \leqslant x_k$ for all $k \geqslant k^*$. We say that K *satisfies condition G* if it satisfies condition G at every $x \in K$. If K has a nonempty interior, then K satisfies condition G at every point in its interior. If K is a closed cone in a Hausdorff topological vector space X, it is proved in [**7**, lemma 3·3] that K is a polyhedral cone in X if and only if X is finite dimensional and K satisfies condition G.

Lemma 6·5. *Let K be a closed cone in a metrizable topological vector space X and let $D \subset K$ be such that $\lambda D \subset D$ for all $0 < \lambda < 1$. Suppose that $f: D \to D$ is order preserving and let $x \in D$. If at every periodic point $\eta \in \omega(x; f)$ of f condition G is satisfied and there exists $\delta = \delta(\eta) > 0$ such that $\lambda f^m(\eta) \leqslant f^m(\lambda\eta)$ for all $m \geqslant 1$ and $1-\delta \leqslant \lambda < 1$, then for every $y \in \omega(x; f)$ and for every periodic point $\xi \in \omega(x; f)$ of f there exists $j \geqslant 0$ such that $f^j(\xi) \leqslant y$. Moreover, if there exists a periodic point ξ in $\omega(x; f)$, then $\mathcal{O}(\xi; f)$ is the only periodic orbit of f in $\omega(x; f)$.*

Before proving this lemma we remark that if K is a polyhedral cone in a finite dimensional vector space, then K satisfies condition G at every point in D. Furthermore, the condition concerning the existence of δ in Lemma 6·5 holds for every $y \in D$ if f is subhomogeneous.

Proof of Lemma 6·5. Assume that $\xi \in \omega(x; f)$ is a periodic point of f with period p. By definition, there exists a sequence $(k_i)_i$ such that $f^{k_i}(x) \to \xi$, as $i \to \infty$. By taking a subsequence we may assume that there exists $0 \leqslant \sigma < p$ such that $k_i \equiv \sigma \bmod p$ for all $i \geqslant 1$. Take λ with $1 - \delta(\xi) \leqslant \lambda < 1$. As K satisfies condition G at ξ, we have that $\lambda\xi \leqslant f^{k_i}(x)$ for all sufficiently large i. Suppose that $y \in \omega(x; f)$ and let $(m_i)_i$ be such that $f^{m_i}(x) \to y$ as $i \to \infty$. By taking a subsequence we may assume that $m_i > k_i$ for all $i \geqslant 1$ and that there exists an integer $0 \leqslant \tau < p$ such that $m_i - k_i \equiv \tau \bmod p$ for all $i \geqslant 1$. For sufficiently large i we now find that

$$f^{m_i}(x) = f^{m_i - k_i}(f^{k_i}(x)) \geqslant f^{m_i - k_i}(\lambda\xi) \geqslant \lambda f^{m_i - k_i}(\xi) = \lambda f^{\tau}(\xi).$$

Letting i go infinity on the left-hand side we find that $\lambda f^{\tau}(\xi) \leqslant y$. Subsequently by letting λ approach 1 we deduce that $f^{\tau}(\xi) \leqslant y$, which proves the first assertion.

To show the second assertion we suppose that ξ and η in $\omega(x; f)$ are periodic points of f with period p and q, respectively. We need to show that $\mathcal{O}(\eta; f) = \mathcal{O}(\xi; f)$. It follows from the first assertion that there exist $0 \leqslant \mu < p$ and $0 \leqslant \nu < q$ such that $f^{\mu}(\xi) \leqslant \eta$ and $f^{\nu}(\eta) \leqslant \xi$. Since f is order preserving, it follows that $f^{\mu+k}(\xi) \leqslant f^k(\eta)$

and $f^{\nu+k}(\eta) \leqslant f^k(\xi)$ for all $k \geqslant 0$. This implies that

$$f^{\mu+\nu}(\xi) \leqslant f^{\nu}(\eta) \leqslant \xi.$$

By Lemma 3·1, we know that $\mathcal{O}(\xi; f)$ is an antichain, so that $\xi = f^{\mu+\nu}(\xi)$ and hence $\xi = f^{\nu}(\eta)$. As η and ξ are both periodic points of f, it follows that $\mathcal{O}(\eta; f) = \mathcal{O}(\xi; f)$ and this completes the proof.

The following two lemmas tell us that we can reduce the problem to the case where the ω-limit set is contained in a part of the cone.

Lemma 6·6. *Let K be a polyhedral cone in a finite dimensional vector space X. If $f: K \to K$ is d_T-nonexpansive, then there exists $m \geqslant 1$ such that $f^{2m}(x) \sim f^m(x)$ for all $x \in K$.*

Proof. Let $F: P(K) \to P(K)$ be the map in Corollary 3·5. Since $P(K)$ is a finite set, we know for each $P \in P(K)$ that the sequence $(F^k(P))_k$ is eventually periodic, i.e., there exist $r \geqslant 0$ and $p \geqslant 1$ such that $F^r(P) = F^{r+kp}(P)$ for all $k \geqslant 0$. By the pigeonhole principle, we can take $r + p \leqslant 2^N$, where N is the number of facets of K, because $|P(K)| \leqslant 2^N$. Now put $m = \operatorname{lcm}(1, \ldots, 2^N)$. Clearly, $r \leqslant m$ and p divides m. Therefore $F^m(P) = F^{2m}(P)$ for each $P \in P(K)$. By taking $P = [x]$, we find that $[f^m(x)] = F^m(P) = F^{2m}(P) = [f^{2m}(x)]$ and from this we conclude that $f^m(x) \sim f^{2m}(x)$ for all $x \in K$.

Lemma 6·7. *Let K be a polyhedral cone in a finite dimensional vector space X and let $g: K \to K$ be a continuous order preserving subhomogeneous map. If $x \in K$ is such that $\mathcal{O}(x; g)$ is bounded and $\mathcal{O}(x; g)$ is contained in a part of K, then $\omega(x; g)$ is contained in a part of K.*

Proof. Assume that $\mathcal{O}(x; g)$ is contained in a part P of K. If $P = \{0\}$, then $\mathcal{O}(x; g) = \omega(x; g) = \{0\}$ and hence the result is trivial in that case. Now assume that $P \neq \{0\}$, so that $I(P)$ is nonempty. We first show that there exists $c \geqslant 1$ such that $y \leqslant cx$ for all $y \in \mathcal{O}(x; g)$. As $\mathcal{O}(x; g) \subset P$, we get that $I_y = I_x = I(P)$ for all $y \in \mathcal{O}(x; g)$. This implies that $\psi_i(y) > 0$ if and only if $i \in I(P)$. Define a number c by

$$c = \sup\{\psi_i(y)/\psi_i(x) : y \in \mathcal{O}(x; g) \text{ and } i \in I(P)\}.$$

The number c is finite, because $\psi_i(x) > 0$ for all $i \in I(P)$ and $\mathcal{O}(x; g)$ is a bounded subset of X. Moreover, $c \geqslant 1$, as $x \in \mathcal{O}(x; g)$ and $I(P)$ is nonempty. Now let $y \in \mathcal{O}(x; g)$. By definition of c we have that $\psi_i(y - cx) \leqslant 0$ for all $i \in I(P)$. Since $\psi_i(y) = \psi_i(x) = 0$ for all $i \notin I(P)$, we deduce that $y \leqslant cx$.

We remark that $\{y \in K : y \leqslant cx\}$ is a closed set that contains $\mathcal{O}(x; g)$ and hence it also contains $\omega(x; g)$. As g is an order preserving subhomogeneous map and $c \geqslant 1$, we find that $g^k(y) \leqslant g^k(cx) \leqslant cg^k(x)$ for all $y \in \omega(x; g)$ and $k \geqslant 0$. The map g maps $\omega(x; g)$ onto itself, because g is continuous and $\omega(x; g)$ is bounded. Therefore $y \leqslant cg^k(x)$ for all $y \in \mathcal{O}(x; g)$ and $k \geqslant 0$. As the set $\{z \in K : c^{-1}y \leqslant z\}$ is closed, this implies that $y \leqslant cz$ for all $y, z \in \omega(x; g)$. Therefore $y \sim z$ for all $y, z \in \omega(x; g)$, which completes the proof.

Equipped with these lemmas we can now prove the following theorem.

Theorem 6·8. *Let K be a polyhedral cone in a finite dimensional vector space X. If $f: K \to K$ is a continuous order preserving subhomogeneous map and $x \in K$ has a bounded orbit under f, then $\omega(x; f)$ is finite.*

Proof. It follows from Lemma 3·4 that f is d_T-nonexpansive. Let m be as in Lemma 6·6 and put $g = f^m$ and $P = [g(x)]$. Clearly, $g(P) \subset [f^{2m}(x)] = [f^m(x)] = P$ and hence $\mathcal{O}(g(x); g) \subset P$. As $\mathcal{O}(g(x); g) \subset \mathcal{O}(x; f)$, the orbit $\mathcal{O}(g(x); g)$ is bounded. It is easy to verify that

$$\omega(x; f) = \omega(g(x); f) = \bigcup_{j=0}^{m-1} f^j(\omega(g(x); g)).$$

Therefore it suffices to show that $\omega(x'; g)$ is finite, whenever $\mathcal{O}(x'; g)$ is bounded and contained in a part P, such that $g(P) \subset P$.

So, suppose that $\mathcal{O}(x'; g)$ is a bounded orbit that is contained in a part P of K and $g(P) \subset P$. It follows from Lemma 6·7 that $\omega(x'; g)$ is included in a part of K, say Q. Since $\omega(x'; g)$ is a bounded closed set in Q, we have that $(\omega(x'; g), d_T)$ is a compact metric space. The map g is d_T-nonexpansive on Q and g maps $\omega(x'; g)$ onto itself. Therefore we can apply Corollary 6·4 and conclude that each point in $\omega(x'; g)$ is a periodic point of g. As g is an order preserving subhomogeneous maps and K is a polyhedral cone, it follows from Lemma 6·5 that there is at most one periodic orbit in $\omega(x'; g)$. This implies that $\omega(x'; g)$ is finite and hence the proof is complete.

Knowing Theorem 6·8 it is now straightforward to prove Theorem 2·1.

Proof of Theorem 2·1. Let $f: K \to K$ be a continuous order preserving subhomogeneous map, where K is a polyhedral cone with N facets in a finite dimensional vector space X. Suppose that the orbit of $x \in K$ is bounded. Then it follows from Theorem 6·8 that $\omega(x; f)$ is finite. Therefore Lemma 6·1 implies that there exists a periodic point $\xi \in K$ of f, with period p, such that $(f^{kp}(x))_k$ converges to ξ. To finish the proof we remark that it follows from Corollary 5·3 that p is bounded by β_N, where β_N is given in (2·2).

7. *A lower bound for the maximal period*

In this section a proof of Theorem 2·2 is presented. Indeed, given $1 \leqslant m \leqslant n$ we construct for every $1 \leqslant p \leqslant \binom{m}{\lfloor m/2 \rfloor}$ and $1 \leqslant q \leqslant \binom{n}{m}$ a continuous order preserving homogeneous map $f: \mathbb{R}^n_+ \to \mathbb{R}^n_+$ that has a periodic point with period $\operatorname{lcm}(p, q)$. In the proof of Theorem 2·2 we use the following consequence of an observation of Gunawardena and Sparrow (see [**13**, p. 152]).

Lemma 7·1 ([**13**]). *For each $1 \leqslant p \leqslant \binom{n}{\lfloor n/2 \rfloor}$ there exists a continuous order preserving homogeneous map $h: \mathbb{R}^n_+ \to \mathbb{R}^n_+$ that has a periodic point x, with period p, such that $\mathcal{O}(x; h) \subset \operatorname{int}(\mathbb{R}^n_+)$.*

Indeed, Gunawardena and Sparrow [**13**] constructed for every $1 \leqslant p \leqslant \binom{n}{\lfloor n/2 \rfloor}$ a so called topical map $f : \mathbb{R}^n \to \mathbb{R}^n$ that has a periodic point u with period p. By defining $h = \mathrm{E} \circ f \circ \mathrm{L}$, where $\mathrm{L}: \operatorname{int}(\mathbb{R}^n_+) \to \mathbb{R}^n$ and $\mathrm{E}: \mathbb{R}^n \to \operatorname{int}(\mathbb{R}^n_+)$ are respectively given in (3·5) and (3·6), we obtain an order preserving homogeneous map $h: \operatorname{int}(\mathbb{R}^n_+) \to \operatorname{int}(\mathbb{R}^n_+)$, which has $\mathrm{E}(u)$ as a periodic point with period p. To derive the conclusion of Lemma 7·1 we take a continuous extension of h to $\mathbb{R}^n_+$ that is order preserving and homogeneous. Such extensions always exist (see [**7**]). Indeed, in our case it is straightforward to find one.

The proof of Theorem 2·2 is quite technical. For the reader's convenience we have therefore worked out an illustrative example in the paragraph directly following the proof. It may be helpful to read the two in parallel. Before we start the proof it useful to introduce the following notation: for $a, b \in \mathbb{R}$ we write $a \wedge b$ to denote $\min\{a, b\}$ and $a \vee b$ to denote $\max\{a, b\}$.

Proof of Theorem 2·2. Consider a collection of q distinct vectors $\{v^1, \ldots, v^q\}$ in $\{0,1\}^n$, each with m nonzero coordinates, so that $q \leqslant \binom{n}{m}$. Put $v^{q+1} = v^1$. Further let $g\colon \mathbb{R}^m_+ \to \mathbb{R}^m_+$ be a continuous order preserving homogeneous map and assume that there exists $C > 0$ such that

$$g(z)_i \leqslant C(z_1 \wedge z_2 \wedge \cdots \wedge z_m) \quad \text{for all } 1 \leqslant i \leqslant m \text{ and } z \in \mathbb{R}^m_+.$$

Assume also that g has a periodic point y with period p, where $1 \leqslant p \leqslant \binom{m}{\lfloor m/2 \rfloor}$ and $\mathcal{O}(y; g) \subset \operatorname{int}(\mathbb{R}^m_+)$. The existence of such a map g and a periodic point y is guaranteed by taking a map h as in Lemma 7·1 and defining $g(z)_i = h(z)_i \wedge C(z_1 \wedge \cdots \wedge z_m)$ for $1 \leqslant i \leqslant m$, with C large enough.

For $1 \leqslant k \leqslant q$ and $1 \leqslant i \leqslant m$, we let $\nu(k, i)$ be the index of the ith nonzero coordinate of v^k. Further for each $x \in \mathbb{R}^n_+$, we let $x_{|v^k}$ be the vector in $\mathbb{R}^m_+$ given by $(x_{|v^k})_i = x_{\nu(k,i)}$ for all $1 \leqslant i \leqslant m$. Subsequently, we define $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ in the following manner:

$$f(x)_i = \bigvee_{(k,r):\nu(k+1,r)=i} g(x_{|v^k})_r \quad \text{for each } 1 \leqslant i \leqslant n \text{ and } x \in \mathbb{R}^n_+. \tag{7·1}$$

It easy to see that $f\colon \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a continuous order preserving homogeneous map. Furthermore it has a periodic point with period $\operatorname{lcm}(p, q)$. Indeed, for $0 \leqslant a \leqslant p-1$ and $1 \leqslant b \leqslant q$ let $y^{a,b} \in \mathbb{R}^n_+$ be given by

$$y_i^{a,b} = \begin{cases} 0 & \text{if } v_i^b = 0 \\ g^a(y)_r & \text{if } i = \nu(b, r). \end{cases}$$

As $\{g^k(y)\colon 0 \leqslant k < p\} \subset \operatorname{int}(\mathbb{R}^m_+)$, it is evident that $y^{a,b} = y^{c,d}$ if and only if $a = c$ and $b = d$, so that they are all distinct. To complete the proof we now show that $f(y^{a,b}) = y^{a+1,b+1}$, where the indices a and b are counted modulo p and modulo q, respectively. As $g(z)_i \leqslant C(z_1 \wedge z_2 \wedge \cdots \wedge z_m)$ for each $1 \leqslant i \leqslant m$ and $z \in \mathbb{R}^m_+$, we have that

$$g\big(y^{a,b}_{|v^k}\big) = \begin{cases} 0 & \text{if } k \neq b \\ g(g^a(y)) & \text{if } k = b. \end{cases}$$

Therefore

$$f(y^{a,b})_i = \begin{cases} 0 & \text{if } v_i^{b+1} = 0 \\ g^{a+1}(y)_r & \text{if } i = \nu(b+1, r), \end{cases}$$

for $1 \leqslant i \leqslant n$. Thus, $f(y^{a,b}) = y^{a+1,b+1}$ and hence $y^{0,1}$ is a periodic point of f with period $\operatorname{lcm}(p, q)$.

To illustrate the construction in the proof of Theorem 2·2, we consider the following example. Let $m = 2$, $n = 3$, $p = 2$, and $q = 3$. Put

$$v^1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \; v^2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } v^3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Further let $g\colon \mathbb{R}^2_+ \to \mathbb{R}^2_+$ be given by

$$g\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 3z_1 \wedge z_2 \\ z_1 \wedge 3z_2 \end{pmatrix},$$

and take $y = (1, 2)$. It is easy to see that y is a periodic point of g with period 2. The map $f\colon \mathbb{R}^3_+ \to \mathbb{R}^3_+$ defined in (7·1) is then given by

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} g(x_1, x_2)_1 \vee g(x_2, x_3)_1 \\ g(x_1, x_3)_1 \vee g(x_2, x_3)_2 \\ g(x_1, x_2)_2 \vee g(x_1, x_3)_2 \end{pmatrix} = \begin{pmatrix} (3x_1 \wedge x_2) \vee (3x_2 \wedge x_3) \\ (3x_1 \wedge x_3) \vee (x_2 \wedge 3x_3) \\ (x_1 \wedge 3x_2) \vee (x_1 \wedge 3x_3) \end{pmatrix}.$$

Now it is easy to verify that

$$y^{0,1} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix},\ y^{1,2} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix},\ y^{0,3} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix},$$
$$y^{1,1} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix},\ y^{0,2} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix},\ y^{1,3} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix},$$

is a periodic orbit of f with period $\operatorname{lcm}(2, 3) = 6$.

We would also like to point out that if we take $g(z)_i = z_1 \wedge z_2 \wedge \cdots \wedge z_m$ for all $1 \leqslant i \leqslant m$ in the proof, then we recover the construction of Gunawardena and Sparrow. In particular, the maps f and g are so-called min-max maps.

It follows directly from Theorem 2·2 that α_N given in (2·3) is a lower bound for the maximal period of periodic points of continuous order preserving subhomogeneous maps $f\colon K \to K$, where K is a polyhedral cone with N facets in a finite dimensional vector space X. By using the prime number theorem we now show that α_N has the same asymptotics as the upper bound β_N given in (2·2). From Lemma 5·2 and equation (5·2) it follows that

$$\beta_N = \max_{\lfloor N/2 \rfloor \leqslant m \leqslant N} \binom{N}{m}\binom{m}{\lfloor m/2 \rfloor} = \max_{1 \leqslant m \leqslant N} \binom{N}{m}\binom{m}{\lfloor m/2 \rfloor}. \tag{7·2}$$

For a given N let m^* be the m that attains the maximum in the right-hand side of (7·2). From the proof of Lemma 5·2 we know that $m^* = \lfloor \frac{N+1}{3} \rfloor + \lfloor \frac{N+2}{3} \rfloor$. Now for each $k \geqslant 1$ let $\rho(k)$ be the largest prime not exceeding k. It then follows from the prime number theorem that

$$\lim_{k \to \infty} \frac{\rho(k)}{k} = 1. \tag{7·3}$$

Indeed, let p_N denote the Nth prime and let $\pi(k)$ be the number of primes not exceeding k. Then $\rho(k) = p_{\pi(k)}$ for each $k \geqslant 1$. It is known that

$$\lim_{N \to \infty} \frac{p_N}{N \log N} = 1 \quad \text{and} \quad \lim_{k \to \infty} \frac{\pi(k) \log \pi(k)}{k} = 1$$

are equivalent to the prime number theorem (see [**2**, p. 80]). Thus, the prime number theorem implies that

$$\lim_{k \to \infty} \frac{\rho(k)}{k} = \lim_{k \to \infty} \frac{p_{\pi(k)}}{\pi(k) \log \pi(k)} \cdot \frac{\pi(k) \log \pi(k)}{k} = 1.$$

We now observe that

$$\alpha_N \geqslant \rho\left(\binom{N}{m^*}\right)\binom{m^*}{\lfloor m^*/2\rfloor},$$

if $\rho(\binom{N}{m^*})$ and $\binom{m^*}{\lfloor m^*/2\rfloor}$ are coprime. As $m^* = \lfloor \frac{N+1}{3}\rfloor + \lfloor \frac{N+2}{3}\rfloor$, we can use Stirling's formula to show that there exists $M \geqslant 1$ such that

$$2\binom{m^*}{\lfloor m^*/2\rfloor} \leqslant \binom{N}{m^*} \quad \text{for all } N \geqslant M \quad \text{and} \quad \binom{N}{m^*} \to \infty, \qquad \text{as } N \to \infty.$$

Therefore (7·3) implies that $\rho(\binom{N}{m^*}) > \binom{m^*}{\lfloor m^*/2\rfloor}$ for all N sufficiently large and hence they are coprime. Thus, we derive that

$$\lim_{N\to\infty} \frac{\alpha_N}{\beta_N} \geqslant \lim_{N\to\infty} \frac{\binom{m^*}{\lfloor m^*/2\rfloor}\rho\left(\binom{N}{m^*}\right)}{\binom{m^*}{\lfloor m^*/2\rfloor}\binom{N}{m^*}} = 1.$$

As $\alpha_N \leqslant \beta_N$ for each $N \geqslant 1$, we find that $\lim_{n\to\infty} \alpha_N/\beta_N = 1$.

We conclude the paper with some remarks. Given a polyhedral cone K, let $\Gamma(K)$ be the set of integers $p \geqslant 1$ for which there exists a continuous order preserving subhomogeneous map $f\colon K \to K$ that has a periodic point with period p. From Theorem 5·1 it follows that $\Gamma(K)$ is a finite set. In fact, Theorem 5·1 implies that if K has N facets, then $\Gamma(K) \subset B(N)$, where $B(N)$ is the set of $p \geqslant 1$ for which there exist integers q_1 and q_2 such that $p = q_1 q_2$, $1 \leqslant q_1 \leqslant \binom{N}{m}$, and $1 \leqslant q_2 \leqslant \binom{m}{\lfloor m/2\rfloor}$ for some $1 \leqslant m \leqslant N$. In particular, it follows that $\Gamma(\mathbb{R}^3_+) \subset \{1, 2, 3, 4, 6\}$, so that 5 is not in $\Gamma(\mathbb{R}^3_+)$. By Theorem 2·2 we know that $\Gamma(\mathbb{R}^N_+) \supset A(N)$, where $A(N)$ is the set of $p \geqslant 1$ for which there exist $1 \leqslant m \leqslant N$, $1 \leqslant q_1 \leqslant \binom{N}{m}$, and $1 \leqslant q_2 \leqslant \binom{m}{\lfloor m/2\rfloor}$ such that $p = \operatorname{lcm}(q_1, q_2)$. For instance, $\Gamma(\mathbb{R}^3_+) \supset A(3) = \{1, 2, 3, 6\} \neq B(3)$. Thus, for each $N \geqslant 1$ we have the following inclusions:

$$A(N) \subset \Gamma(\mathbb{R}^N_+) \subset B(N).$$

Knowing these inclusions it is natural to ask if there exists a characterization of $\Gamma(\mathbb{R}^N_+)$ in terms of arithmetical and (or) combinatorial constraints. In particular, one might wonder if $\Gamma(\mathbb{R}^N_+) = A(N)$ for all $N \geqslant 1$, or, if $\Gamma(\mathbb{R}^N_+) = B(N)$ for all $N \geqslant 1$. This question is investigated by Bas Lemmens and Colin Sparrow in a forthcoming paper.

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