BLOCKS WITH A CYCLIC DEFECT GROUP

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PART C APPLICATIONS TO THE SYMMETRIC, ALTERNATING AND MATHIEU GROUPS

SYNOPSIS

This thesis is concerned with the block theory of a finite group over an algebraically closed field of prime characteristic. We study the indecomposable modules in such a block $\underset{\sim}{B}$ with a cyclic defect group, and then the results that we get are applied to the principal p-blocks of the symmetric group S_p, the alternating group A_p and the five Mathieu groups, p being an odd prime.

There are three parts to this thesis, each one dealing with the following topics. Part A is concerned with finding the full submodule lattice of each one of the projective indecomposable modules in $\underset{\sim}{B}$. These lattices turn out to have simple diamond shapes, and depend upon parameters r(i), s(i). In part B we generalise the methods used in the previous part to get a good description of a full set of non-projective indecomposables in $\underset{\sim}{B}$ (though we do not manage to get their full submodule lattices). Finally in part C we show how to use ordinary character theory, and in particular the Brauer tree, to work out these positive integers r(i), s(i) and then we apply such methods to work out the lattices of those examples listed above.

STANDARD NOTATION

Throughout this thesis, G will be a finite group, p a prime that divides the order of G and k an algebraically closed field which has characteristic p. Also 1 will denote the identity subgroup where appropriate and all modules are to be taken as finitely generated right modules. Indeed if U is a kG-module, $\ell(U)$ will denote its composition length, $\Sigma(U)$ its socle and $\Phi(U)$ its Frattini submodule.

The reader should note that each part of this thesis contains its own set of references and footnotes, which can be found at the end of the appropriate part.

PART A PROJECTIVE INDECOMPOSABLES IN A BLOCK WITH A CYCLIC DEFECT GROUP

§ 1 Introduction

Throughout this part of the thesis, we will be studying the following situation,

Hypothesis: $\underset{\sim}{B}$ is a kG-block with cyclic defect group D of order $q = p^d$ $(d \geqslant 1)$. We will write the unique chain for D as follows,

$$D = D_o > D_1 > \dots > D_{d-1} > 1$$

Set $N_t = N_G(D_t)$ and $C_t = C_G(D_t)$ for all $t = 0,1,\dots,d-1$; which hence yields the following chains:

$$N_G(D) = N_o \leqslant N_1 \leqslant \dots \leqslant N_{d-1} \leqslant G$$

$$C_G(D) = C_o \leqslant C_1 \leqslant \dots \leqslant C_{d-1} \leqslant G$$

For each t, let B_t be the unique kN_t-block of defect group D with $B_t{}^G = \underset{\sim}{B}$ and suppose that $\varepsilon_t \in Z(kN_t)$ is the corresponding block idempotent. Then it is well known that $\varepsilon_t \in Z(kC_t)$, so let $\varepsilon_t = \varepsilon_{t1} + \dots + \varepsilon_{tn_t}$ be a primitive decomposition of ε_t in $Z(kC_t)$, and let b_{tj} be the kC_t-block corresponding to ε_{tj}. Then Brauer [2] and Dade [1] have proved the following results:

a) The defect group of each b_{tj} is D.

b) For each t, a kC_t-block b satisfies $b^G = \underset{\sim}{B}$ if and only if $b \in \{b_{t1},\dots,b_{tn_t}\}$.

c) N_t acts (by conjugation) transitively on $\{\varepsilon_{t1},\dots,\varepsilon_{tn_t}\}$.

d) If E is the stabiliser of ε_{o1} in N_o, then for all t, EC_t is the stabiliser of ε_{t1} in N_t, and $E/C_o \cong EC_t/C_t$.

e) E/C_o is cyclic of order say $e = e(G,\underset{\sim}{B})$; and e divides p-1.

In [13] R. Brauer described the ordinary character theory of such a block $\underset{\sim}{B}$ for the special case when $d = 1$, and twenty five years later E. C. Dade ([1]) extended all of these results to the general case. Then by making essential use of Dade's results, H. Kupisch ([10]) and

G. J. Janusz ([11]), working independently, examined the kG-modules in $\underset{\sim}{B}$ and in part described the projective indecomposables. In this part of the thesis we will also describe these projective indecomposables in $\underset{\sim}{B}$, but the methods will be purely modular. In fact no character theory at all will be used, and the only essential results from Dade's paper that we need are those stated above.(1) The main result to be proved here is the very explicit description of the complete kG-submodule lattice of each projective indecomposable in $\underset{\sim}{B}$, which is given in the main theorem below.

The methods we employ utilise the "Green correspondence" (see § 2), and were first applied to this problem by W. Feit. I am very much indebted to my Ph.D. supervisor J. A. Green, whose helpful suggestions have made this work much easier to read and understand.

Before stating the main theorem we need some definitions:

<u>Defn:</u> If H is a subgroup of G, then a kH-block B will be called (q,e)-uniserial if it satisfies the following,

a) B contains (up to isomorphism) exactly e simple kH-modules.

b) The defect group of B has order q, with e dividing q-1.

c) The e distinct projective indecomposable kH-modules in B are all uniserial.

d) A full set of simple kH-modules in B can be labelled $S_o,\dots,S_{e-1}$ and a full set of projective indecomposable kH-modules in B can be labelled $T_o,\dots,T_{e-1}$ (with the convention that S_i, T_i are defined for all $i \in \mathbb{Z}$ by reducing i mod e) so that the unique composition series of each T_i has the shape,

$$T_i \bullet\!\xrightarrow{\;S_i\;}\!\bullet\!\xrightarrow{\;S_{i+1}\;}\!\bullet \quad \cdots\cdots \quad \bullet\!\xrightarrow{\;S_{i+q-1}\;}\!\bullet\, 0$$

<u>Remarks</u>: (i) $S_{i+q-1} \cong S_i$ since $q-1 \equiv 0 \mod e$.

(ii) By Nakayama's theorem (see [6]), the set of all the kH-factor-modules of the T_i $(0 \leqslant i \leqslant e-1)$ form a full set of indecomposable kH-modules in a (q,e)-uniserial kH-block B. So these indecomposables can be labelled

$T_{i\alpha}$ $i = 0,1,\dots,e-1$; $\alpha = 1,2,\dots,q$

so that the unique composition series of each $T_{i\alpha}$ has the shape:

$$T_{i\alpha} \overset{S_i}{\text{———}} \overset{S_{i+1}}{\text{———}} \cdots \overset{S_{i+\alpha-1}}{\text{———}} 0$$

Hence $T_i = T_{iq}$ and $S_i = T_{i1}$ for each i.

(iii) We define $T_{i\alpha}$ for each $i \in \mathbb{Z}$ by reducing i mod e. We also set $T_{io} = 0$ for all $i = 0,1,\dots,e-1$.

(iv) $T_{i\alpha}$ has composition length α for all $i;\alpha$.

Consider now the special case when H contains a cyclic normal p-subgroup, say $U = \langle u \rangle$.

For each $h \in H$ define an integer z(h), which is unique mod $|U|$, so that $h^{-1}uh = u^{z(h)}$. Also if 1_k denotes the identity element in k, set $\pi(h) = z(h)1_k$, which defines the "natural" linear representation π of H over k.

Defns: (i) For each $i \in \mathbb{Z}$, let Π^i be a kH-module affording the linear representation π^i.

(ii) Let B be a kH-block with S any simple kH-module in B. For each $i \in \mathbb{Z}$ set $S_i = S \otimes \Pi^i$. Then B will be called special (q,e)-uniserial (with respect to U) if it satisfies both of the following conditions,

a) $S_o,\dots,S_{e-1}$ is a full set of simples in B with $S_{i+e} \cong S_i$ for each $i \in \mathbb{Z}$.

b) B is (q,e)-uniserial with respect to the labelling $S_o,\dots,S_{e-1}$ of the simples.

We can now state our main theorem:

THEOREM: Under the hypothesis described above (see page 1) we have,

(i) For each $t = 0,1,\dots,d-1$ the kN_t-block B_t is special (q,e)-uniserial (with respect to D_t).

(ii) Denote the indecomposable kN_{d-1}-modules in B_{d-1} by $\{T_{i\alpha}\}$, and let f be the Green correspondence $(G,\underset{\sim}{B}) \longrightarrow (N_{d-1},B_{d-1})$

(see §2). Then $\underset{\sim}{B}$ contains (up to isomorphism) exactly e simple kG-modules, which can be labelled $V_0,\ldots,V_{e-1}$; so that on writing $I = \{0,1,\ldots,e-1\}$, the "Frattini factor" $fV_i/\Phi(fV_i) \cong S_i$ for all $i \in I$.

Moreover there exists a permutation δ of I so that the "socle" $\Sigma(fV_i) \cong S_{\delta^{-1}(i)}$ for all $i \in I$.

Adopting this notation, let W_i be a projective cover of V_i and define a new permutation ρ of I by $\rho(i) \equiv \delta^{-1}(i) + 1 \mod e$. Then there exist integers $r = r(i)$, $s = s(i)$ so that each W_i contains exactly $rs + 2$ submodules, with the full kG-submodule lattice as shown in Figure 1.

Also these submodules satisfy the following:

a) $W_i / W_{00} \cong V_i$, $W_{r-1,s-1} \cong V_i$,
$W_{a-1,b} / W_{ab} \cong V_{\rho^a(i)}$ for all $1 \leqslant a \leqslant r-1$; $0 \leqslant b \leqslant s-1$,
$W_{a,b-1} / W_{ab} = V_{\delta^b(i)}$ for all $0 \leqslant a \leqslant r-1$; $1 \leqslant b \leqslant s-1$.

b) $W_{r-1,0}$ and $W_{0,s-1}$ are both uniserial submodules of W_i.

c) $W_{r-1,0} \cap W_{0,s-1} \cong V_i$ and $W_{r-1,0} + W_{0,s-1} = \Phi(W_i)$.

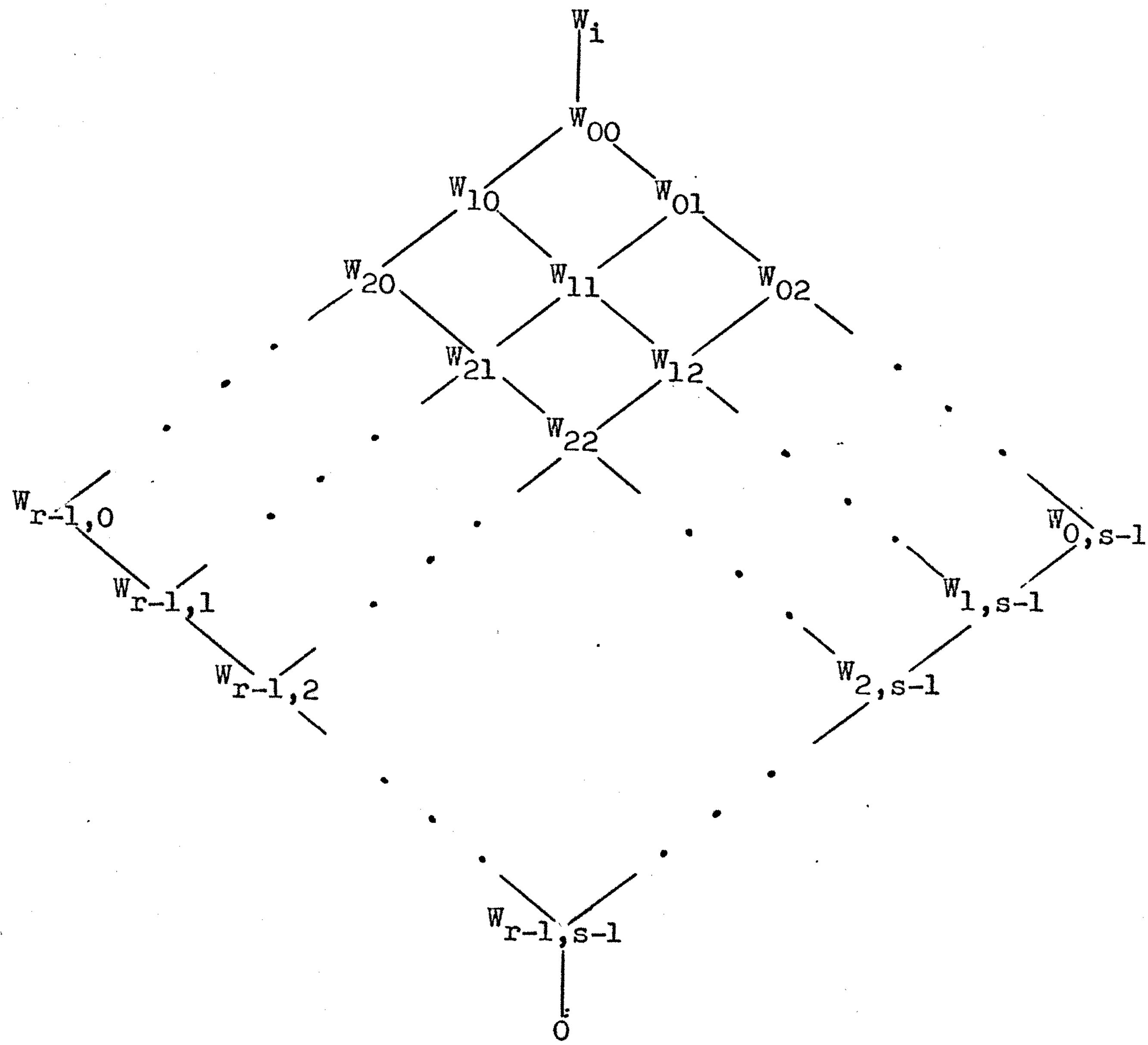
d) $(W_{r-1,0}\,;\, W_{0,s-1})$ is the unique pair of submodules of W_i satisfying b) and c).

Finally we have:

e) For a fixed $i \in I$, W_i is uniserial if and only if either $r(i) = 1$ or $s(i) = 1$; or equivalently if and only if the composition length of fV_i is either q-1 or 1.

f) If W_i is uniserial for all $i \in I$, then either $\delta = 1$ or $\rho = 1$ (1 being the identity permutation on I). Moreover:
In the case $\delta = 1$, $\underset{\sim}{B}$ is (q,e)-uniserial (with respect to this labelling $V_0,\ldots,V_{e-1}$ of simples) and fV_i is simple for all $i \in I$.
In the case $\rho = 1$, $\underset{\sim}{B}$ is (q,e)-uniserial (with respect to this labelling $V_0,\ldots,V_{e-1}$ of simples) and ΩfV_i is simple for all $i \in I$, (see §2 for the definition of Ω).

Remark: Kupisch first proved b), c) and d) above in [10].



W_i
W_{00}
W_{10}
W_{01}
W_{20}
W_{11}
W_{02}
W_{21}
W_{12}
W_{22}
$W_{r-1,0}$
$W_{0,s-1}$
$W_{r-1,1}$
$W_{1,s-1}$
$W_{r-1,2}$
$W_{2,s-1}$
$W_{r-1,s-1}$
0

Figure 1.

§2 Some Assumed Results

Throughout this chapter R is any subgroup of G with $x_1,\dots,x_r$ a right transversal of G by R. Also $\mathcal{S}$ is any set of subgroups of G and U,V,W are kG-modules.

Defns: (i) $(U,V)_R = \mathrm{Hom}_{kR}(U,V)$

(ii) If $\alpha \in (U,V)_R$ then $T_{R,G}(\alpha)$ is the kG-map $\sum_{i=1}^{r} x_i^{-1}\alpha x_i$ which sends $u \in U$ to $\sum (ux_i^{-1})\alpha . x_i$

(iii) We will write $(U,V)_{\mathcal{S},G}$ for $\sum_{S\in\mathcal{S}} T_{S,G}(\ (U,V)_S\ )$ and $(U,V)_G^{\mathcal{S}}$ for the k-space $(U,V)_G/(U,V)_{\mathcal{S},G}$.

(iv) $\theta \in (U,V)_G$ is called an $\mathcal{S}$-projective map if $\theta \in (U,V)_{\mathcal{S},G}$.

(v) U is called $\mathcal{S}$-projective if the identity map on U is $\mathcal{S}$-projective.

(vi) U is called $\mathcal{S}$-projective-free if no indecomposable direct summand of U is $\mathcal{S}$-projective.

Remarks: a) If $\mathcal{S}$ is a single subgroup, say $\mathcal{S} = \{S\}$ we omit the brackets and write S-projective, $(U,V)_{S,G}$, $(U,V)_G^S$ etc...

b) "U is 1-projective" is an equivalent statement to "U is projective" (this is essentially the famous D. G. Higman theorem, see [14]). So analogously we call all maps that are 1-projective just projective.

Lemma 2.1 (see Green [3, §3]): (i) Let W be projective, then:

If $\pi : W \longrightarrow V$ is a kG-epimorphism, $\theta \in (U,V)_G$ is projective if and only if there exist $\phi \in (U,W)_G$ so that $\theta = \phi\pi$.

If $\mu : U \longrightarrow W$ is a kG-monomorphism, $\theta \in (U,V)_G$ is projective if and only if there exist $\psi \in (W,V)_G$ so that $\theta = \mu\psi$.

(ii) $(U,V)_G^1 \cong (U,V)_G$ in both of the following cases:

$$(A)\dots\dots\dots \begin{cases} \text{a) U projective-free and V simple} \\ \text{b) V projective-free and U simple} \end{cases}$$

Consider now an exact sequence $0 \longrightarrow U \longrightarrow W \longrightarrow V \longrightarrow 0$ with W projective. Any sequence of this form is called a projective presentation of V, and if W is minimal (among all other projective presentations of V) then the above sequence is called a minimal projective presentation (mpp). For all kG-modules V, a mpp exists, and we write ΩV for the corresponding "kernel". So any mpp of V yields an exact sequence $0 \longrightarrow \Omega V \longrightarrow W \longrightarrow V \longrightarrow 0$. Schanuel's lemma [8, p.16] now shows that ΩV is unique up to isomorphism. Also,

Theorem 2.2 (Heller, see [7]): If V is non-projective indecomposable, then ΩV is also non-projective indecomposable.

The next theorem is a simple exercise in homological algebra, using 2.1(i):

Theorem 2.3 (Feit):

(B)............ $(U,V)^1_G \cong (\Omega U, \Omega V)^1_G$ as k-spaces.

Now if $0 \longrightarrow \Omega V \xrightarrow{\mu} W \xrightarrow{\pi} V \longrightarrow 0$ is a mpp of V, then $(\Omega V,U)_G \xleftarrow{\mu^*} (W,U)_G \xleftarrow{\pi^*} (V,U)_G \longleftarrow 0$ is exact for all kG-modules U. Also using 2.1(i) we get that $\mathrm{Im}\,\mu^* = (\Omega V,U)_{1,G}$, and so it follows that $\mathrm{Ext}^1_{kG}(V,U) = (\Omega V,U)^1_G$.

Thus by [5, pp.290-292], $(\Omega V,U)^1_G$ is isomorphic to the group of extension classes of V by U. Of particular importance are the following special cases:

Theorem 2.4: a) If $(\Omega V,U)^1_G = 0$ then there exists up to isomorphism only one extension of V by U, namely the split extension $V \oplus U$.

b) If $(\Omega V,U)^1_G \cong k$ then there exists exactly two non-isomorphic extensions of V by U, namely the split extension $V \oplus U$ and one other non-split extension.

Notation: $V \circ U$ will denote any extension of V by U, so that there exists an exact sequence $0 \longrightarrow U \longrightarrow V \circ U \longrightarrow V \longrightarrow 0$.

We now introduce some special subgroups and sets of subgroups of G.

Notation: Let D be any p-subgroup of G and H any subgroup $\geqslant N_G(D)$.

Set $\mathfrak{X} = \{ D^x \cap D : x \in G \setminus H \}$,

and $\mathcal{Y} = \{ D^x \cap H : x \in G \setminus H \}$.

U, V will be kG-modules and L, M kH-modules.

Defns: a) fU is a $\mathcal{Y}$-projective-free kH-module and U' a $\mathcal{Y}$-projective kH-module so that $U_H = fU \oplus U'$

b) gL is an $\mathfrak{X}$-projective-free kG-module and L' an $\mathfrak{X}$-projective kG-module so that $L^G = gL \oplus L'$.

Remarks: (i) By the Krull-Schmidt theorem U', fU; L', gL are all unique up to isomorphism.

(ii) The modules fU, gL are often called the Green correspondents of U, L.

(iii) When $H = G$ the sets $\mathfrak{X}, \mathfrak{Y}$ are both empty. To account for this special case we set $(U,V)_{\emptyset,G} = 0$ for all U,V.
So in particular f,g act as identities when $H = G$, that is $fU = gU = U$ for all U.

Theorem 2.5 (Green, see [9]): If $\mathcal{A}$ is the set of subgroups of D which are not G-conjugate to any subgroup of any X in $\mathfrak{X}$, and if U,L are indecomposable with vertices $D_0, D_1 \in \mathcal{A}$ respectively, then:

a) fU, gL are indecomposable with vertices D_0, D_1 respectively

(C)...... b) $g(fU) \cong U$ and $f(gL) \cong L$.

c) If $\underset{\sim}{B}$ is a kG-block of defect group D with B the unique kH-block of defect group D satisfying $B^G = \underset{\sim}{B}$, then $U \in \underset{\sim}{B}$ if and only if $fU \in B$; $L \in B$ if and only if $gL \in \underset{\sim}{B}$.

Remark: Notice that if U,V are indecomposable with their vertices in $\mathcal{A}$, then it follows from (C) that

$$U \cong V \text{ if and only if } fU \cong fV$$

This will be used quite often in the next chapter.

Theorem 2.6 (Feit, see [9, 4.12]): If U,V; L,M are D-projective then,

$$\text{(D)} \cdots\cdots\cdots\cdots \begin{cases} (U,V)_G^{\mathfrak{X}} \cong (fU,fV)_H^{\mathfrak{X}} \\ (L,M)_H^{\mathfrak{X}} \cong (gL,gM)_G^{\mathfrak{X}} \end{cases}$$

Theorem 2.7 (Green, see [3, 4.5]): Ω commutes with f and g, that is

(E)................ $\Omega fU \cong f\Omega U$; $\Omega gL \cong g\Omega L$

§3 (q,e)-uniserial Blocks

Here we set up the machinery needed to prove our main theorem inductively. Throughout this chapter $D \neq \{1\}$ is a p-subgroup of G with $H \geqslant N_G(D)$ so that $\mathfrak{X} = \{D^x \cap D : x \in G \setminus H\} = \{1\}$.
Let $\underset{\sim}{B}$ be a kG-block of defect group D with B the unique kH-block of defect group D satisfying $B^G = \underset{\sim}{B}$. We assume throughout that B is (q,e)-uniserial (and hence we adopt the notation in §1 for the indecomposable modules in B). The aim is to study $\underset{\sim}{B}$. Our first two lemmas are easy and hence the proofs are left as an exercise.

Lemma 3.1: $\Omega T_{i\alpha} \cong T_{i+\alpha, q-\alpha}$; $\Omega^2 T_{i\alpha} \cong T_{i+1,\alpha}$.

Lemma 3.2 (A generalised form of Schur's lemma):

(i) If $\ell(T_{j\beta}) \leqslant e$ then $(T_{j\beta}, T_{j\beta})_H \cong k$.

(ii) If $\ell(T_{j\beta}) \leqslant e$ then for each $i;\alpha$

$$(T_{j\beta}, T_{i\alpha})_H \cong k \text{ or } 0; \quad (T_{i\alpha}, T_{j\beta})_H \cong k \text{ or } 0.$$

Defn: If $\theta \in (T_{i\alpha}, T_{j\beta})_H$ then write $r(\theta)$ for the composition length $\ell(\text{Im }\theta)$.

Lemma 3.3 (A generalised form of Passman's lemma, see [4, lemma 4]):

$0 \neq \theta \in (T_{i\alpha}, T_{j\beta})_H$ is projective if and only if $r(\theta) \leqslant \alpha + \beta - q$.

Proof: Let $\pi : T_j \longrightarrow T_{j\beta}$ be the natural kH-epimorphism.

a) If $0 \neq \theta \in (T_{i\alpha}, T_{j\beta})_H$ is projective, then by 2.1 there exists $\phi \in (T_{i\alpha}, T_j)_H$ so that $\theta = \phi\pi$.
Hence $r(\theta) = r(\phi\pi) = \ell((\text{Im }\phi)\pi) > 0$, and so we have,
$r(\theta) = \ell(\text{Im }\phi) - \ell(\text{Ker }\pi) = \ell(\text{Im }\phi) - (q-\beta)$, which gives,
$r(\theta) \leqslant \ell(T_{i\alpha}) - (q-\beta) = \alpha + \beta - q$.

b) If $0 \neq \theta \in (T_{i\alpha}, T_{j\beta})_H$ then there exists an inclusion $T_{i,r(\theta)} \overset{\theta'}{\hookrightarrow} T_{j\beta}$. Hence there exists an inclusion $T_{i,r(\theta)+q-\beta} \overset{\phi'}{\hookrightarrow} T_j$ so that $\theta' = \phi'\pi$.
Now if $r(\theta) \leqslant \alpha + \beta - q$, then $r(\theta) + q - \beta \leqslant \alpha$.
Hence there exists $\phi \in (T_{i\alpha}, T_j)_H$ such that $\theta = \phi\pi$, which by 2.1 shows that θ is projective.

Corollary 3.4: By 3.3 and (B),

$$(F)\ldots\ldots\ldots\ldots(T_{i\alpha},T_{j\beta})^1_H \cong \begin{cases}(T_{i\alpha},T_{j\beta})_H & \text{if } \alpha+\beta \leqslant q,\\ (\Omega T_{i\alpha},\Omega T_{j\beta})_H & \text{if } \alpha+\beta > q.\end{cases}$$

Theorem 3.5: Up to isomorphism $\underset{\sim}{B}$ contains exactly e simples, which can be labelled $V_0,\ldots,V_{e-1}$ so that for all $0 \leqslant i,j \leqslant e-1$

a) fV_j is a non-projective indecomposable in B, gS_i is a non-projective indecomposable in $\underset{\sim}{B}$.

b) $(fV_j,S_i)_H \cong (V_j,gS_i)_G \cong$ k if $i = j$, 0 if $i \neq j$.

c) There exists a permutation δ on $I = \{0,1,\ldots,e-1\}$ so that $(S_i,fV_j)_H \cong (gS_i,V_j)_G \cong$ k if $\delta(i) = j$, 0 if $\delta(i) \neq j$.

Proof: Adapt the argument in [3, § 6], noting that the only part of Dade's results that is necessary for Green's proof, is that B (which corresponds to $\underset{\sim}{B}'$ in [3]) is (q,e)-uniserial.

Remark: We will adopt the above notation, and hence for all j:

$$fV_j \,/\, \Phi(fV_j) \cong S_j, \quad \Sigma(fV_j) \cong S_{\delta^{-1}(j)};$$
$$\Omega fV_j \,/\, \Phi(\Omega fV_j) \cong S_{\delta^{-1}(j)+1}, \quad \Sigma(\Omega fV_j) \cong S_j.$$

Theorem 3.6 (Feit): Each fV_j is either "long" i.e. $\ell(fV_j) \geqslant q-e$, or "short" i.e. $\ell(fV_j) \leqslant e$.

Proof: Clearly we can take $q > 2e$.

Case 1 Suppose $e < \ell(fV_i) \leqslant q/2$ for some i.

Then by Schur's lemma, (A),(D) and (F):

$$k \cong (V_i,V_i)_G \cong (V_i,V_i)^1_G \cong (fV_i,fV_i)^1_H \cong (fV_i,fV_i)_H$$

Hence the only maps $fV_i \longrightarrow fV_i$ are the "scalar maps". But $\ell(fV_i) > e$ means that there exists an inclusion $T_{i,\ell(fV_i)-e} \overset{\theta'}{\hookrightarrow} fV_i$, which induces a non-scalar $\theta : fV_i \longrightarrow fV_i$.

This contradiction shows that this case never occurs.

Case 2 Suppose $q/2 < \ell(fV_i) < q-e$ for some i.

Then by Schur's lemma, (A),(D) and (F):

$$k \cong (V_i,V_i)_G \cong (V_i,V_i)^1_G \cong (fV_i,fV_i)^1_H \cong (\Omega fV_i,\Omega fV_i)_H$$

Hence the only maps $\Omega fV_i \longrightarrow \Omega fV_i$ are the "scalar maps".

But $\ell(fV_i) < q-e$ means that $\ell(\Omega fV_i) > e$. So in a similar way to Case 1 we get a contradiction, showing that Case 2 never occurs.

This completes the proof of 3.6

Corollary 3.7: If $T \in B$ is indecomposable, then for all j:

a) $(T, fV_j)^1_H \cong k$ or 0; b) $(fV_j, T)^1_H \cong k$ or 0.

Proof: We will prove a); b) being analogous.

Case 1 Suppose that $\ell(T) + \ell(fV_j) \leqslant q$.

Then if fV_j is long, it is clear that T must be short. So by 3.6 at least one of T, fV_j is short. Hence by (F) and 3.2:

$$(T, fV_j)^1_H \cong (T, fV_j)_H \cong k \text{ or } 0$$

Case 2 Suppose that $\ell(T) + \ell(fV_j) > q$, i.e. $\ell(\Omega T) + \ell(\Omega fV_j) < q$. Then if ΩfV_j is long, it is clear that ΩT must be short. But ΩfV_j is always either short or long, since by 3.6 fV_j always is. So at least one of $\Omega T, \Omega fV_j$ is short. Hence by (F) and 3.2:

$$(T, fV_j)^1_H \cong (\Omega T, \Omega fV_j)_H \cong k \text{ or } 0$$

A Remark on Extensions in B: If $\theta \in (\Omega T_{i\alpha}, T_{j\beta})_H$ is not projective, then by 3.3 $r(\theta) > \beta - \alpha$ and the reader will easily verify that this implies that there exists a non-split extension $T_{i\alpha} \circ T_{j\beta}$ of the form $T_{i,\alpha+r(\theta)} \oplus T_{j,\beta-r(\theta)}$. So by 3.7 applied to 2.4 we have:

(G)...
- a) If $(\Omega T_{i\alpha}, fV_j)^1_H = 0$ then up to isomorphism there is only one extension $T_{i\alpha} \circ fV_j$ namely $T_{i\alpha} \oplus fV_j$.
- b) If $(\Omega T_{i\alpha}, fV_j)^1_H \neq 0$ then there are exactly two non-isomorphic extensions $T_{i\alpha} \circ fV_j$ namely:

  $T_{i\alpha} \oplus fV_j \cong T_{i\alpha} \oplus T_{j,\ell(fV_j)}$ (the split extension) and

  $T_{i,\alpha+r(\theta)} \oplus T_{j,\ell(fV_j)-r(\theta)}$

  (θ is any non-projective map $\Omega T_{i\alpha} \longrightarrow fV_j$)

Lemma 3.8: For all $i, j; \alpha$ there exists up to isomorphism at most one non-split extension $T_{i\alpha} \circ fV_j$ of the form (indecomposable) $\oplus$ (projective) or (indecomposable) namely:

$T_i \oplus T_{j,\alpha+\ell(fV_j)-q}$ if and only if $\alpha + \ell(fV_j) > q$ and $i \equiv \delta^{-1}(j)$

$T_{i,\alpha+\ell(fV_j)}$ if and only if $\alpha + \ell(fV_j) \leqslant q$ and $i + \alpha \equiv j$

where the above congruences (and unless otherwise specified all the congruences in the rest of this thesis) are to be taken mod e.

Proof: Using (G) it is clear that these are the only possible extensions of the required form. The conditions given beside each extension, are those necessary and sufficient for that extension to exist.

Notation: We will call an extension "unique" if it is unique up to isomorphism.

The next theorem is the fundamental connecting link between extensions in $\underset{\sim}{B}$ and B.

Theorem 3.9: If $X, Y \in \underset{\sim}{B}$ are non-projective indecomposables affording a non-projective indecomposable extension $X \circ Y$, then there exists an extension $fX \circ fY \in B$ so that:

$$f(X \circ Y) \oplus (\text{projective}) \cong fX \circ fY.$$

Proof: Reductions preserve extensions, so we may write:

$$(X \circ Y)_H \cong f(X \circ Y) \oplus (\mathcal{Y}\text{-projective})$$
$$= [fX \oplus (\mathcal{Y}\text{-projective})] \circ [fY \oplus (\mathcal{Y}\text{-projective})].$$

Now a $\mathcal{Y}$-projective module in B is both $\mathcal{Y}$ and D-projective, and hence by [12, 4.14] such modules are $\mathcal{X} = \{1\}$-projective.

Moreover projective modules over group algebras are also injective, and so all projective "parts" in the above extensions break off into direct sums (by injectivity). So we get:

$$\left.\begin{aligned} (X \circ Y)_H &\cong f(X \circ Y) \oplus (\text{projective}) \oplus (\text{modules} \notin B) \\ &\cong \{[fX \oplus (\text{modules} \notin B)] \circ [fY \oplus (\text{modules} \notin B)]\} \oplus (\text{projective}) \end{aligned}\right\} (*)$$

Now if $\varepsilon \in Z(kH)$ is an idempotent, then it is easy to check that:

$0 \to A \to E \to B \to 0$ kH-exact (kH-split-exact) implies that

$0 \to A\varepsilon \to E\varepsilon \to B\varepsilon \to 0$ is kH-exact (kH-split-exact).

So multiplication by ε preserves extensions and direct sums.

Also $X, Y, X \circ Y$ are non-projective indecomposables in $\underset{\sim}{B}$, and so they have their vertices in $\mathcal{A} = \{S : 1 < S \leq D\}$ (remember that $\mathcal{X} = \{1\}$!).

Thus by 2.5 $fX, fY, f(X \circ Y)$ are all non-projective indecomposables in B.

Hence on multiplying (*) by the block idempotent of B we get:

$$f(X \circ Y) \oplus (\text{projective}) \cong fX \circ fY \oplus (\text{projective})$$

So as $f(X \circ Y)$ is a non-projective indecomposable there exists an extension $fX \circ fY$ so that:

$$f(X \circ Y) \oplus (\text{projective}) \cong fX \circ fY$$

<u>Note:</u> $fX \circ fY$ is not the split extension.

<u>Theorem 3.10:</u> Let W be a non-projective indecomposable in $\underset{\sim}{B}$ with $V = W/\Phi(W)$ simple, and suppose $fW \cong T_{i\alpha}$ (see 2.5). Then,

a) There exists a non-projective indecomposable extension $W \circ V_j$ if and only if either $\alpha + \ell(fV_j) < q$ and $\alpha + i \equiv j$, when $f(W \circ V_j) \cong T_{i,\alpha + \ell(fV_j)}$;

or $\alpha + \ell(fV_j) > q$ and $i \equiv \delta^{-1}(j)$, when $f(W \circ V_j) \cong T_{j,\alpha + \ell(fV_j) - q}$.

b) There exists a projective indecomposable extension $W \circ V_j$ if and only if $\alpha + \ell(fV_j) = q$ and $\alpha + i \equiv j$, when $W \circ V_j$ is a projective cover of V_j.

c) All such extensions $W \circ V_j$ are "unique", and satisfy

$$W \circ V_j / \Phi(W \circ V_j) \cong V$$

<u>Proof:</u> If $0 \longrightarrow V_j \xrightarrow{\mu} W \circ V_j \xrightarrow{\pi} W \longrightarrow 0$ defines any non-split extension, then it is easy to verify that $V_j \cong (V_j)\mu \leqslant \Phi(W \circ V_j)$.

Hence: $W \circ V_j / \Phi(W \circ V_j) \cong W/\Phi(W) = V$ .... (*)

So in particular $W \circ V_j$ is indecomposable for all non-split $W \circ V_j$ ... (**)

Now by (D),(E) and 3.7 $(\Omega W, V_j)^1_G \cong (\Omega fW, fV_j)^1_H \cong k$ or 0. So by 2.4:

(H).. $\left\{\begin{array}{l}\text{For all } j, \text{ there exists up to isomorphism, at most one non-split} \\ \text{extension } W \circ V_j, \text{ and at most one non-split extension } fW \circ fV_j. \text{ Also} \\ \text{a non-split } W \circ V_j \text{ exists if and only if a non-split } fW \circ fV_j \text{ exists.}\end{array}\right.$

Now (*) and (H) certainly prove part c), and indeed by (**) and (H) $W \circ V_j$ is projective indecomposable if and only if $V_j \cong \Omega W$, i.e. if and only if $fV_j \cong \Omega fW$ (use 2.5 and (E)), which is equivalent to saying that the "unique" non-split $fW \circ fV_j$ exists and is a projective indecomposable. Moreover this result, (**) and (H) show that $W \circ V_j$ is non-projective indecomposable if and only if the "unique" non-split $fW \circ fV_j$ exists and is not a projective indecomposable.

Indeed by 3.9 such extensions must satisfy $f(W \circ V_j) \oplus (\text{proj}) \cong fW \circ fV_j$. So 3.8 now proves part a).

Finally, we know that $W \circ V_j$ is a projective indecomposable if and only if $fV_j \cong \Omega fW$. But a necessary and sufficient condition for this to hold (and hence for $W \circ V_j$ to be projective indecomposable) is that $\alpha + \ell(fV_j) = q$ and $\alpha + i \equiv j$.

Indeed it is clear that when $W \circ V_j$ is such a projective indecomposable its socle is simple, and hence isomorphic to V_j, which says that $W \circ V_j$ is a projective cover of V_j. This completes the proof of 3.10

From now on we make the usual convention that V_i is defined for all $i \in \mathbb{Z}$ by taking i mod e. We focus attention on an arbitrary but fixed i.

Notation: a) W_i will denote a projective cover of V_i.

b) $\mathcal{W}_i = \{ W \in \underset{\sim}{B} \ : \ W$ is isomorphic to a proper factor of $W_i \}$

c) If T is any indecomposable in B with $1 \leqslant u \leqslant \ell(T)$; write $(T)_u$, $(T)^u$ respectively for the unique submodule and factor-module of T of composition length u.

Remark: $W \in \mathcal{W}_i$ if and only if W is a non-projective indecomposable with $W / \Phi(W) \cong V_i$.

Lemma 3.11: $0 \neq W \in \mathcal{W}_i$ if and only if there exists a factor U of W_i, a simple module $V_j \in \underset{\sim}{B}$ and a non-projective indecomposable extension $U \circ V_j$ such that $W \cong U \circ V_j$.

Proof: If $W \in \mathcal{W}_i$ then the existance of U, V_j and $U \circ V_j$ with the required properties is trivial. So assume that $W \cong U \circ V_j$, a non-projective indecomposable extension.

Then by 2.4a) $(\Omega U, V_j)^1_G \neq 0$, and hence by (D) and 3.7a)

$$(\Omega U, V_j)^1_G \cong (f\Omega U, fV_j)^1_H \cong k$$

So by 2.4b) W is the "unique" non-split extension of U by V_j.

Also $(\Omega U, V_j)_G \neq 0$, so we may take $X \leqslant \Omega U \leqslant W_i$ with $U/X \cong V_j$.

Therefore W_i/X is an (indecomposable) extension of U by V_j, and hence it is the "unique" non-split extension. So $W_i/X \cong W$ as required.

We shall classify the elements of $\mathcal{W}_i$ via integers $r(i)$, $s(i)$; but first we need the following easy lemma,

Lemma 3.12: a) $W \in \mathcal{W}_i$ with $(fW)_1 \cong S_{j-1}$ and $\ell(fW) + \ell(fV_j) > q$ implies that $i \equiv j$.

b) $W \in \mathcal{W}_i$ with $(fW)^1 \cong S_{\delta^{-1}(h)}$ and $\ell(fW) + \ell(fV_h) \leqslant q$
implies that $i \equiv h$.

Proof: As $W \in \mathcal{W}_i$, $(W,V_m)_G \cong k$ if $m \equiv i$, 0 if $m \not\equiv i$.
Also by (A),(D); $(W,V_m)_G \cong (W,V_m)^1_G \cong (fW,fV_m)^1_H$ for all m ....($\dagger$)

a) $\ell(fW) + \ell(fV_j) > q$, so on applying (F) to ($\dagger$) with $m = j$, we get

$$(W,V_j)_G \cong (\Omega fW, \Omega fV_j)_H$$

But $(\Omega fW)^1 \cong (\Omega fV_j)_1 \cong S_j$, and hence $(\Omega fW, \Omega fV_j)_H \neq 0$.
Thus $(W,V_j)_G \neq 0$, i.e. $i \equiv j$.

b) $\ell(fW) + \ell(fV_h) \leqslant q$, so on applying (F) to ($\dagger$) with $m = h$, we get

$$(W,V_h)_G \cong (fW,fV_h)_H$$

But $(fW)^1 \cong (fV_h)_1 \cong S_{\delta^{-1}(h)}$, and hence $(fW,fV_h)_H \neq 0$.
Thus $(W,V_h)_G \neq 0$, i.e. $i \equiv h$.

Defns: (i) ρ is the permutation on $I = \{0,1,\dots,e-1\}$ defined by

$$\rho(i) \equiv \delta^{-1}(i) + 1.$$

(ii) For any integers $a,b \geqslant 0$ set

$$\gamma(a,b) = \sum_{j=0}^{a} \ell(fV_{\rho^j(i)}) + \sum_{j=1}^{b} \ell(fV_{\delta^j(i)}) - bq$$

(If $b = 0$ omit the second sum).

(iii) $r = r(i)$, $s = s(i)$ are the smallest integers greater than zero satisfying

$$\gamma(r,0) \geqslant q; \quad \gamma(0,s) \leqslant 0$$

(iv) $W \in \mathcal{W}_i$ is said to be of type (a,b) if $fW \cong T_{\delta^b(i),\gamma(a,b)}$.

Lemma 3.13: a) For all $0 \leqslant a \leqslant r-1$ there exists in $\mathcal{W}_i$ a "unique" extension $P_a = V_i \circ V_{\rho(i)} \circ \dots \circ V_{\rho^a(i)}$.
Moreover each P_a is of type $(a,0)$.

b) For all $0 \leqslant b \leqslant s-1$ there exists in $\mathcal{W}_i$ a "unique" extension $\Delta_b = V_i \circ V_{\delta(i)} \circ \dots \circ V_{\delta^b(i)}$.
Moreover each Δ_b is of type $(0,b)$.

c) $\rho^r(i) = \delta^s(i) = i$ and $\ell(fP_{r-1}) = \gamma(r-1,0) = q-1$,
$\ell(f\Delta_{s-1}) = \gamma(0,s-1) = 1$.

Proof: a) & b): By 3.10 applied inductively there exists "unique" indecomposable non-projective extensions of the forms P_a, Δ_b.

Moreover these are of types $(a,0)$, $(0,b)$ respectively, and 3.11 applied inductively shows that such extensions are isomorphic to factor-modules of W_i, and hence in $\mathcal{W}_i$.

c) $P_{r-1}, \Delta_{s-1} \in \mathcal{W}_i$ so on applying 3.12a) and 3.10b) with $W = P_{r-1}$ we get that $\rho^r(i) = i$; and on applying 3.12b) with $W = \Delta_{s-1}$ we get that $\delta^s(i) = i$. Now $\rho^r(i) = i$ means that there exists an integer $m \geqslant 0$ so that $\ell(fP_{r-1}) = q-me-1$.

But if fV_i is long then $\ell(fP_{r-1}) \geqslant \ell(fV_i) \geqslant q-e$, and so $m = 0$.

Also if fV_i is short then $q-ne-1 + \ell(fV_i) < q$ for all $n \geqslant 1$, and so from the definition of r, bearing in mind that $i = \rho^r(i)$, we get that $m = 0$ again.

Hence $\ell(fP_{r-1}) = q-1$, and similarly $\ell(f\Delta_{s-1}) = 1$.

<u>Corollary 3.14</u>: a) For all $1 \leqslant a \leqslant r-1$; $1 \leqslant b \leqslant s-1$ there exists in $\mathcal{W}_i$ "unique" extensions of the following forms,

$$X_a = \Delta_{s-1} \circ V_{\rho(i)} \circ \dots \circ V_{\rho^a(i)} \quad \text{which is of type } (a, s-1);$$
$$Y_b = P_{r-1} \circ V_{\delta(i)} \circ \dots \circ V_{\delta^b(i)} \quad \text{which is of type } (r-1, b).$$

b) There exists "unique" projective indecomposable extensions

$$X_r = \Delta_{s-1} \circ V_{\rho(i)} \circ \dots \circ V_{\rho^r(i)}; \quad Y_s = P_{r-1} \circ V_{\delta(i)} \circ \dots \circ V_{\delta^s(i)},$$

which are projective covers of V_i, and hence isomorphic to W_i.

<u>Proof</u>: $fP_{r-1} \cong T_{i,q-1}$; $f\Delta_{s-1} \cong S_{\delta^{-1}(i)}$ by 3.13c).

The corollary now follows from inductive applications of 3.10, 3.11.

<u>Remark</u>: Each of X_r, Y_s provide a composition series of $W_i \cong X_r \cong Y_s$, and hence $\ell(W_i) = r + s$. To examine all composition series of W_i we explore $\mathcal{W}_i$. We need the following critical lemma:

<u>Lemma 3.15</u>: If $a+b < r+s$ then $\gamma(a,b) < q$ if and only if $a \leqslant r-1$; and $\gamma(a,b) > 0$ if and only if $b \leqslant s-1$.

<u>Proof</u>: (i) If $a \leqslant r-1$ then $\gamma(a,b) \leqslant \gamma(r-1,b) \leqslant \gamma(r-1,0) = q-1 < q$.

(ii) If $b \leqslant s-1$ then $\gamma(a,b) \geqslant \gamma(a,s-1) \geqslant \gamma(0,s-1) = 1 > 0$.

(iii) If $a \geqslant r$ then $b \leqslant s-1$ and hence,

$$\gamma(a,b) \geqslant \gamma(r,s-1) = \gamma(r-1,0) + \gamma(0,s-1) \qquad \text{as } \rho^r(i) = i$$
$$= q-1+1 = q.$$

(iv) If $b \geqslant s$ then $a \leqslant r-1$ and hence,

$$\gamma(a,b) \leqslant \gamma(r-1,s) = \gamma(r-1,0) + \gamma(0,s-1) - q \quad \text{as } \delta^s(i) = i$$
$$= q-1 + 1 - q = 0.$$

Theorem 3.16: A kG-module W lies in $\mathcal{W}_i$ if and only if there exists $0 \leqslant a \leqslant r-1$; $0 \leqslant b \leqslant s-1$ such that W is of type (a,b) and $\ell(W) = a+b+1$. Moreover such a W has in $\mathcal{W}_i$ at most two extensions of the form $W \circ V_j$ namely,

(J)... A "unique" extension $W \circ V_{\rho^{a+1}(i)}$ of type $(a+1,b)$ if and only if $a+1 \leqslant r-1$;
& a "unique" extension $W \circ V_{\delta^{b+1}(i)}$ of type $(a,b+1)$ if and only if $b+1 \leqslant s-1$.

Proof: Induction on $\ell(W)$.

If $\ell(W) = 1$ then $W \in \mathcal{W}_i$ if and only if $W \cong V_i$, i.e. if and only if $fW \cong fV_i \cong T_{i,\gamma(0,0)}$ (by 2.5).

But this merely says that $W \in \mathcal{W}_i$ if and only if W is of type $(0,0)$.

Also by 3.10, 3.11 there exists in $\mathcal{W}_i$ at most two extensions of the form $V_i \circ V_j$ namely:

A "unique" extension $V_i \circ V_{\rho(i)}$ of type $(1,0)$ if and only if $\gamma(1,0) = \ell(fV_i) + \ell(fV_{\rho(i)}) < q$ i.e. if and only if $1 \leqslant r-1$ (by 3.15)
& a "unique" extension $V_i \circ V_{\delta(i)}$ of type $(0,1)$ if and only if $\gamma(0,1) = \ell(fV_i) + \ell(fV_{\delta(i)}) - q > 0$ i.e. if and only if $1 \leqslant s-1$ (by 3.15).

This proves the theorem for $\ell(W) = 1$.

So now assume inductively that it is true for $\ell(W) = n$.

Let W' be a kG-module with $\ell(W') = n+1$.

Then $W' \in \mathcal{W}_i$ if and only if there exists $W \in \mathcal{W}_i$ with $\ell(W) = n$ and a non-projective indecomposable extension of the form $W \circ V_j$ with $W' \cong W \circ V_j$ (see 3.11).

Now by induction such W 's are precisely those of type (a,b) for some $0 \leqslant a \leqslant r-1$; $0 \leqslant b \leqslant s-1$ with $a+b+1 = \ell(W) = n$.

Moreover their extensions in $\mathcal{W}_i$ of the form $W \circ V_j$ are given inductively by (J).

Hence $W' \in \mathcal{W}_i$ if and only if W' is isomorphic to one of these extensions $W \circ V_j$, i.e. if and only if $fW' \cong f(W \circ V_j)$ for one of these extensions $W \circ V_j$ (by 2.5).

Hence by induction $W' \in \mathcal{W}_i$ if and only if

$$fW' \cong \begin{cases} T_{\delta^b(i),\gamma(a+1,b)} & \text{for some } 0 \leqslant a+1 \leqslant r-1;\ 0 \leqslant b \leqslant s-1 \text{ with } a+b+1 = n \text{ OR} \\ T_{\delta^{b+1}(i),\gamma(a,b+1)} & \text{for some } 0 \leqslant a \leqslant r-1;\ 0 \leqslant b+1 \leqslant s-1 \text{ with } a+b+1 = n \end{cases}$$

i.e. if and only if $fW' \cong T_{\delta^b(i),\gamma(a,b)}$ for some $0 \leqslant a \leqslant r-1$; $0 \leqslant b \leqslant s-1$ with $a+b+1 = n+1 = \ell(W')$.

Moreover by 3.10, 3.11 such a W' has in $\mathcal{W}_i$ at most two extensions of the form $W' \circ V_j$ namely:

A "unique" extension $W' \circ V_{\rho^{a+1}(i)}$ of type $(a+1,b)$ if and only if $\gamma(a+1,b) = \ell(fW') + \ell(fV_{\rho^{a+1}(i)}) < q$, i.e. if and only if $a+1 \leqslant r-1$ (by 3.15);

& a "unique" extension $W' \circ V_{\delta^{b+1}(i)}$ of type $(a,b+1)$ if and only if $\gamma(a,b+1) = \ell(fW') + \ell(fV_{\delta^{b+1}(i)}) - q > 0$, i.e. if and only if $b+1 \leqslant s-1$ (by 3.15).

This completes the induction and hence proves 3.16.

To apply these results to find the structure of W_i we need to account for multiplicities. This is done by:

<u>Lemma 3.17</u>: Let W,U be factor-modules of W_i. Then $W \cong U$ if and only if $W = U$.

<u>Proof</u>: Induction on $\ell(W) = \ell(U)$.

As $W_i/\Phi(W_i) \cong V_i$, the lemma is trivial for $\ell(W) = \ell(U) = 1$.

So assume inductively that the lemma is true for factors of length n.

Let W',U' be factors of W_i with $\ell(W') = \ell(U') = n+1$ and $W' \cong U'$.

Then there exists $j \in I$ and factors W,U of W_i so that:

a) W' is an extension of W by a copy of V_j,

b) U' is an extension of U by a copy of V_j,

c) $W \cong U$.

So W,U are factor-modules of W_i of composition length n, and hence by induction $W \cong U$ implies that $W = U$.

Thus W' and U' are both extensions of W by copies of V_j.

But by (A),(D) and 3.7

$$(\Omega W, V_j)_G \cong (\Omega W, V_j)^1_G \cong (f\Omega W, fV_j)^1_H \cong k$$

So ΩW has only one copy of V_j in its "head", which means that only one extension $W \circ V_j$ is a factor-module of W_i. Thus $W' = U'$.

Theorem 3.18: a) There exists a one-to-one correspondence between the set of all proper factor-modules of W_i and the set $\{(a,b) \;:\; 0 \leqslant a \leqslant r-1;\ 0 \leqslant b \leqslant s-1\}$, which is given by: $W \longleftrightarrow (a,b)$ if and only if W is of type (a,b).

b) If W_{ab} denotes the unique submodule of W_i so that W_i / W_{ab} is of type (a,b) then the full submodule lattice of W_i is of the shape given in Figure 1.

c) $W_i / W_{00} \cong V_i$, $W_{r-1,s-1} \cong V_i$,
$W_{a-1,b} / W_{ab} = V_{\rho^a(i)}$ for all $1 \leqslant a \leqslant r-1;\ 0 \leqslant b \leqslant s-1$,
$W_{a,b-1} / W_{ab} = V_{\delta^b(i)}$ for all $0 \leqslant a \leqslant r-1;\ 1 \leqslant b \leqslant s-1$.

d) $W_{r-1,0}$ and $W_{0,s-1}$ are both uniserial submodules of W_i.

e) $W_{r-1,0} \cap W_{0,s-1} \cong V_i$ and $W_{r-1,0} + W_{0,s-1} = \Phi(W_i)$.

f) $(W_{r-1,0}\ ;\ W_{0,s-1})$ is the unique pair of submodules of W_i satisfying d) and e).

Proof: a) For all $1 \leqslant h \leqslant r-1$, $\ell(fV_i) + \ell(fV_{\rho^h(i)}) \leqslant \gamma(r-1,0) = q-1$ and
for all $1 \leqslant j \leqslant s-1$, $\ell(fV_i) + \ell(fV_{\delta^j(i)}) \geqslant \gamma(0,s-1) + q = q+1$.
Hence $V_{\rho^h(i)} \not\cong V_{\delta^j(i)}$ for all such h,j .... (*)
Now if W is a proper factor-module of W_i, then by 3.16 there exists $0 \leqslant a \leqslant r-1$, $0 \leqslant b \leqslant s-1$ so that W is of type (a,b). Moreover it follows easily from (*) and (J) that (a,b) is unique.
Hence the map which sends W to (a,b) is well-defined, and 3.16, 3.17 show it to be the required one-to-one correspondence.
b),c) These both follow from 3.16 and a).
d) This is immediate from Figure 1.
e) Using (*), $W_{r-1,0} \cap W_{0,s-1} \cong V_i$; and hence:

$$\ell(W_{r-1,0} + W_{0,s-1}) = \ell(W_{r-1,0}) + \ell(W_{0,s-1}) - \ell(W_{r-1,0} \cap W_{0,s-1}) = r+s-1$$

Thus $\ell(W_{r-1,0} + W_{0,s-1}) = \ell(\Phi(W_i))$, and so $W_{r-1,0} + W_{0,s-1} = \Phi(W_i)$.
f) This is clear from Figure 1.

Corollary 3.19: (i) For a fixed $i \in I$, W_i is uniserial if and only if either $r(i) = 1$ or $s(i) = 1$, or equivalently if and only if $\ell(fV_i) = q-1$ or 1.

(ii) If W_i is uniserial for all $i \in I$, then either $\delta = 1$ or $\rho = 1$ (1 being the identity permutation on I). Moreover:

In the case $\delta = 1$, $\underset{\sim}{B}$ is (q,e)-uniserial and fV_i is simple for all $i \in I$.

In the case $\rho = 1$, $\underset{\sim}{B}$ is (q,e)-uniserial and ΩfV_i is simple for all $i \in I$.

Proof: (i) This follows from Figure 1.

(ii) Suppose that W_i is uniserial for all $i \in I$.

Then by (i) $\ell(fV_i) = q-1$ or 1 for all $i \in I$.

Hence if $e = 1$, then trivially either $\delta = 1$ or $\rho = 1$.

So assume that $e > 1$.

Now if there exists $j \in I$ with $\ell(fV_j) = q-1$, then $(fV_{j-1})^1 \cong (fV_j)_1$, which gives $(fV_{j-1}, fV_j)_H \neq 0$.

Hence if $\ell(fV_{j-1}) = 1$, then by (A) and (D)

$$(V_{j-1}, V_j)_G \cong (V_{j-1}, V_j)_G^1 \cong (fV_{j-1}, fV_j)_H^1 \cong (fV_{j-1}, fV_j)_H \neq 0,$$

and therefore $j-1 \equiv j \mod e$, implying $e = 1$, a contradiction.

So $\ell(fV_j) = q-1$ implies $\ell(fV_{j-1}) = q-1$.

But $j \in I$ was arbitrary, so by induction we get that:

$\ell(fV_j) = q-1$ implies $\ell(fV_{j-m}) = q-1$ for all $m \in \mathbb{Z}$, so that certainly $\ell(fV_i) = q-1$ for all $i \in I$.

Thus there exists only two possibilities, namely,

a) $\ell(fV_i) = 1$ for all $i \in I$, which implies that $\delta = 1$, OR

b) $\ell(\Omega fV_i) = 1$ for all $i \in I$, which implies that $\rho = 1$.

Suppose firstly that case a) holds. Then if we look at any fixed i we get that:

$\gamma(0,b) > 0$ if and only if $b \leq s-1$ implies $s = s(i) = 1$, and

$\gamma(a,0) < q$ if and only if $a \leq r-1$ implies $r = r(i) = q-1$.

Also $\delta = 1$ gives $\rho(i) \equiv i+1$ for all $i \in I$.

But when $r = q-1$, $s = 1$ and $\rho^a(i) \equiv i+a$ for all $i \in I$, $a \geq 0$, the lattice structure of each W_i given in Figure 1 reduces to:

$$W_i \overset{V_i}{\rule{2cm}{0.4pt}} W_{00} \overset{V_{i+1}}{\rule{2cm}{0.4pt}} W_{10} \overset{V_{i+2}}{\rule{2cm}{0.4pt}} W_{20} \;\cdots\cdots\; W_{q-2,0} \overset{V_i \cong V_{i+q-1}}{\rule{3cm}{0.4pt}} 0$$

This shows that $\underset{\sim}{B}$ is (q,e)-uniserial (with fV_i simple for all i). Now suppose that case b) holds. Then we can similarly show that the following hold for all $i \in I$,

$$s(i) = q-1, \; r(i) = 1, \; \delta(i) \equiv i+1$$

But when $r = 1$, $s = q-1$ and $\delta^b(i) \equiv i+b$ for all $i \in I$, $b \geqslant 0$, the lattice structure of each W_i given in Figure 1 reduces to:

$$W_i \overset{V_i}{\rule{2cm}{0.4pt}} W_{00} \overset{V_{i+1}}{\rule{2cm}{0.4pt}} W_{01} \overset{V_{i+2}}{\rule{2cm}{0.4pt}} W_{02} \;\cdots\cdots\; W_{0,q-2} \overset{V_i \cong V_{i+q-1}}{\rule{3cm}{0.4pt}} 0$$

This shows that $\underset{\sim}{B}$ is (q,e)-uniserial (with ΩfV_i simple for all i).

§4 The Proof Of The Main Theorem

Throughout this chapter we will work with the situation described in §1 under the heading "hypothesis". Hence since D is cyclic we may write $D = \langle \partial \rangle$ and then on setting $\partial_t = \partial^{p^t}$ we get that $D_t = \langle \partial_t \rangle$ for all $t \in \{0,1,\ldots,d-1\}$. We will firstly consider an arbitrary but fixed t in this range $0 \leqslant t \leqslant d-1$.

Notation: a) Set $\widetilde{N}_t = N_t/D_t$, $\widetilde{C}_t = C_t/D_t$, $\widetilde{D} = D/D_t$ etc...

b) For each $n \in N_t$ define an integer $z_t(n)$, which is unique mod p^{d-t}, so that $n^{-1}\partial_t n = \partial_t^{z_t(n)}$. Also set $\pi_t(n) = z_t(n)1_k$ which defines the "natural" linear representation π_t of N_t over k. For all $i \in \mathbb{Z}$ let $\Pi_t^{\,i}$ be a kN_t-module affording $\pi_t^{\,i}$.

Lemma 4.1: If we regard kD_t as a kN_t-module with N_t acting by conjugation, then for all $h \geqslant 0$:

$$\Pi_t^{\,h} \cong kD_t(\partial_t - 1)^h \,/\, kD_t(\partial_t - 1)^{h+1}$$

Proof: From $kD_t > kD_t(\partial_t - 1) > \ldots > kD_t(\partial_t - 1)^{|D_t|} = 0$, it follows that $\dim\left[kD_t(\partial_t - 1)^h \,/\, kD_t(\partial_t - 1)^{h+1}\right] = 1$ for all $h \geqslant 0$.

Also for each $n \in N_t$:

$$\begin{aligned}\left[(\partial_t - 1)^h\right]^n &= (n^{-1}\partial_t n - 1)^h = (\partial_t^{z_t(n)} - 1)^h \\ &= (\partial_t - 1)^h.(\partial_t^{z_t(n)-1} + \ldots + 1)^h \\ &\equiv (\partial_t - 1)^h.z_t(n)^h \quad \bmod\ kD_t(\partial_t - 1)^{h+1}\end{aligned}$$

Thus $kD_t(\partial_t - 1)^h \,/\, kD_t(\partial_t - 1)^{h+1}$ affords $\pi_t^{\,h}$, for all $h \geqslant 0$.

Lemma 4.2: If W is a projective kN_t-module then for all $h \geqslant 0$

$$W/W(\partial_t - 1) \otimes kD_t(\partial_t - 1)^h \,/\, kD_t(\partial_t - 1)^{h+1} \cong W(\partial_t - 1)^h \,/\, W(\partial_t - 1)^{h+1}$$

Proof: W_{D_t} is projective, and hence free. So we may select a kD_t-basis for it, say $\{w_\lambda : \lambda \in \Lambda\}$.

Also let $\{u_i : i \in I\}$ be a k-basis of kD_t, arranged so that there exists a sequence $I = I_0 \supset I_1 \supset \ldots$, with $\{u_i : i \in I_h\}$ a k-basis of $kD_t(\partial_t - 1)^h$ for all $h \geqslant 0$. We can also arrange that $I_0 \setminus I_1 = \{1\}$.

Now it is clear that each $W(\partial_t - 1)^h \,/\, W(\partial_t - 1)^{h+1}$ has as a k-basis $\{ w_\lambda u_i + W(\partial_t - 1)^{h+1} \ : \ \lambda \in \Lambda,\ i \in I_h \setminus I_{h+1} \}$.

Hence there is a k-isomorphism,

$$\phi : W/W(\partial_t - 1) \otimes kD_t(\partial_t - 1)^h / kD_t(\partial_t - 1)^{h+1} \longrightarrow W(\partial_t - 1)^h / W(\partial_t - 1)^{h+1}$$

$$[w_\lambda + W(\partial_t - 1)] \otimes [u_i + kD_t(\partial_t - 1)^{h+1}] \longmapsto w_\lambda u_i + W(\partial_t - 1)^{h+1}$$

Indeed this is a kN_t-isomorphism, since for all $n \in N_t$,

$$[w_\lambda n + W(\partial_t - 1)] \otimes [u_i^n + kD_t(\partial_t - 1)^{h+1}] \longmapsto w_\lambda n . u_i^n + W(\partial_t - 1)^{h+1}$$
$$= (w_\lambda u_i)n + W(\partial_t - 1)^{h+1}$$

<u>Lemma 4.3</u>: a) $W(\partial_t - 1)^h / W(\partial_t - 1)^{h+1} \cong W/W(\partial_t - 1) \otimes \Pi_t^{\ h}$

for all $0 \leq h \leq |D_t| - 1$.

b) $k\widetilde{N}_t \cong kN_t / kN_t(\partial_t - 1)$.

<u>Proof</u>: a) is an immediate corollary to 4.2 and b) is straightforward.

We now look at the blocks b_{t1} and B_t, which are defined in §1.

<u>Lemma 4.4</u>: a) If T is any kN_t-module in B_t, then there exists a $k(EC_t)$-module J with $J_{C_t} \in b_{t1}$ and $J^{N_t} \cong T$.

b) Conversely if J is any $k(EC_t)$-module with $J_{C_t} \in b_{t1}$ then $T = J^{N_t}$ lies in B_t.

c) In both cases T is indecomposable (simple) if and only if J is indecomposable (simple).

<u>Proof</u>: N_t acts transitively on $\{\varepsilon_{t1}, \dots, \varepsilon_{tn_t}\}$ with EC_t the stabiliser of ε_{t1}. Hence there exists a right transversal $x_1, \dots, x_{n_t}$ of N_t by EC_t so that $\varepsilon_{tj} = \varepsilon_{t1}^{\ x_j}$ for all j. Thus in $Z(kC_t)$ we have an orthoganol decomposition $\varepsilon_t = \varepsilon_{t1}^{\ x_1} + \dots + \varepsilon_{t1}^{\ x_{n_t}}$ .... (*)

a) Set $J = T\varepsilon_{t1}$; this is a $k(EC_t)$-module with $J_{C_t} \in b_{t1}$. Also

$$J^{N_t} \cong \sum_{j=1}^{n_t} (T\varepsilon_{t1})^{x_j} = \sum_{j=1}^{n_t} T.\varepsilon_{t1}^{\ x_j} = T\varepsilon_t = T.$$

b) $(J \otimes x_j)\varepsilon_{t1}^{\ x_j} = J \otimes x_j$ for all j, since $J\varepsilon_{t1} = J$.

Hence $(J \otimes x_j)\varepsilon_t = J \otimes x_j$ for all j (use (*) above), which implies that $J^{N_t}.\varepsilon_t = J^{N_t}$ i.e. $J^{N_t} \in B_t$.

c) This is trivial.

<u>Lemma 4.5</u>: Let χ be any faithful linear representation of EC_t/C_t

and let k_j be a $k(EC_t)$-module corresponding to χ^j for all $j \in \mathbb{Z}$. Suppose also that b_{t1} contains (up to isomorphism) a unique simple kC_t-module, say F.

Then there exists a simple $k(EC_t)$-module J, so that on writing $J_j = J \otimes k_j$ for each $j \in \mathbb{Z}$, the following all hold:

a) J_j is simple with $(J_j)_{C_t} \cong F \in b_{t1}$ for all $j \in \mathbb{Z}$.

b) $F^{EC_t} \cong J_o \oplus \dots \oplus J_{e-1}$.

c) For any $j, j' \in \mathbb{Z}$; $J_j \cong J_{j'}$ if and only if $j \equiv j' \bmod e$.

d) If J^* is any simple $k(EC_t)$-module with $(J^*)_{C_t} \in b_{t1}$, then $J^* \cong J_h$ for some $0 \leqslant h \leqslant e-1$.

Proof: The techniques needed to prove 4.5 are well known and can be found in [1, p.31] for example.

Corollary 4.6: Suppose again that b_{t1} contains a unique simple (up to isomorphism). Then,

a) The number of isomorphism classes of simple kN_t-modules in B_t is e.

b) If S is any simple kN_t-module in B_t, with $S_j = S \otimes \Pi_t{}^j$ for all $j \in \mathbb{Z}$; the modules $S_o, \dots, S_{e-1}$ form a full set of simples in B_t with $S_{j+e} \cong S_j$ for each integer j.

Proof: Take $k_1 = \Pi_t|_{EC_t}$ and hence $J_j = J \otimes \Pi_t{}^j|_{EC_t}$ in 4.5, which shows that:

$$\left.\begin{array}{l} J_{j+e} \cong J_j \text{ for each } j \in \mathbb{Z}\text{, with } J_o, \dots, J_{e-1} \text{ a full set of the} \\ \text{simple } k(EC_t)\text{-modules which on restriction to } C_t \text{ lie in } b_{t1} \end{array}\right\} \dots (**)$$

Now take any simple kN_t-module S in B_t. Then by 4.4 and (**) above, there exists $0 \leqslant h \leqslant e-1$ with $S \cong J_h{}^{N_t}$.

For each integer j set $S_j = S \otimes \Pi_t{}^j \cong J_h{}^{N_t} \otimes \Pi_t{}^j \cong (J_h \otimes \Pi_t{}^j|_{EC_t})^{N_t}$

$$= J_{h+j}{}^{N_t}$$

The corollary now follows by again applying 4.4 to (**).

We can now prove our main theorem stated in §1 by induction on $|G|$. Corollary 4.6 is vital to the proof.

CASE $G = N_o$, $\underset{\sim}{B} = B_o$ If T is a projective indecomposable in B_o then

$T/\Phi(T)$ is completely reducible and so $D \trianglelefteq N_0$ acts trivially on it.
Hence $\Phi(T) \geqslant T(\partial - 1)$.
But every indecomposable component of $T/T(\partial - 1)$ can be regarded as a $k\tilde{N}_0$-module in a block of defect group $1 = \tilde{D}$. Thus $T/T(\partial - 1)$ is completely reducible, and so $T(\partial - 1) \geqslant \Phi(T)$.
Therefore $S = T/\Phi(T) = T/T(\partial - 1)$ is simple.
So $T > T(\partial - 1) > \ldots > T(\partial - 1)^q = 0$ is a composition series with $T(\partial - 1)^h / T(\partial - 1)^{h+1} \cong S \otimes \Pi_0^{\,h}$ for all $0 \leqslant h \leqslant q-1$.
(Hence $\dim T = q.\dim S$)

<u>Lemma 4.7</u>: a) If S is any simple in B_0, set $S_j = S \otimes \Pi_0^{\,j}$ for all $j \in \mathbb{Z}$. Then $S_0, \ldots, S_{e-1}$ is a full set of simples in B_0, and for each integer j, $S_{j+e} \cong S_j$.

b) For all $j \in \mathbb{Z}$, let T_j be a projective cover of S_j. Then each T_j has a unique composition series, namely

$$T_j > T_j(\partial - 1) > \ldots > T_j(\partial - 1)^q = 0$$

where $T_j(\partial - 1)^h / T_j(\partial - 1)^{h+1} \cong S_{j+h}$ for all $0 \leqslant h \leqslant q-1$.

<u>Proof</u>: a) Because $D = D_0$ is in $Z(C_0)$ the natural k-algebra epimorphism $kC_0 \longrightarrow k\tilde{C}_0$ sends $\varepsilon_{01} \in Z(kC_0)$ to a primitive idempotent $\widetilde{\varepsilon_{01}} \in Z(k\tilde{C}_0)$, see [15, p.390]. Moreover the $k\tilde{C}_0$-block corresponding to $\widetilde{\varepsilon_{01}}$ (which we will call $\widetilde{b_{01}}$) has defect group $1 = \tilde{D}$, and hence it contains a unique simple (up to isomorphism). But $D \trianglelefteq C_0$ acts trivially on every simple in b_{01}, and so the simples of b_{01} can be identified with the simples in $\widetilde{b_{01}}$. So up to isomorphism b_{01} contains exactly one simple.
Hence by 4.6a) B_0 contains exactly e classes of simple kN_0-modules, and these are of the required form by 4.6b).
b) Let s denote the dimension of the simples in B_0 (an invariant).
Suppose that X is a submodule of a fixed T_j.
Choose $h > 0$ so that $T_j(\partial - 1)^{h-1} \nleqslant X$ but $T_j(\partial - 1)^h \leqslant X$.
Then from $T_j/X > (T_j/X)(\partial - 1) > \ldots > (T_j/X)(\partial - 1)^h = 0$ it follows that $\dim (T_j/X) \geqslant hs$, i.e. $\dim X \leqslant (q-h)s$.
But $T_j(\partial - 1)^h \leqslant X$ and $\dim T_j(\partial - 1)^h = (q-h)s$

Hence $X = T_j(\partial - 1)^h$.

But we have already shown that T_j has a composition series of the form $T_j > T_j(\partial - 1) > \dots > T_j(\partial - 1)^q = 0$, which has factors $T_j(\partial - 1)^h / T_j(\partial - 1)^{h+1} \cong S_{j+h}$ for all $0 \leqslant h \leqslant q-1$.

Thus this is the unique composition series of T_j, as required.

Corollary 4.8: The main theorem is true for $G = N_o$, $\underset{\sim}{B} = B_o$.

Proof: It suffices to show that B_o is special (q,e)-uniserial (with respect to $D = D_o$); and this follows from 4.7

CASE $G = N_t$, $\underset{\sim}{B} = B_t$ $(1 \leqslant t \leqslant d-1)$ Here we are assuming inductively that the theorem is true for all groups of order less than $|N_t|$. Now if $j < t$ satisfies $N_j < N_t$ then this induction hypothesis shows that B_j is special (q,e)-uniserial (with respect to D_j). Hence in order to prove the theorem for this case, it suffices to show that B_t is special (q,e)-uniserial (with respect to D_t).

Now as in the case $G = N_o$, $\varepsilon_{t1} \in Z(kC_t)$ yields a primitive idempotent $\widetilde{\varepsilon_{t1}} \in Z(k\widetilde{C_t})$, see [15, p.390]. Moreover the $k\widetilde{C}_t$-block $\widetilde{b_{t1}}$ corresponding to $\widetilde{\varepsilon_{t1}}$ has defect group $\widetilde{D}$, and by a trivial calculation $e(\widetilde{C}_t, \widetilde{b_{t1}}) = 1$ (see [1, p.27]).

So as $|\widetilde{C_t}| < |N_t|$ we may apply induction and hence certainly deduce that $\widetilde{b_{t1}}$ contains only one simple (up to isomorphism). But $D_t \trianglelefteq C_t$ acts trivially on every simple in b_{t1}, and so the simples in b_{t1} can be identified with the simples in $\widetilde{b_{t1}}$. So b_{t1} contains a unique simple (up to isomorphism).

Thus by 4.6a) B_t contains exactly e classes of simple kN_t-modules. Let V be any simple in B_t, then 4.6b) now shows that on setting $V_j = V \otimes \Pi_t^{\,j}$ for all $j \in \mathbb{Z}$; $V_o, \dots, V_{e-1}$ is a full set of simples in B_t with $V_{j+e} \cong V_j$ for each j.

Notation: As $D_t \trianglelefteq N_t$ regard these simple kN_t-modules as $k\widetilde{N}_t$-modules.

Set $\tilde{q} = |\widetilde{D}| = p^t$ and write $\widetilde{N} = N_{\widetilde{N}_t}(D_{t-1}/D_t)$

Let f be the Green correspondence $\widetilde{N}_t \longrightarrow \widetilde{N}$.

Lemma 4.9: a) For all $j \in \mathbb{Z}$, there exists a 1-1 correspondence between the submodules of $V_{\tilde{N}}$ and the submodules of $(V_j)_{\tilde{N}}$, which is given by $L \longleftrightarrow L_j = L \otimes \Pi_t{}^j|_{\tilde{N}}$.

b) For every subgroup S of $\tilde{N}$, L is S-projective if and only if L_j is S-projective.

Proof: a) Obvious, since $\Pi_t{}^j$ is one-dimensional with $V_j = V \otimes \Pi_t{}^j$.

b) Let $g_1, \ldots, g_s$ be a right transversal of $\tilde{N}$ by S, and write 1_L, 1_{L_j}, $1_{\Pi_t{}^j}$ for the identity maps on L, L_j, $\Pi_t{}^j$ respectively.

Suppose firstly that L is S-projective.

Then there exists $\alpha \in (L,L)_S$ so that $1_L = \sum_{i=1}^{s} g_i{}^{-1} \alpha g_i$.

But this implies $1_{L_j} = \sum_{i=1}^{s} g_i{}^{-1}(\alpha \otimes 1_{\Pi_t{}^j}) g_i$, which shows that L_j is S-projective.

Conversely if L_j is S-projective we can similarly show that $L_j \otimes \Pi_t{}^{-j} \cong L$ is S-projective.

Corollary 4.10: For all $j \in \mathbb{Z}$, $fV_j \cong fV \otimes \Pi_t{}^j|_{\tilde{N}}$;

$$\text{and } \Omega fV_j \cong \Omega fV \otimes \Pi_t{}^j|_{\tilde{N}}.$$

Lemma 4.11: If $\tilde{\Pi}$ denotes the natural $k\tilde{N}$-module, then $\tilde{\Pi} \cong \Pi_t|_{\tilde{N}}$

Proof: Let N denote the inverse image of $\tilde{N}$, and suppose $\tilde{\Pi}$ affords the natural representation $\tilde{\pi}$ defined for all $n \in N$ by $\tilde{\pi}(\tilde{n}) = z(n)1_k$ where $z(n)$ is an integer satisfying $n^{-1}\partial_{t-1} n \equiv \partial_{t-1}{}^{z(n)} \mod D_t$

So for some $m \in \mathbb{Z}$, $n^{-1}\partial_{t-1} n = \partial_{t-1}{}^{z(n)} . \partial_t{}^m$

Hence we have $\quad n^{-1}\partial_{t-1}{}^p n = \partial_{t-1}{}^{pz(n)} . \partial_t{}^{pm}$

i.e. $\quad n^{-1}\partial_t n = \partial_t{}^{(z(n) + pm)}$

But for all $n \in N$, the natural representation π_t of N_t over k satisfies $\pi_t(n) = z_t(n)1_k$ where $z_t(n)$ is any integer so that $n^{-1}\partial_t n = \partial_t{}^{z_t(n)}$.

Hence it follows that $z_t(n) \equiv z(n) + pm \mod p^{d-t}$, for all $n \in N$.

So $\pi_t(n) = \tilde{\pi}(\tilde{n})$ for all $n \in N$, i.e. $\tilde{\Pi} \cong \Pi_t|_{\tilde{N}}$.

Remark: On applying 4.11 to 4.10 we get that for all $j \in \mathbb{Z}$,

$$fV_j \cong fV \otimes \tilde{\Pi}^j \quad \text{and} \quad \Omega fV_j \cong \Omega fV \otimes \tilde{\Pi}^j$$

here for each integer j, $\widetilde{\Pi}^j$ is a $k\widetilde{N}$-module affording $\widetilde{\pi}^j$.

Notation: Let $\widetilde{B}_t$ be a $k\widetilde{N}_t$-block of defect group $\widetilde{D}$ corresponding to a primitive component of the idempotent $\widetilde{\varepsilon_t} \in Z(k\widetilde{N}_t)$ arising from $\varepsilon_t \in Z(kN_t)$.

Choose the simple module V so that $V \in \widetilde{B}_t$, and set $\widetilde{e} = e(\widetilde{N}_t, \widetilde{B}_t)$.

Finally let $\widetilde{B}$ be the unique $k\widetilde{N}$-block of defect group $\widetilde{D}$ so that $\widetilde{B}^{\widetilde{N}_t} = \widetilde{B}_t$.

Now by induction the main theorem is true for $(\widetilde{N}_t, \widetilde{B}_t)$. So in particular $\widetilde{B}_t$ contains exactly $\widetilde{e}$ classes of simples, $\widetilde{B}$ is special $(\widetilde{q},\widetilde{e})$-uniserial (with respect to D_{t-1}/D_t) and $fV/\Phi(fV)$ is a simple in $\widetilde{B}$.

Write $S = fV/\Phi(fV)$ and for each $j \in \mathbb{Z}$, set $S_j = S \otimes \widetilde{\Pi}^j$.

Then since $\widetilde{\Pi}$ is one-dimensional, it follows that for each j,

$$S_j = S \otimes \widetilde{\Pi}^j = (fV/\Phi(fV)) \otimes \widetilde{\Pi}^j \cong (fV \otimes \widetilde{\Pi}^j)/(\Phi(fV) \otimes \widetilde{\Pi}^j)$$
$$= fV_j/\Phi(fV_j)$$

Now since $\widetilde{B}$ is special $(\widetilde{q},\widetilde{e})$-uniserial $S_o, \dots, S_{\widetilde{e}-1}$ is a full set of simples in $\widetilde{B}$.

In particular since $S_j = fV_j/\Phi(fV_j)$, it follows that fV_j is an indecomposable in $\widetilde{B}$ for all $j \in \mathbb{Z}$.

Now for the Green correspondence $(\widetilde{N}_t, \widetilde{B}_t) \longrightarrow (\widetilde{N}, \widetilde{B})$ notice that the set $\mathfrak{X} = \{\widetilde{D}^x \cap \widetilde{D} \ : \ x \in \widetilde{N}_t \setminus \widetilde{N}\} = \{1\}$, and hence $\mathcal{A} = \{S \ : \ 1 < S \leqslant \widetilde{D}\}$.

So we can apply 2.5 to each V_j (the V_j are non-projective since every $fV_j \neq 0$, and so they each have vertex in $\mathcal{A}$).

This tells us that for all $j \in \mathbb{Z}$, $V_j \in \widetilde{B}_t$ since $fV_j \in \widetilde{B}$.

But $V_o, \dots, V_{e-1}$ are distinct simples, and so it follows from this that $\widetilde{e} = e$.

Moreover since this implies that every simple in B_t, when regarded as a $k\widetilde{N}_t$-module, lies in $\widetilde{B}_t$; it is clear that the idempotent $\widetilde{\varepsilon_t}$ must be the block idempotent of $\widetilde{B}_t$.

Hence to summarise, we have shown that $V_o, \dots, V_{e-1}$ is a full set of simples in the $k\widetilde{N}_t$-block $\widetilde{B}_t$ corresponding to $\widetilde{\varepsilon_t}$. Moreover for each j, $fV_j/\Phi(fV_j) \cong S_j$.

So if we again apply induction to $(\tilde{N}_t, \tilde{B}_t)$ we can now deduce that there exists a permutation δ of $I = \{0,1,\ldots,e-1\}$ so that for all $i \in I$, $\Sigma(fV_i) \cong S_{\delta^{-1}(i)}$.

Moreover if either $\ell(fV_i) = \tilde{q}-1$ for all $i \in I$ or $\ell(fV_i) = 1$ for all $i \in I$, then $\tilde{B}_t$ is $(\tilde{q},e)$-uniserial (with respect to the labelling $V_o,\ldots,V_{e-1}$ of simples).

However we know from the remark on page 27 that there exists a fixed integer $m > 0$ so that $\ell(fV_j) = m$ for all $j \in \mathbb{Z}$. This gives two cases:

(i) Suppose that $m \leqslant \tilde{q}/2$.

Then if $e = 1$ 3.6 shows that $\ell(fV) = 1$ and hence $\tilde{B}_t$ is $(\tilde{q},1)$-uniserial. So assume now that $e > 1$. By Schur's lemma, (A),(D), and (F) we have,

$$0 = (V_1,V_o)_{\tilde{N}_t} \cong (V_1,V_o)^1_{\tilde{N}_t} \cong (fV_1,fV_o)^1_{\tilde{N}} \cong (fV_1,fV_o)_{\tilde{N}}$$

But clearly $(fV_1,fV_o)_{\tilde{N}} \neq 0$ unless $m = 1$.

So $\ell(fV_j) = 1$ for all j, which means that $\tilde{B}_t$ is $(\tilde{q},e)$-uniserial (with respect to the labelling $V_o,\ldots,V_{e-1}$ of simples) and that $fV_j \cong S_j$ for all integers j.

(ii) Suppose that $m > \tilde{q}/2$.

Then if $e = 1$ 3.6 shows that $\ell(fV) = \tilde{q}-1$ and hence $\tilde{B}_t$ is $(\tilde{q},1)$-uniserial. So assume now that $e > 1$. By Schur's lemma, (A),(D), and (F) we have,

$$0 = (V_1,V_o)_{\tilde{N}_t} \cong (V_1,V_o)^1_{\tilde{N}_t} \cong (fV_1,fV_o)^1_{\tilde{N}} \cong (\Omega fV_1,\Omega fV_o)_{\tilde{N}}$$

But clearly $(\Omega fV_1,\Omega fV_o)_{\tilde{N}} \neq 0$ unless $m = \tilde{q}-1$.

So $\ell(fV_j) = \tilde{q}-1$ for all j, which means that $\tilde{B}_t$ is $(\tilde{q},e)$-uniserial (with respect to the labelling $V_o,\ldots,V_{e-1}$ of simples) and that $\Omega fV_j \cong S_j$ for all integers j.

<u>Corollary 4.12</u>: $\tilde{\varepsilon}_t \in Z(k\tilde{N}_t)$ is always primitive and the corresponding $k\tilde{N}_t$-block $\tilde{B}_t$ is $(\tilde{q},e)$-uniserial with respect to the labelling $V_o,\ldots,V_{e-1}$ of simples.

Now return to the block B_t. For each $j \in \mathbb{Z}$ let $W_j \in B_t$ be a projective cover of V_j, and write $\tilde{W}_j = W_j / W_j(\vartheta_t - 1)$.

Then $W_o \oplus \ldots \oplus W_{e-1}$ is a component of kN_t. But $k\tilde{N}_t \cong kN_t / kN_t(\vartheta_t - 1)$ by 4.3b), and so if we regard the $\tilde{W}_j$ as $k\tilde{N}_t$-modules, it follows that

$\widetilde{W}_o \oplus \ldots \oplus \widetilde{W}_{e-1}$ is a component of $k\widetilde{N}_t$.

However $\widetilde{W}_j/\Phi(\widetilde{W}_j) \cong W_j/\Phi(W_j) \cong V_j$ for all j. Hence up to isomorphism the e projective indecomposables in $\widetilde{B}_t$ are $\widetilde{W}_o,\ldots,\widetilde{W}_{e-1}$. So by 4.12 each $k\widetilde{N}_t$-module $\widetilde{W}_j$ has a unique composition series, namely:

$$\widetilde{W}_j \; \bullet\overset{V_j}{\rule{2cm}{0.4pt}}\bullet\overset{V_{j+1}}{\rule{2cm}{0.4pt}}\bullet \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \; \bullet\overset{V_{j+\tilde{q}-1} \cong V_j}{\rule{2cm}{0.4pt}}\bullet \; 0$$

Hence since by 4.3a) $W_j(\partial_t - 1)^h / W_j(\partial_t - 1)^{h+1} \cong \widetilde{W}_j \otimes \Pi_t{}^h$ for all $0 \leqslant j \leqslant e-1$; $0 \leqslant h < |D_t|$, and since $V_j = V \otimes \Pi_t{}^j$ with $V_{j+e} \cong V_j$ for all $j \in \mathbb{Z}$, it follows that each W_j has a composition series,

$$W_j \bullet\overset{V_j}{\rule{1.5cm}{0.4pt}}\bullet \cdots \bullet\overset{V_{j+\tilde{q}-1} \cong V_j}{\rule{2cm}{0.4pt}}\underset{W_j(\partial_t-1)}{\bullet}\overset{V_{j+1}}{\rule{1.5cm}{0.4pt}}\bullet \; \cdot \; \cdot \; \cdot \; \bullet\overset{V_{j+\tilde{q}} \cong V_{j+1}}{\rule{2cm}{0.4pt}}\underset{W_j(\partial_t-1)^2}{\bullet} \cdots$$

$$\cdot \quad \cdot \quad \cdot \quad \underset{W_j(\partial_t-1)^{|D_t|-1}}{\bullet}\overset{V_{j+|D_t|-1} \cong V_j}{\rule{2.5cm}{0.4pt}}\bullet \; \cdot \; \cdot \; \cdot \; \bullet\overset{V_j}{\rule{1.5cm}{0.4pt}}\bullet \; 0$$

So since V was an arbitrary simple in $\widetilde{B}_t$, and hence in B_t, it will follow that B_t is special (q,e)-uniserial (with respect to D_t) if we can show that each W_j is uniserial (has a unique composition series). We will now prove this vital result.

Firstly recall that b_{t1} contains (up to isomorphism) a unique simple kC_t-module, F say. Let W be a projective cover of F.
Then W^{EC_t} is projective with a factor-module $F^{EC_t} \cong J_o \oplus \ldots \oplus J_{e-1}$ where $\{J_o,\ldots,J_{e-1}\}$ is a full set of the simple $k(EC_t)$-modules J that satisfy $J_{C_t} \in b_{t1}$ (see 4.5).
Hence it follows that $W^{EC_t} \cong Z_o \oplus \ldots \oplus Z_{e-1}$ where Z_j is a projective cover of J_j and $(Z_j)_{C_t} \cong W$ for all $j = 0,1,\ldots,e-1$. Thus:
If W is uniserial then Z_j is uniserial for all j .... (*)
Moreover, by 4.4 $\{W_j = Z_j{}^{N_t} : j = 0,1,\ldots,e-1\}$ is a full set of projective indecomposable kN_t-modules in B_t, and:

Z_j is uniserial for all j if and only if W_j is uniserial for all j
.... (**)

Hence from (*) and (**) it is sufficient to show that W is uniserial.

The methods used below to prove this are due to G. Michler, who has kindly given me permission to publish them here.

Notation: Let $A = \varepsilon_{t1} kC_t$ (the block ideal corresponding to b_{t1}), and notice that any kC_t-module in b_{t1} can be regarded as a (right) A-module.

Also if R is any ring and if $m \in \mathbb{Z}$, $m \geqslant 1$, we will denote the ring of all $m \times m$ matrices over R by R_m.

Now as b_{t1} contains (up to isomorphism) a unique projective indecomposable W, A (when regarded as a right kC_t-module) is a direct sum of n copies of W for some $n \in \mathbb{Z}$, $n \geqslant 1$. Hence by the famous Wedderburn-Fitting theorem, $A \cong L_n$ where $L = \mathrm{End}_A(W) = (W,W)_{C_t}$ and L is a local k-algebra.

Defn: A k-algebra R is said to have finite type if there exists (up to isomorphism) only finitely many indecomposable R-modules.

Lemma 4.13: a) For a fixed $m \in \mathbb{Z}$, $m \geqslant 1$; if R_m is of finite type then R is also of finite type.

b) If R is of finite type, so too is every homomorphic image of R.

Proof: Left as an easy exercise.

Lemma 4.14: $L = \mathrm{End}_A(W)$ is of finite type.

Proof: If M is any indecomposable A-module, then M lies in b_{t1} and hence is D-projective. Thus M is a component of U^{C_t} for some indecomposable kD-module U (see [14]). But kD is of finite type, and hence it follows that A ($= L_n$) is of finite type. Thus by 4.13a) L is of finite type.

Now let $J = \mathrm{Jac}(L)$ (the Jacobson radical). Then as L is local, L/J is a division k-algebra, and hence as k is algebraically closed, $L/J \cong k$. Now J/J^2 is a two-sided L-module that is annihilated on both sides by J. Thus every k-subspace of J/J^2 is a two-sided submodule of J/J^2.

Lemma 4.15: $\dim_k(J/J^2) = 1$.

Proof: Certainly $\dim_k(J/J^2) > 0$ since otherwise we would have $J = J^2 = 0$ implying $L \cong k$ and A simple, which would mean that b_{t1} had

defect zero, contradicting $d > 0$. So suppose now that $\dim_k(J/J^2) > 1$.
Let $\overline{L} = L/J^2$ and $\overline{J} = J/J^2$.
Then $\overline{J}$ is an ideal of $\overline{L}$ and every k-subspace of $\overline{J}$ is an ideal of $\overline{L}$.
But $\overline{J}$ has a subspace of co-dimension 2, N say, and hence L has a homomorphic image $L^* = \overline{L}/N$ of dimension 3 which has a k-basis $\{1,a,b\}$ that satisfies $a^2 = ab = ba = b^2 = 0$.
Now L^* is not of finite type since for all $m \in \mathbb{Z}$, $m \geqslant 1$ there is a 2m-dimensional indecomposable representation defined by:

$$a \longmapsto \begin{bmatrix} 0 & I_m \\ 0 & 0 \end{bmatrix}, \qquad b \longmapsto \begin{bmatrix} 0 & I_m^{+} \\ 0 & 0 \end{bmatrix}$$

where 0 is the $m \times m$ zero matrix, I_m is the $m \times m$ identity matrix, and I_m^{+} is the $m \times m$ matrix with 1's down the superdiagonal and zeros elsewhere.

But this implies by 4.13b) that L is not of finite type, contradicting 4.14.
Hence $\dim_k(J/J^2) = \dim_k(\overline{J}) = 1$.

Now pick $\alpha \in J \setminus J^2$. Then the right L-module αL satisfies

$$J^2 < \alpha L + J^2 \leqslant J$$

Thus by 4.15 $\alpha L + J^2 = J$. But $J^2 = \Phi(J)$ is redundant in any set of generators of J, and so $\alpha L = J$.
Similarly by considering the left L-module $L\alpha$, we can show that $L\alpha = J$.
Hence since $\mathrm{Jac}(A) = \mathrm{Jac}(L_n) = J_n$, if we set $c = \alpha I_n \in L_n$, we see that

$$cA = Ac = J_n = \mathrm{Jac}(A)$$

So by [16, theorem 1] A is "generalised uniserial", which means that W is uniserial.
This completes the proof of the main theorem for $G = N_t$, $\underset{\sim}{B} = B_t$ $(1 \leqslant t \leqslant d-1)$.

<u>CASE $G > N_{d-1}$, $\underset{\sim}{B}$</u> Part (i) of the main theorem is true, by induction applied to (N_{d-1}, B_{d-1}).

Hence it remains to prove (ii).

Set $H = N_{d-1}$. Then in this set-up $\mathfrak{X} = \{ D^x \cap D : x \in G \setminus H \} = \{1\}$. So the block $\underset{\sim}{B}$ is as described in §3, which contains all the information that is necessary to prove the second part of the main theorem.

This completes the induction and hence the proof of all parts of the main theorem.

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Footnotes

(1) on page 2: For the special case when $d = 1$, Feit (in unpublished notes) was the first person to prove, without using ordinary character theory, that $\underset{\sim}{B}$ contains exactly e isomorphism classes of simple kG-modules. The first person to prove this result in the general case without using character theory was G. Michler (in a paper shortly to be published). I am grateful to Prof. Michler for allowing me to reproduce some of his work in §4 of part A.

PART B NON-PROJECTIVE INDECOMPOSABLES IN A BLOCK WITH A CYCLIC DEFECT GROUP

§ 1 Introduction

Throughout this part of the thesis, we will be studying the following situation,

Hypothesis: $\underset{\sim}{B}$ is a kG-block with cyclic defect group D of order $q = p^d$ $(d \geqslant 1)$. D_{d-1} is the unique subgroup of D of order p, $H = N_G(D_{d-1})$ and $C = C_G(D_{d-1})$.
Let B be the unique kH-block of defect group D with $B^G = \underset{\sim}{B}$, and let b be any kC-block of defect group D with $b^G = \underset{\sim}{B}$. EC will denote the stabiliser in H of b and Eb the k(EC)-block corresponding to b. Finally set $e = e(G,B) = |EC : C|$, which by [3, (1.1) and 1.4] divides p-1.

In the introduction to part A, we have already mentioned how R. Brauer in [1] described the ordinary character theory of such a block $\underset{\sim}{B}$ for the special case when $d = 1$, and how E. C. Dade ([3]) extended these results to the general case. Indeed by making essential use of Dade's results, H. Kupisch ([11] and [12]) and G. J. Janusz ([9]), working independently, examined all the indecomposable kG-modules in $\underset{\sim}{B}$ and obtained information about their structures.

In part A ([13]) we investigated the projective indecomposables in such a block $\underset{\sim}{B}$ using purely modular techniques. In particular no character theory was used at all, and the information obtained was sufficiently detailed to enable the complete submodule lattice of these projective indecomposable kG-modules to be described.

As a result of this, if W is any (fixed) projective indecomposable in $\underset{\sim}{B}$, then the factor-modules of W are completely determined by a set of co-ordinates (a,b). The aim of this part is to generalise the techniques used in [13] to get a detailed description of the non-projective indecomposable kG-modules in $\underset{\sim}{B}$ via co-ordinates of the form $(i;a_1,b_1;\ \dots\ ;a_m,b_m)$.

The main tools used in this work are the "Green correspondence" and the extension functor "Ext". A knowledge of [13] is essential as the results and notation developed there will be used frequently.

I would like to thank sincerely my Ph.D. supervisor J. A. Green whose ideas and inspiration have been a great help to me throughout.

Before stating in detail the main theorem to be proved here, we need to recall some results from [13] and introduce some new definitions.

Firstly recall that in the situation described by the hypothesis above, B is special (q,e)-uniserial (with respect to D_{d-1}) : see [13, §1]. Indeed if we adopt the usual notation $T_{i\alpha}$, T_i, S_i for the indecomposable kH-modules in B and if the Green correspondence $(G,\underset{\sim}{B}) \longrightarrow (H,B)$ is denoted by f (see §2), then the main theorem in [13] proves the following result of Green:

$\underset{\sim}{B}$ contains (up to isomorphism) exactly e simple kG-modules, which can be labelled $V_0,\dots,V_{e-1}$ so that for all $0 \leqslant j \leqslant e-1$,

$$fV_j/\Phi(fV_j) \cong S_j$$

Moreover there exists a permutation δ of $I = \{0,1,\dots,e-1\}$ so that for all $0 \leqslant j \leqslant e-1$,

$$\Sigma(fV_j) = S_{\delta^{-1}(j)}$$

Let ρ be the permutation of I defined by $\rho(j) \equiv \delta^{-1}(j) + 1 \mod e$. Then, as in [13], δ and ρ play an important rôle.

<u>Defns</u>: a) $\underline{a} = (a_1,\dots,a_m)$ and $\underline{b} = (b_1,\dots,b_m)$ will denote m-vectors over $\mathbb{Z}$ $(m \geqslant 1)$.

b) A <u>co-ordinate</u> is a triple $c = (i;\underline{a},\underline{b}) = (i;a_1,b_1;\ \dots\ ;a_m,b_m)$ for any $i \in I$, $\underline{a}$, $\underline{b}$.

c) For a fixed co-ordinate $c = (i;\underline{a},\underline{b})$ define

$$i_1 = i, \quad i_{t+1} = \delta^{-b_{t+1}} \rho^{a_t + 1}(i_t) \qquad 1 \leqslant t \leqslant m$$

and set $\ell(c) = m + \sum_{t=1}^{m} (a_t + b_t)$: the length of c.

d) A kG-module U is said to be n-headed if $U/\Phi(U)$ is a direct sum of n simple kG-modules and n-footed if $\Sigma(U)$ is a direct sum of n simple kG-modules. ($U/\Phi(U)$, $\Sigma(U)$ are often called the head and foot of U respectively.)

We can now state our main theorem about the block $\underset{\sim}{B}$, in which $\mathcal{F}$ is a full set of non-projective indecomposable kG-modules in $\underset{\sim}{B}$.

<u>THEOREM</u>: There exist positive integers $r(i), s(i)$ $(0 \leqslant i \leqslant e-1)$ so that if we set $\mathcal{G} = \dot{\bigcup}_{m=1}^{\infty} \mathcal{G}_m$ where,

$$\mathcal{G}_m = \Big\{ c = (i;\underline{a},\underline{b}) \ : \ 0 \leqslant i \leqslant e-1;\ 0 \leqslant a_t \leqslant r(i_t) - 2 + \delta_{tm},^{(1)} \\ 1 - \delta_{t1} \leqslant b_t \leqslant s(i_t) - 1 \quad \text{for all } 1 \leqslant t \leqslant m \Big\}$$

then:

a) There is a 1-1 correspondence between the sets $\mathcal{F}$ and $\mathcal{G}$. (We will write $W \sim c$ if $W \in \mathcal{F}$ and $c \in \mathcal{G}$ correspond.)

b) $|\mathcal{F}| = |\mathcal{G}| = (q-1)e$, and hence $\underset{\sim}{B}$ contains (up to isomorphism) exactly qe indecomposable kG-modules.

Moreover for a fixed $W \in \mathcal{F}$, if $W \sim c \in \mathcal{G}_m$ then:

c) $\ell(W) = \ell(c)$.

d) $W/\Phi(W) \cong V_{i_1} \oplus \dots \oplus V_{i_m}$.
(The correspondence between 1-headed W's and $\mathcal{G}_1$ is exactly that defined in [13, 3.18].)

e) If
$$\mathcal{S}(W) = \begin{cases} \{V_{\delta^{b_1}(i_1)}, \dots, V_{\delta^{b_m}(i_m)}, V_{\rho^{a_m}(i_m)}\} & \text{when } b_1, a_m \neq 0 \\ \{V_{\delta^{b_2}(i_2)}, \dots, V_{\delta^{b_m}(i_m)}, V_{\rho^{a_m}(i_m)}\} & \text{when } b_1 = 0,\ a_m \neq 0 \\ \{V_{\delta^{b_1}(i_1)}, \dots, V_{\delta^{b_m}(i_m)}\} & \text{when } b_1 \neq 0,\ a_m = 0 \\ \{V_{\delta^{b_2}(i_2)}, \dots, V_{\delta^{b_m}(i_m)}\} & \text{when } b_1 = a_1 = 0,\ m \geqslant 2 \\ \{V_i\} & \text{when } c = (i;0,0) \in \mathcal{G}_1 \end{cases}$$
then $\Sigma(W) \cong \sum_{V \in \mathcal{S}(W)}^{\oplus} V$

f) The head and foot of W are multiplicity-free (this was first proved by Janusz in [9] and Kupisch in [12]). Also W is m-headed and m'-footed, where $m+1 \geqslant m' \geqslant m-1$ and $m, m' \leqslant q/2$.

g) Set $a_t' = r(i_t) - a_t - 2$, $b_t' = s(i_t) - b_t - \delta_{t1}$ for all $1 \leqslant t \leqslant m$ and define a new co-ordinate Ωc as follows:

$$\Omega c = \begin{cases} (\rho^{a_m+1}(i_m); a_m', 0; a_{m-1}', b_m'; \; \ldots \; ; a_1', b_2'; 0, b_1') & \text{if } a_m \leqslant r(i_m) - 2, \; b_1 \leqslant s(i_1) - 2; \\ (\delta^{b_m+\delta_{m1}}(i_m); a_{m-1}', b_m'; \; \ldots \; ; a_1', b_2'; 0, b_1') & \text{if } a_m = r(i_m) - 1, \; b_1 \leqslant s(i_1) - 2; \\ (\rho^{a_m+1}(i_m); a_m', 0; a_{m-1}', b_m'; \; \ldots \; ; a_1', b_2') & \text{if } a_m \leqslant r(i_m) - 2, \; b_1 = s(i_1) - 1; \\ (\delta^{b_m}(i_m); a_{m-1}', b_m'; \; \ldots \; ; a_1', b_2') & \text{if } a_m = r(i_m) - 1, \; b_1 = s(i_1) - 1, \; m \geqslant 2; \\ (i; 0, 0) & \text{if } c = (i; r(i)-1, s(i)-1) \in \mathcal{G}_1. \end{cases}$$

Then $\Omega c \in \mathcal{G}$ and if ΩW is the module defined in §2,

$$\Omega W \sim \Omega c$$

h) Let $O_t \in \mathcal{F}$ be so that $O_t \sim (i_t; a_t, b_t - 1 + \delta_{tm}) \in \mathcal{G}_1$ for all $1 \leqslant t \leqslant m$. (The O_t are all 1-headed and hence described in [13].)

Then if O_t —— O_{t+1} denotes an indecomposable extension of $O_t \oplus O_{t+1}$ by $V_{\delta^{b_t}(i_t)}$, W can be described by a graph

O_1 —— O_2 —— $\ldots$ —— O_{m-1} —— O_m

(For more details see §8.)

i) A complete set of composition factors of W has the form

$$\left\{ V_{\rho^j(i_t)} : 0 \leqslant j \leqslant a_t, \; 1 \leqslant t \leqslant m \right\} \cup \left\{ V_{\delta^j(i_t)} : 1 \leqslant j \leqslant b_t, \; 1 \leqslant t \leqslant m \right\}$$

§2 Preliminary Results

Throughout this chapter U,V,W are kG-modules.

Defns: a) If R is a subgroup of G, $(U,V)_R = \mathrm{Hom}_{kR}(U,V)$.

b) If $\theta \in (U,V)_G$, $r(\theta) = \pm(\mathrm{Im}\ \theta)$.

c) U is called projective-free if no direct summand of U is projective.

d) U^* is the contragradient module derived from U.

The results 2.1-2.4 below are all vital, but easy to check:

Lemma 2.1: (i) $\Sigma(U \oplus V) \cong \Sigma(U) \oplus \Sigma(V)$

(ii) $(U^*)^* \cong U$ and $\Sigma(U^*) \cong (U/\Phi(U))^*$

(iii) From (i) and (ii) it follows that

$$U \oplus V/\Phi(U \oplus V) \cong U/\Phi(U) \oplus V/\Phi(V)$$

Lemma 2.2: If $U_1,\dots,U_n$ are kG-modules so that for all $1 \leqslant i \neq j \leqslant n$ no simple submodule of U_i is isomorphic to a simple submodule of U_j, then if V denotes a simple kG-module,

$V \leqslant U_1 \oplus \dots \oplus U_n$ if and only if $V \leqslant U_t$ for some $1 \leqslant t \leqslant n$.

Corollary 2.3: Let $V_1,\dots,V_n$ be all of the simple kG-submodules of U. Then if $\Sigma(U)$ is multiplicity-free, $\Sigma(U) \cong V_1 \oplus \dots \oplus V_n$.

Lemma 2.4: Suppose $U_i \leqslant U$ for all $i = 1,2,\dots,n$ with $U = U_1 + \dots + U_n$. Then if U is not projective-free, U_t is not projective-free for some $1 \leqslant t \leqslant n$.

Defn: $\theta \in (U,V)_G$ is projective if there is a k-space homomorphism $\alpha : U \to V$ so that for all $u \in U$ $u\theta = \sum_{g\in G} (ug^{-1})\alpha . g$

Set $(U,V)_{1,G} = \{ \theta \in (U,V)_G : \theta \text{ is projective} \}$ and $(U,V)_G^1 = (U,V)_G / (U,V)_{1,G}$.

Theorem 2.5 (see Green [6, §3]): Let W be projective. Then,

a) If $\pi : W \to V$ is a kG-epimorphism, $\theta \in (U,V)_G$ is projective if and only if there exist $\phi \in (U,W)_G$ so that $\theta = \phi\pi$.

b) If $\mu : U \to W$ is a kG-monomorphism, $\theta \in (U,V)_G$ is projective if and only if there exist $\psi \in (W,V)_G$ so that $\theta = \mu\psi$.

c) $(U,V)_G^1 \cong (U,V)_G$ in both of the following cases:

U projective-free and V simple,

V projective-free and U simple.

Consider now a projective presentation of V, that is an exact sequence $0 \longrightarrow U \longrightarrow W \longrightarrow V \longrightarrow 0$ with W projective. Such a sequence is called a <u>minimal projective presentation</u> (mpp) of V if W is minimal (among all projective presentations of V). A mpp exists for all kG-modules V, and we write ΩV for the corresponding "kernel". Hence any mpp of V yields an exact sequence $0 \longrightarrow \Omega V \longrightarrow W \longrightarrow V \longrightarrow 0$. Schanuel's lemma [10, p.16] now shows that ΩV is unique up to isomorphism. The two theorems we need are:

<u>Theorem 2.6</u> (Heller, see [8]): Let V be projective-free, then:

a) Write $\Omega^{-1}V = (\Omega V^*)^*$ and we then have,

$$\Omega^{-1}\Omega V \cong \Omega\,\Omega^{-1}V \cong V.$$

b) V is non-projective indecomposable if and only if ΩV is non-projective indecomposable.

c) $\Sigma(\Omega V) \cong V/\Phi(V)$.

<u>Theorem 2.7</u> (Feit, a simple exercise in homological algebra using 2.5):

$$(U,V)^1_G \cong (\Omega U, \Omega V)^1_G \quad \text{as k-spaces.}$$

Now if $0 \longrightarrow \Omega V \xrightarrow{\mu} W \xrightarrow{\pi} V \longrightarrow 0$ is a mpp of V, then for all kG-modules U we can form another exact sequence,

$$(\Omega V,U)_G \xleftarrow{\mu^*} (W,U)_G \xleftarrow{\pi^*} (V,U)_G \longleftarrow 0$$

Moreover using 2.5b) we see that $\operatorname{Im}\mu^* = (\Omega V,U)_{1,G}$ and hence $\operatorname{Ext}^1_{kG}(V,U) = (\Omega V,U)^1_G$. This is a very important result for us.

<u>Notation</u>: $V\circ U$ will denote any extension of V by U, so that there exists an exact sequence $0 \longrightarrow U \longrightarrow V\circ U \longrightarrow V \longrightarrow 0$.

Now return to the situation described by the hypothesis in §1. U,V will denote kG-modules in $\underset{\sim}{B}$; L,M kH-modules in B.

Notice that $\mathfrak{X} = \{D^x \cap D : x \in G \setminus H\} \leqslant \{1\}$.

Let $(G,\underset{\sim}{B}) \underset{g}{\overset{f}{\rightleftarrows}} (H,B)$ denote the Green correspondence defined in [13, §2] for example. Then:

<u>Theorem 2.8</u>: a) fU is a projective-free kH-module in B with

$$U_H \cong fU \oplus (\text{proj}) \oplus (\text{modules} \notin B)$$

b) gL is a projective-free kG-module in $\underset{\sim}{B}$ with

$$L^G \cong gL \oplus (\text{proj})$$

c) $f(\Omega U) \cong \Omega(fU)$, $g(\Omega L) \cong \Omega(gL)$.

d) If U,L are projective-free, $f(gL) \cong L$, $g(fU) \cong U$.

e) If U,L are non-projective indecomposable so too are fU,gL.

f) $(U,V)^1_G \cong (fU,fV)^1_H$, $(L,M)^1_H \cong (gL,gM)^1_G$: as k-spaces.

Proof: a),b) follow from the definitions of fU,gL when $\mathfrak{X} \leqslant \{1\}$, along with [7, 4.14].

c) See [6, 4.5].

d),e) See [5].

f) See [4, 4.12] and [5].

Remark: Throughout this part of the thesis "unique" will mean unique up to isomorphism. (This notation was also used in part A.)

§3 Results About B

In [13, §1] the notion of a special (q,e)-uniserial block was introduced. Examples of such blocks are given in:

Lemma 3.1: a) B, Eb are special (q,e)-uniserial (with respect to D_{d-1}) and b is (q,1)-uniserial.

b) If we adopt the following block notation for indecomposables

$$B : T_{i\alpha}\,,\ Eb : J_{i\alpha}\,,\ b : F_\alpha$$

then for all $i;\alpha$ $T_{i\alpha} \cong J_{i\alpha}{}^H$, $(J_{i\alpha})_C \cong F_\alpha$.

Proof: [13, §4] shows that up to isomorphism B, Eb both contain exactly e simples whilst b contains a unique simple. So by applying the main theorem of [13] to (H,B), (EC,Eb), (C,b) in turn we get 3.1a). Part b) now follows by [13, 4.4 and 4.5].

Theorem 3.2: For a fixed $j;\beta$ there is a kC-endomorphism σ of $T_{j\beta}$ so that $T_{j\beta}\,\sigma^m \cong T_{j+m,\,\beta-m}$ for all $0 \leqslant m \leqslant \beta$.

Proof: Without loss of generality assume that for all $i;\alpha$

$$T_{i\alpha} = J_{i\alpha}{}^H,\ (J_{i\alpha})_H = F_\alpha$$

Moreover as b is (q,1)-uniserial, we can further assume that

$$0 = F_0 < F_1 < \ \dots\ < F_{q-1} < F_q$$

Then there exists a kC-endomorphism θ of F_q so that $F_\alpha\theta = F_{\alpha-1}$ for all α. Hence for a fixed $j;\beta$, since $(J_{i\alpha})_C = F_\alpha$ for all $i;\alpha$, it follows that $J_{j\beta}\,\theta^m = J_{j+m,\,\beta-m}$ for all $0 \leqslant m \leqslant \beta$.

Now let X be a right transversal of H by EC; then θ induces a kC-endomorphism σ on $T_{j\beta}$ where

$$(y \otimes x)\sigma = y\theta \otimes x \qquad \text{for all } y \in J_{j\beta},\ x \in X$$

Moreover $T_{j\beta}\,\sigma^m = \sum_{x\in X} J_{j\beta}\theta^m \otimes x = \sum_{x\in X} J_{j+m,\,\beta-m} \otimes x = T_{j+m,\,\beta-m}$ for all $0 \leqslant m \leqslant \beta$.

Remark: We (unambiguously) use the same symbol σ to denote the kC-endomorphism associated with any indecomposable in B.

Lemma 3.3: a) Every $T_{i\alpha}$ is a cyclic kH-module.

b) $T_{j\beta} = \langle t \rangle$ implies $T_{j+m,\,\beta-m} = \langle t\sigma^m \rangle$ for all $0 \leqslant m \leqslant \beta$.

c) If $T = \langle t \rangle \in B$ is indecomposable, then $\ell(T)$ is the smallest positive integer m such that $t\sigma^m = 0$

Proof: a) follows as each $T_{i\alpha}$ is uniserial.

b) $T_{j\beta} = \langle t \rangle$ implies that $t \notin T_{j+1,\beta-1}$, and hence for all $0 \leqslant m < \beta$:

$$t\sigma^m \notin T_{j+m+1,\beta-m-1}, \quad t\sigma^\beta = 0.$$

Thus $T_{j+m,\beta-m} = \langle t\sigma^m \rangle$ for all $0 \leqslant m \leqslant \beta$.

c) follows from part b).

Notation: As usual, set $T_i = T_{iq}$, $S_i = T_{i1}$, $0 = T_{i0}$ for all i.

Also let $V_0, \ldots, V_{e-1}$ be a full set of simple kG-modules in $\underset{\sim}{B}$ labelled so that for all $0 \leqslant j \leqslant e-1$,

$$fV_j/\Phi(fV_j) \cong S_j, \quad \Sigma(fV_j) \cong S_{\delta^{-1}(j)} \quad \text{(see §1).}$$

Finally let W_j be a projective cover of V_j for each j.

Lemma 3.4: a) $\theta \in (T_{i\alpha}, T_{j\beta})_H$ is projective if and only if $r(\theta) \leqslant \alpha+\beta-q$ (This is a generalised form of Passman's lemma, see [4, lemma 4])

b) If $T \in B$ is indecomposable, then for all $0 \leqslant j \leqslant e-1$:

$$(T, fV_j)^1_H \cong k \text{ or } 0; \quad (fV_j, T)^1_H \cong k \text{ or } 0$$

Proof: a) see [13, 3.3]; b) see [13, 3.7].

Notation: For the rest of this chapter, let $T = T_{i_1,\alpha_1} \oplus \ldots \oplus T_{i_n,\alpha_n}$ be any projective-free module in B (and so $1 \leqslant \alpha_\nu < q$ for all ν).

Take $\phi_\nu \in (\Omega T_{i_\nu,\alpha_\nu}, fV_j)_H$ $(0 \leqslant j \leqslant e-1,\ 1 \leqslant \nu \leqslant n)$ so that for a fixed j, each ϕ_ν is either zero or not projective.

Write $\phi = \phi_1 + \ldots + \phi_n \in (\Omega T, fV_j)_H$

Set $R(\phi) = \left\{ (x_1\phi_1 + \ldots + x_n\phi_n, -x_1, \ldots, -x_n) : x_\nu \in \Omega T_{i_\nu,\alpha_\nu} \text{ for all } 1 \leqslant \nu \leqslant n \right\}$

and $E(\phi) = \left(fV_j \oplus T_{i_1} \oplus \ldots \oplus T_{i_n} \right) \Big/ R(\phi)$

Now recall from §1 that $\mathrm{Ext}^1_{kH}(T, fV_j) \cong (\Omega T, fV_j)^1_H$. Indeed using [2, pp.290-292] every extension $T \cdot fV_j$ is isomorphic to $E(\phi)$ for some ϕ and vice-versa.

Defns: (i) Set $r_\nu = r(\phi_\nu) = \ell(\mathrm{Im}\ \phi_\nu)$ for all $1 \leqslant \nu \leqslant n$.

(ii) We say $E(\phi)$ is monic if it is of the form

(non-projective indecomposable) or

(non-projective indecomposable) $\oplus$ (projective)

<u>Remarks:</u> Without loss of generality take $r_1 \leq r_2 \leq \dots \leq r_n$.

Also notice that if $r_\gamma = 0$, then $\phi_\gamma = 0$, and hence $T_{i_\gamma,\alpha_\gamma}$ is isomorphic to a direct summand of $E(\phi)$. So if $E(\phi)$ is monic, $r_n \neq 0$, and therefore ϕ_n is not projective, which by 3.4a) means that $\beta = \alpha_n + r_n - \ell(fV_j) > 0$.

Since we will be looking for all monic extensions $E(\phi)$, if β is as defined above, we can assume throughout that $\beta > 0$.

<u>Notation:</u> Let $fV_j = \langle u \rangle$ and $T_{i_\gamma} = \langle t_\gamma \rangle$ $(1 \leq \gamma \leq n)$. The t_γ can be chosen so that $(t_\gamma \sigma^{\alpha_\gamma})\phi_\gamma = -u\sigma^{\ell(fV_j)-r_\gamma}$ for all $1 \leq \gamma \leq n$.

Define $\tau_\gamma \in E(\phi)$ $(1 \leq \gamma \leq n)$ and $\omega \in E(\phi)$ as follows,

set $s_\gamma = (0;0,\dots,0,t_\gamma,0,\dots,0)$, $\tau_\gamma = s_\gamma + R(\phi)$;

and $v = (u;0,\dots,0,t_n\sigma^\beta)$, $\omega = v + R(\phi)$.

<u>Proposition 3.5:</u> a) $\langle \omega \rangle \cong T_{j,\ell(fV_j)-r_n}$.

b) $\langle \tau_\gamma \rangle \cong T_{i_\gamma,\alpha_\gamma + r_\gamma}$ $(1 \leq \gamma \leq n)$.

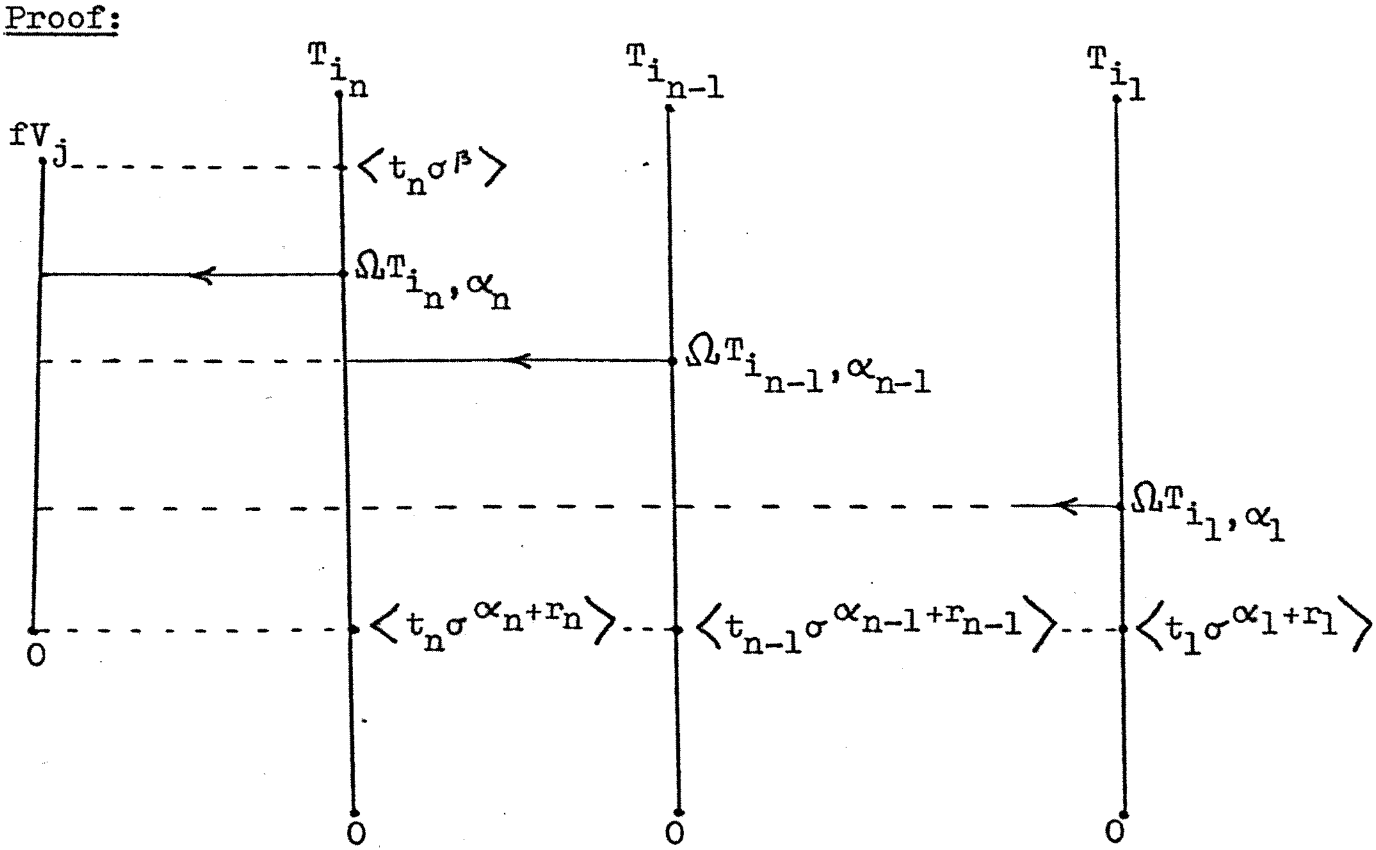
c) $E(\phi) = \langle \omega \rangle \oplus \langle \tau_1, \dots, \tau_n \rangle$.

d) If $\langle \tilde{\tau}_\gamma \rangle = \langle \tau_\gamma \rangle / \langle \tau_1, \dots, \hat{\tau}_\gamma, \dots, \tau_n \rangle \cap \langle \tau_\gamma \rangle^{(2)}$ for all $1 \leq \gamma \leq n$, then $\ell(\langle \tilde{\tau}_\gamma \rangle) \geq \alpha_\gamma$.

Moreover when $\gamma = n$, $\ell(\langle \tilde{\tau}_n \rangle) = \alpha_n + r_n - r_{n-1}$.

e) $\ell(E(\phi)) = \sum_{\gamma=1}^{n} \alpha_\gamma + \ell(fV_j)$.

<u>Proof:</u>



a) Looking at the diagram, it is clear that we can take

$$u = t_n\sigma^{\beta} + \langle t_n\sigma^{\alpha_n}\rangle$$

Hence $(t_n\sigma^{\alpha_n})\phi_n = -u\sigma^{\ell(fV_j)-r_n} = -t_n\sigma^{\alpha_n}$. Using this, and the fact that $\langle v\rangle \cap R(\phi) \cong \langle v\sigma^{\alpha_n+r_n}\rangle$, it is easy to show that $\langle v\rangle/\langle v\rangle \cap R(\phi) \cong \langle t_n\sigma^{\beta}\rangle/\langle t_n\sigma^{\alpha_n+r_n}\rangle \cong T_{j,\ell(fV_j)-r_n}$.

Hence $\langle\omega\rangle \cong T_{j,\ell(fV_j)-r_n}$.

b) For each r, $\langle s_r\rangle \cap R(\phi) = \langle s_r\sigma^{\alpha_r+r_r}\rangle$, and therefore $\langle s_r\rangle/\langle s_r\rangle \cap R(\phi) \cong \langle t_r\rangle/\langle t_r\sigma^{\alpha_r+r_r}\rangle \cong T_{i_r,\alpha_r+r_r}$.

Thus $\langle\tau_r\rangle \cong T_{i_r,\alpha_r+r_r}$.

c) From part a), as $\ell(fV_j) - r_n = \alpha_n - \beta$, $\omega\sigma^{\alpha_n-\beta} = 0$.

So if $\gamma \in kH$ with $u\gamma \in \mathrm{Im}\,\phi_n$ then $\omega\gamma = 0$.

But if $x \in \langle\omega\rangle \cap \langle\tau_1,\dots,\tau_n\rangle$, then there exists $v\gamma \in \langle v\rangle$, $s \in \langle s_1,\dots,s_n\rangle$ so that $x = v\gamma + R(\phi) = s + R(\phi)$, which implies that $v\gamma - s \in R(\phi)$, and hence that $u\gamma \in \mathrm{Im}\,\phi_1 + \dots + \mathrm{Im}\,\phi_n = \mathrm{Im}\,\phi_n$.

Thus by the above $\omega\gamma = 0$, which means that $x = 0$.

So $\langle\omega\rangle \cap \langle\tau_1,\dots,\tau_n\rangle = 0$. But $E(\phi)$ is certainly generated by the set $\{\omega,\tau_1,\dots,\tau_n\}$ and hence this proves c).

d) If $\tilde{\tau}_r\sigma^j = 0$ then $\tau_r\sigma^j \in \langle\tau_1,\dots,\hat{\tau}_r,\dots,\tau_n\rangle$, and hence there exists $s \in \langle s_1,\dots,\hat{s}_r,\dots,s_n\rangle$ so that $\tau_r\sigma^j = s_r\sigma^j + R(\phi) = s + R(\phi)$. But this implies that $s_r\sigma^j - s \in R(\phi)$, which means that $t_r\sigma^j \in \Omega T_{i_r,\alpha_r}$.

Hence $j \geqslant \alpha_r$, which by 3.3c) shows that $\ell(\langle\tilde{\tau}_r\rangle) \geqslant \alpha_r$.

Indeed for $r = n$, we can similarly prove the stronger result that $\tilde{\tau}_n\sigma^j = 0$ if and only if $t_n\sigma^j \in \Omega T_{i_n,\alpha_n+r_n-r_{n-1}}$, which is equivalent to saying if and only if $j \geqslant \alpha_n + r_n - r_{n-1}$. 3.3c) now finishes off the proof of part d).

e) This follows since $E(\phi)$ is an extension of the form $T\circ fV_j$.

<u>Corollary 3.6</u>: If $\omega \neq 0$, then S_j is in the head of $E(\phi)$.

We now examine $E(\phi)$ in detail using the above results.

<u>Case 1 Suppose $\omega \neq 0$</u> By 3.6 and 3.5, if $E(\phi)$ is monic then $\langle\tau_1,\dots,\tau_n\rangle$ is projective.

Let $\ell_r = \ell(\langle\tau_r\rangle) = \alpha_r + r_r$ $(1 \leqslant r \leqslant n)$. Then, there is a permutation

$r \mapsto r'$ of $\{1,2,\ldots,n\}$ so that $\ell_{1'} \geqslant \ell_{2'} \geqslant \ldots \geqslant \ell_{n'}$.

Also notice that for all $1 \leqslant r \leqslant n$, if $\tilde{\tau}_r$ is as defined in 3.5 then

$\langle \tilde{\tau}_r \rangle \cong \langle \tau_1,\ldots,\tau_n \rangle / \langle \tau_1,\ldots,\hat{\tau}_r,\ldots,\tau_n \rangle \neq 0$ .... (*)

Now suppose that $E(\phi)$ is monic, and hence $\langle \tau_1,\ldots,\tau_n \rangle$ is projective.

Then by 2.4 $\ell_{1'} = q$, and therefore:

$$\langle \tau_1,\ldots,\tau_n \rangle \cong T_{i_{1'}} \oplus \langle \tau_1,\ldots,\tau_n \rangle / \langle \tau_{1'} \rangle$$

So if $n \geqslant 2$, $\langle \tau_1,\ldots,\tau_n \rangle / \langle \tau_{1'} \rangle \neq 0$ and is projective (use (*)).

Thus we can now apply 2.4 and (*) to this module in a similar way to the above. Repeating this process inductively gives:

$$\langle \tau_1,\ldots,\tau_n \rangle \cong T_{i_{1'}} \oplus T_{i_{2'}} \oplus \ldots \oplus T_{i_{n'}}$$

So $\ell(E(\phi)) = \sum_{r=1}^{n} \alpha_r + \ell(fV_j) = \ell(\langle \omega \rangle) + nq = \ell(fV_j) - r_n + nq$ (use 3.5)

Hence $\sum_{r=1}^{n} (q - \alpha_r) = r_n$, and so $\sum_{r=1}^{n} r_r \leqslant \sum_{r=1}^{n} (q - \alpha_r) = r_n$.

Thus $n = 1$ (and when $n = 1$, such an $E(\phi)$ is monic if and only if $\alpha_1 + r_1 = q$, when $E(\phi) \cong T_{j,\ell(fV_j)-r_1} \oplus T_{i_1} \cong \langle \omega \rangle \oplus \langle \tau_1 \rangle$).

Case 2 Suppose $\omega = 0$ Here $E(\phi) = \langle \tau_1,\ldots,\tau_n \rangle$.

Subcase $n = 1$: Such an $E(\phi)$ is monic (by 3.5b)) if and only if $\alpha_1 + r_1 < q$, when $E(\phi) \cong T_{i_1,\alpha_1 + r_1} \cong \langle \tau_1 \rangle$.

Subcase $n = 2$: Suppose that $E(\phi)$ is monic, and let $r \mapsto r'$ be the permutation on $\{1,2\}$ defined in case 1.

Now (*) above with $r = 1,2$ shows that $E(\phi)$ is at least 2-headed, and hence not indecomposable. Thus $E(\phi)$ is not projective-free, and so by 2.4 $\ell_{1'} = q$, which implies that:

$$E(\phi) \cong T_{i_{1'}} \oplus \langle \tau_1,\tau_2 \rangle / \langle \tau_{1'} \rangle \cong T_{i_{1'}} \oplus \langle \tilde{\tau}_{2'} \rangle$$

So by 3.5d), such an $E(\phi)$ is monic if and only if

either $\alpha_1 + r_1 = q$, when $E(\phi) \cong T_{i_1} \oplus T_{i_2,\alpha_2+\alpha_1+r_2-q} \cong \langle \tau_1 \rangle \oplus \langle \tilde{\tau}_2 \rangle$, and $2' = 2$;

or $\alpha_2 + r_2 = q$, when $E(\phi) \cong T_{i_1,\alpha_1} \oplus T_{i_2} = \langle \tau_2 \rangle \oplus \langle \tilde{\tau}_1 \rangle$, and $2' = 1$.

Subcase $n \geqslant 3$: Suppose that $E(\phi)$ is monic, and let $r \mapsto r'$ be the permutation defined in case 1. Then if we repeatedly use 2.4 and (*)

in an analogous way to subcase 2, we get:

$$E(\phi) \cong T_{i_{1'}} \oplus \ldots \oplus T_{i_{(n-1)'}} \oplus T$$

where $T = \langle \tau_1, \ldots, \tau_n \rangle / \langle \tau_1, \ldots, \hat{\tau}_{n'}, \ldots, \tau_n \rangle \cong \langle \tilde{\tau}_{n'} \rangle$, which is hence indecomposable.

So $\ell(E(\phi)) = \sum_{\nu=1}^{n} \alpha_\nu + \ell(fV_j) = (n-1)q + \ell(\langle \tilde{\tau}_{n'} \rangle)$ (see 3.5).

But $\omega = 0$, and so by 3.5a) $\ell(fV_j) = r_n$. Hence we get:

$$\sum_{\nu=1}^{n} (q - \alpha_\nu) = q - \ell(\langle \tilde{\tau}_{n'} \rangle) + r_n \quad \ldots\ldots\ldots\ldots\ldots\ldots (\dagger)$$

Now by 3.5d) $\ell(\langle \tilde{\tau}_{n'} \rangle) \geqslant \alpha_{n'}$, so from $(\dagger)$ we get:

$$\sum_{\nu \neq n'} r_\nu \leqslant \sum_{\nu \neq n'} (q - \alpha_\nu) \leqslant r_n$$

Thus as $n \geqslant 3$, we must have $n' = n$. But then $(\dagger)$ and 3.5d) imply:

$$\sum_{\nu=1}^{n-1} r_\nu \leqslant \sum_{\nu=1}^{n-1} (q - \alpha_\nu) \leqslant r_{n-1}$$

which is impossible, since for all $\nu \neq n = n' \geqslant 3$, $\ell_\nu = \alpha_\nu + r_\nu = q$, and hence $r_\nu > 0$.

So for $n \geqslant 3$, there does not exist any monic extensions $E(\phi)$.

Remark: To use these results we must remember that we have assumed throughout that $r_1 \leqslant r_2 \leqslant \ldots \leqslant r_n$, and so we will get other monic extensions when $n = 2$ and $r_2 \leqslant r_1$. However these will all be given by permuting the above results via $1 \longmapsto 2$, $2 \longmapsto 1$.

Theorem 3.7: Given any projective-free $T \in B$ (as above), and any $0 \leqslant j \leqslant e-1$, then there exists (up to isomorphism) at most one monic extension $T \circ fV_j$ given as follows:

$n = 1$ a) A "unique" extension $T_{j, \ell(fV_j) + \alpha_1 - q} \oplus T_{i_1}$ if and only if $j \equiv \delta(i_1)$ and $\alpha_1 + \ell(fV_j) > q$.

b) A "unique" extension $T_{i_1, \alpha_1 + \ell(fV_j)}$ if and only if $j \equiv i_1 + \alpha_1$ and $\alpha_1 + \ell(fV_j) < q$.

$n = 2$ c) A "unique" extension $T_{i_2, \alpha_2 + \alpha_1 + \ell(fV_j) - q} \oplus T_{i_1}$ if and only if $j \equiv \delta(i_1) \equiv i_2 + \alpha_2$, $\alpha_2 + \ell(fV_j) \leqslant q$ and $\alpha_1 + \ell(fV_j) \geqslant q$.

d) A "unique" extension $T_{i_1, \alpha_1} \oplus T_{i_2}$ if and only if

$j \equiv i_2 + \alpha_2$ and $\alpha_2 + \ell(fV_j) = q$.

e) A "unique" extension $T_{i_1,\alpha_1+\alpha_2+\ell(fV_j)-q} \oplus T_{i_2}$ if and only if $j \equiv \delta(i_2) \equiv i_1 + \alpha_1$, $\alpha_1 + \ell(fV_j) \leqslant q$ and $\alpha_2 + \ell(fV_j) \geqslant q$.

f) A "unique" extension $T_{i_2,\alpha_2} \oplus T_{i_1}$ if and only if $j \equiv i_1 + \alpha_1$ and $\alpha_1 + \ell(fV_j) = q$.

<u>$n \geqslant 3$</u> No extension is monic.

(Note: Recall that we are making the convention that all congruences, unless otherwise specified are to be taken mod e.)

<u>Proof</u>: All possible monic extensions are given above, along with necessary and sufficient conditions for each one to exist. We merely rewrite these via the following easy to check results:

$n=1$ a) $\omega \neq 0$, $\alpha_1 + r_1 = q$ if and only if $j \equiv \delta(i_1)$, $\alpha_1 + \ell(fV_j) > q$.

(In this situation $r_1 = q - \alpha_1$.)

b) $\omega = 0$, $\alpha_1 + r_1 < q$ if and only if $j \equiv i_1 + \alpha_1$, $\alpha_1 + \ell(fV_j) < q$.

(In this situation $r_1 = \ell(fV_j)$.)

$n=2$ c) $\omega = 0$, $\alpha_1 + r_1 = q$ if and only if $j \equiv \delta(i_1) \equiv i_2 + c_2$, $\alpha_1 + \ell(fV_j) \geqslant q$, $\alpha_2 + \ell(fV_j) \leqslant q$.

(In this situation $r_1 \leqslant r_2 = \ell(fV_j)$, $\alpha_1 + r_1 \geqslant \alpha_2 + r_2$.)

d) $\omega = 0$, $\alpha_2 + r_2 = q$ if and only if $j \equiv i_2 + \alpha_2$, $\alpha_2 + \ell(fV_j) = q$.

(In this situation $r_1 \leqslant r_2$, $\alpha_2 + r_2 \geqslant \alpha_1 + r_1$.)

e),f) These are analogous to c),d) with $r_2 \leqslant r_1$.

<u>Defn</u>: $U \in \underset{\sim}{B}$ is <u>p/s -free</u> if no indecomposable direct summand of U is isomorphic to any W_k or to any $W_k/\Sigma(W_k)$.

<u>Lemma 3.8</u>: Let $U \in \underset{\sim}{B}$ be indecomposable. Then:

(i) U is projective if and only if $fU = 0$.

(ii) $U \cong W_k/\Sigma(W_k)$ if and only if $fU \cong T_{i\alpha}$ where $i \equiv \delta^{-1}(k)$ and $\alpha = q - \ell(fV_k)$.

(iii) $U \cong W_k/\Sigma(W_k)$ if and only if $fU \cong T_{i\alpha}$ where $i + \alpha \equiv k$ and $\alpha = q - \ell(fV_k)$.

<u>Proof</u>: (i) Follows from 2.8a),d).

(ii),(iii) Using 2.6a) and 2.8c),d) we see that:

$$U \cong W_k/\Sigma(W_k) \cong \Omega^{-1}V_k \text{ if and only if } fU \cong f\Omega^{-1}V_k \cong \Omega^{-1}fV_k.$$

The required results now follow using [13, 3.1].

Lemma 3.9: a) If $U \cong W_k/\Sigma(W_k)$, then for all $0 \leq j \leq e-1$ there are no monic extensions $fU \circ fV_j$.

b) If $U = U_1 \oplus U_2 \in \underset{\sim}{B}$ with U_1, U_2 non-projective indecomposables and $U_\gamma \cong W_k/\Sigma(W_k)$ for at least one γ, then for all $0 \leq j \leq e-1$ there exists at most one monic extension $fU \circ fV_j$. Moreover any such a monic extension is of the form

$$fU_1 \oplus (\text{proj}) \quad \text{or} \quad fU_2 \oplus (\text{proj})$$

Proof: a) By 3.8 $fU \cong T_{i\alpha}$ where $\delta(i) \equiv i+\alpha \equiv k$, $\alpha = q - \ell(fV_k)$. The result now follows from 3.7.

b) This similary follows from 3.8, 3.7.

Lemma 3.10: Let $U = U_1 \oplus \ldots \oplus U_n \in \underset{\sim}{B}$ be p/s-free (each U_γ being indecomposable with $fU_\gamma \cong T_{i_\gamma, \alpha_\gamma}$). Then for a fixed j, $0 \leq j \leq e-1$ there exists (up to isomorphism) at most one monic extension $fU \circ fV_j$ given as follows:

n=1 $fU \circ fV_j \cong T_{j, \alpha_1 + \ell(fV_j) - q} \oplus T_{i_1}$ if and only if $j \equiv \delta(i_1)$ and $\alpha_1 + \ell(fV_j) > q$;

$fU \circ fV_j \cong T_{i_1, \alpha_1 + \ell(fV_j)}$ if and only if $j \equiv i_1 + \alpha_1$ and $\alpha_1 + \ell(fV_j) < q$.

n=2 $fU \circ fV_j \cong T_{i_2, \alpha_2 + \alpha_1 + \ell(fV_j) - q} \oplus T_{i_1}$ if and only if $j \equiv \delta(i_1) \equiv i_2 + \alpha_2$, $\alpha_2 + \ell(fV_j) < q$ and $\alpha_1 + \ell(fV_j) > q$;

$fU \circ fV_j \cong T_{i_1, \alpha_1 + \alpha_2 + \ell(fV_j) - q} \oplus T_{i_2}$ if and only if $j \equiv \delta(i_2) \equiv i_1 + \alpha_1$, $\alpha_1 + \ell(fV_j) < q$ and $\alpha_2 + \ell(fV_j) > q$.

$n \geq 3$ No monic extensions occur.

Proof: Follows from 3.7, noting that by 3.8, since U is p/s-free

a) $\alpha_\gamma + \ell(fV_j) = q$ with $j \equiv i_\gamma + \alpha_\gamma$ cannot occur,

b) $\alpha_\gamma + \ell(fV_j) = q$ with $j \equiv \delta(i_\gamma)$ cannot occur.

§4 Extensions In $\underset{\sim}{B}$

Throughout this chapter let $U = U_1 \oplus \ldots \oplus U_n \in \underset{\sim}{B}$ (the U_r being indecomposable), and let $0 \leqslant j \leqslant e-1$.

Lemma 4.1: a) If $U \circ V_j$ is any extension, there exists an extension $fU \circ fV_j$ with $f(U \circ V_j) \oplus (\text{proj}) \cong fU \circ fV_j$.

b) If $fU \circ fV_j$ is any extension, there exists an extension $U \circ V_j$ with $g(fU \circ fV_j) \oplus (\text{proj}) \cong U \circ V_j$.

Proof: a) Modify [13, 3.9] (in which $U = X$, $V_j = Y$). The argument there works without the assumptions that X, $X \circ Y$ are non-projective indecomposables.

b) This is a "dual" version of a).

Lemma 4.2: If $U \circ V_j$ is non-projective indecomposable then U is p/s-free.

Proof: Certainly if $U \circ V_j$ is non-projective indecomposable, then U is projective-free. Hence suppose for a contradiction, that U is projective-free but not p/s-free and that U affords a non-projective indecomposable extension $U \circ V_j$ for some j.
Then by 4.1, there is a monic extension $fU \circ fV_j$, and hence by 3.7 $n \leqslant 2$.
So if $n = 1$ we can apply 3.9a), and if $n = 2$ we can apply 3.9b).
But by 3.9a) there exist no monic extensions $fU \circ fV_j$ if $n = 1$.
Hence $n = 2$ and the monic extension $fU \circ fV_j$ has, by 3.9b), the form:

$$fU_1 \oplus (\text{proj}) \quad \text{or} \quad fU_2 \oplus (\text{proj})$$

Thus by 4.1 $f(U \circ V_j) \cong fU_1$ or fU_2.
But $U \circ V_j$, U_1, U_2 are non-projective indecomposables, so by 2.8d) we must have $U \circ V_j \cong U_1$ or U_2, both of which are impossible.
This completes the proof of 4.2.

Lemma 4.3: a) For all non-split extensions $0 \longrightarrow V_j \xrightarrow{\mu} U \circ V_j \xrightarrow{\pi} U \longrightarrow 0$, with μ the inclusion map, $V_j \leqslant \Phi(U \circ V_j)$.
(And hence $U/\Phi(U) \cong U \circ V_j / \Phi(U \circ V_j)$.)

b) If U is p/s-free, such a $U \circ V_j$ is projective-free.

Proof: a) If there is a maximal submodule M of $U \circ V_j$ with $V_j \nleqslant M$, then $M \cap V_j = 0$ and $M + V_j = U \circ V_j$. Hence $U \circ V_j = M \oplus V_j$, which implies that $U \circ V_j$ is the split extension : contradiction.
Hence $V_j \leqslant M$ for all maximal submodules M of $U \circ V_j$, and so $V_j \leqslant \Phi(U \circ V\ )$.

b) Suppose that such a non-split $U \circ V_j$ has a projective indecomposable submodule W, with $U \circ V_j = W \oplus X$. There are two cases to consider.

(i) If $V_j \nleqslant W$, then π maps W monomorphically onto $W\pi \leqslant U$. Hence $W\pi$ is a projective submodule of U, and therefore a projective direct summand of U, which shows that U is not projective-free.

(ii) If $V_j \leqslant W$, then $V_j = \Sigma(W)$ and $V_j \nleqslant X$. Hence:

$$U = (W \oplus X)\pi \cong W/V_j \oplus X = W/\Sigma(W) \oplus X$$

which shows that U is not p/s-free.

This proves 4.3.

Theorem 4.4: Let $U \in \underset{\sim}{B}$ be p/s-free. Then for a fixed j:

a) There exists (up to isomorphism) at most one non-projective indecomposable extension $U \circ V_j$. Moreover such a non-projective indecomposable $U \circ V_j$ exists if and only if the "unique" monic $fU \circ fV_j$ exists (see 3.10) and these extensions are related by

$$f(U \circ V_j) \oplus (\text{proj}) \cong fU \circ fV_j$$

b) Whenever a non-projective indecomposable $U \circ V_j$ exists,

$$U/\Phi(U) \cong U \circ V_j/\Phi(U \circ V_j)$$

Proof: a) If $U \circ V_j$ is a non-projective indecomposable extension, then by 4.1a) there is a monic extension $fU \circ fV_j$ (which is "unique" by 3.10) so that $f(U \circ V_j) \oplus (\text{proj}) \cong fU \circ fV_j$.

Hence if E_1, E_2 are two non-projective indecomposable extensions of the form $U \circ V_j$, $fE_1 \oplus (\text{proj}) \cong fE_2 \oplus (\text{proj})$.

Thus $fE_1 \cong fE_2$, which implies that $E_1 \cong E_2$ (by 2.8d)).

This proves the "uniqueness" of such extensions $U \circ V_j$.

Conversely suppose there exists a monic extension $fU \circ fV_j$ (which hence must be "unique").

Then by 4.1b) there is an extension $U \circ V_j$ with

$$g(fU \circ fV_j) \oplus (\text{proj}) \cong U \circ V_j$$

Now 4.3b) shows that $U \circ V_j$ is projective-free, since U is p/s-free.

Thus $g(fU \circ fV_j) \cong U \circ V_j$, which implies that $U \circ V_j$ is non-projective indecomposable, since $fU \circ fV_j$ is monic. Moreover we can immediately deduce from these two facts that $fU \circ fV_j \cong f(U \circ V_j) \oplus (\text{proj})$ (use

2.8d)). This completes the proof of 4.4a).

b) This follows directly from 4.3a).

<u>Corollary 4.5</u>: Let $U \in \underset{\sim}{B}$ be projective-free with $fU_\gamma \cong T_{i_\gamma,\alpha_\gamma}$ for all $1 \leq \gamma \leq n$. Then for all j, there exists at most one non-projective indecomposable extension $U \circ V_j$ namely:

<u>$n=1$</u> A "unique" extension $U \circ V_j$ if and only if

either $j \equiv \delta(i_1)$ and $\alpha_1 + \ell(fV_j) > q$,

when $f(U \circ V_j) \cong T_{j,\alpha_1 + \ell(fV_j) - q}$;

or $j \equiv i_1 + \alpha_1$ and $\alpha_1 + \ell(fV_j) < q$,

when $f(U \circ V_j) \cong T_{i_1,\alpha_1 + \ell(fV_j)}$.

<u>$n=2$</u> A "unique" extension $U \circ V_j$ if and only if

either $j \equiv \delta(i_1) \equiv i_2 + \alpha_2$, $\alpha_1 + \ell(fV_j) > q$ and $\alpha_2 + \ell(fV_j) < q$,

when $f(U \circ V_j) \cong T_{i_2,\alpha_2 + \alpha_1 + \ell(fV_j) - q}$;

or $j \equiv \delta(i_2) \equiv i_1 + \alpha_1$, $\alpha_2 + \ell(fV_j) > q$ and $\alpha_1 + \ell(fV_j) < q$,

when $f(U \circ V_j) \cong T_{i_1,\alpha_1 + \alpha_2 + \ell(fV_j) - q}$.

<u>$n \geq 3$</u> There does not exist any non-projective indecomposable extensions $U \circ V_j$.

<u>Proof</u>: If U is not p/s-free, there exists γ, k where $1 \leq \gamma \leq n$, $0 \leq k \leq e-1$ so that $U_\gamma \cong W_k/\Sigma(W_k)$.

But then by 3.8, if $j \equiv \delta(i_\gamma)$ then $\alpha_\gamma + \ell(fV_j) = q$;

and if $j \equiv i_\gamma + \alpha_\gamma$ then $\alpha_\gamma + \ell(fV_j) = q$.

Hence if U is not p/s-free 4.5 reduces to the statement "There exists no non-projective indecomposable extensions $U \circ V_j$" : which is 4.2.

If U is p/s-free the theorem follows from 4.4 and 3.10.

§5 Co-ordinates

If $T \in B$ is indecomposable, then $(T)^1$, $(T)_1$ will denote respectively the unique top and bottom composition factors of T. For a fixed i, $0 \leqslant i \leqslant e-1$ recall the following from [13]:

Defns: a) For all $a,b \in \mathbb{Z}$ with $a,b \geqslant 0$

$$\gamma_i(a,b) = \gamma(a,b) = \sum_{j=0}^{a} \ell(fV_{\rho^j(i)}) + \sum_{j=1}^{b} \ell(fV_{\delta^j(i)}) - bq$$

(For $b=0$ make the convention $\gamma_i(a,0) = \sum_{j=0}^{a} \ell(fV_{\rho^j(i)})$.)

b) A 1-headed module $W \in \underset{\sim}{B}$ is said to be of type (a,b) if

$$fW \cong T_{\delta^b(i),\gamma_i(a,b)}$$

c) r(i), s(i) are the smallest positive integers such that

$$\gamma_i(r(i),0) \geqslant q, \quad \gamma_i(0,s(i)) \leqslant 0$$

Theorem 5.1: a) If $T = T_{\delta^b(i),\gamma_i(a,b)}$ then $(T)^1 \cong S_{\delta^{-1}(i)}$ and $(T)_1 \cong S_{\delta^{-1}\rho^a(i)}$. (Hence $\delta^b(i) + \gamma_i(a,b) \equiv \rho^{a+1}(i)$.)

b) $\rho^{r(i)}(i) = \delta^{s(i)}(i) \equiv i$.

c) $\gamma_i(r(i)-1,0) = q-1$, $\gamma_i(0,s(i)-1) = 1$.

d) If $a+b < r(i)+s(i)$, then we have the following:

$$\gamma_i(a,b) < q \text{ if and only if } a \leqslant r(i)-1,$$

$$\gamma_i(a,b) > 0 \text{ if and only if } b \leqslant s(i)-1.$$

e) There exists a 1-1 correspondence between a full set of(3) proper factor-modules of W_i and the set

$$\{(a,b) : 0 \leqslant a \leqslant r(i)-1,\ 0 \leqslant b \leqslant s(i)-1\}$$

given by type.

Proof: a) Left as an exercise (the reader may find [13, 3.10] helpful).
b),c) See [13, 3.13c)].
d) See [13, 3.15].
e) See [13, 3.18].

The aim of this chapter is to generalise 5.1e) to give a description of all the non-projective indecomposables in $\underset{\sim}{B}$. 5.1a) – d) are vital for this generalisation.

Now recall the definitions of m-vectors $\underline{a}, \underline{b}$ and co-ordinates $c = (i:\underline{a},\underline{b})$ introduced in §1. For a fixed co-ordinate c, we also defined the elements $i_1,\dots,i_m$ and the length $\ell(c)$. All of these notions will be used here. We also need the following:

Defns: a) For any m-vectors $\underline{a},\underline{b}$ with $a_t, b_t \geqslant 0$ for all $1 \leqslant t \leqslant m$, set

$$\gamma_i(\underline{a},\underline{b}) = \sum_{t=1}^{m} \gamma_{i_t}(a_t,b_t).$$

b) Let $\mathcal{G} = \bigcup_{m=1}^{\infty} \mathcal{G}_m$ where,

$$\mathcal{G}_m = \left\{ c = (i;\underline{a},\underline{b}) \;:\; \begin{array}{l} 0 \leqslant i \leqslant e-1 \text{ and } \underline{a},\underline{b} \text{ are m-vectors with} \\ 0 \leqslant a_t \leqslant r(i_t) - 2 + \delta_{tm} \text{ and} \\ 1 - \delta_{t1} \leqslant b_t \leqslant s(i_t) - 1 \text{ for all } 1 \leqslant t \leqslant m \end{array} \right\}$$

c) A co-ordinate c is called good if $c \in \mathcal{G}$.

d) $\mathcal{F}$ is a full set of non-projective indecomposables in $\underset{\sim}{B}$.

e) If $W \in \mathcal{F}$ and $c = (i;\underline{a},\underline{b}) \in \mathcal{G}$, then W is said to be of type $(i;\underline{a},\underline{b})$ if $fW \cong T_{\delta^{b_1}(i),\gamma_i(\underline{a},\underline{b})}$.

(We will write $W \sim c$ for brevity.)

Lemma 5.2 a) Let $T = T_{\delta^{b_1}(i),\gamma_i(\underline{a},\underline{b})} \in B$ for some $(i;\underline{a},\underline{b})$. Then,

$(T)^1 \cong S_{\delta^{b_1}(i)}$ and $\delta^{b_1}(i) + \gamma_i(\underline{a},\underline{b}) \equiv \rho^{a_m+1}(i_m)$.

(Hence $(T)_1 \cong S_{\delta^{-1}\rho^{a_m}(i_m)}$.)

b) If $\underline{a},\underline{b}$ satisfy $a_t \geqslant 0$, $b_t \geqslant 1 - \delta_{t1}$ for all $1 \leqslant t \leqslant m$, then

$$\gamma_i(\underline{a},\underline{b}) = \sum_{t=1}^{m} \gamma_{i_t}(a_t + 1 - \delta_{tm}, b_t - 1 + \delta_{t1}) - (m-1)q$$

Proof: a) That $(T)^1 \cong S_{\delta^{b_1}(i)}$ is obvious.

Now let $h = \delta^{b_1}(i) + \gamma_i(\underline{a},\underline{b}) = \delta^{b_1}(i) + \sum_{t=1}^{m} \gamma_{i_t}(a_t,b_t)$.

$$\begin{aligned} \text{Then } h &= \sum_{t=1}^{m} (\delta^{b_t}(i_t) + \gamma_{i_t}(a_t,b_t)) - \sum_{t=2}^{m} \delta^{b_t}(i_t) \\ &\equiv \sum_{t=1}^{m} \rho^{a_t+1}(i_t) - \sum_{t=2}^{m} \delta^{b_t}(i_t) \qquad \text{(use 5.1a))} \\ &= \rho^{a_m+1}(i_m) + \sum_{t=1}^{m-1} \delta^{b_{t+1}}(i_{t+1}) - \sum_{t=2}^{m} \delta^{b_t}(i_t) \\ &= \rho^{a_m+1}(i_m) \end{aligned}$$

b) Follows from the definition of the elements $i_1,\dots,i_m$.

Lemma 5.3: If $\underline{a},\underline{b}$ satisfy $0 \leqslant a_t \leqslant r(i_t) - 2$ for all $1 \leqslant t \leqslant m-1$, $a_m \geqslant 0$,
$1 \leqslant b_t \leqslant s(i_t) - 1$ for all $2 \leqslant t \leqslant m$, $b_1 \geqslant 0$,
and $a_m + b_1 < r(i_m) + s(i_1)$;

then $\gamma_i(\underline{a},\underline{b}) < q$ if and only if $a_m \leqslant r(i_m) - 1$,

$\gamma_i(\underline{a},\underline{b}) > 0$ if and only if $b_1 \leqslant s(i_1) - 1$.

Proof: We apply 5.1a) – c) and 5.2b).

(i) If $a_m \leqslant r(i_m) - 1$, then $a_t + 1 - \delta_{tm} \leqslant r(i_t) - 1$ for all $1 \leqslant t \leqslant m$.

Hence $\gamma_{i_t}(a_t + 1 - \delta_{tm}, b_t - 1 + \delta_{t1}) \leqslant \gamma_{i_t}(r(i_t)-1,0) = q-1$ for all t.

$$\text{So } \gamma_i(\underline{a},\underline{b}) = \sum_{t=1}^{m} \gamma_{i_t}(a_t + 1 - \delta_{tm}, b_t - 1 + \delta_{t1}) - (m-1)q$$
$$\leqslant m(q-1) - (m-1)q = q - m < q.$$

(ii) If $b_1 \leqslant s(i_1) - 1$, then $b_t \leqslant s(i_t) - 1$ for all $1 \leqslant t \leqslant m$.

Hence $\gamma_{i_t}(a_t,b_t) \geqslant \gamma_{i_t}(0,s(i_t)-1) = 1$ for all t.

$$\text{So } \gamma_i(\underline{a},\underline{b}) = \sum_{t=1}^{m} \gamma_{i_t}(a_t,b_t) \geqslant m > 0.$$

(iii) If $a_m \geqslant r(i_m)$, then $b_1 \leqslant s(i_1) - 1$, and so $b_t \leqslant s(i_t) - 1$ for all t.

$$\text{Hence } \gamma_i(\underline{a},\underline{b}) = \sum_{t=1}^{m} \gamma_{i_t}(a_t,b_t) \geqslant \sum_{t=1}^{m-1} \gamma_{i_t}(0,s(i_t)-1) + \gamma_{i_m}(a_m,b_m)$$
$$= (m-1) + \gamma_{i_m}(a_m,b_m) \geqslant (m-1) + \gamma_{i_m}(r(i_m),s(i_m)-1)$$

But $\rho^{r(i_m)}(i_m) \equiv i_m$, and therefore:

$$\gamma_{i_m}(r(i_m),s(i_m)-1) = \gamma_{i_m}(r(i_m)-1,0) + \gamma_{i_m}(0,s(i_m)-1)$$
$$= (q-1) + 1 = q$$

Thus $\gamma_i(\underline{a},\underline{b}) \geqslant (m-1) + q \geqslant q$.

(iv) If $b_1 \geqslant s(i_1)$, then $a_m \leqslant r(i_m) - 1$, and so

$a_t + 1 - \delta_{tm} \leqslant r(i_t) - 1$ for all $1 \leqslant t \leqslant m$.

$$\text{Hence } \gamma_i(\underline{a},\underline{b}) = \sum_{t=1}^{m} \gamma_{i_t}(a_t + 1 - \delta_{tm}, b_t - 1 + \delta_{t1}) - (m-1)q$$
$$\leqslant \gamma_{i_1}(a_1 + 1 - \delta_{1m}, b_1) + \sum_{t=2}^{m} \gamma_{i_t}(r(i_t) - 1, 0) - (m-1)q$$
$$= \gamma_{i_1}(a_1 + 1 - \delta_{1m}, b_1) + (m-1)(q-1) - (m-1)q$$
$$\leqslant \gamma_{i_1}(r(i_1)-1, s(i_1)) - (m-1)$$

But $\delta^{s(i_1)}(i_1) \equiv i_1$, and therefore:

$$\gamma_{i_1}(r(i_1)-1,s(i_1)) = \gamma_{i_1}(r(i_1)-1,0) + \gamma_{i_1}(0,s(i_1)-1) - q$$
$$= (q-1) + 1 - q = 0$$

Thus $\gamma_i(\underline{a},\underline{b}) \leqslant -(m-1) \leqslant 0$.

Corollary 5.4: If $(i;\underline{a},\underline{b}) \in \mathcal{G}_m$, then $m \leqslant q/2$.

Proof: From (i) and (ii) in the proof of 5.3, we see that $m \leqslant \gamma_i(\underline{a},\underline{b}) \leqslant q-m$, and hence we must have $m \leqslant q/2$.

Notation: For all co-ordinates $(i;\underline{a},\underline{b})$ define $\underline{a}^+,\underline{b}^+$ as follows,

$$\underline{a}^+ = (a_1,\ldots,a_{m-1},a_m+1), \quad \underline{b}^+ = (b_1+1,b_2,\ldots,b_m).$$

Lemma 5.5: If $(i;\underline{a},\underline{b}) \in \mathcal{G}_m$, then:

a) $(i;\underline{a}^+,\underline{b}) \in \mathcal{G}_m$ if and only if $\gamma_i(\underline{a}^+,\underline{b}) < q$.

b) $(i;\underline{a},\underline{b}^+) \in \mathcal{G}_m$ if and only if $\gamma_i(\underline{a},\underline{b}^+) > 0$.

Proof: If $(i;\underline{a},\underline{b}) \in \mathcal{G}_m$, then $(i;\underline{a}^+,\underline{b})$ and $(i;\underline{a},\underline{b}^+)$ both satisfy the hypotheses of 5.3. The result follows directly from this.

Lemma 5.6: If $(i;\underline{a},\underline{b}) \in \mathcal{G}_m$, $(h;\underline{c},\underline{d}) \in \mathcal{G}_k$ with $\rho^{a_m+1}(i_m) \equiv \delta^{d_1+1}(h_1)$,

then $(i;\underline{a},\underline{b};\underline{c},\underline{d}^+) = (i;a_1,b_1;\ \ldots\ ;a_m,b_m;c_1,d_1+1;\ \ldots\ ;c_k,d_k)$ satisfies the following,

$(i;\underline{a},\underline{b};\underline{c},\underline{d}^+) \in \mathcal{G}_{m+k}$ if and only if $\gamma_i(\underline{a}^+,\underline{b}) < q$, $\gamma_h(\underline{c},\underline{d}^+) > 0$.

Proof: $(i;\underline{a},\underline{b};\underline{c},\underline{d}^+) \in \mathcal{G}_{m+k}$ if and only if $a_m \leqslant r(i_m)-2$, $d_1+1 \leqslant s(h_1)-1$.

But $(i;\underline{a}^+,\underline{b})$ and $(h;\underline{c},\underline{d}^+)$ satisfy the hypotheses of 5.3, and hence

$\gamma_i(\underline{a}^+,\underline{b}) < q$ if and only if $a_m+1 \leqslant r(i_m)-1$,

$\gamma_h(\underline{c},\underline{d}^+) > 0$ if and only if $d_1+1 \leqslant s(h_1)-1$.

This gives the result.

Theorem 5.7: Let $U = U_1 \oplus \ldots \oplus U_n \in \underset{\sim}{B}$ be projective-free (each U_r being indecomposable). Then for each j, $0 \leqslant j \leqslant e-1$:

a) If $U = U_1 \sim (i;\underline{a},\underline{b}) \in \mathcal{G}_m$, there exists (up to isomorphism) at most one non-projective indecomposable extension $U \circ V_j$ namely

A "unique" extension of type $(i;\underline{a},\underline{b}^+)$ if and only if

$$j \equiv \delta^{b_1+1}(i_1) \text{ and } (i;\underline{a},\underline{b}^+) \in \mathcal{G}_m;$$

and a "unique" extension of type $(i;\underline{a}^+,\underline{b})$ if and only if

$$j \equiv \rho^{a_m+1}(i_m) \text{ and } (i;\underline{a}^+,\underline{b}) \in \mathcal{G}_m.$$

b) If $U = U_1 \oplus U_2$ with $U_1 \sim (i;\underline{a},\underline{b}) \in \mathcal{G}_m$ and $U_2 \sim (h;\underline{c},\underline{d}) \in \mathcal{G}_k$, there exists (up to isomorphism) at most one non-projective indecomposable extension $U \circ V_j$ namely

A "unique" extension of type $(h;\underline{c},\underline{d};\underline{a},\underline{b}^+)$ if and only if

$$j \equiv \delta^{b_1+1}(i_1) \equiv \rho^{c_k+1}(h_k) \text{ and } (h;\underline{c},\underline{d};\underline{a},\underline{b}^+) \in \mathcal{G}_{m+k};$$

and a "unique" extension of type $(i;\underline{a},\underline{b};\underline{c},\underline{d}^+)$ if and only if
$j \equiv \delta^{d_1+1}(h_1) \equiv \rho^{a_m+1}(i_m)$ and $(i;\underline{a},\underline{b};\underline{c},\underline{d}^+) \in \mathcal{G}_{m+k}$.

Proof: a) By 4.5, when $n = 1$ there exists (up to isomorphism) at most one non-projective indecomposable extension $U \circ V_j$ namely:

A "unique" extension with $f(U \circ V_j) \cong T_{j,\gamma_i(\underline{a},\underline{b})+\ell(fV_j)-q}$ if and only if $j \equiv \delta^{b_1+1}(i_1)$ and $\gamma_i(\underline{a},\underline{b}) + \ell(fV_j) - q > 0$;

and a "unique" extension with $f(U \circ V_j) \cong T_{\delta^{b_1}(i_1),\gamma_i(\underline{a},\underline{b})+\ell(fV_j)}$ if and only if $j \equiv \delta^{b_1}(i_1) + \gamma_i(\underline{a},\underline{b}) \equiv \rho^{a_m+1}(i_m)$ (see 5.2a)) and $\gamma_i(\underline{a},\underline{b}) + \ell(fV_j) < q$.

But these reduce to the following:

A "unique" extension with $f(U \circ V_j) \cong T_{\delta^{b_1+1}(i_1),\gamma_i(\underline{a},\underline{b}^+)}$ if and only if $j \equiv \delta^{b_1+1}(i_1)$ and $\gamma_i(\underline{a},\underline{b}^+) > 0$;

and a "unique" extension with $f(U \circ V_j) \cong T_{\delta^{b_1}(i_1),\gamma_i(\underline{a}^+,\underline{b})}$ if and only if $j \equiv \rho^{a_m+1}(i_m)$ and $\gamma_i(\underline{a}^+,\underline{b}) < q$.

But from the definition of type, and 5.5 these are equivalent to:

A "unique" extension of type $(i;\underline{a},\underline{b}^+)$ if and only if
$j = \delta^{b_1+1}(i_1)$ and $(i;\underline{a},\underline{b}^+) \in \mathcal{G}_m$;
and a "unique" extension of type $(i;\underline{a}^+,\underline{b})$ if and only if
$j = \rho^{a_m+1}(i_m)$ and $(i;\underline{a}^+,\underline{b}) \in \mathcal{G}_m$.

b) Similarly apply 4.5 with 5.2a) and 5.6.

c) This follows directly from 4.5.

Theorem 5.8: If $W \in \mathfrak{F}$, then there exists a co-ordinate $c = (i;\underline{a},\underline{b})$ so that $W \sim c$ and $\ell(W) = \ell(c)$.

Proof: Induction on $\ell(W)$.

If $\ell(W) = 1$, then $W \cong V_i$ for some i, and so $W \sim (i;0,0) \in \mathcal{G}_1$. This shows that the theorem is true for $\ell(W) = 1$. Hence let $N > 1$, and assume inductively that 5.8 is true for all lengths less than N. Take $W \in \mathfrak{F}$ with $\ell(W) = N$. Choose any minimal submodule V of W, and set $U = W/V$. Then if $V \cong V_j$ there is an extension $U \circ V_j \cong W$, and so U is certainly projective-free.

Let $U = U_1 \oplus \dots \oplus U_n$ (each U_γ being indecomposable).

Then by 4.5 $n \leqslant 2$.

Case $n=1$ By induction there exists $c = (i;\underline{a},\underline{b}) \in \mathcal{G}$ with $U = U_1 \sim c$ and $\ell(U) = \ell(c)$. Hence by 5.7a):

either $c_1 = (i;\underline{a},\underline{b}^+) \in \mathcal{G}$ with $W \sim c_1$,

or $c_2 = (i;\underline{a}^+,\underline{b}) \in \mathcal{G}$ with $W \sim c_2$.

(Observe that $\ell(c_1) = \ell(c_2) = \ell(c) + 1 = \ell(W) = N$.)

Case $n=2$ By induction there exists $c = (i;\underline{a},\underline{b})$, $c' = (h;\underline{c},\underline{d}) \in \mathcal{G}$ with $U_1 \sim c$, $U_2 \sim c'$ and $\ell(c) + \ell(c') = N-1$. Hence by 5.7b):

either $s_1 = (i;\underline{a},\underline{b};\underline{c},\underline{d}^+) \in \mathcal{G}$ with $W \sim s_1$,

or $s_2 = (h;\underline{c},\underline{d};\underline{a},\underline{b}^+) \in \mathcal{G}$ with $W \sim s_2$.

(Observe that $\ell(s_1) = \ell(s_2) = \ell(c) + \ell(c') + 1 = \ell(W) = N$.)

This proves 5.8.

Lemma 5.9: Let W, W' be indecomposables in $\underset{\sim}{B}$ and suppose $c = (i;\underline{a},\underline{b}) \in \mathcal{G}$ with $W \sim c$ and $W' \sim c$. Then:

a) W, W' are non-projective, b) $W \cong W'$.

Proof: Since $c \in \mathcal{G}$, $0 < \gamma_i(\underline{a},\underline{b}) < q$ by 5.3.

Hence $T = T_{\delta b_1(i),\gamma_i(\underline{a},\underline{b})}$ is a non-projective indecomposable in B.

Thus $fW \cong fW' \cong T$ (see 2.8), which shows that W, W' are non-projective.

Moreover by 2.8d) $W \cong g(fW) \cong g(fW') \cong W'$.

Lemma 5.10: For any $c = (i;\underline{a},\underline{b}) \in \mathcal{G}$ there exists $W \in \mathcal{J}$ so that $W \sim c$.

Proof: Since $c \in \mathcal{G}$, $0 < \gamma_i(\underline{a},\underline{b}) < q$ by 5.3.

So if we set $W = g(T_{\delta b_1(i),\gamma_i(\underline{a},\underline{b})})$, 2.8 shows that W is a non-projective indecomposable in $\underset{\sim}{B}$ with $fW \cong T_{\delta b_1(i),\gamma_i(\underline{a},\underline{b})}$.

Remark: In §7 we will prove the following result,

"If $W \in \mathcal{J}$; $c, c' \in \mathcal{G}$ with $W \sim c$ and $W \sim c'$, then $c = c'$."

This will show that there is a well defined map $\mathcal{J} \longrightarrow \mathcal{G}$ given by type, and from 5.8, 5.9 and 5.10 this map is a 1-1 correspondence.

§ 6 Some Properties Of A Co-ordinate

Throughout this chapter, take $W \in \mathcal{J}$, $c = (i;\underline{a},\underline{b}) \in \mathcal{C}_m$ so that $W \sim c$ and $\ell(W) = \ell(c)$ (see 5.8).

Lemma 6.1: The head and foot of W are multiplicity-free.

Proof: This follows since for all j, using 2.5a), 2.8f) and 3.4b):

$$(W,V_j)_G \cong (W,V_j)^1_G \cong (fW,fV_j)^1_H \cong k \text{ or } 0;$$

$$(V_j,W)_G \cong (V_j,W)^1_G \cong (fV_j,fW)^1_H \cong k \text{ or } 0.$$

Theorem 6.2: $W/\Phi(W) \cong V_{i_1} \oplus V_{i_2} \oplus \dots \oplus V_{i_m}$

Proof: Induction on $\ell(W)$.

If $\ell(W) = 1$, then $\ell(c) = 1$ and hence $c = (i;0,0) \in \mathcal{C}_1$. Thus $W \cong V_i = V_{i_1}$ and so the theorem is trivially true when $\ell(W) = 1$.

Hence let $N > 1$, and assume inductively that 6.2 is true for all lengths less than N. Choose W,c so that $W \sim c$ and $\ell(W) = \ell(c) = N$.

Now if $m=1$, then by [13, 3.18] W is isomorphic to a factor-module of W_i, and hence $W/\Phi(W) \cong V_i = V_{i_1}$ as required.

So we may as well assume that $m \geqslant 2$.

Case $b_1 = 0$ By 5.7 there exists a non-projective indecomposable $U \sim s = (i;a_1,b_1-1;a_2,b_2;\ \dots\ ;a_m,b_m)$ and a non-split extension $U \circ V_{\delta^{b_1}(i_1)} \cong W$.

(Notice that certainly $s \in \mathcal{C}$ and also $\ell(s) = \ell(U) = N-1$.)

Now by 4.3a) $W/\Phi(W) \cong U/\Phi(U)$, and by induction, $U/\Phi(U) \cong V_{i_1} \oplus V_{i_2} \oplus \dots \oplus V_{i_m}$. Hence:

$$W/\Phi(W) \cong V_{i_1} \oplus V_{i_2} \oplus \dots \oplus V_{i_m}$$

Case $b_1 = 0$ By 5.7 there exists non-projective indecomposables $U_1 \sim s_1 = (i;a_1,0)$, $U_2 \sim s_2 = (i_2,a_2,b_2-1;a_3,b_3;\ \dots\ ;a_m,b_m)$ and a non-split extension $(U_1 \oplus U_2) \circ V_{\delta^{b_2}(i_2)} \cong W$.

(Notice that certainly $s_1,s_2 \in \mathcal{C}$ and also $\ell(s_1) + \ell(s_2) = N-1$.)

Now by 4.3a) $W/\Phi(W) \cong U_1 \oplus U_2 / \Phi(U_1 \oplus U_2)$.

Hence $W/\Phi(W) \cong U_1/\Phi(U_1) \oplus U_2/\Phi(U_2)$ (see 2.1).

Also by [13, 3.18] U_1 is uniserial with $U_1/\Phi(U_1) \cong V_i = V_{i_1}$ and $\ell(U_1) = a_1 + 1 = \ell(s_1)$.

Hence since $\ell(U_1) + \ell(U_2) = N-1 = \ell(s_1) + \ell(s_2)$, it follows that $\ell(U_2) = \ell(s_2) < N$.

Thus by induction $U_2/\Phi(U_2) \cong V_{i_2} \oplus \dots \oplus V_{i_m}$, and so

$$W/\Phi(W) \cong V_{i_1} \oplus V_{i_2} \oplus \dots \oplus V_{i_m}$$

This completes the proof of 6.2.

Theorem 6.3: Let Ωc be the co-ordinate defined in the main theorem (see §1). Then $\Omega c \in \mathcal{G}$ and $\Omega W \sim \Omega c$.

Proof: We will consider only the case when $a_m \leq r(i_m) - 2$, $b_1 \leq s(i_1) - 2$ (the remaining cases being similar to verify). Hence assume this and set $\Omega c = (h;\underline{c},\underline{d}) = (\rho^{a_m+1}(i_m); a_m', 0; a_{m-1}', b_m'; \ \dots \ ; a_1', b_2'; 0, b_1')$ Recall all the properties of $\gamma(\ ,\ )$; $r(\ )$; $s(\ )$ stated in §5, since we will be using these throughout the proof without specific mention.

Notice firstly that:

$$h = h_1 = \rho^{a_m+1}(i_m),$$

$$h_2 = \delta^{-b_m'}\rho^{a_m'+1}(h_1) = \delta^{b_m}(i_m) \equiv \rho^{a_{m-1}+1}(i_{m-1}),$$

$$h_3 = \delta^{-b_{m-1}'}\rho^{a_{m-1}'+1}(h_2) = \delta^{b_{m-1}}(i_{m-1}) \equiv \rho^{a_{m-2}+1}(i_{m-2})$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$h_m = \delta^{-b_2'}\rho^{a_2'+1}(h_{m-1}) = \delta^{b_2}(i_2) \equiv \rho^{a_1+1}(i_1),$$

$$h_{m+1} = \delta^{-b_1'}\rho^{a_1'+1}(h_m) = \delta^{b_1+1}(i_1).$$

$$\left.\begin{array}{ll} \text{So } \delta^{b_t+\delta_{t1}}(i_t) \equiv h_{m-t+2} & \text{for all } 1 \leq t \leq m, \\ \rho^{a_t+1}(i_t) \equiv h_{m-t+1} & \text{for all } 1 \leq t \leq m. \end{array}\right\} \dots\dots\dots\dots \quad (1)$$

Now $a_t \leq r(i_t) - 2$, $b_t \leq s(i_t) - 1 - \delta_{t1}$ for all $1 \leq t \leq m$, and hence:

$$a_t' \geq 0, \ b_t' \geq 1 \quad \text{for all } 1 \leq t \leq m \ \dots\dots\dots\dots \quad (2)$$

Also as $\sum_{j=0}^{r(i_t)-1} \ell(fV_{\rho^j(i_t)}) = q-1$, $\sum_{j=a_t+1}^{r(i_t)-1} \ell(fV_{\rho^j(i_t)}) < q-1$.

Hence by (1) $a_t' < r(h_{m-t+1}) - 1$ for all $1 \leq t \leq m$, and therefore

$$a_t' \leq r(h_{m-t+1}) - 2 \quad \text{for all } 1 \leq t \leq m \ \dots\dots\dots\dots \quad (3)$$

Moreover by a dual process using the other part of (1), we get

$$b_t' \leqslant s(h_{m-t+2}) - 1 \quad \text{for all } 1 \leqslant t \leqslant m \text{ ....} \quad (4)$$

(2), (3) and (4) hence show that $\Omega c \in \mathcal{G}$.

It remains to prove that $\Omega fW \cong T_{h,\gamma_h(\underline{c},\underline{d})}$.

But by [13, 3.1] $\Omega fW \cong T_{\delta^{b_1}(i)+\gamma_i(\underline{a},\underline{b}),\, q-\gamma_i(\underline{a},\underline{b})}$ so we show that

a) $\delta^{b_1}(i) + \gamma_i(\underline{a},\underline{b}) \equiv \rho^{a_{m+1}}(i_m) \equiv h$, b) $\gamma_i(\underline{a},\underline{b}) + \gamma_h(\underline{c},\underline{d}) = q$.

a) This follows directly from 5.2a).

b) This follows since:

$$\begin{aligned}
\gamma_i(\underline{a},\underline{b}) + \gamma_h(\underline{c},\underline{d}) &= \sum_{t=1}^{m} \gamma_{i_t}(a_t,b_t) + \sum_{w=1}^{m+1} \gamma_{h_w}(c_w,d_w) \\
&= \sum_{t=1}^{m} \gamma_{i_t}(r(i_t)-1, s(i_t)) + q \\
&= \sum_{t=1}^{m} \left[\gamma_{i_t}(r(i_t)-1, 0) + \gamma_{i_t}(0, s(i_t)-1)\right] - mq + q \\
&= \sum_{t=1}^{m} ((q-1)+1) - mq + q = q
\end{aligned}$$

<u>Lemma 6.4</u>: $\mathcal{I}(\Omega W) = \mathcal{I}(\Omega c)$ and $\Sigma(\Omega W) \cong V_{i_1} \oplus \dots \oplus V_{i_m}$

<u>Proof</u>: That $\mathcal{I}(\Omega W) = \mathcal{I}(\Omega c)$ is left as an exercise. The second half of the lemma follows since $\Sigma(\Omega W) \cong W/\Phi(W)$ by 2.6c).

<u>Theorem 6.5</u>: Let $\mathcal{S}(W)$ be the set of simples described in the main theorem (see §1), and set $S(W) := \sum^{\oplus}_{V \in \mathcal{S}(W)} V$. Then $\Sigma(W) \cong S(W)$.

<u>Proof</u>: Since $\Omega : \mathcal{f} \longrightarrow \mathcal{f}$ is a bijection (see 2.6) it suffices to prove the result for ΩW, Ωc. But this is easy to check using 6.4 and (1) in the proof of 6.3, along with its analogies for the three other cases omitted in this proof.

<u>Theorem 6.6</u>: a) $W/\Phi(W) \cong V_{i_1} \oplus \dots \oplus V_{i_m}$; $\Sigma(W) \cong S(W)$.

b) $\{V_{i_t} : 1 \leqslant t \leqslant m\}$ contains pairwise non-isomorphic simples, $\{V : V \in \mathcal{S}(W)\}$ contains pairwise non-isomorphic simples.

c) W is m-headed and m'-footed where $m+1 \geqslant m' \geqslant m-1$ and $m, m' \leqslant q/2$.

<u>Proof</u>: a) This is 6.2 and 6.5.

b) This is 6.1 applied to a).

c) This follows from a) and 5.4 applied to both W and ΩW.

§ 7 Uniqueness Of Co-ordinates

Assume throughout that $W \sim c = (i;\underline{a},\underline{b}) \in \mathcal{G}_m$ and $W \sim c' = (h;\underline{c},\underline{d}) \in \mathcal{G}_k$.

Notation: a) If $b_1 > 0$ let $s_1 = (i;a_1,b_1-1;a_2,b_2; \ldots ;a_m,b_m) \in \mathcal{G}$ and choose $U_1 \in \mathcal{F}$ so that $U_1 \sim s_1$.

If $m \geqslant 2$, $2 \leqslant t \leqslant m$ let $s_{t1} = (i;a_1,b_1; \ldots ;a_{t-1},b_{t-1}) \in \mathcal{G}$, $s_{t2} = (i_t;a_t,b_t-1;a_{t+1},b_{t+1}; \ldots ;a_m,b_m) \in \mathcal{G}$ and choose $U_{t1},U_{t2} \in \mathcal{F}$ so that $U_{t1} \sim s_{t1}$, $U_{t2} \sim s_{t2}$.

If $a_m > 0$ let $s = (i;a_1,b_1; \ldots ;a_{m-1},b_{m-1};a_m-1,b_m)$ and choose $U \in \mathcal{F}$ so that $U \sim s$.

b) For c' analogously define U_1',s_1'; U_{t1}',s_{t1}'; U_{t2}',s_{t2}' and U',s'.

Lemma 7.1: a) There exists extensions of the form:

$$\left.\begin{array}{ll} U_1 \circ V_\delta{}^{b_1}(i_1) \cong W & (\text{if } b_1 > 0), \\ (U_{t1} \oplus U_{t2}) \circ V_\delta{}^{b_t}(i_t) \cong W & (\text{if } m \geqslant 2,\ 2 \leqslant t \leqslant m), \\ U \circ V_\rho{}^{a_m}(i_m) \cong W & (\text{if } a_m > 0). \end{array}\right\} \ldots\ldots\ldots \text{(A)}$$

Moreover if $m \geqslant 2$, $2 \leqslant t \leqslant m$ then:

$$\ell(fU_{t1}) + \ell(fV_\delta{}^{b_t}(i_t)) < q,\ \ \ell(fU_{t2}) + \ell(fV_\delta{}^{b_t}(i_t)) > q \ \ldots \text{(B)}$$

Also similar formulae (A'), (B') hold for c'.

b) $\ell(c) = \ell(c') = \ell(W)$.

Proof: a) (A) follows from 5.7, (B) from 4.5.

b) A simple induction on $\ell(W)$ using (A) shows that if $W \in \mathcal{F}$, $c \in \mathcal{G}$ with $W \sim c$ then $\ell(W) = \ell(c)$. This proves 7.1b).

Lemma 7.2: $m = k$, and if $c,c' \in \mathcal{G}_1$ then $c = c'$.

Proof: By 7.1b) $\ell(W) = \ell(c)$, and hence by 6.2 W is m-headed. Also by considering c', we similarly see that W is k-headed. Hence $m = k$. Also if $c,c' \in \mathcal{G}_1$ then $c = c'$ follows directly from [13, 3.18].

Theorem 7.3: $c = c'$ always.

Proof: Induction on $\ell(W) = \ell(c) = \ell(c')$.

If $\ell(W) = 1$, then $c = (i;0,0) \in \mathcal{G}_1$ and $c' = (h;0,0) \in \mathcal{G}_1$.

Hence $W \cong V_i \cong V_h$ which implies that $i = h$ and thus $c = c'$.

So the theorem is trivially true for $\ell(W) = 1$.

Hence let $N > 1$, and assume inductively that 7.3 is true for all lengths less than N. Choose $W \in \mathfrak{J}$ so that $\ell(W) = N$.

By 7.2 it is clear that we may as well assume that $m = k \geq 2$.

Now $\Sigma(W) \cong S(W)$ is multiplicity-free (by 6.1), and therefore by 2.3 W contains exactly $|\mathfrak{S}(W)|$ simple submodules.

Hence the decomposable factor-modules of W of composition length N-1 can be taken to be:

$$\{U_{t1} \oplus U_{t2} \;:\; 2 \leq t \leq m\}$$

Analogously with respect to the co-ordinate c', the decomposable factor-modules of W of composition length N-1 can be taken to be:

$$\{U_{w1}' \oplus U_{w2}' \;:\; 2 \leq w \leq m\}$$

Hence for a fixed t ($2 \leq t \leq m$) there is some w ($2 \leq w \leq m$) so that

$$U_{t1} \oplus U_{t2} \cong U_{w1}' \oplus U_{w2}'$$

(And thus by (A),(A') $W \cong (U_{t1} \oplus U_{t2}) \circ V_\delta b_{t(i_t)} \cong (U_{w1}' \oplus U_{w2}') \circ V_\delta d_{w(h_w)}$, which means that $V_\delta b_{t(i_t)} \cong V_\delta d_{w(h_w)}$.)

So either a) $U_{t1} \cong U_{w1}'$, $U_{t2} \cong U_{w2}'$ and $V_\delta b_{t(i_t)} \cong V_\delta d_{w(h_w)}$;

or b) $U_{t1} \cong U_{w2}'$, $U_{t2} \cong U_{w1}'$ and $V_\delta b_{t(i_t)} \cong V_\delta d_{w(h_w)}$.

But b) is inconsistent with (B),(B') and hence a) must occur.

However it now follows from induction that $s_{t1} = s_{w1}'$, $s_{t2} = s_{w2}'$; which implies that $t = w$ and that $c = c'$.

This completes the induction and hence the proof of 7.3.

<u>Corollary 7.4</u>: There exists a 1-1 correspondence between $\mathfrak{J}$ and $\mathcal{G}$ given by type.

<u>Proof:</u> This follows from 7.3 and the remark at the end of §5.

§8 The Proof Of The Main Theorem

Lemma 8.1: There is a 1-1 correspondence between $\mathfrak{F}$ and the set

$$\{T_{j\beta} : 0 \leqslant j \leqslant e-1,\ 1 \leqslant \beta \leqslant q-1\} \text{ given by } W \longleftrightarrow fW.$$

Proof: Use 2.8d),e).

Corollary 8.2: a) $|\mathfrak{F}| = |\mathfrak{G}| = (q-1)e$.

b) If $0 \leqslant j \leqslant e-1$, $1 \leqslant \beta \leqslant q-1$ then there is a unique co-ordinate $c = (i;\underline{a},\underline{b}) \in \mathfrak{G}$ so that $T_{j\beta} = T_{\delta^{b_1}(i),\gamma_i(\underline{a},\underline{b})}$

c) A full set of indecomposables in $\underset{\sim}{B}$ has order qe.

Proof: a),b) These follow from 8.1 and 7.4.

c) This follows from 8.1 and the fact that up to isomorphism $\underset{\sim}{B}$ contains exactly e projective indecomposables (see [13]).

Notation: Fix $W \in \mathfrak{F}$ and $c = (i;\underline{a},\underline{b}) \in \mathfrak{G}_m$ so that $W \sim c$.

If $1 \leqslant \mu < \nu \leqslant m$ define a co-ordinate $s_{\mu\nu} \in \mathfrak{G}$ by

$$s_{\mu\nu} = (i_\mu; a_\mu, b_\mu; \ \dots\ ; a_{\nu-1}, b_{\nu-1}; a_\nu, b_\nu - 1 + \delta_{\nu m})$$

and let $U_{\mu\nu} \in \mathfrak{F}$ be so that $U_{\mu\nu} \sim s_{\mu\nu}$.

Finally if $1 \leqslant \mu \leqslant m-1$ set $O_\mu = U_{\mu,\mu+1}$, and notice that each O_μ is 1-headed and hence described in [13].

Remark: If $1 \leqslant \lambda < \mu < \nu \leqslant m$ then $U_{\lambda\nu}$ is the "unique" indecomposable extension $(U_{\lambda\mu} \oplus U_{\mu+1,\nu}) \circ V_{\delta^{b_\mu}(i_\mu)}$ (see 5.7).

To demonstrate this we set $U_{\lambda\nu} = U_{\lambda\mu}$ —— $U_{\mu+1,\nu}$

Also if $1 \leqslant \kappa < \lambda < \mu < \nu \leqslant m$ we denote $U_{\kappa\lambda} \oplus U_{\mu\nu}$ by $\overset{U_{\kappa\lambda}}{\bullet}$ $\overset{U_{\mu\nu}}{\bullet}$

Defn: A build of W is an extension $X_1 \circ X_2 \circ \dots \circ X_n \cong W$.

Notation: The diagram

$$O_1 —— O_2 —— O_3 —— \quad \dots \quad —— O_{m-1} —— O_m$$

will be denoted by G(W) and called the graph of W.

A build of G(W) will mean a way of drawing G(W) step by step, where one step consists of adding a vertex O_t or a line joining two adjacent vertices O_w, O_{w+1} that are already drawn.

Lemma 8.3: W is described by G(W) in the sense that any build of G(W) corresponds naturally to a build of W.

Proof: Use 5.7.

Remarks: a) 8.3 gives a rich source of composition series of W. Indeed it is possible to obtain many more by applying at least one "connecting simple" $V_\delta b_{t(i_t)}$ before the full lattices of O_t, O_{t+1} are built up. Unfortunately we get multiplicities occuring (in the sense that if X,Y are factor-modules of W with $X \cong Y$, then it does not follow that $X = Y$ except in the special case when W is 1-headed) and this seems to make it impossible to describe the full submodule lattice of W by these methods.

b) For all $1 \leqslant t \leqslant m$, let M_t be a submodule of W so that

$$W/M_t = \underset{\bullet}{O_1} \text{———} \underset{\bullet}{O_2} \text{—} \cdots \text{—} \underset{\bullet}{O_{t-1}} \qquad \underset{\bullet}{O_{t+1}} \text{—} \cdots \text{—} \underset{\bullet}{O_{m-1}} \text{———} \underset{\bullet}{O_m}$$

Then the modules $M_1, M_2, \ldots, M_m$ form a V-system of W in the sense of Kupisch (see [12]).

Theorem 8.4: All parts of the main theorem (stated in §1) are true.

Proof: a) This is 7.4.

b) This is 8.2a),c).

c) This follows from 7.1b).

d),e),f) These are 6.6.

g) This is 6.3.

h) This is 8.3.

i) This follows from 8.3, and a knowledge of the composition factors of each O_t, which can be obtained from [13].

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[13] PART A of this thesis (to be published in the J. of Algebra under the title Blocks with a cyclic defect group).

Footnotes

(1) on page 3: Throughout this thesis δ_{ij} will mean the "Kronecker delta", and hence,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

(2) on page 10: $\langle \tau_1, \ldots, \hat{\tau}_r, \ldots, \tau_n \rangle$ denotes the submodule of $E(\phi)$ generated by the following set of elements,

$$\{\tau_1, \ldots, \tau_{r-1}, \tau_{r+1}, \ldots, \tau_n\}$$

(3) on page 19: [13, 3.17] shows that if W,U are factor-modules of W_i with $W \cong U$, then $W = U$.
Hence a full set of proper factor-modules of W_i is precisely the set of all proper factor-modules of W_i.

PART C APPLICATIONS TO THE SYMMETRIC, ALTERNATING AND MATHIEU GROUPS

§ 1 Introduction

Throughout this part of the thesis, we will be studying the following situation,

Hypothesis: $\underset{\sim}{B}$ is a kG-block with cyclic defect group D of order $q = p^d$ $(d \geqslant 1)$. D_{d-1} is the unique subgroup of D of order p, $H = N_G(D_{d-1})$ and $C = C_G(D_{d-1})$.
Let B be the unique kH-block of defect group D with $B^G = \underset{\sim}{B}$, and let b be any kC-block of defect group D with $b^G = \underset{\sim}{B}$. EC will denote the stabiliser in H of b, and we set $e = e(G,\underset{\sim}{B}) = |EC : C|$, which by [3, (1.1) and 1.4] divides p-1.

The main theorem in part A ([6]) shows that B is special (q,e)-uniserial (with respect to D_{d-1}) and so we can adopt the usual notation for a full set of indecomposable kH-modules in B, namely $\{T_{i\alpha} : 0 \leqslant i \leqslant e-1,\ 1 \leqslant \alpha \leqslant q\}$. Set $I = \{0,1,\ldots,e-1\}$ and for all $i \in I$ write S_i for T_{i1}. Then if f denotes the Green correspondence $(G,\underset{\sim}{B}) \longrightarrow (H,B)$ (see [6, §2]), then [6] also proves the following result of Green:

> $\underset{\sim}{B}$ contains (up to isomorphism) exactly e simple kG-modules, which can be labelled $V_0,\ldots,V_{e-1}$ so that for all $i \in I$ $fV_i/\Phi(fV_i) \cong S_i$. Moreover there exists a permutation δ of I so that for all $i \in I$ $\Sigma(fV_i) \cong S_{\delta^{-1}(i)}$.

Now let W_i be a projective cover of V_i for all $i \in I$, and define a new permutation ρ of I by $\rho(i) \equiv \delta^{-1}(i) + 1 \bmod e$. Then the main theorem in [6] shows that there exists positive integers $r = r(i)$, $s = s(i)$ $(0 \leqslant i \leqslant e-1)$ so that the full kG-submodule lattice of each W_i has the following form:

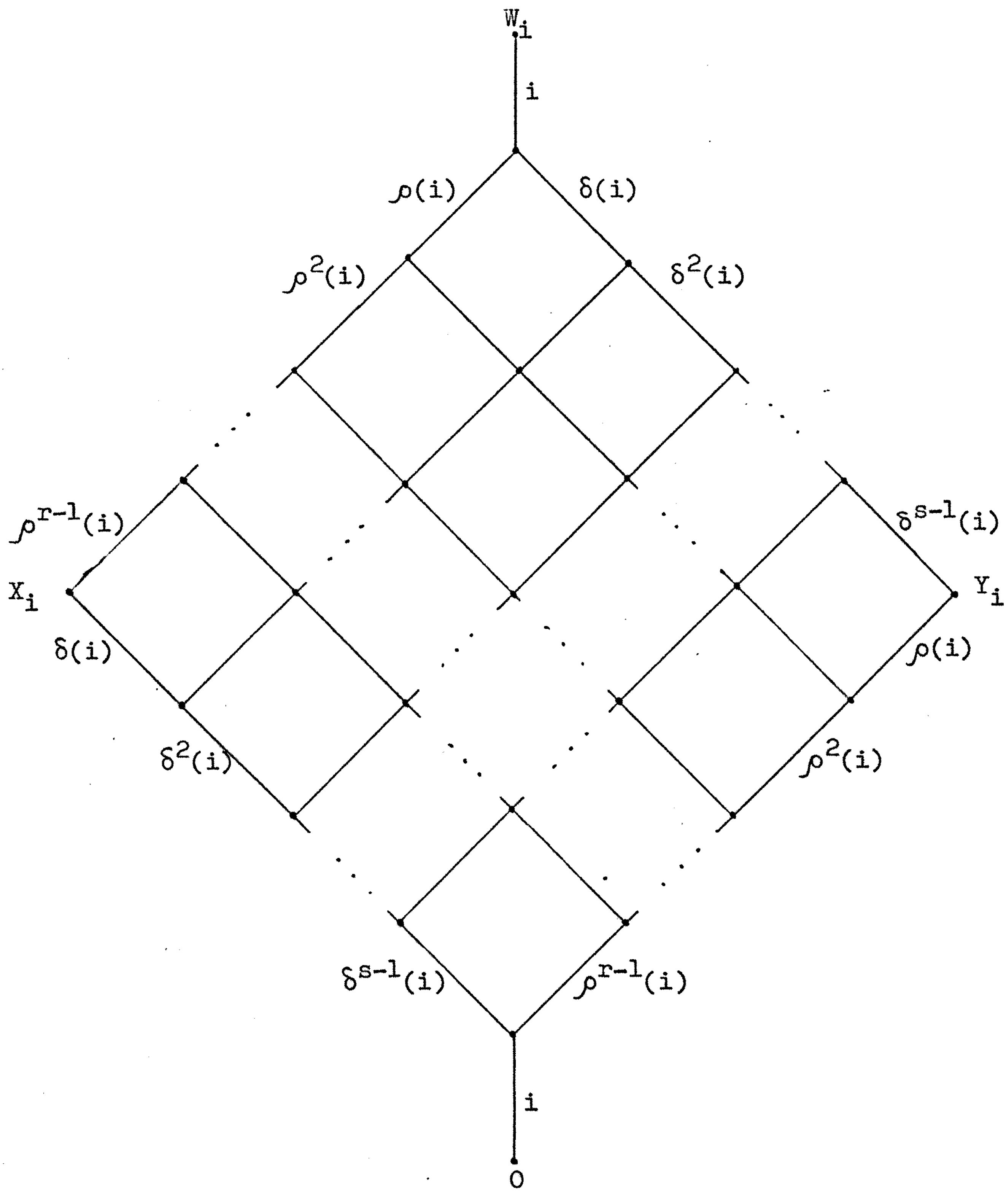


Figure 2

Note: $\rho^a(i)$, $\delta^b(i)$ denote the composition factors $V_{\rho^a(i)}$, $V_{\delta^b(i)}$ $(0 \leqslant a \leqslant r-1,\ 0 \leqslant b \leqslant s-1)$.

The information that we require about the $r(i), s(i)$ is as follows,

Lemma 1.1: a) $r = r(i)$, $s = s(i)$ are the unique positive integers such that $\sum_{j=0}^{r-1} \ell(fV_{\rho^j(i)}) = q-1$, $\sum_{j=0}^{s-1} \ell(fV_{\delta^j(i)}) - (s-1)q = 1$.

b) For all $i \in I$, $\rho^{r(i)}(i) = i$ and $\delta^{s(i)}(i) = i$.

Proof: See [6, §3].

Now for some blocks (like the principal p-blocks of S_p, A_p or PSL(2,p)) we can calculate these integers directly, using ordinary character theory (in §3 we will look at S_p and A_p). However for other blocks this becomes more difficult (certainly we cannot generally apply 1.1a) since the lengths $\ell(fV_i)$ are usually difficult to evaluate). The aim of §4 is to considerably simplify such vital calculations by answering the following question, which arises naturally from 1.1b),

"When are $r(i), s(i)$ the smallest positive integers such that

$$\rho^{r(i)}(i) = i, \quad \delta^{s(i)}(i) = i \text{ ?"}$$

Our answer will show that a knowledge of δ or of the Brauer tree (see §2) is almost sufficient to give all of the integers $r(i), s(i)$; and hence the lattice structures of all of the W_i. We will then (in §5) apply these results to the Mathieu groups, using the information given by G. D. James in [3], where all the possible Brauer trees are constructed.

§2 Connection With Dade's Theory

Notation: Let $(G,\underset{\sim}{B}) \overset{f}{\underset{g}{\rightleftarrows}} (H,B)$ denote the Green correspondence and Ω the Heller functor (see [6, §2]).

Defn: By an ordinary simple character we will mean a $\mathbb{C}G$-character afforded by a simple $\mathbb{C}G$-module.

If χ is an ordinary simple character, χ° will denote χ restricted to the p-regular elements (and hence χ° can be regarded as a modular character). Also we will write $\chi \in \underset{\sim}{B}$ if any kG-module affording χ° lies in $\underset{\sim}{B}$.

Lemma 2.1: Looking at Figure 2, the following hold for all $i \in I$,

a) $fX_i \cong S_i$, $fY_i \cong \Omega S_{\delta^{-1}(i)}$; b) $gS_i \cong X_i$, $\Omega g S_i \cong Y_{\delta(i)}$.

Proof: a) $fX_i \cong f(\Omega(W_i/X_i)) \cong \Omega f(W_i/X_i) \cong \Omega T_{i,q-1} \cong S_i$,

$fY_i \cong f(\Omega(W_i/Y_i)) \cong \Omega f(W_i/Y_i) \cong \Omega S_{\delta^{-1}(i)}$.

(Use [6, 2.7 and 3.13c)].)

b) This follows from a) and [6, 2.5b)].

Theorem 2.2 (Dade, see [1]): $\underset{\sim}{B}$ contains exactly $e + (q-1)/e$ ordinary simple characters, which fall into two classes:

e "non-exceptional" characters χ_i $(0 \leqslant i \leqslant e-1)$ and

$(q-1)/e$ "exceptional" characters χ_λ ($\lambda \in \Lambda$: a suitable index set).

The χ_λ are all equal on p-regular elements, and we set $\chi_e = \sum_{\lambda\in\Lambda} \chi_\lambda$ (called the exceptional character).

Moreover if Φ_i is the modular character of W_i $(0 \leqslant i \leqslant e-1)$, then there exists a unique distinct pair of elements for each $i \in I$, say $i(1), i(2) \in \{0,1,\ldots,e-1,e\}$ so that $\Phi_i = \chi_{i(1)}^{\circ} + \chi_{i(2)}^{\circ}$ for all $i \in I$.

Hence we can construct a graph with e edges (the W_i) and e+1 vertices (the χ_i). Indeed this graph is a tree (it is called the Brauer tree).

Defns: a) The vertex of the Brauer tree corresponding to χ_e is called the exceptional vertex.

b) The valence of a vertex is the number of edges joining it.

Theorem 2.3 (Janusz, see [4, 7.1] or [2, 6.1(iii), 7.1 and 7.2]):

For all $i \in I$; X_i, Y_i afford $\chi_{i(1)}{}^{o}$, $\chi_{i(2)}{}^{o}$ in some order.

Lemma 2.4: Suppose that $D \in Syl_p(G)$, then if $\dim S_i = n$ $(i \in I)$, the following all hold for each $i \in I$,

a) $\dim X_i \equiv n \mod p$,

b) $\dim Y_i \equiv -n \mod p$,

c) $\dim W_i \equiv 0 \mod q$,

d) If $\underset{\sim}{B}$ is the principal p-block $\dim X_i \equiv -\dim Y_i \equiv 1 \mod p$.

Proof: Since D is a Sylow p-subgroup, all projective kG-modules have dimension divisible by q. Also by Sylow's theorem $|G : H| \equiv 1 \mod p$. So a) and b) follow, since by 2.1,

$$S_i{}^G \cong gS_i \oplus (\text{proj}) \cong X_i \oplus (\text{proj}),$$

$$\Omega S_{\delta^{-1}(i)}{}^G \cong \Omega gS_{\delta^{-1}(i)} \oplus (\text{proj}) \cong Y_i \oplus (\text{proj}).$$

c) This follows since W_i is projective.

d) If $\underset{\sim}{B}$ is the principal p-block, then the one dimensional trivial kG-module k_G lies in $\underset{\sim}{B}$.

Thus $fk_G = k_H \in B$ (see [6, 2.5c)]).

This shows that $n = 1$ (it also shows that B is the principal p-block of kH).

§3 Applications to S_p, A_p

In this chapter, we look at the principal p-blocks of kS_p, kA_p which have cyclic defect groups of order p. Assume throughout that $p \neq 2$.

Firstly let $G = S_p$ and let $\underset{\sim}{B}$ be the principal p-block of kG.

Theorem 3.1: a) There exists exactly p ordinary simple characters in $\underset{\sim}{B}$ (these are the "p-hooks"), and hence $e = p-1$.

b) If $\chi \in \{\chi_0, \ldots, \chi_{e-1}, \chi_e\}$ (see 2.2) then $\chi|_{S_{p-1}}$ is a sum of at most two ordinary simple $\mathbb{C}S_{p-1}$-characters, and hence χ° is a sum of at most two modular simple characters.

Proof: a) See [5, 7.2] (that $e = p-1$ follows from 2.2).

b) See [5, 4.52].

Corollary 3.2: For all $i \in I$, $r(i), s(i) \leqslant 2$, and hence $\delta = \delta^{-1}$, $\rho = \rho^{-1}$.

Proof: From 3.1b) and 2.3 we see that $r(i), s(i) \leqslant 2$ for all $i \in I$. That $\delta = \delta^{-1}$, $\rho = \rho^{-1}$ now follows from 1.1b).

Remarks: (i) As B is special (p,e)-uniserial, and since $k_H \in B$, we can choose $S_o = k_H$ (and hence $V_o = k_G$).

(ii) Since $p \neq 2$, $e = p-1$ is always even.

(iii) Recall that we are making the convention that all congruences, unless otherwise specified are to be taken mod e.

Theorem 3.3: a) For all $j \in I$, $\delta(j) \equiv e-j$ (and hence $\rho(j) \equiv e-j+1$).

b) $$\mathfrak{r}(fV_j) = \begin{cases} 1 & \text{if } j = 0 \\ p-2j & \text{if } 1 \leqslant j \leqslant e/2 \\ 2h+1 & \text{if } j = e-h,\ e/2 < j \leqslant e-1 \end{cases}$$

c) For all $j \in I$, $r(j) = 2$, $s(j) = 1$ if $j = 0$ or $e/2$,
$s(j) = 2$ otherwise.

Proof: Order the set $I = \{0,1,\ldots,e-1\}$ as follows,

$0,1,e-1,2,e-2,3,e-3,4, \ \ldots \ ,e/2+3,e/2-1,e/2$ .... (*)

We will prove the theorem by induction with respect to this ordering.

Now as $V_o = k_G$, $fV_o = k_H = S_o$. Thus $\ell(fV_o) = 1$ and $\delta(0) = 0$.

So from 1.1 and 3.2 we can immediately deduce that $s(0) = 1$, $r(0) = 2$.

Thus the theorem is true for $j = 0$.

Hence now suppose that $j \neq e/2$, and that 3.3 is true for j.

We look at the next element j^+ (in the ordering (*)).

Case 1 Suppose that $1 \leqslant j < e/2$, and hence that $j^+ = e-j$.

By induction $\ell(fV_j) = p-2j$, $\delta(j) = e-j$ and $s(j) = 2$.

Now by 1.1, as $s(j) = 2$ and $\delta(j) = e-j$, $\ell(fV_j) + \ell(fV_{e-j}) = p+1$.

So $\ell(fV_{e-j}) = (p+1) - (p-2j) = 2j+1$.

Thus $r(e-j) = s(e-j) = 2$ (using 1.1 and 3.2), and also since

$$\ell(fV_{e-j}) + (e-j-1) \equiv \delta(e-j)$$

we get $\delta(e-j) \equiv (2j+1) + (e-j-1) \equiv j$.

Case 2 Suppose that $e/2 < j \leqslant e-1$ or that $j = 0$.

Write $j = e-h$ (if $j \neq 0$) and $j = h$ (if $j = 0$). Then $0 \leqslant h < e/2$ and $j^+ = h+1$.

By induction $\ell(fV_j) = 2h+1$, $\delta(j) = h$ and $r(j) = 2$.

Now by 1.1, as $r(j) = 2$ and $\rho(j) = h+1$, $\ell(fV_j) + \ell(fV_{h+1}) = p-1$.

So $\ell(fV_{h+1}) = (p-1) - (2h+1) = p-2(h+1)$.

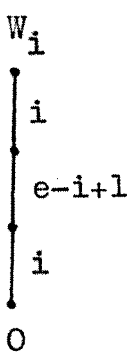
Thus $r(h+1) = 2$, $s(h+1) = 1$ if $h = e/2 - 1$ and $s(h+1) = 2$ if $h \neq e/2 - 1$ (using 1.1 and 3.2), and also since

$$\ell(fV_{h+1}) + h \equiv \delta(h+I)$$

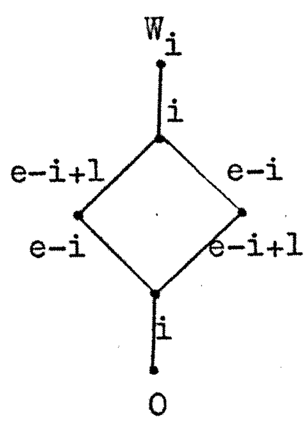
we get $\delta(h+1) \equiv p - 2(h+1) + h = (p-1) - (h+1) = e - (h+1)$.

This completes the proof of the theorem.

Corollary 3.4: The projective indecomposables in $\underset{\sim}{B}$ have the shapes,



$i = 0, e/2$



$i \neq 0, e/2$

Now let $G = A_p$ and let $\underset{\sim}{B}$ be the principal p-block of kG.

Theorem 3.5: a) There exists exactly $(p+3)/2$ ordinary simple characters in $\underset{\sim}{B}$, and hence $e = (p-1)/2$.

b) If $\chi \in \{\chi_0, \dots, \chi_{e-1}, \chi_e\}$ (see 2.2) then $\chi|_{A_{p-1}}$ is a sum of at most two ordinary simple $\mathbb{C}A_{p-1}$-characters, and hence χ^0 is a sum of at most two modular simple characters.

Proof: a) See [5, 4.54 and 7.6] (that $e = (p-1)/2$ follows from 2.2).
b) See [5, 4.52 and 4.54].

Corollary 3.6: For all $i \in I$, $r(i), s(i) \leqslant 2$, and hence $\delta = \delta^{-1}$, $\rho = \rho^{-1}$.

Proof: Analogous to 3.2.

Remarks: (i) Again we can take $S_0 = k_H$, $V_0 = k_G$.

(ii) For convenience set $[e/2] = e/2$ if e is even,
$[e/2] = (e+1)/2$ if e is odd.

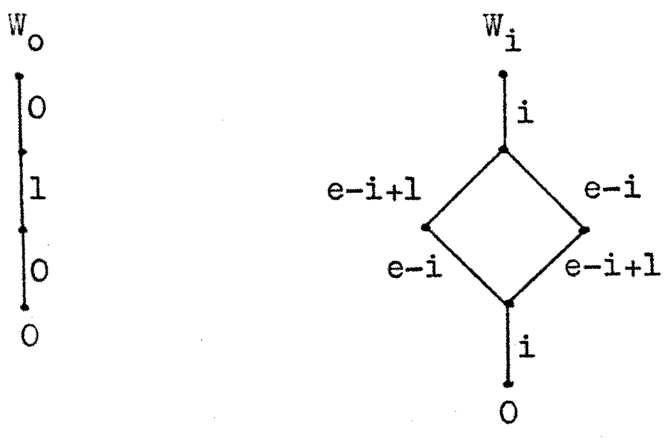
Theorem 3.7: a) For all $j \in I$, $\delta(j) \equiv e-j$ (and hence $\rho(j) \equiv e-j+1$).

b) $$\dim(fV_j) = \begin{cases} 1 & \text{if } j = 0 \\ p-2j & \text{if } 1 \leqslant j \leqslant [e/2] \\ 2h+1 & \text{if } j = e-h, \ [e/2] < j \leqslant e-1 \end{cases}$$

c) For all $j \in I$, $r(j) = 2$, $s(j) = 1$ if $j = 0$,
$s(j) = 2$ if $j \neq 0$.

Proof: Analogous to 3.3.

Corollary 3.8: The projective indecomposables in $\underset{\sim}{B}$ have the shapes,(1)



$i \neq 0$

§4 Results About r(i), s(i)

Defns: (i) $a = (q-1)/e$.

(ii) $\chi = \chi_{\lambda_1} + \dots + \chi_{\lambda_a}$ is the exceptional character in $\underset{\sim}{B}$ (this was denoted χ_e in 2.2).

(iii) Recall the notions of "long" and "short" used in [6]: An indecomposable T is called long if $\ell(T) \geqslant q-e$, and short if $\ell(T) \leqslant e$.

Remarks: a) The χ_{λ_i} are all equal on any p-regular element of G.

b) The structure of each projective indecomposable kG-module in $\underset{\sim}{B}$ is given in Figure 2. We will be using the notation in this figure throughout and without further mention.

Using 1.1, notice that since $\rho^{r(i)}(i) = \delta^{s(i)}(i) = i$ for all i, $r(\rho^j(i)) = r(i)$, $s(\delta^j(i)) = s(i)$ for all $i \in I$, $j \in \mathbb{N}$. Hence for a fixed $i \in I$, define $P(i) = \{\rho^j(i) : j \in \mathbb{N}\}$, $\Delta(i) = \{\delta^j(i) : j \in \mathbb{N}\}$. We call P(i) a ρ-set, and $\Delta(i)$ a δ-set.

Observe that $I = \{0, 1, \dots, e-1\}$ is partitioned by the ρ-sets and also by the δ-sets. Moreover we can talk about the r, s values on each P(i), $\Delta(i)$; and these determine the structure of all of the kG-modules W_i ($i \in I$) in $\underset{\sim}{B}$.

Now X_i, X_j have the same character if and only if $j \in \Delta(i)$, and Y_i, Y_j have the same character if and only if $j \in P(i)$.

Moreover X_i, Y_j never have the same character, since in the proof of [6, 3.18] it was shown that,

$$V_{\rho^h(i)} \not\cong V_{\delta^j(i)} \quad \text{for all } 1 \leqslant h \leqslant r(i)-1,\ 1 \leqslant j \leqslant s(i)-1;\ i \in I.$$

So in the Brauer tree, each vertex can be uniquely labelled by either a ρ-set P(i) or a δ-set $\Delta(i)$, and then,

a) The edge W_j joins the vertex P(i) if and only if $j \in P(i)$, the edge W_j joins the vertex $\Delta(i)$ if and only if $j \in \Delta(i)$.

b)
$$\left.\begin{array}{l} |P(i)| = \text{the valence of the vertex } P(i), \\ |\Delta(i)| = \text{the valence of the vertex } \Delta(i). \end{array}\right\} \dots\dots\dots\dots\dots (*)$$

Remark: Hence each vertex carries a label in the set $\{P, \Delta\}$. Also each edge joins a P-vertex to a Δ-vertex, and so since any tree is by definition connected, if we know the label of one vertex, we know the rest too. This labelling is very important for us.

Lemma 4.1: The number of distinct ρ-sets plus the number of distinct δ-sets is $e+1$.

Proof: This follows since the number of vertices in the Brauer tree is $e+1$.

Additional Hypothesis: Assume, until further notice, that $2e < q = p^d$.

Lemma 4.2: For all $i \in I$ the following both hold,

a) $\{fV_i, fV_{\rho(i)}, \ldots, fV_{\rho^{r(i)-1}(i)}\}$ contains at most one long fV.

b) $\{fV_i, fV_{\delta(i)}, \ldots, fV_{\delta^{s(i)-1}(i)}\}$ contains at most one short fV.

Proof: a) If $fV_{\rho^u(i)}$ and $fV_{\rho^h(i)}$ are both long $(0 \leqslant u < h \leqslant r(i)-1)$ then,

$$\ell(fV_{\rho^u(i)}) + \ell(fV_{\rho^h(i)}) \leqslant \sum_{j=0}^{r(i)-1} \ell(fV_{\rho^j(i)}) = q-1 \quad \text{(see 1.1);}$$

and so $2(q-e) \leqslant q-1$, which implies that $2e > q$, a contradiction.

b) This is analogous.

Lemma 4.3: There is an injection $\mu : I \to \{P(i)\} \cup \{\Delta(i)\}$ given by $\mu : i \mapsto P(i)$ if fV_i is long, $\mu : i \mapsto \Delta(i)$ if fV_i is short.

Proof: Use 4.2.

Notation: Using 4.1 and 4.3 let the complement of $\operatorname{Im}\mu$ be $\{\underset{\sim}{E}\}$.

Theorem 4.4: Let $i \in I$, then:

a) If $P(i) \neq \underset{\sim}{E}$ then $r(i)$ is the smallest positive integer r so that $\rho^r(i) = i$.

b) If $\Delta(i) \neq \underset{\sim}{E}$ then $s(i)$ is the smallest positive integer s so that $\delta^s(i) = i$.

Proof: a) By [6, 3.13] $f(W_i/X_i)$ is an extension of the form,

$$fV_i \circ fV_{\rho(i)} \circ \cdots \circ fV_{\rho^{r(i)-1}(i)} \cong T_{i,q-1}$$

Now if $P(i) \neq \underset{\sim}{E}$ then fV_i is long, and so $U = fV_{\rho(i)} \circ \cdots \circ fV_{\rho^{r(i)-1}(i)}$ satisfies $\ell(U) < e$, $\Sigma(U) \cong S_{i-1}$.

Thus $\rho^j(i) \neq i$ for all $j = 1, 2, \ldots, r(i)-1$.

This (and 1.1b)) proves part a).

b) This is left as an exercise (the methods required are similar).

Corollary 4.5: Let $i \in I$, then in the Brauer tree the following hold:

a) If $P(i) \neq \underset{\sim}{E}$, then the valence of $P(i)$ is $r(i)$.

b) If $\Delta(i) \neq \underset{\sim}{E}$, then the valence of $\Delta(i)$ is $s(i)$.

Proof: Use 4.4 and (*) above.

Lemma 4.6: $\underset{\sim}{E}$ corresponds to the exceptional vertex.

Proof: Suppose $P(i)$ corresponds to the exceptional vertex. Then Y_i affords the modular character $\chi^{\circ} = \chi_{\lambda_1}{}^{\circ} + \ldots + \chi_{\lambda_a}{}^{\circ}$ (see 2.3). Moreover all the $\chi_{\lambda_i}{}^{\circ}$ are equal.

So each composition factor of Y_i is repeated at least a times .. (**)
Thus $r(i)$ is not the smallest positive integer r such that $\rho^r(i) = i$ (since $a \geqslant 2$), and so by 4.4, $P(i) = \underset{\sim}{E}$.
Similarly if $\Delta(i)$ corresponds to the exceptional vertex, $\Delta(i) = \underset{\sim}{E}$.

Remark: If $P(i) = \underset{\sim}{E}$, then as $f(W_i/X_i)$ is an extension of the form,

$$fV_i \circ fV_{\rho(i)} \circ \ldots \circ fV_{\rho^{r(i)-1}(i)} \cong T_{i,q-1}$$

it follows from (**) that every composition factor of $f(W_i/X_i)$ is repeated at least a times. But since $f(W_i/X_i) \cong T_{i,q-1}$, this number is exactly a times. Hence the composition factors of W_i/X_i (and therefore those of Y_i) are repeated exactly a times.
Moreover a similar result holds for X_i if $\Delta(i) = \underset{\sim}{E}$, and hence we have the following,

Theorem 4.7: a) If $P(i) = \underset{\sim}{E}$, then $r(i)$ is the smallest positive integer r so that a divides r and $\rho^{r/a}(i) = i$.

b) If $\Delta(i) = \underset{\sim}{E}$, then $s(i)$ is the smallest positive integer s so that a divides s and $\delta^{s/a}(i) = i$.

Corollary 4.8: a) If $P(i) = \underset{\sim}{E}$, then a.(the valence of $P(i)$) = $r(i)$.

b) If $\Delta(i) = \underset{\sim}{E}$, then a.(the valence of $\Delta(i)$) = $s(i)$.

Now let e be unrestricted, and denote the exceptional vertex by $\underset{\sim}{E}$ (as before). Notice that if $e = q-1$, then $a = 1$, and so $\underset{\sim}{E}$ corresponds to an ordinary simple character, like all the other vertices.

Theorem 4.9: The following all hold for every $i \in I$,

a) $r(i)$ is the smallest positive integer r so that

$$\rho^{r}(i) = i \qquad \text{if } P(i) \neq \underset{\sim}{E},$$

$$a \text{ divides } r \text{ and } \rho^{r/a}(i) = i \qquad \text{if } P(i) = \underset{\sim}{E};$$

$s(i)$ is the smallest positive integer s so that

$$\delta^{s}(i) = i \qquad \text{if } \Delta(i) \neq \underset{\sim}{E},$$

$$a \text{ divides } s \text{ and } \delta^{s/a}(i) = i \qquad \text{if } \Delta(i) = \underset{\sim}{E}.$$

b) In the Brauer tree,

$$\text{the valence of } P(i) = r(i) \qquad \text{if } P(i) \neq \underset{\sim}{E},$$

$$a.(\text{the valence of } P(i)) = r(i) \qquad \text{if } P(i) = \underset{\sim}{E};$$

$$\text{the valence of } \Delta(i) = s(i) \qquad \text{if } \Delta(i) \neq \underset{\sim}{E},$$

$$a.(\text{the valence of } \Delta(i)) = s(i) \qquad \text{if } \Delta(i) = \underset{\sim}{E}.$$

Proof: If $e \neq q-1$ we have already proved this.
If $e = q-1$, then $a = 1$ and hence (using (*)) it suffices to show that for all $i \in I$ the following both hold,

$r(i)$ is the smallest positive integer r so that $\rho^{r}(i) = i$;

$s(i)$ is the smallest positive integer s so that $\delta^{s}(i) = i$.

But both of these follow using techniques similar to those employed in 4.4, bearing in mind that B is $(q,q-1)$-uniserial in this case.

Abbreviations: B.T. = the Brauer tree.

L.B.T. = the Brauer tree with the edges correctly labelled by W_i's.

I.B.T. = the Brauer tree with the exceptional vertex identified (via the label $\underset{\sim}{E}$) and at least one vertex (and hence all vertices) labelled correctly from the set $\{P, \Delta\}$.

L.I.B.T. = L.B.T. $\cup$ I.B.T.

I.δ = δ along with the exceptional vertex identified correctly (via its ρ-set $P(i)$ or δ-set $\Delta(i)$).

The full structure of $\underset{\sim}{B}$ will mean the e lattices fully labelled with the W_j and the V_h. The lattice structure of $\underset{\sim}{B}$ will mean just the e lattices (with no labelling).(2)

Deductions from 4.9: a) I.δ is sufficient to give the full structure of B. Moreover so too is L.I.B.T. $\cup\ \delta$.

b) I.B.T. will give the lattice structure of $\underset{\sim}{B}$ (but not necessarily any labelling at all). It will also give the cycle types of δ and ρ.

c) L.I.B.T. will give the lattice structure of $\underset{\sim}{B}$ fully labelled with the W_j (but not necessarily with the V_h). It will also give the cycle types of δ and ρ.

<u>Remarks</u>: a) Identification of the exceptional vertex (in some form) is vital.

b) We cannot necessarily deduce δ from B.T., L.B.T. or even L.I.B.T.

c) If $\underset{\sim}{B}$ is the principal p-block, then the vertex $\underset{\sim}{1}$ which corresponds to the one-dimensional trivial $\mathbb{C}G$-character, must be a Δ-vertex. So in this case we can get the lattice structure of $\underset{\sim}{B}$ from the following,

B.T. + identification of $\underset{\sim}{E}$ + identification of $\underset{\sim}{1}$

We will use this in §5.

d) If $\underset{\sim}{B}$ is the principal p-block, and if we are given B.T. in which every vertex has the dimension of the simple character to which it corresponds listed (for the exceptional vertex this means the dimension of any one of the simple χ_λ's); then if $e \neq 1$ or p-1, we can identify $\underset{\sim}{E}$, since using 2.4d) it will be the unique vertex whose corresponding ordinary simple character does not have dimension congruent to +1 or -1 mod p. (If $q = p$ and $e = p-1$, then identification of $\underset{\sim}{E}$ is in any case redundant, see 4.9.)

Moreover if the one-dimensional trivial $\mathbb{C}G$-character is the unique ordinary simple character in $\underset{\sim}{B}$ of dimension one, then we can of course also identify $\underset{\sim}{1}$.

These are the techniques that we will be using in §5.

§5 Applications to The Mathieu Groups

Let G be one of the Mathieu groups, in which a Sylow p-subgroup of G has order p. We examine the principal p-block $\underset{\sim}{B}$ of kG, which has cyclic defect group of order p.

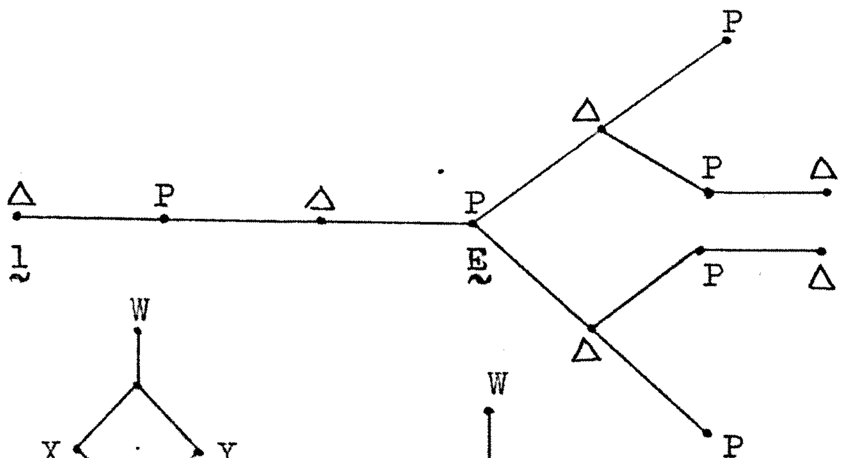
The Brauer trees of all such blocks are given, by James, in [3]. Also the dimensions of the ordinary simple characters corresponding to each vertex are listed, from which we can identify $\underset{\sim}{1}$, and when $e \neq q-1$, $\underset{\sim}{E}$ (see the remark d) on page 13). We can hence (by the remark c) on page 13) use 4.9 to work out the lattice structure of $\underset{\sim}{B}$.

Notation: a) If σ is a permutation, $\sigma \in 1^a 2^b 3^c \ldots$ will mean that σ has cycle type $1^a 2^b 3^c \ldots$

b) An integer in brackets under a lattice will denote the number of projective indecomposables in $\underset{\sim}{B}$ of this type. Also these lattices will labelled with W,X,Y which correspond to W_i, X_i, Y_i in Figure 2 for some $i \in I$.

The Brauer trees and lattice structures for each possible $\underset{\sim}{B}$ are as follows,

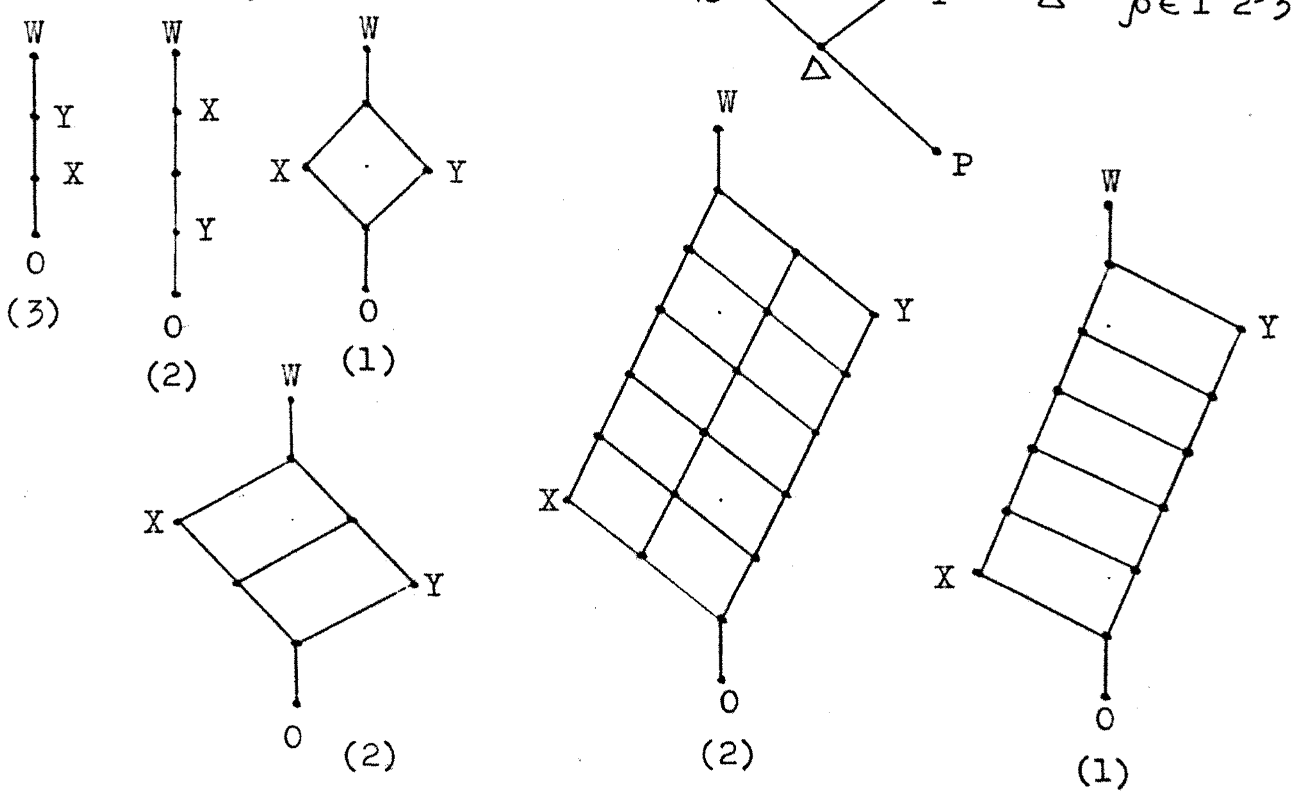
1 p = 23 M_{23}



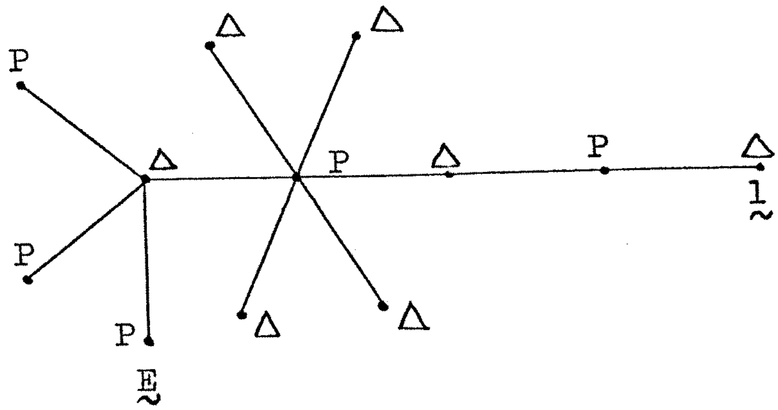
$e = 11$

$\delta \in 1^3 23^2$

$\rho \in 1^2 2^3 3$



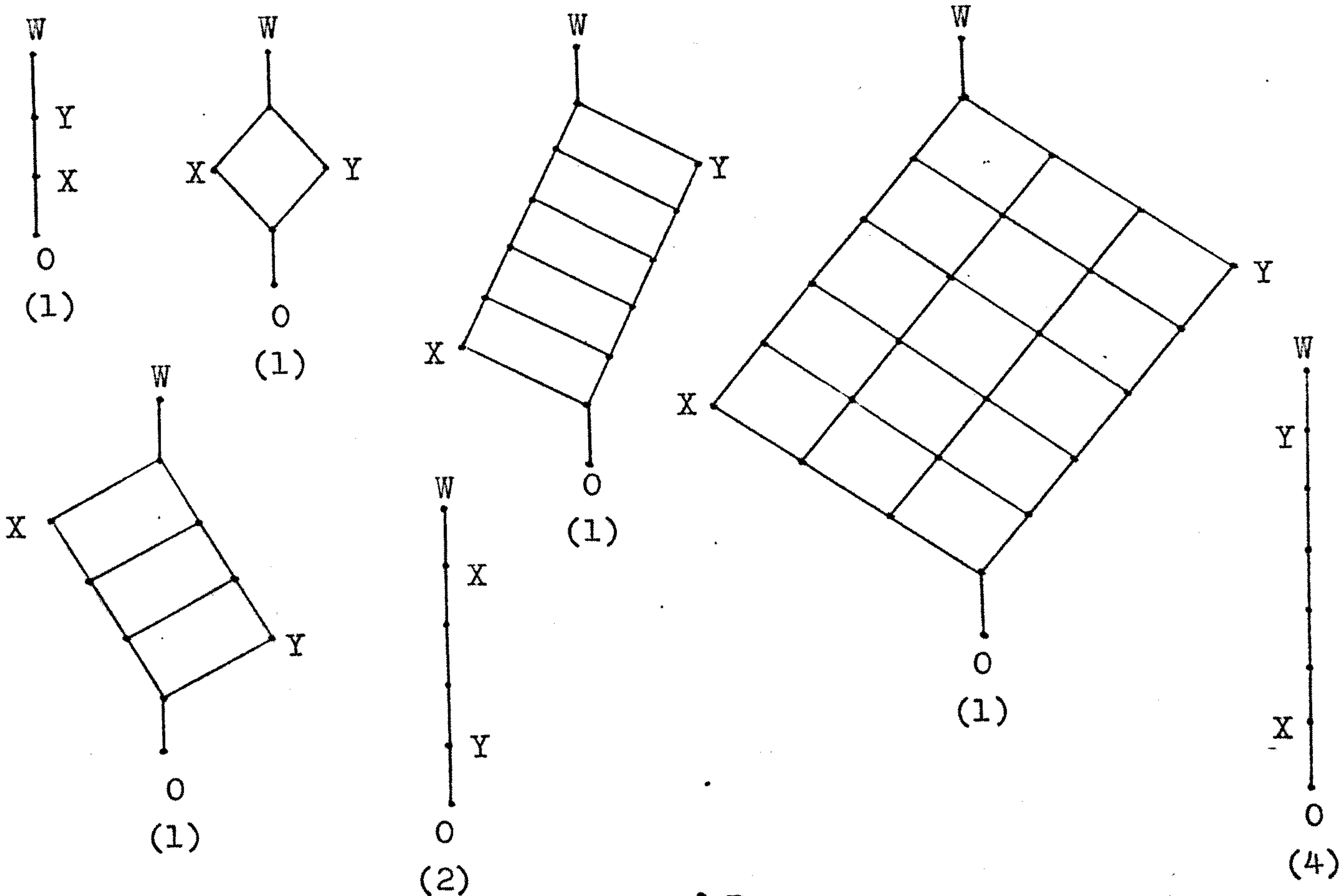
M_{22}



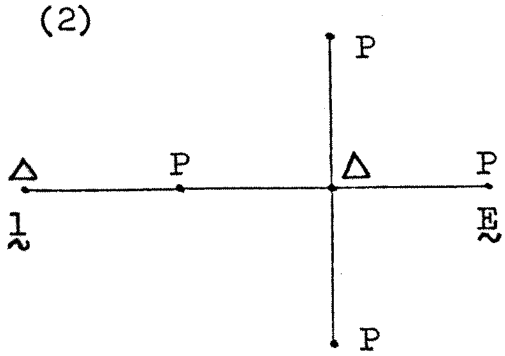
$e = 11$

$\delta \in 1^5 2 4$

$\rho \in 1^3 2 6$



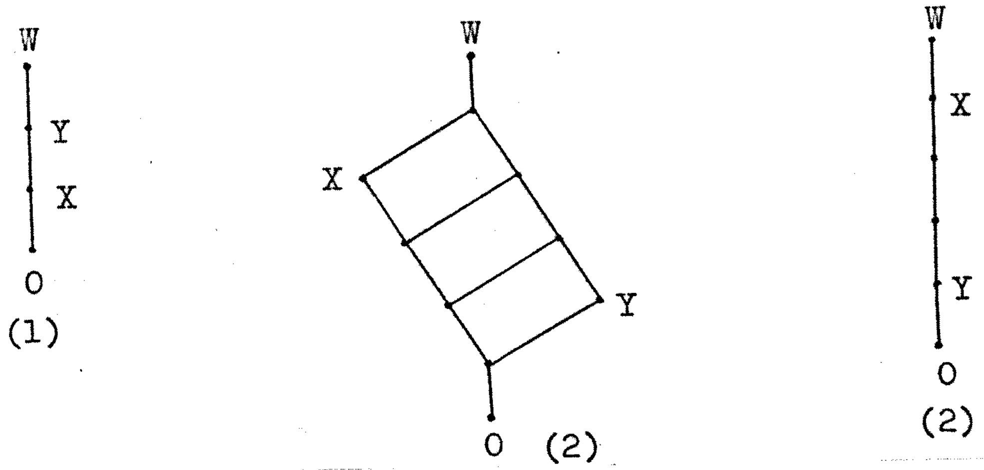
2 $p = 11$ M_{11}



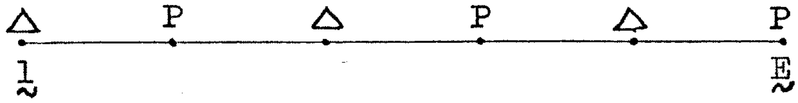
$e = 5$

$\delta \in 14$

$\rho \in 1^3 2$



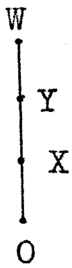
$\underline{M_{12}}$



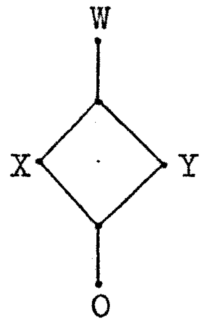
$e = 5$

$\delta \in 12^2$

$\rho \in 12^2$

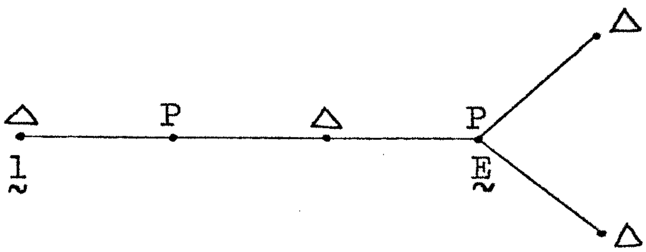


(1)



(4)

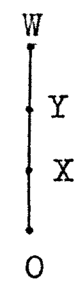
$\underline{M_{22}, M_{23}}$



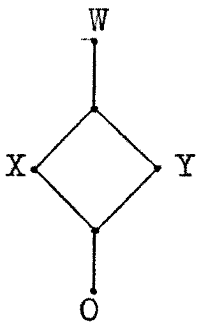
$e = 5$

$\delta \in 1^3 2$

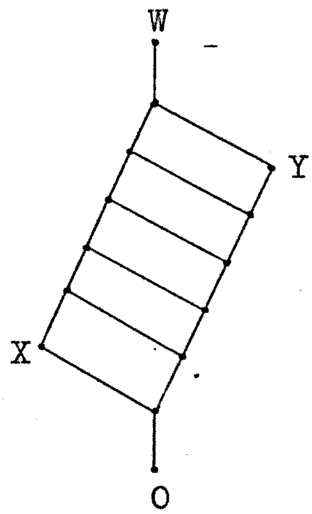
$\rho \in 23$



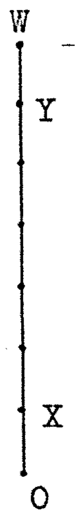
(1)



(1)

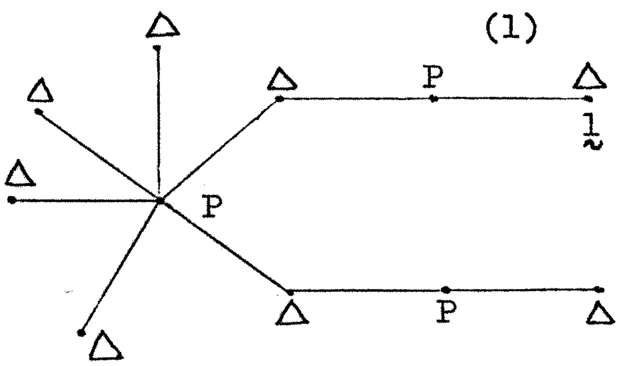


(1)



(2)

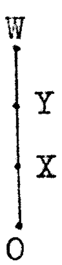
$\underline{M_{24}}$



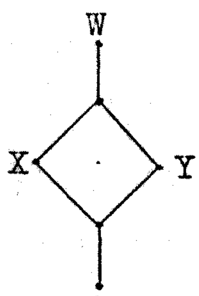
$e = 10$

$\delta \in 1^6 2^2$

$\rho \in 2^2 6$



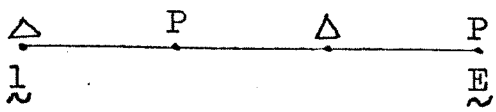
(2)



(2)

(6)

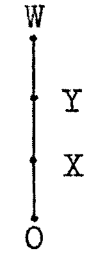
3 $p = 7$ M_{22}, M_{23}, M_{24}



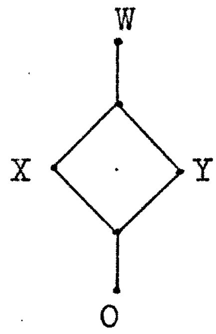
$e = 3$

$\delta \in 12$

$\rho \in 12$

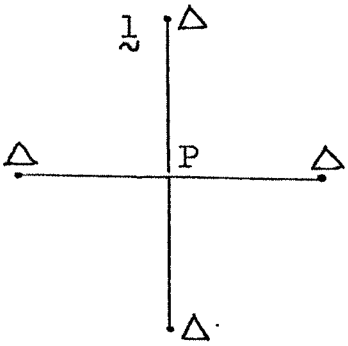


(1)



(2)

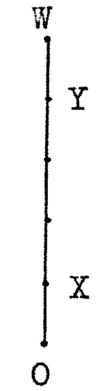
4 $p = 5$ M_{23}, M_{11}



$e = 4$

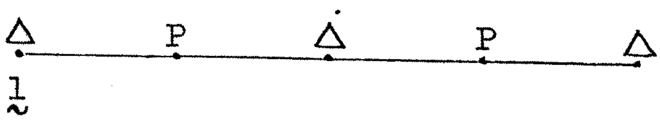
$\delta \in 1^4$

$\rho \in 4$



(4)

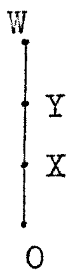
M_{12}, M_{22}, M_{24}



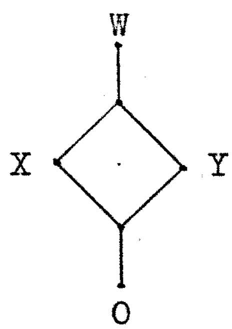
$e = 4$

$\delta \in 1^2 2$

$\rho \in 2^2$



(2)



(2)

Remarks: a) $PSL(2,7) \hookrightarrow A_7$, $M_{11} \hookrightarrow A_{11}$ and $M_{23} \hookrightarrow A_{23}$. However it is interesting to note that although the principal p-blocks of M_p and A_p do not have the same δ for $p = 11$ or 23; the principal 7-blocks of $PSL(2,7)$ and A_7 do have the same δ (indeed they have the same block structure, see footnote 1). Unfortunately these methods do not seem to shed any light on the following well known conjecture,

"If $D \in Syl_p(A_p)$, $H = N_{A_p}(D)$ and $H < K < A_p$ then

$$\begin{array}{lll} \text{either} & p = 7, & K = PSL(2,7) \\ \text{or} & p = 11, & K = M_{11} \\ \text{or} & p = 23, & K = M_{23} \text{"} \end{array}$$

b) For some of the above blocks (for example the principal 11-blocks of M_{22} or M_{23}) it is possible to get the full structure of $\underset{\sim}{B}$ from the given data; but for others (like the principal 11-block of M_{11}) it is not.

References

[1] E. C. DADE, Blocks with cyclic defect groups, Ann. of Math. 84 (1966) 20-48.

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[3] G. D. JAMES, The modular characters of the Mathieu groups, J. of Algebra 27 (1973) 57-111.

[4] G. J. JANUSZ, Indecomposable modules for finite groups, Ann. of Math. 89 (1969) 209-241.

[5] A. KERBER, "Representations of Permutation Groups", Springer-Verlag notes Vol 240 (1971).

[6] PART A of this thesis (to be published in the J. of Algebra under the title Blocks with a cyclic defect group).

Footnotes

(1) on page 8: Starting from the ordinary character table of PSL(2,p), it is possible to work out the r(i), s(i) for the principal p-block, and it can be shown that 3.7 (and hence 3.8) still hold for this block (and $e = (p-1)/2$ still). So for p odd, the block structures of the principal p-blocks of PSL(2,p) and A_p are effectively the same. This is rather suprising, since in general the two groups are not related in any way.

(2) on page 12: All of our lattice diagrams have X_i on the left and Y_i on the right, so we can at least label any lattice with W,X,Y to correspond to W_i, X_i, Y_i in Figure 2 for some $i \in I$.