Clustering and the hyperbolic geometry of complex networks

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Abstract. Clustering is a fundamental property of complex networks and it is the mathematical expression of a ubiquitous phenomenon that arises in various types of self-organized networks such as biological networks, computer networks or social networks. In this paper, we consider what is called the *global clustering coefficient* of random graphs on the hyperbolic plane. This model of random graphs was proposed recently by Krioukov et al. [22] as a mathematical model of complex networks, implementing the assumption that hyperbolic geometry underlies the structure of these networks. We do a rigorous analysis of clustering and characterize the global clustering coefficient in terms of the parameters of the model. We show how the global clustering coefficient can be tuned by these parameters, giving an explicit formula.

1 Introduction

The theory of complex networks was developed during the last 15 years mainly as a unifying mathematical framework for modeling a variety of networks such as biological networks or large computer networks among which is the Internet, the World Wide Web as well as social networks that have been recently developed over these platforms. A number of mathematical models have emerged whose aim is to describe fundamental characteristics of these networks as these have been described by experimental evidence – see for example [1]. Among the most influential models was the Watts-Strogatz model of small worlds [30] and the Barabási-Albert model [3], that is also known as the preferential attachment model. The main typical characteristics of these networks have to do with the distribution of the degrees (e.g., power-law distribution), the existence of clustering as well as the typical distances between vertices (e.g., the small world effect).

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Loosely speaking, the notion of a complex network refers to a class of large networks which exhibit the following characteristics:

1. they are *sparse*, that is, the number of their edges is proportional to the number of nodes;
2. they exhibit the *small world phenomenon*: most pairs of vertices which belong to the same component are within a short distance from each other;
3. *clustering*: two nodes of the network that have a common neighbour are somewhat more likely to be connected with each other;
4. the tail of their degree distribution follows a *power law*: experimental evidence (see [1]) indicates that many networks that emerge in applications follow power law degree distribution with exponent between 2 and 3.

The books of Chung and Lu [13] and of Dorogovtsev [15] are excellent references for a detailed discussion of these properties.

The models that we described above as well as other known models, such as the Chung-Lu model (defined by Chung and Lu [11], [12]) fail to capture *all* the above features simultaneously or if they do so, they do it in a way that is difficult to tune these features independently. For example, the Barabasi-Albert model (when suitably parametrized) exhibits a power law degree distribution with exponent between 2 and 3, and average distance of order $O(\log\log N)$, but it is locally tree-like around a typical vertex (cf. [8], [16]). On the other hand, the Watts-Strogatz model, although it exhibits clustering and small distances between the vertices, has degree distribution that decays exponentially [4].

The notion of clustering formalizes the property that two nodes of a network that share a neighbor (for example two individuals that have a common friend) are more likely to be joined by an edge (that is, to be friends of each other). In the context of social networks, sociologists have explained this phenomenon through the notion of *homophily*, which refers to the tendency of individuals to be related with similar individuals, e.g. having similar socioeconomic background or similar educational background. There have been numerous attempts to define models where clustering is present – see for example the work of Coupechoux and Lelarge [14] or that of Bollobás, Janson and Riordan [9] where this is combined with the general notion of inhomogeneity. In that context, clustering is *planted* in a sparse random graph. Also, it is even more rare to quantify clustering precisely (as for example in random intersection graphs [5]). This is the case as the presence of clustering is the outcome of heavy dependencies between the edges of the random graphs and, in general, these are not easy to handle.

However, clustering is naturally present on random graphs that are created on a metric space, as is the case of a random geometric graph on the Euclidean plane. The theory of random geometric graphs was initiated by Gilbert [18] already in 1961 and started taking its present form later by Hafner [20]. In its standard form a geometric random graph is created as follows: N points are sampled within a subset of $\mathbb{R}^d$ following a particular distribution (most usually this is the uniform distribution or the distribution of the point-set of a Poisson point process) and any two of them are joined when their Euclidean distance is smaller than some threshold value which, in general, is a function of N. During the last two decades, this kind of random graphs was studied in depth by several researchers – see the monograph of Penrose [29] and the references therein. Numerous typical properties of such random graphs have been investigated, such as the chromatic number [25], Hamiltonicity [2] etc.

There is no particular reason why a random geometric graph on a Euclidean space would be intrinsically associated with the formation of a complex network. Real-world networks consist of heterogeneous nodes, which can be classified into groups. In turn, these groups can be classified into larger groups which belong to bigger subgroups and so on. For example, if we consider the network of citations, whose set of nodes is the set of research papers and there is a link from one paper to another if one cites the other, there is a natural classification of the nodes according to the scientific fields each paper belongs to (see for example [10]). In the case of the network of web pages, a similar classification can be considered in terms of the similarity between two web pages: the more similar two web pages are, the more likely it is that there exists a hyperlink between them [24].

This classification can be approximated by tree-like structures representing the hidden hierarchy of the network. The tree-likeness suggests the hypothesis that the geometry of this hierarchy is *hyperbolic*. One of the basic features of a hyperbolic space is that the volume growth is exponential which is also the case, for example, when one considers a k-ary tree, that is, a rooted tree where every vertex has k children. Let us consider for example the Poincaré unit disc model (which we will discuss in more detail in the next section). If we place the root of an infinite k-ary tree at the centre of the disc, then the hyperbolic metric provides the necessary room to embed the tree into the disc so that every edge has unit length in the embedding.

Recently Krioukov et al. [22] introduced a model which implements this idea. In this model, a random network is created on the hyperbolic plane (we will see the detailed definition shortly). In particular, Krioukov et al. [22] determined the degree distribution for *large* degrees showing that it is *scale free* and its tail follows a power law, whose exponent is determined by some of the parameters of the model. Furthermore, they consider the clustering properties of the resulting random network. A numerical approach in [22] suggests that the (local) clustering coefficient[3] is positive and it is determined by one of the parameters of the model. In fact, as we will discuss in Section 2, this model corresponds to the sparse regime of random geometric graphs on the hyperbolic plane and hence is of independent interest within the theory of random geometric graphs.

This paper investigates *rigorously* the presence of clustering in this model, through the notion of the *clustering coefficient*. Our first contribution is that we manage to determine exactly the value of the clustering coefficient as a function of the parameters of the model. More importantly, our results imply that in fact the exponent of the power law, the density of the random graph and the amount of clustering can be tuned *independently* of each other, through the parameters of the random graph. Furthermore, we should point out that the clustering coefficient we consider is the so-called *global clustering coefficient*.

[3] This is defined as the average density of the neighbourhoods of the vertices.

Its calculation involves tight concentration bounds on the number of triangles in the random graph. Hence, our analysis initiates an approach to the small subgraph counting problem in these random graphs, which is among the central problems in the general theory of random graphs [7],[21] and of random geometric graphs [29].

1.1 Random geometric graphs on the hyperbolic plane

The most common representations of the hyperbolic plane are the upper-half plane representation $\{z \in \mathbb{C} \ : \ \Im z > 0\}$ as well as the Poincaré unit disc which is simply the open disc of radius one, that is, $\{(u,v) \in \mathbb{R}^2 \ : \ 1 - u^2 - v^2 > 0\}$. Both spaces are equipped with the hyperbolic metric; in the former case this is $\frac{1}{(\zeta y)^2} dy^2$ whereas in the latter this is $\frac{4}{\zeta^2} \ \frac{du^2+dv^2}{(1-u^2-v^2)^2}$, where ζ is some positive real number. It can be shown that the (Gaussian) curvature in both cases is equal to $-\zeta^2$ and the two spaces are isometric, i.e., there is a bijection between the two spaces that preserves (hyperbolic) distances. In fact, there are more representations of the 2-dimensional hyperbolic space of curvature $-\zeta^2$ which are isometrically equivalent to the above two. We will denote by $\mathbb{H}^2_\zeta$ the class of these spaces.

In this paper, following the definitions in [22], we shall be using the *native* representation of $\mathbb{H}^2_\zeta$. Here, the ground space of $\mathbb{H}^2_\zeta$ is $\mathbb{R}^2$ and every point $x \in \mathbb{R}^2$ whose polar coordinates are (r,θ) has hyperbolic distance from the origin equal to r. More precisely, the native representation can be viewed as a mapping of the Poincaré unit disc to $\mathbb{R}^2$, where the origin of the unit disc is mapped to the origin of $\mathbb{R}^2$ and every point v in the Poincaré disc is mapped to a point $v' \in \mathbb{R}^2$, where $v' = (r,\theta)$ in polar coordinates: r is the hyperbolic distance of v from the origin of the Poincaré disc and θ is its angle.

An elementary but tedious calculation can show that a circle of radius r around the origin has length equal to $\frac{2\pi}{\zeta} \sinh \zeta r$ and area equal to $\frac{2\pi}{\zeta^2}(\cosh \zeta r - 1)$.

We are now ready to give the definitions of the two basic models introduced in [22]. Consider the native representation of the hyperbolic plane of curvature $K = -\zeta^2$, for some $\zeta > 0$. For some constant $\nu > 0$, we let $N = \nu e^{\zeta R/2}$ – thus R grows logarithmically as a function of N. We shall explain the role of ν shortly. We create a random graph by selecting randomly N points from the disc of radius R centered at the origin O, which we denote by $\mathcal{D}_R$.

The distribution of these points is as follows. Assume that a random point u has polar coordinates (r,θ). The angle θ is uniformly distributed in $(0, 2\pi]$ and the probability density function of r, which we denote by $\rho_N(r)$, is determined by a parameter $\alpha > 0$ and is equal to

$$\rho_N(r) = \begin{cases} \alpha \frac{\sinh \alpha r}{\cosh \alpha R - 1}, & \text{if } 0 \leq r \leq R \\ 0, & \text{otherwise} \end{cases}. \tag{1}$$

Note that when $\alpha = \zeta$, this is simply the uniform distribution.

An alternative way to define this distribution is as follows. Consider $\mathbb{H}^2_\alpha$ and the Poincaré representation of it. Let O' be the centre of the disc. Consider the disc $\mathcal{D}'_R$ of radius R around O' and select N points within $\mathcal{D}'_R$ uniformly at random. Subsequently, the selected points are projected onto $\mathcal{D}_R$ preserving their polar coordinates. The projections of these points, which we will be denoting by V_N, will be the vertex set of the random graph. We will be also treating the vertices as points in the hyperbolic space indistinguishably.

Note that the curvature in this case determines the rate of growth of the space. Hence, when $\alpha < \zeta$, the N points are distributed on a disc (namely $\mathcal{D}'_R$) which has smaller area compared to $\mathcal{D}_R$. This naturally increases the density of those points that are located closer to the origin. Similarly, when $\alpha > \zeta$ the area of the disc $\mathcal{D}'_R$ is larger than that of $\mathcal{D}_R$, and most of the N points are significantly more likely to be located near the boundary of $\mathcal{D}'_R$, due to the exponential growth of the volume.

Given the set V_N on $\mathcal{D}_R$ we define the following two models of random graphs.

1. *The disc model*: this model is the most commonly studied in the theory of random geometric graphs on Euclidean spaces. We join two vertices if they are within (hyperbolic) distance R from each other. Figure 1 illustrates a disc of radius R around a vertex $v \in \mathcal{D}_R$.

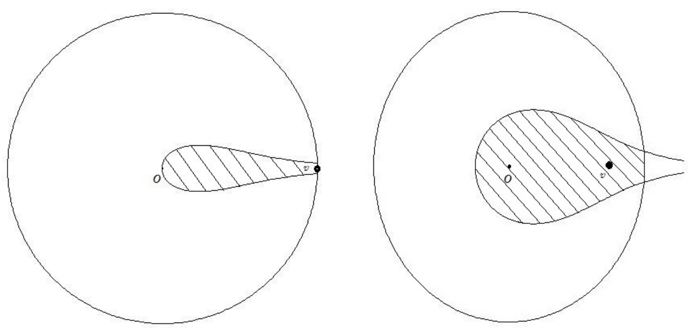

Fig. 1. The disc of radius R around v in $\mathcal{D}_R$

2. *The binomial model*: we join any two distinct vertices u, v with probability

$$p_{u,v} = \frac{1}{\exp\left(\beta\ \frac{\zeta}{2}(d(u,v) - R)\right) + 1},$$

independently of every other pair, where $\beta > 0$ is fixed and $d(u,v)$ is the hyperbolic distance between u and v. We denote the resulting random graph by $\mathcal{G}(N; \zeta, \alpha, \beta, \nu)$.

The binomial model is in some sense a *soft* version of the disc model. In the latter, two vertices become adjacent if and only if their hyperbolic distance is at most R. This is *approximately* the case in the former model. If $d(u,v) = (1+\delta)R$, where $\delta > 0$ is some small constant, then $p_{u,v} \to 0$, whereas if $d(u,v) = (1-\delta)R$, then $p_{u,v} \to 1$, as $N \to \infty$.

Also, the disc model can be viewed as a limiting case of the binomial model as $\beta \to \infty$. Assume that the positions of the vertices in $\mathcal{D}_R$ have been realized. If $u, v \in V_N$ are such that $d(u,v) < R$, then when $\beta \to \infty$ the probability that u and v are adjacent tends to 1; however, if $d(u,v) > R$, then this probability converges to 0 as β grows.

Müller [26] has shown that the disc model is in fact determined by the ratio ζ/α. In that case, one may set $\zeta = 1$ and keep only α as the parameter of the model.

Krioukov et al. [22] provide an argument which indicates that in both models the degree distribution has a power law tail with exponent that is equal to $2\alpha/\zeta + 1$. Hence, when $0 < \zeta/\alpha < 2$, any exponent greater than 2 can be realised. This has been shown rigorously by Gugelmann et al. [19], for the disc model, and by the second author [17], for the binomial model. In the latter case, the average degree of a vertex depends on all four parameters of the model. For the disc model in particular, having fixed ζ and α, which determine the exponent of the power law, the parameter ν determines the average degree. In the binomial model, there is an additional dependence on β. Our results focus on the binomial model and show that clustering *does not depend* on ν. Therefore, in the binomial model the "amount" of clustering and the average degree can be tuned *independently*.

1.2 Notation

Let $\{X_N\}_{N\in\mathbb{N}}$ be a sequence of real-valued random variables on a sequence of probability spaces $\{(\Omega_N, \mathbb{P}_N)\}_{N\in\mathbb{N}}$. For a real number a, we write $X_N \xrightarrow{p} a$ or else X_N converges to *a in probability*, if for every $\varepsilon > 0$, we have $\mathbb{P}_N(|X_N - a| > \varepsilon) \to 0$ as $N \to \infty$. If $\mathcal{E}_N$ is a measurable subset of Ω_N, for any $N \in \mathbb{N}$, we say that the sequence $\{\mathcal{E}_N\}_{N\in\mathbb{N}}$ occurs *asymptotically almost surely (a.a.s.)* if $\mathbb{P}(\mathcal{E}_N) = 1 - o(1)$, as $N \to \infty$. However, with a slight abuse of terminology, we will be saying that an *event occurs a.a.s.* implicitly referring to a sequence of events.

For two functions $f, g : \mathbb{N} \to \mathbb{R}$ we write $f(N) \ll g(N)$ if $f(N)/g(N) \to 0$ as $N \to \infty$. Similarly, we will write $f(N) \asymp g(N)$, meaning that there are positive constants c_1, c_2 such that for all $N \in \mathbb{N}$ we have $c_1 g(N) \le f(N) \le c_2 g(N)$. Analogously, we write $f(N) \lesssim g(N)$ (resp. $f(N) \gtrsim g(N)$) if there is a positive constant c such that for all $N \in \mathbb{N}$ we have $f(N) \le c g(N)$ (resp. $f(N) \ge c g(N)$). These are shorthands for the standard Landau notation, but we chose to express them as above in order to make our presentation more readable.

2 Some geometric aspects of the two models

The disc model on the hyperbolic plane can be also viewed within the framework of random geometric graphs. Within this framework, the disc model may be defined for *any* threshold distance r_N and not merely for threshold distance equal to R. However, only taking $r_N = R$ yields a random graph with constant average degree that is bounded away from 0. More specifically for any $\delta \in (0,1)$, if $r_N = (1-\delta)R$, then the resulting random graph becomes rather trivial and most vertices have no neighbours. On the other hand, if $r_N = (1+\delta)R$, the resulting random graph becomes too dense and its average degree grows polynomially fast in N.

The proof of these observations relies on the following lemma which provides a characterization of what it means for two points u, v to have $d(u,v) \leq (1+\delta)R$, for $\delta \in (-1,1)$, in terms of their *relative angle*, which we denote by $\theta_{u,v}$. For this lemma, we need the notion of the *type* of a vertex. For a vertex $v \in \mathrm{V}_N$, if r_v is the distance of v from the origin, that is, the radius of v, then we set $t_v = R - r_v$ – we call this quantity the *type* of vertex v. It is not very hard to see that the type of a vertex is approximately exponentially distributed. If we substitute $R - t$ for r in (1), then assuming that t is fixed that expression becomes asymptotically equal to $\alpha e^{-\alpha t}$.

The lemma that connects the hyperbolic distance between two vertices with the relative angle between them is a generalisation of a similar lemma that appears in [6].

Lemma 1. *Let $\delta \in (-1,1)$ be a real number. For any $\varepsilon > 0$ there exists an $N_0 > 0$ and a $c_0 > 0$ such that for any $N > N_0$ and $u, v \in \mathcal{D}_R$ with $t_u + t_v < (1-|\delta|)R - c_0$ the following hold.*

- *If $\theta_{u,v} < 2(1-\varepsilon)\exp\left(\frac{\zeta}{2}(t_u + t_v - (1-\delta)R)\right)$, then $d(u,v) < (1+\delta)R$.*
- *If $\theta_{u,v} > 2(1+\varepsilon)\exp\left(\frac{\zeta}{2}(t_u + t_v - (1-\delta)R)\right)$, then $d(u,v) > (1+\delta)R$.*

Let us consider temporarily the (modified) disc model, where we assume that two vertices are joined precisely when their hyperbolic distance is at most $(1+\delta)R$. Let $u \in \mathrm{V}_N$ be a vertex and assume that $t_u < C$ (by the above observation on the distribution of the type of a vertex, it is not hard to see that most vertices will satisfy this, if C is chosen large). We will show that if $\delta < 0$, then the expected degree of u, in fact, tends to 0. Let us consider a simple case where $0 < \zeta/\alpha < 2$ and δ satisfies $\frac{\zeta}{2\alpha} < 1 - |\delta| < 1$. It can be shown that a.a.s. there are no vertices of type much larger than $\frac{\zeta}{2\alpha}R$. Hence, since $t_u < C$, if N is sufficiently large, then we have $\frac{\zeta}{2\alpha}R < (1-|\delta|)R - t_u - c_0$. By Lemma 1, the probability that a vertex v has type at most $\frac{\zeta}{2\alpha}R$ and it is adjacent to u (that is, its hyperbolic distance from u is at most $(1+\delta)R$) is proportional to $e^{\frac{\zeta}{2}(t_u + t_v - (1-\delta)R)}/\pi$. If we average this over t_v we obtain

$$\begin{aligned}\Pr\left[u \text{ is adjacent to } v | t_u\right] &\asymp \frac{e^{\zeta t_u/2}}{e^{\frac{\zeta}{2}(1-\delta)R}} \int_0^{\frac{\zeta}{2\alpha}R} e^{\zeta t_v/2} \frac{\alpha \sinh(\alpha(R-t_v))}{\cosh(\alpha R) - 1} dt_v \\ &\lesssim \frac{e^{\zeta t_u/2}}{e^{\frac{\zeta}{2}(1-\delta)R}} \int_0^{R} e^{\zeta t_v/2} \, \frac{e^{\alpha(R-t_v)}}{\cosh(\alpha R) - 1} dt_v \\ &\asymp \frac{e^{\zeta t_u/2}}{e^{\frac{\zeta}{2}(1-\delta)R}} \int_0^{R} e^{(\zeta/2-\alpha)t_v} dt_v \overset{0<\zeta/\alpha<2}{\asymp} \frac{e^{\zeta t_u/2}}{N^{1-\delta}} \overset{\delta\leq 0}{=} o\left(\frac{1}{N}\right).\end{aligned}$$

Hence, the probability that there is such a vertex is $o(1)$. Markov's inequality implies that with high probability most vertices will have no neighbors.

A similar calculation can actually show that the above probability is $\Omega\left(\frac{e^{\zeta t_u/2}}{N^{1-\delta}}\right)$. Thereby, if $0 < \delta < 1$, then the expected degree of u is of order N^{δ}. A more detailed argument can show that the resulting random graph is too dense in the sense that the number of edges is *no longer proportional* to the number of vertices but grows much faster than that.

3 The clustering coefficient

The theme of this work is the study of clustering in $\mathcal{G}(N;\zeta,\alpha,\beta,\nu)$. The notion of clustering was introduced by Watts and Strogatz [30], as a measure of the local density of the graph. In the context of biological or social networks, this measures the likelihood of two vertices that have a common neighbor to be joined with each other. This is expressed by the density of the neighborhood of each vertex. More specifically, for each vertex v of a graph, the *local clustering coefficient* is defined to be the density of the neighborhood of v. In [30], the *clustering coefficient* of a graph G, which we denote by $C_1(G)$, is defined as the average of the local clustering coefficients over all vertices of G. The clustering coefficient $C_1(\mathcal{G}(N;\zeta,\alpha,\beta,\nu))$, as a function of β is discussed in [22], where simulations and heuristic calculations indicate that C_1 can be tuned by β. For the disc model, Gugelmann et al. [19] have shown rigorously that this quantity is asymptotically with high probability bounded away from 0 when $0 < \zeta/\alpha < 2$.

The case where $\beta > 1$ and $0 < \zeta/\alpha < 2$ is of particular interest. More specifically, in this regime $\mathcal{G}(N;\zeta,\alpha,\beta,\nu)$ has constant (i.e., not depending on N) average degree that depends on ν,ζ,α and β, whereas the degree distribution follows the tail of a power law with exponent $2\alpha/\zeta + 1$. This has been shown by the second author in [17]. Note that since $2\alpha/\zeta > 1$, the exponent of the power law may take any value greater than 2. When $1 < \zeta/\alpha < 2$, this exponent is between 2 and 3. In [17] we also show that when $\beta \leq 1$, the average degree of the random graph grows at least logarithmically in N.

As we mentioned above, there has been significant experimental evidence which shows that many networks which arise in applications have degree distributions that follow a power law usually with exponent between 2 and 3 (cf. [1] for example). Also, such networks are typically sparse with only a few nodes of very high degree which are the *hubs* of the network. Thus, in the regime where $\beta > 1$ and $0 < \zeta/\alpha < 2$ the random graph $\mathcal{G}(N;\zeta,\alpha,\beta,\nu)$ appears to exhibit these characteristics. In this work, we explore further the potential of this random graph model as a suitable model for complex networks focusing on the notion of *global clustering* and how this is determined by the parameters of the model.

A first attempt to define this notion was made by Luce and Perry [23], but it was rediscovered more recently by Newman, Strogatz and Watts [27]. Given a graph G, we let $T = T(G)$ be the number of triangles of G and let $\Lambda = \Lambda(G)$ denote the number of *incomplete triangles* of G; this is simply the number of (not necessarily induced) paths having length 2. Then the *global clustering coefficient* $C_2(G)$ of a graph G is defined as

$$C_2(G) := \frac{3T(G)}{\Lambda(G)}. \tag{2}$$

This parameter measures the likelihood that two vertices which share a neighbor are themselves adjacent.

The present work has to do with the value of $C_2(\mathcal{G}(N;\zeta,\alpha,\beta,\nu))$. Our results show exactly how clustering can be tuned by the parameters β,ζ and α only. More precisely, our main result states that this undergoes an abrupt change as β crosses the critical value 1.

Theorem 1. *Let $0<\zeta/\alpha<2$. If $\beta>1$, then*

$$C_2(\mathcal{G}(N;\zeta,\alpha,\beta,\nu)) \xrightarrow{p} \begin{cases} L_\infty(\beta,\zeta,\alpha), & \text{if } 0<\zeta/\alpha<1 \\ 0, & \text{if } 1\le\zeta/\alpha<2 \end{cases},$$

where

$$L_\infty(\beta,\zeta,\alpha) =$$
$$\frac{3}{2}\,\frac{(\zeta-2\alpha)^2(\alpha-\zeta)}{(\pi C_\beta)^2}\int_{[0,\infty)^3} e^{\frac{\zeta}{2}(t_u+t_v)+\zeta t_w} g_{t_u,t_v,t_w}(\beta,\zeta)e^{-\alpha(t_u+t_v+t_w)}dt_u dt_v dt_w,$$

with

$$g_{t_u,t_v,t_w}(\beta,\zeta) = \int_{[0,\infty)^2} \frac{1}{z_1^\beta+1}\frac{1}{z_2^\beta+1}\frac{1}{\left(e^{\frac{\zeta}{2}(t_w-t_v)}z_1+e^{\frac{\zeta}{2}(t_w-t_u)}z_2\right)^\beta+1}dz_1dz_2$$

and $C_\beta := \frac{2}{\beta\sin(\pi/\beta)}$.
If $\beta\le 1$, then

$$C_2(\mathcal{G}(N;\zeta,\alpha,\beta,\nu)) \xrightarrow{p} 0.$$

The fact that the global clustering coefficient asymptotically vanishes when $\zeta/\alpha\ge 1$ is due to the following: when ζ/α crosses 1 vertices of very high degree appear, which incur an abrupt increase on the number of incomplete triangles with no similar increase on the number of triangles to counterbalance that.

Recall that for a vertex $u\in \mathrm{V}_N$, its *type* t_u is defined to be equal to $R-r_u$ where r_u is the radius (i.e., its hyperbolic distance from the origin) of u in $\mathcal{D}_R$. When $1\le\zeta/\alpha<2$, vertices of type larger than $R/2$ appear, which affect the tail of the degree sequence of $\mathcal{G}(N;\zeta,\alpha,\beta,\nu)$. For $\beta>1$, it was shown in [17] that when $1\le\zeta/\alpha<2$ the degree sequence follows approximately a power law with exponent in $(2,3]$. More precisely, asymptotically as N grows, the degree of a vertex $u\in\mathrm{V}_N$ conditional on its type follows a Poisson distribution with parameter equal to $Ke^{\zeta t_u/2}$, where $K=K(\zeta,\alpha,\beta,\nu)>0$. As we have pointed out in Section 2, when $\zeta/\alpha<1$, a.a.s. all vertices have type less than $R/2$.

Let us consider, for example, more closely the case $\zeta=\alpha$, where the N points are uniformly distributed on $\mathcal{D}_R$. In this case, the type of a vertex u

is approximately exponentially distributed with density $\zeta e^{-\zeta t_u}$. Hence, there are about $N e^{-\zeta(R/2-\omega(N))} \asymp e^{\zeta\omega(N)}$ vertices of type between $R/2-\omega(N)$ and $R/2+\omega(N)$; here $\omega(N)$ is assumed to be a slowly growing function. Now, each of these vertices has degree that is (up to multiplicative constants) at least $e^{\frac{\zeta}{2}\ (\frac{R}{2}-\omega(N))} = N^{1/2}e^{-\zeta\omega(N)/2}$. Therefore, these vertices' contribution to Λ is at least $e^{\zeta\omega(N)} \times \left(N^{1/2}e^{-\zeta\omega(N)/2}\right)^2 = N$.

Now, if vertex u is of type less than $R/2-\omega(N)$, its contribution to Λ in expectation is proportional to

$$\int_0^{R/2-\omega(N)} \left(e^{\zeta t_u/2}\right)^2 e^{-\zeta t_u} dt_u \asymp R. \tag{3}$$

As most vertices are indeed of type less than $R/2-\omega(N)$, it follows that these vertices contribute RN on average to Λ.

However, the amount of triangles these vertices contribute is asymptotically much smaller. Recall that for any two vertices u, v the probability that these are adjacent is bounded away from 0 when $d(u,v) < R$. By Lemma 1 having $d(u,v) < R$ can be expressed saying that the relative angle between u and v is $\theta_{u,v} \lesssim e^{\frac{\zeta}{2}(t_u+t_v-R)}$. Consider three vertices w, u, v which, without loss of generality, satisfy $t_v < t_u < t_w < R/2-\omega(N)$. Since the relative angle between u and v is uniformly distributed in $[0,\pi]$, it turns out that the probability that u is adjacent to w is proportional to $e^{\frac{\zeta}{2}(t_w+t_u-R)}$; similarly, the probability that v is adjacent to w is proportional to $e^{\frac{\zeta}{2}(t_w+t_v-R)}$. Note that these events are independent. Now, conditional on these events, the relative angle between u and w is approximately uniformly distributed in an interval of length $e^{\frac{\zeta}{2}(t_w+t_u-R)}$. Similarly, the relative angle between v and w is approximately uniformly distributed in an interval of length $e^{\frac{\zeta}{2}(t_w+t_v-R)}$. Hence, the (conditional) probability that u is adjacent to v is bounded by a quantity that is proportional to $e^{\frac{\zeta}{2}(t_u+t_v)}/e^{\frac{\zeta}{2}(t_v+t_w)} = e^{\frac{\zeta}{2}(t_u-t_w)}$. This implies that the probability that u, v and w form a triangle is proportional to $e^{\frac{\zeta}{2}t_w+\zeta t_u+\frac{\zeta}{2}t_v}/N^2$. Averaging over the types of these vertices we have

$$\frac{1}{N^2}\int_0^{R/2-\omega(N)}\int_0^{t_w}\int_0^{t_u} e^{\frac{\zeta}{2}t_w+\zeta t_u+\frac{\zeta}{2}t_v-\zeta(t_v+t_u+t_w)} dt_v dt_u dt_w \asymp \frac{1}{N^2}.$$

Hence the expected number of triangles that have all their vertices of type at most $R/2-\omega(N)$ is only proportional to N. Note that if we take $\alpha > \zeta$, then the above expression is still proportional to N, whereas (3) becomes asymptotically constant giving contribution to Λ that is also proportional to N. This makes the clustering coefficient be bounded away from 0 when $\zeta/\alpha < 1$. Our analysis makes the above heuristics rigorous.

It turns out that the situation is somewhat different if we do not take into consideration high-degree vertices (or, equivalently, vertices that have large type). For any fixed $t > 0$, we will consider the global clustering coefficient of the subgraph of $\mathcal{G}(N;\zeta,\alpha,\beta,\nu)$ that is induced by those vertices that have type at most t. We will denote this by $\widehat{C_2}(t)$. We will show that when $\beta > 1$ then for all $0 < \zeta/\alpha < 2$, the quantity $\widehat{C_2}(t)$ remains bounded away from 0 with high probability. Moreover, we determine its dependence on ζ, α, β.

Theorem 2. *Let $0 < \zeta/\alpha < 2$ and let $t > 0$ be fixed. If $\beta > 1$, then*

$$\widehat{C_2}(t) \xrightarrow{p} L(t; \beta, \zeta, \alpha), \tag{4}$$

where

$$L(t; \beta, \zeta, \alpha) := \frac{6 \int_{[0,t)^3} e^{\frac{\zeta}{2}(t_u+t_v)+\zeta t_w} g_{t_u,t_v,t_w}(\beta, \zeta) e^{-\alpha(t_u+t_v+t_w)} dt_u dt_v dt_w}{(\pi C_\beta)^2 \int_{[0,t)^3} e^{\frac{\zeta}{2}(t_u+t_v)+\zeta t_w} e^{-\alpha(t_u+t_v+t_w)} dt_u dt_v dt_w},$$

where $g_{t_u,t_v,t_w}(\beta, \zeta)$ and C_β are as in Theorem 1.

The most involved part of the proofs, which may be of independent interest, has to do with counting triangles in $\mathcal{G}(N; \zeta, \alpha, \beta, \nu)$, that is, with estimating $T(\mathcal{G}(N; \zeta, \alpha, \beta, \nu))$. In fact, most of our effort is devoted to the calculation of the probability that three vertices form a triangle. Thereafter, a second moment argument, together with the fact that the degree of high-type vertices is concentrated around its expected value, implies that $T(\mathcal{G}(N; \zeta, \alpha, \beta, \nu))$ is close to its expected value.

4 Conclusions

In this contribution, we study the presence of clustering as a result of the hyperbolic geometry of a complex network. We consider the model of Krioukov et al. [22], where the resulting random graph is sparse and its degree distribution follows a power law. We quantify the existence of clustering and, furthermore, for the part of the random network that consists of typical vertices, we show that the clustering coefficient is bounded away from 0. More importantly, we find how does this quantity depend on the parameters of the random graph and show that this can be determined independently of the average degree.

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