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ACCESSIBILITY AND SINGULAR
FOLIATIONS

by

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Thesis submitted for the
degree of Doctor of Philosophy
at the University of Warwick,
December 1973
ABSTRACT

In Part One we study the partition of a finite-dimensional manifold $M$ into the accessible sets of an arbitrary system $A$ of isotopy families of local diffeomorphisms of $M$ and, in particular, into the accessible sets of an arbitrary system of differentiable vectorfields on $M$.

In Part Two we generalize the methods of Part One to study the integrability of singular distributions on infinite-dimensional manifolds.

In Part Three we return to finite-dimensional manifolds and use the results of Part One to study in detail the contrasting properties of integrability and irreducibility of systems of vectorfields on $M$. 
ACKNOWLEDGEMENTS

I want to thank my supervisor, Professor James Eells, for his help, encouragement, good humour and patience.

I also want to thank Professor Jaroslav Kurzweil who taught me my first real-life mathematics and told me that every man has to ruin his life in his own way.

Theorem 3 in Part One answers a question put by Professor S.A. Robertson in connection with his work on mobility and co-mobility. I am grateful to him for asking it and for taking an interest in my work.

I want to thank David Chillingworth for taking the trouble to read and answer my long letters and for many hours of stimulating discussions and much constructive criticism. In particular, his original proof [14] convinced me that the methods of Part One can be modified to fit the infinite dimensional case and inspired the work which is presented in Part Two.

I have benefited very much from the discussions I have had with Professor R. Brown in Bangor and David Elworthy at Warwick, and many other mathematicians in these two places and elsewhere.

I am grateful to the Mathematics Institute at the University of Warwick for giving me an opportunity to continue my research there, and to the University College of North Wales for employing me as a lecturer.

I want to thank Miss S. Rhodes for typing this thesis and all my friends who are not going to read it.
1. RESULTS. The contents of this thesis is divided into three parts. Part One is (except in the Introduction) identical to a paper which will appear in the Proceedings of the London Mathematical Society. Part Two generalizes the results of Part One to infinite-dimensional (Banach) manifolds and Part Three consists of two shorter papers which could be considered as applications of the results of Part One.

The main result of Part One is to show that the accessible sets of an arbitrary collection A of 'arrows' (= isotopy families of local diffeomorphisms) on a manifold M possess a differentiable structure which makes them into connected immersed submanifolds of M. We also show that this differentiable structure is unique and that, more generally, every differentiable function \( N \rightarrow M \) which factors set-theoretically through an accessible set \( L \) of A factors differentiably through \( L \). Moreover, a similar result holds if we replace \( N \) by a locally connected topological space and substitute 'continuous' for 'differentiable'. In short, the accessible sets of A are almost as well-behaved as embedded submanifolds.

As for the 'differentiability transversally to the accessible sets' we show that the latter fit together to form a foliation with singularities (a regular foliation if they happen to be all of the same dimension). For example, the partition of the plane into the graphs of the functions \( y^3 = (x-c)^3 \) does not represent the collection of accessible sets of any set A of arrows on \( \mathbb{R}^2 \).
There is a simple description of the tangent spaces of the accessible sets of \( A \) which allows us to compute their dimension and to give a necessary and sufficient condition (homogeneity) that \( A \) be integrable (that is, that it span the tangent spaces of its accessible sets). This condition is later shown to imply the 'classical' integrability conditions such as the Frobenius theorem and Nagano's results on integrability of (possibly singular) real-analytic distributions [7].

In Theorem 2 we show that, if \( \sim \) is an arbitrary equivalence relation on \( M \), then there exists the greatest foliation with singularities whose leaves are inscribed in the equivalence classes of \( \sim \). If \( \phi \) is a local diffeomorphism of \( M \) such that \( x \sim y \) implies \( \phi(x) \sim \phi(y) \), then \( \phi \) is a local diffeomorphism for the differentiable structure defined by this foliation. The collection of \( \sim \)-preserving vectorfields on \( M \) is closed under formation of the Lie bracket (§5).

In Theorem 3 we show that every subgroup \( G \) of Diff \( M \) defines a foliation with singularities whose leaves are the orbits of the isotopy component \( G_o \) of \( \text{id}_M \) in \( G \). The orbits of \( G \) are unions of \( G_o \)-orbits of constant dimension. In Theorems 4 and 5 we state a similar result for groupoids of germs of local diffeomorphisms and for arbitrary collections of differentiable vectorfields.

In Theorem 6 we re-formulate our integrability condition in terms of Lie brackets. We also show how it implies various other integrability conditions and give some examples illustrating their relationship.
In §6 of Part One we introduce the concept of 'multiarrows'. This is a convenient gadget for replacing Lie brackets (infinitesimal commutators) by 'finite' commutators. It is used here to give a direct proof of the so-called Chow's theorem.

In Part Two we define the differentiability of (possibly singular) distributions on infinite-dimensional (Banach) manifolds and show that it can be described in terms of vector-valued one-forms. The main result (Theorem 1) states that a weakly differentiable, possibly singular, distribution is integrable if and only if it is homogeneous. We give some other necessary and sufficient conditions of integrability and show that an integrable distribution B defines a unique differentiable structure σ on M such that (M, σ) is an integral submanifold of B. Further, σ is a foliation with singularities and the connected components of σ are the accessible sets of B.

In §8 of Part Two we use Lie derivatives to give some necessary and sufficient conditions that a vectorfield X respects a distribution B and hence deduce the corresponding conditions of homogeneity. We also show how these conditions imply the standard Frobenius theorem and prove that a real-analytic (possibly singular) distribution is integrable if and only if it is involutive and locally everywhere defined.

In the last section of Part Two we introduce the concepts of a neat leaf of a distribution and a neat submanifold of M and discuss a related unsolved problem.
In Part Three, §1, we return to the integrability of a system of vectorfields on a finite dimensional manifold and answer in full some of the problems which were left open in Part One. In particular, we show that, contrary to the claims in [6], [10] and [11], the condition that a set $S$ of vectorfields be 'locally of finite type' is not sufficient for its integrability. We give some related necessary and sufficient conditions of integrability for the $C^\infty$ case and show that they are not sufficient in the real-analytic case.

Finally, in §2 of Part Three we prove that the set of irreducible pairs of $C^k$-vectorfields on $M$ is $C^k$-generic for every $k \geq 1$. (A pair $S = \{X, Y\}$ of vectorfields on $M$ is irreducible if the accessible sets of $S$ coincide with the connected components of $M$; the result has been known for $k \geq 2n$ [19]).

2. CONTEXT. Although the motif of this thesis goes back to Caratheodory's work on Thermodynamics (cf. Math. Annalen 67, 1909) the main reference is undoubtedly the 1939 paper of Wei-Liang Chow [2]. Chow's results can be summarized in our notation as follows. Let $S$ be a system of $C^1$ vectorfields on a manifold $M$ and assume that

(1) \[ \dim \tilde{S}(y) = \text{const} \quad \text{for} \quad y \in \tilde{\Omega}, \]

where $\tilde{\Omega}$ is a neighbourhood of a point $x$ in $M$ (cf. Part One, §4.3). Then the accessible sets of $S$ define a regular foliation on a (possibly smaller) neighbourhood $\tilde{\Omega}'$ of $x$, which is tangent to the distribution $(\tilde{S}(y) : y \in \tilde{\Omega}')$. The regularity condition (1) is
of course a major restriction on the class of the admissible systems of vectorfields, whereas the results of Part One are valid for an arbitrary system $S$ (and, more generally, for an arbitrary collection of 'arrows' on $M$).

As far as I know, the first proof of a 'Frobenius theorem' for singular (but real-analytic) distributions was published by Tadashi Nagano in 1966 [7]. I have also benefited a great deal from reading the papers [5] and [21] of Robert Hermann and from Claude Lobry's 1970 paper [6].

A preliminary version of Part One appeared as [11]. It has since transpired that some of the results (notably much of §5 and, to a lesser extent, the assertion of Theorem 5) partially overlap with the recent work of Héctor J. Sussmann [9], [10].
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*) §1 represents an abbreviated version of the pre-print [12].
PART ONE

Accessible Sets, Orbits, and Foliations

with Singularities
INTRODUCTION

In §§1-3 we prove a general theorem on accessible sets of collections of 'arrows' on the C^q-manifold M, where 1 ≤ q ≤ ω.

In §4 we apply this result to the following situations:

(1) the partition of M into G-orbits, where G is an arbitrary subgroup of Diff(M);

(2) a similar situation, in which G is replaced by an arbitrary groupoid Γ of germs of local diffeomorphisms of M; and

(3) the partition of M into the accessible sets of an arbitrary collection of vectorfields on M.

In §5 we study the tangent spaces of accessible sets and obtain various generalizations of the Frobenius theorem. Finally in §6 we introduce the concept of a multiarrow and give a direct proof of the so-called Chow's theorem.
§1. ACCESSIBLE SETS FORM A FOLIATION WITH SINGULARITIES

Let $M$ be a finite-dimensional paracompact $C^q$-manifold, $1 \leq q \leq \omega$. The word 'differentiable' always refers to this fixed class $C^q$. A subset $L$ of $M$ is said to be a $k$-leaf of $M$ if there exists a differentiable structure $\sigma$ on $L$ such that (i) $(L, \sigma)$ is a connected $k$-dimensional immersed submanifold of $M$ and (ii) if $N$ is an arbitrary locally connected topological space and $f : N \to M$ is a continuous function such that $f(N) \subset L$, then $f : N \to (L, \sigma)$ is continuous.

It follows from the properties of immersions that if $f : N \to M$ is a differentiable mapping of manifolds such that $f(N) \subset L$, then $f : N \to (L, \sigma)$ is also differentiable. In particular, $\sigma$ is the unique differentiable structure on $L$ which makes $L$ into an immersed $k$-dimensional submanifold of $M$. Since $M$ is paracompact, every connected immersed submanifold of $M$ is separable, and so $L$ does not admit a differentiable structure of a connected immersed submanifold of $M$ of a dimension other than $k$.

Every embedded connected submanifold of $M$ is a leaf, and so is the flow-line of the irrational flow if $M$ is the two-torus.

We say that $\mathcal{F}$ is a $C^q$-foliation of $M$ with singularities if $\mathcal{F}$ is a partition of $M$ into $C^q$-leaves of $M$, such that, for every $x \in M$, there exists a local $C^q$-chart $\psi$ of $M$ with the following properties:

(a) The domain of $\psi$ is of the form $U \times W$, where $U$ is an open neighbourhood of $0$ in $\mathbb{R}^k$, $W$ is an open neighbourhood of $0$ in $\mathbb{R}^{n-k}$ and $k$ is the dimension of the leaf through $x$. 

(b) \( \psi(0,0) = x. \)

(c) If \( L \) is a leaf of \( \mathcal{F} \), then \( L \cap \psi(U \times \mathbb{W}) = \psi(U \times L) \), where

\[ L = \{ w \in \mathbb{W} : \psi(0,w) \in L \}. \]

Let \( d(x,F) \) denote the dimension of the leaf of \( \mathcal{F} \) which contains the point \( x \). It follows immediately from the above definition that the function \( x \mapsto d(x,F) \) is semi-continuous below. We write \((M,F)\) for the \( C^q \)-manifold with the same underlying set as \( M \) and with the \( C^q \)-structure of the disjoint sum of the leaves of \( \mathcal{F} \); in general, this is a disconnected manifold whose components are not necessarily all of the same dimension.

By a \textit{local diffeomorphism} of \( M \) we mean a diffeomorphism of one open subset of \( M \) onto another. We say that a differentiable function \( a : \mathbb{R} \times M \to M \) is an \textit{arrow} if its domain is an open subset of \( \mathbb{R} \times M \) and if it satisfies the following two conditions: (i) for every \( t \in \mathbb{R} \), \( a^t = a(t,\cdot) \) is a local diffeomorphism of \( M \) (possibly with the empty domain) and (ii) if \((t,x)\) belongs to the domain of \( a \), then so does \((s,x)\) for every \( s \) between \( 0 \) and \( t \) and \( a(0,x) = x \). We write \( a(t,x) \) for the tangent vector at \( t \) of the curve \( a(-,x) \), and \( (a^t)^*(x) \) for the differential at \( x \) of the function \( a^t : M \to M \). If \( y = a(t,x) \), then \( a(t,x) \in T_y M \) and \((a^t)^*(x)\) is a linear mapping \( T_x M \to T_y M \).

An example of an arrow is the \textit{flow} of a differentiable vectorfield (§4.3). In general, an arrow \( a \) does not necessarily satisfy the condition \( a^{t+s} = a^t \circ a^s \).
Let $A$ be a collection of arrows on $M$. We write $\Theta A$ for the set of all local diffeomorphisms $\phi$ of $M$ such that $\phi = a^t$ for some $a \in A$ and $t \in \mathbb{R}$, and let $\mathcal{V} A$ denote the set which consists of the identity mapping of $M$ and all the local diffeomorphisms of the form $\phi_1 \circ \phi_2 \circ \ldots \circ \phi_p$, where $p$ is an arbitrary positive integer and $\phi_i$ or $\phi_i^{-1}$ belongs to $\Theta A$ for $1 \leq i \leq p$. We write $y = x \bmod A$ if $y = \phi(x)$ for some $\phi \in \mathcal{V} A$. This is clearly an equivalence relation on $M$; its equivalence classes are termed the accessible sets of $A$.

Given $x \in M$, we consider two vector subspaces, $A(x)$ and $\bar{A}(x)$, of the tangent space $T_x M$, spanned respectively by the sets $\{a(t,y) : a \in A, a(t,y) = x\}$ and $\{\phi^*(y) : \phi \in \mathcal{V} A, \phi(y) = x, w \in A(y)\}$.

If $\phi \in \mathcal{V} A$ and $\phi(x) = y$, then clearly $\phi^*(x) \bar{A}(x) = \bar{A}(y)$. Thus, if $x = y \bmod A$, then $\dim \bar{A}(x) = \dim \bar{A}(y)$.

**THEOREM 1.** Let $\Xi$ be the partition of $M$ into the accessible sets of $A$. Then $\Xi$ is a foliation with singularities and, for every $x \in M$, $T_x (M, \Xi) = \bar{A}(x)$.

In particular, every accessible set of $A$ is a leaf of $M$ and thus admits a unique differentiable structure of a connected immersed submanifold of $M$.

Let $\sim$ be an equivalence relation on $M$. We say that a local diffeomorphism $\phi$ preserves $\sim$ if $\phi(x) \sim x$ whenever $x$ belongs to the domain of $\phi$. We say that $\phi$ respects $\sim$ if $\phi(x) \sim \phi(y)$ whenever $x \sim y$ and both $x$ and $y$ belong to the domain of $\phi$. An arrow $a$ preserves (or respects) $\sim$ if so does the local diffeomorphism $a^t$ for every $t \in \mathbb{R}$. 
THEOREM 2. Let ~ be an equivalence relation on M and let A be the collection of all the arrows on M which preserve ~. If \( \phi \in \text{Loc Diff } M \) and \( \phi \) respects ~, then \( \phi \in \text{Loc Diff } (M, \mathcal{E}) \), where \( \mathcal{E} = \mathcal{E}(A) \) is the partition of M into the accessible sets of A.

§2 HOMOGENEOUS AND SYMMETRIC ENVELOPES

A collection A of arrows on M is said to be homogeneous if \( A(x) = \tilde{A}(x) \) for every \( x \in M \), that is if \( \phi^*(x) \cdot A(x) \subseteq A(y) \) whenever \( \phi \) or \( \phi^{-1} \) belongs to \( \Theta A \) and \( \phi(x) = y \); A is symmetric if \( \phi \in \Theta A \) implies that \( \phi^{-1} \) is a composition of finitely many members of \( \Theta A \).

We write \( A_0(x) \) for the vector subspace of \( T_x M \) spanned by the set \( \{ a(0, x) : a \in A, (0, x) \in \text{domain}(a) \} \).

PROPOSITION 2.1. If A is a collection of arrows on M, then there exists a symmetric homogeneous collection of arrows B such that the accessible sets of A and B are the same and \( \tilde{A}(x) = B(x) = B_0(x) \) for every \( x \in M \).

LEMMA 2.2. Let \( A^# \) be the collection of all arrows which can be written as \( \tau^s \cdot (a^s)^{-1} \) or \( s^t \cdot (a^s)^{-1} \) for some \( a \in A \) and \( s \in R \). Then

(a) \( \forall A^# = \forall A \);
(b) \( A^# \) is symmetric;
(c) \( A_0^#(x) = A_0^#(x) = A(x) \) for every \( x \in M \);
(d) the accessible sets of \( A^# \) and A are the same; and
(e) if A is homogeneous, then so is \( A^# \).
PROOF. Note that the assertion (d) follows at once from (a) and that (e) follows from (a) and (c). Let \( b^t = a^{t+s} \circ (a^s)^{-1} \) and \( c^t = a^{s-t} \circ (a^s)^{-1} \). If \( x \) lies in the domain of \( b^t \), then it lies in the domain of \( b^\tau \) for every \( \tau \) between 0 and \( t \), and so \( b \) (and similarly \( c \)) is indeed an arrow. Taking \( s = 0 \) \((t = s)\) we get \( b^t = a^t \left( c^s = (a^s)^{-1} \right) \) and so \( \Theta A \subset \Theta A^# \subset \forall A \) and \( (\Theta A)^{-1} \subset \Theta A^# \), proving the assertions (a) and (b). Finally the equations

\[
\begin{align*}
\dot{b}(t,x) &= \dot{a}(t+s, (a^s)^{-1}.x), \\
\dot{c}(t,x) &= -\dot{a}(s-t, (a^s)^{-1}.x)
\end{align*}
\]

and

\[
\dot{b}(0, a^s.x) = \dot{a}(s, x)
\]

show that \( A^\#(x) \subset A(x) \subset A^\#_o(x) \), which proves the assertion (c).

LEMMA 2.3. Let \( A^* \) be the collection of all the arrows \( a \) on \( M \) such that:

(i) the domain of \( a \) is of the form \( J \times V \), where \( J \) is an open interval in \( \mathbb{R} \) and \( V \) is an open subset of \( M \).

(ii) there exists \( b \in A^# \) and \( \phi \in \forall A \) such that, for every \( t \) in \( J \) and \( x \) in \( V \),

\[
a(t,x) = \phi \circ b^t \circ \phi^{-1}.x.
\]

Then

(a) every \( \phi \in \forall A^* \) is a restriction of some local diffeomorphism in \( \forall A \);

(b) the accessible sets of \( A^* \) and \( A \) are the same;

(c) \( A^*(x) = \tilde{A}(x) \) for every \( x \in M \); and

(d) \( A^* \) is a homogeneous set of arrows.
PROOF. Note that the assertion (d) follows at once from (a) and (c). Since (a), (b) and the inclusion $A^*(x) \subseteq \tilde{A}(x)$ are obvious, it remains to show that $\tilde{A}(x) \subseteq A^*(x)$. Let $w \in A(y)$, $\phi \in \mathcal{A}$ and $\phi(y) = x$. By Lemma 2.2c, $w = \hat{b}(0,y)$ for some $b \in A^\#$. There exists $\delta > 0$ such that $a^t = \phi^t \phi^{-1}$ is defined for $|t| < \delta$ on a sufficiently small neighbourhood of $x$. Hence $\phi^*(y).w = \hat{a}(0,x) \in A^*(x)$, Q.E.D.

Proposition 2.1 is now proved by taking $B = (A^*)^\#$.

§3. THE PROOF OF THEOREMS 1 AND 2

LEMMA 3.1 Let $L$ be a subset of $M$. For every $x \in L$, let $L(x)$ be a vector subspace of $T_x M$. Assume that $\dim L(x) = k$ for every $x \in L$ and that, for every $x \in L$, there exists a local chart $\psi$ of $M$ such that

(a) the domain of $\psi$ is $U \times W$, where $U$ and $W$ are open neighbourhoods of the origin in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$ respectively;

(b) $\psi(0,0) = x$;

(c) $L \cap \psi(U \times W) = \psi(U \times \mathcal{L})$, where $\mathcal{L} = \{s \in W : \psi(0,s) \in L\}$; and

(d) $D_i \psi(t,s) \in L(\psi(t,s))$ for $1 \leq i \leq k$ and all $(t,s) \in \psi^{-1}(L)$.

Then there exists a differentiable structure $\sigma$ on $L$ with the following properties:
(i) \((L,\sigma)\) is an immersed submanifold of \(M\) and \(T_x(L,\sigma) = L(x)\) for every \(x \in L\);

(ii) If \(f : N \to M\) is a differentiable mapping of manifolds such that \(f(N) \subseteq L\) and \(f^*(\xi) \cdot T_{\xi}N \subseteq L(f(\xi))\) for every \(\xi \in N\), then \(f : N \to (L,\sigma)\) is differentiable.

(iii) Every connected component of \((L,\sigma)\) is a leaf of \(M\).

REMARKS. (1) It follows from (ii) that \(\sigma\) is the unique differentiable structure on \(L\) which satisfies the condition (i).

(2) Apart from (iii), the assertions of this lemma do not depend on the paracompactness of \(M\).

The proof of Lemma 3.1 consists in piecing together some *) of the arguments in [1] and we give it here merely for the sake of completeness.

PROOF. A function \(\psi : U \times W \to M\) is said to be a privileged chart of \(M\) if it is a local chart of \(M\) and if it satisfies the conditions (a), (c) and (d) above.

(A) Let \(\psi : U \times W \to M\) be a privileged chart of \(M\) and let \(f : N \to M\) be a differentiable mapping of a connected manifold \(N\) into \(M\) such that \(f(N) \subseteq \psi(U \times W)\) and, for every \(\xi \in N\), \(f^*(\xi) \cdot T_{\xi}N \subseteq L(f(\xi))\). Then there exists a constant \(w \in W\) such that \(f(N) \subseteq \psi(U \times \{w\})\).

*) See pp. 91-95; it is assumed there that \(L = M\), and the assertion (ii) is not formulated.
To prove this, let \( p : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k} \) be the projection on the second coordinate and let \( g = p \circ \psi^{-1} \). As \( g \circ \psi \cdot (t,s) = s \), we have \( D_i(g \circ \psi)(t,s) = g^*(\psi(t,s)) \cdot D_i\psi(t,s) = 0 \) for \( 1 \leq i \leq k \). If \( \psi(t,s) \in L \), then \( D_1\psi(t,s), D_2\psi(t,s), \ldots, D_k\psi(t,s) \) span \( L(\psi(t,s)) \), which proves that \( g^*(x) \cdot L(x) = 0 \) for every \( x \in L \cap \psi(U \times W) \). Hence \( (g \circ f)^*(\xi) = 0 \) for every \( \xi \in N \). As \( N \) is connected, \( g \circ f \) is constant on \( N \), Q.E.D.

(B) Let \( \Psi \) be the collection of all the functions of the form \( \psi_w = \psi(-,w) : U_\psi \to L \), where \( \psi \) is some privileged chart of \( M \) with the domain \( U_\psi \times W_\psi \), \( w \in W_\psi \), and \( \psi(0,w) \in L \). Let \( f : N \to M \) be a differentiable mapping of a manifold \( N \) into \( M \) such that \( f(N) \subset L \) and, for every \( \xi \in N \), \( f^*(\xi) \cdot T_N L = L(f(\xi)) \). If \( \psi_w \in \Psi \), then \( G = f^{-1}(\psi_w(U_\psi)) \) is an open subset of \( N \), and \( (\psi_w)^{-1} \circ f : G \to \mathbb{R}^k \) is a differentiable function.

Indeed, it follows immediately from (A) that \( G \) is the union of some of the connected components of the open set \( f^{-1}(\psi(U_\psi \times W_\psi)) \), and it is obvious that \( (\psi_w)^{-1} \circ f = \psi^{-1} \circ f \big|_G \).

(C) \( \Psi \) is an atlas of a differentiable structure \( \sigma \) on \( L \) which satisfies the assertions (i) and (ii).

For it follows immediately from (B) that the charts of \( \Psi \) are mutually compatible; as their ranges cover the whole of \( L \), \( \Psi \) is an atlas of a differentiable structure \( \sigma \), which obviously satisfies the assertion (i). The assertion (ii) follows from (B).
(D) By paracompactness of $M$, every connected component $L_0$ of $(L, \sigma)$ is separable in the topology $\tau(\sigma)$ of the differentiable structure $\sigma$. Let $\psi : U \times W \to M$ be a privileged chart of $M$ and let $t_0 = \{ s \in W : \psi(0, s) \in L_0 \}$. Since $\{ \psi(U, s) : s \in t_0 \}$ is a collection of mutually disjoint $\tau(\sigma)$-open subsets of $L_0$, $\psi(\{0\} \times t_0)$ is an isolated subset of $L_0$. Hence $t_0$ is countable and, therefore, completely disconnected subset of $W$. It follows that $U \times \{0\}$ is a connected component of $U \times t_0$ in the product topology of $U \times W$, and so $\psi(U \times \{0\})$ is a connected component of $\psi(U \times t_0) = \psi(U \times W) \cap L_0$ in the induced topology $\tau(L_0, M)$. Since $U$ can be taken arbitrarily small, we have proved that every point of $L_0$ has a fundamental system of $\tau(\sigma)$-neighbourhoods that are $\tau(L_0, M)$-connected components of $\tau(L_0, M)$-neighbourhoods.

It follows trivially from the definitions and the above assertion that $L_0$ is a leaf of $M$, which concludes the proof of the lemma.

PROOF OF THEOREM 1. By Proposition 2.1, we may assume that $A$ is homogeneous and that $A(x) = A_0(x)$ for every $x \in M$.

Let $L$ be an accessible set of $A$ and let $x \in L$. Choose $a_i \in A$, $1 \leq i \leq k$, such that $a_i(0, x)$ form a basis of $A(x)$ and let $\phi(t_1, t_2, ..., t_k, y) = a_1^{t_1} \circ a_2^{t_2} \circ ... \circ a_k^{t_k}(y)$. Then $\phi$ is a
differentiable function $\mathbb{R}^k \times M \to M$ and we may assume that the domain of $\phi$ is of the form $U \times V$, where $U$ is an open neighbourhood of the origin in $\mathbb{R}^k$ and $V$ is a neighbourhood of $x$ in $M$. It is obvious that

(i) for every $t \in U$ and $y \in V$, $\phi(t,y) = y \pmod{A}$ and $\phi(0,y) = y$;

(ii) $D_i \phi(0,x) = a_i(0,x)$ for $1 \leq i \leq k$. We claim that

(iii) for every $t \in U$, $y \in V$ and $i$ between 1 and $k$, $D_i \phi(t,y) \in A(\phi(t,y))$.

To prove (iii) note that, for example, $D_2 \phi(t,y) = (a_1^t)^*(z) \cdot a_2(t_2,w)$ where $a_2(t_2,w) = z$ and $a_1(t_1,z) = \phi(t,y)$. Since $a_2(t_2,w) \in A(z)$, the result follows from the homogeneity of $A$.

Let now $n = \dim M$ and let $Q$ be an $(n-k)$-dimensional submanifold of $M$ such that $x \in Q$ and $T_x M = T_x Q + A(x)$. Let $f : W \to Q$ be a local chart for $Q$ such that $f(0) = x$ and $f(W) \subset V$ and let $\psi : U \times W \to M$ be defined by $\psi(t,s) = \phi(t,f(s))$. Since the rank of $\psi$ at $(0,0)$ is $n$, $\psi$ is a local chart of $M$ for sufficiently small $U$ and $W$. It is easy to check that $\psi$ satisfies the conditions of Lemma 3.1 with $L(y) = A(y)$ and that the condition (c) remains valid if $L$ is replaced by an arbitrary accessible set of $A$.

Let $\sigma$ be the differentiable structure on $L$ whose existence is asserted in Lemma 3.1. It remains to show that $(L, \sigma)$ is a connected immersed submanifold of $M$. Let $u(t) = a(t,x)$, where $a \in A$ and $x \in L$. Then $u : \mathbb{R} \to M$ is differentiable, the range of $u$ is contained in $L$ and $u(t) \in A(u(t))$ for every $t$ in the domain.
of $u$. By Lemma 3.1 (ii), the 'arrow-path' $u : R \to (L, \sigma)$ is differentiable and it remains to note that any two points of $L$ can be joined by a succession of such arrow-paths (traced forwards or backwards).

**Lemma 3.2.** Let $\sim$ be an arbitrary equivalence relation on $M$ and let $A$ be the set of all arrows which preserve $\sim$. Then $A$ is symmetric and homogeneous and $A(x) = A_0(x)$ for every $x$ in $M$.

**Proof.** It is clear that $A^\# \subseteq A$ and $A^* \subseteq A$. (See Lemmas 2.2 and 2.3.)

**Proof of Theorem 2.** Consider $\phi \in \text{Loc Diff}(M)$ such that $\phi$ respects $\sim$. Let $x \in \text{domain}(\phi)$. Let $L$ be the accessible set of $A$ through $x$ and let $L'$ be the accessible set of $A$ through $\phi(x)$. Let $k = \dim L$ and let $a_1, a_2, \ldots, a_k$ be members of $A$ such that $a_i^1(0, x)$ form a basis of $A(x)$. If $b_i^t = a_i^{t^i - 1}$, then $b_i \in A$, $1 \leq i \leq k$. If $y = f(t_1, t_2, \ldots, t_k) = a_1^{t_1} a_2^{t_2} \cdots a_k^{t_k}(x)$ and $|t_i|$ are sufficiently small, then $\phi(y) = b_1^{t_1} b_2^{t_2} \cdots b_k^{t_k}(\phi(x))$ and so $\phi(y) \in L'$. Since the rank of $f : \mathbb{R}^k \to L$ is $k$, we have proved that there exists a neighbourhood $U$ of $x$ in $L$ such that $\phi(U) \subseteq L'$. Since $\phi : U \to M$ is differentiable and $L'$ is a leaf of $M$, it follows that $\phi : U \to L'$ is differentiable, Q.E.D.
§4 COROLLARIES OF THEOREMS 1 AND 2

4.1 Let $G$ be a subgroup of $\text{Diff}(M)$. Two elements $g$ and $h$ of $G$ are said to be $G$-isotopic if there exists a differentiable mapping $a : \mathbb{R} \times M \to M$ such that $a^t \in G$ for every $t \in \mathbb{R}$, $a^t = g$ for $t \leq 0$ and $a^t = h$ for $t \geq 1$. This is an equivalence relation and the component $G_o$ of the identity is a normal subgroup of $G$.

THEOREM 3. (a) Let $E = E(G_o)$ be the partition of $M$ into $G_o$-orbits. Then $E$ is a foliation with singularities.

(b) $G \subseteq \text{Diff}(M, E)$ and every $G$-orbit consists of $G_o$-orbits of constant dimension.

(c) If $G/G_o$ is countable, then every $G$-orbit admits a unique structure of a separable immersed submanifold of $M$.

PROOF. Let $A$ be the set of all differentiable mappings $R \times M \to M$ such that $a^t \in G$ for every $t \in \mathbb{R}$ and $a^t = \text{id}_M$ for $t \leq 0$. Then $A$ is a symmetric set of arrows and the accessible sets of $A$ are the orbits of $G_o$. The assertion (b) follows at once from Theorem 2 if we take $\sim$ to be the equivalence relation defined by the action of $G_o$ and use the fact that $G_o$ is normal. The assertion (c) follows easily from (b) and the fact that every $G_o$-orbit is a leaf of $M$.

4.2 If $\phi \in \text{Loc Diff}(M)$ and $x$ belongs to the domain of $\phi$, let $\gamma(x, \phi)$ denote the germ of $\phi$ at $x$. Let $e(x) = \gamma(x, \text{id}_M)$ and let $\Delta = \Delta(M)$ be the groupoid of all germs of local diffeomorphisms of $M$. Let $\alpha : \Delta \to M$ and $\omega : \Delta \to M$ be the projections onto the initial and final points respectively, so that $\alpha(\gamma(x, \phi)) = x$ and
Let $\Gamma$ be a subgroupoid of $\Delta$. We say that $g \in \Gamma$ and $h \in \Gamma$ are $\Gamma$-isotopic if $a(g) = a(h)$ and if there exists an open neighbourhood $U$ of $a(g)$ and a differentiable mapping $a : \mathbb{R} \times U \to M$ such that (i) $\gamma(a(g), a^t) \in \Gamma$ for every $t \in \mathbb{R}$, (ii) $\gamma(a(g), a^t) = g$ for $t \leq 0$ and (iii) $\gamma(a(g), a^t) = h$ for $t \geq 1$. Let $\Gamma_0 = \{g \in \Gamma : g$ is $\Gamma$-isotopic to $e(a(g))\}$. Then $\Gamma_0$ is a subgroupoid of $\Gamma$ and $g$ and $h$ are $\Gamma$-isotopic if and only if $\Gamma_0 g = \Gamma_0 h$.

**Theorem 4.** The assertions of Theorem 3 remain valid if we replace $G$ by $\Gamma$, $G_0$ by $\Gamma_0$ and $G \subseteq \text{Diff}(M, \mathbb{R})$ by $\tilde{\Gamma} \subseteq \text{Loc Diff}(M, \mathbb{R})$, where $\phi \in \tilde{\Gamma}$ if and only if $\gamma(x, \phi) \in \Gamma$ for every $x$ in the domain of $\phi$.

4.3 If $X$ is a differentiable vector field on $M$, $\exp X$ denotes the flow of $X$, so that $t \to \exp X \cdot (t, x)$ is the integral curve of $X$ passing through $x$ at $t = 0$. If $S$ is a set of vector fields on $M$ we put $\exp S = \{\exp X : X \in S\}$. It is clear that $\exp S$ is a symmetric set of arrows; the accessible sets of $S$ are, by definition, the accessible sets of $\exp S$.

We write $\Theta S$, $\Psi S$, $S(x)$ and $\tilde{S}(x)$ instead of $\Theta \exp S$, $\Psi \exp S$, $(\exp S)(x)$ and $(\exp S)(x)$, so that $S(x)$ is the vector subspace of $T_x M$ spanned by $\{X(x) : X \in S\}$, and $\tilde{S}(x)$ is spanned by all the vectors of the form $\phi*(y) \cdot X(y)$, where $\phi \in \Psi S$, $\phi(y) = x$ and $X \in S$. 
THEOREM 5. Let $\mathcal{F} = \mathcal{F}(S)$ be a partition of $M$ into the accessible sets of $S$. Then $\mathcal{F}$ is a foliation with singularities*) and $T_x(M,\mathcal{F}) = \tilde{S}(x)$ for every $x$ in $M$.

COROLLARY. $T_x(M,\mathcal{F}) = S(x)$ for every $x \in M$ if and only if $\exp S$ is a homogeneous set of arrows, that is if and only if $(\exp X^t)^*(x).S(x) \subset S(y)$ whenever $X \in S$ and $\exp X(t,x) = y$.

§5. LIE BRACKETS AND SUFFICIENT CONDITIONS OF HOMOGENEITY

LEMMA 5.1. Assume that $\mathcal{F}$ is a partition of $M$ into immersed submanifolds and let $S$ be the collection of all vectorfields on $M$ that leave $\mathcal{F}$ invariant. Then

(a) $X \in S$ if and only if $X(x) \in T_x(M,\mathcal{F})$ for every $x \in M$;
(b) If $X$ and $Y$ belong to $S$, then $[X,Y](x) \in T_x(M,\mathcal{F})$ for every $x \in M$; and so
(c) if $q = \infty$ or $\omega$, then $S$ is closed under formation of the Lie bracket.

*) See also [10], where it is proved that the accessible sets of $S$ are immersed submanifolds of $M$ (but not that they fit together to form a foliation with singularities). The 'D-invariance' in [10] is equivalent to our 'homogeneity'.
PROOF. The assertion (a) depends on the existence and uniqueness theorem for ordinary differential equations (see [8], Lemma 2.4) and (b) follows from the fact that \((M,E)\) is an immersed submanifold of \(M\) (see [4], (17.14.3.5)).

COROLLARY 5.1. Let \(\sim\) be an arbitrary equivalence relation on \(M\) and let \(S\) be the set of all vectorfields on \(M\) that leave the equivalence classes of \(\sim\) invariant. If \(q = \infty\) or \(\omega\), then \(S\) is closed under formation of the Lie bracket.

PROOF. If \(A\) is the collection of all the arrows on \(M\) that preserve \(\sim\) and \(F = F(A)\), then clearly \(\exp S \subseteq A\) and \(X \in S\) if and only if \(X\) leaves \(F\) invariant.
THEOREM 6. Let $q \geq 2$ and let $S$ be a set of $C^1$-vectorfields on $M$. The following assertions are equivalent:

(i) $\exp S$ is homogeneous.

(ii) Given $X \in S$ and $x \in M$, there exists $\epsilon > 0$, a finite set \{${X_1,X_2,\ldots,X_p}$\} \subset S and continuous functions $\lambda_{ij} : \mathbb{R} \to \mathbb{R}$ (1 \leq i, j \leq p) such that (a) the vectors $X_1(x), X_2(x), \ldots, X_p(x)$ span $S(x)$, (b) for every $t \in ]-\epsilon,\epsilon[\text{ and } j$ between 1 and $p$

$$[X,X_j](u(t)) = \sum_{i=1}^{p} \lambda_{ij}(t)X_i(u(t)),$$

where $u(t) = \exp X(t,x)$ and (c) $X_i(u(t))$ span $S(u(t))$.

PROOF. The implication (i) $\Rightarrow$ (ii) follows easily from Theorem 5 and Lemma 5.1. Assume (ii), and let $\phi(t) = (\exp X^t)^*(x)$.

If $y = u(t)$, we must show that $\phi(t).S(x) = S(y)$. By a compactness argument, it is sufficient to prove this for $|t| < \epsilon$. Put

$$V_j(t) = \phi(t)^{-1}.X_j(u(t)).$$

Then $V_j(t) \in T_xM$ and (using, for example, the formula (17.14.3.2) of [4])

$$V_j(t) = \phi(t)^{-1}.[X,X_j](u(t)) = \phi(t)^{-1}\sum_{i=1}^{p} \lambda_{ij}(t)X_i(u(t)) = \sum_{i=1}^{p} \lambda_{ij}(t)V_i(t).$$

Let $H : T_xM \to \mathbb{R}$ be a linear functional and let $h_i(t) = \langle H, V_i(t) \rangle$ and $h(t) = (h_1(t), h_2(t), \ldots, h_p(t)) \in \mathbb{R}^p$. Then $h(t) = \Lambda(t)h(t)$, where $\Lambda(t)$ is the $p \times p$ matrix with entries $\lambda_{ij}$. Thus $h(t) = 0$ if and only if $h(0) = 0$, and it follows that the vectors $V_1(t), V_2(t), \ldots, V_p(t)$ span the same subspace of $T_xM$ as the vectors $V_1(0), V_2(0), \ldots, V_p(0)$.

Since $V_i(0) = X_i(x)$ span $S(x)$ and $V_i(t) = \phi(t)^{-1}.V_i(u(t))$ span

*) It is sufficient to assume that $\lambda_{ij}$ are Lebesgue integrable.
\( \phi(t)^{-1}.S(y) \), we have \( S(x) = \phi(t)^{-1}.S(y) \), Q.E.D.

Let \( R^X \) denote the ring of germs at \( x \) of real-valued \( C^q \)-functions and let \( S^X \) denote the module over \( R^X \) generated by the vectorfields in \( S \). Following Lobry [6], we say that \( S \) is locally of finite type if, for every \( x \in M \), there exists a finite set \( F \subseteq S \) such that (i) \( F(x) = S(x) \) and (ii) \( [S,F]^X \subseteq F^X \). Consider the following conditions on \( S \) (where we assume for simplicity that \( q = \infty \) or \( \omega \)):

(H) \( \exp S \) is a homogeneous set of arrows;

(1) \( S \) is locally of finite type;

(2) \( S \) is closed under formation of the Lie bracket and, for every \( x \in M \), \( S^X \) is a finitely generated module over \( R^X \);

(3) \( S \) is closed under formation of the Lie bracket and \( \dim S(x) \) is locally constant on \( M \);

(4) \( S \) is closed under formation of the Lie bracket, \( q = \omega \), and the vectorfields in \( S \) are defined everywhere on \( M \).

(5) \( \dim S(x) \leq 1 \) for every \( x \in M \).

(6) \( S \) is closed under formation of the Lie bracket.

(7) \( \tilde{S} \) is locally of finite type and \( q \neq \omega \). Here \( \tilde{S} \) is the set of all \( C^\infty \) vectorfields \( X \) such that \( X(y) \in S(y) \) for every \( y \in \text{domain } X \).

**PROPOSITION 5.2.** The conditions (3), (4), (5) and (7) imply (H).

The assertions (5) \( \Rightarrow \) (H) and (3) \( \Rightarrow \) (H) are easily deduced from Theorem 6 and are left to the reader. The proof of (4) \( \Rightarrow \) (H) is given in Nagano's paper [7]; a simpler proof (of a slightly stronger result) using Theorem 6 is given in [12]. The only proof of (7) \( \Rightarrow \) (H) known to the author is given in [12].
It is claimed in [6] (and also in [10] and [11]) that $1 \Rightarrow (H)$, but this assertion is false [12].

REMARKS. (a) The examples below show that $(6) \not\Rightarrow (H)$, $(H) \not\Rightarrow (1)$ and $(5)$ and $(6) \not\Rightarrow (2)$.

(b) Note that $(3) \Rightarrow (H)$ in combination with Theorem 5 gives the classical Frobenius theorem and $(4) \Rightarrow (H)$ together with Theorem 5 give Nagano's theorem on integrability of real-analytic distributions with singularities [7]. Thus Theorems 5 and 6 and Proposition 5.2 taken together can be regarded as a generalization of the Frobenius theorem.

COROLLARY 5.2. Let $q = \infty$ and let $[S]$ denote the smallest set of vectorfields on $M$ which contains $S$ and is closed under formation of the Lie bracket. Let $L$ be an accessible set of $S$. Then $[S](x) \subseteq T_x L$ for every $x \in L$. If $\dim [S](x)$ is constant on $L$, then $[S](x) = T_x L$ for every $x \in L$.

PROOF. The first part follows at once from Lemma 5.1. Let $\dim [S](x)$ be constant on $L$. Without loss of generality, we may assume that $L = M$. Since $S \subseteq [S]$, $L$ is an accessible set of $[S]$. By Proposition 5.2, $[S]|_L$ is homogeneous, and so the assertion follows from the Corollary of Theorem 5.
EXAMPLE 5.3. Let \( \phi : R \to R \) be defined by \( \phi(x) = 0 \) for \( x \leq 0 \) and \( \phi(x) = e^{-1/x} \) for \( x > 0 \). Let \( M = R^3 \) and let \( S = \{X,Y\} \), where \( X = \frac{3}{3x} \) and \( Y = \frac{3}{3x} + \phi(x-1) \frac{3}{3y} + \phi(-1-x) \frac{3}{3z} \). It is easy to check that \( L = R^3 \) and that \( \dim [S](x,y,z) = \dim S(x,y,z) = 1 \) if \(-1 \leq x \leq 1 \) and 2 otherwise. In particular, \( \exp[S] \) is not homogeneous.

EXAMPLE 5.4. Let \( M = R^2 \) and let \( S \) consist of the vectorfield \( \frac{3}{3x} \) and all the vectorfields of the form \( \phi(y)e^{x\frac{3}{3y}} \), where \( \phi : R \to R \) is a differentiable function such that \( \phi(x) = P(1/x). \exp(-1/x^2) \) for some real polynomial \( P \) and all \( x \neq 0 \). If \( \mathcal{F} \) is the partition of \( R^2 \) into the accessible sets of \( S \), then clearly \( \mathcal{F} = \{ \text{upper half-plane, x-axis, lower half-plane} \} \) and \( S(x) = T_x(R^2, \mathcal{F}) \) for every \( x \in R^2 \). By the Corollary of Theorem 5, \( \exp S \) is homogeneous. We claim that \( S \) is not locally of finite type.

Indeed, assume that \( F = \{X_1,X_2,...,X_p\} \) satisfies the assumptions (i) and (ii) of Lobry's condition at the origin of \( R^2 \). It follows from (i) that \( \frac{3}{3x} \in \{X_1,X_2,...,X_p\} \), say \( \frac{3}{3x} = X_1 \). Let \( X_i(x,y) = \phi_i(y)e^{x\frac{3}{3y}} \), \( 2 \leq i \leq p \), and let \( X(x,y) = \phi(y)e^{x\frac{3}{3y}} \in S \). By (ii),

\[
\phi(y)e^{x\frac{3}{3y}} = [\frac{3}{3x}, \phi(y)e^{x\frac{3}{3y}}] = [X_1,X] = \lambda_1(x,y)\frac{3}{3x} + \sum_{i=2}^{p} \lambda_i(x,y)\phi_i(y)e^{x\frac{3}{3y}}
\]

for \( x \) and \( y \) sufficiently near the origin. Comparing the coefficients at \( \frac{3}{3y} \) and setting \( x = 0 \), we see that \( \phi(y) = \sum_{i=2}^{p} a_i(y)\phi_i(y) \). There exists an integer \( k > 0 \) such that, for \( 2 \leq i \leq p \), \( \phi_i(y).y^k. \exp(1/y^2) \) is continuous at \( y = 0 \), and so \( \phi(y).y^k. \exp(1/y^2) \) is also continuous. Setting \( \phi(y) = (1/y)^{k+1}. \exp(-1/y^2) \) for \( y \neq 0 \), we arrive at a contradiction.
EXAMPLE 5.5. Let $M = \mathbb{R}$ and let $S$ be the set of all vectorfields of the form $\phi(x)\frac{\partial}{\partial x}$, where $\phi$ is as in Example 5.4. Then $S$ satisfies the conditions (5) and (6) of Proposition 5.2, but the condition (2) breaks down at the origin.

§6. MULTIAROWS. A DIRECT PROOF OF CHOW'S THEOREM

6.1. Throughout this section, we assume that $q = \infty$ or $\omega$ and let $[S]$ denote the smallest set of vectorfields on $M$ which contains $S$ and is closed under formation of the Lie bracket. The following theorem follows from the results of Chow [2] and is usually referred to under his name ([5], [6], [8]).

THEOREM (Chow). Let $L$ be an accessible set of $S$ and let $x \in L$. If $\dim[S](x) = \dim M$, then $L$ is an open subset of $M$.

PROOF. This follows immediately from Theorem 5 and Lemma 5.1.

In this section we give an alternative proof of Chow's theorem, which is based on the concept of a multiarow (see 6.3).

6.2. Let $f : \mathbb{R}^k \to \mathbb{R}^n$ be a smooth function defined in a neighbourhood of the origin of $\mathbb{R}^k$. Let $x \in \mathbb{R}^n$ and assume that $f(t) = x$ whenever at least one component of $t = (t_1, t_2, \ldots, t_k)$ is zero. Clearly, $D_{p_1}^1 D_{p_2}^2 \ldots D_{p_k}^k f(0,0,\ldots,0) = 0$ whenever some $p_i = 0$. In particular,
where $\bar{D} = D_1D_2 \ldots D_k$, $\omega(t)$ is a symmetric $(k+1)$-linear mapping $(R^k)^{k+1} \rightarrow R^n$ which depends differentiably on $t$, and $t^{(k+1)} = (t, t, \ldots, t) \in (R^k)^{k+1} ([3], (8.14.3), (8.12.7))$. If $\phi : R^n \rightarrow R^m$ is a smooth function defined in a neighbourhood of $x$, it is easily checked that

$$D(\phi \circ f)(0) = \phi(0) \cdot Df(0).$$

In particular, $\bar{D}f(0)$ is a well-defined vector in $T_xM$ if $R^n$ above is replaced by a smooth manifold $M$.

Note also that $(t_2, t_3, \ldots, t_k) \cdot D^1f(0, t_2, t_3, \ldots, t_k)$ is a smooth function from $R^{k-1}$ into the vector space $T_xM$ and that $\bar{D}f(0)$ is its $(k-1)^{st}$ mixed partial derivative. As

$$D^1f(0, t_2, t_3, \ldots, t_k) = 0$$

whenever one of the components of $(t_2, t_3, \ldots, t_k)$ is zero, the above arguments show that

$$D^1f(0, t_2, t_3, \ldots, t_k) = t_2t_3 \ldots t_ka(t),$$

where $a(t) = Df(0)$ as $\tilde{t} + \sigma$.

6.3 A smooth function $a : R^k \times M \rightarrow M$ is a multiarrow of order $k$ if

(a) $a$ is defined on an open neighbourhood of $0 \times M$;

(b) for every $t$ in $R^k$, $a^t = a(t,-)$ is a diffeomorphism of an open subset of $M$ onto an open subset of $M$;

(c) $a(t, x) = x$ whenever $(t, x)$ belongs to the domain of $a$ and at least one component of $t = (t_1, t_2, \ldots, t_k)$ is zero.
If \( a(t,x) = y \), \( 1 \leq i \leq k \), we write \( D_i a(t,x) \in T y \) for the partial derivative of the function \( a(\cdot,x) : \mathbb{R}^k + M \) and \( a^*(t,x) : T_x M \to T_y M \) for the differential of \( a^t = a(t,\cdot) : M + M \). It follows from (6.2) that \( \bar{D}a(0,x) = D_1 D_2 \ldots D_k a(0,0,\ldots,0,x) \) is a well-defined element of \( T_x M \) and that, for \( \bar{t} = (t_2, t_3, \ldots, t_k) \in \mathbb{R}^{k-1} \), \( t_2 t_3 \ldots t_k \neq 0 \),

\[
D_1 a(0,\bar{t},x) = t_2 t_3 \ldots t_k X(\bar{t},x), \quad \lim_{\bar{t} \to 0} X(\bar{t},x) = \bar{D}a(0,x).
\]

We write \( A_k \) for the set of all \( k \)-multiarrows on \( M \) and put

\[ A = \bigcup_{k=1}^{\infty} A_k. \]

If \( a \in A \), \( \bar{D}a \) denotes the smooth vectorfield \( M \to TM : x \mapsto \bar{D}a(0,x) \). The bracket of \( a \in A_k \) and \( b \in A_\ell \) is the \( k+\ell \)-multiarrow \([a,b]\) defined by

\[
[a,b](t,s,x) = (b^s)^{-1} \circ (a^t)^{-1} \circ (b^s) \circ (a^t)(x).
\]

**LEMMA 6.3.**

\[
\bar{D}[a,b] = [\bar{D}a, \bar{D}b].
\]

**PROOF.** Let \( f(t,x) = (a^t)^{-1}(x) \) and \( g(s,x) = (b^s)^{-1}(x) \). Fixing \( x \in M \) and the local coordinates, we assume that \( M = \mathbb{R}^n \). Let

\[
\phi(t,s) = [a,b](t,s,x) = g(s,f(t,b(s,a(t,x))))
\]

so that

\[
\bar{D}[a,b](x) = \bar{D}_x \bar{D}_s \phi(0,0).
\]

By (6.2.(b)),

\[
\bar{D}_s \phi(t,0) = \bar{D}_s g(0,f(t,a(t,x))) + g^*(0,f(t,a(t,x))).f^*(t,a(t,x)) \bar{D}_s b(0,a(t,x)) = \bar{D}_s g(0,x) + f^*(t,a(t,x)) \bar{D}_s b(a(t,x)).
\]

If \( \psi(t) = f^*(t,a(t,x)) \) and \( \bar{t} = (t_2,\ldots,t_k) \), then

\[
D_1 \psi(0,\bar{t}) = D_1 f^*(0,\bar{t},x) + f^{**}(0,\bar{t},x).D_1 a(0,\bar{t},x)
\]

\[
= D_1 f^*(0,\bar{t},x)
\]
because \( f^*(O, \bar{t}, x) = \text{id} \) and so \( f^{**}(O, \bar{t}, x) = 0 \). Hence \( Df(0) = Df^*(O, x) \) and

\[
\begin{align*}
D_t D_{\bar{s}} f(O, O) &= (Df)^*(O, x). Db(x) + f^*(O, x)(D\bar{b})^*(x). Da(x) = \\
&= (D\bar{b})^*(x). Da(x) + (Df)^*(O, x). Db(x).
\end{align*}
\]

Differentiating the equation \( f(t, a(t, x)) = x \), we immediately see that \( Df(0, x) = -Da(x) \), and so (cf. [4], 17.14.3.2)

\[
\begin{align*}
D_t D_{\bar{s}} f(O, O) &= (D\bar{b})^*(x). Da(x) - (Da)^*(x) Db(x) \\
&= \{Da, Db\}(x),
\end{align*}
\]

Q.E.D.

6.4 Let \( \sim \) be an equivalence relation on \( M \). We say that a
multiarrow \( a \) preserves \( \sim \) if \( a(t, x) \sim x \) for all \((t, x)\) in the domain
of \( a \). Let \( \tilde{A} \) denote the set of all the multiarrows on \( M \) which
preserve \( \sim \).

**Lemma 6.4.1.** Let \( V = DA \) be the collection of all the
vectorfields of the form \( Da, a \in \tilde{A} \). Then \( V \) is closed under
formation of the Lie bracket.

**Proof.** This follows at once from Lemma 6.3

**Lemma 6.4.2.** If \( \dim V(x) = d \), then there exists an immersion
\( \psi: R^d \to M \) such that

(a) \( \psi \) is defined on a neighbourhood of \( 0 \in R^d \) and \( \psi(0) = x \).

(b) \( \psi(t) \sim x \) for every \( t \) in the domain \( \psi \).

**Proof.** Choose \( a_i \in \tilde{A} \) so that the vectors \( \tilde{D}a_1(x), \tilde{D}a_2(x), \ldots, \tilde{D}a_d(x) \)
are linearly independent. It follows from (6.3.(d)) that there
exist \( \lambda_i \in R \) such that the vectors \( D_j a_i(0, \lambda_i, x) \) are also linearly
independent. Consider the arrows \( b_i \) defined by \( b_i^t = a_i(t, \lambda_i, -) \)
and let $\psi(t_1, t_2, \ldots, t_d) = b_1^{t_1} \circ b_2^{t_2} \circ \ldots \circ b_d^{t_d}(x)$. It is clear that
\psi satisfiess the conditions (a) and (b) of the lemma and that
$D_i\psi(0) = D_i a_i(0, \lambda_i, x)$, so that $\psi$ restricts to an immersion on a
sufficiently small neighbourhood of the origin.

**PROOF OF CHOW'S THEOREM.** Let $\sim$ be the relation
\[ x \sim y \pmod{\exp S}. \]
It is clear that $S \subseteq \overline{V}$ and so, by Lemma 6.4.1,
$[S] \subseteq \overline{V}$. Hence $\dim \overline{V}(x) = \dim M$ and Lemma 6.4.2 implies that there
is an open set $U$ in $M$ such that $x \in U \subseteq L$. If $y \in L$, then
\[ y = \phi(x) \] for some $\phi \in \mathcal{V}S$. Let $W$ be a neighbourhood of $x$ contained
in $U \cap \text{domain}(\phi)$. Then $y \in \phi(W) \subseteq L$, which proves the assertion of
the Theorem.
PART TWO

Integrability of singular distributions on
infinite-dimensional manifolds
Let $M$ be a $C^q$ Banach manifold, where $2 \leq q \leq \omega$. To simplify the notation, we assume that $M$ is modelled on a single Banach space $E$. The word differentiable refers to a fixed class $C^r$, where $1 \leq r \leq q - 1$ and we take $\infty = \infty - 1$ and $\omega = \omega - 1$.

A distribution $B$ on $M$ is a family $(B_x : x \in M)$, where each $B_x$ is a topological direct summand of the tangent space $T_xM$. $B$ is regular if it defines a differentiable subbundle of the tangent bundle $TM$, otherwise $B$ is singular. If $B$ is singular, then $B_x$ need not be isomorphic to $B_y$ even if $x$ and $y$ lie in the same connected component of $M$.

By an immersion we always mean a split immersion, so that a differentiable function $f : N \to M$ is an immersion if and only if, for every $x$ in $N$, the differential $f^*(x)$ is an isomorphism of $T_xN$ onto a topological direct summand of $T_{f(x)}M$.

An immersed submanifold of $M$ is a subset $L$ of $M$ together with a differentiable structure $\sigma$ on $L$ such that the inclusion mapping of $L$ into $M$ is an immersion. We identify the tangent space $T_{x}(L,\sigma)$ at $x \in L$ with the corresponding subspace of $T_xM$. $L$ is an integral manifold of the distribution $B$ if $T_{x}(L,\sigma) = B_x$ for every $x \in L$.

We say that $B$ is an integrable distribution if there exists a differentiable structure $\sigma$ on $M$ such that $(M,\sigma)$ is an integral manifold of $B$. 

(0.1)

(0.2)

(0.3)
In §§4 and 5 we define the differentiability of (possibly singular) distributions and show that it can be described in terms of vector-valued one-forms on $M$.

Our main results are collected in §7, where we show that a differentiable (possibly singular) distribution is integrable if and only if it is homogeneous (0.6) and give some other necessary and sufficient conditions of integrability. We also prove that the differentiable structure $\sigma$ which makes $M$ into an integral manifold of $B$ is unique, that every integral manifold of $B$ is an open submanifold of $(M,\sigma)$ and that $\sigma$ is a foliation with singularities as defined in §2.

A differentiable vectorfield $X$, defined on an open subset of $M$, is said to lie in $B$ of $X(x) \in B_x$ for every $x$ in the domain of $X$. We say that a vectorfield $X$ (which does not necessarily lie in $B$) respects $B$ if

$$(X^t)^*(x).B_x = B_y$$

whenever $X^t.x = y$. $B$ is said to be homogeneous if every vectorfield in $B$ respects $B$.

In §8 we give some necessary and sufficient conditions that $X$ respect $B$, formulated in terms of Lie brackets, and deduce the corresponding conditions for the homogeneity (and so integrability) of $B$, which generalise Theorem 6 of Part One. In particular, we recover the standard Frobenius theorem (SFT) on the integrability of regular distributions, as stated, for example, in [13] or [16].
We also prove that a real analytic (possibly singular) distribution is integrable if and only if it is involutive and locally everywhere defined.

Finally, in §9, we introduce the concept of a neat leaf and discuss a related unsolved problem.

Just as is the case with SFT, the proofs of our results are fairly simple and have a 'coordinate free', rather than a true 'functional-analytic' flavour. I hope that they will pave the way for some future 'hard' theorems.
§1. TOPOLOGICAL DIRECT SUMMANDS

1.1 Given two topological vector spaces $E$ and $F$ (over $\mathbb{R}$), $L(E,F)$ denotes the vector space of all continuous linear mappings $E \rightarrow F$. If $E$ and $F$ are normed (or normable), we shall always think of $L(E,F)$ as a normed (normable) space with the usual 'sup over the unit ball of $E$' norm (or the corresponding topology). LIS$(E,F)$ denotes the subspace of $L(E,F)$ consisting of the top linear isomorphisms, and we write End$(E)$ and GL$(E)$ instead of $L(E,E)$ and LIS$(E,E)$.

Recall that a vector subspace $F$ of a Hausdorff topological vector space $E$ is a **direct summand** of $E$ if it satisfies one of the following equivalent conditions:

1. there exists a topological vector space $G$ and a top linear isomorphism $\alpha : E \rightarrow F \times G$ such that

   $\text{id}_F = p \circ \alpha \circ i,$

   where $i : F \rightarrow E$ is the inclusion and $p : F \times G \rightarrow F$ is the coordinate projection;

2. there exists a subspace $G$ of $E$ such that the mapping $F \times G \rightarrow E$:

   $(x,y) \rightarrow x+y$ is a top linear isomorphism;

3. there exists a continuous linear projection $P \in \text{End}(E)$ such that $F = \ker P$;

4. there exists a continuous linear projection $Q \in \text{End}(E)$ such that $F = \text{Im} Q.$
It is easy to check that a closed, finite codimensional subspace of $E$ is always a direct summand.

If $E$ is locally convex, then, by the Hahn-Banach theorem, every finite-dimensional subspace of $E$ is a direct summand.

If $E$ is a Fréchet space, then, by the closed graph theorem, $F$ is a direct summand of $E$ if and only if

\[(5) \ F \text{ is closed and there exists a closed subspace } G \text{ of } E \text{ such that } F \cap G = 0 \text{ and } F + G = E.\]

1.2 Let $\text{GL}(E|F)$ denote the subset of $\text{GL}(E)$ consisting of those topolinar automorphisms of $E$ which map $F$ into itself, and let $\text{GL}_0(E|F)$ consist of those members of $\text{GL}(E|F)$ that restrict to an automorphism of $F$.

**PROPOSITION 1.2.** Let $E$ be a Banach space and let $F$ be a direct summand of $E$. Then $\text{GL}_0(E|F)$ is open and closed in $\text{GL}(E|F)$. If $F$ is finite dimensional or finite codimensional, then $\text{GL}_0(E|F)$ coincides with $\text{GL}(E|F)$.

**PROOF.** Let $F_1 = F$ and let $F_2$ be some topological complement of $F$. Let $i_k$ be the inclusion mapping $F_k + E$ and let $p_1$ denote the projection of $E$ onto $F_1$ along $F_2$, and $p_2$ the complementary projection of $E$ onto $F_2$. An operator $a$ in $\text{End}(E)$ is represented by a matrix.
where \( a_{k\ell} = p_k a_{i\ell} \) belongs to \( L(F_\ell, F_k) \). It is clear that \( GL_0(E|F) \) is open in \( GL(E|F) \), being an inverse image of the open subset \( GL(F) \) of \( \text{End}(F) \) under the continuous mapping \( a \rightarrow a_{11} \). We prove that \( GL_0(E|F) \) is closed in \( GL(E|F) \) by showing that it is the inverse image of zero under the mapping

\[
GL(E|F) \rightarrow L(F_1, F_2) : a \rightarrow p_2 \circ (a)^{-1} \circ i_1.
\]

To check this, let

\[
a = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in GL(E|F)
\]

and

\[
a^{-1} = b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.
\]

From \( ab = ba = \text{id}_E \) we obtain

\[
\begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} = \begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} & b_{21}a_{12} + b_{22}a_{22} \end{pmatrix} = \begin{pmatrix} \text{id}_{F_1} & 0 \\ 0 & \text{id}_{F_2} \end{pmatrix}.
\]
If $b_{21} = 0$, then $a_{11}b_{11} = b_{11}a_{11} = \text{id}_{F_1}$, and $a$ belongs to $\text{GL}_0(E|F)$. If $a \in \text{GL}_0(E|F)$, then $a_{11}$ is surjective, and so $b_{21}a_{11} = 0$ implies $b_{21} = 0$.

If $F_1$ is finite co-dimensional, then $F_2$ is finite-dimensional and $a_{22}b_{22} = \text{id}_{F_2}$ implies that $a_{22}$ is injective. Since $a_{22}b_{21} = 0$, we again obtain $b_{21} = 0$ and so $a \in \text{GL}_0(E|F)$.

§2. BOXES, SLICES AND FOLIATIONS

If $E$ is a normed space, we put $E^\epsilon = \{ x \in E : \|x\| < \epsilon \}$, so that $E^1$ is the open unit ball of $E$. A $C^r$ box of the manifold $M$ is a triple $(\psi, U, W)$, where $U$ and $W$ are Banach spaces and $\psi$ is a $C^r$ diffeomorphism of $U^1 \times W^1$ onto an open subset of $M$. By a slice of $(\psi, U, W)$ we mean any of the mappings

$$\psi(-, w) : U^1 \to M, \ w \in W^1.$$  

A differentiable structure $\sigma$ on the underlying set of $M$ is a $C^r$ foliation of $M$ if, given $x \in M$, there exists a $C^r$ box $(\psi, U, W)$ of $M$ such that

(1) $\psi(0, 0) = x$;

(2) $\psi(-, 0) : U^1 \to M$ is a chart of $\sigma$; and

(3) every slice of $(\psi, U, W)$ is a differentiable function of $U^1$ into $(M, \sigma)$.  

(Here and below, \((M,\sigma)\) denotes the underlying set of \(M\) equipped with the differentiable structure \(\sigma\).) A foliation \(\sigma\) is regular if, given \(x\) in \(M\), there exists a \(C^r\) box \((\psi, U, W)\) of \(M\) such that 
\[
\psi(0,0) = x
\]
and

(4) every slice of \((\psi, U, W)\) is a chart of \(\sigma\).

A foliation \(\sigma\) which fails to be regular is said to be singular.

More generally, let \((\psi, U, W)\) be a differentiable box of \(M\) and let \(N\) be an arbitrary immersed submanifold of \(M\). We say that \((\psi, U, W)\) cuts \(N\) in open slices if

(a) \(N_w = \psi(-,w)^{-1}(N)\) is an open subset of \(U\) for every \(w \in W\);

and

(b) \(\psi(-,w) : N_w \to N\) is differentiable for every \(w \in W\).

We say that the box \((\psi, U, W)\) is parallel to \(N\) if the tangent spaces to the slices are contained in the tangent spaces of \(N\) at every point of intersection, that is if

\[
D_{\psi(u,w)} U \subset T_{\psi(u,w)} N
\]

whenever \(\psi(u,w) \in N\).

**Proposition 2.1.** A box of \(M\) cuts an immersed submanifold \(N\) of \(M\) in open slices if and only if it is parallel to it.

**Proof.** Assume that \((\psi, U, W)\) is parallel to \(N\) and let \(H\) be a neighbourhood of \(\bar{h} = \psi(\bar{u}, \bar{w})\) in \(N\). It is sufficient to show that \(\psi(\bar{u} + u, \bar{w}) \in H\) for \(u\) in some neighbourhood \(U^c\) of the origin in \(U\), for this implies that \(N_w\) is an open subset of \(U\) and that
\( \psi(\cdot, \bar{w}) : N_{\bar{w}} \to N \) is continuous. The differentiability of this function then follows from the fact that \( \psi(\cdot, \bar{w}) : U^1 \to M \) is differentiable and \( N \) is an immersed submanifold of \( M \).

Taking a suitable local chart, we may identify a neighbourhood of \( \bar{h} \) in \( M \) with an open subset \( \Omega \) of \( E \) and assume that \( \bar{h} \) is the origin of \( E \) and that \( H \) is the open unit ball \( F^1 \) of some direct summand \( F \) of \( E \).

We may also assume that \((\bar{u}, \bar{w})\) is the origin of \( U \times W \) and that \( \psi(u, w) \in \Omega \) for all \((u, w)\) in \( U^1 \times W^1 \). Let
\[
a = D\psi(0, 0) \in \text{LIS}(U \times W, E).
\]

Since \((\psi, U, W)\) is parallel to \( N \), we have

1. \( D_1 \psi(u, w).U \subset F \) whenever \( \psi(u, w) \in H \), and in particular
2. \( a = aj_wp_w \)

where \( j_w \) is the inclusion \( W \to U \times W : w \to (0, w) \) and \( p_w \) is the coordinate projection.

Let now \( G \) be a complement of \( F \) in \( E \), \( p_G \in \text{End}(E) \) the projection onto \( G \) along \( F \) and \( p_F \in \text{End}(E) \) the complementary projection onto \( F \). Put
\[
\pi_G = p_wa^{-1}p_Gaj_w \in \text{End}(W)
\]
and
\[
\pi_F = p_wa^{-1}p_Faj_w \in \text{End}(W).
\]

Using (2), it is easily checked that
\[
\pi_G^2 = \pi_G, \quad \pi_F^2 = \pi_F \quad \text{and} \quad \pi_G + \pi_F = \text{id}_W.
\]
Hence \( \pi_G \) and \( \pi_F \) are complementary projections and \( W \) is the direct sum of

\[
V = \pi_F W \quad \text{and} \quad Z = \pi_G W.
\]

Moreover, it is easy to show that

\[
(3) \quad b = p_G \circ j_z \in \text{LIS}(Z,G),
\]

where \( j_z \) is the inclusion \( Z \to U \times W: z \to (0, z) \) and \( p_G \) is now considered as a map \( E \to G \).

Let now

\[
\theta : U \times V \times Z \to G
\]

be defined by

\[
\theta(u,v,z) = p_G \psi(u,v+z).
\]

Then

\[
(4) \quad \theta(u,v,z) = 0 \text{ if and only if } \psi(u,v+z) \in H, \text{ and so, by (1)},
\]

\[
(5) \quad \theta(u,v,z) = 0 \text{ implies } D_1 \theta(u,v,z) = 0.
\]

Since

\[
(6) \quad D_3 \theta(0,0,0) = b \in \text{LIS}(Z,G), \text{ if follows from the implicit function theorem ([3], 10.2.1) that there exists }\epsilon > 0, \delta > 0, \text{ and}
\]

a differentiable function

\[
\phi : U^\epsilon \times V^\epsilon \to Z^\delta
\]

such that, for every \( (u,v,z) \) in \( U^\epsilon \times V^\epsilon \times Z^\delta \),

\[
(7) \quad D_3 \theta(u,v,z) \in \text{LIS}(Z,G)
\]

and

\[
(8) \quad \theta(u,v,z) = 0 \text{ if and only if } z = \phi(u,v).
\]

Thus \( \theta(u,v,\phi(u,v)) = 0 \) for all \( (u,v) \) in \( U^\epsilon \times V^\epsilon \). Differentiating by \( u \), we obtain
and so, by (5), (7) and (8),

\[ D_1 \theta(u,v,\phi(u,v)) + D_3 \theta(u,v,\phi(u,v)).D_1 \phi(u,v) = 0 \]

\[ D_1 \phi(u,v) = 0 \]

for every \((u,v)\) in \(U^e \times V^e\). Hence

\[ \phi(u,v) = \phi(0,v), \]

and it follows from (8) and (4) that

\[ \psi(u,v + \phi(0,v)) \in H \]

for every \((u,v)\) in \(U^e \times V^e\). Since \(\phi(0,0) = 0\), we have

\[ \psi(u,0) \in H \]

for every \(u\) in \(V^e\), Q.E.D.

REMARKS. 1) A shorter proof, based on the existence theorem for ordinary differential equations, can be given if \(\psi\) and \(N\) are at least of the order \(C^2\), or if \(N\) is finite-dimensional. If \(\psi\) or \(N\) are of order \(C^1\), then the right-hand sides of the corresponding differential equations are generally only continuous, and if \(N\) is infinite-dimensional, then the existence theorem no longer applies.

2) A cautionary example against relaxing the assumptions of Proposition 2.1 is given by \(M = \mathbb{R}^2\), \(N = \mathbb{R} \times 0\), and

\[ \psi(u,w) = (u,(u-w)^3) \]

(see Fig. 1).
§3. $C^1$ SUBMANIFOLDS AND FLOWS

PROPOSITION 3.1. Let $N$ be a $C^1$ immersed submanifold of $M$ and let $X$ be a $C^1$ vectorfield on $M$ such that $X(x) \in T_xN$ for every $x$ in $N$. Let

$$\Delta = \{(t,x) \in \mathbb{R} \times N : X^t.x \in N \text{ and } X^s.x \in N \text{ for all } s \text{ between } 0 \text{ and } t\}.$$ 

Then $\Delta$ is open in $\mathbb{R} \times N$ and the function

$$\Delta \to N : (t,x) \mapsto X^t.x$$

is differentiable.
REMARK. We cannot use the usual existence, uniqueness and 'dependence on initial conditions' theorems because the 'restriction' of \(X\) to \(N\) is generally only a \(C^0\) vector field. The proof is similar to the proof of Proposition 2.1 and is given here only for the sake of completeness.

**Lemma 3.2.** Assume that a \(C^1\) function \(\phi: \mathbb{R} \times M \to M\) satisfies the following properties:

(i) the domain of \(\phi\) is an open neighbourhood of \(0 \times M\) and 
   \(\phi(0,x) = x\) for every \(x \in M\);

(ii) \(D_1 \phi(t,x) \in T_y N\) whenever \(\phi(t,x) = y \in N\).

If \(x \in N\), then there exists a neighbourhood \(H\) of \(x\) in \(N\) and \(\varepsilon > 0\) such that

(a) \(\phi(t,y) \in N\) for every \((t,y) \in \mathbb{R}^\varepsilon \times H\) and

(b) \(\phi: \mathbb{R}^\varepsilon \times H \to N\) is differentiable.

**Proof.** Using a suitable chart to identify a neighbourhood of \(x\) in \(M\) with an open subset \(\Omega\) of \(E\), we may assume that \(x\) is the origin of \(E\) and that there exist two complementary closed subspaces \(F\) and \(G\) of \(E\) such that \(\Omega = F^1 + G^1\) and \(F^1\) is a neighbourhood of \(x\) in \(N\).

Let \(p_F\) be the projection of \(E\) onto \(F\) along \(G\) and let \(p_G\) be the complementary projection of \(E\) onto \(G\). Since \(\phi(t,y) \in F^1\) implies \(\phi(t,y) = p_F \phi(t,y)\) and \(p_F \circ \phi\) is a differentiable function into \(F\), it is sufficient to find \(\varepsilon > 0\) such that \(\phi(t,y) \in F^1\) for every \((t,y) \in \mathbb{R}^\varepsilon \times F^\varepsilon\).
If \( h : R \times F \times G \) is defined by \( h(t, \xi, \eta) = p_G \phi(t, \xi + \eta) \), then, by (ii),

(1) \( D_1 h(t, \xi, \eta) = 0 \) whenever \( h(t, \xi, \eta) = 0 \).

Since

(2) \( D_3 h(0, \xi, \eta) = p_G \circ D_2 \psi(0, \xi + \eta) \circ i_G = p_G \circ i_G = id_G \), (where \( i_G \) is the inclusion \( G \hookrightarrow E : \eta \mapsto (0, \eta) \)) and since \( h(0, 0, 0) = 0 \), the inverse function theorem implies the existence of \( \varepsilon > 0, \delta > 0 \) and a \( C^1 \) function \( \theta : R^E \times F^E \rightarrow G^\delta \) such that, for all \( (t, \xi, \eta) \in R^E \times F^E \times G^\delta \),

(3) \( D_3 h(t, \xi, \eta) \in GL(G) \) and

(4) \( h(t, \xi, \eta) = 0 \) is equivalent to \( \eta = \theta(t, \xi) \).

Differentiating by \( t \) the equation \( h(t, \xi, \theta(t, \xi)) = 0 \), and using (1) and (3), we deduce that

\( D_1 \theta(t, \xi) = 0 \)

and so, for \( (t, \xi) \in R^E \times F^E \),

\( \theta(t, \xi) = \theta(0, \xi) \).

Hence \( \theta(t, \xi) = p_G \phi(0, \xi + \theta(t, \xi)) = h(0, \xi, \theta(t, \xi)) = h(0, \xi, \theta(0, \xi)) = 0 \),

and therefore

\( h(t, \xi, 0) = h(t, \xi, \theta(t, \xi)) = 0 \),

which proves that \( \phi(t, \xi) \in F \) for every \( (t, \xi) \in R^E \times F^E \), Q.E.D.

PROOF OF PROPOSITION 3.1. Let \( (t, x) \in \Delta \) and assume, for example, that \( t \geq 0 \). Assume that \( \tau \in [0, t] \) has the following property:

(5) there exists a neighbourhood \( H \) of \( x \) in \( N \) and \( \varepsilon > 0 \) such that

\( \phi((t-\varepsilon, t+\varepsilon) \times H) \subset N \) and \( \phi : [t-\varepsilon, t+\varepsilon] \times H \rightarrow N \) is differentiable.

Using Lemma 3.2 and the equation \( \phi(t+s, y) = \phi(\delta, \phi(t+s-\delta, y)) \), it is easy to show that the set \( I_0 \) of those \( \tau \in [0, t] \) which satisfy (5) is open and closed in \( I = [0, t] \) and contains 0. Hence \( I_0 = I \), Q.E.D.
§4. DIFFERENTIABLE DISTRIBUTIONS

We say that the distribution $B$ is differentiable at a point $x \in M$, if there exists a differentiable section $f$ of the bundle $\text{LIS}(T_x M, TM)$, defined on a neighbourhood $\Omega$ of $x$, such that $f(x)B_x = B_x$ and $f(y)B_x \subset B_y$ for every $y \in \Omega$. We call such a section a stem of $B$ at $x$ and we usually assume that $f(x) = \text{id}_{T_x M}$.

A stem $f$ can also be described as a vector-bundle isomorphism

$$\Omega \times T_x M \rightarrow TM|\Omega$$

which is the identity on $(x) \times T_x M$ and maps $\Omega \times B_x$ into $B|\Omega$. The stem $f$ is said to be regular if $f(y)B_x = B_y$ for every $y \in \Omega$.

Locally\(^*)\), $f$ is represented by a differentiable function

$$\Omega \rightarrow \text{GL}(E).$$

There is a weaker definition of differentiability, where we only ask for the function

$$\Omega \times T_x M \rightarrow TM : (y,v) \rightarrow f(y).v$$

to be differentiable (implying that $f$ is $C^{r-1}$). Such $f$ is called a weak stem of $B$ and $B$ is then said to be weakly differentiable at $x$.

(4.1) The distribution $B$ is differentiable (or weakly differentiable) if it is differentiable (weakly differentiable) at every $x \in M$. $B$ is regular (0.1) if and only if it has a regular differentiable stem at every $x \in M$.

*\) This word always signals that we are using some local chart to identify a neighbourhood of a point in $M$ with an open subset of $E$, tangent spaces with $E$, vectorfields with their principal parts,...
If $B_x$ is finite dimensional, then $B$ is differentiable at $x$ if and only if there exist differentiable vector fields $X_1, X_2, \ldots, X_k$ in $B$ whose values at $x$ form a basis of $B_x$.

To see this, assume that $TM|_\Omega = \Omega \times E$, and put,

$$g(y).v = \ell_1(v)X_1(y) + \ell_2(v)X_2(y) + \ldots + \ell_k(v)X_k(y),$$

where $\ell_i : E \to \mathbb{R}$ are continuous linear functionals such that

$$\ell_i(x_j(x)) = \delta_{ij}.$$  

It is then sufficient to put

$$f(y) = \text{id}_E + g(y) - g(x)$$

and to note that $f(y) \in \text{GL}(E)$ for $y$ sufficiently near to $x$.

If $\text{codim} B_x < \infty$, then, as we show in the next section, the differentiability of $B$ at $x$ can be described in terms of finitely many real-valued differentiable one-forms.

§5. DIFFERENTIABILITY AND VECTOR-VALUED ONE-FORMS

We recall that, given a Banach space $F$, an $F$-valued one-form on $M$ is a differentiable section of the vector bundle

$$L(TM,F) = \bigcup_{x \in M} L(T_x M,F).$$

Locally, such a form is represented by a $C^r$-mapping $\Omega \to L(E,F)$ (cf. [13], 8.3.1).
PROPOSITION 5.1. A distribution $B$ is differentiable at $x$ if and only if there exists a Banach space $F$ and $F$-valued differentiable one-form $\omega$, defined on a neighbourhood $\Omega$ of $x$, such that

(a) $\omega(x) : T_x M \rightarrow F$ is surjective and $\ker \omega(x) = B_x$; and

(b) for every $y$ in $\Omega$, $\ker \omega(y) \subset B_y$.

PROOF. We may assume that $TM|_\Omega = \Omega \times E$. Let $G$ be a complement of $B_x$ in $E$, and let $j : G \rightarrow E$ be the inclusion mapping. By the closed graph theorem, $\omega(x) \circ j \in \text{LIS}(G,F)$, and, taking $\Omega$ sufficiently small, we may assume that $\omega(y) \circ j \in \text{LIS}(G,F)$ for every $y \in \Omega$. Let

$$\gamma(y) = j \circ (\omega(y) \circ j)^{-1} : F \rightarrow E$$

and put

$$p(y) = \gamma(y) \omega(y) : E \rightarrow E.$$  

Since $\omega(y) \gamma(y) = \text{id}_F$, we have $\omega(y)p(y) = \omega(y)$ and $p^2(y) = p(y)$. Put

$$f(y) = \text{id}_E + p(x) - p(y).$$

Taking $\Omega$ smaller if necessary, we may assume that $f(y) \in \text{GL}(E)$ for every $y$ in $\Omega$. If $g \in G$, then $p(x)g = g$ and

$$\omega(y)f(y)g = 2\omega(y)g - \omega(y)p(y)g = \omega(y)g.$$  

If $b \in B_x$, then $p(x)b = 0$ and

$$\omega(y)f(y)b = \omega(y)b - \omega(y)p(y)b = 0.$$  

Hence

$$\omega(y)f(y)(b+g) = \omega(y)g = 0$$  

if and only if $g = 0$,

and therefore

$$f(y)B_x = \ker \omega(y) \subset B_y,$$

so that $f$ is a differentiable stem*) of $B$.

*) Note that $f$ is regular if $\ker \omega(y) = B_y$ for every $y$ in $\Omega$.  

Conversely, given a stem $f$ of $B$ at $x$, it is sufficient to put

$$F = p(T_x M) \text{ and } \omega(y) = p \circ (f(y))^{-1},$$

where $p \in \text{End}(T_x M)$ is a projection and $B_x = \text{Ker} p$.

**COROLLARY 5.1.** Suppose that $\text{codim} B_x = k < \infty$. Then $B$ is differentiable at $x$ if and only if there exist differentiable real-valued one-forms $\omega_1, \omega_2, \ldots, \omega_k$ on a neighbourhood $\Omega$ of $x$ such that

(a) $\omega_1(x), \omega_2(x), \ldots, \omega_k(x)$ is a basis of the annihilator $B_x^0$ of $B_x$ in $T_x^* M$; and,

(b) for every $y \in \Omega$, $B_y^0$ is contained in the span of $\omega_1(y), \omega_2(y), \ldots, \omega_k(y)$.

**PROOF.** Take $F = \mathbb{R}^k$ and $\omega = (\omega_1, \omega_2, \ldots, \omega_k)$.

---

### §6. HOMOGENEOUS STEMS AND LOCAL INTEGRABILITY

Throughout this section we assume that $f$ is a weak stem ($\S 4$) of the distribution $B$ at a point $x \in M$. We say that $f$ is **homogeneous** if the vectorfields

$$y + f(y) \cdot v, \quad v \in B_x$$

respect the distribution $B$ (see Introduction, (0.5)).

A differentiable box $(\psi, U, W)$ is said to be **parallel** to $B$ if

$$D_1 \psi(u, w). U \subset \psi(u, w)$$

for every $(u, w) \in U^1 \times W^1$. Recall ($\S 2$) that a **slice** of $(\psi, U, W)$ is
any of the functions\[\psi(\cdot, w) : U^1 \to M, \ w \in W^1.\]

By abuse of language, the set \(\psi(U^1, w)\), together with its differentiable structure of an embedded submanifold of \(M\), is also called a slice of \(\psi\).

**PROPOSITION 6.1.** Let \(f\) be a homogeneous weak stem of \(B\) at the point \(x\). There exists a differentiable box \((\psi, U, W)\) of \(M\) such that

1. \(\psi(0, 0) = x;\)
2. \((\psi, U, W)\) is parallel to \(B;\)
3. the slice \(\psi(U^1, 0)\) of \(\psi\) is an integral manifold of \(B;\) and
4. every point of \(\psi(U^1, 0)\) can be reached from \(x\) along an integral curve of a vectorfield in \(B\).

**COROLLARY 6.1.** Let \(N\) be an integral manifold of \(B\). Then

5. \((\psi, U, W)\) cuts \(N\) in open slices and
6. if \(Z_x = \psi(U^1, 0)\), then \(Z_x \cap N\) is an open subset of both \(Z_x\) and \(N\) and inherits the same differentiable structure from \(Z_x\) and \(N\).

**PROOF OF THE COROLLARY.** The assertion (5) follows from (2) above and from Proposition 2.1. The assertion (6) follows at once from (5), (3), and the inverse function theorem.
PROOF OF PROPOSITION 6.1. We may assume that the domain $\Omega$ is an open subset of $E$ and that

$$f : \Omega \to \text{GL}(E),$$

where the function

$$\Omega \times E \to E : (y,v) \mapsto f(y).v$$

is differentiable. Let

$$\xi(t,y,u)$$

be the value at $t$ of the maximal solution of the equations

$$(6.1.1) \quad \dot{\xi} = f(\xi)u, \quad \xi(0) = y.$$

Then

\begin{enumerate}
  \item[(A)] $\text{dom}\xi$ is an open neighbourhood of $(0,x,0)$ in $\mathbb{R} \times \Omega \times E$;
  \item[(B)] $\xi : \text{dom}\xi \to \Omega$ is differentiable;
  \item[(C)] if $\xi(t,y,u) \in \text{dom}\xi$ and $s \neq 0$ is a real number, then $(t/s,y,su) \in \text{dom}\xi$ and $\xi(t/s,y,su) = \xi(t,y,u)$.
\end{enumerate}

Here (A) and (B) are deduced easily from, say [3], (10.7.4) and (C) follows from the uniqueness theorem because $\eta(t) = \xi(t/s,y,su)$ is a solution of (6.1.1).

$$\text{(D)} \quad D_3\xi(t,y,u)_{B_x} \subset B_z, \quad z = \xi(t,y,u).$$

To prove this, fix $u \in B_x$ and $y \in \Omega$ and put $y_t = \xi(t,y,u)$, $\alpha_t = D_3\xi(t,y,u)$ and $\gamma_t = D_2\xi(t,y,u)$. Let the function $g : \Omega \to E$ be defined by

$$g(y) = f(y).u.$$

Then $\alpha_t \in \text{End}(E)$, $\gamma_t \in \text{GL}(E)$ and (cf. [3], (10.7.3.1), (10.8.4.1))
\[
\dot{y}_t = g(y), \quad y_0 = y; \\
\text{(6.1.D.1)}
\]

\[
\dot{y}_t = Dg(y_t)\gamma_t, \quad \gamma_0 = \text{id}_E; \\
\dot{a}_t = Dg(y_t)a_t + f(y_t), \quad a_0 = 0.
\]

Hence
\[
\text{(6.1.D.2)}
\]

\[
\alpha_t = \gamma_t \int_0^t (\gamma_s)^{-1}f(y_s)ds.
\]

By homogeneity of \(f\), the vectorfield \(f(-)u\) respects \(B\) and we have

\[
(\gamma_s)^{-1}B_x = B_y
\]

(cf. Introduction, (0.5)) and so

\[
(\gamma_s)^{-1}f(y_s)B_x \subseteq B_y.
\]

Hence

\[
\int_0^t (\gamma_s)^{-1}f(y_s)ds
\]

decreases \(B_x\) into \(B_y\), and \(\alpha_t\) maps \(B_x\) into \(\gamma_tB_y\). Using the homogeneity of \(f\) once more, we have

\[
\alpha_tB_x \subseteq \gamma_tB_y = B_y = B_z,
\]

which proves (d).

Note also that \(u = 0\) implies \(y_t = y_0 = y\), \(\gamma_t = \gamma_0 = \text{id}_E\), and therefore

\[
\alpha_t = \int_0^t f(y)ds = tf(y),
\]

or

\[
\text{(6.1.E)}
\]

\[
D_3\xi(t,y,0) = tf(y)
\]

It follows from (A) and (C) that there exists \(\varepsilon > 0\) such that

\(\xi(1,y,u)\) is defined for \(\|x-y\| < \varepsilon\) and \(\|u\| < \varepsilon\). Since

\[
D_3\xi(1,x,0) = f(x) = \text{id}_E,
\]

we may assume that

\[
D_3\xi(1,x,u) \in \text{GL}(E) \quad \text{for} \quad \|u\| < \varepsilon.
\]
We claim that, for $u \in B_x$ and $\|u\| < \epsilon$,

\[(F) \quad D_3\xi(1,x,u).B_x = B_z, \text{ where } z = \xi(1,x,u).\]

To prove $(F)$, put $z_u = \xi(1,x,u)$, $\beta_u = D_3\xi(1,x,u)$ and $\delta_u = D_2\xi(1,x,u)$. The homogeneity of $f$ implies that

\[\delta_u B_x = B_{z_u} \quad \text{for } u \in B_x.\]

Now, by $(D)$, $u \in B_x$ implies that

\[\beta_u B_x \subset B_{z_u},\]

and so

\[(\delta_u)^{-1}\beta_u B_x \subset B_x',\]

or (cf. §1.2)

\[\theta(u) = (\delta_u)^{-1}\beta_u \in \text{GL}(E|B_x)\]

Since $B_x^c = \{u \in B_x : \|u\| < \epsilon\}$ is connected, $\theta : B_x^c \rightarrow \text{GL}(E|B_x)$ is continuous, and $\theta(0) = (\delta_0)^{-1}\beta_0 = \text{id}_E \in \text{GL}_o(E|B_x)$, it follows from Proposition 1.2 that

\[\theta(u) \in \text{GL}_o(E|B_x)\]

for every $u \in B_x^c$. Hence $\theta(u)B_x = B_x$ and so

\[\beta_u B_x = \delta_u \theta(u) B_x = \delta_u B_x = B_{z_u}\]

for $u \in B_x^c$, which proves the assertion $(F)$.

\[(G) \quad \text{Taking } s = t \text{ in } (C) \text{ we see that, for } u \in B_x,\]

\[\xi(1,x,tu) = \xi(t,x,u)\]

is an integral curve of a differentiable vectorfield in $B$. 
Let now $Q$ be a submanifold of $\Omega$ such that $x \in Q$ and $T_x Q$ is a complement of $B_x$ in $E$, and let
\[ \phi : W^1 \to Q, \quad \phi(0) = x \]
be a diffeomorphism of the unit ball $W^1$ of a Banach space $W$ onto a neighbourhood of $x$ in $Q$. Let $U = B_x$ and let $\psi$ from $U \times W$ into $M$ be defined by
\[ \psi(u,w) = \xi(1,\phi(w),u). \]
Since $D_2 \xi(1,x,0) = \text{id}_E = D_3 \xi(1,x,0)$ (see (F)), it follows from the closed graph theorem that $D\psi(0,0) \in \text{LIS}(U \times W, E)$ and we may assume that $\psi$ is a diffeomorphism of $U^c \times W^c$ onto an open subset of $\Omega$. Multiplying the norms of $U$ and $W$ by $1/\epsilon$, we turn $U^c$ and $W^c$ into $U^1$ and $W^1$ and $(\psi,U,W)$ into a differentiable box of $M$. The assertions (2), (3) and (4) of Proposition 6.1 follow now from (D), (F) and (G).

§7. INTEGRABLE DISTRIBUTIONS

Recall that a distribution $B$ on $M$ is homogeneous if every differentiable vectorfield in $B$ respects $B$ (see Introduction, (0.6) and (0.7)) and that $B$ is said to be integrable if there exists a differentiable structure $\sigma$ on $M$ such that $(M,\sigma)$ is an integral manifold of $B$. 
THEOREM 1. Let B be a weakly differentiable distribution on M.

(a) The following conditions are equivalent:

(1) B is integrable.

(2) For every x in M, there exists an integral manifold of B containing x.

(3) B is homogeneous.

(4) For every x in M, there exists a homogeneous weak stem of B at x.

(b) If B is integrable, then there exists a unique differentiable structure \(\sigma\) on M such that \((M,\sigma)\) is an integral manifold of B.

Furthermore,

(5) Every integral manifold of B is an open submanifold of \((M,\sigma)\).

(6) \(\sigma\) is a foliation of M in the sense of §2.

(7) Two points of M belong to the same connected component of \((M,\sigma)\) if and only if they can be joined by finitely many integral curves of differentiable vectorfields in B.

PROOF OF THEOREM 1. To show that \((2) \implies (3)\), let \(X\) be a differentiable vectorfield in B, let \(x \in \text{domain of } X\) and let \(Z\) be an integral manifold of B through the point \(x\). By Proposition 3.1, there exists a neighbourhood \(H_1\) of \(x\) in \(Z\) and \(\varepsilon_1 > 0\) such that \(X^t.y \in Z\) for \(|t| < \varepsilon\) and \(y \in H^1\) and \((t,y) \to X^t.y : R \times Z \to Z\).
is differentiable. Choose $\varepsilon > 0$ and a neighbourhood $H$ of $x$ such that $X^t \cdot y \in H_1$ for $|t| < \varepsilon$ and $y \in H$, $H \subset H_1$ and $\varepsilon \leq \varepsilon_1$.

Differentiating the equations $X^{-t} (X^t \cdot y) = y = X^{-t} (X^{-t} \cdot y)$ with respect to $y \in Z$ we prove that, for $|t| < \varepsilon$, $(X^t)^*(x)$ is an isomorphism of $T_xZ = B_x$ onto $T_yZ = B_y$. The result now follows from Lemma 1.1, Part Three.

Let us now assume that $B$ satisfies (4). For every $x \in M$, let $Z_x$ be the integral manifold of $B$ described in Corollary 6.1. By (6.1.6), the differentiable structures of $Z_x$ and $Z_y$ coincide on $Z_x \cap Z_y$ and hence define a differentiable structure $\sigma$ on $M$ such that each $Z_x$ is an open submanifold of $(M, \sigma)$ ([13], 5.2.4).

It is clear that $(M, \sigma)$ is an integral manifold of $B$ and so (4) $\Rightarrow$ (1). Furthermore, it follows from (6.1.6) that every integral manifold of $B$ is an open submanifold of $\sigma$. Hence (4) $\Rightarrow$ (5). In particular, $\sigma$ is the unique differentiable structure which makes $M$ into an integral manifold of $B$.

The assertion (6) now follows at once from (6.1.2), (6.1.3) and Proposition 2.1, where we take $N = (M, \sigma)$.

Finally let $\xi$ be the equivalence relation ' $x$ and $y$ can be joined by finitely many integral curves of differentiable vector fields in $B$'. By Proposition 3.1, the integral curves of vector fields in $B$ are continuous as functions $R \rightarrow (M, \sigma)$, and so the equivalence classes of $\xi$ are connected. On the other hand, (6.1.4) shows that each equivalence class of $\xi$ is open in $(M, \sigma)$, which proves the assertion (7).
§8. THE USE OF LIE BRACKETS

8.1. NOTATION. If $X$ and $Y$ are differentiable vectorfields on $M$ and $x_t = x^t \cdot x$, then locally,

$$[X, Y](x_t) = D_Y(x_t)X(x_t) - D_X(x_t)Y(x_t) = \frac{d}{dt} Y(x_t) - D_X(x_t)Y(x_t).$$

This leads to the following definitions.

(8.1.1) A vectorfield over a curve $\sigma : I \rightarrow M$ (where $I$ is an interval in $\mathbb{R}$) is a curve $Y : I \rightarrow TM$ such that $Y(t) \in T_{\sigma(t)}M$ for all $t \in I$.

(8.1.2) If $\sigma(t) = x_t = x^t \cdot x$ for some differentiable vectorfield $X$ on an open subset of $M$, and if $Y'$ is a differentiable vectorfield over $\sigma$, we define the vectorfield $[X, Y]$ over $\sigma$ by the local coordinate formula

$$[X, Y](t) = \frac{d}{dt} Y(t) - D_X(x_t)Y(t).$$

(To show that this formula behaves well under $C^2$ changes of coordinates, let $\phi$ be a $C^2$ local diffeomorphism of $E$ and let

$$\tilde{Y}(t) = D\phi(x_t).Y(t), \quad \tilde{X}(x) = D\phi(\phi^{-1}(x)).X(\phi^{-1}(x)).$$

Then

$$D\tilde{X}(x).v = D^2\phi(\phi^{-1}(x)).(D\phi^{-1}(x).v).X(\phi^{-1}(x)) +$$

$$+ D\phi(\phi^{-1}(x)).DX(\phi^{-1}(x)).D\phi^{-1}(x).v,$$

and it is easily checked that

$$\frac{d}{dt} \tilde{Y}(t) - D\tilde{X}(x_t).\tilde{Y}(t) = D\phi(x_t).[X, Y](t).$$)
(8.1.3) Let \( F \) be a Banach space and let
\[
f : I \to L(F, TM)
\]
be a \( C^1 \) curve in the vector-bundle \( L(F, TM) \) such that \( f(t) \in L(F, T_{\sigma(t)}M) \)
where \( \sigma(t) = x_t = x^t.x \) as above. If \( v \in F \), we write
\[
(8.1.3.1) \quad v : I + TM : t \to f(t).v
\]
for the corresponding vectorfield over \( \sigma \). The Lie derivative of \( f \) with respect to \( X \) is the curve
\[
L_X f : I \to L(F, TM)
\]
(which again covers \( \sigma \)), defined by the formula
\[
(8.1.3.2) \quad L_X f(t) : F \to T_{\sigma(t)}M : v \to [X, v](t), \text{ or, locally,}
\]
\[
(8.1.3.3) \quad L_X f(t) = \dot{f}(t) - D_X(x_t) \circ f(t).
\]

8.2 A CONDITION OF RESPECTABILITY. The next theorem is a straight generalization of Theorem 6 in Part One.

THEOREM 2. Let \( X \) be a \( C^1 \) vectorfield on an open subset of \( M \), \( x \in \text{domain } X \), \( x_t = x^t.x \), and \( y_t = (x^t) \ast (x) \). Let \( B \) be an arbitrary (not necessarily differentiable) distribution on \( M \). Assume that there exist \( \epsilon > 0 \), a Banach space \( F \), a differentiable function
\[
f : \mathbb{R}^\epsilon \to L(F, TM)
\]
and a continuous function
\[
\Lambda : \mathbb{R}^\epsilon \to \text{End}(F),
\]
such that
\[
(8.2.1) \quad f \text{ covers the integral curve } t \to x_t; \text{ and, for all } t \in \mathbb{R}^\epsilon,
\]
\[
(8.2.2) \quad f(t)F = B_{x_t} \quad \text{and}
\]
\[
(8.2.3) \quad L_X f(t) = \dot{f}(t) \Lambda(t).
\]
Then, for all \( t \in \mathbb{R} \),

\[
(8.2.4) \quad \gamma_t B_x = B_{x_t}.
\]

**COROLLARY 2.** If the assumptions of Theorem 2 are satisfied at every \( x \in \text{domain } X \), then \( X \) respects the distribution \( B \) (Lemma 1.1, Part Three).

In particular, a weakly differentiable distribution \( B \) on \( M \) is integrable if (and, as we shall see in the next section, only if) the assumptions of Theorem 2 are satisfied for every differentiable vector field \( X \) in \( B \) and every \( x \in \text{domain } X \).

**REMARK.** It will be seen from the proof of Theorem 2 that its assertion is valid even if the vector spaces in the distribution \( B \) are not direct summands of the tangent spaces. We have to assume, however, that they are closed.

**PROOF OF THEOREM 2.** Let

\[
\alpha_t = (\gamma_t)^{-1} f(t) : F \to T_x M.
\]

Locally,

\[
\dot{\alpha}_t = -\gamma_t^{-1} \gamma_t^{-1} f(t) + \gamma_t^{-1} \dot{f}(t) = (\gamma_t)^{-1} (\dot{f}(t) - DX(x_t)f(t))
\]

and so (8.1.3.3)

\[
\dot{\alpha}_t = (\gamma_t)^{-1} L_x f(t) = (\gamma_t)^{-1} f(t) \Lambda(t) = \alpha_t \circ \Lambda(t).
\]

Let now \( h \in (T_x M)^* \) and let

\[
h_t = h \alpha_t = (\alpha_t)^* h
\]

Then

\[
h_t = h \Lambda(t) = \Lambda^*(t) h_t,
\]
and, by the uniqueness theorem, \( h_t = 0 \) if and only if \( h_0 = 0 \).

This means that \( h \) vanishes on \( B_x = \alpha_0 F \) if and only if it vanishes on \( \alpha_t F = (\gamma_t)^{-1} f(t) F = (\gamma_t)^{-1} B_{x_t} \), and so, by the Hahn-Banach theorem,

\[
B_x = (\gamma_t)^{-1} B_{x_t},
\]

Q.E.D.

8.3. COVARIANT STEMS. Let \( X \) be a \( C^1 \) vectorfield on an open subset of \( M \) and let \( x \in \text{domain } X \) and \( x_t = x^t x \). If \( B \) is a distribution on \( M \), we define an \( X \)-covariant \( B \)-stem at \( x \) as a \( C^1 \) function

\[
f : R^c \rightarrow \text{LIS (} T_x M, TM)\]

which covers the integral curve \( t + x_t \) and satisfies the following conditions:

(8.3.1) \( f(0) = \text{id}_{T_x M} \), and, for all \( t \in R^c \),

(8.3.2) \( f(t) B_x = B_{x_t} \) and

(8.3.3) \( L_X f(t) B_x \subset B_{x_t} \),

or equivalently,

(8.3.3.a) \( [X, \bar{v}] (t) \in B_{x_t} \) for all \( v \in B_x \),

where the vectorfield \( \bar{v} \) over \( t \) is defined by (8.1.3.1)

Let now

\[
i : B_x \rightarrow T_x M \quad \text{and} \quad p : T_x M \rightarrow B_x
\]

be the inclusion mapping and the projection along some complement of \( B_x \) in \( T_x M \). Let

\[
\Lambda(t) = f(t)^{-1} L_X f(t) \in \text{End}(T_x M)
\]
and let
\[ \bar{f} : R^e \rightarrow L(B_x, TM) \]
and
\[ \bar{\Lambda} : R \rightarrow \text{End}(B_x) \]
be defined by \( \bar{f}(t) = f(t) \circ i \) and \( \bar{\Lambda}(t) = p \circ \Lambda(t) \circ i \). As these functions obviously satisfy the conditions of Theorem 2, we have proved the following lemma.

**LEMMA 8.3.** Let \( \gamma_t(x) = (x^t)^*(x) \) and assume that there exists and \( X \)-covariant \( B \)-stem at \( x \) defined for \( |t| < \varepsilon \). Then, for \( |t| < \varepsilon \),

\[ \gamma_t B_x = B_{x_t} \]

**THEOREM 3.** Let \( X \) be a \( C^1 \) vectorfield defined on an open subset of \( M \) and let \( B \) be an arbitrary (not necessarily differentiable) distribution on \( M \). Then \( X \) respects \( B \) if and only if, for every \( x \) in the domain of \( X \), there exists an \( X \)-covariant \( B \)-stem at \( x \).

**PROOF.** If \( X \)-covariant \( B \)-stems exist, then the assertion follows at once from Lemma 8.3 and from Lemma 1.1 in Part Three. Conversely, if \( X \) respects \( B \) and \( x \) belongs to the domain of \( X \), we put
\[ f(t) = \gamma_t(x) = (x^t)^*(x) \in \text{LIS}(T_x M, T_x M). \]
Locally,
\[ L_X f(t) = \dot{\gamma}_t - DX(x_t)\gamma_t = 0 \]
and so \( f \) obviously satisfies the conditions (8.3.1) - (8.3.3), Q.E.D.
COROLLARY 3. A weakly differentiable distribution $B$ is integrable if and only if, for every differentiable vectorfield $X$ in $B$, and for every $x \in \text{domain } X$, there exists an $X$-covariant $B$-stem at $x$.

This follows at once from Theorems 1 and 2.

8.4. THE STANDARD FROBENIUS THEOREM. Recall that a distribution $B$ is said to be involutive if the Lie bracket of any two differentiable vectorfields in $B$ lies in $B$. The following proposition follows at once from ([4], 17.14.3.5).

PROPOSITION 8.4. Every integrable distribution is involutive.

We can now state the 'standard' Frobenius theorem.

THEOREM 4 (Frobenius). A regular distribution $B$ is integrable if and only if it is involutive.

PROOF. Let $X$ be a vectorfield in $B$, $x$ a point in the domain of $X$ and $x_t = x^t \cdot x$. Let $\Omega$ be a neighbourhood of $x$ in $M$ and let $f : \Omega \rightarrow \text{LIS}(T_x M, TM)$ be a regular $C^r$ stem of $B$ at the point $x$. By regularity, $f(y)_{B_x} = B_y$ for every $y$ in $U$ and so, if $B$ is involutive, $t \rightarrow f(x_t)$ is an $X$-covariant $B$-stem at $x$, and the integrability of $B$ follows at once from Corollary 3.
REMARKS. 1) Note that the stem f in the proof above has to be differentiable (rather than weakly differentiable, cf. the definition of the regular distribution in (4.1)).

2) There exist involutive non-integrable $C^\infty$ distributions on $\mathbb{R}^2$ (cf. Example 8.5.2). Hence, the regularity of $B$ in Theorem 4 is, in general, essential. We shall see below that the situation is simpler in the real-analytic case.

8.5. REAL ANALYTIC DISTRIBUTIONS. The basic facts about real analytic functions on Banach spaces and real analytic manifolds are collected in [13], §§3 and 5. We need the result that the integral curves of real analytic vector fields are real analytic functions of time ([13], 9.1.8).

If $B = (B_x : x \in M)$ is a distribution on $M$ and $T$ is a subset of $M$, we say that $B|T$ is spanned by a set $S$ of vector fields on $M$ if $B_y = \text{span}\{X(y) : X \in S\}$ for every $y \in T$.

We write $C^\omega(B,T)$ for the set of real analytic vector fields in $B$ whose domain includes $T$, and we say that $B$ is locally everywhere defined if, for every vector field $X$ in $B$, and every $x \in \text{domain } X$, there exists $\epsilon > 0$ such that $B|\{x_t : |t| < \epsilon\}$ is spanned by $C^\omega(B,\{x_t : |t| < \epsilon\})$, where $x_t = X^t.x$. Note that the word 'locally' refers here to a small portion of an integral curve, rather than to a neighbourhood of $x$ in $M$ (cf. Example 8.5.1).
The next theorem generalizes a result of Nagano [7].

THEOREM 5. A real analytic distribution is integrable if and only if it is involutive and locally everywhere defined.

PROOF. Let $B$ be an involutive, locally everywhere defined real analytic distribution on $M$, $X$ a real analytic vectorfield in $B$, $x \in \text{domain } X$, $x_t = x^t \cdot x$ and $\gamma_t = (x^t)^*(x)$.

Let $\Omega$ be a neighbourhood of $x$ and let

$$f : \Omega \to L(T_x M, TM)$$

be a real analytic stem of $B$ at $x$. If $B$ is the distribution on $\Omega$ given by

$$B_y = f(y)B_x,$$

then, clearly, $t \mapsto f(x_t)$ is an $X$-covariant $B$-stem at $x$, and so, by Lemma 8.3, there exists $\varepsilon > 0$ s.t.

$$\gamma_t B_x = \tilde{B}_{x_t} \subset B_{x_t} \quad \text{for } |t| < \varepsilon.$$  \hfill (8.5.0)

Let now $v \in B_{x_t}$. Then $v = Y(x_t)$ for some real analytic vectorfield $Y$ in $B$. Since $B$ is locally everywhere defined, we may assume that $x_s \in \text{domain } Y$ for all $s$, $|s| < \varepsilon$. Writing $Y(s)$ instead of $Y(x_s)$, we put

$$v(s) = (\gamma_s)^{-1}Y(s) \in T_x M.$$  

A simple computation of the usual kind shows that

$$\dot{v}(s) = (\gamma_s)^{-1}[X,Y](s) = (\gamma_s)^{-1}(\text{ad}_X Y)(s).$$
Hence, by induction,
\[ \frac{d^n}{ds^n}v(s) = (\gamma_s)^{-1}((\text{ad}X)^n.Y)(s), \]
and in particular, as B is involutive,
\[ \frac{d^n}{ds^n}v(0) = ((\text{ad}X)^n.Y)(x) \in B_x. \]

Let now h be an arbitrary linear functional in (T_x M)^* and let
\[ H(s) = h.v(s). \]
Then H is a real analytic function of s; if h vanishes at B_x, then all the derivatives of H vanish at s = 0 and so H(s) ≡ 0.

Hence, by the Hahn-Banach theorem,
\[ v(t) = (\gamma_t)^{-1}Y(x_t) \in B_x, \]
and so \((\gamma_t)^{-1}B_x \subset B_x\) or \(B_x \subset \gamma^* B_x\). Combining this with (8.5.0), we see that
\[ \gamma_t B_x = B_x \quad \text{for} \quad |t| < \epsilon \]
and the result now follows from Theorem 1 and from Lemma 1.1 in Part Three.

**EXAMPLE 8.5.1.** Let M = R and let B be the real analytic distribution on R spanned by the vectorfields \( X_1 \) and \( X_2 \), where \( X_1 = 0 \) and \( X_2 \) is defined on \( R^+ \) by \( X_2(x) = (1/x) \partial/\partial x \). Then B is clearly integrable, and the origin of R has no neighbourhood \( \Omega \) such that
\[ B|\Omega \text{ is spanned by } C^\omega(B,\Omega). \]
EXAMPLE 8.5.2. Let $M = \mathbb{R}^2$ and let $B$ be the real analytic distribution on $\mathbb{R}^2$ spanned by the vectorfields $X_1$ and $X_2$, where $X_1 = \partial / \partial \xi$ and $X_2$ is defined for $\xi > 0$ by $X_2(\xi, \eta) = (1/\xi) \partial / \partial \eta$.

(See Fig. 2)

If $x \in (\eta$-axis) and $X$ is a real analytic vectorfield in $B$ defined on a connected neighbourhood of $x$, then

$$X = \alpha \frac{\partial}{\partial \xi} + \beta \frac{\partial}{\partial \eta}$$

where $\beta$ is a real analytic function vanishing for $\xi < 0$. Hence $\beta = 0$, and it is easily seen that $B$ is an example of a non-integrable involutive real analytic distribution on $\mathbb{R}^2$. 
§9. NEAT LEAVES

Throughout this section, \( B \) is an integrable \( C^1 \) distribution on the \( C^2 \) manifold \( M \) and \( \sigma \) is the \( C^1 \) structure which makes \((M,\sigma)\) into an integral manifold of \( B \) (§7). Unless otherwise stated, the words 'differentiable', 'diffeomorphism' etc. refer to the class \( C^1 \).

9.1 LOCAL AUTOMORPHISMS. By a local automorphism of \( B \) we understand a bijection of an open subset of \( M \) onto another open subset of \( M \) which is a local diffeomorphism for both \( M \) and \((M,\sigma)\). The set of local automorphisms of \( B \) is denoted by \( \text{Loc Aut } B \).

**LEMMA 9.1.1.** A local diffeomorphism \( \phi \) of \( M \) belongs to \( \text{Loc Aut } (B) \) if and only if

\[(9.1.a) \quad \phi^*(x)B_x = B_y \text{ whenever } \phi(x) = y.\]

**PROOF.** Assume that \( \phi \) satisfies (9.1.a). It is clearly sufficient to show that \( \phi \) is differentiable relatively to \( \sigma \). If \( x \) belongs to the domain of \( \phi \) and \( Z \) is an integral manifold of \( B \) which contains \( x \) and is contained in the domain of \( \phi \), then \( \phi(Z) \), with the differentiable structure defined by the bijection \( \phi| : Z \to \phi(Z) \), is an immersed submanifold of \( M \) passing through \( y = \phi(x) \). The result now follows from Theorem 1 (5).
Let $\Theta B$ denote the set of all the local diffeomorphisms $\phi$ of $M$ such that $\phi: x \to X^t.x$ for some differentiable vectorfield $X$ in $B$ and some $t \in \mathbb{R}$, and let $\Sigma B$ denote the set which consists of $i_d_M$ and all the finite compositions of members of $\Theta B$.

**Lemma 9.1.2.** $\Sigma B$ is contained in Loc Aut $B$ and closed under the operations of restricting the domain, composition, and taking an inverse. Two points $x$ and $y$ in $M$ belong to the same connected component of $(M, \sigma)$ if and only if $y = \phi(x)$ for some $\phi \in \Sigma B$.

This follows at once from Theorem 1 ((3) and (7)) and from Lemma 9.1.1.

We define the normaliser $N\Sigma B$ of $\Sigma B$ as the set of all local automorphisms $\phi \in$ Loc Aut $B$ such that

$$(9.1.b) \quad \phi \Sigma B \phi^{-1} \subseteq \Sigma B \quad \text{and} \quad \phi^{-1} \Sigma B \phi \subseteq \Sigma B.$$

These inclusions are to be understood as follows: If $\psi \in \Sigma B$ and both the domain and the range of $\psi$ are included in the domain of $\phi$, then $\phi \psi \phi^{-1}$ is in $\Sigma B$; if both the domain and the range of $\psi$ are in the range of $\phi$, then $\phi^{-1} \psi \phi$ is in $\Sigma B$ (Fig. 3).

\[ \gamma_2 = \phi \gamma_1 \phi^{-1} \]
\[ \gamma_1 = \phi^{-1} \gamma_2 \phi \]

Figure 3
LEMMA 9.1.3.

\[ \forall B \cup (\text{Loc Aut } B \cap \text{Diff}^2 M) \subset \forall \forall B. \]

PROOF. It is clear that
\[ \forall B \subset \forall \forall B. \]
If \( \phi \in \text{Loc Aut } B \cap \text{Diff}^2 M \) and \( X \) is a \( C^1 \) vectorfield in \( B \), then \( \phi \circ X^\phi \circ \phi^{-1} = Y^\phi \), where
\[ Y(\xi) = \phi(\phi^{-1}(\xi)).X(\phi^{-1}(\xi)) \in B^\phi. \]
Hence \( \forall B \phi^{-1} \subset \forall B \) and it is easily deduced that \( \forall B \phi^{-1} \subset \forall B \) (there are no problems with the domains because domain \( \phi = \text{range } \phi = M \)).

LEMMA 9.1.4. Let \( \sim \) be the equivalence relation given by the partition of \( M \) into the connected components of \( (M, \sigma) \). If \( \phi \in \forall \forall B \), then \( \phi \) respects \( \sim \) (i.e. \( x \sim y \) and \( x \) and \( y \in \text{domain } \phi \) implies \( \phi(x) = \phi(y) \)).

PROOF. Let \( \phi \in \forall \forall B \) and let \( x \) and \( y \) lie in domain \( \phi \). If \( x \sim y \), then \( x = \psi(y) \) for some \( \psi \in \forall B \). We may assume that the domain and range of \( \psi \) are contained in the domain of \( \phi \), so that \( \phi \psi \phi^{-1} \in \forall B \) and \( \phi(x) = \phi \psi \phi^{-1}(\psi(y)) \), which proves that \( \phi(x) \sim \phi(y) \).

9.2. NEAT LEAVES. A leaf is a connected component of \( (M, \sigma) \).
A \( C^1 \) box \( (\psi, U, W) \) of \( M \) is admissible if \( \psi(-, w) : U^1 \rightarrow (M, \sigma) \) is differentiable for every \( w \in W^1 \) and if the slice \( \psi(-, 0) : U^1 \rightarrow (M, \sigma) \) is a local chart. An admissible box is neat if \( \psi(-, w) \) is a local chart for \( (M, \sigma) \) whenever \( \psi(0, w) \) belongs to the same leaf as \( \psi(0, 0) \) (Fig. 4, p. 68).

A point \( x \in M \) is neat if there exists a neat box \( (\psi, U, W) \) such that \( \psi(0, 0) = x \).
LEMMA 9.2.1. a) If \( \phi \in \text{Loc Aut} B \) and \( (\psi, U, W) \) is an admissible box such that \( \psi(U^1 \times W^1) \subseteq \text{dom} \phi \), then \( (\phi \circ \psi, U, W) \) is an admissible box.

b) If, in addition, \( \phi \in N\Psi B \) and \( (\psi, U, W) \) is neat, then \( (\phi \circ \psi, U, W) \) is neat.

This follows at once from the definitions and Lemma 9.1.4.

If now \( x \) and \( y \) belong to the same leaf, then \( x = \phi(y) \) for some \( \phi \in \Psi B \). Since \( \Psi B \subseteq N\Psi B \), we have the following result.

COROLLARY 9.2.1. If a leaf \( L \) of \((M,\sigma)\) contains a neat point, then every point of \( L \) is neat.

Such leaves will be referred to as neat leaves.

LEMMA 9.2.2. a) Every finite dimensional or finite codimensional leaf is neat.

b) If \( B \) is a regular distribution, then every leaf of \((M,\sigma)\) is neat.

To prove b), set
\[
\delta(u, w) = \psi^*(0,0)^{-1}f^{-1}(\psi(u, w))\psi^*(u, w),
\]
where \( f \) is some regular stem of \( B \) at \( x = \psi(0,0) \) and we assume that \( \psi(U^1 \times W^1) \subseteq \text{domain} f \). Then \( \delta(u, w) \in \text{GL}(E|U \times 0) \), \( E = U \times W \), and so, by Proposition 1.2, \( \delta(u, w) \in \text{GL}_0(E|U \times 0) \) for every \( (u, w) \in U^1 \times W^1 \).

Hence \( D_1\psi(u, w).U = f(\psi(u, w))\psi^*(0,0)\delta(u, w).U \times 0 = f(\psi(u, w))\psi^*(0,0)(U \times 0) = f(\psi(u, w))B_x = B_{\psi(u, w)} \). (This argument shows that every admissible box of a regular distribution is neat.)
9.3. NEAT SUBMANIFOLDS. An immersed submanifold \( L \) of \( M \) is **neat** if, for every \( x \in L \), there exists a \( C^1 \) box \( (\psi, U, W) \) of \( M \) such that

(i) \( \psi(0, O) = x \);

(ii) \( L \cap \psi(U^1 \times W^1) = \psi(U^1 \times L) \), where \( L = \{ w \in W^1 : \psi(0, w) \in L \} \);

and

(iii) for every \( w \in L \), \( \psi(\_ , w) : U^1 \to L \) is a local chart for \( L \).

For example, the union of all the leaves of \( (M, \sigma) \) of a given finite dimension (or codimension) is a neat submanifold of \( M \), and so is \( (M, \sigma) \) if \( B \) is a regular distribution, or any single neat leaf of \( (M, \sigma) \) in general.

A \( C^1 \) box \( (\psi, U, W) \) which satisfies the conditions (ii) and (iii) above is called a **neat box for \( L \)**.

**PROPOSITION 9.3.1.** Let \( L \) be a neat submanifold of \( M \) and let \( \psi : N \to M \) be a continuous mapping such that \( \psi(N) \subset L \).

(a) If \( \psi : N \to M \) is a differentiable mapping between manifolds and if, for every \( \xi \in N \), \( \psi^*(\xi)T_N \subset T_{\psi(\xi)}L \), then \( \psi : N \to L \) is differentiable.

(b) If \( \psi : N \to M \) is a differentiable mapping between manifolds and \( L \) is separable, then \( \psi : N \to L \) is differentiable.

(c) More generally, if \( N \) is a locally connected topological space and \( L \) is separable, then \( \psi : N \to L \) is continuous.

The proof follows the same lines as the proof of Lemma 3.1 in Part One and is therefore omitted. Note that the assertion (b) follows from (c).
The next proposition is probably a special case of a more general result.

PROPOSITION 9.3.2. Let \( L \) be a connected neat submanifold of \( M \). If \( M \) is paracompact and modelled on a separable Banach space \( E \), then \( L \) is separable*).

LEMMA 9.3.3. (See [1], Ch. 111, §9, Lemma 1.) If a topological space \( T \) admits a locally countable covering by open separable subsets, then each connected component of \( T \) is separable.

PROOF OF PROPOSITION 9.3.2. Since \( M \) is paracompact and locally separable, each connected component of \( M \) is separable by Lemma 9.3.3. Hence, there exist a countable family of boxes \( (\psi_n, U_n, W_n) \) for \( L \) such that \( L \subset \bigcup H_n \), where \( H_n = \psi_n(U_n^1, W_n^1) \). Let \( S_{nw} = \psi_n(U_n^1, w) \) be a slice of \( H_n \). It is easily checked that, if \( w \in \lambda_n = \{ w \in W_n^1 : \psi_n(0, w) \in L \} \), then each connected component of the open subset \( S_{nw} \cap H_m \) of \( S_{nw} \) is contained in a slice of \( \psi_m \). Since \( S_{nw} \) is separable, we see that \( S_{nw} \) meets \( \bigcup_{\lambda_n} \) for at most countably many \( w \in \lambda_m \). Hence the family \( \{ S_{nw} : n \in N, w \in \lambda_n \} \) is a locally countable cover of \( L \) and the assertion follows from Lemma 9.3.3.

9.4. AN UNSOLVED PROBLEM. A leaf of \( (M, \sigma) \) is wild if it contains no neat points.

*) by 'separable' we mean: with a countable basis of open sets.
QUESTION 9.4.1. Do wild leaves exist? More precisely: does there exist an integrable $C^1$ distribution $B$ on a (separable) $C^2$ manifold $M$ such that the corresponding foliation of $M$ has a wild leaf?

The author made several attempts to construct such a foliation, but failed. The examples below illustrate the difficulties encountered.

LEMMA 9.4.2. Let $H$ be a separable Hilbert space and let $(A_n)_{n \in \mathbb{Z}}$ be a doubly infinite sequence of members of $\text{GL}(H)$. There exists a $C^\infty$ function $\gamma : \mathbb{R} \to \text{GL}(H)$ such that

(i) $f(n) = A_n$ for all $n \in \mathbb{Z}$; and
(ii) $f$ is constant on each of the intervals $[n - \frac{1}{3}, n + \frac{1}{3}]$.

PROOF. Since $\text{GL}(H)$ is contractible [15], there exists a continuous path $[0,1] \to \text{GL}(H)$ joining any two elements. Since $\text{GL}(H)$ is an open subset of the Banach space $\text{End}(H)$, this path can be replaced by a broken straight line with finitely many segments. The corners can be smoothed off in the two dimensional space spanned by the two adjacent edges. Property (ii) follows on re-parametrization.

From now on, $H = l_2(\mathbb{Z})$ is the space of all doubly infinite real sequences $(x_n)_{n=-\infty}^\infty$ such that $\sum x_n^2 < \infty$, $F$ is the closed subspace of $H$ given by the equations $x_n = 0$ for $n \geq 1$, and $S$ is the right shift on $H$:

$$(Sx)_n = x_{n-1}.$$ 

We note that $S : H \to H$ is an isometric isomorphism.
EXAMPLE 9.4.2. We take
\[ L = \phi(R \times F), \]
where
\[ \phi : R \times F \rightarrow N \times H : (x, v) \rightarrow \left( \theta(x), \gamma(x) \cdot v \right), \]
\( N \) is a manifold, \( \theta : R \rightarrow N \) is an immersion and \( \gamma : R \rightarrow \text{GL}(H) \) is a differentiable function. It is easily checked that \( \phi \) is an immersion, so that \( L \) is an immersed submanifold of \( N \times H \).

(9.4.2.a) Chose \( N = \mathbb{R}^2 \), \( \gamma(t) = \text{id}_H \) for \(-\frac{1}{3} \leq t \leq \frac{1}{3}\), \( \gamma(t) = S \) for \(\frac{2}{3} \leq t \leq \frac{4}{3}\) and \( \theta : R \rightarrow \mathbb{R}^2 \) as in Fig. 4.

\[ \begin{array}{c}
\begin{tikzpicture}
\draw[->] (-2,0) -- (2,0);
\draw[->] (0,-2) -- (0,2);
\node at (0,2) {\( \eta \)};
\node at (-2,0) {\( \xi \)};
\filldraw[black] (0,0) circle (2pt);
\filldraw[black] (2,0) circle (2pt);
\filldraw[black] (0,-2) circle (2pt);
\filldraw[black] (-2,0) circle (2pt);
\node at (0,-1) {\( t=-1/3 \)};
\node at (2,0) {\( t=2/3 \)};
\node at (0,2) {\( t=4/3 \)};
\node at (-2,0) {\( t=1/3 \)};
\end{tikzpicture}
\end{array} \]

Figure 4

It is easily checked that \( \phi \) is then an embedding and that \( L \) is a leaf of the distribution \( B \) given by \( B_x = T_xL \) for \( x \in L \) and
\[ B_x = T_xM = T_x(R^2 \times H) \] for \( x \notin L \).
An admissible box with the range $Q \times H$, where $Q$ is the square indicated in Fig. 4 is obtained from the identity mapping of $\mathbb{R}^2 \times H$, with $(\xi\text{-axis}) \times F$ for the first coordinate and $(\eta\text{-axis}) \times F^1$ for the second coordinate. It is clear that this box is not neat and that it restricts to a neat box on a neighbourhood of $\phi(0,0)$.

(9.4.2.b) Take $N = \mathbb{R}^2$, $\gamma(t) = \text{id}_H$ for $t \leq 0$, $\gamma(n) = s^n$ for $n \geq 1$, and $\gamma(t)$ constant on each of the intervals $[n - \frac{1}{3}, n + \frac{1}{3}]$. Let $\theta : \mathbb{R} \rightarrow \mathbb{R}^2$ be as in Figure 5.

![Figure 5](image-url)
Using the identity of $\mathbb{R}^2 \times H$, we can define an admissible box at $A = \phi(-1,0)$ which does not restrict to a neat box for $L$. Note, however, that $L$ is not a leaf of a foliation with singularities. (C is a neat point and if $L$ were a leaf of a foliation this would imply that every point of $L$ is neat (Corollary 9.2.1). There are no admissible charts at $B$.)

(9.4.2.c) The attempts to use $N = \text{two-torus}$, $\theta : R \to N$ an integral curve of a fixed irrational flow, and a suitable $\gamma : R \to \text{GL}(H)$ failed for the following reasons: 1) the differentiability of the distribution $B$ demands that every $x \in N \times H$ has a neighbourhood $\Omega$ such that $B_y$ is 'larger' than $B_x$ for $y \in \Omega$; 2) Proposition 1.2.

**Example 9.4.3.** Let $H_0 = \bigcup_{n=1}^{\infty} S^n_F$ and let the distribution $B$ on $H$ be defined by taking $B_x = S^n_{F_y}$ for $x \in S^n_F \setminus S^{n-1}F$ and $B_x = 0$ if $x = 0$ or if $x \in H \setminus H_0$. (The projection $S^{n+2}_F \to S^n_{F_y} / S^n_F = \mathbb{R}^2$ takes $B$ into the distribution $\tilde{B}$ illustrated in Fig. 6.)

---

**Figure 6**
It is clear that $B$ is integrable. Note, however, that $B$ is not differentiable. ($H_0$ is a dense subset of $H$ of the first category, so we are in a similar difficulty as with the flow-line of the irrational flow on the torus in (9.4.2.c.).)

We now construct an integrable distribution on $T \times H$, where $T$ is the circle. Consider $[0,1] \times H$ as a subspace of $R \times H$ and let

$$\widetilde{B}(t,x) = V(t,x) + \gamma_t B(\gamma_t^{-1} x),$$

where $\gamma : [0,1] \to GL(H)$ is a $C^\infty$ function such that $\gamma_t = id_H$ for $0 \leq t \leq \frac{1}{3}$, $\gamma(t) = S$ for $\frac{2}{3} \leq t \leq 1$, and $V(t,x)$ is the one-dimensional subspace of $R \times H$ spanned by the vector $(1, \gamma(t)x)$ (Fig. 7)

![Figure 7](image)

Note that $\widetilde{B}(t,x) = R \times B$ for $0 \leq t \leq \frac{1}{3}$ and for $\frac{2}{e} \leq t \leq 1$, so that $\widetilde{B}$ defines an integrable distribution $\tilde{B}$ on $T \times H$. If $\ell$ is the point of $T$ obtained by identifying the endpoints of the interval $[0,1]$, then, clearly, $\{\ell\} \times (H_0 \setminus \{0\})$ is contained in a single wild leaf of $\tilde{B}$. 
PART THREE

Integrability and irreducibility of systems of vectorfields
§1. ON INTEGRABILITY OF SYSTEMS OF VECTORFIELDS

1.1 GENERALITIES. Let $S$ be a set of smooth vectorfields on a paracompact finite-dimensional manifold $M$. (For simplicity, we assume that $M$ and the vectorfields in $S$ are of the class $C^\infty$ or $C^\omega$.) Recall that the accessible sets of $S$ (or orbits in the terminology of [10]) are the equivalence classes of the relation '$x$ and $y$ can be joined by finitely many (unoriented) pieces of integral curves of vectorfields in $S$'. It is proved in Part One and in [10] that the accessible sets of $S$ are immersed submanifolds of $M$.

$S$ is said to be **homogeneous** if every vectorfield in $S$ respects the distribution $B(S) = (S(x) : x \in M)$, where $S(x)$ is the vector subspace of $T_x M$ spanned by the values at $x$ of the vectorfields in $S$.

**THEOREM 1.** The following conditions are equivalent.

(a) For every $x \in M$, there exists an integral manifold of $B(S)$ which contains the point $x$.

(b) $S$ is homogeneous.

(c) $S$ spans the tangent spaces of its accessible sets.

We say that $S$ is **integrable** if it satisfies either of the conditions in Theorem 1. The non-trivial step in the proof of Theorem 1 is the proof of (b) $\rightarrow$ (c), given in Part One and in [10]. The assertion (a) $\rightarrow$ (b) follows from Theorem 1 in Part Two. If the integral manifolds in question are at least of the class $C^2$, then (a) $\rightarrow$ (b) can be deduced from the existence and uniqueness theorem for ordinary differential equations and the following lemma.
LEMMA 1.1. A vector field $X$ on $M$ respects a distribution $B$ if and only if, for every $x$ in the domain of $X$, there exists $\epsilon > 0$ such that

$$(x^t)^*(x)_*B_x = B_{x^t}$$

over $|t| < \epsilon$ and $x^t.x = y$.

PROOF. Let $x_t = x^t.x$, $y_t = (x^t)^*(x)$ and $\gamma_{t_s}$ be the domain of the integral curve $t \to x_t$ and let

$I_0 = \{ t \in I : y_{t_x} = B_{x_t} \}$.

Then $y_{t+s} = y_{t-s} y_s$ and $I_0$ is easily shown to be both open and closed.

REMARKS. 1) If, for every $x \in$ domain $X$,

.a) $x^t.x = y \Rightarrow (x^t)^*(x)_*B_x \subset B_{x^t}$,

then $X$ respects $B$. This follows at once from (1.1.a) since $x = x$ and $(x^{-t})^*(y)$ is the inverse of $(x^t)^*(y)$.

2) Consider the following property of $X$:

.b) For every $x \in$ domain $X$, there exists $\epsilon > 0$ such that $|t| < \epsilon$ and $x^t.x = y$ implies $(x^t)^*(x)_*B_x \subset B_{x^t}$.

This example shows that (1.1.b) does not imply that $X$ respects distribution $B$.

EXAMPLE 1. Let $M = \mathbb{R}^2$ and let $B$ be the distribution spanned by vector fields $\partial/\partial x$ and $\partial/\partial y$ (cf. Fig. 1).
Figure 1.

X be an arbitrary vectorfield such that $X(x) \in B_x$ for every domain $X$. It is easily checked that $X$ satisfies the condition (1.1.b) at every $x \in \mathbb{R}^2$. (If $x \in \eta$-axis and $X(x) \neq 0$ this follows from $(X^t)^*(x).X(x) = X(y) \in B_y$, where $y = X^t.x$.)

no vectorfield which moves a point $x$ on the $\eta$-axis onto the $\eta$-axis can respect $B$ since then $\dim((X^t)^*(x).B_y) = \dim B = 1$ and $\dim B_y = 2$.

CONDITION L FOR $S$. Recall [6] that $S$ is locally of finite type at $x$ (Lx) if there exist finitely many vectorfields $X_1, X_2, \ldots, X_p$ such that

1) The vectors $X_i(x)$ ($i = 1, \ldots, p$) span $S(x)$;

2) For every $Y \in S$, there exists a neighbourhood $\Omega$ of $x$ and the continuous real-valued function $\lambda_{ij}$ defined on $\Omega$ such that

$$[Y, X_i](y) = \sum_{j=1}^p \lambda_{ij}(y)X_j(y)$$

for every $y \in \Omega$. 


is said to be locally of finite type \( (L) \) if it is locally of finite type at every \( x \) in \( M \).

It is claimed in [6] \(^*)\) that every set \( S \) of smooth vector fields which is locally of finite type is homogeneous, and therefore integrable. However, the proofs given show only that the vector fields in \( S \) satisfy (1.1.b) (with \( B_x = S(x) \)). The next example shows that, in fact, \( L \) for \( S \) does not imply that \( S \) is integrable.

**EXAMPLE 2.** Let \( M = \mathbb{R}^2 \) and let \( S \) be the set of all vector fields of the form

\[
\frac{\partial}{\partial \xi} + \phi(\xi, \eta) \frac{\partial}{\partial \eta},
\]

where \( \phi \) is an arbitrary function such that \( \phi(0,0) = 0 \) and \( \partial \phi / \partial \xi = 0 \) in some neighbourhood of the origin depending on \( \phi \).

If \( x \neq 0 \), then there exists a neighbourhood \( \Omega \) of \( x \) and a vector field \( X_1 \) in \( S \) such that \( X_1 = \frac{\partial}{\partial \eta} \) on \( \Omega \). Taking \( X_2 = \frac{\partial}{\partial \xi} \in S \), it is easy to see that \( S \) satisfies the condition \( (Lx) \) with \( X_1, X_2 \) and the same \( \Omega \) for every \( Y \) in \( S \). If, on the other hand, \( x \) is the origin, we may take \( \{ X_1, X_2, \ldots, X_p \} = \{ \partial / \partial \xi \} \), as

\[
[\partial / \partial \xi + \phi \frac{\partial}{\partial \eta}, \partial / \partial \xi] = -\partial \phi / \partial \xi \frac{\partial}{\partial \eta} = 0
\]

in a sufficiently small neighbourhood of \( 0 \). This shows that \( S \) is locally of finite type, and it is clear that \( S \) is not integrable.

---

\(^*)\) And repeated in [10],[11] and several other places.
REMARK. We say that \( S \) is involutive if \([X,Y] \in S\) whenever \( X \) and \( Y \) are in \( S \). Lobry [6] proves that an involutive set of real-analytic vector fields on a real-analytic manifold \( M \) is locally of finite type. He then comes to the (false) conclusion that every involutive set of real-analytic vector fields is integrable (cf. Part Two, Theorem 5 and Example 8.5.2.).

.3. CONDITION L for \( \tilde{S} \). Let \( \tilde{S} \) denote the set of all smooth vector fields \( X \) on \( M \) such that \( X(x) \in S(x) \) for every \( x \) in the domain of \( X \). We observe that if \( S \) is defined as in Example 2, then \( \tilde{S} \) is not locally of finite type. It is therefore natural to ask the following question.

QUESTION 1.3. Assume that \( \tilde{S} \) is locally of finite type. Does it follow that \( S \) is integrable?

Example 8.5.2 in Part Two shows that the answer is NO in the real-analytic case (unless the vector fields in \( S \) be locally everywhere defined). On the other hand, Theorem 2 below shows the answer is YES in the \( C^\infty \) case. The example in (1.5) shows that this is only a sufficient condition: \( S \) may be integrable even if \( \tilde{S} \) is not locally of finite type.

1.4. CONDITION K. Let \( x \in M \). We say that \( S \) satisfies the condition \((Kx)\) if there exist finitely many vector fields \( X_1, X_2, \ldots, X_p \) in \( S \), defined on a neighbourhood \( \Omega \) of \( x \), and continuous*) functions
\( \lambda_{ijk} : \Omega \to \mathbb{R} \) such that

*) It is sufficient to assume that \( \lambda_{ijk} \) are bounded measurable.
(K1) \( X_i(x) \) span \( S(x) \) and, for every \( y \in \mathbb{R} \),

\[
[X_i, X_j](y) = \sum_{k=1}^{p} \lambda_{ijk}(y)X_k(y), \quad 1 \leq i, j \leq p;
\]

(K2) for every \( X \) in \( S \), such that \( x \in \text{domain} \ X \), there exists \( \varepsilon > 0 \) and continuous function \( \lambda_{ij} : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R} \) such that, for every \( t \in [-\varepsilon, \varepsilon] \), and every \( i, 1 \leq i \leq p \),

\[
[X, X_i](x_t) = \sum_{i=1}^{p} \lambda_{ij}(t)X_j(x_t),
\]

where \( x_t = x^t.x \).

It is clear that \( (Lx) \implies (Kx) \). We say that \( S \) satisfies condition \( K \), if it satisfies the condition \( (Kx) \) for every \( x \in \mathbb{R} \).

**THEOREM 2.** Let \( S \) be a set of \( C^\infty \) vector fields on a \( C^\infty \) manifold \( M \) let \( \tilde{S} \) be defined as in 1.3. Then \( S \) is integrable if and if \( \tilde{S} \) satisfies the condition \( K \).

In particular, \( S \) is integrable if \( \tilde{S} \) is locally of finite type.

**LEMMA 1.4.1.** Assume that \( S \) satisfies the condition \( (Kx) \) and \( \lim S(x) = d \). Then there exists a box \((\psi, U, W)\) of \( M \) such that

(a) \( \psi(0, 0) = x \) and \( \dim U = d \);

(b) \( \psi^*(u, w).(U \times 0) \subseteq S(\psi(u, w)) \) for every \( (u, w) \) in the domain of \( \psi \).

(c) If \( Y \) is an arbitrary vector field in \( S \) such that \( x \in \text{domain} \ Y \) and if \( \sigma(t) = y^t.x \), then there exists \( \varepsilon > 0 \) such that \( \sigma(t) \in \psi(U, 0) \) for \( |t| < \varepsilon \).
PROOF. Assume that the vectorfields $X_1, \ldots, X_p$ satisfy the condition (K1) and (K2) on a neighbourhood $\Omega$ of $x$, and let $S^* = \{X_1|\Omega, \ldots, X_p|\Omega\}$. It follows at once from Theorem 6, Part One, that $S^*$ is homogeneous (and hence integrable) on $\Omega$. Since $S^*(x) = S(x)$ and $S^*(y) \subseteq S(y)$ for every $y$ in $\Omega$, there exists a box $(\psi, U, W)$ of $M$ which satisfies the conditions (a) and (b) (see Theorem 5 and its Corollary in Part One). Moreover, we may assume that $\psi(U, 0)$ is a neighbourhood of $x$ in $L^*$, where $L^*$ is the accessible set of $S^*$ through the point $x$.

Let now $\sigma(t) = Y^t(x)$ for some vectorfield $Y$ in $S$ such that $x \in \text{domain } Y$ and let $\gamma^t = (Y^t)^*(x)$. By Theorem 2, Part Two (or by the proof of Theorem 6, Part One), (K2) implies that, for $|t| \leq \varepsilon$,

$$\gamma^t S^*(x) = S^*(\sigma(t)).$$

Hence, for $|t| \leq \varepsilon$,

1. $\dim S^*(\sigma(t)) = \dim S^*(x) = k$ and
2. $\dot{\sigma}(t) = Y(\sigma(t)) = \gamma^t Y(x) \in S^*(\sigma(t))$.

Let $L$ be the union of the $k$-dimensional accessible sets of $S^*$. Then $L$ is a neat submanifold of $\Omega$ (Part Two, §9.3) and the tangent spaces of $L$ are spanned by $S^*$ because $S^*$ is homogeneous. By (1), $\sigma(t) \in L$ for $|t| \leq \varepsilon$ and so, by (2) and Proposition 9.3.1, Part Two

$$\sigma : ]-\varepsilon, \varepsilon[ \rightarrow L$$

is differentiable. Since $L^*$ is the connected component of $x$ in $L$, $\sigma(t) \in L^*$ for $|t| < \varepsilon$, whence follows the assertion (c) of our lemma.
LEMMA 1.4.2. If $\tilde{S}$ satisfies the condition $K$, then $\dim S(x)$ remains constant along the integral curves of vectorfields in $S$.

PROOF. Let $\sigma(t) = x^t_x$ for some $X$ in $S$ and some $x \in \text{domain } X$. Let $t_1 < t_2$ and assume that $\dim S(\sigma(t_1)) < \dim S(\sigma(t_2))$. Let $\ell = \max(\dim S(\sigma(t)) : t_1 \leq t \leq t_2)$. The set $I_0 = \{t \in [t_1, t_2] : \dim S(\sigma(t)) = \ell\}$ is relatively open in $I$. Let $J$ be a connected component of $I_0$ and let $t_0$ be the left-hand end-point of $J$. It is easily seen that $t_0 \notin I_0$. Without loss of generality, we assume that $t_0 = 0$, so that, for some $\varepsilon > 0$ and all $t, 0 < t < \varepsilon$,

$$\dim S(x) = k < \ell = \dim S(\sigma(t)).$$

Let now $(\psi, U, W)$ satisfy the conditions of Lemma 1.4.1 with $\tilde{S}$ in the place of $S$, and let $Z = \psi(U^1, 0)$. Using $\psi$ to identify a neighbourhood of $x$ in $M$ with an open subset $\Omega$ of $\mathbb{R}^n$, we may assume that

1. $x$ is the origin in $\mathbb{R}^n$;
2. $Z$ is the open unit ball in a $k$-dimensional subspace $E$ of $\mathbb{R}^n$;
3. every vectorfield $Y : \Omega \to E$ belongs to $\tilde{S}$;
4. If $Y$ is a vectorfield in $\tilde{S}$ and $x \in \text{domain } Y$, then there exists $\delta > 0$ such that, for $|t| < \delta$,

$$Y^t.x \in E.$$ 

Let $P$ be the orthogonal projection of $\mathbb{R}^n$ onto $E$ and let $Q = \text{id} - P$ be the projection onto $E^\perp$. If $Y : \Omega \to \mathbb{R}^n$ is an arbitrary vectorfield, then $PY \in \tilde{S}$ by (3). If $Y \in \tilde{S}$, then $QY = Y - PY \in \tilde{S}$. Using this, the fact that $\dim S(\sigma(t)) = \ell > \dim E$ for $0 < t < \varepsilon$, and suitable 'bump functions', it is easy to construct a vectorfield $Y$ in $\tilde{S}$ such that
(5) \( Y(y) \perp E \) for all \( y \in \Omega \) and, 

(6) for every \( \delta > 0 \), there exists \( t \) such that \( 0 < t < \delta \) and \( Y(\sigma(t)) \neq 0 \).

Let now \( u(t) = (PX + Y)^t.x \). By (4), there exists \( \delta > 0 \) such that

(7) \( u(t) \in E \) and \( \sigma(t) \in E \) for \( |t| < \delta \). Hence \( \dot{u}(t) = \\
= PX(u(t)) + Y(u(t)) \in E \) and so

(8) \( Y(u(t)) = 0 \) and

(9) \( \dot{u}(t) = PX(u(t)) \) for \( |t| < \delta \).

Since \( \delta(t) = X(\sigma(t)) \in E \), we have \( X(\sigma(t)) = PX(\sigma(t)) \) and

(10) \( \dot{\sigma}(t) = PX(\sigma(t)) \) for \( |t| < \delta \).

By (9) and (10), \( \sigma(t) = u(t) \) for \( |t| < \delta \) and so, by (8), \( Y(\sigma(t)) = 0 \) for \( |t| < \delta \), in contradiction with (6).

PROOF OF THEOREM 2. Assume that \( \vec{S} \) satisfies the condition \( \mathcal{K} \).

Let \( X \) be a vectorfield in \( S \), \( x \in \text{domain } X \), \( \sigma(t) = X^t.x \), and 
\( \gamma^t = (X^t)^*x(x) \). It follows easily from (K2) and from Theorem 6, 
Part One, that there exists \( \varepsilon > 0 \) such that, for \( |t| < \varepsilon \), 
\( \gamma^tS(x) \subset S(\sigma(t)) \).

Hence, by Lemma 1.4.2, \( \gamma^tS(x) = S(\sigma(t)) \) for \( |t| < \varepsilon \). By Lemma 1.1, 
\( X \) respects the distribution \( (S(x) : x \in M) \), which proves that \( S \) is 
homogeneous and, therefore, integrable.
Conversely, assume that $S$ is integrable and let $L$ be the accessible set of $S$ through the point $x$. By the Corollary of Theorem 5, Part One, there exists a coordinate system $(\xi_1, \ldots, \xi_n)$ on a neighbourhood $\Omega_M$ of $x$ in $M$ such that the vector fields $X_i = \partial/\partial \xi_i$ belong to $\tilde{S}$ for $1 \leq i \leq k$ and span $T_y M$ for $y \in \Omega_L$, where $\Omega_L$ is some neighbourhood of $x$ in $L$. If $X$ is a vector field in $\tilde{S}$ and $x \in$ domain $X$, then there exists $\epsilon > 0$ such that $x_t = x^t \cdot x \in \Omega_L$ for $|t| < \epsilon$. Hence, for any vector field $X$ in $\tilde{S}$ defined on a neighbourhood of $x$, $|t| < \epsilon$ lies

$$X(x_t) = \sum_{i=1}^{p} \lambda_i(t) X_i(x_t)$$

for some differentiable functions $\lambda_i : (-\epsilon, \epsilon) \to \mathbb{R}$. Since $S$ is integrable, $\tilde{S}$ is involutive and (*) applies in particular to vector fields $[X, X_j]$, $1 \leq j \leq p$, which proves that $\{X_1, X_2, \ldots, X_k\}$ satisfy the condition (K2). The condition (K1) follows at once from $[X_i, X_j] = 0$ ($1 \leq i, j \leq p$).

**AN EXAMPLE.** In Part One, Example 5.4, we have shown that an integrable system of vector fields need not be locally of finite type. We shall now show that the integrability of $S$ does not imply that $\tilde{S}$ (as defined in 1.3) is locally of finite type.

Let $\phi : \mathbb{R} \to \mathbb{R}$ be the $C^\infty$ function defined by $\phi(\xi) = e^{-1/\xi}$ for $\xi > 0$ and $\phi(\xi) = 0$ for $\xi \leq 0$. Let the vector field $X_2$ on $\mathbb{R}$ be defined by

$$X_2(\xi_1, \xi_2) = \phi(\xi_2) \partial/\partial \xi_2,$$

d let $X_1 = \partial/\partial \xi_1$ and $S = \{X_1, X_2\}$. Let $\tilde{S}$ be the set of all $C^\infty$ vector fields in the distribution $S(x) : X \in \mathbb{R}^2$ (Fig. 2).
To prove that $S$ is not locally of finite type, we argue by contradiction and assume that the vector fields $Y_1, \ldots, Y_p$ in $\tilde{S}$ satisfy the conditions (L.1) and (L.2) at the origin, with $\tilde{S}$ in the place of $S$. Let

$$Y_1 = \alpha_1 \frac{\partial}{\partial \xi_1} + \beta_1 \frac{\partial}{\partial \xi_2},$$

where $\beta_1(\xi_1, \xi_2) = 0$ for $\xi_2 < 0$. We may assume that the $C^\infty$ function $\alpha_1$ is $> 0$ on a neighbourhood of the origin and introduce a local diffeomorphism

$$\psi : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) : (\xi_1, \xi_2) \to Y_1(0, \xi_2).$$

Since $\psi$ maps the upper half-plane ($\xi_2 > 0$), $\xi_1$-axis and lower half-plane into themselves and since the conditions (L.1) and (L.2) are invariant under $C^\infty$ changes of coordinates, we may assume $Y_1 = \partial / \partial \xi_1$. 
Let $\theta : \mathbb{R} \to \mathbb{R}$ be an arbitrary $C^\infty$ function such that $\theta(\xi) = 0$ for $\xi \leq 0$ and let the vector field $X$ on $\mathbb{R}^2$ be defined by

$$X(\xi_1, \xi_2) = e^{\xi_1} \theta(\xi_2) \cdot \frac{\partial}{\partial \xi_2}.$$ 

Then $X \in \tilde{S}$, $[Y_1, X] = X$ and so (L.2) implies that

$$e^{\xi_1} \theta(\xi_2) = \sum_{j=1}^{p} \lambda_j(\xi_1, \xi_2) \theta_j(\xi_1, \xi_2)$$

for all $\xi_1$ and $\xi_2$ in a sufficiently small neighbourhood of the origin (depending on the function $\theta$). But this is absurd, since, on setting $\xi_1 = 0$, it contradicts the following lemma.

**LEMMA 5.3.** Let $A$ be the ring of germs at the origin of the continuous function $\mathbb{R} \to \mathbb{R}$. Let $B$ be the subring of $A$ generated by the $C^\infty$ functions $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(\xi) = 0$ for $\xi < 0$. Then $B$ is not contained in any ideal of $A$ generated by finitely many members of $B$.

**PROOF No. 1.** Let $S$ be the collection of all $C^\infty$ vector fields

$$X = \alpha \frac{\partial}{\partial \xi_1} + \beta \frac{\partial}{\partial \xi_2}$$

on $\mathbb{R}^2$ such that $\beta(\xi_1, \xi_2) = 0$ for $\xi_1 \leq 0$ (cf. Example 8.5.2, Part Two). Then $S = \tilde{S}$. If $B$ were contained in some ideal of $A$ generated by finitely many members of $B$, then $\tilde{S}$ would be locally of finite type and therefore, by Theorem 2, integrable, which is absurd.
PROOF No. 2. Let $B^*$ be the set of all the $C^\infty$ function $\theta : \mathbb{R} \to \mathbb{R}$ such that $\theta(t) = 0$ for $t \leq 0$. If $B$ is contained in some ideal of $A$ which is generated by finitely many members of $B$, then there exists functions $\beta_1, \beta_2, \ldots, \beta_p$ in $B^*$ such that, given any $\theta \in B^*$, there exist $\varepsilon > 0$ and continuous functions $\lambda_1, \lambda_2, \ldots, \lambda_p$ such that, for $|t| < \varepsilon$,

$$\theta(t) = \sum_{i=1}^{p} \lambda_i(t) \beta_i(t).$$

In particular, there exists $\varepsilon > 0$ and continuous functions $\lambda_{ij}$ such that, for $|t| < \varepsilon$,

$$\dot{\beta}_i(t) = \sum_{j=1}^{p} \lambda_{ij}(t) \beta_j(t).$$

As this is a homogeneous system of linear differential equations and as $\beta_i(0) = 0$ for $1 \leq i \leq p$, we have $\beta_i(t) = 0$ for all $t$, $|t| \leq \varepsilon$ and all $i$, $1 \leq i \leq p$, which is absurd.
§2. ALMOST ALL PAIRS OF VECTORFIELDS ARE IRREDUCIBLE

.1 INTRODUCTION. A set $S$ of vectorfields on $M$ is irreducible if the accessible sets of $S$ coincide with the connected components of $M$. By theorem 5, Part One, this is equivalent to saying that $\mathcal{S}(x) = T_xM$ for every $x$ in $M$.

In [18], Lobry probed that the set of irreducible pairs of $C^k$ vectorfields on a $C^\infty$ manifold $M$ is $C^k$-generic for $k \geq n^2 + n$. Later [19], he improved this to $k \geq 2n$. We shall show in this section that the set of irreducible pairs of $C^k$ vectorfields on a $C^{k+1}$ manifold is $C^k$-generic for all $k \geq 1$.

Lobry's proof can be roughly described as follows. Consider a differentiable function

$$\rho : A \times M \to Q,$$

where $A$ is the space of pairs of vectorfields and $Q$ is a manifold, and a stratified subset $W$ of $Q$ such that

$$\rho_a^{-1}(W) = \emptyset \Rightarrow a \text{ is irreducible},$$

where $\rho_a = \rho(a, -) : M \to Q$. If now $\rho$ is transversal to $W$ and $\text{codim} W > \text{dim} M$, Thom's transversality theorem implies that almost every pair $a$ is irreducible.

Our proof follows a similar pattern, with the difference that the results of Part One allow us to take a simpler $\rho$, $Q$ and $W$. We also go through the details of the transversality argument, so that our proof is more self-contained than Lobry's (and, in particular, independent of [20]).

*) Héctor Sussmann tells me that he has recently obtained some similar
For the sake of simplicity, we assume that \( k \) is finite and that \( M \) is a compact manifold. A similar, but a little more involved, proof shows that the assertion of Theorem 2.6 holds for any separable finite-dimensional \( C^{k+1} \) manifold \( M \) and for \( 1 \leq k \leq \infty \), with the Whitney \( C^k \) topology on the space of \( C^k \) vectorfields on \( M \).

2.2. A TRANSVERSALITY THEOREM. Recall that by an immersion we always mean a split immersion. By a submanifold of a differentiable manifold \( Q \) we mean an immersed submanifold such that the inclusion mapping is an embedding. A subset \( W \) of \( Q \) is of codimension \( \geq c \) if it is contained in a countable union of submanifolds of codimension \( \geq c \):

\[
(2.2.a) \quad W \subset \bigcup_n W_n, \quad W_n \text{ a submanifold of } Q, \quad \text{codim } W_n \geq c, \quad n = 1, 2, \ldots
\]

The next result is stated in such generality as we need in what follows; the proof is adapted from [17], §§18 and 19.

**PROPOSITION 2.2.** Let \( A, M \) and \( Q \) be \( C^r \) manifolds and let

\[
\alpha : A \times M \to Q \quad \text{and} \quad \beta : A \times M \to Q
\]

be \( C^r \) functions. Let \( W \) be a subset of \( Q \) of codimension \( \geq c \) and put

\[
A_W = \{ a \in A : \alpha_a^{-1} W \cap \beta_a^{-1} W \text{ is of codimension } \geq c \},
\]

where \( \alpha_a = \alpha(a,-) \) and \( \beta_a = \beta(a,-) : M \to Q \). Assume that

1. \( M \) has finite dimension \( d \);
2. \( A \) and \( M \) are second countable;
3. \( r > \max(0,d-c) \);
4. for every \( (a,x) \in A \times M \), at least one of the derivatives
\[ \alpha^*(a,x) : T_a A \times T_x M \to T_{\alpha(a,x)} Q \]

\[ \beta^*(a,x) : T_a A \times T_x M \to T_{\beta(a,x)} Q \]

split surjective.

Then \( A_w \) is a residual \(^*\) subset of \( A \).

**COROLLARY 2.2.** If \( c \geq d + 1 \), we can take \( r = 1 \) and

\[ A_w = \{ a \in A : \alpha_{-1}^{-1} w \cap \beta_{-1}^{-1} w = \emptyset \}. \]

**PROOF OF PROPOSITION 2.2.** Let \( w_n \) satisfy the condition (2.2.a)

\[ L_n = w_n \times Q, \quad R_n = Q \times w_n \text{ and } Z_n = w_n = \bigcap_{n=1}^{\infty} L_n \cap R_n. \]

Set \( f = (\alpha, \beta) : A \times M \to Q \times Q \) and let

\[ A_n = \{ a \in A : \mathrm{codim} f^{-1}(Z_n) \geq c \}. \]

\( \bigcap_{n=1}^{\infty} A_n \subset A_w \), it is sufficient to show that each \( A_n \) is residual.

From now on, \( n \) is assumed to be fixed.

Since \( A \times M \) is second countable, the hypothesis (4) implies that there exist a countable open cover \((H_m)_{m=1}^{\infty}\) of \( A \times M \) such that, for very \( m \), at least one of the functions

\[ \alpha|H_m \text{ or } \beta|H_m \]

is a submersion and therefore \( f|H_m \) is transversal to at least one of the manifolds \( L_n \) and \( R_n \). Let \( \Lambda = f^{-1}(L_n) \), \( P = f^{-1}(R_n) \) and let

\[ Q_m = \begin{cases} 
\Lambda \cap H_m & \text{if } f|H_m \text{ is transversal to } L_n \\
R \cap H_m & \text{if } f|H_m \text{ is not transversal to } L_n.
\end{cases} \]

\(^*\) We say that a set is residual if it contains a countable intersection of open dense sets.
Then $Q_m$ is a submanifold of $A \times M$ of codimension $q \geq c$. Let 
$p : A \times M \rightarrow A$ be the coordinate projection and let $p_m = p|Q_m$.

By ([17], Lemma 19.3), $p_m : Q_m \rightarrow A$ is a Fredholm map of constant
index $d - q \leq d - c < r$. Since $Q_m$ is embedded in $A \times M$, it is
second countable and so, by Smale's density theorem ([17], §16) the
set $B_m$ of regular values of $p_m$ is residual. If now $a \in B_m$,
then the mapping

$$\theta_a : M \rightarrow A \times M : x \rightarrow (a,x)$$

is transversal to $Q_m$ and so $\theta_a^{-1}(Q_m)$ is a submanifold of $M$ of
codimension $q \geq c$. Since $f^{-1}(Z_n) \subset \bigcup_{m=1}^{\infty} Q_m$ and $f_a = f \circ \theta_a$, we
have $f_a^{-1}(Z_n) \subset \bigcup_{m=1}^{\infty} \theta_a^{-1}(Q_m)$ and thus $A_n \supset \bigcap_{m=1}^{\infty} B_m$, Q.E.D.

2.3. A TRANSVERSALITY LEMMA. We need the following result.

**LEMMA 2.3.** Let $E$ be a separable Banach space, $M$ a compact
$n$-dimensional $C^1$ manifold and $Q$ a finite dimensional $C^k$ vector
bundle over $M$, where $k = 0$ or $1$. Let $W$ be a closed subset of $Q$;
if $k = 1$, assume that $\text{codim} W \geq n + 1$. Let

$$\rho : E^2 \times M \rightarrow Q$$

be a $C^k$ mapping and

$$E^2_W = \{(a,b) \in E^2 : \rho_{ab}^{-1}W \cap \rho_{ba}^{-1}W = \emptyset\},$$

where $\rho_{ab} = \rho(a,b,-) : M \rightarrow Q$. Assume that:
(1) for every $a \in E$, $a^\#: M \times E \to Q$ : $(x,b) \mapsto \rho(a,b,x)$ is a vector-bundle morphism;

(2) there exists an open dense subset $A$ of $E^2$ such that, for every $(a,b,x) \in A \times M$, at least one of the linear functions $a_x^\#: E \to Q_x$ and $b_x^\#: E \to Q_x$ is surjective.

Then $E^2_W$ is an open subset of $E^2$. If $k = 1$, then $E^2_W$ is open dense.

**PROOF.** The openness of $E^2_W$ follows at once from the fact that $W$ is closed and the function $E^2 \to C^0(M,Q^2) : (a,b) \mapsto (\rho_{ab},\rho_{ba})$ is continuous for the compact-open topology on $C^0(M,Q^2)$. The fact that, for $k = 1$, $E^2_W$ is residual follows at once from Proposition 2.2 on setting $\alpha = \rho|A \times M$, $\beta(a,b,x) = \alpha(b,a,x)$ and observing that, locally, the derivative of $\alpha$ at $(a,b,x)$ is given by the matrix

$$
\begin{pmatrix}
\ast & a_x^\# & \ast \\
0 & 0 & \text{id}_M
\end{pmatrix}
$$

where the columns represent the partial derivatives of $\alpha$ with respect to the first, second and third coordinates, the top row corresponds to the 'fibre coordinate' of $Q$ and the bottom row to the 'base' coordinate.

2.4. A STRATIFIED SET. The following result is elementary and the proof is given here only for the sake of completeness.

**LEMMA 2.4.** Let $T$ be an $n$-dimensional $C^1$ vector-bundle over a manifold $M$ and let $Q = \bigoplus^p T$ be the Whitney sum of $p$ copies of $T$, $p \geq n$. For each $x \in M$, let $W_x$ be the subset of $Q_x = T_x^p$ consisting of those $p$-tuples $(v_1,v_2,\ldots,v_p) \in Q_x$ which do not contain a basis of $T$, and let
Then $W$ is a closed subset of $Q$ of codimension $p - n + 1$. More precisely, there exists a finite sequence $(V_i)$ of submanifolds of $M$ such that

$$W = \bigcup_{i=1}^{\tau} V_i,$$

where

$$\tau = \sum_{k=0}^{n-1} \binom{p}{k},$$

and there are exactly $\binom{p}{k}$ $V_i$'s of codimension $(p-k)(n-k)$ for each $k$, $0 \leq k \leq n-1$.

**Proof.** It is clear that, locally, $W$ is a product of an open subset of $M$ with $W_x$ and so it is sufficient to prove the lemma if $M$ is a point and $T$ is a single $n$-dimensional vector space.

Let now $\alpha$ be a subset of $\{1,2,\ldots,p\}$ and let $V_\alpha$ be the set of all $p$-tuples $(v_1,v_2,\ldots,v_p)$ in $T^p$ such that the vectors $\{v_i : i \in \alpha\}$ form a basis of the vector space spanned by $\{v_1,v_2,\ldots,v_p\}$. It is clear that

$$W = \bigcup_{0 \leq |\alpha| \leq n-1} V_\alpha,$$

and so it is sufficient to show that each $V_\alpha$ is a submanifold of $T^p$ of codimension $(p-|\alpha|)(n-|\alpha|)$.

To see this, assume that $\alpha = \{1,2,\ldots,k\}$ and let $A$ be the set of all linearly independent $k$-tuples (k-frames) in $T^k$. Then $A$ is an open subset of $T^k$ and $V_\alpha$ is the kernel of the vector-bundle morphism

$$\theta : A \times T^{p-k} + A \times \bigoplus_{i=0}^{p-k} k+1 T.$$
where, for each \( a = (a_1, a_2, \ldots, a_k) \in A \), we put \( \tilde{a} = a_1 \wedge a_2 \wedge \ldots \wedge a_k \)
and
\[
\theta_a (v_{k+1}, v_{k+2}, \ldots, v_p) = (\tilde{a} \wedge v_{k+1}, \tilde{a} \wedge v_{k+2}, \ldots, \tilde{a} \wedge v_p).
\]

The result now follows at once from the fact that \( \text{rank} \theta_a = (p-k)(n-k) \)
for every \( a \in A \).

2.5. DIFFERENTIABILITY. From now on, \( M \) is an \( n \)-dimensional
compact \( C^{k+1} \) manifold and \( k \) is a finite integer \( \geq 1 \). Let \( E = \Gamma^k \)
be the vector space of \( C^k \) vector fields on \( M \) together with its
\( C^k \)-topology. Fixing a Finsler structure on the \( k \)-jet bundle
of sections of \( TM \) and taking the corresponding \( \text{sup-norm} \) we make
\( E \) into a separable Banach space ([17], Theorem 12.2).

**LEMMA 2.5.** The mapping
\[
\phi : E \times E \times M \times R \rightarrow TM,
\]
\[
(X, Y, x, t) \rightarrow ((X^t) \ast (x))^{-1} Y(X^t x)
\]
is of the class \( C^{k-1} \).

**PROOF.** Let \( v \in T^*_x M \) and let
\[
u(t) = u(t, x, v) = (X^t) \ast (x) \cdot v.
\]
Then \( u \) is the integral curve of the differential equation
\[
(1) \quad \ddot{u}(t) = \xi(u(t), X), \quad u(0) = v
\]
on \( TM \), where the parametrized vector field
\[
\xi : TM \times E \rightarrow T(TM)
\]
is defined by the equation \( \xi(u, X) = \omega TX(u) \) and \( \omega : T^2 M \rightarrow T^2 M \) is the
involution which 'interchanges the second and third coordinates'
(cf. [17], p. 17).
Locally, the equation (1) is given by the system of 2n equations
\[ \dot{x} = X(x), \quad \dot{u} = DX(x)u. \]

Since
\[ \xi : TM \times \Gamma^k \xrightarrow{\delta} TM \times \Gamma^{k-1} \xrightarrow{\text{ev}} T^2M \xrightarrow{\text{ev}} T^2M, \]
where \( \delta = \text{id} \times T \), \( \xi \) is of the class \( C^{k-1} \) ([17], Theorem 12.3). By the theorem on dependence of the solutions of differential equations on initial conditions and parameters [3],
\[ u : R \times E \times TM \rightarrow TM \]
\[ (t, X, v) \mapsto u(t, X, v) \]
is of the class \( C^{k-1} \). Similarly, we have \( C^k \)-functions
\[ \lambda : E \times R \times M \rightarrow M : (X, t, x) \mapsto x^Tm \]
and
\[ \mu : E \times (E \times R \times M) \xrightarrow{\text{id} \times \lambda} E \times M \xrightarrow{\text{ev}} M, \]
\[ (Y, X, t, x) \mapsto Y(x^Tm), \]
and so it is sufficient to note that
\[ \phi(X, Y, x, t) = u(-t, X, \mu(Y, X, t)). \]

2.6. RESULT. We are now in a position to prove the following theorem.

THEOREM 2.6. Let \( 1 \leq k < \infty \) and let \( M \) be a compact \( C^{k+1} \) manifold.
Let \( E \) be the Banach space of \( C^k \) vectorfields on \( M \) and let \( P \subset E^2 \)
be the set of all irreducible pairs of \( C^k \) vectorfields on \( M \). Then \( P \) contains an open dense subset of \( E^2 \).
PROOF. Let $T = TM$, $Q = \bigoplus_{n=1}^{2n} T$ be the Whitney sum of $2n$ copies of $T$ and

$$W = \bigcup_{x \in M} W_x,$$

where $W_x$ consists of those $2n$-tuples $(v_1, v_2, \ldots, v_{2n}) \in Q_x = T_{x}^{2n}$ which do not contain a basis of $T_x$. By Lemma 2.4, $W$ is a closed subset of $Q$ of codimension $n + 1$.

Let $\phi$ be the mapping in Lemma 2.5,

$$\phi : E^2 \times M \times R \to T : (X, Y, x, t) \mapsto ((X^t)(x))^{-1} Y(X^t, x),$$

and let

$$\phi_k = \phi(-, -, -, k) : E^2 \times M \to T.$$

The range of the $C^{k-1}$ mapping

$$\psi = (\phi_0, \phi_1, \ldots, \phi_{2n-1}) : E^2 \times M \to \bigoplus_{n=1}^{2n} T = T^{2n}$$

is contained in the $C^{k-1}$ submanifold $Q$ of $T^{2n}$ and so $\psi$ defines a $C^{k-1}$ mapping $\rho$ such that the diagram

$$\begin{array}{ccc}
E^2 \times M & \xrightarrow{\psi} & T^{2n} \\
\rho \downarrow & & \downarrow \text{inclusion} \\
Q & & 
\end{array}$$

commutes. It is easily seen that, for every $X \in E$,

$$X^\# : M \times E \to Q : (x, Y) \mapsto \rho(x, Y, x)$$

is a vector-bundle morphism. Let now $X \in E$ and $x \in M$ be fixed and let

$$(1) \quad x_t = X^t \cdot x \quad \text{and} \quad \gamma_t = ((X^t)(x))^{-1}.$$

Then

$$X^\#_x : E \to Q_x : y \mapsto (Y(x), \gamma_1 Y(x_1), \ldots, \gamma_{2n-1} Y(x_{2n-1})).$$

If $X(x) \neq 0$, then the points $x, x_1, \ldots, x_{2n-1}$ are mutually distinct and so $X^\#_x : E \to Q_x$ is surjective.
Let $A \subset E^2$ be the set of pairs $(X,Y)$ such that, for every $x \in M$, at least one of the vectors $X(x), Y(x)$ is non-zero. A simple transversality argument applied to the evaluation map 

$$E^2 \times M \to \bigoplus_T$$

shows that $A$ is an open dense subset of $E^2$. If $(X,Y) \in A$ then, for every $x \in M$, at least one of the functions 

$$X^x_\#: E \to Q_x, \quad Y^x_\#: E \to Q_x$$

is surjective. We claim that 

$$E^w = \{ (X,Y) \in E^2 : \rho^{-1}_X \cap \rho^{-1}_Y = \emptyset \}$$

is an open dense subset of $E$. For $k \geq 2$ this follows at once from Lemma 2.3. If $k = 1$, then Lemma 2.3 implies that $E^w$ is open; the density of $E^w$ follows from the result for $k = 2$ and from the fact that the space $\Gamma^2$ of $C^2$ vectorfields on $M$ is dense in $E = \Gamma^1$. (Note that we may assume without loss of generality that $M$ is a $C^\infty$ manifold; cf. [15], p. 15.)

If $S = \{X,Y\}$ and $(X,Y) \in E^w$, then, for every $x \in M$, the collection of vectors 

$$Y(x), Y_1(x_1), \ldots, Y_{2n-1}(x_{2n-1});$$

$$X(x), \delta_1(x_1), \ldots, \delta_{2n-1}(x_{2n-1})$$

(where $x_t$ and $y_t$ are as in (1), $y_t = Y^t.x$ and $\delta_t = ((Y^t) \star (x))^{-1}$) contains a basis of $T_xM$. Hence $S(x) = T_xM$ for every $x \in M$ and, by Theorem 5 of Part One, $(X,Y)$ is an irreducible pair of vectorfields.
REFERENCES
PART ONE


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PART TWO


PART THREE


