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TJURINA AND MILNOR NUMBERS OF MATRIX SINGULARITIES

V. GORYUNOV AND D. MOND

Abstract

To gain understanding of the deformations of determinants and Pfaffians resulting from deformations of matrices, the deformation theory of composites $f \circ F$ with isolated singularities is studied, where $f : Y \longrightarrow \mathbf{C}$ is a function with (possibly non-isolated) singularity and $F : X \longrightarrow Y$ is a map into the domain of f, and F only is deformed. The corresponding $T^1(F)$ is identified as (something like) the cohomology of a derived functor, and a canonical long exact sequence is constructed from which it follows that

$$\tau = \mu(f \circ F) - \beta_0 + \beta_1,$$

where τ is the length of $T^1(F)$ and β_i is the length of $\mathrm{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_Y/J_f, \mathcal{O}_X)$. This explains numerical coincidences observed in lists of simple matrix singularities due to Bruce, Tari, Goryunov, Zakalyukin and Haslinger. When f has Cohen–Macaulay singular locus (for example when f is the determinant function), relations between τ and the rank of the vanishing homology of the zero locus of $f \circ F$ are obtained.

0. *Introduction*

In [**1**], Bill Bruce classified simple singularities of symmetric matrix families with respect to a natural equivalence relation (see Section 2 below). The very first look at his tables reveals a rather unexpected relation peculiar to two-parameter families: the dimension of the base of a matrix miniversal deformation coincides with the Milnor number of the determinant of the family. This observation was the main motivation for the paper [**3**], where equality of the Tjurina and Milnor numbers for hypersurface sections of an isolated hypersurface singularity was proved. That provided a partial explanation for the symmetric matrix problem, in the case of 2×2 matrices.

In the present paper, we prove that $\tau = \mu$ for two-parameter families of symmetric matrices of any order. We also prove similar statements for two other closely related matrix classification problems: for arbitrary square matrices depending on three parameters (as conjectured in [**14**]), and for skew-symmetric $2k \times 2k$ matrices in five variables. The key to the proof is to switch from the Koszul complex used in [**3**] to an appropriate free resolution of a determinantal or Pfaffian variety. Very suprisingly, these resolutions (introduced in [**15**, **17**, **18**]) have all been known for many years and use exactly the Lie algebras involved in the matrix classifications introduced recently in [**1**–**3**, **14**, **16**].

The coincidence of τ and μ in the three classifications is a particular case of our main result, Theorem 1.5, on the relation between these two numbers for sections, with isolated singularity, of a possibly non-isolated hypersurface singularity

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$(V, 0) \subset (\mathbf{C}^n, 0)$, obtained by pulling back V by a map $F : (\mathbf{C}^m, 0) \longrightarrow (\mathbf{C}^n, 0)$:

$$\tau = \mu - \beta_0 + \beta_1, \tag{1}$$

where the β_i are the Betti numbers of the pull-back of a free resolution of the jacobian algebra of the hypersurface singularity (and thus are the ranks of certain Tor modules). The main result of [**3**], on sections of an isolated hypersurface singularity, is also a special case.

The formula (1) has a topological interpretation when the singular subspace of $V = f^{-1}(0)$ is Cohen–Macaulay. In the case of families of matrices (symmetric, general, skew) in two or three, three or four, and five or six variables respectively, the right-hand side is in fact the rank of the vanishing homology of $V(\det(S_t))$ (or the Pfaffian $V(\mathrm{Pf}(S_t))$ in the skew-symmetric case) for a generic perturbation S_t of the matrix S. This is the generalised Milnor number, in the sense of the rank of the vanishing homology of the nearby stable object, appropriate to the problem considered. In two, three and five variables, $V(\det(S))$, or $V(\mathrm{Pf}(S))$ in the skew case, is smoothed in a deformation of S; in dimensions 3, 4 and 6 it is not. In all these cases, however, one has

$$\text{rank of the vanishing homology} = \text{Tjurina number.}$$

Closely related results on sections of free divisors were obtained in a number of papers by Damon, for example [**9**, **10**], and in the paper [**12**] by Damon and Mond. Free divisors in $\mathbf{C}^n$ are characterised by having a singular subspace which is Cohen–Macaulay of dimension $n - 2$. Our topological results here concern sections with isolated singularity of hypersurfaces whose singular subspace is Cohen–Macaulay of arbitrary dimension. In [**12**] the freeness of the divisors was important in allowing an easy proof of conservation of multiplicity (analogous to conservation of the Milnor number in a deformation of a function singularity), which was the basis for the comparison between the vanishing homology and the module of first-order deformations. Here we use the conservation of the Milnor number itself.

The structure of the paper is as follows.

The proof of our main theorem here is a modification of the arguments used (not very explicitly) in [**3**] to prove (1) for sections of isolated hypersurface singularities. We generalise this in Section 1 by identifying the T^1 of a section of a singular hypersurface as something rather like (the homology of) a derived functor associated to the comparison between the jacobian algebra of the section and the pull-back of the jacobian algebra of the hypersurface singularity. This allows us to embed the T^1 in a canonical long exact sequence, from which (1) follows in case the section has isolated singularity.

Section 2 deals in detail with various notions of equivalence of matrix families, and Section 3 describes the commutative algebra which allows us to relate the various notions of equivalence to one another, and to prove conservation of multiplicity. Section 4 builds on the material of the previous sections to relate τ to the rank of the vanishing homology, and in Section 5 we comment on Cohen–Macaulay properties of the relative T^1's and on some related results which suggest that the discriminants in the matrix versal base spaces are free divisors.

0.1. *Notation*

At the urging of Kyoji Saito, in this paper we have harmonised our notation with standard notation in algebraic geometry. Given a divisor V in a complex space X, it has been usual in singularity-theory papers to denote by $\mathrm{Der}(\log V)$ (or $\mathrm{Derlog}(V)$) the $\mathcal{O}_X$-module of vector fields on X which are tangent to V at its smooth points, and by $\Omega^1(\log V)$ the $\mathcal{O}_X$-module of 1-forms with logarithmic poles along V. Since these two modules are mutually dual, the conventions of algebro-geometric notation would insist that one of them have a $-\log V$ in parentheses rather than $\log V$. As a logarithmic pole is after all a pole, and k-forms with a (first-order) pole along V are denoted $\Omega^k(V)$, it is clear that it has to be $\mathrm{Der}(\log V)$ that accepts the minus sign and henceforth becomes $\mathrm{Der}(-\log V)$. Similarly, we have replaced the notation $\mathrm{Der}(\log f)$, for the module of vector fields tangent to all the level sets of a function f, by $\mathrm{Der}(-\log f)$.

1. *Equivalence and deformations of sections of hypersurfaces and functions*

Consider a pair consisting of a function $f : (\mathbf{C}^n, 0) \longrightarrow (\mathbf{C}, 0)$ and a map $F : (\mathbf{C}^m, 0) \longrightarrow (\mathbf{C}^n, 0)$. Let $V = f^{-1}(0)$. We seek to describe the deformations of F in relation to f, with a view to understanding the deformations of $F^{-1}(V)$. Many of our formulae will involve both $\mathcal{O}_{\mathbf{C}^m}$ and $\mathcal{O}_{\mathbf{C}^n}$; where possible, we will use $\mathcal{O}$ to abbreviate $\mathcal{O}_{\mathbf{C}^m}$, but *never* $\mathcal{O}_{\mathbf{C}^n}$.

DEFINITION 1.1. Two map-germs $F, F' : (\mathbf{C}^m, 0) \longrightarrow (\mathbf{C}^n, 0)$ are called $\mathcal{K}_f$-*equivalent* if there exist diffeomorphisms Φ of $(\mathbf{C}^m \times \mathbf{C}^n, 0)$ and φ of $(\mathbf{C}^m, 0)$, such that the following hold.

(1) $\pi_1 \circ \Phi = \varphi \circ \pi_1$ (that is, Φ lifts φ).
(2) $f \circ \pi_2 \circ \Phi = f \circ \pi_2$ (that is, Φ preserves f).
(3) Φ induces a diffeomorphism $\mathrm{graph}(F) \longrightarrow \mathrm{graph}(F')$.

This equivalence was introduced in [**12**]. It is closely related to $\mathcal{K}_V$-equivalence, introduced by Damon in [**8**], in which (2) is replaced by

(2′) Φ sends $\mathbf{C}^m \times \{f = 0\}$ to itself.

If F and F' are $\mathcal{K}_f$-equivalent then $f \circ F$ and $f \circ F'$ are right-equivalent, and if F and F' are $\mathcal{K}_V$ -equivalent then $f \circ F$ and $f \circ F'$ are contact-equivalent.

The extended tangent space to the $\mathcal{K}_f$-orbit of F is

$$T\mathcal{K}_f F = tF(\theta_{\mathbf{C}^m}) + F^*(\mathrm{Der}(-\log f)),$$

where the following hold.

(i) $\theta_{\mathbf{C}^m}$ is the space of germs of holomorphic vector fields on $(\mathbf{C}^m, 0)$.
(ii) $tF : \theta_{\mathbf{C}^m} \longrightarrow \theta(F) := F^*(\theta_{\mathbf{C}^n})$ is the sheafification of the derivative dF.
(iii) $\mathrm{Der}(-\log f) \subset \theta_{\mathbf{C}^n}$ is the $\mathcal{O}_{\mathbf{C}^n}$-module of vector fields annihilating f, and $F^*(\mathrm{Der}(-\log f))$ is the $\mathcal{O}_{\mathbf{C}^m}$-submodule of $\theta(F)$ generated by the composites with F of the vector fields in $\mathrm{Der}(-\log f)$.

The extended tangent space to the $\mathcal{K}_V$ orbit of F is

$$T\mathcal{K}_f F = tF(\theta_m) + F^*(\mathrm{Der}(-\log V)),$$

where $\mathrm{Der}(-\log V)$ is the $\mathcal{O}_{\mathbf{C}^n}$-module of vector fields on $\mathbf{C}^n$ which are tangent to V at its smooth points. Denote $\theta(F)/T\mathcal{K}_f F$ and $\theta(F)/T\mathcal{K}_V F$ respectively by $T^1_{\mathcal{K}_f}F$ and $T^1_{\mathcal{K}_V}F$.

Notice that

$$\frac{\theta(F)}{T\mathcal{K}_V F}\bigotimes_{\mathcal{O}_{\mathbf{C}^m,x}}\mathbf{C} = \frac{T_{F(x)}\mathbf{C}^n}{d_x F(T_x\mathbf{C}^m) + T^{\log}_{F(x)}V},$$

where for any point $y \in \mathbf{C}^n$, $T^{\log}_y V = \{\zeta(y) : \zeta \in \mathrm{Der}(-\log V)_y\}$ is the *logarithmic tangent space* to V at y. Thus $T^1_{\mathcal{K}_V}F$ measures failure of 'logarithmic transversality' (or algebraic transversality, in Damon's terminology) of F to V. The geometric interpretation of $T^1_{\mathcal{K}_f}F \otimes \mathbf{C}$ is less clear (see Remark 4.7(iii) below) although $T^1_{\mathcal{K}_f}F$ in some sense measures failure of logarithmic transversality of F to the level sets of f.

Both $\mathcal{K}_V$ and $\mathcal{K}_f$ are 'geometric subgroups' of the group of all diffeomorphism-germs, and so by Damon's general theory [**7**] the usual theorems of singularity theory apply: finite determinacy, infinitesimal criterion for versality, and so on. In particular $T^1_{\mathcal{K}_f}F$ and $T^1_{\mathcal{K}_V}F$ are the tangent spaces at 0 to the (smooth) miniversal base-spaces of F for the two equivalences.

Now we describe another approach to these two deformation theories, which identifies the two T^1 as something resembling a derived functor. By this means we are able to locate them in long exact sequences which provide solutions to the problems that prompted this paper. (Of course this is not how the solution was first found! Nevertheless this seems to be the most canonical presentation.)

Consider the two surjective comparison maps

$$\frac{\mathcal{O}_{\mathbf{C}^m}}{J_{f\circ F}} \longrightarrow \frac{\mathcal{O}_{\mathbf{C}^m}}{F^*(J_f)}$$

and

$$\frac{\mathcal{O}_{\mathbf{C}^m}}{(f\circ F) + J_{f\circ F}} \longrightarrow \frac{\mathcal{O}_{\mathbf{C}^m}}{F^*((f) + J_f)}.$$

By considering free resolutions of the modules involved, we are going to incorporate these maps into long exact sequences also involving $T^1_{\mathcal{K}_f}F$ and $T^1_{\mathcal{K}_V}F$.

Let $\mathbf{L}_\bullet$ and $\widetilde{\mathbf{L}}_\bullet$ be $\mathcal{O}_{\mathbf{C}^n}$-free resolutions of $\mathcal{O}_{\mathbf{C}^n}/J_f$ and $\mathcal{O}_{\mathbf{C}^n}/(f) + J_f$, let $\mathcal{K}_\bullet(f)$ be the Koszul complex on the first-order partials of f, and let $\widetilde{\mathbf{K}}_\bullet(f)$ be $\mathbf{K}_\bullet(f)$ augmented by another generator in degree 1, mapping onto f in degree 0 (so that $H_0(\widetilde{\mathbf{K}}_\bullet(f)) = \mathcal{O}_{\mathbf{C}^n}/(f)+J_f$). By lifting the identity maps on $\mathcal{O}_{\mathbf{C}^n}/J_f$ and $\mathcal{O}_{\mathbf{C}^n}/(f)+J_f$, we obtain morphisms of complexes

$$\mathbf{K}_\bullet(f) \longrightarrow \mathbf{L}_\bullet$$

and

$$\widetilde{\mathbf{K}}_\bullet(f) \longrightarrow \widetilde{\mathbf{L}}_\bullet.$$

These complexes and morphisms can be pulled back by F (in other words, tensored over $\mathcal{O}_{\mathbf{C}^n}$ with $\mathcal{O}_{\mathbf{C}^m}$). There is a natural morphism

$$\bigwedge^\bullet tF : \mathbf{K}_\bullet(f\circ F) \longrightarrow F^*(\mathbf{K}_\bullet(f)),$$

and by taking the direct sum of tF with the identity map $\mathcal{O} \longrightarrow \mathcal{O}$ in degree 1 (recall that we frequently abbreviate $\mathcal{O}_{\mathbf{C}^m}$ simply to $\mathcal{O}$) we also obtain a morphism

$$\bigwedge^\bullet \widetilde{tF} : \widetilde{\mathbf{K}}_\bullet(f\circ F) \longrightarrow F^*(\widetilde{\mathbf{K}}_\bullet(f)).$$

By composing these with the pulled-back morphisms mentioned above, we obtain morphisms of complexes

$$\phi_f : \mathbf{K}_\bullet(f\circ F) \longrightarrow F^*(\mathbf{L}_\bullet)$$

and

$$\phi_V : \widetilde{\mathbf{K}}_\bullet(f\circ F) \longrightarrow F^*(\widetilde{\mathbf{L}}_\bullet).$$

Let $C_\bullet(\phi_f)$ and $C_\bullet(\phi_V)$ be the cones (cf. [**13**, pp. 153–158], but the definition is recalled below) on these morphisms of complexes. Then our main technical result is the following.

THEOREM 1.2.

$$T^1_{\mathcal{K}_f} F \simeq H_1(C_\bullet(\phi_f))$$

and

$$T^1_{\mathcal{K}_V} F \simeq H_1(C_\bullet(\phi_V)).$$

Proof. Both statements are straightforward consequences of the definitions. If $\psi : A_\bullet \longrightarrow B_\bullet$ is a morphism of complexes then the cone $C_\bullet(\psi)$ has $C_n(\psi) = A_{n-1} \oplus B_n$ and differential taking (a_{n-1}, b_n) to $(-d(a_{n-1}), d(b_n) - \psi(a_{n-1}))$.

The following diagram shows the morphism ϕ_f.

$$\begin{array}{ccccccccccc}
\mathbf{K}_\bullet(f\circ F) & : & \dots & \xrightarrow{d(f\circ F)} & \bigwedge^2 \mathcal{O}^m & \xrightarrow{d(f\circ F)} & \mathcal{O}^m & \xrightarrow{d(f\circ F)} & \mathcal{O} & \longrightarrow & 0 \\
\phi_f \downarrow & & \dots & & \downarrow & & dF\downarrow & & \mathrm{id}\downarrow & & \\
F^*(\mathbf{L}_\bullet) & : & \dots & \longrightarrow & \mathcal{O}^r & \xrightarrow{F^*(\alpha_1)} & F^*(\theta_{\mathbf{C}^n}) & \xrightarrow{F^*(df)} & \mathcal{O} & \longrightarrow & 0
\end{array} \qquad (2)$$

The cone is the total complex of this (rather small) double complex; its modules are direct sums along the south-west to north-east parallels, and its differential runs south-east. Note that the image of $F^*(\alpha_1)$ is $F^*(\mathrm{Der}(-\log f))$.

We have

$$Z_1(C_\bullet(\phi_f)) = \{(a,\xi) \in \mathcal{O}\oplus F^*(\theta_{\mathbf{C}^n}) : F^*(tf)(\xi) = a\} = \{(F^*(tf)(\xi),\xi) : \xi \in F^*(\theta_{\mathbf{C}^n})\},$$

and $H_1(C_\bullet(\phi_f))$ is the quotient of this by

$$\{(-t(f\circ F)(\eta), \zeta - tF(\eta)) : \eta \in \theta_{\mathbf{C}^m}, \zeta \in F^*(\mathrm{Der}(-\log f))\}.$$

Under projection to $F^*(\theta_{\mathbf{C}^n})$, forgetting the first component, this quotient maps isomorphically to

$$\frac{F^*(\theta_{\mathbf{C}^n})}{\{\zeta - tF(\eta) : \eta \in \theta_{\mathbf{C}^m}, \zeta \in F^*(\mathrm{Der}(-\log f))\}} = T^1_{\mathcal{K}_f} F.$$

In the case of $C_\bullet(\phi_V)$, we can choose $\widetilde{L}_\bullet$ to take the form

$$\dots \longrightarrow \widetilde{L}_2 \xrightarrow{d_2} \theta_{\mathbf{C}^n} \oplus \mathcal{O}_{\mathbf{C}^n} \longrightarrow \mathcal{O}_{\mathbf{C}^n} \longrightarrow 0,$$

with the image of d_2 consisting of pairs (ζ, b) such that $tf(\zeta) + b\cdot f = 0$, so that $\zeta \in \mathrm{Der}(-\log V)$. Thus

$$Z_1(C_\bullet(\phi_V)) = \{(a,\eta,b) \in \mathcal{O}\oplus F^*(\theta_{\mathbf{C}^n}) \oplus \mathcal{O} : a = t(f\circ F)(\eta) + (f\circ F)b\}.$$

This projects isomorphically to $F^*(\theta_{\mathbf{C}^n}) \oplus \mathcal{O}$ by forgetting the first component. Also $B_1(C_\bullet(\phi_V))$ consists of sums of terms of the form $(0,\zeta,b) \in \mathcal{O} \oplus F^*(\theta_{\mathbf{C}^n}) \oplus \mathcal{O}$ such that $\zeta \in F^*(\mathrm{Der}(-\log V))$ and $F^*(tf)(\zeta) + b \cdot f \circ F = 0$, coming from $F^*(\widetilde{L}_2)$, together with terms of the form $(--, tF(\eta), a)$ coming from $\widetilde{K}_1$ (we do not care what is in the first component). By projecting this into $F^*(\theta_{\mathbf{C}^n}) \oplus \mathcal{O}$, forgetting the first component, we see that

$$H_1(C_\bullet(\phi_V)) = \frac{F^*(\theta_{\mathbf{C}^n}) \oplus \mathcal{O}}{\{(\zeta - tF(\eta), b-a) :\ F^*(tf)(\zeta) + b \cdot f \circ F = 0, \eta \in \theta_{\mathbf{C}^m}, a \in \mathcal{O}\}}.$$

This is isomorphic to

$$\frac{F^*(\theta_{\mathbf{C}^n})}{tF(\theta_{\mathbf{C}^m}) + F^*(\mathrm{Der}(-\log V))},$$

that is, to $T^1_{\mathcal{K}_V} F$. □

If $C_\bullet(\phi)$ is the cone on a map of complexes $\phi : A_\bullet \longrightarrow B_\bullet$, there is a long exact sequence of homology

$$\ldots \longrightarrow H_k(A_\bullet) \longrightarrow H_k(B_\bullet) \longrightarrow H_k(C_\bullet(\phi)) \longrightarrow H_{k-1}(A_\bullet) \longrightarrow \ldots$$

constructed by completely standard means (see for example [**13**]). Thus from Theorem 1.2 we deduce the following.

Corollary 1.3. *There are exact sequences*

$$\ldots \longrightarrow H_1(\mathbf{K}_\bullet(f \circ F)) \longrightarrow H_1(F^*(\mathbf{L}_\bullet)) \longrightarrow T^1_{\mathcal{K}_f} F \longrightarrow \frac{\mathcal{O}_{\mathbf{C}^m}}{J_{f\circ F}} \longrightarrow \frac{\mathcal{O}_{\mathbf{C}^m}}{F^*(J_f)} \longrightarrow 0 \tag{3}$$

and

$$\begin{aligned} \ldots \longrightarrow\ & H_1(\widetilde{\mathbf{K}}_\bullet(f \circ F)) \longrightarrow H_1(F^*(\widetilde{\mathbf{L}}_\bullet)) \longrightarrow T^1_{\mathcal{K}_V} F \\ \longrightarrow\ & \frac{\mathcal{O}_{\mathbf{C}^m}}{(f \circ F) + J_{f\circ F}} \longrightarrow \frac{\mathcal{O}_{\mathbf{C}^m}}{F^*((f) + J_f)} \longrightarrow 0 \end{aligned} \tag{4}$$

in which the maps $T^1_{\mathcal{K}_f} F \longrightarrow \mathcal{O}_{\mathbf{C}^m}/J_{f\circ F}$ and $T^1_{\mathcal{K}_V} F \longrightarrow \mathcal{O}_{\mathbf{C}^m}/(f \circ F) + J_{f\circ F}$ are induced by $F^*(tf) : \theta(F) \longrightarrow \mathcal{O}_{\mathbf{C}^m}$.

Remark 1.4. Let $g = f \circ F$. Then $\mathcal{O}_{\mathbf{C}^m}/J_g$ is the T^1 of g for right-equivalence. The morphism $T^1_{\mathcal{K}_f} F \longrightarrow T^1 g$ in (3) is a map between deformation functors, telling us which of the first-order deformations of g we can get by deforming F alone. A similar statement holds for $T^1_{\mathcal{K}_V} F \longrightarrow \mathcal{O}_{\mathbf{C}^m}/(g) + J_g$, with contact-equivalence in place of right-equivalence.

In most of what follows we use (3) to compute $\tau_{\mathcal{K}_f} F := \dim_{\mathbf{C}} T^1_{\mathcal{K}_f} F$ in cases where $f \circ F$ has isolated singularity. In such cases the Koszul complex $\mathbf{K}_\bullet(f \circ F)$ is acyclic. The homology of $F^*(\mathbf{L}_\bullet)$ computes $\mathrm{Tor}^{\mathcal{O}_{\mathbf{C}^n}}_\bullet(\mathcal{O}_{\mathbf{C}^n}/J_f, \mathcal{O}_{\mathbf{C}^m})$; denoting the rank of the jth Tor module by β_j, from the exact sequence (3) we obtain the following.

Theorem 1.5. *If $f \circ F$ has isolated singularity then $\tau_{\mathcal{K}_f} F = \mu(f \circ F) - \beta_0 + \beta_1$.*

This equality is the key to the comparisons between $\tau_f(F)$ and the rank of the vanishing homology of $f \circ F$ under deformations of F alone, which occupy Section 4.

Note that acyclicity of the Koszul complex $\mathbf{K}_\bullet(f \circ F)$ means that for $k \geqslant 2$, $H_k(C_\bullet(\phi_f)) \simeq \mathrm{Tor}_k^{\mathcal{O}_{\mathbf{C}^n}}(\mathcal{O}_{\mathbf{C}^m}, \mathcal{O}_{\mathbf{C}^n,0}/J_f)$.

1.1. *Is this the key to any door?*

When $f \circ F$ has isolated singularity, by taking the alternating sum of the lengths of the modules in the exact sequences (3) and (4) we obtain the formulae

$$\chi(C_\bullet(\phi_f)) = \mu(f \circ F) - \chi(\mathcal{O}_{\mathbf{C}^m,0}, \mathcal{O}_{\mathbf{C}^n}/J_f) \tag{5}$$

and

$$\chi(C_\bullet(\phi_V)) = \tau(f \circ F) - \chi\big(\mathcal{O}_{\mathbf{C}^m,0}, \mathcal{O}_{\mathbf{C}^n}/(f) + J_f\big). \tag{6}$$

Here, the last term on the right is Serre's intersection multiplicity [**21**]. This is defined, for modules M, N over the ring R ($= \mathcal{O}_{\mathbf{C}^n}$ in our case) by

$$\chi(M, N) = \sum_j (-1)^j \ell\big(\mathrm{Tor}_j^R(M, N)\big).$$

The right-hand side makes sense only if $M \bigotimes_R N$ has finite length, and in fact this is a sufficient condition for finiteness of all the other summands.

When $\dim M + \dim N < \dim R$ then $\chi(M, N) = 0$ [**21**]. If also $\mathrm{Tor}_j^{\mathcal{O}_{\mathbf{C}^n}}(\mathcal{O}_{\mathbf{C}^m}, \mathcal{O}_{\mathbf{C}^n}/J_f) = 0$ for $j > 1$, then $\beta_0 = \beta_1$, and from Theorem 1.5 it follows that $\tau_{\mathcal{K}_f} F = \mu(f \circ F)$. As we shall see in Lemma 4.3 below, this explains the surprising equality referred to in the opening paragraph of the introduction.

When $f \circ F$ has non-isolated singularity the Koszul complex is no longer acyclic, and one might wish to replace it by a free resolution $\mathbf{F}_\bullet$ of $\mathcal{O}/J_{f\circ F}$. However, in general the comparison morphism $\mathcal{O}/J_{f \circ F} \longrightarrow \mathcal{O}/F^*(J_f)$ will not lift to a morphism of complexes $\mathbf{F}_\bullet \longrightarrow F^*(\mathbf{L}_\bullet)$ if $F^*(\mathbf{L}_\bullet)$ is not acyclic. In particular, it is not clear that $tF(\mathrm{Der}(-\log f \circ F)) \subset F^*(\mathrm{Der}(-\log f))$, which is required in order to have such a lift in degree 1. The functoriality of the Koszul complex seems to be playing an important rôle here. In order to progress towards an understanding of the relation between $\tau_f(F)$ and the vanishing homology of $V(f \circ F)$ when $f \circ F$ has non-isolated singularity, we will need some understanding of the first Koszul homology of $\mathcal{O}_{\mathbf{C}^m}/J(f \circ F)$.

1.2. *Almost free divisors*

Some of the most interesting developments in the theory of sections of hypersurface singularities have concerned sections of free divisors (see for example [**9**–**12**]), and here the condition of isolated singularity is very far from being fulfilled. A singular free divisor in $\mathbf{C}^n$ has singular subspace of dimension $n-2$, so that among reduced spaces, free divisors have the biggest possible singular set. An *almost free divisor* is a section of a free divisor $V = f^{-1}(0)$ by a map F which is logarithmically transverse to V outside the origin (this definition is due to Damon), so an almost free divisor is also singular in codimension 1. Thus both the modules $\mathcal{O}_{\mathbf{C}^m,0}/F^*(J_f)$ and $\mathcal{O}_{\mathbf{C}^m,0}/J_{f\circ F}$ have $(m-2)$-dimensional support. On the other hand in this case $\mathrm{Tor}_j^{\mathcal{O}_{\mathbf{C}^n}}(\mathcal{O}_{\mathbf{C}^m,0}, \mathcal{O}_{\mathbf{C}^n}/J_f) = 0$ for $j > 0$, and so the sequence (3) reduces to the short exact sequence

$$0 \longrightarrow T^1_{\mathcal{K}_f} F \longrightarrow \frac{\mathcal{O}_{\mathbf{C}^m}}{J_{f\circ F}} \longrightarrow \frac{\mathcal{O}_{\mathbf{C}^m}}{F^*(J_f)} \longrightarrow 0, \tag{7}$$

together with a collection of isomorphisms $H_k(C_\bullet(\phi_f)) \simeq H_{k-1}(\mathbf{K}_\bullet(f \circ F))$ for $k \geqslant 2$. It is interesting to note that (7) allows us to give $T^1_{\mathcal{K}_f}F$ the multiplicative structure of the quotient of two ideals:

$$T^1_{\mathcal{K}_f}F \simeq \frac{F^*(J_f)}{J_{f\circ F}}. \tag{8}$$

The acyclicity of $F^*(\mathbf{L}_\bullet)$ for almost free divisors, versus the acyclicity of $\mathbf{K}_\bullet(f\circ F)$ for sections with isolated singularity, shows that these two cases are at opposite corners of the field one might wish to survey.

2. *Singularities of matrix families and their determinants*

In papers [**1**–**3**, **16**], parametrised families of $n \times n$ matrices are classified up to coordinate changes in the parameter space and parametrised versions of the natural action of $\mathrm{Sl}_n(\mathbf{C})$ and $\mathrm{Gl}_n(\mathbf{C})$. In this section we show that these equivalence relations are in fact the same as the relations $\mathcal{K}_f$ and $\mathcal{K}_V$ when f is the determinant or Pfaffian function on matrix space.

First we recall the definition of the equivalence relations. There are three cases, corresponding to symmetric, skew symmetric and arbitrary square matrices, each with two flavours, special and general. We remind the reader that we use $\mathcal{O}$ to abbreviate $\mathcal{O}_{\mathbf{C}^m}$.

Definition 2.1. (1) For *symmetric matrices*, symmetric matrix families

$$S_1, S_2 : (\mathbf{C}^m, 0) \longrightarrow \mathrm{Sym}_n(\mathbf{C})$$

are *Sl_n-symmetric equivalent* if there is a matrix family $A \in \mathrm{Sl}_n(\mathcal{O})$ and a germ of biholomorphic diffeomorphism $\psi : (\mathbf{C}^m, 0) \longrightarrow (\mathbf{C}^m, 0)$ such that

$$S_2 = A^t(S_1 \circ \psi)A,$$

and Gl_n-*symmetric equivalent* if A is allowed to be in $\mathrm{Gl}_n(\mathcal{O})$ rather than $\mathrm{Sl}_n(\mathcal{O})$.

(2) For *skew symmetric matrices*, Sl_n- and Gl_n-*skew-equivalence* are defined by the same formulae.

(3) *Arbitrary square matrix* families $M_1, M_2 : (\mathbf{C}^m, 0) \longrightarrow \mathrm{Mat}_n(\mathbf{C})$ are Sl_n-*equivalent* if there are matrix families $A, B \in \mathrm{Sl}_n(\mathcal{O})$ and a germ of biholomorphic diffeomorphism $\psi : (\mathbf{C}^m, 0) \longrightarrow (\mathbf{C}^m, 0)$ such that

$$M_2 = A(M_1 \circ \psi)B,$$

and *Gl_n-equivalent* if A and B are allowed to be in $\mathrm{Gl}_n(\mathcal{O})$.

It is an immediate consequence of the definitions that the extended tangent-spaces to the special and general orbits are

$$T_{\mathrm{ss}}S = tS(\theta_{\mathbf{C}^m}) + \{A^tS + SA : A \in \mathrm{sl}_n(\mathcal{O})\},$$

$$T_{\mathrm{gs}}S = tS(\theta_{\mathbf{C}^m}) + \{A^tS + SA : A \in \mathrm{gl}_n(\mathcal{O})\},$$

for symmetric and skew symmetric matrices, and

$$T_{\mathrm{sg}}M = tM(\theta_{\mathbf{C}^m}) + \{AM + MB : A, B \in \mathrm{sl}_n(\mathcal{O})\},$$

$$T_{\mathrm{gg}}M = tM(\theta_{\mathbf{C}^m}) + \{AM + MB : A, B \in \mathrm{gl}_n(\mathcal{O})\},$$

for arbitrary square matrices. Here the first letter in the subscript refers to the flavour, special or general, and the second letter refers to the type of the matrix, symmetric, skew or general. We do not need to distinguish, in our notation, between the symmetric and skew-symmetric cases, since the equivalence relation is the same.

We denote the codimension of these tangent spaces (in $S^*(\mathrm{sym}_n(\mathbf{C})), S^*(\mathrm{sk}_n(\mathbf{C}))$ and $M^*(\mathrm{mat}_n(\mathbf{C}))$ respectively) by $\tau_{\mathrm{ss}}, \tau_{\mathrm{gs}}, \tau_{\mathrm{sg}}$ and τ_{gg}.

Let det denote the determinant function on $\mathrm{Mat}_n(\mathbf{C})$ in the general case, and on $\mathrm{Sym}_n(\mathbf{C})$ in the symmetric case, and let $V=\{\det =0\}$ (in which space will be clear from the context). Similarly, let $\mathrm{Pf} : \mathrm{Sk}_n(\mathbf{C}) \longrightarrow \mathbf{C}$ be the Pfaffian function, and in this context let V denote its zero-locus. Since $\mathrm{Pf} \equiv 0$ if n is odd, from now on when we are discussing skew-symmetric $n \times n$ matrices, n will be assumed to be even.

THEOREM 2.2. (i) *For a symmetric matrix family S,*

$$T_{\mathrm{ss}}S = T\mathcal{K}_{\det}S, \quad T_{\mathrm{gs}}S = T\mathcal{K}_V S.$$

(ii) *For a general matrix family M,*

$$T_{\mathrm{sg}}M = T\mathcal{K}_{\det}M, \quad T_{\mathrm{gg}}M = T\mathcal{K}_V M.$$

(iii) *For a skew-symmetric matrix family S,*

$$T_{\mathrm{ss}}S = T\mathcal{K}_{Pf}S, \quad T_{\mathrm{gs}}S = T\mathcal{K}_V S.$$

Proof. In each of the three cases, the first equality, concerning Sl_n-equivalence, can be read off from the well known free resolutions mentioned in the introduction. These are described in detail after this proof. For now, we show only that the right hand of each of the three pairs of equalities follows from the left.

In fact we show it only for symmetric matrices; the other two cases are essentially identical.

By comparing the formulae for the tangent spaces, we see that

$$T_{\mathrm{gs}}S = T_{\mathrm{ss}}S + \{\lambda S : \lambda \in \mathcal{O}\} \tag{9}$$

since $\mathrm{gl}_n = \mathrm{sl}_n + \{\lambda \cdot \mathrm{id}_n : \lambda \in \mathbf{C}\}$, where id_n is the $n \times n$ identity matrix, from which $\mathrm{gl}_n(\mathcal{O}) = \mathrm{sl}_n(\mathcal{O}) + \{\lambda \cdot \mathrm{id}_n : \lambda \in \mathcal{O}\}$ follows. On the other hand, because det is a homogeneous function, there is a splitting

$$\mathrm{Der}(-\log V) = \mathrm{Der}(-\log \det) \oplus \mathcal{O}_{\mathrm{Mat}_n(\mathbf{C})} \cdot \chi_e,$$

where χ_e is the Euler vector field $\sum_{i,j} x_{ij}\partial/\partial x_{ij}$. Hence

$$T\mathcal{K}_V S = T\mathcal{K}_{\det}S + S^*(\mathcal{O}_{\mathrm{Mat}_n(\mathbf{C})} \cdot \chi_e) = T\mathcal{K}_{\det}S + \{\lambda \cdot S : \lambda \in \mathcal{O}\}. \tag{10}$$

From (9) and (10) the equality $T_{\mathrm{gs}}S = T\mathcal{K}_V S$ follows. □

The equalities of the tangent spaces, together with uniqueness of solutions of ordinary differential equations, imply that the equivalences themselves coincide, as stated at the beginning of the section. However we will only need equality of the tangent spaces in what follows.

3. *Free resolutions*

In this section we describe complexes which give free resolutions of the jacobian algebras of det : $\mathrm{Mat}_n(\mathbf{C}) \longrightarrow \mathbf{C}$, det : $\mathrm{Sym}_n(\mathbf{C}) \longrightarrow \mathbf{C}$, and the Pfaffian function $\mathrm{Pf} : \mathrm{Sk}_n(\mathbf{C}) \longrightarrow \mathbf{C}$ for n even. Surprisingly (to us), the original papers where they appeared make no mention of partial derivatives and vector fields; in each of the three cases, the jacobian ideal is identified as a purely algebraic object, namely the ideal I_{n-1} generated by the submaximal minors in the case of det, and the ideal Pf_{n-2} generated by the $(n-2) \times (n-2)$ sub-Pfaffians in the case of Pf.

3.1. *The Gulliksen–Negård resolution*

For a family $M \in \mathrm{Mat}_n(\mathcal{O})$, Gulliksen and Negård constructed in [**15**] a complex which is a free resolution of $\mathcal{O}/I_{n-1}(M)$ in case the codimension of the variety $V(I_{n-1}(M))$ in $\mathbf{C}^m$ is 4 (its greatest possible value). Their complex is

$$0 \to \mathcal{O} \xrightarrow{d_4} \mathrm{mat}_n(\mathcal{O}) \xrightarrow{d_3} \mathrm{sl}_n(\mathcal{O}) \oplus \mathrm{sl}_n(\mathcal{O}) \xrightarrow{d_2} \mathrm{mat}_n(\mathcal{O}) \xrightarrow{d_1} \mathcal{O} \longrightarrow \mathcal{O}/I_{n-1}(M) \longrightarrow 0, \tag{11}$$

where the following hold.

(i) $d_1(U) = \mathrm{trace}(M^*U)$, where M^* is, the adjugate of M, that is, the matrix of signed cofactors.

(ii) $d_2(X, Y) = MX - YM$.

(iii) $d_3(Z) = (ZM - (\mathrm{tr}(ZM)/n)\ I_n,\ MZ - (tr(MZ)/n)\ I_n)$.

(iv) $d_4(a) = aM^*$.

(v) We use lower case $\mathrm{mat}(\mathcal{O})$ rather than upper case $\mathrm{Mat}(\mathcal{O})$ because we are thinking of the (pull-back of the) tangent sheaf on the vector space $\mathrm{Mat}(\mathbf{C})$. It is of course indistinguishable from $\mathrm{Mat}(\mathcal{O})$ as $\mathcal{O}$-module.

Several aspects of this resolution come unexpectedly to our aid. The first is that the i, jth signed cofactor M^*_{ij} of a matrix M is equal to the partial derivative of det with respect to the i, jth entry M_{ij}, so in the generic case (where the entries are the variables), (11) is a resolution of $\mathcal{O}_{\mathrm{Mat}_n(\mathbf{C})}/J_{\det}$. Moreover

$$\mathrm{trace}\,(M^*X) = \sum_{ij} M^*_{ij} X_{ij} = \sum_{ij} X_{ij} \frac{\partial \det}{\partial M_{ij}} = t(\det)\left(\sum_{ij} X_{ij} \frac{\partial}{\partial M_{ij}}\right).$$

The second is that thanks to this fact, we can interpret the module of relations among the sub-maximal minors as the module of vector fields annihilating the function det, and the Gulliksen–Negård resolution shows that the relations among them are precisely those induced by the action of sl_n. This fact, which was already noted by Bill Bruce in [**1**], is precisely what is needed to prove Theorem 2.2 for matrix families.

Moreover, the Gulliksen–Negård complex $\mathbf{GN}_\bullet(S)$ is equal to $M^*(\mathbf{GN}_\bullet)$, where $\mathbf{GN}_\bullet$ is the generic complex, over $\mathcal{O}_{\mathrm{Mat}(\mathbf{C})}$. Thus it can play the rôle of the complex $F^*(L_\bullet)$ of Theorem 1.2. A similar remark holds for the other two complexes we now describe.

Once again we have slightly modified the description of the complex from that found, for example, in [**5**], in order to adapt it to our situation.

3.2. *Józefiak's resolution*

In [**17**], Józefiak constructed a complex of free $\mathcal{O}$-modules which gives a resolution of $\mathcal{O}/I_{n-1}(S)$ provided that the codimension of the zero variety $V(I_{n-1}(S))$ of $I_{n-1}(S)$ in $\mathbf{C}^m$ is 3 (its greatest possible value). His complex is

$$0 \longrightarrow \mathrm{sk}_n(\mathcal{O}) \xrightarrow{d_3} \mathrm{sl}_n(\mathcal{O}) \xrightarrow{d_2} \mathrm{sym}_n(\mathcal{O}) \xrightarrow{d_1} \mathcal{O} \longrightarrow \mathcal{O}/I_{n-1}(S) \longrightarrow 0. \tag{12}$$

Here the following hold.

(i) $\mathrm{sk}_n(\mathcal{O})$ is the space of order n skew-symmetric matrices over $\mathcal{O}$.

(ii) $d_1(X) = \mathrm{trace}\,(S^*X)$, where S^* is the adjugate matrix of S.

(iii) $d_2(Y) = SY + Y^T S$.

(iv) $d_3(Z) = ZS$.

In fact in [**17**], $\mathrm{gl}_n(\mathcal{O})/\mathrm{sk}_n(\mathcal{O})$ appears in place of $\mathrm{sym}_n(\mathcal{O})$, and d_2 has a different (equivalent) description.

As with the Gulliksen–Negård resolution, in the generic case d_1 is equal to $t(\det)$: $\theta_{\mathrm{Sym}(\mathbf{C})} \longrightarrow \mathcal{O}_{\mathrm{Sym}(\mathbf{C})}$, and so the acyclicity of (12) shows that the module of vector fields annihilating det is generated by the infinitesimal sl_n action.

3.3. *The Józefiak–Pragacz resolution*

A complex giving a free resolution of $\mathcal{O}/\mathrm{Pf}_{n-2}(S)$ in case $V(\mathrm{Pf}_{n-2}(S))$ has codimension 6 (its greatest possible value) in $\mathbf{C}^m$ is due to Józefiak and Pragacz [**18**]. In slightly modified form it is

$$\begin{aligned}0 \longrightarrow \mathcal{O} \xrightarrow{d_6} \mathrm{sk}_n(\mathcal{O}) \xrightarrow{d_5} \mathrm{sl}_n(\mathcal{O}) \xrightarrow{d_4} \mathrm{sym}_n(\mathcal{O}) \oplus \mathrm{sym}_n(\mathcal{O}) \xrightarrow{d_3}\\ \mathrm{sl}_n(\mathcal{O}) \xrightarrow{d_2} \mathrm{sk}_n(\mathcal{O}) \xrightarrow{d_1} \mathcal{O} \longrightarrow \mathcal{O}/\mathrm{Pf}_{n-2}(S) \longrightarrow 0,\end{aligned} \tag{13}$$

where the following hold.

(i) $d_1(U) = \frac{1}{2}\mathrm{trace}(S^*U)$, where S^* is the matrix of order $(n-2)$ signed Pfaffians of S, which satisfies $S^*S = SS^* = \mathrm{Pf}(S)I_n$.

(ii) $d_2(V) = SV + V^T S$.

(iii) $d_3(W, X) = S^*W - XS$.

(iv) $d_4(Y) = (SY + (SY)^T,\ YS^* + (YS^*)^T)$.

(v) $d_5(Z) = ZS - (\mathrm{tr}(ZS)/n)\ I_n$.

(vi) $d_6(a) = aS^*$.

In the generic case $\partial\mathrm{Pf}/\partial S_{ij} = S^*_{ij}$, so $\mathrm{Pf}_{n-2} = J_{\mathrm{Pf}}$, $d_1 = t(\mathrm{Pf})$, (13) is a resolution of the jacobian algebra of Pf, and by acyclicity of (13) it follows that the vector fields annihilating Pf are generated by the sl_n action.

3.4. *The morphisms ϕ_f for matrix families*

In the case of matrix families, the morphism of complexes $\mathbf{K}_\bullet(f \circ F) \xrightarrow{\phi_f} F^*(\mathbf{L}_\bullet)$ constructed in Section 1 embodies some non-trivial rules of differentiation. For a symmetric or skew-symmetric $n \times n$ matrix family S,

$$(\phi_f)_2 : \bigwedge^2 \theta_{\mathbf{C}^m} \longrightarrow \mathrm{sl}_n(\mathcal{O})$$

is given by

$$(\partial/\partial x_i \wedge \partial/\partial x_j) \longmapsto (S^*_{x_i} S_{x_j} - S^*_{x_j} S_{x_i})/2 \tag{14}$$

(where the subscript x_i indicates partial derivative), and for a general $n\times n$ matrix family M,

$$(\phi_f)_2 : \bigwedge^2 \theta_{\mathbf{C}^m} \longrightarrow \mathrm{sl}_n(\mathcal{O}) \oplus \mathrm{sl}_n(\mathcal{O})$$

is given by

$$(\partial/\partial x_i \wedge \partial/\partial x_j) \longmapsto (M_{x_j} M^*_{x_i} - M_{x_i} M^*_{x_j}, M^*_{x_j} M_{x_i} - M^*_{x_i} M_{x_j})/2. \tag{15}$$

It would be interesting to obtain explicit formulae for the remaining $(\phi_f)_j$.

3.5. *Cohen–Macaulay and Gorenstein properties of determinantal varieties*

In each of the three cases, let m_0 denote the length of the resolution (12), (11) and (13). Since m_0 is also the codimension of $V(I_{n-1}(S))$, $V(I_{n-1}(M))$ and $V(\mathrm{Pf}_{n-2}(S))$ in the generic case, the three resolutions show that these varieties are Cohen–Macaulay, and the same conclusion also holds for any matrix family provided the codimension of the respective variety is m_0, which is its maximal possible value.

If $\mathscr{S}$ is a deformation over base B of a three-parameter matrix family S meeting this requirement in (say) the symmetric case, then the codimension in $\mathbf{C}^3\times B$ of $V(I_{n-1}(\mathscr{S}))$ is also 3. Since $V(I_{n-1}(S))$ is the fibre of $V(I_{n-1}(\mathscr{S}))$ over $0\in B$, $V(I_{n-1}(\mathscr{S}))$ is finite over B, and therefore $\mathcal{O}_{\mathbf{C}^3\times B}/I_{n-1}(\mathscr{S})$ is $\mathcal{O}_B$-free. This implies that $\dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^3,0}/I_{n-1}(S)$ is conserved in a deformation; it is equal to $\sum_x \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^3,x}/I_{n-1}(S_t)$ (where the sum is over the points x into which the isolated zero of $I_{n-1}(S)$ splits), since both are equal to the rank of the free sheaf $\pi_*\,\mathcal{O}_{\mathbf{C}^3\times B}/I_{n-1}(\mathscr{S})$. It is the index of intersection of the image of $\mathbf{C}^3$ under S with the set of symmetric matrices of corank greater than 1.

Similar arguments prove conservation of multiplicity in the other two cases, when $m=m_0$.

Both (11) and (13) are self-dual complexes, and so if the codimension of $V(I_{n-1}(M))$ or $V(\mathrm{Pf}_{n-2}(S))$ is m_0, then both are Gorenstein varieties. This has a consequence for the relation between the Betti numbers of $\mathbf{GN}_\bullet(M)$ and $\mathbf{JP}_\bullet(S)$ when m is less than m_0, which we now explain.

LEMMA 3.1. *Let R be a noetherian local ring, let $\mathbf{F}_\bullet =: 0\longrightarrow F_M \longrightarrow \ldots \longrightarrow F_0 \longrightarrow 0$ be a finite complex of free R-modules, and let $\mathbf{F}^\bullet$ be its R-dual. There is a spectral sequence with $E_2^{p,q}=\mathrm{Ext}^q_R(H_p(\mathbf{F}_\bullet), R)$ converging to $H^{p+q}(\mathbf{F}^\bullet)$.*

Proof. Form the double complex $\mathrm{Hom}(\mathbf{F}_\bullet, I^\bullet)$, where $I^\bullet$ is an injective resolution of R. Writing the arrow of $\mathbf{F}_\bullet$ pointing from left to right and the arrow of the injective resolution pointing up, the double complex has arrows pointing up and from right to left.

Taking first the horizontal differential, we get

$$H_p(\mathrm{Hom}(F_\bullet, I^q)),$$

which is equal to

$$\mathrm{Hom}(H_p(F_\bullet), I^q)$$

by injectivity of I^q. Now, taking the vertical differential we get

$$E_2^{p,q} = \mathrm{Ext}^q(H_p(F_\bullet), R).$$

If we take first the vertical differential in the double complex we get

$$\mathrm{Ext}^q(F_p, R),$$

which is equal to F^p for $q=0$ and is zero otherwise (since Ext can also be calculated by using a projective resolution of F_p). Now, taking the horizontal differential, we get

$$H^p(\mathbf{F}^\bullet).$$

Since both means of calculating the homology of the double complex must give the same answer, we conclude that the first spectral sequence must also converge to $H^p(\mathbf{F}^\bullet)$. □

LEMMA 3.2. *Suppose in addition that each of the homology modules $H_p(\mathbf{F}_\bullet)$ has finite length and that R is a regular local ring of dimension m. Then*

$$H^p(\mathbf{F}^\bullet) \simeq \mathrm{Ext}^m(H_{p-m}(\mathbf{F}_\bullet), R).$$

Proof. Each module $H_p(\mathbf{F}_\bullet)$ now has a free resolution of length m. By a lemma of Ischebeck [**19**, p. 133],

$$\mathrm{Ext}^q(H_p(F_\bullet), \mathcal{O}) = 0.$$

except when $q=m$. It follows that the spectral sequence of Lemma 3.1 collapses at E_2, and the lemma follows. □

Now suppose that the complex $\mathbf{F}_\bullet$ is self-dual, in the sense that there is an integer m_0 (the length of the complex) such that $H^p(F^\bullet) = H_{m_0-p}(F_\bullet)$. This is the case for the complexes $\mathbf{GN}_\bullet(M)$ and $\mathbf{JP}_\bullet(S)$, since they are the pull-backs of free resolutions of Gorenstein quotients. Then Lemma 3.2 gives

$$H_{m_0-p}(\mathbf{F}_\bullet) \simeq \mathrm{Ext}^m(H_{p-m}(\mathbf{F}_\bullet), R). \tag{16}$$

PROPOSITION 3.3. *Suppose that $\mathbf{F}_\bullet$ is a self-dual complex of free modules, of length m_0, over the regular local ring R of dimension m, with all homology modules of $\mathbf{F}_\bullet$ having finite length. Then $H_{m_0-p}(F_\bullet)$ has the same length as $H_{p-m}(\mathbf{F}_\bullet)$.*

Proof. This is immediate from (16) and the following lemma. □

LEMMA 3.4. *Let M be an R-module of finite length. Then $\mathrm{Ext}^m(M, R)$ has the same length as M.*

Proof. Because R is Gorenstein, local duality (see for example [**4**]) gives us

$$\mathrm{Ext}^m(M, R) = \mathrm{Ext}^m(M, \omega_R) = \mathrm{Hom}\big(H^0_{\{0\}}(M), E(k)\big),$$

where ω_R is the dualising module of R and $E(k)$ is the injective hull of the residue field of R. Since M is supported only at 0, this gives

$$\mathrm{Ext}^m(M, R) = \mathrm{Hom}(M, E(k)).$$

Finally, an easy induction [**4**, p. 102] shows that the length of $\mathrm{Hom}(M, E(k))$ is equal to the length of M for any R-module M of finite length. □

PROPOSITION 3.5. (i) *Let M be a general matrix family on $m < 4$ parameters, and suppose that* $\det(M)$ *has isolated singularity. Then*

$$\beta_k(\mathbf{GN}_\bullet(M)) = \beta_{k+(4-m)}(\mathbf{GN}_\bullet(M)).$$

(ii) *Let S be a skew-symmetric matrix family on $m < 6$ parameters, and suppose that* $\mathrm{Pf}(S)$ *has isolated singularity. Then*

$$\beta_k(\mathbf{JP}_\bullet(S)) = \beta_{k+(6-m)}(\mathbf{JP}_\bullet(S)).$$

4. *τ, μ and the vanishing homology of sections with isolated singularity*

We will suppose throughout this section that $\mathcal{O}_{\mathbf{C}^n}/J_f$ is Cohen–Macaulay, of dimension $n - m_0$. Then the dimension of $\mathcal{O}_{\mathbf{C}^m}/F^*(J_f)$ is at least $m - m_0$, and if it is $m - m_0$ then $\mathcal{O}_{\mathbf{C}^m}/F^*(J_f)$ is Cohen–Macaulay. Moreover in this case if $\mathbf{L}_\bullet$ is a free $\mathcal{O}_{\mathbf{C}^n}$-resolution of $\mathcal{O}_{\mathbf{C}^n}/J_f$ then $F^*(\mathbf{L}_\bullet)$ is a free $\mathcal{O}_{\mathbf{C}^m}$-resolution of $\mathcal{O}_{\mathbf{C}^m}/F^*(J_f)$. From Theorem 1.5 we therefore obtain the following.

THEOREM 4.1. *If $m = m_0$ and $f \circ F$ has isolated singularity then*

$$\tau_{\mathcal{K}_f} F = \mu(f \circ F) - \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^m}/F^*(J_f).$$

Applying this to the matrix families we considered in Section 2, we have the following.

COROLLARY 4.2. (i) *Symmetric case with $m = 3$:*

$$\tau_{\mathrm{ss}}(S) = \mu(\det(S)) - \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^3}/I_{n-1}(S).$$

(ii) *General case with $m = 4$:*

$$\tau_{\mathrm{sg}}(M) = \mu(\det(M)) - \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^4}/I_{n-1}(M).$$

(iii) *Skew-symmetric case with $m = 6$:*

$$\tau_{\mathrm{ss}}(S) = \mu(Pf(S)) - \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^6}/\mathrm{Pf}_{n-2}(S).$$

If $m = m_0 - 1$ or $m = m_0 - 2$ a surprising phenomenon occurs.

LEMMA 4.3. (i) *Suppose that $m = m_0 - 1$, and $f \circ F$ has isolated singularity. Then the numbers β_0 and β_1 in Theorem 1.5 are finite and equal to one another, and $\beta_j = 0$ for $j \geqslant 2$. In consequence, $\tau_{\mathcal{K}_f} F = \mu(f \circ F)$.*

(ii) *If $m = m_0 - 2$ and $f \circ F$ has isolated singularity, then β_0, β_1 and β_2 are finite and $\beta_j = 0$ for $j \geqslant 3$. Moreover $\beta_0 + \beta_2 = \beta_1$, so that $\tau_{\mathcal{K}_f} F = \mu(f \circ F) + \beta_2$.*

Proof. We remarked in Subsection 1.1 that given the vanishing of higher Tor modules, the relations between the β_j follow from the vanishing of $\chi(\mathcal{O}_{\mathbf{C}^n}/J_f, \mathcal{O}_{\mathbf{C}^m})$. Since we need this vanishing, however, we give a self-contained proof.

(i) Choose a deformation $\mathscr{F}$ of F on one extra parameter t, so that $\mathcal{O}_{\mathbf{C}^{m_0-1}\times\mathbf{C}}/\mathscr{F}^*(J_f)$ has dimension 0. Then $\mathscr{F}^*(\mathbf{L}_\bullet)$ is acyclic. The short exact sequence of complexes

$$0 \longrightarrow \mathscr{F}^*(\mathbf{L}_\bullet) \xrightarrow{t\cdot} \mathscr{F}^*(\mathbf{L}_\bullet) \longrightarrow F^*(\mathbf{L}_\bullet) \longrightarrow 0$$

gives rise to a long exact sequence of homology; since $\mathscr{F}^*(\mathbf{L}_\bullet)$ is acyclic, this ends

$$\begin{aligned} 0 = H_1(\mathscr{F}^*(\mathbf{L}_\bullet)) &\longrightarrow H_1(F^*(\mathbf{L}_\bullet)) \longrightarrow H_0(\mathscr{F}^*(\mathbf{L}_\bullet)) \longrightarrow H_0(\mathscr{F}^*(\mathbf{L}_\bullet)) \\ &\longrightarrow H_0(F^*(\mathbf{L}_\bullet)) \longrightarrow 0. \end{aligned}$$

As $H_0(\mathscr{F}^*(\mathbf{L}_\bullet)) = \mathcal{O}_{\mathbf{C}^{m_0-1}\times\mathbf{C}}/\mathscr{F}^*(J_f)$, it has finite length, and the conclusion follows from the fact that the alternating sum of the lengths of the modules in an exact sequence is 0.

(ii) Let $\mathscr{F}_1$ be a deformation of F on the parameter t_1 and let $\mathscr{F}_2$ be a deformation of $\mathscr{F}_1$ on the parameter t_2, such that $V(\mathscr{F}_2^*(J_f))$ has codimension m_0. The argument of (i) applied to the long exact sequence arising from

$$0 \longrightarrow \mathscr{F}_2^*(\mathbf{L}_\bullet) \xrightarrow{t_2\cdot} \mathscr{F}_2^*(\mathbf{L}_\bullet) \longrightarrow \mathscr{F}_1^*(\mathbf{L}_\bullet) \longrightarrow 0$$

shows that $\beta_2(\mathscr{F}_1^*(\mathbf{L}_\bullet)) = 0$ and $\beta_1(\mathscr{F}_1^*(\mathbf{L}_\bullet)) < \infty$. An analogous argument, applied to the (longer) exact sequence arising from

$$0 \longrightarrow \mathscr{F}_1^*(\mathbf{L}_\bullet) \xrightarrow{t_1\cdot} \mathscr{F}_1^*(\mathbf{L}_\bullet) \longrightarrow F^*(\mathbf{L}_\bullet) \longrightarrow 0$$

then gives the result. □

In case (i) it is curious that despite the equality of μ and τ, the natural map from $T^1_{\mathcal{K}_f}F$ to the jacobian algebra is not an isomorphism.

For our matrix families we conclude the following from Theorem 1.5 and Lemma 4.3(i).

COROLLARY 4.4. (i) *Symmetric case with* $m=2$*:*

$$\tau_{\rm ss}(S) = \mu(\det(S)).$$

(ii) *General case with* $m=3$*:*

$$\tau_{\rm sg}(M) = \mu(\det(M)).$$

(iii) *Skew-symmetric case with* $m=5$*:*

$$\tau_{\rm ss}(S) = \mu(Pf(S)).$$

Also somewhat surprisingly, under a mild assumption on the distributions $\mathrm{Der}(-\log V)$ and $\mathrm{Der}(-\log f)$, the two disparate phenomena described in Theorem 4.1 and Lemma 4.3 are both subsumed into the same phenomenon when we consider the vanishing homology of $F^{-1}(V)$ under deformation of F.

The assumption is that the following hold.

(1) There exist perturbations of F which are logarithmically transverse to V.

(2) At each point $x \in V$ where $\dim_{\mathbf{C}} T_x^{\log}V \geqslant n-m$,

$$T_x^{\log}V = \mathrm{Der}(-\log f)(x). \tag{17}$$

This equality holds, for example, if $f \in m_x J_f$. For if $f \in J_f$ then $\mathrm{Der}(-\log V)$ splits as a direct sum $\mathrm{Der}(-\log f) \oplus \mathcal{O}_{\mathbf{C}^n}\cdot\chi$ where χ is a vector field such that $\chi\cdot f = f$. If in addition we can choose χ to vanish at x (that is if $f \in m_x J_f$) then (17) follows. In fact if $f \in m_x J_f$ and u is any unit then $uf \in m_x J_{uf}$, so (17) holds for *every* choice of equation. This has a partial converse.

Proposition 4.5. *Suppose that $T_x^{\log}V \neq 0$. Then* (17) *holds for every choice of equation for V if and only if $f \in m_x J_f$.*

Proof. 'If' is already shown. Choose $\chi \in \mathrm{Der}(-\log f)$ such that $\chi(x) \neq 0$. If $\chi(x) \in \mathrm{Der}(-\log uf)(x)$, then there must exist $\eta \in m_x\theta_{\mathbf{C}^n,x}$ such that $(\chi+\eta)\cdot uf = 0$. This gives

$$u(\eta \cdot f) + f\big((\chi+\eta)\cdot u\big) = 0.$$

Now choose a unit u such that $d_x u(\chi(x)) \neq 0$. Then $(\chi+\eta)\cdot u$ is a unit in $\mathcal{O}_{\mathbf{C}^n,x}$, so $f \in m_x J_f$. □

We will say that m is *in the range of holonomy* with respect to f if (17) holds at all points of V where $\dim_{\mathbf{C}} T_x^{\log}V \geqslant n-m$.

Theorem 4.6. *Suppose that $m = m_0$ or $m = m_0 - 1$, and is in the range of holonomy with respect to f, and let F_t be a perturbation of F which is logarithmically transverse to V. Let $X_t = V(f \circ F_t)$ and $X_0 = V(f \circ F)$. Then X_t has the homotopy type of a wedge of $\tau_{\mathcal{K}_f}F$ copies of the $(m-1)$-sphere.*

Proof. When $m = m_0 - 1$ the argument is straightforward; X_t is a Milnor fibre of X_0, since F can only meet V at regular points, and there transversely. Hence X_t has the homotopy type of a wedge of μ spheres. By Lemma 4.3(i), $\mu = \tau$.

When $m = m_0$, X_t is no longer a smoothing of X_0. Instead, it has an isolated singular point at each zero of $F_t^*(J_f)$. However, $T^1_{\mathcal{K}_V}F_t$ is everywhere 0, so by the assumption that m is in the range of holonomy of f, $T^1_{\mathcal{K}_f}F_t$ vanishes also at each point of $V(F_t^*(J_f))$ (these points all lie on $F_t^{-1}(V)$), and so by Theorem 1.5, the Milnor number of $f \circ F_t$ at each singular point x is equal to the local contribution $\beta_0(F_t, x)$. By smoothing each singularity, and so obtaining a Milnor fibre for the isolated singularity $(X_0, 0)$, we would increase the rank of the middle homology by the local Milnor number of X_t at x. It follows that the rank of $H_{m-1}(X_t)$ is $\mu - \sum_x \beta_0(F_t, x)$. However, $\sum_x \beta_0(F_t, x) = \beta_0$, by the well-known argument sketched at the start of Subsection 3.5.

It is also well known that every fibre of a deformation of an isolated hypersurface singularity, whether smooth or not, has the homotopy type of a wedge of spheres. □

Remark 4.7. (i) The existence of a perturbation F_t of F which is logarithmically transverse to V is not always assured (see (ii) below); however, the logarithmic stratification of matrix space is finite, and an argument of Damon [**10**, 2.4] using Sard's theorem shows that for families of matrices the required perturbations do exist. Moreover, standard row-reduction arguments show that the varieties $\det = 0$ and $\mathrm{Pf} = 0$ are everywhere locally quasihomogeneous, so that the range of holonomy of det and Pf has no upper bound. In fact, at the singular points of the determinant of a generic matrix in four parameters or of a generic symmetric matrix in three parameters, and at a singular point of the Pfaffian of a generic skew-symmetric matrix in six parameters, the families are Sl_n-equivalent, respectively, to

$$\begin{pmatrix} x_1 & x_2 & 0 \\ x_3 & x_4 & 0 \\ 0 & 0 & M_{n-2} \end{pmatrix}, \quad \begin{pmatrix} x_1 & x_2 & 0 \\ x_2 & x_2 & 0 \\ 0 & 0 & S_{n-2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & x_1 & x_2 & x_3 & 0 \\ -x_1 & 0 & x_4 & x_5 & 0 \\ -x_2 & -x_4 & 0 & x_6 & 0 \\ -x_3 & -x_5 & -x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{n-4} \end{pmatrix}, \tag{18}$$

where M_{n-2}, S_{n-2} and A_{n-4} are constant matrices with non-vanishing determinants. The determinants and Pfaffians of the families in (18) are equal to

$$x_1x_4 - x_2x_3, \quad x_1x_3 - x_2^2 \quad \text{and} \quad x_1x_6 - x_2x_5 + x_3x_4, \tag{19}$$

each of which has a non-degenerate critical point at 0. The ideals I_{n-1} and Pf_{n-2} are in each case equal to the maximal ideal.

(ii) Consider the hypersurface $V = \{y(x+y)(x-y)(x+zy) = 0\}$ in $\mathbf{C}^3$ and the map $F(x,y) = (x,y,0)$. This is the total space of a family of quadruple lines in the plane, with parameter z. As the cross-ratio varies with z, V is not analytically trivial along the z-axis. We claim that F has no perturbation which is logarithmically transverse to V. For on the one hand at every point P on the z-axis, $T_P^{\log} V = 0$, so that no map from $\mathbf{C}^2$ to $\mathbf{C}^3$ can meet V transversely at P, while on the other hand since F meets the z-axis in an isolated point, every perturbation F_t of F will also meet the z-axis. In this example, V is neither globally weighted homogeneous, nor locally quasihomogeneous at any point on the z-axis.

(iii) In contrast, the hypersurface (cf. [**6**]) with equation

$$f(x,y,z) = x^5z + x^3y^3 + y^5z$$

is globally homogeneous, but not locally quasihomogeneous at any point of the z-axis outside 0. As a *Macaulay* calculation readily shows, $\mathrm{Der}(-\log f)$ is generated by vector fields which vanish everywhere on the z-axis, so $\mathrm{Der}(-\log f)(0,0,z) = 0$. On the other hand $\mathrm{Der}(-\log V)(0,0,z) \neq 0$ for $z \neq 0$, since it contains the value of the Euler vector field. The section $F(x,y) = (x,y,x+y)$, with $\tau_{\mathcal{K}_f(F)} = 10$, has a perturbation $F_t(x,y) = (x,y,x+y+t)$ which is logarithmically transverse to V (in fact it is transverse to the distribution spanned by the Euler vector field) but nevertheless we have $F_t(0,0) \in V$ and $(T^1_{\mathcal{K}_f} F_t)_{(0,0)} \neq 0$.

We must admit that here $\mathcal{O}_{\mathbf{C}^3}/J_f$ is not Cohen–Macaulay, so that the conclusion of Theorem 4.6 fails for other reasons too. Indeed the conclusion of Theorem 4.1 fails also; one calculates that $\mu(f \circ F) = 25$ and $\dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^2}/F^*(J_f) = 19$, so that here

$$\tau_{\mathcal{K}_f}(F) \neq \mu(f \circ F) - \dim_{\mathbf{C}} \mathcal{O}_{\mathbf{C}^3}/F^*(J_f).$$

From Theorem 4.6 and Remark 4.7(i) we conclude the following.

THEOREM 4.8. *In each of the three kinds of matrix families, if $m = m_0$ or $m = m_0 - 1$ then the relevant Tjurina number τ is equal to the rank of the vanishing homology of the determinant or Pfaffian.*

One further comparison between $\tau_{\mathcal{K}_f}(F)$ and the vanishing homology follows immediately from Theorem 1.5, Proposition 3.3 and Lemma 4.3(ii) in case $\mathcal{O}_{\mathbf{C}^n}/J_f$ is Gorenstein.

PROPOSITION 4.9. *Suppose that $\mathcal{O}_{\mathbf{C}^n}/J_f$ is Gorenstein of dimension $n-m_0$, and that $F:(\mathbf{C}^{m_0-2},0)\longrightarrow(\mathbf{C}^n,0)$ has $\tau_{\mathcal{K}_f}F<\infty$. Then*

$$\tau_{\mathcal{K}_f}F=\mu(f\circ F)+\dim_{\mathbf{C}}\mathcal{O}_{\mathbf{C}^{m_0-2}}/F^*(J_f).$$

COROLLARY 4.10. (i) *If M is a general matrix family on two parameters, then*

$$\tau_{\mathrm{sg}}(M)=\mu(\det(M))+\dim_{\mathbf{C}}\mathcal{O}/I_{n-1}(M).$$

(ii) *If S is a skew-symmetric matrix family on four parameters, then*

$$\tau_{\mathrm{ss}}(S)=\mu(Pf(S))+\dim_{\mathbf{C}}\mathcal{O}/Pf_{n-2}(S).$$

REMARK 4.11. Since $\mathbf{J}_\bullet$ is not self-dual, an analogue of Corollary 4.10 does not hold in general for symmetric matrices. According to [**1**], any 1-parameter family of such matrices with finite Tjurina number is equivalent to $S=\mathrm{diag}\{x^{a_1},x^{a_2},\dots,x^{a_n}\}$, for some non-decreasing sequence of integers $0\leqslant a_1\leqslant a_2\leqslant\dots\leqslant a_n$. For such a family S,

$$\tau_{\mathrm{ss}}(S)=\sum_{i=1}^{n}(n-i+1)a_i-1,\quad \mu(\det S)=\sum_{i=1}^{n}a_i-1,\quad \beta_0(\mathbf{J}_\bullet)=\sum_{i=1}^{n-1}a_i.$$

Therefore, $\tau_{\mathrm{ss}}(S)=\mu(\det S)+\beta_0$ if and only if the corank of the matrix $S(0)$ is at most 2, in which case $S^*(J_{\det})$ is actually a complete intersection ideal and Theorem 4.12(2) (below) applies.

4.1. *Sections of isolated hypersurface singularities*

Now suppose that $f:(\mathbf{C}^n,0)\longrightarrow(\mathbf{C},0)$ has an isolated singularity at the origin. Then $m_0=n$, and there are three values of m when the number β_1 in the right-hand side of (1.5) is easy to calculate: $\beta_1=0,\ \beta_0,\ 2\beta_0$ if $m=n,\ n-1,\ n-2$ respectively. From Theorem 4.1, Lemma 4.3 and Proposition 4.9 we have the following.

THEOREM 4.12 [**3**]. (0) *$\tau_{\mathcal{K}_f}(F)=\mu(f\circ F)-\beta_0$ if $m=n$.*
(1) *$\tau_{\mathcal{K}_f}(F)=\mu(f\circ F)$ if $m=n-1$.*
(2) *$\tau_{\mathcal{K}_f}(F)=\mu(f\circ F)+\beta_0$ if $m=n-2$.*

If $m=n-1$, a generic perturbation F_t of F will be transverse to $V(f)$ and miss $0\in\mathbf{C}^n$ altogether. Hence $V(f\circ F_t)$ is a smoothing of $V(f\circ F)$. If $m=n$, than $\beta_0=\mu(f)\cdot\deg F$; moreover if F_t is a generic perturbation of F, it will cover $0\in\mathbf{C}^n$ $\deg F$ times, and thus $F_t^{-1}(V(f))$ will have $\deg F$ singular points, each with Milnor number $\mu(f)$. Smoothing these to get a Milnor fibre of $f\circ F$, we increase the rank of the middle homology of $F_t^{-1}(V(f))$ by $\deg F\cdot\mu(f)$; hence we have the following.

COROLLARY 4.13. *If $m=n$ or $m=n-1$, then*

$$\mathrm{rank}\ H_{m-1}(V(f\circ F_t))=\tau_{\mathcal{K}_f}(F).$$

5. *Cohen–Macaulay properties of the relative T^1*

In the last section we showed that given a diagram

$$(\mathbf{C}^m,0)\xrightarrow{F}(\mathbf{C}^n,0)\xrightarrow{f}(\mathbf{C},0)$$

with $\mathcal{O}_{\mathbf{C}^n}/J_f$ Cohen–Macaulay of codimension m_0, $\tau_{\mathcal{K}_f} F < \infty$, and $m = m_0$ or $m = m_0 - 1$, then

$$\text{rank(vanishing homology of } V(f \circ F)) = \tau_{\mathcal{K}_f} F.$$

In each case, this formula was proved by placing $T^1_{\mathcal{K}_f} F$ in an exact sequence which related it to other modules whose lengths are conserved in a deformation. Now suppose that $\mathscr{F} : (\mathbf{C}^m \times B, (0,0)) \longrightarrow (\mathbf{C}^n, 0)$ is a deformation of F over the smooth base B. A slight modification of this argument gives us information about the relative T^1:

$$T^1_{\mathcal{K}_f/B}\mathscr{F} := \frac{\theta(\mathscr{F})}{t\mathscr{F}(\theta_{\mathbf{C}^m \times B/B}) + \mathscr{F}^*(\mathrm{Der}(-\log f))}.$$

THEOREM 5.1. *Suppose, in addition, that $\tau_{\mathcal{K}_f} F < \infty$. Then the following hold.*

(i) *If $m = m_0$ then $T^1_{\mathcal{K}_f/B}\mathscr{F}$ is Cohen–Macaulay over $\mathcal{O}_{\mathbf{C}^m \times B}$, of dimension equal to* $\dim B$. *Moreover, it is free of rank $\tau_{\mathcal{K}_f} F$ over $\mathcal{O}_B$.*

(ii) *If $m = m_0 - 1$ and in addition we suppose that* $\operatorname{codim} \mathcal{O}_{\mathbf{C}^m \times B}/\mathscr{F}^*(J_f) = m_0$, *then the same conclusions hold as in* (i).

Proof. (i) From the relative version of Theorem 1.2, we obtain the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_{\mathbf{C}^n}}(\mathcal{O}_{\mathbf{C}^n}/J_f, \mathcal{O}_{\mathbf{C}^m \times B}) \longrightarrow T^1_{\mathcal{K}_f/B}\mathscr{F} \longrightarrow \mathcal{O}_{\mathbf{C}^m \times B}/J^{\mathrm{rel}}_{f\circ\mathscr{F}} \\ \longrightarrow \mathcal{O}_{\mathbf{C}^m \times B}/\mathscr{F}^*(J_f) \longrightarrow 0. \end{aligned}$$

As the absolute Tor, $\mathrm{Tor}_1^{\mathcal{O}_{\mathbf{C}^m}}(\mathcal{O}_{\mathbf{C}^n}/J_f, \mathcal{O}_{\mathbf{C}^m})$, vanishes, so does the parametrised Tor, and this sequence reduces to a short exact sequence. The second and third modules in this short exact sequence are finite over $\mathcal{O}_B$, and are both of dimension equal to $\dim B$. It follows that they are $\mathcal{O}_B$-free. Hence, by the depth lemma, so is the first.

(ii) The argument is almost identical. The only differences are that we have explicitly to require that $\operatorname{codim} V(\mathbf{F}^*(J_f)) = m_0$, in order to guarantee the vanishing of the relative Tor, and that instead of $\mathcal{O}_{\mathbf{C}^m \times B}/\mathscr{F}^*(J_f)$ being free over $\mathcal{O}_B$, it is finite of dimension $\dim B - 1$. Once again, the depth lemma guarantees that $T^1_{\mathcal{K}_f/B}\mathscr{F}$ is $\mathcal{O}_B$-free. $\square$

From this one can prove Theorem 4.8 by the argument of [**12**, Section 5].

Recall from Section 1 the Damon module $T^1_{\mathcal{K}_V} S$, in which $\mathrm{Der}(-\log\det)$ is replaced by $\mathrm{Der}(-\log V)$, and its relative version $T^1_{\mathcal{K}_V/B}\mathscr{S}$. Their support is the set of points where S is not logarithmically transverse to V (respectively, $\mathscr{S}$ is not logarithmically transverse to V relatively to B). The *discriminant* $\mathscr{D} \subset B$ is defined to be the projection to B of the support of $T^1_{\mathcal{K}_V/B}\mathscr{S}$.

It turns out that in all simple singularities of symmetric matrix families in two, three *and* four variables, $\mathscr{D}$ is a free divisor. For simple families in two variables, this follows from the explicit description of the discriminant given in [**14**, Proposition 3.3]; as implied also by [**1**], for three variables it follows from the fact that all of the discriminants are the discriminants of simple singularities of functions on a manifold with boundary. The result in dimension 4 (observed empirically by computer calculation) is surprising; also surprising in these (rather few) examples of four-parameter simple symmetric matrix families is that the conclusion of

Theorem 5.1 (and therefore Theorem 4.6) holds for them. We note that by the argument of Damon [**10**], the freeness of $\mathscr{D}$ is closely related to the (experimentally verified) fact that $T^1_{\mathcal{K}_V/B}\mathscr{S}$ is Cohen–Macaulay of dimension $\dim B - 1$, and that Damon's condition on the existence of 'Morse-type singularities' holds.

Freeness of the discriminant in the base of a family of general matrices in two variables would follow from [**14**, Conjecture 3.5].

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