# Totally Geodesic Foliations

by

Andrew Wellard Wadsley

A thesis submitted for the degree of Doctor of Philosophy to the University of Warwick, September 1974 .

Abstract:

Theorem A of Chapter I states that a periodic flow on a riemannian manifold with each trajectory geodesic is equivalent to a circle action with the same orbits. Using a similar method of proof we obtain a theorem on pointwise periodic homeomorphisms of immersed submanifolds. This generalises a result of N. Weaver. As an application, we show that if M is a two-dimensional riemannian manifold with all closed geodesics then the geodesic loops of M are all of equal length.

In Chapter II, our main theorem asserts that a foliated riemannian manifold which is foliated by totally geodesic compact leaves has finite holonomy. This result has some application to isometric immersions of riemannian manifolds in spaces of constant curvature.

to Helen

Contents page

## 0.0 Introduction

In this thesis we will be concerned with a manifold $M$ foliated by compact leaves. The elucidation of the structure of such foliations is a difficult problem in general. For example, much is known when the foliation arises as the decomposition of $M$ by the orbits of some compact Lie group action (see, for example, (BR)) but not all foliations by compact leaves have a regularity similar to such a model situation. Indeed, in the paper (E1) an analytic foliation of a non-compact manifold $M$ by circles is given for which there is no circle action on $M$ with that orbit structure. However, this type of 'bad' foliation does not have finite holonomy; in fact, a foliation with finite holonomy possesses a high degree of structural regularity. In (E2) D.B.A. Epstein discusses this condition in detail; a version of his main theorem is reviewed here in Chapter II , Theorem 1.1 .

For a manifold $M$ foliated by compact leaves, it is not difficult to see that there is a saturated, open dense subset $M_o$ of $M$ such that the foliated manifold $M_o$ is the total space of a locally trivial fibre bundle, with each fibre homeomorphic to a leaf of $M_o$ . In general, if the foliated manifold $M$ has finite holonomy then it is the total space of a V-bundle over some V-manifold $Q$ . Moreover, Corollary 1 of Chapter II asserts that a foliated riemannian manifold $M$ , foliated by totally geodesic

compact leaves, not only has finite holonomy but is the total space of a V-bundle with structure group a Lie group of isometries of a leaf. Conversely, a foliated manifold M with this V-bundle structure can be given a riemannian metric so that each leaf is totally geodesic in M . (For precise statements, see Chapter II .) In Chapter II, §5 we briefly indicate how this result applies to isometric immersions of riemannian manifolds in spaces of constant curvature.

In Chapter I , we specialise our methods to the case of pointwise periodic flows on manifolds and hypersurfaces. In this manner, we reprove an old theorem of D. Montgomery (M) and generalise a result of N. Weaver (W) . Indeed, the idea behind the proofs of the theorems in both Chapters I and II is a mild extension of an argument due to Montgomery . A version of Theorem A , Chapter I was presented previously by the author and appears in (WA) . In Chapter I, §8 we give a simple application of Theorem A to the class of riemannian manifolds all of whose geodesics are closed.

Finally, I should like to acknowledge the patient supervision given me over the past three years by Professor D.B.A. Epstein. I am also grateful to the other mathematicians of the Mathematics Institute who taught me all I know about differential geometry. Thanks are due to the British Council who supported me financially whilst in the United Kingdom.

## 0.1 Notation

Let $M$ be a differentiable manifold. We denote by $TM$ the tangent bundle of $M$; and for $x \in M$ we denote by $T_xM$ the tangent space to $M$ at $x$. If $f : M \to N$ is a differentiable mapping then $f_* : T_xM \to T_{fx}N$ is the induced map on the tangent space at $x$. $T^*M$ denotes the cotangent bundle of $M$.

Generally, we use $M$ to denote an n-dimensional manifold which may or may not be compact. All manifolds are assumed to be Hausdorff and completely separable; in particular, they are paracompact.

If $f : X \to Y$ and $g : Y \to Z$ are mappings, then their composition is denoted by $g \circ f : X \to Z$.

# Chapter I

## Periodic Flows and Homeomorphisms

### 1.0 Introduction

The impetus for this chapter arose from a paper by David Epstein (E1) where smooth foliations of compact three-manifolds by circles were analysed. Essentially, he showed that all such foliations arise as a decomposition of the manifold by the orbits of some smooth circle action. Crucial to the proof of this difficult result were the well-known theorem of D. Montgomery (M) and a recent theorem of N. Weaver (W). The theorems presented in this chapter are generalisations of those results.

A $C^r$ flow $(0 \leq r \leq \infty)$ on a $C^r$ manifold $M$ is a $C^r$ action $\mu : \mathbb{R} \times M \to M$ of the additive reals on $M$. That is, for each $x \in M$ and $s,t \in \mathbb{R}$ we have $\mu(o,x) = x$ and $\mu(s+t,x) = \mu(s,\mu(t,x))$. A point $x \in M$ is said to be <u>periodic</u> under $\mu$ if there is some positive number $\lambda$ (in general, $\lambda$ depends upon the point $x$) such that $\mu(\lambda,x) = x$. For any set $A \subset M$, define the <u>orbit of A</u> under $\mu$, <u>orb A</u>, by

$$\text{orb}\,A = \{x \in M : x = \mu(t,a) \text{ for any } t \in \mathbb{R}, a \in A\} .$$

Thus $x$ is periodic if and only if orb $\{x\}$ is compact; in which

case, the orbit of $x$ is either the set $\{x\}$ ( $x$ is then a <u>fixed point</u>) or is homeomorphic to a circle. Moreover, if $A$ is an open subset of $M$ then $\mathrm{orb}\,A$ is open also.

It will be convenient to define for every $t \in \mathbb{R}$ the $C^r$ homeomorphism $\mu_t : M \to M$ by $\mu_t(x) = \mu(t,x)$ , $x \in M$ . If the flow $\mu$ is at least $C^1$ then the tangent vector field $T$ can be defined as $T_x = (d/dt)\mu_t(x)|_{t=0}$ at $x \in M$ . The Lie derivative $L_T\alpha$ of an one-form $\alpha$ is defined by

$$(L_T\alpha)_x = \lim_{t-0} ((\mu_t^*\alpha)_x - \alpha_x)/t \quad .$$

The following are the theorems to be proved. Montgomery's theorem (Theorem B) is included here because it can be obtained easily from the proof of Theorem A and also because it is required in the second chapter.

<u>Theorem A</u>

Let $\mu : \mathbb{R} \times M \to M$ be a $C^r$ flow $(2 \leqslant r \leqslant \infty)$ with every orbit a circle on a connected $C^r$ manifold $M$ . Let the vector field $T$ generate the flow $\mu$ . Then there is a $C^r$ action $\rho : S^1 \times M \to M$ of the circle group $S^1$ with the same orbits as $\mu$ if and only if there exists a one-form $\alpha$ on $M$ with $\alpha(T) > 0$ and $T \lrcorner\, d\alpha = 0$ .

(Where $T \lrcorner$ denotes contraction by $T$ and $d\alpha$ is the exterior derivative of $\alpha$ .)

1.1 Remark: Define another vector field $T'$ on $M$ by $T' = T/\alpha(T)$. Then $T' \lrcorner\, d\alpha = 0$ and $\alpha(T') = 1$. It follows from the formula $L_{T'}\alpha = d(T' \lrcorner\, \alpha) + T' \lrcorner\, d\alpha$ that the Lie derivative $L_{T'}\alpha = 0$; moreover, the flow $\mu'$ generated by $T'$ has the same orbits as $\mu$. Thus the condition of the theorem implies the existence of some reparametrised flow $\mu'$ which preserves a transverse one-form $\alpha$.

1.2 Remark: Suppose $P : M \to Q$ is a principal circle bundle and the orbits of $\mu$ are the fibres of $P$. Then the form $\alpha$ can be taken to be the connection form of some connection on the bundle $P$.

1.3 Remark: For such flows the existence of such a one-form $\alpha$ is equivalent to the existence of some riemannian metric on $M$ with respect to which the orbits of $\mu$ are geodesic in $M$ (see §3, Proposition 3.4).

Theorem B (Montgomery)

Let $M$ be a connected topological manifold and $h : M \to M$ a pointwise periodic homeomorphism. Then $h$ has global period.

Theorem C

Let $\mu : \mathbb{R} \times M \to M$ be an orientation preserving $C^r$ flow $(2 \leqslant r \leqslant \infty)$ on an oriented manifold $M$. Let the vector field $T$ generate the flow $\mu$. Further, let $N$ be a connected $C^r$ mani-

fold with $\dim N = \dim M - 1$ and $\varphi : N \to M$ a $C^r$ immersion. Set $\hat{N} = \varphi N$ ; assume $\operatorname{orb}\hat{N} = \hat{N}$ and $\mu$ restricted to $\hat{N}$ is pointwise periodic without fixed points. If there exists a one-form $\alpha$ on $M$ with $\alpha|_{\hat{N}}(T) = 1$ and $L_T\alpha|_{\hat{N}} = 0$ then $\mu$ restricted to $\hat{N}$ has global period; that is, there is some $c > 0$ such that $\mu_c|_{\hat{N}} = \mathrm{id}$ .

<u>1.4 Remark</u>: If the immersion $\varphi : N \to M$ is 1-1 then the conclusion of the theorem follows directly from Theorem A .

<u>1.5 Remark</u>: Given $N$ , $M$ and $\mu$ as in Theorem C with $\mu|_{\hat{N}}$ having global period, then the existence of some such one-form is necessary if the flow is at least $C^2$ (see §7) .

<u>Theorem D</u>

Let $M$ be an oriented $C^r$ manifold $(1 \leq r \leq \infty)$ and $h : M \to M$ an orientation preserving $C^r$ diffeomorphism. Further assume that $N \subset M$ is a connected immersed codimension-one $C^r$ submanifold of $M$ with $h$ restricted to $N$ pointwise periodic. Then $h$ restricted to $N$ has global period.

<u>1.6 Remark</u>: Note that we do not assume that $h(N) = N$ . Also, Theorem D should be compared with Weaver's theorem (W) where the same conclusion holds with similar hypotheses on $h$ but with $\dim M = 2$ and $N$ an arbitary invariant continuum.

<u>1.7 Remark</u>: Theorems C and D are not true when the codimension of $N$ in $M$ is greater than unity. The following is a $C^\infty$ counter-example.

<u>1.8 Example</u>: Using the $C^\infty$ function $e : \mathbb{R} \to \mathbb{R}$ defined by $e(x) = 0$ for $x \leqslant 0$ and $e(x) = \exp(-1/x^2)$ for $x > 0$ we can easily construct differentiable functions $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$ for each integer $n \geqslant 1$ with the following properties:

$$f_n(x) = 2^{-(n+1)} \qquad \text{for } x \leqslant \frac{(2n+1)}{2n(n+1)} \quad ;$$

$$2^{-(n+1)} < f_n(x) < 2^{-n} \qquad \text{for } \frac{2n+1}{2n(n+1)} < x < \frac{1}{n} \quad ;$$

$$f_n(x) = 2^{-n} \qquad \text{for } x \geqslant \frac{1}{n} \quad ;$$

and for each positive integer $k$ $\sup f_n^{(k)}(x) \to 0$ as $n \to \infty$ :

$$g_n(x) = 0 \qquad \text{for } x \leqslant \frac{1}{n+1} \quad ;$$

$$g_n(x) > 0 \qquad \text{for } \frac{1}{n+1} < x < \frac{2n+1}{2n(n+1)} \quad ;$$

$$g_n(x) = 0 \qquad \text{for } x \geqslant \frac{2n+1}{2n(n+1)} \quad ;$$

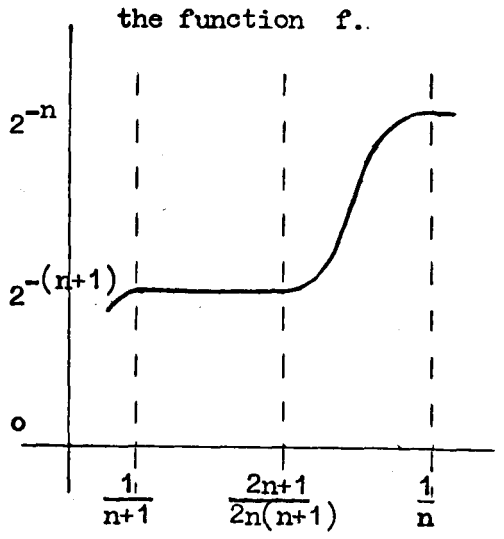
and for each positive integer $k$ $\sup g_n(x) \to 0$ and $\sup g_n^{(k)}(x) \to 0$ as $n \to \infty$.

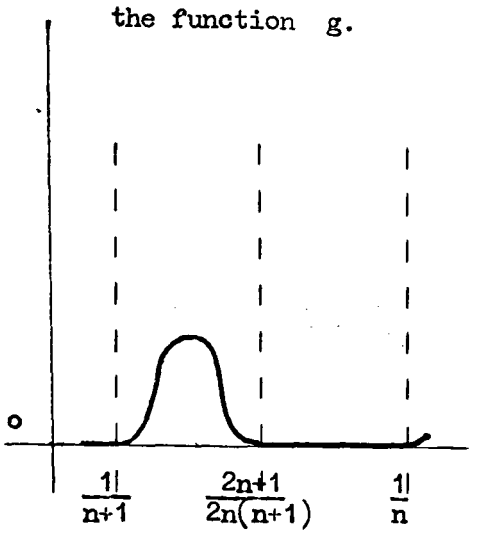
Glueing these functions together we obtain $C^\infty$ functions $f$ and $g$ with

$$f(x) = g(x) = 0 \qquad \text{for } x \leqslant 0 \quad ;$$

$$f(x) = f_n(x) \ \& \ g(x) = g_n(x) \qquad \text{for } \frac{1}{n+1} \leqslant x \leqslant \frac{1}{n} \quad ;$$

and $$f(x) = \tfrac{1}{2} \ \& \ g(x) = 0 \qquad \text{for } x \geqslant 1 \quad .$$




Set the manifold $N \subset \mathbb{R}^3$ to be the graph of $g$ in the $(x,y)$ plane of $\mathbb{R}^3$. We define a $C^\infty$ diffeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ by rotating the plane $P_a = \{(a,y,z) \in \mathbb{R}^3 ;\ y,z \in \mathbb{R}\}$ through an angle $2\pi f(a)$. Thus for planes $P_a$ with $\frac{1}{n+1} \leqslant a \leqslant \frac{2n+1}{2n(n+1)}$ the rotation is through the constant angle $2\pi 2^{-(n+1)}$. Thus the points $(a,g(a),0)$ of $N$ with $\frac{1}{n+1} < a < \frac{2n+1}{2n(n+1)}$ have period $2^{n+1}$ whilst the points $(b,g(b),0)$ with $\frac{2n+1}{2n(n+1)} \leqslant b \leqslant \frac{1}{n}$ are fixed under $h$. Therefore $h|_N$ is pointwise periodic but the period is not locally bounded at the origin; thus $h$ has no global period.

## 2.0 Preliminaries

$M$ shall denote an n-dimensional $C^r$ manifold $(0 \leqslant r \leqslant \infty)$. Let $A \subset M$ and suppose we are given a $C^r$ flow $\mu : \mathbb{R} \times M \to M$ such that every orbit intersecting $A$ is a circle.

Let $V_o$ be a small open disk in $M$ transverse to the flow with $\mathrm{cl}V_o$ compact. Then there is an $\varepsilon > 0$ such that $\mu$ defines a homeomorphism of $[-\varepsilon, \varepsilon] \times \mathrm{cl}V_o$ into $M$. By a <u>flat neighbourhood</u> in $M$ (resp. of a point $x$ in $M$) we shall mean an open subset $V$ of $M$ (resp. an open neighbourhood $V$ of $x$ such that $V = \mu((-\varepsilon, \varepsilon) \times V_o)$ for some open disk $V_o$ (resp. for some open disk $V_o$ with $x \in V_o$) as above. Let $\pi : V \to V_o$ be the projection map.

Suppose $m \in A$ and $V = \mu((-\varepsilon, \varepsilon) \times V_o)$ is a flat neighbourhood of $m$ such that the orbit of $m$ intersects the closure of the disk $V_o$ in precisely one point; that is, $\mathrm{cl}V_o \cap \mathrm{orb}\{m\} = \{m\}$. We now define the Poincaré map $S : V_1 \to V_o$ for some smaller disk $V_1 \subset V_o$. For more detail, the reader is referred to (E1). Precisely, there exists a connected neighbourhood $V_1$ of $m$ in $V_o$ such that the map $f : V_1 \to \mathbb{R}$, given by the conditions

1) $fx > 0$

2) $\mu_t x \notin V_o$ for $0 < t < fx$

3) $\mu_{fx} x \in V_o$

is well-defined and $C^r$ on $V_1$. The <u>Poincaré Map</u> $S : V_1 \to V_o$ is defined by $Sx = \mu_{fx} x$ at $x \in V_1$. The point $m \in V_o$ is invariant under $S$. Now define the <u>periodicity function</u> $\lambda : A \to \mathbb{R}$ by the conditions

1) $\lambda x > 0$

2) $\mu_t x \neq x$ for $0 < t < \lambda x$

3) $\mu_{\lambda x} x = x$ .

The function $\lambda$ is invariant under the flow; that is, if $x,y \in A$ with $y = \mu(s,x)$ for some $s \in \mathbb{R}$, then $\lambda x = \lambda y$.

2.1 Proposition

The function $\lambda : A \to \mathbb{R}$ giving the period of a point is lower semi-continuous; moreover, the set of points of continuity of $\lambda$ is open and dense in the induced topology on $A$.

Proof: Let $x \in A$ and let $V = \mu((-\varepsilon,\varepsilon) \times V_0$ be a flat neighbourhood of $x$ such that the orbit through $x$ intersects $V_0$ in only the single point $x$. Then $\lambda \geqslant f$ on $A \cap V_1$, where the $C^r$ function $f : V_1 \to \mathbb{R}$ is defined as for the Poincaré map above. This proves lower-semicontinuity.

To prove the second statement it suffices to observe, that if $\lambda$ is continuous at $x$, then $V_0$ can be made sufficiently small to ensure that any orbit of $\mu$ intersecting $A \cap V_0$ will intersect $V_0$ in only a single point. But then we have $\lambda = f$ on $A \cap V_0$; and moreover, $f$ is continuous. Thus $\lambda$ is continuous on $A \cap V$. This together with the following well-known result shows that the points of continuity of $\lambda$ are dense in $A$.

2.2 Lemma

Let $\lambda : A \to \mathbb{R}$ be a lower semi-continuous function on a space $A$. Then the set $B = \{ x \in A : \lambda$ is not continuous at $x \}$ is of the first category in $A$.

Proof: For each rational number $r$, define $B_r = \{ x : \lambda x > r \}$. Then $B \subset \bigcup_{\mathbb{Q}} (\mathrm{cl}B_r - B_r)$ and each $\mathrm{cl}B_r - B_r$ is nowhere dense.

## 3.0 g-Flows

Let $N$, $M$ be $C^r$ manifolds $(0 \leqslant r \leqslant \infty)$ where $\dim N \leqslant \dim M$. Let $\varphi : N \to M$ be a $C^r$ immersion (that is, $\varphi : N \to \varphi N$ is a local $C^r$ homeomorphism); setting $\hat{N} = \varphi N$, we further assume that $\hat{N}$ is connected. Additionally, let $\mu : \mathbb{R} \times M \to M$ be a $C^r$ flow so that $\mathrm{orb}\hat{N} = \hat{N}$; that is, $\hat{N}$ is invariant under the flow

In the remainder of this chapter we will be concerned with two situations in which we impose further conditions on the manifold $M$ and flow $\mu : \mathbb{R} \times M \to M$; namely,

1) we assume the flow $\mu$ is differentiable of class $C^2$ and the manifold $M$ is given a riemannian metric;

2) alternatively, in the topological case, we assume there exists a codimension-one foliation transverse to the orbits of the flow $\mu$.

Under either assumption 1) or 2) we can introduce a well-defined notion of 'orthogonal' curve. Precisely, when $M$ is a riemannian manifold and the flow $\mu : \mathbb{R} \times M \to M$ is differentiable, a smooth curve $\sigma : [0,1] \to M$ is said to be 'orthogonal' to the flow if, for $s \in [0,1]$, each tangent vector $\sigma'(s)$ is orthogonal to the

vector $T_{\sigma(s)}$ (with respect to the inner product on $T_{\sigma(s)}M$ induced by the riemannian structure on $M$ ); here, the vector field $T$ generates the flow $\mu$ . This notion of 'orthogonality' coincides with that in common usuage. Alternatively, in the topological case of 2) , any curve $\tau : [0,1] \to M$ lying entirely within a single leaf of the transverse foliation will be called 'orthogonal' .

Fix $x \in N$ . Let $U \subset N$ be an open connected neighbourhood of $x$ in $N$ such that $\varphi|_U : U \to \varphi U$ is a homeomorphism. Setting $\hat{U} = \varphi U$ , let $V = \mu((-\varepsilon, \varepsilon) \times V_o)$ be a flat neighbourhood of $\hat{x} = \varphi x$ in $M$ such that $V \cap \hat{U} = \hat{U}$ . Let $\pi : V \to V_o$ be the projection mapping.

3.1 <u>Lemma</u>

Assume either condition 1) or 2) for the flow $\mu : \mathbb{R} \times M \to M$ ; so we have a well-defined concept of 'orthogonality' for curves transverse to the flow. Then for each point $y \in U$ sufficiently near $x$ and $C^r$ curve $\sigma : [0,1] \to U$ with $\sigma(0) = x$ and $\sigma(1) = y$ , there exists a unique 'orthogonal' curve $\hat{\sigma} : [0,1] \to V$ such that $\hat{\sigma}(0) = \hat{x}$ and $\pi \circ \hat{\sigma} = \pi \circ (\varphi \circ \sigma)$ .

Proof: More generally, it is easy to see that if $\sigma_1 : [0,1] \to V$ and $\sigma_2 : [0,1] \to V$ are 'orthogonal' $C^r$ curves in the flat neighbourhood $V$ such that $\pi \circ \sigma_1 = \pi \circ \sigma_2$ and $\sigma_1(0) = \sigma_2(0)$ , then $\sigma_1 = \sigma_2$ . (Note: in general, the lemma is false for a $C^1$ flow on a riemannian manifold. The difficulty arises because the orthogonal distribution may only be continuous.)

3.2 Definition

Let $\mu : \mathbb{R} \times M \to M$ be a $C^r$ flow such that either 1) $r \geqslant 2$ and $M$ is a riemannian manifold, or 2) $r \geqslant 0$ and there is a codimension-one foliation of $M$ transverse to the orbits of $\mu$ ; that is, we have a well-defined notion of 'orthogonality' for curves transverse to the flow. Let $\varphi : N \to M$ be a $C^r$ immersion such that the submanifold $\hat{N} = \varphi N$ is invariant under the flow $\mu$ . Then the flow $\mu : \mathbb{R} \times M \to M$ is called a g-flow (w.r.t. the immersion $\varphi : N \to M$ ) if for each 'orthogonal' curve $\hat{\sigma} : [0,1] \to \hat{N}$ we have $\mu_t \circ \hat{\sigma}$ orthogonal for each $t \in \mathbb{R}$ .

3.3 Remark: Lemma 3.1 ensures that we have a plentiful supply of 'orthogonal' curves lying entirely within the submanifold $\hat{N}$ .

If the flow $\mu : \mathbb{R} \times M \to M$ is differentiable, then g-flows arise in many natural settings. Specifically, suppose we are given a one-form $\alpha$ on $M$ with $\alpha|_{\hat{N}}(T) = 1$ and $L_T\alpha|_{\hat{N}} = 0$ , where the vector field $T$ generates the flow $\mu$ . Without loss of generality we shall suppose that $\alpha(T) > 0$ on the whole of the manifold $M$ . These are the hypotheses of Theorems C and A (using Remark 1.1) . Define the vector subspaces $Q_x$ and $P_x$ by

$$Q_x = \{ X \in T_xM : \alpha(X) = 0 \}$$

and $$P_x = \{ X \in T_xM : X = cT , c \in \mathbb{R} \} .$$

Then the tangent bundle of $M$ splits as $TM = Q \oplus P$ . Furthermore,

a straighforward construction defines a riemannian metric on $M$ such that $Q_x$ is orthogonal to $P_x$ at each $x \in M$. The subspace $P_x$ is tangent to the flow $\mu$. The condition on the Lie derivative, $L_T\alpha|_{\hat{N}} = 0$, implies that for each point $x \in \hat{N} = \varphi N$ and vector $X \in Q_x \cap \varphi_* T_y N$ (where $y \in \varphi^{-1}x$) we have $\mu_{t*}X$ orthogonal for all time $t$. Clearly the action $\mu$ is a g-flow (w.r.t. the immersion $\varphi : N \to M$).

In Theorem A we are given a one-form $\alpha$ and vector field $T$ such that $\alpha(T) > 0$ and $T \lrcorner d\alpha = 0$. Such one-forms arise naturally in the study of contact manifolds, as defined by Boothby and Wang (B-W). In this case, the manifold $M$ (of class $C^\infty$) is assumed to have dimension $2n+1$ with a globally defined one-form $\omega$ such that $\omega \wedge (d\omega)^n \neq 0$ on $M$ ( $(d\omega)^n = d\omega \wedge \ldots \wedge d\omega$). On the subspace $P_x = \{X \in T_xM : X \lrcorner d\omega = 0\}$ we have $\omega \neq 0$; moreover, $P_x$ has dimension one and is complementary to the subspace of dimension $2n$ on which $\omega$ is zero. Let $Z_x$ be that element of $P_x$ for which $\omega(Z_x) = 1$. Then the vector field $Z$ and one-form $\omega$ satisfy the conditons of the preceding paragraph; whence the flow generated by $Z$ is a g-flow.

Indeed, in their paper (B-W), Boothby and Wang prove a special case of our theorem. They consider the case where the manifold $M$ is compact and the induced foliation of $M$ by the trajectories of $Z$ is <u>regular</u> in the sense of Palais (P1). That is, about each point $x$ of $M$ there is an open neighbourhood $U$ of $x$ so that any non-empty intersection of a trajectory with $U$ is an open set. In this situation, each trajectory is closed and hence compact; so each orbit is a circle. They deduce that there is a <u>free</u> circle action on $M$ with the same orbits as the $\mathbb{R}$-action

generated by $Z$. In fact, the periodicity function $\lambda : M \to \mathbb{R}$ defined as in §2 is constant on $M$. Two examples of contact manifolds with natural contact forms are an arbitary odd-dimensional sphere $S^{2n+1} \subset \mathbb{R}^{2n+2}$ (coordinates $x_1, \ldots, x_{2n+2}$) with form

$$\omega = \sum_{i=1}^{n+1}(x_{2i-1}dx_{2i} - x_{2i}dx_{2i-1}) \quad \text{restricted to } S^{2n+1},$$

and the cotangent bundle of the unit tangent bundle of a riemannian manifold with its canonical one-form.

For a $C^2$ flow $\mu$ with each trajectory a geodesic in a riemannian manifold $M$ we can prove Proposition 3.4 below thus showing such flows are g-flows. In fact, the existence of some riemannian metric on $M$ with respect to which the trajectories of $\mu$ are geodesic is equivalent to the existence of some invariant one-form $\alpha$ as discussed above. Only the proof of sufficiency will be given here because the converse is proved by the reverse argument and is quite straightforward.

3.4 <u>Proposition</u>

Let $T$ be a vector field on a riemannian manifold $M$ which generates a flow $\mu : \mathbb{R} \times M \to M$; further assume that each trajectory is a geodesic parametrised by arc-length. Fix $m \in M$; let $X_m \in T_mM$ and suppose $X_m$ is orthogonal to $T_m$. Then the vector $\mu_{t*}X_m$ in the tangent space to $M$ at $y = \mu_t m$ is orthogonal to $T_y$ for every $t \in \mathbb{R}$. That is, the flow maps orthogonal vectors into orthogonal vectors for all time.

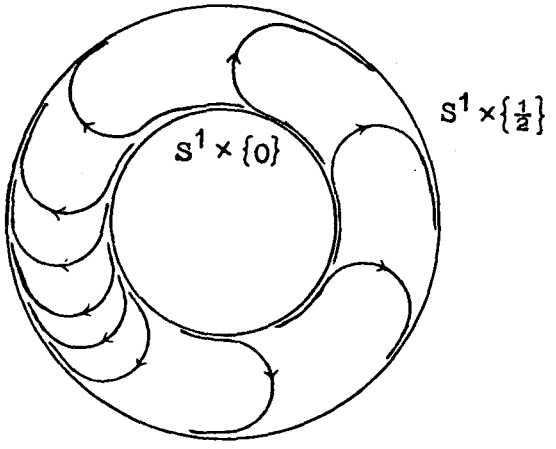
Proof: Let $m$ and $X_m \in T_mM$ be as in the statement of the prop-

osition. Let $V = \mu((-\varepsilon,\varepsilon)\times V_o)$ be a flat neighbourhood of $m$ with $X_m$ tangent to $V_o$ at $m$. Furthermore, we may assume there are defined on $V_o$ coordinate functions $x^2,\ldots,x^n$ with $x^i(m) = 0$ and $(\partial/\partial x^n)_m = X_m$ (where $\dim M = n$). On $V$ we may define coordinate functions $y^1,\ldots,y^n$ as follows: for $y = \mu_t(z)$, $z \in V_o$, $-\varepsilon < t < \varepsilon$, set $y^1(y) = t$ and $y^i(y) = x^i(z)$, $2 \leqslant i \leqslant n$. Then $(\partial/\partial y^1)_y = T_y$ and $(\partial/\partial y^i)_y = \mu_{t*}(\partial/\partial x^i)_z$; in particular, if $y = \mu_t(m)$ then $(\partial/\partial y^n)_y = \mu_{t*}X_m$. Define the vector field $X$ on $V$ by $X = (\partial/\partial y^n)$.

Our hypotheses imply (i) $\nabla_T T = 0$, since the trajectories are geodesics (where $\nabla$ is the Levi-Civita connection of the metric), (ii) $\langle T,T\rangle = 1$ and (iii) $0 = [T,X] = \nabla_T X - \nabla_X T$. Therefore we have $T\langle T,X\rangle = \langle\nabla_T T,X\rangle + \langle T,\nabla_T X\rangle = \langle T,\nabla_T X\rangle = \langle T,\nabla_X T\rangle = \frac{1}{2}X\langle T,T\rangle = 0$. That is, the inner product $\langle T,X\rangle$ is constant along the orbit of $T$ through $m$. In particular, we have $\langle\mu_{t*}X_m,T\rangle = 0$ for $-\varepsilon < t < \varepsilon$. This completes the proof. In general, the flow $\mu$ need not be metric preserving.

Generally, given a flow on a manifold it is a difficult task to determine whether there is some reparametrisation and riemannian metric with respect to which the trajectories are geodesic. The counter-example in Epstein's paper (E1) presents a flow with every orbit periodic for which this is not possible! Moreover, it is easy to construct an example of a flow on the 2-torus which cannot preserve a transverse one-form (see Example 3.5 below); nevertheless, there may exist general conditions (e.g. recurrence?) on a non-singular flow on a <u>compact</u> manifold which guarantee this type of regularity.

<u>3.5 Example</u>: We identify the torus as $T^2 = S^1 \times \mathbb{R}/Z$. On the annular region $A = S^1 \times [0,\frac{1}{2}]$ construct a flow with trajectories as shown in the diagram:




In the complement, put any flow. Thus we have constructed some global flow $\mu : \mathbb{R} \times T^2 \to T^2$. Give the torus $T^2$ any riemannian metric; then the distribution $\mathcal{D}^\perp$ of vectors orthogonal to the flow is one-dimensional and hence integrable. Since $\mathcal{D}^\perp$ is everywhere transverse, a leaf $\ell$ of $\mathcal{D}^\perp$ entering the region $A$ through the boundary circle $S^1 \times \{0\}$ will be unable to escape; that is, a particle which starts on a point of the intersection of $\ell$ with $S^1 \times \{0\}$ and travels along the leaf $\ell$ in a fixed direction into the region $A$, will not meet either boundary circle $S^1 \times \{\frac{1}{2}\}$ or $S^1 \times \{0\}$ again. Now suppose that the orthogonal distribution $\mathcal{D}^\perp$ is preserved by $\mu_t$ for all time $t$. Let $\ell$ be an orthogonal leaf which intersects the circle $S^1 \times \{0\}$ in a point $x_0$. Let $x_1 \in A \cap \ell$ be a point on $\ell$ inside the region $A$ such that the open segment $(x_0, x_1)$ of points between $x_0$ and $x_1$ in $\ell$ lies entirely within the region $A$. Because the circle $S^1 \times \{0\}$ is an orbit of the flow $\mu$, each point $z$ of $S^1 \times \{0\}$ will be periodic under $\mu$. Let $c > 0$ be the least period of $\mu | S^1 \times \{0\}$; thus $\mu_c z = z$ for each $z \in S^1 \times \{0\}$. In particular, $\mu_{kc} x_0 = x_0$ for every integer $k$; moreover, for each $k$ the leaf $\ell$ is invariant under the transformation $\mu_{kc}$. However, the sequence of points

$\{\mu_{kc}x_1\}_{k=1,2,..}$ tends towards the circle $S^1 \times \{\frac{1}{2}\}$ . Thus there exists an integer $K \geqslant 1$ such that the point $\mu_{Kc}x_1$ meets some leaf $\ell'$ which is orthogonal to the trajectories of $\mu$ and which intersects the circle $S^1 \times \{\frac{1}{2}\}$ transversely . Clearly, the leaves $\ell$ and $\ell'$ must coincide . But this is impossible by our choice of leaf $\ell$ . Thus the trajectories of the flow $\mu$ are not geodesic in any metric.

However, an interesting class of flows for which this is true is the class of geodesic flows in the unit tangent bundle $\Sigma M$ to a riemannian manifold $(M,g)$ . Specifically, we can construct a canonical riemannian metric $g$ on $\Sigma M$ induced from the metric $g$ on $M$ using the following theorem of J. Vilms (V) .

3.6 <u>Theorem</u> (Vilms)

Let $B$ be a riemannian manifold and let $p : E \to B$ be a fibre bundle with a connection (in the sense of Ehresmann (EH)) and with structure group $H$ a Lie group of isometries of a fibre. Then there exists a riemannian metric on $E$ such that $p$ is a riemannian submersion and the horizontal lift of a geodesic in $B$ is geodesic in $E$ ; moreover, with respect to this metric the fibres of $p : E \to B$ are totally geodesic submanifolds of $E$ .

Outline proof: Because the structure group $H$ is a Lie group of isometries of a fibre we can give each fibre of the bundle $E$ a well-defined riemannian metric. Moreover, parallel translation with

respect to the connection is an H-isomorphism of the fibres. The riemannian metric for the manifold $E$ is defined as the product of the metric on a fibre and the metric induced on the horizontal distribution by projection onto the riemannian manifold $B$.

If we generalise to the bundle $p : \Sigma M \to M$ of unit vectors over a complete riemannian manifold $(M,g)$ then the distribution of horizontal vectors in $\Sigma M$ defined by the Levi-Civita connection of $g$ defines a connection in $\Sigma M$ such that the hypotheses of Theorem 3.6 are satisfied. Because the canonical lift of a geodesic in $M$ to $\Sigma M$ is also horizontal, we may use Proposition 3.4 to deduce that the geodesic flow in $\Sigma M$ is indeed a g-flow (with respect to the identity immersion $id : \Sigma M \to \Sigma M$.

Another class of g-flows arises directly from consideration of $C^r$ homeomorphisms $h : M \to M$ $(0 \leqslant r \leqslant \infty)$. Define the manifold $M_h$ as the quotient manifold obtained from $M \times (-\frac{1}{2},\frac{1}{2})$ by the identification $i : (x, -\frac{1}{2}+s) \sim (hx, \frac{1}{2}-s)$ for all $x \in M$ and $0 < s < \frac{1}{3}$. Then the flow $\mu_h$ is generated by the vector field $d/ds$ where $(-\frac{1}{2},\frac{1}{2})$ is parametrised by $s$. This construction is called the <u>suspension</u> of the homeomorphism $h$. If a point $x \in M$ is periodic under $h$ then the flow $\mu_h$ on $M_h$ has a closed orbit through the point corresponding to $(x,0)$ under the identification $i : M \times (-\frac{1}{2},\frac{1}{2}) \to M_h$. Furthermore, the flow $\mu_h$ maps each section $i(M \times \{s\}) \subset M_h$ into some other such section; these sections foliate the manifold $M_h$ and the flow $\mu_h$ maps leaves into leaves for all time. Clearly, such an 'orthogonal' leaf structure makes $\mu_h$ a g-flow. Thus each homeomorphism $h : M \to M$ induces a natural

g-flow $\mu_h$ (w.r.t. the identity immersion id: $M_h \to M_h$) on a manifold $M_h$ ; moreover, if $A \subset M$ then $h|A$ is pointwise periodic if and only if $\mu_h | i(A \times (-\frac{1}{2},\frac{1}{2}))$ is pointwise periodic.

## 4.0 The locally bounded case

### 4.1 Lemma

Let $\mu : \mathbb{R} \times M \to M$ be a $C^r$ flow $(0 \leqslant r \leqslant \infty)$. Let $\varphi : N \to M$ be a $C^r$ immersion and suppose that $\mu$ is a g-flow (w.r.t. to $\varphi$). Let $U$ be an open connected subset of $N$. Setting $\hat{U} = \varphi U$, further assume there exists a continuous invariant map $h : \hat{U} \to \mathbb{R}$ such that $\mu_{hx}x = x$ for all $x \in \hat{U}$. Then $h$ is a constant map.

Proof: Fix $x \in \hat{U}$. Let $V = \mu((-\varepsilon,\varepsilon) \times V_o)$ be a flat neighbourhood of $x$ in $M$. Note that each point of $U$ is periodic under $\mu$, so we may define the periodicity function $\lambda : \hat{U} \to \mathbb{R}$ as in §2. On $V \cap \hat{U}$ we have $\lambda \geqslant 2\varepsilon$. Choose another neighbourhood $W$ of $x$, $W = \mu((-\varepsilon,\varepsilon) \times W_o)$, $x \in W_o \subset V_o$ such that for $y \in W \cap \hat{U}$ we have $|hx - hy| < \varepsilon$. For $p' \in W$, by taking a smaller neighbourhood if need be, we may further suppose that there exists an orthogonal curve $\hat{\sigma} : [0,1] \to W \cap \hat{U}$ with $\hat{\sigma}(0) = x$ and $\hat{\sigma}(1) = p$, where $p$ and $p'$ lie on the same orbit of $\mu$. Now $\mu_{hx} \circ \hat{\sigma}$ is orthogonal and its image is contained in $W \cap \hat{U}$ ; furthermore, it is easy to see that

$\pi \circ \hat{\sigma} = \pi \circ (\mu_{hx} \circ \hat{\sigma})$ where $\pi : W \to W_0$ is projection. In particular, $\mu_{hx} \circ \hat{\sigma}(1) = \mu_{hx}(p) = p$. Clearly $hp = k_1 \lambda p$ where $k_1$ is an integer; similarly, we have $hx = k_2 \lambda p$. As $|hx - hp| < \varepsilon$ and $\lambda p \geqslant 2\varepsilon$ we obtain $|k_1 - k_2| < \frac{1}{2}$, which implies $k_1 = k_2$. Whence $hx = hp = hp'$. As $p' \in W$ was arbitary and $\hat{U}$ is connected the lemma is proved.

Given our immersion $\varphi : N \to M$, we now assume that the flow $\mu$ restricted to $\hat{N} = \varphi N$ is pointwise periodic. If the action $\mu$ is a non-trivial g-flow (w.r.t. $\varphi : N \to M$) then we see immediately that the flow on $\hat{N}$ is fixed-point free. Thus each orbit intersecting $\hat{N}$ is a circle. We now define the <u>periodicity function</u>

$$\lambda : \hat{N} \to \mathbb{R}$$

as was done in §2. Thus for each $x \in N$ we have i) $\lambda x > 0$, ii) $\mu_t x \neq x$ for $0 < t < \lambda x$, and iii) $\mu_{\lambda x} x = x$.

The following construction is due to Montgomery (M). We define the set $B_1 \subset N$ as

$$B_1 = \{x \in N : \lambda : \hat{N} \to \mathbb{R} \text{ is not continuous at } \hat{x} = \varphi x\} .$$

Clearly $\hat{B}_1 = \varphi B_1$ is invariant; moreover, by Proposition 2.1 $B_1$ is closed and has null interior in $N$. Thus the components of $N - B_1$ are open connected subsets of $N$. From Lemma 4.1 we can easily deduce the following corollary.

4.2 Lemma

Let $\lambda : \hat{N} \to \mathbb{R}$ be the periodicity function, and let $C$ be a connected component of $N - B_1$. Then $\lambda | \hat{C} = c$, a constant (where $\hat{C} = \varphi C$).

Define the set $K \subset N$ by

$$K = \{ x \in N : \lambda : \hat{N} \to \mathbb{R} \text{ is unbounded in any neighbourhood of } \hat{x} = \varphi x \} .$$

$K$ is a closed subset of $N$; by Lemma 4.2, the periodicity function $\lambda$ is locally constant on $\hat{N} - \hat{B}_1$ (where $\hat{B}_1 = \varphi B_1$). Thus we have $K \subset B_1$. To obtain our theorems we will show that the set $K$ is empty. In later sections we assume $K$ is non-empty and prove a contradiction.

Let $D$ be a connected component of $N - K$; thus $\hat{D} = \varphi D$ is an open invariant subset of $\hat{N}$. Fix $m \in D$ and let $A \subset \hat{D}$ be the orbit of $\mu$ through $\hat{m} = \varphi m$. Let $V = \mu((-\varepsilon, \varepsilon) \times V_0)$ be a flat neighbourhood of $\hat{m}$ with $\mathrm{cl}V_0$ compact, such that $V \cap \hat{N} \subset \hat{N} - \hat{K}$. Then $\lambda \geqslant 2\varepsilon$ on $V \cap \hat{D}$. Because $\lambda$ is locally bounded on $\hat{D}$ we may assume, by making $V$ smaller if necessary, that on $V \cap \hat{D}$ we have $\lambda \leqslant \Lambda$, for some constant $\Lambda \in \mathbb{R}$. Additionally, it may be supposed that the disk $V_0$ is sufficiently small to ensure that the orbit $A$ intersects $\mathrm{cl}V_0$ in only the single point $m$. We define the Poincaré map $S : V_1 \to V_0$ for some smaller disk $V_1 \subset V_0$ as in §2; recall that $S(\hat{m}) = \hat{m}$.

Let $R = [(\Lambda/2\varepsilon) + 1]$ and define by induction neighbourhoods $V_i$ of $\hat{m}$ in $V_o$ such that $SV_{i+1} \subset V_i$ for $1 \leqslant i \leqslant R!$. Because $\lambda \geqslant 2\varepsilon$ on the invariant set $\mathrm{orb}(V_o \cap \hat{D})$ and because $\lambda \leqslant \Lambda$ here, it is easy to see that for each point $x \in V_q \cap \hat{D}$, where $q = R!$, we have $S^r x = x$ for some $r$, $1 \leqslant r \leqslant R!$. Hence $S^q = \mathrm{id}$ on $V_q \cap \hat{D}$. Set $W = \cap_{i=1}^{q} S^i V_q$ and define the function $h : W \to \mathbb{R}$ by $h(x) = \Sigma_{i=1}^{q} f \circ S^i(x)$ where $f : V_1 \to \mathbb{R}$ is defined as in the construction of the Poincaré map of §2. That is, for each $x \in V_1$ we have i) $fx > 0$, ii) $\mu_t x \notin V_o$ for $0 < t < fx$, and iii) $\mu_{fx} x \in V_o$.

Thus the set $\mathrm{orb}(W \cap \hat{D})$ is invariant and open in $\hat{D}$. Furthermore, $h|W \cap \hat{D}$ is continuous and invariant under $S$, whence we may assume that it is defined on all of $\mathrm{orb}(W \cap \hat{D})$. Now $S^q = \mathrm{id}$ on $W \cap \hat{D}$. Therefore we have $\mu_{hx} x = x$ for $x \in (W \cap \hat{D})$. It then follows that $\mu_{hx} x = x$ for all $x \in \mathrm{orb}(W \cap \hat{D})$. Now let $U$ be a connected neighbourhood of $m$ in $D$ such that $\varphi|U$ is a homeomorphism onto its image. Then $\mathrm{orb}\hat{U}$ (where $\hat{U} = \varphi U$) is a connected invariant set in $M$ which intersects $\mathrm{orb}(W \cap \hat{D})$ non-trivially; and we may further assume that $\mathrm{orb}\hat{U} \cap \mathrm{orb}(W \cap \hat{D}) = \mathrm{orb}\hat{U}$. Thus $h|\hat{U}$ satisfies the hypotheses of Lemma 4.1, whence $h$ is a constant map. Thus we have shown

4.3 <u>Proposition</u>

Let $K = \{x \in N : \lambda : \hat{N} \to \mathbb{R}$ is unbounded near $\hat{x} = \varphi x\}$. Let $D$ be a connected component of $N - K$, and fix $m \in D$. Then there exists a connected open neighbourhood $U$ of $m$ in $D$ and a constant $d > 0$ such that $\mu_d|\mathrm{orb}\hat{U} = \mathrm{id}$ (where $\hat{U} = \varphi U$).

## 5.0 Periodic Homeomorphisms

A vital ingredient in the proofs of the Theorems is the following theorem of M.H.A. Newman (N).

### 5.1 Theorem (Newman)

Let $h : M \to M$ be a periodic homeomorphism of a manifold $M$. If the set of fixed points of $h$ contains an interior point, then $h$ is the identity map.

For a modern proof of this result along the lines of Newman's original proof, see the paper by A. Dress (D).

Unfortunately, there is no version of Newman's theorem for homeomorphisms of immersed submanifolds. Overcoming this difficulty requires the following lemma for differentiable mappings which appears in (W).

### 5.2 Lemma

Let $U$ be a neighbourhood of the origin in $\mathbb{R}^n$, $n \geqslant 1$. Let $f : U \to \mathbb{R}$ be a $C^1$ map with the origin as a fixed point. Let $p \geqslant 1$ be an integer and let $\{a_i\}$ be a sequence converging to $0$ consisting of non-zero periodic points with $p$ as least period. Then there exists a vector $v \neq 0$ whose least period is $p$ with respect to the map $f_* : T_0\mathbb{R}^n \to T_0\mathbb{R}^n$.

The proof is straighforward (see (W)). In fact, if $p > 1$ is prime then the vector $v$ may be taken as the limit of some convergent subsequence of the sequence $\{ |fa_i - a_i|^{-1}(fa_i - a_i) \}$. If the period $p$ is strictly greater than one then it should be noted that the sequence $\{ |a_i|^{-1}a_i \}$ need not necessarily converge to a vector $v$ of period $p$. However, the period of $v$ will divide $p$.

5.3 <u>Lemma</u>

Let $\mu : \mathbb{R} \times M \to M$ be a $C^r$ flow $(0 \leqslant r \leqslant \infty)$ such that every orbit of $\mu$ is a circle. Let $\lambda : M \to \mathbb{R}$ be the periodicity function; and let $C$, $D$ be components of the sets $M - B_1$ and $M - K$ respectively such that $C \cap D \neq \emptyset$. (Where $B_1$ is the set of points of discontinuity of $\lambda$, and $K \subset B_1$ is that set near which $\lambda$ is locally unbounded.) Let $\mu$ be a g-flow (w.r.t. the identity immersion $id : M \to M$); by Lemma 4.2, there exists a constant $c > 0$ such that $\lambda | C = c$. Under these hypotheses, $\mu_c | \mathrm{cl} D = id$.

Proof: Fix $m \in D$. By Proposition 4.3, there exists an invariant, open connected neighbourhood $U$ of $m$ in $D$ and a constant $d > 0$ such that $\mu_d | U = id$. Let $C$ be some component of $M - B_1$ which intersects the set $U$ non-trivially; by Lemma 4.2 there exists $c > 0$ such that $\lambda | C = c$. It is easily seen that $d = kc$ on $C \cap U$ where $k \geqslant 1$ is an integer; thus $\mu_{kc} = id$ on all of $U$. Thus the transformation $\mu_c | U$ is periodic and is the identity on the interior set $C \cap U$. Because $U$ is open in $M$, Theorem 5.1 implies $\mu_c | U = id$. Straightforward use of a covering of $D$ by such neighbourhoods $U$ and the fact that $D$ is connected completes the proof of the proposition.

The next lemma is considerably more difficult than Lemma 5.3 because we can not apply Newman's theorem to the image of an immersion.

5.4 Lemma

Let $\varphi : N \to M$ be a differentiable $C^r$ immersion $(1 \leq r \leq \infty)$ Let the $C^r$ flow $\mu : \mathbb{R} \times M \to M$ be a g-flow (w.r.t. the immersion $\varphi$) such that each orbit of $\mu$ intersecting $\hat{N} = \varphi N$ is a circle. Let $\lambda : \hat{N} \to \mathbb{R}$ be the periodicity function; and let $C$, $D$ be components of the sets $N - B_1$ and $N - K$ respectively such that $C \cap D \neq \emptyset$. (Setting $\hat{B}_1 = \varphi B_1$ and $\hat{K} = \varphi K$, then $\hat{B}_1 \subset \hat{N}$ is the set of points of discontinuity of $\lambda$ and $\hat{K} \subset \hat{B}_1$ is that subset of $\hat{N}$ near which $\lambda$ is locally unbounded.) By Lemma 4.2, there exists a constant $c > 0$ such that $\lambda|\hat{C} = c$ (where $\hat{C} = \varphi C$). If we further assume that $M$ is oriented (in which case $\mu$ is orientation preserving) and that $\dim M = \dim N + 1$, then $\mu_c|\mathrm{cl}D = \mathrm{id}$ (where $\hat{D} = \varphi D$).

Proof: Fix $m \in D$. By Proposition 4.3, there exists a connected open neighbourhood $U$ of $m$ in $D$ and a constant $d > 0$ such that $\hat{U} = \varphi U$ is invariant and $\mu_d|U = \mathrm{id}$. Without loss of generality we may assume that $\lambda$ is bounded below on $\hat{U}$. Because $\mu_d|U = \mathrm{id}$, for every $x \in \hat{U}$ we have $d/\lambda x$ an integer. Thus the range of possible values of $\lambda|\hat{U}$ is finite.

Let $c_1 < \ldots < c_l$ be those values of $\lambda|\hat{U}$ for which there is some component $C_\alpha$ of $N - B_1$ such that $\hat{C}_\alpha$ intersects $\hat{U}$ non-trivially and $\lambda|\hat{C}_\alpha = c_i$ for some $i$, $1 \leq i \leq l$ (where

$\hat{C}_\alpha = \varphi C_\alpha$ ) . Define

$$E_j = \mathrm{int}(\mathrm{cl} \bigcup_\alpha \{\hat{C}_\alpha \cap U : C_\alpha \text{ is a component of } N - B_1 \text{ and } \lambda|\hat{C}_\alpha = c_j\} )$$

with closure taken in $\hat{U}$ . If for some $i$ $(1 \leq i \leq l)$ we have $E_i \supset \hat{U}$ then we have nothing to prove. So we assume at least two distinct $c_j$ occur. Taking closures in $\hat{U}$ , this assumption implies that $\mathrm{bdy}E_1 \neq \emptyset$ . (Note that $E_i$ and $E_j$ are disjoint for $i \neq j$ .)

Now $N$ is a differentiable manifold, so there exist small (metric) balls $B$ in $N$ whose boundary $\mathrm{bdy}B$ is a smooth $(n-2)$ - sphere. A point $x \in \mathrm{bdy}E_1$ is said to be <u>tangentially accessible</u> from $E_1$ if there exists some such small ball $B$ with $\mathrm{int}B \subset E_1$ and $x \in \mathrm{bdy}B$ . Such points are easily seen to be dense in the boundary of $E_1$ . Let $z$ be such a tangentially accessible point of $\mathrm{bdy}E_1$ . Thus there are at least $n-1$ paths $\sigma_k$ in $\mathrm{cl}E_1$ with initial point $z$ such that their initial vectors $v_k$ in $T_zM$ are independent. Because $\mu_{c_1}|E_1 = \mathrm{id}$ the vectors $v_k$ will be fixed by the transformation $\mu_{c_1*} : T_zM \to T_zM$ . (Note that $\mu_{c_1}z = z$ because $z \in \mathrm{bdy}E_1$ .

However, every neighbourhood of $z$ in $\hat{N}$ contains points of at least one other $E_j$ . Because the possible values of the $c_j$ are finitely many there exists a sequence $\{b_q\}$ converging to $z$ with $\lambda b_q = c_j$ for some fixed $c_j > c_1$ and all $q$ . Because $c_j|d$ and $c_1|d$ the points $b_q$ have finite least period greater than unity under the transformation $\mu_{c_1}$ . By Lemma 5.2 , there exists a non-fixed periodic vector $v$ in $T_zM$ under the transformation $\mu_{c_1*}$. Thus we have $n-1$ independent fixed vectors in $T_zM$ and at least one other non-fixed periodic vector; consequently, $\mu_{c_1*}$ is

not orientation preserving. This contradicts our hypotheses. Thus for any component $C$ of $N - B_1$ with $\lambda|C = c$ and $C \cap U \neq \emptyset$ we have $\mu_c|U = \mathrm{id}$. Clearly this completes the proof.

## 6.0 The locally unbounded case

Firstly, let us assume the hypotheses of Lemma 5.4. Consider the set of points $\hat{K} \subset \hat{N}$ where the periodicity function $\lambda : \hat{N} \to \mathbb{R}$ is locally unbounded. By Lemma 5.4, if $D$ is a component of $N - K$ then there is some $d > 0$ with $\sup\{\lambda x : x \in \hat{D}\} = d$ and $\mu_d|\mathrm{cl}\hat{D} = \mathrm{id}$. Because $\hat{D}$ is a codimension-one differentiable submanifold of $M$, if $x \in \mathrm{cl}\hat{D}$ then there is a codimension one hyperplane $H_x$ in $T_xM$ on which the transformation $\mu_{d*} : T_xM \to T_xM$ is the identity; that is, $\mu_{d*}|H_x = \mathrm{id}$. Set

$$\underline{D} = \bigcup_{\alpha} \{D_\alpha : D_\alpha \text{ is a component of } N - K \text{ with } \sup\{\lambda x : x \in \hat{D}_\alpha\} = d\} .$$

Such sets $\underline{D}$ are of countable number; thus we may index them: $\underline{D}_1$, $\underline{D}_2$, ... with $\sup\{\lambda x : x \in D_i\} = d_i$. So $d_i \neq d_j$ for $i \neq j$.

### 6.1 Lemma

If $x \in \mathrm{bdy}\hat{\underline{D}}_i$, then $x \notin \mathrm{bdy}\hat{\underline{D}}_j$ for $i \neq j$.

Proof: Fix $x \in \mathrm{bdy}\hat{\underline{D}}_i$. Because the flow $\mu$ is orientation

preserving and $\hat{\underline{D}}_i$ has codimension-one in $M$, each vector in $T_xM$ which is periodic under the transformation $\mu_{\lambda x *} : T_xM \to T_xM$ will have least period dividing $d_i/\lambda x$. Moreover, the subspace of periodic vectors has dimension at least $n-1$ (where $\dim M = n$); furthermore, the transformation $\mu_{\lambda x *}$ restricted to this subspace has least period $d_i/\lambda x$. (The last statement follows from Lemma 5.2 because there is a sequence of points converging to $x$ in $\hat{\underline{D}}_i$ with least period $d_i/\lambda x$ under the diffeomorphism $\mu_{\lambda x}$.) The proof of the lemma is now clear (since $d_i/\lambda x \neq d_j/\lambda x$ for some $\hat{\underline{D}}_j$ where $\sup\{\lambda x : x \in \hat{\underline{D}}_j\} = d_j$ and $i \neq j$).

Let $W \subset N$ be an open connected set such that $W \cap K = \emptyset$, $\lambda | \hat{W} \cap \hat{K}$ is continuous, some fixed $\underline{D}_i$ intersects $W$ non-trivially and further, $\varphi | W : W \to \hat{W}$ is a homeomorphism where $\hat{W} = \varphi W$. Recall $\mathrm{bdy}\underline{D}_i \subset K$. Taking closures in $W$, choose $z \in \mathrm{bdy}(W - \underline{D}_i)$ to be tangentially accessible from within $\mathrm{int}(W - \underline{D}_i)$. Let $V = \mu((-\varepsilon, \varepsilon) \times V_0)$ be a flat neighbourhood of $\hat{z} = \varphi z$ in $M$ such that $V \cap \hat{W} = \hat{W}$ For some smaller disk $V_1 \subset V_0$ let $S : V_1 \to V_0$ be the Poincaré map with $S(\hat{z}) = \hat{z}$; because $\lambda | \hat{W} \cap \hat{K}$ is continuous, we may further assum that $S | V_1 \cap \hat{K} = \mathrm{id}$. Then there are $n-2$ paths $\sigma_k : [0,1] \to V_1$ with $\sigma_k(0,1] \subset V_1 \cap \mathrm{int}(\hat{W} - \hat{\underline{D}}_i)$ and $\sigma_k(0) = z$ such that their initial vectors $v_k$ in $T_{\hat{z}}V_1$ are independent. Because $\hat{z} \in \mathrm{bdy}\underline{D}_i$ as well, each path $\sigma_k$ must intersect the set $\hat{K}$ arbitarily near $\hat{z}$ (for otherwise, we would have $\hat{z} \in \mathrm{bdy}\hat{\underline{D}}_j$ for some $j \neq i$). Then the $v_k$ are fixed under the transformation $S_* : T_{\hat{z}}V_1 \to T_{\hat{z}}V_1$. If $d_i/\lambda z >$ then $S_*$ cannot be orientation preserving since its fixed point set has codimension at most one and there is a vector with least period $d_i/\lambda z >$ But this would mean $\mu_{\lambda z}$ is not orientation preserving; thus we have $\lambda z = d_i$. Because $W \cap \mathrm{bdy}\underline{D}_i = \mathrm{bdy}(W - \underline{D}_i)$ and tangentially accessil points are dense in the boundary, we have $\lambda x = d_i$ for all $x \in \mathrm{bdy}\hat{\underline{D}}_i \cap \hat{}$

Thus we have proved

6.2 Proposition

Assume the hypotheses of Lemma 5.4 . Let $W$ be an open set in the manifold $N$ such that $W \cap K \neq \emptyset$ and $\lambda | \hat{W} \cap \hat{K}$ is continuous. (Where $\hat{K} = \varphi K$ is that set of points in $\hat{N}$ where the periodicity function $\lambda : \hat{N} \to \mathbb{R}$ is locally unbounded; and $\hat{W} = \varphi W$ .) If $z \in \text{bdy} D \cap W$ , where $D$ is some component of $N - K$ with $\sup \{ \lambda x : x \in \hat{D} \} = d$ , then $\lambda \hat{z} = d$ (where $\hat{D} = \varphi D$ and $\hat{z} = \varphi z$).

6.3 Proposition

Assume the hypotheses of Lemma 5.3 . That is, $\mu : \mathbb{R} \times M \to M$ is a $C^r$ g-flow $(o \leqslant r \leqslant \infty)$ with every orbit a circle. Let $W$ be an open set in $M$ such that $W \cap K \neq \emptyset$ and $\lambda | W \cap K$ is continuous. ( $\lambda : M \to \mathbb{R}$ is the periodicity function and $K$ is the subset of $M$ near which $\lambda$ is locally unbounded.) If $z \in \text{bdy} D \cap W$ , where $D$ is some component of $M - K$ with $\sup \{ \lambda x : x \in D \} = d$ , then $\lambda z = d$ .

Proof: Let $z$ , $W$ and $D$ be as in the statement of the proposition. Recall that $\text{bdy} D \subset K$ . By taking $W$ smaller if necessary we may assume that $\lambda | W \cap \text{bdy} D$ is constant. (Briefly, for each $y \in \text{bdy} D$ we have $\lambda y = d/k_y$ , $k_y \geqslant 1$ an integer. Thus $\lambda | \text{bdy} D$ takes only a discrete range of values. For $W$ sufficiently small, $\lambda | W$ is bounded below; thus $\lambda | \text{bdy} D \cap W$ takes only finitely many values; moreover, $\lambda | \text{bdy} D \cap W$ is continuous.) Thus for $y \in W \cap \text{bdy} D$ we may assume $\lambda y = \lambda z = d/k$ , for $k \geqslant 1$ an integer.

Without loss of generality, we may suppose that $W$ is invariant in $M$; furthermore, $\mathrm{bdy}D \cap W$ separates $D \cap W$ and $W - D$. Define the homeomorphism $h : W \to W$ by

$$\begin{aligned} h(x) &= x && \text{if } x \in W - D \\ &= \mu_{\lambda z} x && \text{if } x \in W \cap D \end{aligned}$$

Clearly, $h$ is well-defined and periodic on $W$ with period $k$; and $h = \mathrm{id}$ on the interior set $W - \mathrm{cl}D$ (which is non-empty since $\lambda$ is unbounded near $z \in \mathrm{bdy}D$ whereas $\lambda$ is locally bounded on $D$). By Theorem 5.1, we must have $h = \mathrm{id}$ on $W$. In particular, $k = 1$. This completes the proof.

We now show that the set $K$ of bad points is empty under the hypotheses of either Lemma 5.3 or Lemma 5.4. For convenience only, we shall demonstrate this using the notation of Lemma 5.4; that is, $\varphi : N \to M$ is an immersion and $\lambda : \hat{N} \to \mathbb{R}$ is the periodicity function where $\hat{N} = \varphi N$. (Alternatively, in the topological situation, set $N = M$ and let the immersion $\varphi$ be the identity $\mathrm{id} : M \to M$.)

Let $W$ be an open connected set in the manifold $N$ such that $W \cap K \neq \emptyset$; set $\hat{W} = \varphi W$. Suppose $\lambda | \hat{W} \cap \hat{K}$ is continuous; and define the function $\lambda^* : \hat{W} \to \mathbb{R}$ by

$$\begin{aligned} \lambda^* x &= \lambda x && \text{if } x \in \hat{K} \cap \hat{W} \\ &= d_j && \text{if } x \in \hat{D}_j \text{ with } \sup \{\lambda x : x \in \hat{D}_j\} = d_j \\ & && \text{and } D_j \text{ is a component of } N - K. \end{aligned}$$

Using Proposotion 6.2 (alternatively, Proposition 6.3) the function

$\lambda^*$ is clearly seen to be continuous on $W$ ; moreover, $\mu_{\lambda^* x} x = x$ for all $x \in W$ . By Lemma 4.1 $\lambda^*$ is constant on $W$ ; in particular, $\lambda$ is bounded on $W$ . But this contradicts the hypothesis that $K$ is non-empty (because the set of points of continuity of $\lambda | K$ are dense in $K$ ) . In fact, we have shown $\lambda$ is locally bounded on all of $\hat{N} = \varphi N$ .

## 7.0 Conclusion of the proof

Under the hypotheses of either Lemma 5.3 or Lemma 5.4 , we have shown in the preceding section that the periodicity function $\lambda$ is locally bounded. From the definition of a g-flow and the discussion of §3 , we see that the conclusions of Lemmas 5.3 and 5.4 are sufficient to imply the conclusions of Theorems A and C . Theorem B and D are obtained by suspending the homeomorphism $h : M \to M$ Thus the proofs of our theorems are complete apart from necessity in Theorem A .

Returning to Theorem A , suppose we have $\rho : S^1 \times M \to M$ a $C^k$ action $(1 \leqslant k \leqslant \infty)$ of the circle group $S^1$ without fixed points. According to Palais (P2) such an action is $C^k$ equivalent to a $C^\infty$ action; by integration over $S^1$ we can assume that preserves some $C^k$ riemannain metric on $M$ . With respect to this invariant metric, the flow obtained by the identification $S^1 = \mathbb{R}/\mathbb{Z}$ leaves the distribution of orthogonal vectors invariant, and hence

the corresponding one-form defined by this distribution.

To show necessity for the existence of such a one-form $\alpha$ in Theorem C is easy of the flow $\mu$ is at least $C^2$. We merely average some riemannian metric on M by integration over the global period of $\mu$, viz.

$$g' = 1/c \int_0^c \mu_t^* g \, dt \qquad \text{where } \mu_c|\hat{N} = \text{id and } g \text{ is a metric tensor on } M .$$

The space of othogonal vectors with respect to $g'$ defines the one-form $\alpha$ which clearly has the desired proerties when restricted to $\hat{N} = \varphi N$.

## 8.0 Manifolds with all closed geodesics

Let M be a connected $C^\infty$ riemannian n-manifold such that every geodesic in M is simply-closed. We see by the discussion of §3 that there is some riemannian metric on the unit tangent bundle $\Sigma M$ for which the canonical lift of a geodesic in M is geodesic in $\Sigma M$. (Note: if $\sigma : [0,1] \to M$ is a curve in M then the canonical lift $\tilde{\sigma}$ of $\sigma$ to TM is defined by $\tilde{\sigma}(t) = (\sigma(t), \sigma'(t))$, where $\sigma'(t) = d/dt\, \sigma(t)$; paths parametrised by arc-length will lift to $\Sigma M$.) This construction defines the geodesic flow on $\Sigma M$, $\rho : \mathbb{R} \times \Sigma M \to \Sigma M$, which is a global flow because our hypothesis on the geodesics implies M is compact; moreover, each trajectory of $\rho$ is a circle. By Remark 1.3, $\rho$ has some global least period $\ell$.

Clearly, the period $\ell$ is the top length of the simply-closed geodesic loops based at any point of $M$.

Such manifolds were considered by R. Bott in the paper (B): given a manifold $\underline{M}$, he supposed there was some point $p \in \underline{M}$ such every geodesic through $p$ was a simple curve in $\underline{M}$, simply-closed at $p$, and with a fixed length $\ell$ (these assumptions have since been weakened by Nagagawa (NA)). He was able to show that the manifold $\underline{M}$ had the homology of a rank one symmetric space (for which manifolds the assumption is obviously the case).

These manifolds have been considered since by various authors: however, at least two questions remain unresolved. 1) Is $\underline{M}$ diffeomorphic to a rank-one symmetric space? (Asked by J. Eells in (PUJ; p. 169)); and 2) if $M$ has closed geodesics of different lengths can $M$ be simply-connected? (See M. Berger (BE; p. )). By Theorem A, the possible lengths of the closed geodesics are $\ell/k$, where $k \geqslant 1$ is an integer. Examples of such manifolds $M$ are some lens spaces; in particular, the quotient of $SO(3)$ (with the usual metric) by the action of any (non-trivial) element of order 2 is such that there is exactly one geodesic of length $\pi/2$ through each point, and with the remaining geodesics having length $\pi$; this quotient manifold has fundamental group $Z_4$. Of course, with additional assumptions on the sectional curvatures of $M$ in relation to the length $\ell$ of the geodesics (for example, $0 \leqslant K \leqslant 1$ with $\ell \leqslant 2\pi$) we can resolve Problem 2). But this procedure is not admissible in general; for example, there exists a 'Zoll surface' (an analytic manifold $M$ all of whose geodesics are closed which is diffeomorphic (but not isometric) to $S^2$) with points of strictly negative curvature (see (BE; p. )).

The purpose of this section is to solve Problem 2) for 2-manifold (surely well-known) by an argument which may indicate a path towards a general solution; unfortunately, in higher dimensions complexity is endemic.

8.1 Proposition

Let $M$ be a simply-connected riemannian 2-manifold such that every geodesic of $M$ is simply closed. Then all the geodesics of $M$ are of the same length.

Proof: The manifold $M$ is simply-connected and two-dimensional; thus it is diffeomorphic to the 2-sphere; also, $M$ is given a riemannian metric so that every geodesic of $M$ is simply closed. Let $\rho : \mathbb{R} \times \Sigma M \to \Sigma M$ be the geodesic flow; by identifying $S^1 = \mathbb{R}/\ell\mathbb{Z}$ we may suppose $\rho$ is, in fact, a locally-free $S^1$ action on $\Sigma M$. On $\Sigma M$ let $a : \Sigma M \to \Sigma M$ be the antipodal map on the fibres ( = 1-spheres) . It is easy to see that this involution is equivariant with respect to $\rho$ : let $v$ be a vector in $\Sigma M$, then we have $\rho(v,-t) = \rho(-v,t)$ , which merely states that travelling backwards in time along a geodesic is the same as travelling forwar ds in time but in the opposite direction; and recall that $a(v) = -v$ . In fact, the joint action of $a$ and $\rho$ induces an $O(2)$-action on $\Sigma M$ such that no orbit of $\rho$ is invariant under $a$ .

Let $\pi : \Sigma M \to \underline{O}$ be the quotient mapping to the orbit space $\underline{O}$ of $\Sigma M$ under the action $\rho$ . The involution $a$ induces a fixed-poin free involution $\underline{a}$ on $\underline{O}$ and let $i : \underline{O} \to \underline{O}/a$ be the identifation map.

Because the manifold $M$ is a 2-sphere we have $\Sigma M \cong \mathbb{RP}^3$. It is well-known that any $SO(2)$ - action on $\mathbb{RP}^3$ can have zero, one or two orbits of shorter length (exceptional orbits) ; moreover, the orbit space of $\mathbb{RP}^3$ by such an action is homeomorphic to the 2-sphere (see (O-R)) ; so $\underline{O} \cong S^2$. Furthermore, because of the existence of the involution $a$ our action $\rho$ will have either zero or two exceptional orbits. If there are no exceptional orbits we are done so assume there are two such orbits which map to the points $x_1$, $x_2$ in $\underline{O}$ under the quotient map $\pi$. Then $\underline{a}(x_1) = x_2$. Let $x_o \in \underline{O}/\underline{a}$ such that $i(x_1) = i(x_2) = x_o$.

Because the action $\underline{a}$ is fixed-point free the space $\underline{O}/\underline{a}$ is homeomorphic to $\mathbb{RP}^2$. Let $\gamma$ be an essential simple closed path in $\underline{O}/\underline{a}$ which does not meet the point $x_o$. Then the lifted path $\underline{\gamma}$ in $\underline{O}$ of $\gamma$ does not meet either $x_1$ or $x_2$ ; moreover, $\underline{O} - \underline{\gamma}$ is the union of the dispoint open disks $D_1$ and $D_2$ where we assume $x_1 \in D_1$ and $x_2 \in D_2$. Then $\pi^{-1}(D_1)$ and $\pi^{-1}(D_2)$ are invariant solid tori; and the map $a : \Sigma M \to \Sigma M$ throws $\pi^{-1}(D_1)$ onto $\pi^{-1}(D_2)$.

Identify $D_1 \cong \{ z_1 \in \mathbb{C} : |z_1|^2 : |z_1|^2 < 1 \}$ and $D_2 = \{ z_2 \in \mathbb{C} : |z_2|^2 < 1 \}$. Then there exist coprime integers $p$ and $q$ such that the action $\rho$ restricted to the solid torus $^{-1}(D_1)$ is equivalent to the suspension (see §3) of the periodic map $\varphi_1 : D_1 \to D_1$ defined by $\varphi_1(z_1) = z_1 \exp 2\pi i p/q$ (see (O-R)). Because the involution $a$ is equivariant with respect to $\rho$ the action of $\rho$ on $\pi^{-1}(D_2)$ is equivalent to the suspension of the periodic map $\varphi_2 : D_2 \to D_2$ defined by $\varphi_2(z_2) = z_2 \exp 2\pi i p/q$ with the opposite orient ation.

Now $\Sigma M$ is homeomorphic to the union of $\pi^{-1}(D_1)$ and $\pi^{-1}(D_2)$ joined equivariantly along their boundaries $= \pi^{-1}(\underline{\gamma})$ . So if either $p > 1$ or $q > 1$ then $\Sigma M \cong S^2 \times S^1$ ; but this is absurd. Thus $p = q = 1$ . That is, all the orbits have the same length.

In higher dimensions, if $M$ is a simply-connected n-manifold with $n \geqslant 3$ , then $\Sigma M$ is also simply-connected. A solution to Problem 2) may involve a partial analysis of our rather special $O(2)$-actions on (odd-dimensional) simply-connected manifolds.

# Chapter II

# Totally Geodesic Foliations

## 1.0 Preliminaries

$M$ denotes a connected $C^\infty$ n-manifold. Henceforth, reference to differentiability will be suppressed wherever possible.

In general, if $x_1,\ldots,x_n$ is a coordinate system on $M$ with domain $U$, then a slice of $U$ is defined to be any subset of $U$ on which $r$ of the functions $x_1,\ldots,x_n$ are constant $(1 \leqslant r \leqslant n)$. Obviously, each slice of $U$ is a submanifold of $U$ (or of $M$).

Let $\mathfrak{F}$ be a p-dimensional foliation of $M$ with tangent distribution $\mathfrak{D} \subset TM$; that is, $\mathfrak{D}$ is a completely integrable, regular p-dimensional distribution on $M$. By a <u>flat neighbourhood</u> in $M$ we shall mean a connected (cubical) coordinate neighbourhood $U$ in $M$, with $\mathrm{cl}U$ compact and coordinate functions $x_1,\ldots,x_p$, $y_1,\ldots,y_q$ $(p+q=n)$ with $|x_i| < \varepsilon$, $|y_j| < \varepsilon$, $(1 \leqslant i \leqslant p\ ;\ 1 \leqslant j \leqslant q)$ for some $\varepsilon > 0$, such that the slices of $U$ for which $y_1,\ldots,y_q$ are constant are integral manifolds of $\mathfrak{D}$. A flat neighbourhood <u>distinguishes</u> such slices $y_1 = c_1, \ldots, y_q = c_q$ where the $c_j$, $|c_j| < \varepsilon$, are constants; and henceforth <u>a slice of a flat neighbourhood</u> shall refer only to a distinguished slice. A flat neigh-

bourhood $U$ has a natural product structure $U \cong U_x \times U_y$ where $U_y$ = submanifold of $U$ on which $x_1 = x_2 = \dots = x_p = 0$, and $U_x$ = submanifold of $U$ on which $y_1 = y_2 = \dots = y_q = 0$; and natural projections $\pi_x : U \to U_x$ and $\pi_y : U \to U_y$ are also given, so that for $m \in U$ we have $\pi_y^{-1}\pi_y m$ = slice through $m$.

Let $m \in M$; a <u>transverse disk</u> $D_o$ through $m$ is given by $D_o = U_y$ where $U \cong U_x \times U_y$ is a flat neighbourhood of $m$. Let $\sigma : [0,1] \to L$ be a path lying in a leaf $L$ of $M$, and let $D_o$, $D_1$ be transverse disks through $\sigma(0)$ and $\sigma(1)$ respectively. The following is a well-known construction (see (H; p. 377), for example): There exists a neighbourhood $V$ of $\sigma(0)$ in $D_o$ and a map

$$H : V \times [0,1] \to M$$

such that

1) $H(\{0\} \times [0,1]) \subset$ single leaf for any $v \in V$;

2) $H|V \times \{0\} = \mathrm{id}$;

3) $H|V \times \{1\} \subset D_1$;

4) $H(\sigma(0),s) = \sigma(s)$ for any $s \in [0,1]$.

Moreover, if $h : D_o \to D_1$ is defined by $h(v) = H(v,1)$ then the germ of $h$ is uniquely determined by 1) - 4). Furthermore, if a path $\tilde{\sigma}$ from $\sigma(0)$ to $\sigma(1)$ is homotopic to $\sigma$ then the germ of $\tilde{h}$, defined for $\tilde{\sigma}$ as above, coincides with the germ of $h$ (cf. the Poincaré map of Chapter I, § 2).

Fixing $m \in M$ and a transverse disk $D_o$ through $m$, each closed loop $\sigma$ with $\sigma(0) = \sigma(1) = m$ induces a germ of a diffeo-

morphism $h_\sigma : D_o \to D_o$ which is independent of the homotopy class of $\sigma$. This constuction defines the <u>holonomy homomorphism</u>

$$\Phi : \pi_1 L \to \{\text{germs of diffeos of } D_o \text{ at } m\} .$$

The image of the group $\pi_1 L$ under $\Phi$, $\Phi(\pi_1 L)$, is known as the <u>holonomy group</u> of the leaf $L$ through $m$. The isomorphism class of $\Phi(\pi_1 L)$ is independent of $m \in L$, each transverse disk $D_o$ through $m$, and depends only upon the leaf $L$ and foliation $\mathcal{F}$.

A foliation $\mathcal{F}$ is said to have <u>finite holonomy</u> if the holonomy group $\Phi(\pi_1 L)$ is finite for every leaf $L$ of the foliation. When each leaf $L$ is compact the reader is referred to (E2) where conditions equivalent to finite holonomy are fully discussed (see Theorem 1.1 below).

Denote by $Q$ the set of leaves of $\mathcal{F}$ and let $\pi_Q : M \to Q$ be the quotient mapping of $M$ onto $Q$ which carries $x \in M$ onto the leaf of $Q$ containing $x$. The <u>quotient topology</u> for $Q$ is the strongest topology which makes $\pi_Q$ continuous. With respect to this topology, the mapping $\pi_Q$ is open or, equivalently, the saturation of an open set is open (P1; p. 12). For $A \subset M$, the <u>saturation of A</u> is the union of all leaves intersecting $A$; that is,

$$\text{sat}\, A = \pi_Q^{-1} \pi_Q A .$$

Theorem 1.1 below is an abbreviated version of the main theorem presented in (E2).

1.1 <u>Theorem</u>

Let $M$ be a foliated manifold, with all leaves compact. The following conditions are equivalent:

1) Each leaf $L$ has arbitarily small saturated neighbourhoods.

2) The holonomy group of each leaf $L$ is finite.

3) For some (For any) riemannian metric on $M$, and for each compact set, there is a bound for the volume of leaves meeting the compact set.

4) The following is a local model for the foliation near a fixed leaf $L$.

Let $q$ be the codimension and let $G$ be a finite subgroup of the orthogonal group $O(q)$. Let $\Phi : \pi_1 L \to G$ be an epimorphism and let $\tilde{L}$ be the covering space corresponding to the kernel of $\Phi$. Then $G$ acts on the open unit disk $D^q$ by orthogonal transformations and on $\tilde{L}$ by covering translations. Let $\tilde{L} \times_G D^q$ be the quotient of $\tilde{L} \times D^q$ by identifying $(\tilde{\ell} g, d)$ with $(\tilde{\ell}, gd)$ for each $\tilde{\ell} \in \tilde{L}$, $g \in G$, and $d \in D^q$; and let $r : \tilde{L} \times D^q \to \tilde{L} \times_G D^q$ be the quotient map. Then $\tilde{L} \times_G D^q$ is foliated by compact leaves of the form $r(\tilde{L} \times \{d\})$.

Condition 4) is the condition that for each leaf $L$ there is a finite subgroup $G$ of $O(q)$ and an epimorphism $\Phi : \pi_1 L \to G$, and a diffeomorphism $\varphi$ of $\tilde{L} \times_G D^q$ onto a neighbourhood of $L$, preserving the leaves.

5) The leaf space $Q$ is Hausdorff.

Let $\mathcal{F}$ be a foliation of a riemannian manifold $M$ with all leaves compact. Set $\mathcal{D}^{\perp}$ to be the distribution of vectors orthogonal to the leaves. Let $L$ be a leaf of $\mathcal{F}$ and let $N(L) = \mathcal{D}^{\perp}|L$ be the normal bundle of $L$ in $M$. Let $D^q(L) \subset N(L)$ be a q-disk sub-bundle of $N(L)$ such that $\varphi : D^q(L) \to M$ (where $\varphi = \exp|D(L)$ ) is well-defined and is a diffeomorphism onto an open neighbourhood of $L$ in $M$; moreover, we assume that the restriction of $\varphi$ to the zero-section of $L$ in $D^q(L)$ is the inclusion of $L$ in $M$. An open <u>tubular neighbourhood</u> $U_L$ of $L$ in $M$ is the image of $\varphi : D^q(L) \to M$. (Such neighbourhoods always exist because $L$ is closed in $M$; see, for example (BR; p. 303).) Thus there is a well-defined projection $\pi_L : U_L \to L$ induced from bundle projection in $D^q(L)$.

<u>1.2 Remark</u>: Condition 4) of Theorem 1.1 gives us a saturated tubular neighbourhood of a leaf $L$ in $M$; furthermore, the epimorphism $\Phi : \pi_1 L \to G$ defines the holonomy group $\Phi(\pi_1 L) = G$. Conversely, given a saturated tubular neighbourhood of a leaf one can show that the holonomy group $\Phi(\pi_1 L)$ is finite; more generally, the following proposition is true:

1.3 <u>Proposition</u>

Let $p : E \to B$ be a differentiable bundle over a compact manifold $B$. Further assume there exists a foliation $\mathcal{F}$ of $E$ with compact leaves transverse to the fibres. Then the foliation $\mathcal{F}$ has finite holonomy.

Proof: See (EH; p.38) and (E2; Thms 1 & 2) .

In fact, Corollary 2 of our main theorem, Theorem E , in § 2 generalises this result.

## 2.0 Statement of the theorem

Let $\mathcal{F}$ be a p-dimensional codimension q foliation of M with tangent distribution $\mathcal{D}$ (thus $p+q=n$ where $n=\dim M$ ) . The classical Frobenius Theorem for involutive distributions states that there exist q independent one-forms $\theta_1,\ldots,\theta_q$ defined locally on M with $\mathcal{D}_m = \{X \in T_mM : \theta_j(X) = 0 \text{ for } 1 \leqslant j \leqslant q\}$ such that $d\theta_j = \Sigma_{k=1}^{q} \theta_{kj} \wedge \theta_k$ for some choice of one-forms $\theta_{kj}$ . Let $W_x$ be that subspace of $T_m^*M$ spanned by $\theta_1,\ldots,\theta_q$ . $W_x$ has dimension q , and there exists a globally defined locally decomposable q-form $\Theta$ on M such that $\Theta_m$ corresponds to $W_m$ under the natural correspondence between q-dimensional subspaces of $T_mM$ and decomposable q-forms on $T_mM$ which are defined up to a scalar factor. Locally, we can find a flat neighbourhood $U \cong U_x \times U_y$ of m so that $\Theta = dy_1 \wedge \cdots \wedge dy_q$ ; the integrability of the distribution $\mathcal{D}$ implies that $d\Theta$ is a multiple of $\Theta$ .

Now, let $\Omega$ be a locally decomposable p-form on M with $\Omega \wedge \Theta$ non-zero everywhere. Constructing such forms is easy; for example, put a riemannian metric on M and let $\Omega$ be a p-form

corresponding to that set of one-forms which are zero on the orthogonal distribution $\mathcal{D}^{\perp}$. However, we can construct $\Omega$ in a more precise fashion: given a riemannian metric on $M$, the restriction of this metric to a leaf $L$ defines a volume form for $L$; specifically, let $\omega_1, \ldots, \omega_p$ be local orthonormal one-forms defined in an open set $U$ of $M$ such that for each leaf $L$ the forms $\omega_i | U \cap L$ form an orthonormal basis for the cotangent bundle $T^*(U \cap L)$. Then the volume form $\omega_L$ of $L$ is given by $\omega_L | U \cap L = \omega_1 \wedge \cdots \wedge \omega_p | U \cap L$; moreover, $\omega_L$ is independent of the choice of the $\{\omega_i\}$. Clearly, the p-form $\Omega$ defined locally by $\Omega = \omega_1 \wedge \cdots \wedge \omega_p$, is globally defined such that $\Omega | L = \omega_L$ for each leaf $L$; furthermore, $\Omega \wedge \Theta \neq 0$ on $M$. Conversely, given a locally decomposable p-form $\Omega$ with $\Omega \wedge \Theta \neq 0$ on $M$ we can construct a riemannian metric for $M$ so that, for each leaf $L$ of $M$, $\Omega | L$ is the volume form with respect to the induced metric on $L$.

Let $U \cong U_x \times U_y$ be a flat neighbourhood in $M$, and $\pi_y : U \to U_y$ projection. Let $X$ be a vector field on $U_y$ and define the <u>orthogonal lift</u> $\tilde{X}$ of $X$ to $U$ to be the (unique) vector field in $U$ such that $\tilde{X}_m \in \mathcal{D}^{\perp}_m$ and $\pi_{y*} X_m = X_{\pi_y m}$ at $m \in U$. Given a locally decomposable p-form $\Omega$ on $M$ with $\Omega \wedge \Theta \neq 0$, we set $\mathcal{D}^{\perp}$ to be the distribution of vectors such that $\Omega$ corresponds to the subspace of one-forms in $T^*M$ which are zero on $\mathcal{D}^{\perp}$; that is, $\mathcal{D}^{\perp}_x = \{ X \in T_x M : X \lrcorner \Omega = 0 \}$ (where $X \lrcorner$ is contraction by $X$).

2.1 <u>Definition</u>

Let $\Theta$ be a locally decomposable q-form on $M$ which defines

a foliation $\mathcal{F}$ of $M$. $\mathcal{F}$ will be called a <u>g-foliation</u> if there exists a locally decomposable p-form $\Omega$ on $M$ $(p+q=n)$ such that

1) $\Omega \wedge \Theta \neq 0$ everywhere on $M$

2) for any flat neighbourhood $U \cong U_x \times U_y$ and vector field $X$ on $U_y$, we have $L_{\tilde{X}}\Omega = 0$ ;

where $\tilde{X}$ is the orthogonal lift of $X$ to $U$ (defined by $\Omega$ as above) and $L_{\tilde{X}}\Omega$ denotes the Lie derivative of $\Omega$ by $\tilde{X}$.

<u>Theorem E</u>

Let $M$ be a connected $C^\infty$ manifold, and let $\mathcal{F}$ be a smooth foliation of $M$ such that each leaf is compact. If $\mathcal{F}$ is a g-foliation then $\mathcal{F}$ has finite holonomy.

<u>Corollary 1</u>

Let $M$ be a connected riemannian manifold together with a foliation $\mathcal{F}$ on $M$ such that each leaf is compact and totally geodesic in $M$. Then $\mathcal{F}$ has finite holonomy; moreover, $\pi_Q : M \to Q$ is a V-bundle with structure group a Lie group of isometries of the fibre (= leaf).

(For the definition of a V-bundle, see §4)

2.2 Remark: If the flow $\mu : \mathbb{R} \times M \to M$ of Theorem A, Chapter I is at least $C^2$, then by Remark 3 there, we have Theorem A as a special case of this Corollary. In this case, a g-flow is a one-dimensional g-foliation.

Corollary 2

Let $M$ be a connected riemannian manifold together with an R-foliation $\mathcal{F}_1$. Further assume that the distribution of vectors orthogonal to the leaves of $\mathcal{F}_1$ is integrable and defines a foliation $\mathcal{F}_2$ of $M$ such that each leaf of $\mathcal{F}_2$ is compact. Then the foliation $\mathcal{F}_2$ has finite holonomy.

(For the definition of an R-foliation, see §4 .)

2.3 Remark: If the leaves of an R-foliation are closed in $M$ then it follows that the foliation has finite holonomy; see (R2) .

2.4 Remark: Given a g-foliation $\mathcal{F}$ we obtain an equivalent formulation of conditions 1) and 2) of the definition: because $\Omega \wedge \Theta \neq 0$ and $\Omega$ is locally decomposable, there exists a riemannian metric on $M$ such that i) $\Omega | L = \omega_L$, the volume form of the induced metric on a leaf $L$, and ii) the distribution of vectors on which $\Omega$ is zero $\mathcal{D}^{\perp}$ is orthogonal to the leaves. Let $U \cong U_x \times U_y$ be a flat neighbourhood in $M$ with $m \in U_y$ and $\sigma : [0,1] \to U_y$ a path with $\sigma(0) = m$. There exists a neighbourhood $V$ of $m$ in $S_m$ = (slice through $m$) such that the unique orthogonal lift $\sigma_b : [0,1] \to U$ of $\sigma$, defined by requiring

$\sigma_b(0) = b$ and $\pi_y \circ \sigma_b = \sigma$, exists for each $b \in V$ (where $\pi_y : U \to U_y$ is projection) ; moreover, the map $h : V \to S_{\sigma(1)}$ defined by $h(b) = \sigma_b(1)$ is a diffeomorphism onto its image (where $S_{\sigma(1)}$ = slice through $\sigma(1)$ ). Condition 2) is equivalent to requiring that the map $h$ preserves the volume form of the leaves; that is, if $L$ is the leaf through $m$ and $L'$ the leaf through $\sigma(1)$ then $h^*(\omega_{L'}|h(V)) = \omega_L|V$ .

## 3.0 Proof of the theorem

The constructions leading up to the proof of Proposition 3.1 is due to (E2) . These follow:

Any open covering of a leaf $L$ has a refinement $\{U_1,\ldots,U_N\}$ with the following properties:

a) $U_i$ is open and connected for each $i$ ;

b) for each $i$ , $L - \bigcup_{j\neq i} U_j$ has non-empty interior in $L$ .

Let $U_L$ be a tubular neighbourhood of $L$ with projection $\pi_L : U_L \to L$ such that the foliation is transverse to the normal disks in the tubular neighbourhood. There is a covering $\{V^1,\ldots,V^N\}$ of $L$ by flat neighbourhoods whose product structure is given by $\pi^i \times \pi_L$ where $\pi^i : V^i \to V^i_y$ is projection. We may assume: c) for each slice $S$ of $V^i$ the projection $\pi_L : S \to {}_LS$ sends the volume form of $L$ restricted to $\pi_L S$ to the volume form of $S$ multiplied by a

continuous function nearly equal to one; d) the covering $\{L \cap V^1, \ldots, L \cap V^N\}$ satisfies conditions a) and b) above, and in particular the leaf $L$ intersects each $V^j$ in only the one slice; e) there is an $\varepsilon > 0$ such that if $P$ is any union of slices of $V^1, \ldots, V^N$ lying in some leaf of the foliation, then the volume of $P$ is increased by at least $\varepsilon > 0$ if we adjoin to $P$ another slice.

Define the <u>volume function</u> $\underline{\mathrm{vol}} : M \to \mathbb{R}$ by $\underline{\mathrm{vol}}\, m =$ (volume of the leaf through $m$), for each $m \in M$.

Now let $A$ be a saturated set in $M$ with $\underline{\mathrm{vol}}\, A$ bounded; and let $L$ be a leaf of $A$. Let the volume function be bounded by $c$ on $A \cap U_L$. Let $T_1 \supset T_2$ be open tubular neighbourhoods of $L$ such that

f) $\mathrm{cl}\, T_1 \subset V^1 \cup V^2 \cup \ldots \cup V^N \subset U_L$;

g) if $P$ is the connected union of $[c/\varepsilon] + 2$ slices of $V^1, \ldots, V^N$ and if $P \cap T_2 \neq \emptyset$, then $P \subset T_2$.

We claim that the saturation of $A \cap T_2$ is contained in $U_L$. Let $P$ be a leaf which meets $A \cap T_2$. Let $k$ be the maximum integer such that there is a connected union $S$ of $k$ distinct slices of $V^1, \ldots, V^N$ inside $P$. By our construction, we have

$$k\varepsilon \leq \mathrm{vol}\, P \leq c \quad .$$

Hence $k \leq [c/\varepsilon] + 1$, so $S \subset T_1$. Since $k$ is maximal, $S$ meets no other slice of $V^1, \ldots, V^N$. This means that $S$ is closed in $P$ by condition f) for $T_1$ and $T_2$. Since $S$ is open in $P$ by construction, we have $S = P$ and $P \subset T_1$. Thus we have proved

3.1 Proposition

Let $A \subset M$ be a saturated set such that vol $A$ is locally bounded. Then each leaf of $A$ has arbitarily small saturated neighbourhoods in $A$ (with respect to the induced topolgy).

The following lemma is obvious.

3.2 Lemma

Let $L$ be a leaf in $M$ and $U_L$, $\{V^1,\ldots,V^N\}$ be as above. Then there is an $\varepsilon > 0$ such that any leaf coming within a distance $\varepsilon$ of $L$ intersects each $V^j$ non-trivially.

3.3 Proposition (cf. Proposition 2.1, Chapter I)

The volume function vol : $M \to \mathbb{R}$ is lower semi-continuous. Moreover, if $A \subset M$ then the set of points of continuity of vol $A$ is open and dense in the induced topology on $A$.

Proof: Given $\eta > 0$, Lemma 3.2 shows that any leaf sufficiently near a fixed leaf $L$ in $M$ has volume $> (\mathrm{vol}L - \eta)$. This proves lower semi-continuity.

If vol is continuous at some point $m$ of $A$, then it is continuous at all points of the leaf $L$ through $m$. Without loss

of generality, we may assume that $A$ is saturated in $M$ ; then there is a neighbourhood $W$ of $L$ in $M$ such that $\underline{\mathrm{vol}}|W\cap A$ is bounded. By Proposition 3.1 we may assume $W\cap A$ is saturated and that $W\cap A$ is contained in some tubular neighbourhood $U_L$ of $L$ , as above. If $L_o\subset W\cap A$ , then $\pi_L|L_o : L_o\to L$ is a covering projection ; it follows from condition c) above that, for $L_o$ sufficiently near $L$ , $\mathrm{vol}L_o$ is close to an integral multiple of $\mathrm{vol}L$ ; but $\underline{\mathrm{vol}}$ is continuous at $L$ . So for each $L_o\subset W\cap A$ sufficiently near $L$ , $\mathrm{vol}L_o$ is close to $\mathrm{vol}L$ . In particular, there is some neighbourhood $W'$ of $L$ such that $W'\cap A$ is saturated, $\pi_L|L_o : L_o\to L$ is a diffeomorphism for each $L_o$ in $W'\cap A$ ; and moreover , $\underline{\mathrm{vol}}|W'\cap A$ is continuous.

That the points of continuity of $\underline{\mathrm{vol}}\,A$ are dense in $A$ follows from Lemma 2.2 of Chapter I .

As in Chapter I , §4 define

$$B_1 = \{ m\in M : \underline{\mathrm{vol}} \text{ is not continuous at } m \} .$$

$B_1$ is a saturated , closed nowhere dense subset of $M$ .

3.4 Lemma

Let $\mathcal{F}$ be a g-foliation and $C$ a connected component of $M-B_1$ . Then $\underline{\mathrm{vol}}|C$ is constant.

Proof: $\underline{\mathrm{vol}}$ : $M \to \mathbb{R}$ is continuous on $C$ . It follows directly from the proof of Proposition 3.3 that about any point in $C$ there exists a flat neighbourhood $U$ such that any leaf intersecting $U$ non-trivially does so in only one slice ; that is, the foliation is regular. According to (P1 ; Corol. 4 p.21) the restriction of $\pi_Q$ to $C$ (where $\pi_Q : M \to Q$ is the quotient map onto the leaf space $Q$ ) induces on $C$ the structure of a locally trivial bundle over a differentiable manifold $\pi_Q C$ with fibre a leaf in $C$ . Because eac leaf is compact, the distribution $\mathcal{D}^{\perp}|C$ of vectors orthogonal to th leaves defines a connection on $C$ (see (EH; p.36)) ; thus for each path $\sigma$ in $\pi_Q C$ and point $m \in \pi_Q^{-1}\sigma(o)$ there exists a unique orthogonal lift $\sigma_m$ of $\sigma$ through $m$ . Furthermore, the mapping $h : \pi_Q^{-1}\sigma(o) \to \pi_Q^{-1}\sigma(1)$ defined by $m \to \sigma_m(1)$ is a diffeomorphism The discussion of Remark 2.4 implies that $h$ is volume preserving . This completes the proof.

Define the set $K \subset M$ by

$$K = \{ m \in M : \underline{\mathrm{vol}} \text{ is not locally bounded at } m \} .$$

$K$ is closed and saturated in $M$ ; moreover, $K \subset B_1$ . In the proo we assume $K \neq \emptyset$ and prove a contradiction. Let $D$ be a component of $M-K$ ; then $\underline{\mathrm{vol}}|D$ is locally bounded. From Theorem 1.1 we hav the local model, Condition 4) , for the foliation near a leaf $L \subset D$ It is immediate from this model situation that the holonomy of $L$ is non-trivial if and only if $L \subset B_1$ ; that is, $\underline{\mathrm{vol}}$ is not continuous at $L$ .

Using the notation of Theorem 1.1 , the map $r : \tilde{L} \times D^q \to \tilde{L}$

is a covering projection ; thus the foliation of $\tilde{L} \times D^q$ by the le
$\tilde{L} \times \{d\}$ will also be a g-foliation with the lifted metric from
$\varphi(\tilde{L} \times_G D^q)$ . By Lemma 3.4 $\widetilde{\text{vol}}$ , the volume function on $\tilde{L} \times D'$
is constant, say $\widetilde{\text{vol}} = c$ . Now $r$ is a riemannian covering wit
each leaf $\tilde{L} \times \{d\}$ covering a leaf $r(\tilde{L} \times \{d\})$ , so each leaf in
$\tilde{L} \times_G D^q$ has volume $c/k$ where $k \geqslant 1$ is an integer. Note that
if and only if $r(\tilde{L} \times \{d\})$ has trivial holonomy. Thus we have p

3.5 <u>Lemma</u>

Let $K = \{ m \in M :$ <u>vol</u> is not locally bounded near $m \}$ .
Let $D$ be a component of $M-K$ . Then there exists $c > 0$ such t
for each leaf $L$ in $D$ we have <u>vol</u> $L = c/k$ , where $k \geqslant 1$ is a
integer (depending on $L$) ; and <u>vol</u>$L = c$ if and only if $L \subset D-B_1$
where $B_1 = \{ m \in M :$ <u>vol</u> is not continuous at $m \}$ .

Now consider such a component $D$ . Because <u>vol</u> is lower
semi-continuous we have <u>vol</u>$|\text{bdy}D \leqslant c$ and $\text{int}(\text{cl}\,D) = D$ .

By Proposition 3.3 , there is a relatively open dense set
points in bdy$D$ where the volume function is continuous ; let $W$ b
a connected open set in $M$ such that <u>vol</u>$|W \cap \text{bdy}D$ is continuous.
Without loss of generality we may assume $W \cap \text{bdy}D$ is saturated. N
<u>vol</u>$|W \cap \text{cl}\,D$ is locally bounded ; so by Proposition 3.1 we may assu
$W \cap \text{cl}\,D$ is saturated and, by taking $W$ smaller, that <u>vol</u>$|W \cap \text{cl}\,D$
is bounded there. Let $L$ be a leaf of $W \cap \text{bdy}D$ ; we may further
assume that there exists a tubular neighbourhood $U_L$ of $L$ and a
covering of $L$ by flat neighbourhoods $\{ V', \ldots, V^N \}$ satisfying co

ditions c) - e) of the proof of Proposition 3.1 , together with $cl(W \cap clD) \subset V^1 \cup \ldots \cup V^N \subset U_L$. We also assume that the volume of e leaf in $W \cap clD$ is close to an integral multiple of volL ; that i $\underline{vol}L \sim d/k$ where $k \geqslant 1$ is an integer.

Let $D_o$ be the transverse disk of $U_L$ through some point $m \in L$ . Let $\beta \in \pi_1 L$ ; then we may find another open transvers disk $D_1 \subset D_o$ such that $m \in D_1$ , $clD_1 \subset D_o$ and $cl(W \cap clD) \cap D_o$ and we may find a map $h : D_1 \to D_o$ which is the holonomy map representing $\beta$ . Clearly, $D_1 \cap W \cap clD$ is invariant under h and $h|D_1 \cap W \cap bdyD = id$ . We may further assume that $W \cap D_1$ is connect Since $bdyD \cap (D_1 \cap W)$ separates the sets $(D_1 \cap W) - D$ and $D \cap (D_1 \cap$ the homeomorphism $h^*: D_1 \cap W \to D_1 \cap W$ constructed below is well-de Set

$$h^*(x) = x \quad \text{for} \quad x \in (D_1 \cap W) - D$$

$$= h(x) \quad \text{for} \quad x \in D \cap (D_1 \cap W) .$$

$h^*|D \cap (D_1 \cap W)$ is pointwise periodic because each leaf is compact and intersects $D_1$ only finitely many times ; thus $h^*$ is pointwise pe iodic on $D_1 \cap W$ . By Montgomery's Theorem (Theorem B of Chapte $h^*$ has finite period (in fact, it is easy to see that $h^*$ will h period at most k , where k is the integer defined at the end of preceding paragraph ) . However, $h^*$ is the identity on the inte set $D_1 \cap W - clD$ (which is non-empty because $\underline{vol}$ is locally unbo at bdyD ) ; thus by Newman's Theorem (Theorem 5.1 , Chapter I) w have $h^* = id$ . Since this is true of all such holonomy maps h r resenting elements of $\Phi(\pi_1 L)$ , it follows from the construction $h^*$ that we must have $\underline{vol}L = c$ (where $c > o$ is as in Lemma 3.5 ;

that is, $\underline{\mathrm{vol}}|D-B_1 = c$ ) .

Recall that $K = \{ m \in M :$ $\underline{\mathrm{vol}}$ is not locally bounded near and assume $K = \emptyset$ . Let $m \in K$ such that there is a neighbourhood $W$ of $m$ in $M$ with $\underline{\mathrm{vol}}|W \cap K$ continuous. We have see above, for each point $x \in \mathrm{bdy} D_i \cap W$ where $D_i$ is a component of $M - K$ with $\underline{\mathrm{vol}}|D_i - B_1 = c_i$ , that $\underline{\mathrm{vol}}\, x = c_i$ . Define the function $w : W \to \mathbb{R}$ by

$$\begin{aligned} w(y) &= c_i \quad \text{if} \quad y \in \mathrm{cl} D_i \cap W \\ &= \underline{\mathrm{vol}}\, y \quad \text{if} \quad y \in W \cap K \quad . \end{aligned}$$

The function $w$ is clearly continuous and $\underline{\mathrm{vol}} \leqslant w$ on $W$ . Thus $\underline{\mathrm{vol}}$ is locally bounded on $W$ ; and in particular $\underline{\mathrm{vol}}$ is locally bounded in a neighbourhood of $m \in K$ . But this contradicts the hypothesis that $K$ is non-empty (because points of continuity of $\underline{\mathrm{vol}}\, K$ are dense in $K$ ). Hence $\underline{\mathrm{vol}}$ is locally bounded on $M$ . By Theorem 1.1 the foliation $\mathcal{F}$ has finite holonomy and the proof of Theorem E is complete.

## 4.0 The proof of the corollaries

Let $M$ be a riemannian manifold and let

$$f : (-\varepsilon, \varepsilon) \times [0,1] \to M$$

be a smooth map such that for each $a \in (-\varepsilon, \varepsilon)$ the curve $f_a(t) = f(a,t)$ is a geodesic parametrised by arc-length. The function $f$

is called a <u>one-parameter family of geodesics</u> in $M$ . The vector field $W_a$ defined by $W_a(t) = \partial f/\partial a(a,t)$ is called a <u>Jacobi fie</u> along the geodesic $f_a$ . As well, define the tangent vector fiel $T_a$ along $f_a$ by $T_a(t) = \partial f/\partial t(a,t)$ ; and for each $t \in [0,1]$ define the curve $f^t : (-\varepsilon, \varepsilon) \to M$ by $f^t(a) = f(a,t)$ . The fol lowing lemma is classical (see, for example, (HI; 10.2); cf. Lem 3.2 of Chapter I) .

4.1 <u>Lemma</u>

Let the function $f$ and vector fields $W_a$ and $T_a$ be as the preceding paragraph. Then the inner product $\langle W_a(t), T_a(t)\rangle$ is constant along the geodesic $f_a(t)$ .

Let $U \cong U_x \times U_y$ be a flat neighbourhood in $M$ with $\pi_y : U \to U_y$ projection, such that $\mathrm{cl}\,U$ is contained within some open convex ball of $M$ . Let

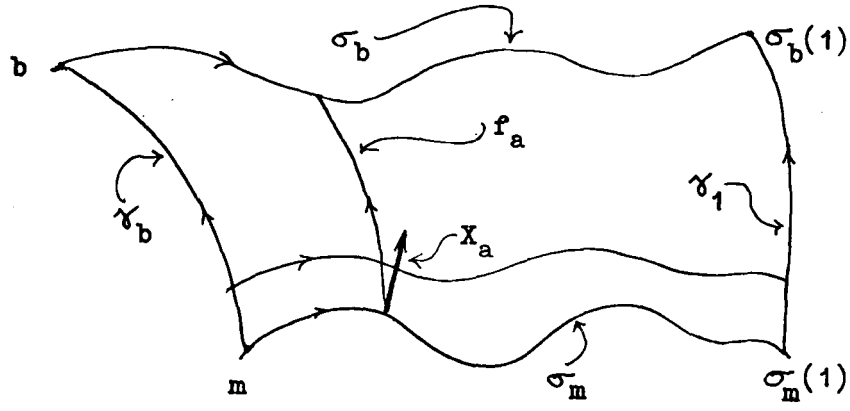
$$\sigma : (-\varepsilon, 1+\varepsilon) \to U_y \qquad \text{for some } \varepsilon > 0$$

be a path in $U_y$ . Further assume that each slice of $U_y$ is totally geodesic in $M$ .

Fix $m \in \pi_y^{-1}\sigma(0)$ . Let $V$ be a neighbourhood of $m$ in $\pi_y^{-1}\sigma(0)$ = (slice through $m$ ) such that, for each point $b \in V$ the orthogonal lift $\sigma_b : (-\varepsilon, 1+\varepsilon) \to U$ of $\sigma$ through $b$ can be defined. (The orthogonal lift is that unique curve through $b$ which is orthogonal to the slices of $U$ and which projects onto $\sigma$

by $\pi_y$ .) We shall show directly that the map $h: V \to \pi_y^{-1}\sigma(1)$ defined by $h(b) = \sigma_b(1)$ is an isometry (cf. (HE)) .

Let $b \neq m$ be in $V$ ; because the slices are totally geodes there exists a unique (minimal) geodesic $\gamma_0 : [0,\delta] \to M$ lying within the slice $\pi_y^{-1}\sigma(0)$ such that $\gamma_0(0) = m$ and $\gamma_0(\delta) = b$ (with parametrisation by arc-length). Moreover, for each $a \in (-\varepsilon, 1+\varepsilon)$ , there exists a unique geodesic segment $\gamma_a$ (parametrised by arc-length) which minimises distance between $\sigma_m(a)$ and $\sigma_b(a)$ ; clearly, for each $a$ , the geodesic $\gamma_a$ will lie in the (totally geodesic) slice $\pi_y^{-1}\sigma(a)$ . Let $X_a \in T_{\sigma_m(a)}M$ b the unit initial vector of the geodesic $\gamma_a$ . Define a one-paramet family of geodesics $f$ in $U$ by $f(a,s) = \exp sX_a$ (for $a \in (-\varepsilon, 1$ and appropriate $s > 0$) . Because the curve $f^0(a) = \sigma_m(a)$ is orthogonal to each vector $X_a$ and because each tangent field $W_a(s) = \partial f/\partial a(a,s)$ along the geodesic $f_a(s)$ is a Jacobi field it follows from Lemma 4.1 that each path $f^s(a)$ (for fixed $s > 0$) is also orthogonal. In particular the curve $f^\delta$ coincides with $\sigma_b$ ; and the segment $f_1(s)$ defined on $[0,\delta]$ coincides with $\gamma_1$ . That is, the diffeomorphism $h$ is distance preserving.



4.2 Proposition

Let $M$ be a riemannian manifold with a foliation $\mathcal{F}$ of $M$ by totally geodesic leaves. Then $\mathcal{F}$ is a g-foliation.

Proof: Immediate from the preceding paragraph and Remark 2.4 .

Using Proposition 4.2 and Theorem E the first part of the statement of Corollary 1 has been proved; that is, the foliation has finite holonomy. It remains to show that the quotient $\pi_Q : M \to Q$ has the structure of a V-bundle with group a Lie group of isometries of the fibre. The following definition of a V-manif (originally due to Satake (S) ) comes from (BA) .

4.3 Definition

By a local uniformizing structure (l.u.s.) $\{U,G,\varphi\}$ for an open subset $\tilde{U}$ of a Hausdorff space $B$ we mean a collection of the following objects:

$U$ : a connected open neighbourhood of the origin in $\mathbb{R}^n$ ;

$G$ : a finite group of $C^\infty$ transformations of $U$ ;

$\varphi$ : a continuous map of $U$ onto $\tilde{U}$ , such that $\varphi \circ g = \varphi$ for all $g \in G$ and such that the induced map of $U/G$ onto $\tilde{U}$ is a homeomorphism.

Let $\{U,G,\varphi\}$ and $\{U',G',\varphi'\}$ be l.u.s.'s for $\tilde{U}$ and

$\tilde{U}'$ respectively such that $\tilde{U} \subset \tilde{U}'$ ; by an injection of $\{U,G,\varphi\}$ into $\{U',G',\varphi'\}$ we mean a $C^\infty$ isomorphism $\lambda$ of $U$ onto $\lambda(U) \subset U'$ such that for any $g \in G$ there exists $g' \in G'$ satifyin the relations $\lambda \circ g = g' \circ \lambda$ and $\varphi = \varphi' \circ \lambda$ . Then a $C^\infty$ <u>V-manifold</u> shall consist of a connected Hausdorff space $B$ and a family $\mathcal{B}$ of l.u.s.'s for open subsets of $B$ satisfying the following conditions:

i) If $\{U,G,\varphi\}$ , $\{U',G',\varphi'\} \in \mathcal{B}$ and $\tilde{U} = \varphi U$ is contained in $\tilde{U}' = \varphi' U'$ , then there exists an injection of $\{U,G,$ into $\{U',G',\varphi'\}$ ;

ii) The open sets $\tilde{U}$ , for which there exists a l.u.s. $\{U,G,\varphi\} \in \mathcal{B}$ , form a basis of the open sets in $B$ .

By a V-bundle $E$ over $B$ with group $H$ and fibre $F$ we mean that there is given for each l.u.s. $\{U,G,\varphi\} \in \mathcal{B}$ a bundle $E_U$ over $U$ with group $H$ and fibre $F$ together with an anti-isomorphism $h_U$ of $G$ onto a group of bundle maps of $E_U$ onto itself such that if $b$ lies in the fibre over $x \in U$ , then $h_U(g)b$ lies in the fibre over $g^{-1}x$ for $g \in G$ ; and moreover, if $\lambda$ is an injection, $\lambda : \{U,G,\varphi\} \to \{U',G',\varphi'\}$ , then we are given a bundle map $\lambda^* : E_{U'}|\lambda(U) \to E_U$ satisfying the requirements that if $g \in G$ and $g'$ is the unique element of $G'$ such that $\lambda \circ g = g' \circ$ then $h_U(g) \circ \lambda^* = \lambda^* \circ h_{U'}(g')$ , and if $\lambda : \{U,G,\varphi\} \to \{U',G'$ and $\lambda' : \{U',G',\varphi'\} \to \{U'',G'',\varphi''\}$ are injections, then $(\lambda \circ \lambda') = \lambda'^* \circ \lambda^*$ . This completes the definition.

Recall the model situation of Condition 4) , Theorem 1.1 .

That is, given a leaf $L$ of $M$ there exists a tubular neighbourhoo $U_L$ of $L$ such that $U_L \cong \tilde{L} \times_G D^q$ (where $q$ = codim.; $D^q$ is an open disk transverse to the leaf $L$; and $\tilde{L}$ is a covering of $L$ corresponding to the kernel of the holonomy epimorphism $\Phi : \pi_1 L \to G$ The natural action of $\Phi(\pi_1 L) = G$ on $\tilde{L} \times D^q$ is given by

$$g(\tilde{\ell}, d) = (\tilde{\ell} g^{-1}, gd)$$

where $g \in G$, $\tilde{\ell} \in \tilde{L}$ and $d \in D^q$; $\tilde{L} \times_G D^q$ is the quotient of $\tilde{L} \times D^q$ by this action.

From Theorem 1.1 we know that the leaf space $Q$ is Hausdorf Moreover, about any leaf $L$ of $Q$ we can find an open set $\tilde{U}$ containing the point $L \in Q$ with l.u.s. given by $(D^q, G, \pi_Q)$ where $D^q$ is a transverse disk as above and $\pi_Q : M \to Q$ is the quotient mapping; so $\pi_Q D^q = \tilde{U}$. Let $L'$ be another leaf in the tubular neighbourhood $\tilde{L} \times_G D^q$ and let $D' \subset D^q$ be a transverse q-disk through $L'$ which is invariant under the holonomy group $G'$ of $L'$; then a l.u.s. containing $L' \in Q$ is given by $(D', G', \pi_Q)$. Because $\pi_Q D' \subset \pi_Q D^q$, the finding of an injection $\lambda$ of $(D', G'$, into $(D, G, \pi_Q)$ will be sufficient to show that $Q$ is a V-manifold; however, the injection $\lambda : D' \subset D^q$ is clearly an injection of l.u.

Over the l.u.s. $(D^q, G, \pi_Q)$ we have a product bundle $\tilde{L} \times D^q \to D^q$ given by projection onto the second factor. Define the anti-isomorphism $h_{D^q}$ of $G$ onto a group of bundle maps of $\tilde{L} \times D^q$ as the inverse of the natural action of $G$ on $\tilde{L} \times D^q$; that is, for $g \in G$ we set $h_{D^q}(g)(\tilde{\ell}, d) = (\tilde{\ell} g, g^{-1} d)$. Clearly the necessary injection maps $\lambda^*$ exist for this construction to define in $M$ the

structure of a V-bundle over $Q$. However, we have not yet specif the structure group $H$ of this bundle; in fact, we will show that the group $H$ is a Lie group of isometries of a fibre $L$.

In (BA), the tangent bundle $TB$ of a V-manifold $B$ is defined as the collection of tangent bundles $TU$ over each l.u.s. $(U,G,\varphi)$ where the bundle map $h_U(g)$ for $g \in G$ is the inverse of the action induced on $TU$ by the action of $g$ on $U$. Furthermore, a $C^\infty$ riemannian metric for $B$ assigns to each l.u.s. $(U,G,\varphi)$ a metric in $TU$ which is compatible with injection of l.u. When $B$ is paracompact, such a riemannian metric always exists.

Choose some riemannian metric for the V-manifold $Q$. Furth assume that the open disk $D^q$ of the l.u.s. $(D^q,G,\pi_Q)$ is geodesically convex with respect to this metric. The collection of such convex l.u.s. form a basis for the open sets of $Q$. Recall that th product bundle $L \times D^q$ over $D^q$ has compact totally geodesic leaves in the metric induced from $M$. (Note: the metric on $D^q$ obtained from the riemannian metric for the V-manifold $Q$ will not coincide in general with the metric induced from $M$ by the inclusion $D^q \subset M$.

Because the fibres of $\pi : L \times D^q \to D^q$ are compact, the distribution of vectors orthogonal to the fibres defines a connection (see (EH)). In particular, parallel translation of the fibre along a curve $\sigma$ in $D^q$ defines a diffeomorphism $h : \pi^{-1}\sigma(0) \to \pi^{-1}\sigma(1$ as we have seen above, $h$ is an isometry. Let $d_o \in D^q$ and fix $\tilde{L} = \pi^{-1}(d_o)$. Let $H$ be the Lie group of isometries of $\tilde{L}$ and for $d \in D^q$ let $H_d$ be the set of isometries of $\tilde{L}$ onto $\pi^{-1}(d)$. Define the set $E$ to be the point-set union of all $H_d$ for $d \in D^q$;

and let $p : E \to D^q$ be the map $H_d \mapsto d$. $H$ acts in an obvious way on $E$ on the left. In the paragraph below we give the set $E$ the $C^\infty$ manifold structure of a (trivial) principal H-bundle over $D^q$

Because the open disk $D^q$ is convex in the metric induced from $Q$ there exists an unique minimal geodesic $\sigma_d$ in $D^q$ from $d_o$ to each point $d$ of $D^q$. Thus we have well-defined isometries $h_d$ from $\tilde{L} = \pi^{-1}(d_o)$ to $\pi^{-1}(d)$ obtained by parallel translation along the path $\sigma_d$. We define a section $s_{D^q} : D^q \to E$ by $s_{D^q}(d) = h_d \in H_d$. Use this section and the left action of $H$ on $E$ to define an isomorphism $E \cong H \times D^q$; in turn, this isomorphism defines the $C^\infty$ manifold structure on $E$ (which is induced from that on $H \times D^q$). Clearly the bundle $\pi : \tilde{L} \times D^q \to D^q$ is an associated H-bundle of $p : E \to D^q$.

Now let $\lambda : (D', G', \pi_Q) \to (D^q, G, \pi_Q)$ be the inclusion injection of a geodesically convex l.u.s. $(D', G', \pi_Q)$ into the l.u.s. $(D^q, G, \pi_Q)$. Construct the principal H-bundle $p' : E' \to D'$ and some section $s_{D'} : D' \to E'$ as above. There is a natural inclusion $E' \subset E$; furthermore, it follows from our constructions that the map $g : D' \to H$ defined by

$$s_{D^q}(d) = g(d)s_{D'}(d) \quad \text{for } d \in D'$$

is $C^\infty$. Thus the bundle $E'$ is an H-subbundle of $E$; so the inclusion $\tilde{L} \times D' \subset \tilde{L} \times D^q$ is an H-bundle morphism. Because the bundle map $\lambda^* : \tilde{L} \times D^q | D' \subset \tilde{L} \times D'$ corresponding to the injection $\lambda : D' \subset D^q$ is defined to be the inverse of the inclusion $\tilde{L} \times D' \subset \tilde{L} \times D^q$, it is clear that $\lambda^*$ is an H-bundle map. Furthermore,

because the group $G = \Phi(\pi_1 L)$ acts as a group of isometric covering translations for the riemannian covering $\tilde{L} \times D^q \to \tilde{L} \times_G D^q$ and preser the product bundle structure $\tilde{L} \times D^q \to D^q$, it will act as a group of H-bundle morphisms on the bundle $\tilde{L} \times D^q$; thus the anti-isomorphism group $h_{D^q}$ is an H-bundle transformation group with respect to the isometry group H and bundle $\pi : \tilde{L} \times D^q \to D^q$. So finally, we ha shown that the V-bundle $\pi_Q : M \to Q$ has structure group a Lie group of isometries of a fibre. This completes the proof of Corollary 1 .

<u>4.4 Remark</u>: We now use Theorem 3.6 of Chapter I (due to J. Vilm to prove the converse to Corollary 1 : namely, given a foliated manifold M and quotient $\pi_Q : M \to Q$ which is a V-bundle with group H a Lie group of isometries of the fibre, then there exists a riemannian metric on M such that each leaf of the folaition is total geodesic in M .

Observe that associated to the V-bundle M with group H there is an associated principal V-bundle P with group and fibre H Precisely, let $(U, G, \varphi)$ be a l.u.s. for the open set $\tilde{U} \subset Q$ . Then we have a bundle $M_U$ over U which defines the V-bundle struct of M over $\tilde{U}$ . Let $P_U$ be the associated principal H-bundle of $M_U$ over U . Using the mapping transformations defined by the bundl map $h_U(g) : M_U \to M_U$ for each $g \in G$ it follows from (ST; lemma 2.6 that we have a well-defined anti-isomorphism $h_{P,U}$ of G onto a group of bundle maps of $P_U$ onto itself. Clearly the bundle maps $h_{P,U}(g) : P_U \to P_U$ will commute with injection of l.u.s.'s .

Recall that a connection on a principal H-bundle $\pi : P \to B$

assigns to $P$ a $q$-dimensional distribution $\mathcal{D}$ (where $\dim B = q$) such that the distribution $\mathcal{D}$ is invariant under left-translation by elements of $H$ and $\pi_* \mathcal{D}_p = T_{\pi_p} B$ for each $p \in P$. Let $(U, G, \varphi)$ be a l.u.s. for the open set $\tilde{U} \subset Q$; and let $\pi_{P,U} : P_U \to U$ be the associated principal V-bundle to the bundle $\pi_U : M_U \to U$ with structure group $H$. It is easily seen that on $P_U$ we can find a connection such that the horizontal distribution $\mathcal{D}_U$ is invariant under the anti-isomorphism action $h_{P,U}$ of the finite group $G$ on $P_U$. Moreover, using a locally finite covering $\{U_i\}$ of the V-manifold $Q$ and a partition of unity subordinate to this covering (see (BA; p.865) ) it is quite straighforward to construct a 'connection' for the whole principal V-bundle $P$ over $Q$ which is compatible with injection of l.u.s.'s. (See (BA; p.865) where a similar construction defines a riemannian metric in the tangent bundle to a V-manifold.)

In the usual way, the 'connection' on $P$ induces a 'connection' on the associated V-bundle $\pi_Q : M \to Q$; that is, each connection $\mathcal{D}_U$ on the principal bundle $P_U$ induces a connection on the H-bundle $M_U$ over $U$ (where $(U, G, \varphi)$ is a l.u.s. for an open set $\tilde{U} \subset Q$). Choosing a riemannian metric for the V-manifold $Q$ fixes a $G$-invariant metric for the open set $U$. By Theorem 3.6 of Chapter I we can find a natural riemannian metric for the total space $M_U$ of the bundle over $U$ such that each fibre $\tilde{L}$ of $M_U$ is totally geodesic with respect to this metric. Furthermore, this metric will be invariant under the anti-isomorphism action $h_U$ of $G$ on $M_U$; and moreover, this construction gives a well-defined riemannian metric for the whole manifold $M$ such that each leaf of $M$ is totally geodesic. This completes the proof of the converse to Coroll

4.5 Remark: A parallel foliation on a riemannian manifold $M$ is such that the distribution $\mathcal{D}$ tangent to the leaves is invariant under parallel translation; equivalently, for $X \in \mathcal{D}_x$ then $\nabla_Y X \in \mathcal{D}_x$ for every $Y \in T_x M$, where $\nabla$ is the Levi-Civita connection of the metric. Obviously, a parallel foliation has totall geodesic leaves. (Note: if $\mathcal{D}$ is a distribution tangent to the leaves of a foliation, then the leaves are totally geodesic if and onl if, for any $X$, $Y \in \mathcal{D}_x$ we have $\nabla_Y X \in \mathcal{D}_x$; this is precisely the condition that the second fundamental form of each leaf be zero.)

Let $M$ be a riemannian n-manifold with a p-dimensional foliation $\mathcal{F}_1$. Locally, we may find a flat neighbourhood $U \cong U_x \times U_y$ in $M$ and one-forms $\omega_1, \ldots, \omega_p$ on $U$ such that $\{\omega_1, \ldots, \omega_p, dy_1, \ldots, dy_q\}$ $(p+q=n)$ is a base for the cotangent space $T^*U$ with dual base $\{\partial/\partial x_1, \ldots, \partial/\partial x_p, v_1, \ldots, y_q\}$.

4.6 Definition

$\mathcal{F}_1$ is called an R-foliation if about every point of $M$ there exists a flat neighbourhood $U \cong U_x \times U_y$ as above with the metr on $U$ given by

$$ds^2 = \Sigma g_{ij}(x,y)\,\omega_i\,\omega_j + \Sigma g_{ab}(y)dy_a dy_b \quad .$$

A manifold $M$ with an R-foliation is said to have a 'bundle-like' metric. Such manifolds were introduced in (R1); equivalent characterisations are given in (PA) and (VR; p.186). (In (VR) s

foliated manifolds $M$ are called Reinhart spaces .) The following two propositions appear in (R1) ; the proofs are quite straightforwa

4.7 Proposition

The following statements are equivalent:

1) $ds^2$ is bundle-like ;

2) any two orthogonal curves in $U$ with the same projection under $\pi_y : U \to U_y$ have the same length ;

3) any two orthogonal vectors with the same projection have the same length.

4.8 Proposition

Let $(M, \mathcal{F}_1)$ be an R-foliated manifold. Then a geodesic in $M$ is orthogonal to a leaf at one point if and only if it is orthogonal at every point.

Examples of R-foliations are given in (R1) ; however, on the total space of a locally trivial fibration $p : E \to B$ we can alway construct a riemannian metric such that the foliation of $E$ by fibres $F$ is an R-foliation. Specifically, take a trivialisation of $E$ , $\{\psi_i : p^{-1}(U_i) \to U_i \times F\}$ where $\{U_i\}$ is a locally finite cover of $B$ and $\{\varphi_i\}$ is a partition of unity subordinate to $\{U_i$

Give B and F riemannian metrics, and put a metric on $p^{-1}(U)$ induced by $\psi_i$ from the product metric on $U_i \times F$ ; then a bundle-like metric for E is obtained by piecing together the metrics on $p^{-1}(U_i)$ via the partition of unity $\{\varphi_i\}$ . Moreover, given a distribution $\mathcal{D}$ in E transverse to the fibres we may ensure that the bundle-like metric is such that $\mathcal{D}$ is orthogonal.

Now, let M have an R-foliation $\mathcal{F}_1$ such that the distribution of vectors orthogonal to the leaves of $\mathcal{F}_1$ is integrable and defines a foliation $\mathcal{F}_2$ . It is immediate from Proposition 4.3 that the leaves of $\mathcal{F}_2$ are totally geodesic in M . Corollary 2 now follows directly from Corollary 1 .

## 5.0 Nullity foliations

This section contains a brief summary of the main properties of the so-called 'nullity foliations' of riemannian manifolds; proofs, in most cases, will be omitted. The ideas inherent here were first introduced by Chern and Kuiper (C-K) where they obtained theorems on isometric immersions of manifolds in flat euclidean space. The presentation here is due to D. Ferus (F) .

Let M be a riemannian manifold with Levi-Civita connection $\nabla$ ; and let $\eta$ be a vector bundle over M with a covariant derivative $\nabla^{\eta}$ . For each integer $p \geqslant 0$ denote by $\Lambda^p(\eta)$ the bundl of (alternating) p-forms on M with values in $\eta$ . Then the

covariant derivatives $\nabla$ and $\nabla^{\eta}$ induce a covariant derivative $\nabla^{p}$ on $\Lambda^{p}(\eta)$ by

$$\nabla^{p}_{X}\omega(X_1,\ldots,X_p) = \nabla^{\eta}_{X}(\omega(X_1,\ldots,X_p)) - \omega(\nabla_X X_1,\ldots,X_p) - \ldots - \omega(X_1,\ldots,\nabla_X X_p) \quad ;$$

where $\omega \in \Gamma\Lambda^{p}(\eta)$ = sections of $\Lambda^{p}(\eta)$, and $X, X_1,\ldots,X_p$ are vector fields on $M$.

Fix a vector bundle $\xi$ over $M$ with covariant derivative $\nabla^{\xi}$. For $p,q > 0$ integers, we define the <u>bundle of double forms</u> of type $(p,q)$ over $M$, $\Lambda^{p,q}(\xi)$, by $\Lambda^{p,q}(\xi) = \Lambda^{p}(\Lambda^{q}(\xi))$. The covariant derivatives $\nabla$ and $\nabla^{\xi}$ induce a covariant derivative $\nabla^{p,q}$ for $\Lambda^{p,q}$ as above. For a double form $A \in \Gamma\Lambda^{p,q}(\xi)$ we define $A \in \Gamma\Lambda^{p+1,q-1}(\xi)$ and $DA \in \Gamma\Lambda^{p+1,q}(\xi)$ by

$$A^{*}(X_o,X_1,\ldots,X_p)(Y_2,\ldots,Y_q) = \Sigma_{j=o}^{p}(-1)^{j}A(X_o,\ldots,\hat{X}_j,\ldots,X_p)(X_j,Y$$

$$(DA)(X_o,\ldots,X_p) = \Sigma_{j=o}^{p}(-1)^{j}(\nabla^{p,q}_{X_j}(X_o,\ldots,\hat{X}_j,\ldots,X_p)$$

for vector fields $X_o,\ldots,X_p$, $Y_2,\ldots Y_q$ on $M$.

5.1 <u>Definition</u>

A double-form $A \in \Gamma\Lambda^{p,p}(\xi)$ is called a <u>riemannian double form</u> of type $(p,p)$ on $M$ with values in $\xi$, if we have

1) $A^{*} = 0$

2) $DA = 0$

3) $A(X_1,\ldots,X_p)(Y_1,\ldots,Y_p) = A(Y_1,\ldots,Y_p)(X_1,\ldots,X_p)$

for all vector fields $X_1,\ldots,X_p$, $Y_1,\ldots,Y_p$ on $M$.

Now define the distribution $\mathcal{D}$ on $M$ by

$$\mathcal{D}_x = \{Y \in T_xM : A(Y,X_2,\ldots,X_p) = 0 \text{ for all } X_2,\ldots,X_p \in T_xM$$

$\mathcal{D}_x$ is called the <u>nullity space</u> of $A$ at $x$ and the dimension $\mu(x)$ of $\mathcal{D}_x$ is called the <u>index of nullity</u> of $A$ at $x$. It is easy to see that the function $\mu$ is upper-semicontinuous. Thus the set $M_o$ where $\mu$ attains its minimum is open in $M$.

5.2 <u>Theorem</u> (Maltz; Abe; Clifton and Maltz; Gray; Ferus)

The distribution $\mathcal{D}^o = \mathcal{D}|M_o$ is integrable and the integral manifolds of $\mathcal{D}^o$ are totally geodesic in $M$. Moreover, if $M$ is complete then the maximal integral manifolds of $\mathcal{D}^o$ are also complete.

Proof: See D. Ferus (F) ; and also Gray (G) .

The foliation induced by $\mathcal{D}^o$ on $M_o$ is known as the <u>nullity foliation</u> of $M$ induced by $A$.

Many examples of riemannian double forms are given in (G) ; however, the following two examples are the most important historicall

5.3 <u>Example</u>: Let $\xi$ be the trivial line bundle on $M$ with its canonical connection. Let $R$ be the riemannian curvature tensor of

M ; that is, $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ for vector fields X, Y and Z on M . Then the double form A defined by

$$A(X_1,X_2)(Y_1,Y_2) = \langle R(X_1,X_2)Y_1, Y_2 \rangle$$

is a riemannian double form on M , Condition 1) of the definition follows from the 1st. Bianchi identity ; 2) follows from the 2nd. Bianchi identity ; and 3) , which may be derived from 1) , expres the symmetry of the components of the curvature tensor. Related to this double form we have the riemannian double form B defined by

$$B(X_1,X_2)(Y_1,Y_2) = k( \langle X_1,Y_1 \rangle \langle X_2,Y_2 \rangle - \langle X_2,Y_1 \rangle \langle X_1,Y_2 \rangle )$$

for a fixed constant k . First notice that B = 0 if and only if the manifold M has constant sectional curvature. Further, we have a riemannian double form given by the difference tensor A - B ; define the tensor field K of type (1,3) on M , regarded as a linear map $K_x(X,Y) : T_xM \to T_xM$ for $X, Y \in T_xM$ , by

$$\langle K(X_1,X_2)Y_1, Y_2 \rangle = (A - B)(X_1,X_2)(Y_1,Y_2) \quad .$$

The tensor K measures how much the curvature of the space M differs from that of a space of constant curvature k . The index of nullity of K , $\mu_k$ , defined by

$$\mu_k(x) = \dim \{ X \in T_xM : K(X,Y) = 0 \text{ for all } Y \in T_xM \}$$

corresponds to the index of nullity of the tensor A - B , and is called the k-nullity index of M .

Following Clifton and Maltz (C-M) we call manifolds M such that

$\mu_k > 0$ is constant <u>quasi-constant curvature</u> manifolds. From (C-M) we have the following examples:

i) for $k = 0$ such manifolds are product spaces of arbitary manifolds by locally flat manifolds;

ii) spaces of <u>recurrent curvature</u> where $\nabla R = R \otimes \alpha$, for some one-form $\alpha$;

iii) riemannian homogenous spaces, symmetric spaces and Lie groups.

5.4 <u>Theorem</u>

Let $M$ be a complete quasi-constant curvature manifold with $k > 0$ and $\mu_k \geqslant 2$. Then $M$ has the structure of a V-bundle over a V-manifold $Q$ with structure group a Lie group of isometries of the fibre.

Proof: From Theorem 5.2 we know that $M$ is foliated by totally geodesic leaves which are complete in the induced metric. By the Gauss equation (for submanifolds), each leaf has constant curvature $k > 0$; thus each leaf is compact. The theorem now follows directly from Corollary 1. (Note that the foliation could have only the one leaf $M$, in which case $Q = *$, a point.)

<u>5.5 Example</u>: Let $f: M \to M_k$ be an isometric immersion of a riemannian manifold $M$ into a space $M_k$ of constant curvature $k$. Let $\xi$ be the normal bundle to $M$ in $M_k$ endowed with the normal connection.

Then the second fundamental form $S_f$ of the immersion $f$ is a riemannian double form of type $(1,1)$ with values in $\xi$. Condition 1) of the definition comes from the symmetry of the tensor $S_f$, whilst 2) is the Codazzi-Mainardi equation. Let $\nu$ be the index of nullity of $S$; then $\nu(x)$ is known as the <u>relative nullity index</u> of $x \in M$ in $M_k$.

5.6 <u>Proposition</u> (Chern-Kuiper)

Let $f : M \to M_k$ be an isometric immersion of a riemannian manifold $M$ into a space $M_k$ of constant curvature $k$. Let $\mu_k$ be the k-nullity index of $M$; let $\nu$ be the relative nullity index of $f$, and let $p$ be the codimension of $f(M)$ in $M_k$. Then

$$\nu(x) \leqslant \mu_k(x) \leqslant \nu(x) + p$$

for all $x \in M$.

If the curvature $k$ of $M_k$ is greater than zero then the leave of the relative nullity foliation (with the dimension of a leaf $\geqslant 2$) will be complete and of constant sectional curvature $k$; thus the leaves are compact. Hence Corollary 1 applies to this situation as well; that is, the relative nullity foliation of the immersed manifold $M$ has finite holonomy.

Using Proposition 5.6 and Frankel's Theorem (FR) on intersections of totally geodesic submanifolds of a manifold of positive curvature, various authors have obtained rigidity theorems and non-

isometric immersion theorems for immersions of spheres in spheres, etc. (See, for example (O'N-S) and (A).) Moreover, the leaves of the relative nullity distribution are totally geodesic in the manifold $M_k$ (O'N-S; Thm.I).

Possibly, the results of this chapter will further elucidate the riemannian structure of such immersions.

BIBLIOGRAPHY

(A) K. Abe, "A characterisation of totally geodesic submanifolds in $S^N$ and $CP^N$ by an inequality", Tohoku Math. J. 23 (1971) pp. 219-244 .

(B) R. Bott, "On manifolds all of whose geodesics are closed", Ann. of Math. 60 (1954) pp. 375-382 .

(BA) W.L. Bailey Jr., "The decomposition theorem for V-manifolds", Amer. J. of Math. 78 (1956) pp. 862-888 .

(BE) M. Berger, "Lectures on Geodesics in Riemannian Geometry", Tata Institute, Bombay 1965.

(BR) G.E. Bredon, "Introduction to Compact Transformation Groups", Academic Press, New York and London 1972 .

(B-W) W.M. Boothby & H.C. Wang, "On contact manifolds", Ann. of Math. 68 (1958) pp. 721-734 .

(C-K) S.S. Chern & N. Kuiper, "Some theorems on the isometric imbedding of a compact riemannian manifold in euclidean space", Ann. of Math. 56 (1952) pp. 422-430 .

(C-M) Y.H. Clifton & R. Maltz, "The K-nullity of the curvature operator", Mich. Math. J. 17 (1970) pp. 85-89 .

(D) A. Dress, "Newman's theorems on transformation groups", Topology 8 (1969) pp. 203-207 .

(E1) D.B.A. Epstein, "Periodic flows on three-manifolds", Ann. of Math. 95 (1972) pp. 68-82 .

(E2) _____, "Foliations with all leaves compact", preprint, University of Warwick 1974 .

(EH) C. Ehresmann, "Les connexions infinitesimales dans un espace fibre differentiable", Colloque de Topologie, Bruxelles (1950) pp. 29-55 .

(F) D. Ferus, "On the completeness of nullity foliations", Mich. Math. J. 18 (1971) pp. 61-64 .

(FR) T. Frankel, "Manifolds with positive curvature", Pac. J. of Math. 11 (1961) pp. 165-174 .

(G) A. Gray, "Some relations between curvature and characteristic classes", Math. Ann. 184 (1970) pp. 257-267 .

(H) A. Haefliger, "Varietes feuilletes", Ann. Scuola Norm. Sup. Pisa (3) 16 (1962) pp. 367-397 .

(HE) R. Hermann, "A sufficient condition that a mapping of riemannian manifolds be a fibre bundle", Proc. Amer. Math. Soc. 11 (1960) pp. 236-242 .

(HI) N.J. Hicks, "Notes on Differential Geometry", Van Nostrand, Princeton 1965 .

(M) D. Montgomery, "Pointwise periodic homeomorphisms", Amer. J. of Math. 59 (1937) pp. 118-120 .

(MA) R. Maltz, "The nullity spaces of the curvature operator", Cahiers de Topologie et Geom. Diff. 8 (1966) pp.1-20 .

(N) M.H.A. Newman, "A theorem on periodic transformations of spaces", Quart. J. of Math. 2 (1931) pp. 1-8 .

(NA) H. Nagakawa, "A note on theorems of Bott and Samuelson", J. Math. Kyoto Univ. 7 (1967) pp. 205-220 .

(O-R) P. Orlik & F. Raymond, "Actions of SO(2) on 3-manifolds", Proc. Conf. on Transf. Groups, New York 1967, pp. 297-318 , Springer-Verlag, Berlin and New York 1968 .

(O'N-S) B. O'Neil & E. Stiel, "Isometric immersions of constant curvature manifolds", Mich. Math. J. 10 (1963) pp. 335-339 .

(P1) R.S. Palais, "A global formulation of the Lie theory of transportation groups", Mem. Amer. Math. Soc. 22 (1957) .

(P2) _____, "$C^1$ -actions of compact Lie groups on compact manifolds are $C^1$ equivalent to $C^\infty$-actions", Amer. J. of Math. 92 (1970) pp. 748-760 .

(PA) J. Pasternack, "Foliations and compact Lie group actions", Comm. Math. Helv. 46 (1971) pp. 467-477 .

(PUJ) Proc. of the United States - Japan Seminar in Diff. Geom. , Kyoto, 1965 .

(R1) B.L. Reinhart, "Foliated manifolds with bundle-like metrics", Ann. of Math. 69 (1959) pp. 119-132 .

(R2) _____, "Closed metric foliations", Mich, Math, J. 8 (1961) pp. 7-9 .

(S) I. Satake, "On a generalisation of the notion of manifolds", Proc. Nat. Acad. Sciences 42 (1956) .

(ST) N. Steenrod, "The Topolgy of Fibre Bundles", Princeton Univ. Press, New Jersey 1951 .

(V) J. Vilms, "Totally geodesic maps", J. Diff. Geom. 4 (1970) pp. 73-79 .

(VR) I. Vaisman, "Cohomology and Differential Forms", Marcel Dekker, New York 1973 .

(W) N. Weaver, "Pointwise periodic homeomorphisms of continua", Ann. of Math. 95 (1972) pp. 83-85 .

(WA) A.W. Wadsley, "Geodesic foliations by circles", preprint, Univ. of Warwick 1974 .