ON THE COHOMOLOGY OF FINITE GROUPS

OVER MODULAR FIELDS

by

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CONTENTS

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INTRODUCTION

The aim of this thesis is to study the cohomology of a finite group G by means of certain of its modular representations. For such an approach to be possible, it is necessary to restrict our attention to the case where the coefficient module is a vector space over some field K of characteristic p dividing the order of G. When this is done, it is possible, in view of Theorem 1.1, to work entirely within the category of finitely-generated KG-modules. The simplifications in the theory which are thus achieved compensate in some measure for the attendant loss of generality. In fact, we hope to put forward a case that the natural way to study the cohomology of a finite group, at least from an algebraic point of view, is within the above-mentioned category. Although much of what follows remains true for representations over p-adic rings, and the loss of generality is not so great in this case, we restrict ourselves to fields of characteristic p for the sake of simplicity. In fact, in view of Remark 2.13, no further loss of generality is sustained by considering only the field of p elements.

The basis of the whole thing is described in Chapter I, which contains a number of results (the key reference being [12]), assembled and supplemented by J.A. Green. These results are used in Chapter II to obtain further preliminary results, many of which, although well-known, do not seem to appear explicitly in the literature. The theme of Chapter III is the analogy between representations and cohomology, the only original result here being Theorem 3.7. The contents of Chapter IV have been submitted to the London Mathematical Society in the form of two papers which should appear sometime in 1969 (see [19] and [20]), the modules described (by way of example) therein being of interest from the point of view of representation theory. The only original result of Chapter V is Theorem 5.2, while Chapter VI consists of the application of classical results in order to define a product structure in the case in hand. As far as we know, the results concerned with the product structure appearing in Chapters VII and VIII are hitherto unknown.

As well as the original results described in the sequel, the ease with which a large number of known results can be proved in this context should provide motivation for studying the cohomology of a finite group by means of modular representations.

NOTATION

The notation and other conventions used in the sequel are either standard, defined in context or included in the following list.

$\mathbb{Z}$: rational integers.

$\mathbb{Z}^+$: rational integers greater than zero.

$<, \subset$: strict inequalities.

All maps are written on the left, and " $\circ$ " stands for composition of maps.

G: a group (with identity e); except in 2.11, all groups considered will be finite.

$\sigma_p(G)$: largest normal p-subgroup of G.

$\sigma_{p'}(G)$: largest normal p'-subgroup of G.

$\sigma_{p'p}(G)$: characterised by the rule $\dfrac{\sigma_{p'p}(G)}{\sigma_{p'}(G)} = \sigma_p\left(\dfrac{G}{\sigma_{p'}(G)}\right)$.

S_n: the symmetric group on n symbols, (unless otherwise stated).

K: a field of characteristic p, usually GF(p).

KG: the group-ring of G over K.

$\mathfrak{U} = \mathfrak{U}(G)$: augmentation ideal of KG.

$\rho = \rho(KG)$: Jacobson radical of KG.

A, B: KG-modules; all modules are left, unital modules, finite-dimensional as K-spaces.

$F_1, \ldots F_t$: a complete list of non-zero pairwise-nonisomorphic, irreducible KG-modules.

$U_1, \ldots U_t$: the corresponding principal indecomposable KG-modules.

K can be made into a KG-module by letting each element of G operate trivially upon it. The result is an irreducible KG-module, usually taken to be F_1.

KG is frequently regarded as a left KG-module and (as with K) it is always clear from the context which meaning of the symbol is intended.

While K is called <u>the</u> trivial KG-module, the KG-module A (say) will be called KG-trivial if it is KG-isomorphic to a direct sum of copies of K.

No distinction will be made between a module and its isomorphism class. If A is a K-space with subspace B, we say that the subset S of A is a complementary K-basis for B in A if:

T is a K-basis for B $\Longrightarrow S \cup T$ is a K-basis for A.

$\oplus, \Sigma\oplus$: internal direct sum.

$\dotplus, \Sigma\dotplus$: external direct sum.

$A \mid B$ means $\exists$ KG-mdoule C such that $B \cong A \dotplus C$.

If $n \in \mathbb{Z}^+$ and A a module, then n.A denotes either the direct sum of n copies of A or the module $\{na \mid a \in A\}$, according to context.

$\mu(A)$: the intersection of the maximal submodules of A.

$\nu(A)$: the sum of the minimal submodules of A.

Let A be an R-module, where R is a ring-with-1, and suppose I is a left ideal of R and S is a subset of A; then we denote by I.S the R-submodule of A composed of all finite sums of elements of the form i.s, $i \in I$, $s \in S$.

$A \otimes B$ means $A \otimes_K B$ (A and B KG-modules), made into a KG-module by: $x.(a \otimes b) = xa \otimes xb$ ($x \in G$, $a \in A$, $b \in B$), extended by linearity. Any other meaning of $\otimes$ will be given in the context.

The obvious meanings of $\bigotimes_{i=1}^{n} A_i$ and $A^{\otimes n}$ will also be used.

If $H \leq G$, A a KG-module and B a KG-module,

A_H is the KH-module obtained from A by restricting the operators, B^G is the induced KG-module, $KG \otimes_{KH} B$ (unless otherwise stated),

$\mathcal{N}_G(H) = \{x \in G \mid H^x = H\}$, $\mathcal{C}_G(H) = \{x \in G \mid h^x = h, \forall h \in H\}$.

For x, y in G, $x^y = yxy^{-1}$.

$\Phi(G)$: the intersection of the maximal subgroups of G (the Frattini subgroup of G).

CHAPTER I: BASIC THEORY

Theorem 1.1

Let $\mathcal{O}$ be a commutative ring-with-1 and G a finite group; then, regarding $\mathcal{O}$ as an $\mathcal{O}$-module on which G acts trivially,

$$H^n(G, \mathcal{O}) \cong Ext^n_{\mathcal{O}G}(\mathcal{O}, \mathcal{O}), \quad \forall n \geq 0.$$

Proof. First, take any $\mathcal{O}G$-projective resolution of $\mathcal{O}$:

$$\mathbf{P}: \quad --- \to P_n \xrightarrow{\partial_n} P_{n-1} \to --- \to P_0 \xrightarrow{\partial_0} \mathcal{O} \to 0, \text{---}(1).$$

Then we claim that the homology in dimension n of $Hom_{\mathcal{O}G}(\mathbf{P}, \mathcal{O})$ is just $Ext^n_{\mathcal{O}G}(\mathcal{O}, \mathcal{O})$ for all $n \geq 0$. This follows from the elementary properties of "Ext" using induction.

Now consider any free $\mathbb{Z}G$-resolution of $\mathbb{Z}$:

$$\underline{F}: --- \to F_n \to F_{n-1} \to --- \to F_0 \to \mathbb{Z} \to 0, \text{---} (2).$$

Applying the functor $\mathcal{O} \underset{\mathbb{Z}}{\otimes} \cdot$ we obtain a free $\mathcal{O}G$-resolution of $\mathcal{O}$:

$$\mathcal{O} \otimes \underline{F}: --- \to \mathcal{O} \otimes F_n \to --- \to \mathcal{O} \otimes F_0 \to \mathcal{O} \to 0, \text{---} (3)$$

Now if F is any free $\mathbb{Z}G$-module and $\alpha: F \to \mathcal{O} \otimes F$ is given by $\alpha(x) = 1 \otimes x$, $\forall x \in F$, then the mapping

$$K: Hom_{\mathcal{O}G}(\mathcal{O} \otimes F, \mathcal{O}) \to Hom_{\mathbb{Z}G}(F, \mathcal{O}),$$

given by $K(\theta) = \theta \circ \alpha$, $\forall \theta \in Hom_{\mathcal{O}G}(\mathcal{O} \otimes F, \mathcal{O})$ is an isomorphism. If we have $\phi: F \to \mathcal{O}$ and define $K'(\phi): \mathcal{O} \otimes F \to \mathcal{O}$ by

$\kappa'(\phi)(a\otimes x) = a\,\phi(x)$, then it is easy to check that $\kappa\kappa'$ and $\kappa'\kappa$ are the respective identities. Applying this to the free modules in (2) and (3), we establish a chain isomorphism between $\mathrm{Hom}_{\mathbb{Z}G}(E, \vartheta)$ and $\mathrm{Hom}_{\vartheta G}(\vartheta\otimes E, \vartheta)$. Thus the cohomologies are isomorphic as ϑ-modules.

We first observe that the above theorem remains true when the coefficients are in an arbitrary ϑG-module A, viz,

$$H^n(G, A) \cong \mathrm{Ext}^n_{\vartheta G}(\vartheta, A)\ ,\ \forall n \geqslant 0.$$

Thus the cohomology of G with coefficients in an ϑG-module can be realculated using a resolution of the form (1).

From now on, we shall consider only the case ϑ = K = GF(p), where p is some prime dividing $|G|$. (If (p, $|G|$) = 1, KG is semisimple, and there exists a resolution of the form (1) with P_0 = K, $\partial_0 = 1_K$ and $P_0 = (0)$ for n $\geqslant$ 1. The cohomology in this case is thus trivial). The situation is hereby simplified, in that KG is a Frobenius algebra and the Krull-Schmidt Theorem holds for finitely-generated KG-modules (see [4]).

An equivalence relation may be defined on KG-modules as follows: A $\sim$ B if there exist projectives P and Q such that A $\dotplus$ P $\cong$ B $\dotplus$ Q. Further, any KG-module A can be decomposed into a direct sum, A = c(A) $\oplus$ p(A), where p(A) is projective and c(A) involves no projective direct summand. $c(A)$ in unique up to isomorphism and

$$A \sim B \Longleftrightarrow c(A) \cong c(B).$$

Again in virtue of the above remarks, we can write Schanuel's Lemma (see [21]) in the following form.

<u>Lemma 1.2.</u> If
$$0 \to R \to P \to A \to 0, \qquad (4)$$
$$0 \to S \to Q \to B \to 0,$$
are short exact sequences of KG-modules with P and Q projective, then

$$A \sim B \Longleftrightarrow R \sim S.$$

Further, in a short exact sequence of the form (4), as is shown in [12],

$$c(R) \text{ indecomposable} \Longleftrightarrow c(A) \text{ indecomposable}.$$

Thus, if A is already indecomposable, we can write $c(R) = \Omega A$, as in [12], which is defined up to isomorphism if A is given. $\Omega^{-1}A$ can be defined in a dual fashion. We now give the other main result of this section.

<u>Theorem 1.3.</u>

For all $n > 0$, $c(\mathfrak{U}^{\otimes n}) \cong \Omega^n K$.

<u>Proof.</u> We induct on n, the case $n = 1$ being just the definition of $\mathfrak{U}$. Apply the exact functor $\mathfrak{U}^{\otimes n} \otimes \cdot$ to the short exact sequence:

$$0 \to \mathfrak{U} \to KG \to K \to 0,$$

to obtain a short exact sequence:

$$0 \to \mathfrak{U}^{\otimes (n+1)} \to F \to U^{\otimes n} \to 0,$$

with F free. The inductive step is completed by application of Lemma 1.2. We conclude this section by giving some lemmas which will be of use in the sequel.

Lemma 1.4. Let A be a non-projective indecomposable KG-module, with G a p-group and $\sigma = \sum_{x \in G} x, \in KG$; then $\sigma . a = 0, \forall a \in A$.

Proof. Let $I_a = \{ \gamma \in KG \mid \gamma . a = 0 \}, \forall a \in A$, a left ideal of KG. Then, since $J = \{ \lambda\sigma \mid \lambda \in K \}$ is the unique minimal left ideal of KG, either $J \leq I_a$ or $I_a = (0), (\forall a \in A)$. If the former case holds $\forall a \in A$, the lemma is proved; otherwise, let $a \in A$ be such that $I_a = (0)$. This implies that the set $\{ x.a \mid x \in G \}$ is linearly independent, so that KG.$\{a\} \cong$ KG.
So in this case, since KG is injective, A involves a projective direct summand, and this is a contradiction.

Now let A be a KG-module, and consider $\mathrm{Hom}_K(A,K)$. Make this into a KG-module - call it A* - by defining $\forall x \in G, a \in A$:

$$(x.f)(a) = f(x^{-1}.a).$$

Further, given a map $\phi : A \rightarrow B$, of KG-modules, we can define a map, $\phi^* : B^* \rightarrow A^*$ in the obvious way. The main properties of the functor .* may be summarised as follows.

Lemma 1.5. The functor $\cdot^*$ is contravariant, exact and preserves projectives and Kronecker products.

A^* is called the contragedient of A.

Now denote by $O_{p'}(G)$ the largest normal subgroup of G of order prime to p, and for any KG-module A, let
Ker A = $\{x \in G \mid x.a = a, \forall a \in A\}$.

Lemma 1.6. If $\mathcal{B}$ denotes the principal block of KG, $\bigcap_{A \in \mathcal{B}} \operatorname{Ker} A = O_{p'}(G)$.

Proof. Let $O_{p'}(G)$ have order n, and consider

$$\varepsilon = \frac{1}{n} \sum_{x \in O_{p'}(G)} x ,$$

a central idempotent of KG. Now $\varepsilon \notin \mathcal{U}$, so we have:

$$K \cong \frac{KG}{\mathcal{U}} \cong \frac{KG.\{\varepsilon\} + \mathcal{U}}{\mathcal{U}} \cong \frac{KG.\{\varepsilon\}}{\mathcal{U} \cap KG.\{\varepsilon\}} .$$

Hence K is a homomorphic image of $KG.\{\varepsilon\}$, establishing that $KG.\{\varepsilon\}$ contains the principal block of KG. But by the construction of ε,

$O_{p'}(G) \leq \operatorname{Ker} KG.\{\varepsilon\}$, and so $O_{p'}(G) \leq \bigcap_{A \in \mathcal{B}} \operatorname{Ker} A$.

Now let H be any Sylow p-subgroup of G, and consider any principal indecomposable, U_i say, in $\mathcal{B}$. Since $(U_i)_H$ is free, it represents H faithfully, and this implies that $\bigcap_{A \in \mathcal{B}} \operatorname{Ker} A$ can have no p-part. This completes the proof.

Some results of a similar nature to the one just proved are given in [2].

CHAPTER II: MINIMAL RESOLUTIONS

Suppose we are given a KG-projective resolution of K:

$$---\to P_n \xrightarrow{\partial_n} P_{n-1} \to ---\to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} K \to 0, ---(1),$$

and let us write $\mathrm{Im}\,\partial_n = R_n$, $n \geqslant 0$. Then from Chapter I, $\forall n \geqslant 0$:

$$c(R_n) \cong \Omega^n(K) \cong c(\mathcal{U}^{\otimes n}),$$

(with the obvious convention when $n = 0$). Also, set $\Omega^n K = c_n(G)$, $n \geqslant 0$, the n^{th} core of G (which also depends on p). The resolution (1) will be called minimal if $R_n \cong c_n(G)$, $\forall n \geqslant 0$. The main object of this chapter is to prove the following result.

Theorem 2.1.

There is a minimal resolution of K corresponding to any G and p. This is proved by means of two lemmas, which are also of interest in their own right.

Lemma 2.2. If A is a KG-module and $\mu(A)$ = the intersection of the maximal submodules of A, then $\mu(A) = \rho\cdot A$. Further, $\dfrac{A}{\rho\cdot A}$ is the maximal completely reducible quotient of A.

Proof. Since this result is reasonably well-known, we merely sketch the proof, which falls naturally into four stages.

(i) $\frac{A}{B}$ completely reducible $\Rightarrow$ B can be written as an intersection of maximal submodules $\Rightarrow$ B $\not\geq \mu(A)$.

(ii) $\frac{A}{\mu(A)}$ is completely reducible. The easiest way to prove this is to dualise (using $\circ^*$) and show that the socle of A, viz. $\nu(A)$ = the sum of the minimal submodules of A, is completely reducible. For this, use the Jordan-Hölder theorem to show that $\nu(A)$ is in fact the sum of a finite number of irreducibles, $A_1, \ldots A_n$, say. By induction, these can be chosen so that:

$A_i \cap \sum_{j=1}^{i-1} A_j = (0)$, for $2 \leq i \leq n$, and a further induction gives the result.

(i) and (ii) $\Rightarrow \frac{A}{\mu(A)}$ is the maximal completely reducible quotient of A.

(iii) To show $\frac{A}{\rho . A}$ completely reducible, it is sufficient to show that if $\rho . A = (0)$, then A is completely reducible. So assume $\rho . A = (0)$ and choose generators $a_1, \ldots a_r$ for A over KG. Present KG. $\{a_i\}$ as a quotient of KG, $1 \leq i \leq r$, and so deduce that it is completely reducible, since $\frac{KG}{\rho}$ is. Thus, $A = \nu(A)$, and the result follows by the proof of (ii). Thus, $\mu(A) \leq \rho . A$.

(iv) If $M \lneq A$ is a maximal submodule, $\rho . (\frac{A}{M}) = (0)$ by definition of ρ, i.e. $\rho . A \leq M$. Hence, $\rho . A \leq \mu(A)$. This completes the proof of the lemma.

<u>Lemma 2.3</u>. Given any short exact sequence of KG-modules

$$0 \to R \to P \to A \to 0,$$

with P projective, there exists a short exact sequence

$$0 \to c(R) \xrightarrow{i} Q \to A \to 0,$$

with Q projective. Further, $\operatorname{Im} i \leq \rho . Q$, so that
$\dfrac{Q}{\rho . Q} \cong \dfrac{A}{\rho . A}$.

<u>Proof</u>. The proof is by means of the following diagram, in terms of which it will be sufficient to prove

(a) Q is projective,

(b) f is onto,

(c) $c(R) \cong \operatorname{Ker} f$.

$$\begin{tikzcd}
 & 0 & & \operatorname{Ker} f \arrow[d] & 0 \\
 & c(R) \arrow[u] & & Q \arrow[ur] \arrow[d, "f"] \arrow[dl, shift left, "\eta"] & \\
0 \arrow[r] & R \arrow[u] \arrow[r, "\alpha"] & P \arrow[r, "\beta"'] \arrow[ur, shift left, "\xi"] & A \arrow[r] & 0 \\
 & p(R) \arrow[u, "\phi"] \arrow[ur, "\theta"'] & & & \\
0 \arrow[ur] & 0 \arrow[u] & & &
\end{tikzcd}$$

The diagram is constructed as follows. The horizontal row is the given short exact sequence, ϕ is any split embedding and $\theta = \alpha \circ \phi$. ξ, which is onto Q, is the cokernel of θ and η is a splitting for ξ. Then $f = \beta \circ \eta$. Q is clearly projective and, since β is onto and $P = \mathrm{Im}\,\theta \oplus \mathrm{Im}\,\eta$, f must be onto. By Schanuel's Lemma, $c(R) \cong c(\mathrm{Ker}\ f)$, but from the diagram, $\dim_K c(R) = \dim_K \mathrm{Ker} f$, and so $c(R) \cong \mathrm{Ker} f$ as required.

Let the short exact sequence thus obtained be

$$0 \longrightarrow c(R) \xrightarrow{i} Q \longrightarrow A \longrightarrow 0,$$

and dualise using the functor $.^*$ to obtain an onto mapping:

$$Q^* \xrightarrow{i^*} c(R)^* .$$

Let U_j with unique minimal submodule F_j, be an indecomposable component of Q^*. Then, since $p(c(R)^*) = (0)$, i^* restricted to U_j cannot be (1 - 1). Hence, $i^*(F_i) = (0)$. Applying this to each of the indecomposable components of Q^* in turn, we deduce that $i^*(\vartheta(Q^*)) = (0)$. Dualising again, this means that the composite:

$$c(R) \xrightarrow{i} Q \xrightarrow{nat.} \frac{Q}{\rho . Q}$$

is zero. Hence, $\mathrm{Im}\, i \leq \rho . Q$ as claimed. Thus, every maximal submodule of Q contains c(R), and the final statement in the lemma follows.

Theorem 2.1 now follows trivially by induction.

Theproblem of the existence of projective presentations is trivial in the case we are considering (the modules are even finite as sets), but we shall prove a useful lemma in this connection, which is of particular interest in the case G a $\mathfrak{p}$-group. To this end, let A be a non-projective indecomposable KG-module and choose one coset representative for each irreducible occurring in decomposition of $\frac{A}{\rho.A}$, denoting the set of these by X.

<u>Lemma 2.4.</u> With the above notation and conditions, KG.X = A, and if G is a $\mathfrak{p}$-group, X is a minimal set of KG-generators for A.

<u>Proof.</u> If $\overline{X} = \{x+\rho.A \mid x \in X\}$, $KG.\overline{X} = \frac{A}{\rho.A}$, by complete reducibility. Thus, $KG.X + \rho.A = A$, and since $\rho.A = \mu(A)$, KG.X = A. The minimality of X when G is a $\mathfrak{p}$-group follows from the fact that in this case, $\overline{X}$ is a K-basis for the vector space $\frac{A}{\rho.A}$.

Suppose now that (1) is in fact a minimal KG-projective resolution of K. Applying the functor $.^*$, we obtain an exact sequence:

$$0 \to K^* \xrightarrow{\partial_0^*} P_0^* \xrightarrow{\partial_1^*} P_1^* \to \cdots \to P_{n-1}^* \xrightarrow{\partial_n^*} P_n^* \to \cdots, \quad \text{----}(2)$$

with P_i^* projective (and so injective) for all $i \geqslant 0$. We also have that:

$$\operatorname{Ker} \partial_n^* \cong c_n(G)^*, \ \forall n \geqslant 0.$$

and we define $c_{-n}(G) = c_n(G)^*$, $\forall n \geqslant 0$. (Note that in either case $c_0(G)$ is just K). Since $K^* \cong K$, we can put the resolutions (1) and (2) together to get a complete resolution of K. The above results dualise to give, for example, that $\nu(c_{-n}(G)) \cong \nu(P_n^*)$, $\forall n \geqslant 0$.

Theorem 2.5.

If A is a completely reducible KG-module, then

$$\hat{H}^n(G, A) \cong \mathrm{Hom}_{KG}(c_n(G), A), \ \forall n \in \mathbb{Z}.$$

Proof. We prove the result for $n > 0$, the isomorphism being obvious if $n = 0, -1$, and the rest will follow by duality. By theorem 1.1., $\hat{H}^n(G, A) \cong \mathrm{Ext}_{KG}(c_{n-1}(G), A)$, using induction and the long exact sequence for "Ext". Thus we are interested in the collection of KG-homomorphisms: $c_n(G) \longrightarrow A$, modulo those which can be factored through P_{n-1}. But if f is any KG-map: $P_{n-1} \longrightarrow A$, the composite: $c_n(G) \xrightarrow{inc} P_{n-1} \xrightarrow{f} A$ is zero by Lemma 2.3. This completes the proof.

We conclude this chapter with a few remarks.

2.6. Using Lemma 2.3. and its dual, we have $\forall n \in \mathbb{Z}$,

$$\frac{c_{-n}(G)}{\rho \cdot c_{-n}(G)} \cong \nu(c_n(G)) \cong \nu(P_{n-1}) \cong \frac{P_{n-1}}{\rho \cdot P_{n-1}} \cong \frac{c_{n-1}(G)}{\rho \cdot c_{n-1}(G)}.$$

From this and theorem 2.5., we deduce the Universal Coefficient Theorem:

$$\hat{H}^n(G, K) \cong \hat{H}^{-(n+1)}(G, K), \ \forall n \geqslant 0.$$

2.7. Again with respect to the minimal resolution (1), let $P_n \cong \sum_{\tau=1}^{t} +1_\tau U_\tau$. Then ∂_n cannot vanish on any of these U_τ since $p(\mathrm{Ker}\,\partial_n) = (0)$. Let ∂_n restricted to some fixed U_τ be denoted by f. Consideration of $\mathrm{Im} f \leq P_{n-1}$ shows that F_τ occurs as a composition factor of P_{n-1}. Induction then shows that all the U_τ occurring in (1) must be in the first block, and thus $\forall n \in \mathbb{Z}$, $c_n(G)$ is in the first block.

2.8. In case G is p-nilpotent, with normal p-complement N and S a Sylow p-subgroup, N is in the kernel of all the representations in the first block, so any KS-projective resolution of K can be made into a KG-projective resolution of K simply by inflation. It follows that, $\forall n \in \mathbb{Z}$, $\hat{H}^n(G,K) \cong \hat{H}^n(S,K)$, a result that follows from the exact sequence in [16], §4, Theorem 2.

2.9. To calculate the integral cohomology (at least, as far as number of generators is concerned), apply the long exact sequence of cohomology to the short exact sequence: $0 \rightarrow \mathbb{Z} \xrightarrow{M_p} \mathbb{Z} \rightarrow GF(p) \rightarrow 0$, (where M_p denotes multiplication by p and the other map is just the cokernel of this). It is then evident that if $h_n = \dim_K \dfrac{\hat{H}^n(G,\mathbb{Z})}{p \cdot \hat{H}^n(G,\mathbb{Z})}$

$\forall n \in \mathbb{Z}$, then $\dim_K \hat{H}^n(G,K) = h_n + h_{n+1}$, $\forall n \in \mathbb{Z}$.

This result appears in [25], p.283.

2.10. $\forall n \in \mathbb{Z}$ and for G any p-group, there is a $\mathbb{Z}$-free $\mathbb{Z}G$-module C_n, such that $\frac{C_n}{p.C_n} \cong c_n(G)$ as KG-modules. This follows at once from Corollary 5.2. of [29] using the above results.

2.11. Let $1 \to R \to F \to G \to 1$ be a short exact sequence of groups, with F free of finite rank. Make $\frac{R}{R'R^p}$ into a KG-module C in the usual way. Then it is shown in [24] that $c(C) \cong c_2(G)$. In case G is a p-group and the above free presentation is minimal, a consideration of ranks and dimensions implies that $C \cong c_2(G)$. (Note: On this topic, cf. [8] and [11]).

2.12. Using the characterisation of $\hat{H}^2(G,A)$ as the group of extensions of the abelian group A (with given G-module structure) by the group G (see [6]), the above methods give a very short proof of the Schur-Zassenhaus Theorem: Let G be a finite group with normal subgroup M whose order and index are coprime; then $\exists N \leq G$ such that $MN = G$ and $M \cap N = \{e\}$.

By the usual group-theoretic induction, it is sufficient to prove the theorem in the case M an elementary abelian p-group. If $K = GF(p)$, M is a $K\frac{G}{M}$-module and $\hat{H}^2(G/M, M)$ is zero by Maschke's Theorem (see remark following proof of Theorem 1.1). This completes the proof.

2.13. It is well-known that if E is an extension of K,

$$\rho(EG) \cong E \otimes_K \rho(KG) \text{ (this being the usual tensor product).}$$

It follows that for any KG-module A, $\dfrac{E \otimes_K A}{\mu(E \otimes_K A)} \cong E \otimes_K \dfrac{A}{\mu(A)}$,

and if:

$$0 \longrightarrow B \longrightarrow P \longrightarrow A \longrightarrow 0$$

$$0 \longrightarrow C \longrightarrow Q \longrightarrow E \otimes_K A \longrightarrow 0$$

are minimal projective presentations of A, $E \otimes_K A$

respectively,

$$\frac{Q}{\mu(Q)} \cong \frac{E \otimes_K A}{\mu(E \otimes_K A)} \cong E \otimes_K \frac{A}{\mu(A)} \cong E \otimes_K \frac{P}{\mu(P)}, \qquad \text{by Lemma 2.3.}$$

Thus, $Q \cong E \otimes_K P$ and so $C \cong E \otimes_K B$. Hence $\Omega(E \otimes_K A) \cong E \otimes_K \Omega A$.

We have shown that if A absolutely indecomposable, then so is ΩA, and it follows by induction that for any G and any $n \in \mathbb{Z}$, $c_n(G)$ is absolutely indecomposable.

2.14. The question of periodicity admits an easy treatment in this setting, the results concerned being illuminating examples.

(i) Cyclic Groups. Since cyclic groups are p-nilpotent, it is sufficient (by 2.8) to consider the case G a cyclic p-group and $K = GF(p)$. With this notation, we have: $c_n(G) \cong \begin{cases} K, & \text{if } n \text{ even} \\ \mathfrak{U}, & \text{if } n \text{ odd} . \end{cases}$

Thus, $\forall n \in \mathbb{Z}$, $\dim_K \hat{H}^n(G,K) = 1$.

This result is derived in [18], where dihedral groups are also discussed.

(ii) Generalised Quaternion Groups. Let G be such a group and $Q = c_2(G)$. Then we have: $c_n(G) \cong K, \mathfrak{U}, Q, \mathfrak{U}^*$ according as $n \equiv 0,1,2,3 \pmod 4$.

It follows that $\dim_K \hat{H}^n(G,K) = \begin{cases} 1, & \text{if } n \equiv 0,3 \pmod 4 \\ 2, & \text{if } n \equiv 1,2 \pmod 4 . \end{cases}$

This result is derived in [3], Chapter XII, §7, where a periodic resolution is constructed.

(iii) Periodicity. The following results are proved in [27] and [28], and can be deduced from (i), (ii) (above), Theorem 3.1, Theorem 3.2 and Chapter IV in this context.

(a) If G has a p-periodic resolution, the G has p-periodic cohomology.

(b) If G is p-nilpotent and has p-periodic cohomology, then G has a p-periodic resolution.

(c) G has a p-periodic resolution if and only if the Sylow p-subgroup of G is cyclic or generalised quaternionic.

(d) If G is a group with cyclic Sylow p-subgroup H, then the p-period of the resolution of G is $2[\mathfrak{N}_G(H):\mathcal{C}_G(H)]$; further, if $H \triangleleft G$ the p-periods of the resolution and of the cohomology coincide.

2.15. We end this chapter with an elementary result, a more complicated analogue of which is involved in Theorem 7.3.

Let G and K be as usual with p an odd prime whose α^{th} power is the largest dividing $|G|$, and fix $n \in \mathbb{Z}$. Define $\varepsilon : c_n(G)^{\otimes 2} \longrightarrow c_n(G)^{\otimes 2}$ by $\varepsilon(x \otimes y) = \dfrac{x \otimes y + y \otimes x}{2}$,

and extend by linearity. Writing $A = \operatorname{Ker} \varepsilon$ (antisymmetric tensors) and $S = \operatorname{Im} \varepsilon$ (symmetric tensors), we have $c_n(G)^{\otimes 2} = A \oplus S$, since ε is a KG-idempotent. It is at once apparent that, modulo p^α,

$$\left.\begin{array}{lll} & \dim_K A \equiv 0, \quad \dim_K S \equiv 1 & \text{if } n \text{ even}, \\ \text{and} & \dim_K A \equiv 1, \quad \dim_K S \equiv 0 & \text{if } n \text{ odd}. \end{array}\right\}$$

Thus by the Krull-Schmidt Theorem,

$$\left.\begin{array}{lll} & A \text{ is projective and } c_{2n}(G) \mid S & \text{if } n \text{ even} \\ \text{and} & S \text{ is projective and } c_{2n}(G) \mid A & \text{if } n \text{ odd}. \end{array}\right\}$$

CHAPTER III: RESTRICTION, INDUCTION AND TRANSFER

§1. Restriction and Induction of Cores.

Before going on to consider applications to cohomology, we apply the standard notions of restriction and induction of modules to the cores defined in the previous chapter.

Theorem 3.1.

If $H \leqslant G$, then $c_n(G)_H \sim c_n(H)$.

Proof. This follows immediately from the following three well-known properties of the functor $\cdot_H$:

(i) $\cdot_H$ is exact;

(ii) If P is a projective KG-module, then P_H is a projective KH-module;

(iii) K_H is the trivial representation of H over K.

Theorem 3.2.

If $H \leqslant G$ and char $K \nmid [G : H]$, $c_n(H)^G$ contains a copy of $c_n(G)$ as a direct summand.

Proof. The theorem follows from the following three observations:

(i) $\cdot^G$ is exact;

(ii) P a projective KH-module $\Rightarrow P^G$ a projective KG-module;

(iii) The theorem is true when $n = 0$, the relevant idempotent being $\frac{1}{[G:H]} \sum_{x \in T} x$, where T is a transversaal for the left cosets of H in G.

Note: More detailed investigations into considerations of this kind are made in [14] and [9].

Definition. If A is a KG-module and τ is an automorphism of G, define a new KG-module $A^{\tau} = \{a^{\tau} \mid a \in A\}$ by $x.a^{\tau} = [\tau(x).a]^{\tau}$, for all $x \in G$.

Theorem 3.3.

For any automorphism τ of G, $c_n(G)^{\tau} \cong c_n(G)$.

Proof. (i) The functor $\cdot^{\tau}$ is exact;

(ii) P projective $\Rightarrow$ P^{τ} projective;

(iii) Theorem true for $n = 0$.

We conclude this section by applying Theorem 3.1. to obtain a well-known classical result - see for example the corollary to Proposition 3 in [16].

Theorem 3.4. If $H \leq G$ and B is any KH-module, then for all $n \in \mathbb{Z}$,

$$H^n(H,B) \cong H^n(G,B^G).$$

Proof. First, let A be a KG-module and B a KH-module, and consider the mapping,

$$\kappa: \quad \mathrm{Hom}_{KH}(A_H, B) \longrightarrow \mathrm{Hom}_{KG}(A, B^G),$$

given by $(\kappa f)(a) = \sum_{i=1}^{n} x_i \otimes f(x_i^{-1}.a)$, where $\{x_i \mid 1 \leq i \leq n\}$ is a transversal for the left cosets of H in G. It is straightforward to check that $\mathbf{K}$ commutes with the elements of G and is independent of the transversal.

If now $f \in \mathrm{Hom}_{KG}(A, B^G)$ is such that $f(a) = \sum_{i=1}^{n} x_i \otimes b_i$, define $K'f : A_H \to B$ by $(K'f)(a) = b_1$. If, for convenience, we let $x_1 = e \in G$, it can be shown that $K'f$ commutes with the elements of H and that $\mathbf{K}\cdot\mathbf{K}'$ and $\mathbf{K}'\cdot\mathbf{K}$ are the respective identities. Further, it is clear from the definition that $\mathbf{K}$ is natural in both variables, so we can deduce that : $\mathrm{Ext}_{KH}(A_H, B) \cong \mathrm{Ext}_{KG}(A, B^G)$.

Thus we have $\forall n \in \mathbb{Z}$, $\hat{H}^n(H, B) \cong \mathrm{Ext}_{KH}(c_{n-1}(H), B)$, by the results of Chapter II,

$\cong \mathrm{Ext}_{KH}(c_{n-1}(G)_H, B)$, by 3.1. and the main result of [15],

$\cong \mathrm{Ext}_{KG}(c_{n-1}(G), B^G)$ by the above,

$\cong \hat{H}^n(G, B^G)$.

§2. Restriction and Transfer of Cohomology.

We shall now, as throughout the sequel, make the identification given by the isomorphism of Theorem 2.5.

<u>Definition</u>. Let $H \leq G$ and for all $n \in \mathbb{Z}$, take a split embedding

$$i_n : c_n(H) \to c_n(G), \text{ ---------(1)};$$

then for any element $\chi \in \hat{H}^n(G, K)$, define the restriction of χ from G to H, $r(\chi) = \chi \cdot i_n \in \hat{H}^n(H, K)$.

Notes. 1. We have defined the mapping r only for coefficients in K, but a generalisation to arbitrary coefficients can be effected in the usual way.

2. The question of uniqueness poses some difficult problems, some of which will be tackled later, but for the moment, let us merely remark that if $\{i'_n \mid n \in \mathbb{Z}\}$ is another set of split embeddings, the Krull-Schmidt theorem guarantees the existence of a set $\{\phi_n : c_n(G) \to c_n(G) \mid n \in \mathbb{Z}\}$ of KH-isomorphisms such that $i'_n = \phi_n \circ i_n$, for all $n \in \mathbb{Z}$.

<u>Definition</u>. Let $H \leq G$ and for all $n \in \mathbb{Z}$, take a splitting of (1),

$$p_n : c_n(G) \longrightarrow c_n(H), \text{-----} (2)$$

Further, let $x_1, \cdots x_s$ be a right transversal for H in G; then for any element $\psi \in \hat{H}^n(H,K)$, define the transfer t of ψ from H to G, $t(\psi)(a) = \psi \circ p_n\left(\sum_{r=1}^{s} x_r . a\right)$ for all $a \in c_n(G)$. If we let $\sum_{r=1}^{s} x_r = \sigma$ and $\mu : c_n(G) \longrightarrow c_n(G)$ be given by $\mu(a) = \sigma . a$, for all $a \in c_n(G)$, then $t(\psi) = \psi \circ p_n \circ \mu \in \hat{H}^n(G,K)$.

The necessary checking (e.g. that $t(\psi)$ is in fact a KG-homomorphism) is straightforward, and a comparable definition for general coefficients can be made.

<u>Lemma 3.5</u>. With the above notation, let $f \in \mathrm{Hom}_{KH}(c_n(G), K)$ some $n \in \mathbb{Z}$; then $f \circ i_n = 0 \Rightarrow f \circ \mu = 0$.

<u>Proof</u>. Suppose $f \circ i_n = 0$ and let $j : c_n(G) \longrightarrow P_{n-1}$, where P_{n-1} is projective over KG, be a minimal KG-embedding. Then $\exists \varepsilon \in \mathrm{Hom}_{KH}(P_{n-1}, K)$ such that $f = \varepsilon \circ j$. Now if

$\mu': P_{n-1} \to P_{n-1}$ is given by $\mu'(a) = \sigma.a$ for all $a \in P_{n-1}$, $j \circ \mu = \mu' \circ j$, since j is a KG-homomorphism. Thus, $f_0 \mu = \varepsilon_0 j_0 \mu = \varepsilon_0 \mu'_0 j$, i.e. $f_0\mu$ can be factored through P_{n-1} by the KG-map $\varepsilon_0 \mu'$. But $\operatorname{Im} j \leq \rho(KG).P_{n-1} \leq \operatorname{Ker}(\varepsilon_0 \mu')$, so $f_0 \mu = 0$ as required.

We use this lemma to prove the main result of [5].

Theorem 3.6.

Let $\mu_s : \hat{H}^n(G,K) \to \hat{H}^n(G,K)$ denote multiplication by $s \in \mathbb{Z}$; then with the above notation, $t_0 r = \mu_s$.

Proof. Let $\chi \in H^n(G,K)$ so that $\chi_0 \mu = \mu_s$. Thus it is sufficient to prove that $\chi_0 \mu = t_0 r(\chi) = \chi_0 i_{n0} p_{n0} \mu$, i.e. $(\chi - \chi_0 i_{n0} p_n)_0 \mu = 0$, and this follows immediately from Lemma 3.5.

§3. Mackey's Formula

This result is well known for modules (see [9]) and occurs in its cohomological form on page 77 of [23] We merely state the result here (again referring to [23]) since its derivation in the present context involves difficulties with uniqueness, and since we shall only be interested in one of its consequences. First, let $H \leq G$ and $i_n : c_n(H) \to c_n(G)$, as in (1). Then $\forall x \in G$, $x.\operatorname{Im} i_n$ is a copy of $c_n(H^x)$, $\forall n \in \mathbb{Z}$, and it has a KH^x-complement in $c_n(G)$, (cf. Theorem 3.3). Thus, given $\chi \in \hat{H}^n(H,K)$, we can define $\chi^x \in \hat{H}^n(H^x, K)$ by:

$$\chi^x(a) = \chi(x^{-1} a), \quad \forall a \in x.\operatorname{Im} i_n.$$

Further, for any such $H \leqslant G$, with r and t denoting restriction and transfer between G and H, write $r(\chi) = \chi_H$ and $t(\psi) = \psi^G$. Now let D and H be subgroups of G, with X a set of representatives for the double cosets DxH in G, then $\forall n \in \mathbb{Z}$ and $\forall \chi \in \hat{H}^n(D, K)$,

$$(\chi^G)_H = \sum_{x \in X} \left[(\chi^x)_{D^x \cap H} \right]^H, \quad \text{-----------(3)}$$

This is the version of Mackey's Formula which we shall now use to obtain an analogue of the main result of [10].

<u>Definition</u>. Let H be a fixed sylow p-subgroup of the finite group G, and $K = GF(p)$. If $D \leqslant G$ denote by t_D the image in $\hat{H}^n(G,K)$ of the transfer from D to G. Write $\hat{H}^n(G,K)' = \sum_{D<H} t_D$.

Note that $t_H = \hat{H}^n(G,K)$, (by Theorem 3.6).

<u>Theorem 3.7</u>.

Let H be a Sylow p-subgroup of G, $N = N_G(H)$ and $\chi \in \hat{H}^n(H,K)$; then $(\chi^G)_H - (\chi^N)_H \in \hat{H}^n(H,K)'$. Further,

$$\frac{\hat{H}^n(G,K)}{\hat{H}^n(G,K)'} \cong \frac{\hat{H}^n(N,K)}{\hat{H}^n(N,K)'}$$

<u>Proof</u>. In (3), take $D = H =$ the H of the theorem, and observe that if X' is a set of left coset representatives of H in N, we can choose $X \supseteq X'$. Thus we have

$$(\chi^G)_H - (\chi^N)_H = \sum_{x \in X \setminus X'} \left[(\chi^x)_{H^x \cap H} \right]^H.$$

Since, $\forall x \in X \setminus X'$, $H^x \cap H < H$, the first part of the theorem is proved. For the remainder, the crucial step is to show that:

$$\chi \in \hat{H}^n(N,K) \text{ and } \chi^G = 0 \Rightarrow \chi \in \hat{H}^n(N,K)'.$$

Suppose χ satisfies these hypotheses and let $\psi \in \hat{H}^n(H,K)$ be such that $\psi^N = \chi$. (ψ exists by Theorem 3.6). Of course, $\psi^G = (\psi^N)^G = \chi^G = 0$. So, by the first part of the theorem applied to ψ, $-(\psi^N)_H \in \hat{H}^n(H,K)'$.

Again by Theorem 3.6.,

$$\hat{H}^n(N,K)' \ni \left[-(\psi^N)_H\right]^N = -[N:H].\chi.$$

$\therefore\ \chi \in \hat{H}^n(N,K)'$, as required.

Now denote by α the composite:

$$\hat{H}^n(N,K) \xrightarrow{\text{transfer}} \hat{H}^n(G,K) \xrightarrow{\text{nat.}} \frac{\hat{H}^n(G,K)}{\hat{H}^n(G,K)'} .$$

Then we have to prove that $\operatorname{Ker}\alpha = \hat{H}^n(N,K)'$.

By transitivity of transfer, $\hat{H}^n(N,K)' \leq \operatorname{Ker}\alpha$. Now let $\chi \in \operatorname{Ker}\alpha$ so that we can write $\chi^G = \sum_{D<H} \chi(D)^G$, where $\chi(D) \in \hat{H}^n(D,K)$ for each D. Again by transitivity,

$$\Big[\sum_{D<H} \chi(D)^N - \chi\Big]^G = 0, \quad \text{so that}$$

$$\sum_{D<H} \chi(D)^N - \chi \in \hat{H}^n(N,K)', \quad \text{by the above.}$$

Thus, $\chi \in \hat{H}^n(N,K)'$, as required.

CHAPTER IV: THE CORES FOR THE GROUP (p,p)

1. Preliminaries

In this chapter we take $K = GF(p)$, $G = Gp\{a, b; a^p = b^p = e, ab = ba\}$ and calculate the modules $\{c_n(G) \mid n \in \mathbb{Z}\}$. We then deduce a classification result in the case $p = 2$ and finally give a number of applications of interest from the points of view of cohomology and representation theory. We need one computational lemma.

Lemma 4.1 Let G be a p-group and K a field of characteristic p; then if $S = \{a_1, \ldots a_s\}$ and $T = \{b_1, \ldots b_t\}$ are subsets of the KG-module A having the properties (i) $KG.S+K.T = A$,

(ii) $s.|G|+t = \dim_K A$,

then $KG.S = F$ is free, and the cosets $\{b_j+F \mid 1 \leqslant j \leqslant t\}$ form a K-basis for $\frac{A}{F}$. If in addition we have that

$$\sigma.b_j \equiv 0 (\text{mod } F),\ 1 \leqslant j \leqslant t, \text{ where } \sigma = \sum_{g \in G} g, \text{ then } \frac{A}{F} \cong c(A).$$

Proof Since KG.S is a quotient of the direct sum of s copies of KG, $\dim_K KG.S \leqslant s.|G|$. The reverse inequality being supplied by (i) and (ii), we have that $\dim_K KG.S = s.|G|$, so that $KG.S = F$ is free. (i) and (ii) now imply that $K.T$ is a K-complement for $KG.S$ in A, and the second part of the lemma follows. Thus, $\frac{A}{F} \cong c(A) \dotplus F'$, with F' free, but the additional hypothesis shows that $\frac{A}{F}$ can have no free direct summand, and this completes the proof.

2. Even Powers

Define for all $n \geqslant 0$ the KG-module Q_n (of K-dimension np^2+1) as follows.

A K-basis for Q_n is: $B_n = \{x_{ij}^{(r)} \mid 1 \leqslant i,j \leqslant p, 1 \leqslant r \leqslant n\} \cup \{x\}$.

If we write $z^{(r)} = \sum_{j=1}^{p} x_{pj}^{(r)}$, the action of G is defined as follows, (reducing subscripts modulo p to one of 1,2,...p):

$$
\begin{aligned}
b.x_{ij}^{(r)} &= x_{i,j+1}^{(r)}, \quad \text{all } i,j,r, \\
b.x &= x\,; \\
a.x_{1j}^{(r)} &= x_{1j}^{(r)} + \sum_{\rho=1}^{r-1} z^{(\rho)}, \quad \text{all } j,r\,; \\
a.x_{ij}^{(r)} &= x_{i+1,j}^{(r)}, \text{all } j,r \text{ and } 2 \leq i \leq p-1, \\
a.x_{pj}^{(r)} &= x_{1,j+1}^{(r)} - \sum_{i=1}^{p} x_{ij}^{(r)}, \text{ all } j,r, \\
a.x &= x + \sum_{\rho=1}^{n} z^{(\rho)}.
\end{aligned}
$$

Then we claim that $Q_n = c(\mathfrak{U}^{\otimes 2n})$, $\forall n \in \mathbb{Z}$, (defining Q_{-n} as Q_n^* for all $n \in \mathbb{Z}^+$). By 1.4 and induction, it will be sufficient to prove that:

(i) $Q_1 \sim \mathfrak{U}^{\otimes 2}$,

(ii) $Q_{n+1} \sim Q_1 \otimes Q_n$, $n \geqslant 1$,

(iii) $\forall n \geqslant 1$, $\sigma = \sum_{g \in G} g$ annihilates Q_n.

(i) A K-basis for $\mathfrak{U}$ is:

$$\{(a^{i-1} - a^i)b^j = y_{ij} \mid 1 \leq i \leq p-1, 1 \leq j \leq p\} \cup \{(b^{j-1} - b^j) = y_{pj} \mid 1 \leq j \leq p-1\}.$$

Let

$$S_1 = \{y_{ij} \otimes y_{11} \mid 2 \leq i \leq p-1,\ 1 \leq j \leq p\},$$

$$S_2 = \{y_{pi} \otimes y_{11} \mid 1 \leq i \leq p-1\},$$

$$S_3 = \{y_{pi} \otimes y_{p1} \mid 2 \leq i \leq p-1\} \text{ and } S = \bigcup_{k=1}^{3} S_k;$$

$$T_1 = \{b^j.(y_{11} \otimes y_{11}) \mid 1 \leq j \leq p\},\ T_2 = \{b^j.(y_{i1} \otimes y_{p1}) \mid 1 \leq j \leq p, 1 \leq i \leq p-1\},$$

$$T_3 = \{y_{p1} \otimes y_{p1}\} \text{ and } T = \bigcup_{k=1}^{3} T_k.$$

It can then be checked that S and T satisfy the conditions (i) and (ii) of Lemma 4.1, and that

$$\frac{\mathfrak{U} \otimes \mathfrak{U}}{KG.S} \cong Q_1.$$

(ii) In $Q_1 \otimes Q_n$, omitting the superscripts from the elements of Q_1, let

$$S_1 = \{x_{2j} \otimes x_{11}^{(r)} \mid 1 \leq r \leq n,\ 1 \leq j \leq p-1\},$$

$$S_2 = \{(x_{11} - x) \otimes x_{11}^{(r)} \mid 1 \leq r \leq n\},$$

$$S_3 = \{x_{ij} \otimes x_{21}^{(r)} \mid 1 \leq r \leq n,\ 2 \leq i \leq p-1,\ 1 \leq j \leq p\},$$

$$S_4 = \{x_{pj} \otimes x_{21}^{(r)} \mid 1 \leq r \leq n,\ 1 \leq j \leq p-1\},$$

$$S_5 = \{(x_{11} - x) \otimes x_{21}^{(r)} \mid 1 \leq r \leq n\} \text{ and } S = \bigcup_{k=1}^{5} S_k;$$

$$T = \{x \otimes b \mid b \in B_n\} \cup \{b \otimes x \mid b \in B_1\}.$$

A straightforward computation then shows that S and T satisfy the conditions of Lemma 4.1, and using the relations:

$$\left.\begin{aligned} x_{ij} \otimes x_{k1}^{(r)} &\equiv 0 \pmod{KG.S},\ 2 \leq i \leq p, \\ x_{1j} \otimes x_{k1}^{(r)} &\equiv x \otimes x_{k1}^{(r)} \pmod{KG.S}, \end{aligned}\right\} \text{ for all } j,k,l,r,$$

it is readily shown that $\dfrac{Q_1 \otimes Q_n}{KG.S} \cong Q_{n+1}$.

(iii) Write $\sigma_a = \sum_{i=0}^{p-1} a^i$ and $\sigma_b = \sum_{j=0}^{p-1} b^j$, so that $\sigma = \sigma_a \cdot \sigma_b$. To show that σ annihilates B_n, note first that $\sigma_b . x = 0$. Further, for all i,j,r, $\sigma_b . x_{ij}^{(r)} = \sum_{j=1}^{p} x_{ij}^{(r)} = \alpha_i^{(r)}$, say. Now if $2 \leq i \leq p$, it is clear that $\sigma_a . \alpha_i^{(r)} = 0$, for all r. Finally, for all r,

$$a . \alpha_1^{(r)} = \sum_{j=1}^{p} a . x_{1j}^{(r)} = \sum_{j=1}^{p} x_{1j}^{(r)} + \sum_{j=1}^{p} \left\{ \sum_{\rho=1}^{r-1} z^{(\rho)} \right\}$$

$$= \alpha_1^{(r)} + 0, \text{ and the result follows.}$$

3. Odd Powers

In this case, we merely describe the modules, observing that the methods of §2 apply equally well here. For $n \geq 1$, let V_n be the KG-module (of K-dimension np^2-1) with K-basis:

$\{x_{ij}^{(r)} \mid 1 \leq i,j \leq p,\ 1 \leq r \leq n\} \setminus \{x_{pp}^{(n)}\}$ and G-action as follows (taking subscripts modulo p):

$$b.x_{p,p-1}^{(n)} = -\sum_{j=1}^{p-1} x_{pj}^{(n)},$$

$$b.x_{ij}^{(r)} = x_{i,j+1}^{(r)}, \qquad \text{in all other cases.}$$

Writing $z^{(r)} = \sum_{j=1}^{p} x_{pj}^{(r)}$, for $1 \leq r \leq n-1$, we have for all j,r:

$$a.x_{ij}^{(r)} = x_{i+1,j}^{(r)},\ 1 \leq i \leq p-2,$$

$$a.x_{p-1,j}^{(r)} = -\sum_{i=1}^{p-1} x_{ij}^{(r)} + z^{(r-1)},$$

$$a.x_{pj}^{(r)} = x_{pj}^{(r)} + \sum_{\rho=1}^{r}\left[x_{p-1,j}^{(\rho)} - x_{p-1,j-1}^{(\rho)}\right].$$

Then we have that $V_n \cong c(\mathcal{U}^{\bullet(2n-1)}),\ \forall n \in \mathbb{Z}^+$.
As in the case of the even powers, $c_n(G)$ for $n \leqslant 0$ is obtained by taking contragredients.

4. The Four-Group

In the case p = 2, the modules described in the preceding two sections have a common form, and this we now describe. For $n \geqslant 0$, define the KG-module W_n to have K-basis $\{x_i, x_i' \mid 1 \leqslant i \leqslant n\} \cup \{x\}$ and G-action given by: $b.x_i = x_i'$, $1 \leqslant i \leqslant n$, $b.x = x$,

$$a.x_1 = x_1,\ ax_i = x_{i-1} + x_i + x_{i-1}',\ 2 \leqslant i \leqslant n,\ ax = x_n + x + x_n',$$

the other defining relations being obtained using the relations in G. Then it can be shown that in this case, $Q_n \cong W_{2n}$ and $V_n \cong W_{2n-1}$, for all $n \geqslant 0$, so that $c_n(G) \cong W_n$, for all $n \geqslant 0$. The other cores are obtained as usual by duality. Now $\dim_K W_n = 2n+1$, and the interesting thing about the set $\{W_n, W_n^* \mid n \geqslant 0\}$ is that it forms an exhaustive list of odd-dimensional indecomposable KG-modules. (This is shown in [1] for K algebraically closed.) Before proving this, we remark that the W_n are absolutely indecomposable (by 2.13) and so we can classify the odd-dimensional, indecomposable representations of the four-group over any field of characteristic 2, (since the results of this section generalise easily to any such field). Let K now be an arbitrary field of characteristic 2.

<u>Lemma 4.2</u> If A is any non-projective indecomposable KG-module, then either $A \cong K$ or $\mathfrak{U}.A = \nu(A)$.

<u>Proof</u> Suppose first that $\nu(A) \nleq \mathfrak{U}.A$. Then since $\nu(A)$ is just the maximal G-trivial submodule of A, we can find a G-trivial element - call it c - in $A \setminus \mathfrak{U}A$. Letting $\frac{M}{\mathfrak{U}.A}$ denote a K-complement for $\frac{K.\{c\}+\mathfrak{U}.A}{\mathfrak{U}.A}$ in $\frac{A}{\mathfrak{U}A}$, M being a KG-submodule of A, we have $A = M \oplus K\{c\}$, so either $M = (0)$ and $A = K$, or we have a contradiction. Assume now that $A \ncong K$. $\mathfrak{U}^2$ consists of scalar multiples of $\sigma = e+a+b+ab, \varepsilon\, KG$, and $\sigma.a = 0$, $\forall a \in A$ (by 1.4). Thus the chain $A \geqslant U.A \geqslant U^2.A = (0)$ has G-trivial factors and hence $\mathfrak{U}.A \leqslant \nu(A)$.

<u>Theorem 4.3</u> Let A be any indecomposable KG-module of odd dimension over K; then $\exists\, n \in \mathbb{Z}$ such that $A \cong \Omega^n K$.

<u>Proof</u> Assume $A \ncong K$ to avoid triviality. By taking a projective presentation and projective embedding of A, we have by 2.3 that,

$$\dim_K A + \dim_K \Omega A = 4.\dim_K \frac{A}{\mathfrak{U}.A},$$

and $\dim_K A + \dim_K \Omega^{-1} A = 4.\dim_K \mathfrak{U}.A$, (using 4.2).

Adding these, we obtain: $\dim_K \Omega A + \dim_K \Omega^{-1} A = 2 \dim_K A$.
Since $\dim_K A$ is odd, $\dim_K \Omega A \neq \dim_K \Omega^{-1} A$ (using the above equations), so we have $\dim_K \Omega A < \dim_K A$, say, the other case being dual. We now claim that $\dim_K \Omega^{r+1} A < \dim_K \Omega^r A$ for all $r \in \mathbb{Z}^+$, provided that no $\Omega^i A$ is isomorphic to K, $1 \leqslant i \leqslant r$. The above method applied to $\Omega^r A$ gives: $\dim_K \Omega^{r+1} A + \dim_K \Omega^{r-1} A = 2 \dim_K \Omega^r A$, under the given proviso. A straightforward induction establishes the claim. Since A is finite-dimensional over K, $\exists\, n \in \mathbb{Z}^+$ such that $\Omega^n A \cong K$, so that $A \cong \Omega^{-n} K$. This n is negative in the dual case, viz. $\dim_K \Omega^{-1} A < \dim_K A$.

<u>Remarks, 4.4</u> Applying the above method in the case $\dim_K A$ even merely yields that $\dim_K \mathfrak{U}.A = \dim_K \frac{A}{\mathfrak{U}.A}$.

<u>4.5</u> For the group (p,p) with p odd neither 4.2 nor 4.3 is valid; indeed any attempt to classify the indecomposables in this case would be abortive, as is shown in [22].

5. General Remarks

4.6 The following properties of the Q's and V's vastly simplified the inductions involved in their computation. In Q_n (V_{n+1}) let the KG-submodules generated over K by $\{x_{ij}^{(\rho)} \mid 1 \leqslant i, j \leqslant p, 1 \leqslant \rho \leqslant r\}$ be denoted by $Q_n^{(r)}$ $(V_{n+1}^{(r)})$, $1 \leqslant r \leqslant n$. Then, with the convention $Q_n^{(0)} = (0) = V_{n+1}^{(0)}$, it can be shown that, for all $0 \leqslant r < s \leqslant n$:

$$\frac{Q_n^{(s)}}{Q_n^{(r)}} \cong Q_{s-r}^{(s-r)} \quad , \quad \frac{V_{n+1}^{(s)}}{V_{n+1}^{(r)}} \cong V_{s-r+1}^{(s-r)} ,$$

$$\frac{Q_n}{Q_n^{(r)}} \cong Q_{n-r} \quad , \quad \frac{V_{n+1}}{V_{n+1}^{(r)}} \cong V_{n-r+1} .$$

Essentially, these relations stem from the equivalences:

$$Q_1 \otimes Q_r^{(r)} \sim Q_r^{(r)} \quad , \quad Q_1 \otimes V_{r+1}^{(r)} \sim V_{r+1}^{(r)} .$$

4.7 Using 1.3, the multiplication table of the Q's and V's is given by:

$$\Omega^n K \otimes \Omega^m K \sim \Omega^{(m+n)} K , \ \forall m,n \in \mathbb{Z}.$$

4.8 Write $Q_n = W_{2n}$, $V_n = W_{2n-1}$, $n \geqslant 1$, $W_o = K$ and $W_{-n} = W_n^*$ for $n \in \mathbb{Z}^+$. Then by the long exact sequence of cohomology, we have for any prime p,

$$\dim_K \hat{H}^m(G, W_n) = \begin{cases} m-n+1 & \text{if } m \geqslant n, \\ n-m & \text{if } m < n. \end{cases}$$

4.9 Fixing $n > 0$, let A and B be the matrices corresponding to the action of a and b respectively on Q_n afforded by the basis B_n, and regard these as matrices over $\mathbb{Z}$, (the only entries are 0, ± 1). Then it is easy to show (by following G-orbits) that $A^p = B^p = I$, the identity matrix, and $AB = BA$. This shows that there exist indecomposable $\mathbb{Z}$-free $\mathbb{Z}G$-modules $\hat{Q}_n$, $n \geqslant 1$, such that $\frac{\hat{Q}_n}{p\cdot\hat{Q}_n} \cong Q_n$ for all $n \geqslant 1$. The same remarks apply to the case n negative or zero, and to the V_n. The $\hat{Q}_n, \hat{V}_n$ are the modules used implicitly in [13] to establish the existence of infinitely many non-isomorphic, indecomposable, $\mathbb{Z}$-free $\mathbb{Z}G$-modules. In a later article [26], an infinite collection of such modules (apparently quite different from the $\hat{Q}_n, \hat{V}_n$) is described, and the paucity of such modules invoked as partial motivation.

CHAPTER V: NORMAL SYLOW p-SUBGROUP

§1. The Cores

Let H be a Sylow p-subgroup of G, $H \triangleleft G$, let $\boldsymbol{\rho} = \boldsymbol{\rho}(KG)$ and $\mathfrak{U} = \boldsymbol{\rho}(KH)$. Let $K = F_1, \ldots F_t$ be the irreducibles of KG; then since $\sigma_p(G) = H$, these F's are just a list of irreducible KM-modules, where M is a complement for H in G (which exists by Schur-Zassenhaus). Further, we know that $\boldsymbol{\rho} = \mathfrak{U}^G$, so that if $U_1, \ldots U_t$ are the principal indecomposables of KG (corresponding in the obvious way to the F's), we can deduce from the dimensions that for $1 \leq \tau \leq t$, $U_1 \otimes F_\tau \cong U_\tau$. Let the irreducibles in the first block be $F_1, \ldots F_s$.

Lemma 5.1. If A is any KG-module, there exists an onto homomorphism,

$$\theta : \quad U_1 \otimes \frac{A}{\rho . A} \longrightarrow A$$

such that c(Ker θ) = (0), i.e. θ is a minimal projective presentation of A.

Proof. We know from Lemma 2.3 that there is a projective presentation:

$$\emptyset : P \longrightarrow A,$$

of A such that c(Ker $\emptyset$) = (0), and that $\frac{P}{\boldsymbol{\rho} \cdot P} \cong \frac{A}{\boldsymbol{\rho} \cdot A}$. Now if we let $Q = U_1 \otimes \frac{A}{\boldsymbol{\rho} \cdot A}$, it follows from the above remarks that $\frac{Q}{\boldsymbol{\rho} \cdot Q} \cong \frac{A}{\boldsymbol{\rho} \cdot A}$, and thus $P \cong Q$, which completes the proof.

Theorem 5.2.

$$\forall n \in \mathbb{Z},\ \dim_K c_n(G) = \dim_K c_n(H).$$

Proof. We prove the stronger result that if A is a non-principal indecomposable KG-module, then

A_H indecomposable $\Rightarrow$ $(\Omega A)_H$ indecomposable,

and the theorem will follow by induction. Let A have the above properties, and

$$0 \longrightarrow \Omega A \longrightarrow U_1 \otimes \frac{A}{\rho . A} \longrightarrow A \longrightarrow 0$$

be the minimal projective presentation of 5.1. If $\alpha = \dim_K \dfrac{A}{\rho . A}$, we have, on restricting this to H:

$$0 \longrightarrow (\Omega A)_H \longrightarrow \alpha . KH \longrightarrow A_H \longrightarrow 0, \quad \text{------}(1),$$

since $\dim_K U_1 = |H|$. If $\beta = \dim_K \dfrac{A_H}{\mathfrak{U} . A_H}$ we also have (by 2.3):

$$0 \longrightarrow \Omega(A_H) \longrightarrow \beta . KH \longrightarrow A_H \longrightarrow 0, \quad \text{------}(2).$$

By 1.2, we have $\Omega(A_H) \,\big|\, (\Omega A)_H$, so that $\dim_K \Omega(A_H) \leq \dim_K (\Omega A)_H$, and thus that $\beta \leq \alpha$. On the other hand, every maximal KG-submodule of A is an intersection of maximal KH-submodules of A_H (since $H = \sigma_p(G)$), and so by 2.2, $\mathfrak{U} . A_H \leq (\rho . A)_H$, so that $\beta \geq \alpha$ (and $\mathfrak{U} . A_H = (\rho . A)_H$). Thus, from (1) and (2), $(\Omega A)_H \cong \Omega(A_H)$, and is thus indecomposable.

Corollary 5.3. $\forall n \in \mathbb{Z},\ \dim_K \dfrac{c_n(G)}{\rho . c_n(G)} = \dim_K \dfrac{c_n(H)}{\mathfrak{U} . c_n(H)}$.

Proof. This is just the assertion $\alpha = \beta$ in the proof of the theorem.

§2. A Useful Isomorphism

It is well-known that if $\mathfrak{V}$ is the augmentation ideal of $\mathbb{Z}G$, the integral group-ring, then $\frac{\mathfrak{V}}{\mathfrak{V}^2} \cong \frac{G}{G^1}$ as groups (see [24]). For $K = GF(p)$, this becomes (with above notation), $\frac{\mathfrak{U}}{\mathfrak{U}^2} \cong \frac{H}{H' \cdot H^p}$, as K-spaces, (see [17]). This remains true for any finite group G, taking $\mathfrak{U}$ to be the augmentation ideal of KG. The purpose of this section is to make $\frac{\mathfrak{U}}{\mathfrak{U}^2}$ and $\frac{H}{H' \cdot H^p}$ into KG-modules in a natural way and to define a KG-isomorphism between them. To do this, we first need to express $\frac{\mathfrak{U}}{\mathfrak{U}^2}$ in a slightly different form, and to this end, set $\varepsilon = \frac{1}{|M|}\sum_{x \in M} x$, so that $U_1 = KG.\varepsilon$. Regarding KH as a subset of KG in the obvious way, we have in fact that $KG.\varepsilon = KH.\varepsilon$. Then $\rho.(KG.\varepsilon) = \mathfrak{U}.\varepsilon$, $\rho^2.(KG.\varepsilon) = \mathfrak{U}^2.\varepsilon$, (using 6.3 with $n = 1$). Thus we have a canonical isomorphism between $\frac{\mathfrak{U}.\varepsilon}{\mathfrak{U}^2.\varepsilon}$ and $\frac{\rho.U_1}{\rho^2.U_1}$, and $\frac{\rho.U_1}{\rho^2.U_1}$ is a KG-module. Make $\frac{H}{H'H^p}$ into a KM-module by defining, $\forall x \in M$, $x.(g.H'H^p) = g^x.H'H^p$, for all $g \in H$. and inflate to a KG-module.

Theorem 5.4.

$\frac{\rho.U_1}{\rho^2.U_1}$ and $\frac{H}{H'H^p}$ are isomorphic as KG-modules, (in accordance with the above definitions and notation).

Proof. Let $g_1, \ldots g_r$ be a minimal set of generators for H; then by the Burnside Basis Theorem, $g_1 H' H^p, \ldots g_r H' H^p$ are a basis for $\frac{H}{H' H^p}$ as a vector space over K, and $(e-g_1)\varepsilon + \mathfrak{U}^2\varepsilon, \ldots (e-g_r) + \mathfrak{U}^2\varepsilon$ are a K-basis for $\frac{\mathfrak{U}\varepsilon}{\mathfrak{U}^2\varepsilon}$. Thus let $\phi : \frac{H}{H'H^p} \longrightarrow \frac{\mathfrak{U}\varepsilon}{\mathfrak{U}^2\varepsilon}$ be given by: $\phi(g_i H'H^p) = (e-g_i)\varepsilon + \mathfrak{U}^2\varepsilon, 1 \leq i \leq r,$ and extend by linearity to an isomorphism of K-spaces. Now both modules are trivial over H, so it is sufficient to prove that ϕ commutes with the elements of M. Let $x \in M$; then $\phi[x.(g_i H'H^p)] = \phi[\prod_{j=1}^{r} g_j^{\lambda_j}.\ H'H^p]$, where

$$g_i^x = \Big(\prod_{j=1}^{r} g_j^{\lambda_j}\Big).h,\ h \in H'H^p, = \sum_{j=1}^{r} \lambda_j (e-g_j)\varepsilon + \mathfrak{U}^2\varepsilon, \text{---------} (3).$$

On the other hand,

$$\begin{aligned} x.\phi(g_i H'H^p) &= (x - xg_i)\varepsilon + \mathfrak{U}^2\varepsilon = (e-g_i^x)x\varepsilon + \mathfrak{U}^2\varepsilon \\ &= \Big(e - \Big[\prod_{j=1}^{r} g_j^{\lambda_j}\Big].h\Big)\varepsilon + \mathfrak{U}^2\varepsilon, \text{ since } x.\varepsilon = \varepsilon, \\ &= \Big(e - \prod_{j=1}^{r} g_j^{\lambda_j}\Big)\varepsilon + (e-h)\varepsilon + \mathfrak{U}^2\varepsilon \\ &= \Big(e - \prod_{j=1}^{r} g_j^{\lambda_j}\Big)\varepsilon + \mathfrak{U}^2\varepsilon, \text{ since } (e-h) \in \mathfrak{U}^2, \text{----} (4). \end{aligned}$$

It is well known that if G is any group and U is its augmentation ideal (over any commutative ring-with-1) then

$$\forall x_1, \cdots x_r \in G,\ \Big(e - \prod_{i=1}^{r} x_i\Big) \equiv \sum_{i=1}^{r} (e - x_i) \ \big(\mathrm{mod}\ \mathfrak{U}^2\big).$$

(This is proved by induction from the relation: $(e-x)(e-y) = (e-x)+(e-y)-(e-xy)$). Thus the right-hand sides of (3) and (4) are equal and the proof is complete.

§3. Some Consequences

Theorem 5.5. If, with the above notation $\frac{\rho.U_1}{\rho^2.U_1}$ is trivial as a KG-module, then G is p-nilpotent.

Proof. If $\dim_K \frac{\rho.U_1}{\rho^2.U_1} = r$, let $\theta : r.U_1 \longrightarrow \rho.U_1$ be a projective presentation of $\rho.U_1$, in accordance with 5.1. This induces an onto mapping, $\theta' : \rho.(r.U_1) \longrightarrow \rho^2.U_1$. It follows that $\frac{\rho^2.U_1}{\rho^3.U_1}$ is a quotient of $\rho.(r.U_1)$ and hence (since it is completely reducible) of $\frac{\rho.(r.U_1)}{\rho^2.(r.U_1)}$. But this last module is G-trivial by hypothesis, and therefore so is $\frac{\rho^2.U_1}{\rho^3.U_1}$. Repeating this argument a finite number of times (or using induction), we deduce that U_1 has only trivial composition factors. Thus, in this case $s = 1$. The intersection of the annihilators of the irreducibles in the first block is thus G, so that $G = \mathcal{O}_{p'p}(G)$, which implies the existence of a normal complement for H in G, as required.

Corollary 5.6. (Burnside). If τ is a p'-automorphism of H which induces the identity on $\frac{H}{H'H^p}$, then τ is the identity on H.

Proof. Let G be H extended by τ and apply 5.4 and 5.5.

The following is a special case of a result in [30].

Corollary 5.7. With the above notation, let the restriction : $\hat{H}^2(G,\mathbb{Z}) \longrightarrow \hat{H}^2(H,\mathbb{Z})$ be onto; then G is p-nilpotent.

Proof. Since $\hat{H}^1(\overline{G}, \mathbb{Z}) = (0), \forall \overline{G}$, the hypothesis implies by 2.9 that $\dfrac{\rho.U_1}{\rho^2.U_1}$ is G-trivial (using 5.2 and the definition of restriction). The result now follows by 5.5.

CHAPTER VI: RING STRUCTURE

§1. Definition of the Product

Denote by $\hat{H}^*(G,K)$ the singly graded abelian group whose n^{th} component, for all $n \in \mathbb{Z}$, is $\hat{H}^n(G,K)$, and let $H^*(G,K)$ denote the subgroup generated by elements of non-negative degree. Suppose further that embeddings $j_n: c_n(G) \longrightarrow \mathfrak{U}^{\otimes n}$, together with splittings $q_n: \mathfrak{U}^{\otimes n} \longrightarrow c_n(G)$, are given for all $n \geq 0$. Now suppose $\psi \in \hat{H}^m(G,K)$ and

$\chi \in \hat{H}^n(G,K)$, $m,n \geq 0$, and define $\psi \cup \chi$ to be the composite: $c_{m+n}(G) \xrightarrow{j_{m+n}} \mathfrak{U}^{\otimes(m+n)} \xrightarrow{q_m \otimes q_n} c_m(G) \otimes c_n(G) \xrightarrow{\psi \otimes \chi} K$ (identifying $K \otimes K$ with K in the usual way); then $\psi \cup \chi$ is an element of $\hat{H}^{m+n}(G,K)$, and we have defined a product on $H^*(G,K)$. This product structure can be extended to $\hat{H}^*(G,K)$ by duality (take $j_{-n} = q_n^*$, $q_{-n} = j_n^*$, $\forall\, n \geq 0$) and by choosing a fixed embedding $j: K \longrightarrow \mathfrak{U} \otimes \mathfrak{U}$ together with a fixed splitting $q: \mathfrak{U} \otimes \mathfrak{U} \longrightarrow K$. That this product is independent of the j's and q's can be shown directly by means of the lemmas in the following section, but this is unnecessary since the relevant justification is given in [3], Chapter XII, §§4 and 5.

Since it is more convenient to do so here, we show the uniqueness of the restriction homomorphism defined in Chapter III. This is followed by the observation that $\hat{H}^*(G,K)$ is a ring (under " + " and " $\cup$ ") and a proof that restriction is a ring homomorphism. Some examples are then given, and we conclude with a consideration of some reduction theorems.

§2. Preliminary Lemmas.

Lemma 6.1. If $C \cong c(A)$ and $P \cong p(A)$ are direct summands of the KG-module A, then $C \cap P = (0)$.

Proof. Let $C+P = B \leq A$, and let P', C' be complements for C, P (respectively) in A; then $C \oplus P' \cap B = B = P \oplus C' \cap B$. Hence C, P are isomorphic to direct summands of $c(B)$, $p(B)$ respectively, so that we have:

$$\dim_K A = \dim_K C + \dim_K P \leq \dim_K c(B) + \dim_K p(B) = \dim_K B.$$

Thus, $C+P = A$ and hence $C \cap P = (0)$.

Lemma 6.2. Let X and Y be direct summands of the KG-module A, each isomorphic to $c(A)$; then, if M is any maximal KG-submodule of A,

$$X \leq M \Leftrightarrow Y \leq M.$$

Proof. Sufficient by symmetry to prove one implication, so assume that $X \nleq M$. Then:

$$p(A) \cong \frac{A}{X} = \frac{X+M}{X} \cong \frac{M}{X \cap M},$$

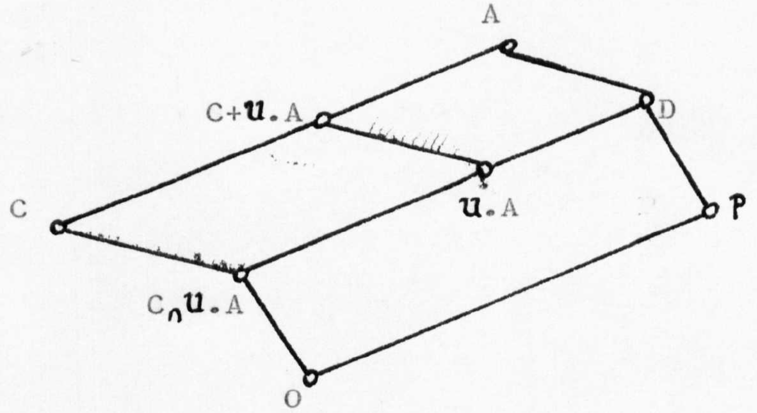
so that $p(A)$ is isomorphic to a direct summand of M. If, however, $Y \leq M$, we deduce that $c(A)$ is isomorphic to a direct summand of M also. Thus, as in Lemma 6.1, we deduce that $M = A$, a contradiction.

Lemma 6.3. Let $H \leq G$, A a KG-module, and let $C \cong c(A_H)$ be a direct summand of A_H; then there exists a complement $P \cong p(A_H)$ for C in A_H such that, for any KG-homomorphism $\chi : A \longrightarrow K$, there are KG-homomorphisms $\alpha, \beta : A \longrightarrow K$ with

$$\operatorname{Ker} \alpha \geq P, \quad \operatorname{Ker} \beta \geq C \text{ and } \chi = \alpha + \beta.$$

[Note: This is true in particular if $H = G$, in which case the proof is trivial, since any complement for C possesses the required property.]

Proof. Let $\mathfrak{U}$ be the augmentation ideal of KG and consider the lattice diagram which follows.



$\dfrac{C+\mathfrak{U}.A}{\mathfrak{U}.A}$ has a complement $\dfrac{D}{\mathfrak{U}.A}$ in $\dfrac{A}{\mathfrak{U}.A}$, since this last module is G-trivial. Further, $\dfrac{D}{C_\cap \mathfrak{U}.A} \cong \dfrac{A}{C} \cong p(A_H)$, so that $C_\cap \mathfrak{U}.A$ is complemented in D, by P say. We claim that P has the property required by the lemma.

Let $\chi : A \longrightarrow K$ be an arbitrary KG-homomorphism and let $\nu : A \longrightarrow \dfrac{A}{\mathfrak{U}.A}$ be the natural map, so that there is a unique $\chi' : \dfrac{A}{\mathfrak{U}.A} \longrightarrow K$ such that $\chi' \circ \nu = \chi$. Regarding

$\frac{A}{\mathbf{U}.A} = \frac{C+U.A}{U.A} \oplus \frac{D}{U.A}$, define α', β': $\frac{A}{\mathbf{U}.A} \longrightarrow K$ by: $\alpha'(x,y) = \chi'(x)$, $b'(x,y) = \chi'(y)$. It is then clear that $\alpha = \alpha'_o \nu$, $\beta = \beta'_o \nu$ are as required.

§3. Uniqueness

Theorem 6.4. Let $H \leq G$ and let $r, r' : \hat{H}^*(G,K) \longrightarrow \hat{H}^*(H,K)$ be the restriction homomorphisms defined with respect to sets of split embeddings $\{i_n : c_n(H) \to c_n(G) \mid n \in \mathbb{Z}\}$, $\{i'_n : c_n(H) \longrightarrow c_n(G) \mid n \in \mathbb{Z}\}$ respectively in accordance with the definition given in Chapter III. Then $\hat{H}^*(H,K)$ has an automorphism $\mathbf{K}$ of degree zero such that $\mathbf{K}_o r = r'$.

Proof. Fix $n \in \mathbb{Z}$, and choose $P_n \leq c_n(G)_H$ with the property described in Lemma 6.3 with respect to Im i_n, and define a mapping, p_n: $c_n(G) \longrightarrow c_n(H)$ by $p_n(x) = 0$ if $x \in P_n$ and $p_n(i_n(y)) = y$, $y \in c_n(H)$. Now define $\mathbf{K}: \hat{H}^*(H,K) \longrightarrow \hat{H}^*(H,K)$ by specifying that its n^{th} component $\mathbf{K}_n$ be given by:

$$\mathbf{K}_n(\psi) = \psi_o p_n \circ i'_n \ , \ \forall \psi \in \hat{H}^n(H,K).$$

Thus we have to show that $\forall \chi \in \hat{H}^n(G,K)$, $\mathbf{K}(\chi_o i_n) = \chi_o i'_n$,

$$\text{i.e. } \chi_o[i_n \circ p_n - 1]_o i'_n = 0,$$

where 1 denotes the appropriate identity mapping. In accordance with Lemma 6.3, let $\chi = \alpha + \beta$, where Im $i_n \leq \mathrm{Ker}\beta$, $P_n \leq \mathrm{Ker}\alpha$, and observe that Im $i'_n \leq \mathrm{Ker}\beta$, by Lemma 6.2. Thus

$$\chi_o[i_n \circ p_n - 1]_o i'_n = (\alpha + \beta)_o[i_n \circ p_n - 1]_o i'_n = \alpha_o[i_n \circ p_n - 1]_o i'_n,$$

and $\mathrm{Im}(i_{n}\circ p_{n}-1) \leq \mathrm{Ker}\ p_{n} = P_{n} \leq \mathrm{Ker}\,\alpha$, establishing that κ has the required form.

Note. The isomorphism κ of Theorem 6.4 preserves products; in fact if "$\cup$" denotes the product in $H^{*}(H,K)$ defined with respect to the j_n, q_n (say) and "$\dot{\cup}$" the product with respect to $j_n \circ (p_n \circ i'_n)$, $(p'_n \circ i_n) \circ q_n$, it can easily be checked that $\kappa(\psi \cup \chi) = \kappa(\psi) \dot{\cup} \kappa(\chi)$, (with usual notation).

§4. Additional Properties of the Product

Theorem 6.5. The restriction homomorphism preserves products.

Proof. Let $H \leq G$ and write $\mathfrak{U}(H)$, $\mathfrak{U}(G)$ for the augmentation ideals of KH, KG (respectively). Suppose further that the following embeddings and their splittings are prescribed.

$$\left.\begin{array}{l} c_n(H) \xrightarrow{i_n} c_n(G), \\ \begin{tikzcd} c_n(G) \arrow[r, "j_n"] & \mathfrak{U}(G)^{\otimes n} \arrow[l, bend left, "q_n"] \end{tikzcd}, \end{array}\right\} \quad n \geq 0,$$

$$\begin{tikzcd} \mathfrak{U}(H) \arrow[r, "i"] & \mathfrak{U}(G) \arrow[l, bend left, "p"] \end{tikzcd}.$$

Now define, for all $n \geq 0$, $j'_n : c_n(H) \longrightarrow \mathfrak{U}(H)^{\otimes n}$ by $j'_n = p^{\otimes n} \circ j_n \circ i_n$, which is easily seen to be a split embedding. For $m,n \geq 0$, let ψ, χ be elements of $\hat{H}^m(G,K)$, $\hat{H}^n(G,K)$, respectively; then we have to prove the equality of the two maps from $c_{m+n}(H)$ into K in the following diagram, (where q'_n is a splitting for j'_n, $\forall n \geq 0$).

$$\begin{tikzcd}
c_{m+n}(H) \arrow[r, "j'_{m+n}"] \arrow[d, "i_{m+n}"'] & \mathfrak{U}(H)^{\otimes(m+n)} \arrow[r, "q'_m \otimes q'_n"] & c_m(H) \otimes c_n(H) \arrow[d, "i_m \otimes i_n"] \\
c_{m+n}(G) \arrow[r, "j_{m+n}"'] & \mathfrak{U}(G)^{\otimes(m+n)} \arrow[r, "q_m \otimes q_n"'] & c_m(G) \otimes c_n(G) \arrow[d, "\psi \otimes \chi"] \\
 & & K
\end{tikzcd}$$

[Note: If $m = 0$, $n = 0$ or both, the theorem is obvious.]

We must show that:

$$\psi \otimes \chi \circ i_m \otimes i_n \circ q'_m \otimes q'_n \circ j'_{m+n} - \psi \otimes \chi \circ q_m \otimes q_n \circ j_{m+n} \circ i_{m+n} = 0,$$

i.e. $\psi \otimes \chi \circ [i_m \otimes i_n \circ q'_m \otimes q'_n \circ p^{\otimes(m+n)} - q_m \otimes q_n] \circ j_{m+n} \circ i_{m+n} = 0,$

i.e. $\mathrm{Im}(j_{m+n} \circ i_{m+n})$ is mapped onto zero by the homomorphism

$\theta = \psi \otimes \chi \circ [i_m \otimes i_n \circ q'_m \otimes q'_n \circ p^{\otimes(m+n)} - q_m \otimes q_n]$ and hence, by Lemma 6.2, that θ sends to zero any direct summand of $\mathfrak{U}(G)^{\otimes(m+n)}$ isomorphic to $c_{m+n}(H)$. Well, $\mathrm{Im}(j_m \otimes j_n \circ i_m \otimes i_n)$ contains such a direct summand, and:

$\theta \circ j_m \otimes j_n \circ i_m \otimes i_n =$

$$\psi \otimes \chi \circ [(i_m \circ q'_m \circ p^{\otimes m} \circ j_m \circ i_m) \otimes (i_n \circ q'_n \circ p^{\otimes n} \circ j_n \circ i_n) - (q_m \circ j_m \circ i_m) \otimes (q_n \circ j_n \circ i_n)]$$

which is zero, as required.

The following result is also derived in [3] Chapter XII, §5.

Theorem 6.6. The product operation defined above makes $\hat{H}^*(G,K)$ into an associative ring-with-1, the identity being the identity mapping: $K \longrightarrow K$, an element of degree zero. Further, if $\phi \in \hat{H}^m(G,K)$ and $\chi \in \hat{H}^n(G,K)$, we have : $\phi \cup \chi = (-1)^{mn} \chi \cup \phi$.

§5. Examples

The necessary verifications in the following examples will not be carried out as they are all easy consequences of the results of 2.14 and Chapter IV.

6.7. The Group of Order 2. Let G be this group and K any field of characteristic 2. If x is an indeterminate and χ is any non-zero element of $\hat{H}^1(G,K)$, then the mapping κ: $K[x,x^{-1}] \longrightarrow \hat{H}^*(G,K)$ defined by $\kappa(x) = \chi$ (and extended in the obvious way) is an isomorphism of graded K-algebras. Note that every (non-zero) homogeneous element in $\hat{H}^*(G,K)$ is invertible.

6.8. The Cyclic Group of Order $p^n \neq 2$. Let the notation correspond to that in 6.7 and χ,ψ be non-zero elements of

$\hat{H}^1(G,K)$, $\hat{H}^2(G,K)$ respectively, where ψ has inverse $\phi \in \hat{H}^{-2}(G,K)$; then the mapping,

$$\nu : K[x, y^2, y^{-2}] \longrightarrow \hat{H}^*(G,K)$$

defined by $\nu(x) = \chi$, $\nu(y^2) = \psi$ is an epimorphism of graded K-algebras with kernel (x^2). Every (non-zero) homogeneous element of even degree is invertible, while the product of any two homogeneous elements of odd degree is zero.

6.9. The Four-Group. Again with similar notation, let $\{\chi, \psi\}$ be a basis for $\hat{H}^1(G,K)$ and let $0 \neq \phi \in \hat{H}^{-1}(G,K)$; then the mapping $\nu : K[x, y, z^{-1}] \longrightarrow \hat{H}^*(G,K)$ defined by $\nu(x) = \chi$, $\nu(y) = \psi$, $\nu(z^{-1}) = \phi$ is an epimorphism of graded K-algebras with kernel $(xz^{-1}, yz^{-1}, z^{-2})$.

§6. Reduction Theorems.

We give analogues of the theorems in [6] for cohomology with coefficients in K, a similar discussion holding for general coefficients. With reference to section 8 of [6], we can regard $C^1_T(G,K)$ (replacing π by G and G by K) as $Hom_K(\mathfrak{U},K)$ with action as described in Chapter I. Then the theorem merely asserts that $\hat{H}^n(G,\mathfrak{U}) \cong \hat{H}^{n+1}(G,K)$, $\forall n \in \mathbb{Z}$, which follows from the long exact sequence of cohomology applied to $0 \to K \to KG \to \mathfrak{U} \to 0$. The iterated reductions are given by $\hat{H}^n(G, \Omega^{-m}K) \cong \hat{H}^{m+n}(G,K)$, which holds $\forall m,n \in \mathbb{Z}$.

In Theorem 10.1 of [6], **Hom**(R,G) is just $\mathrm{Hom}_K(c_2(G),K)$ (by 2.11 above), so that the Cup Product Reduction Theorem becomes, in this case, $\hat{H}^{n+2}(G,K) \cong \hat{H}^n(G, c_{-2}(G))$, which is just the reduction of the previous paragraph iterated twice.

CHAPTER VII: FURTHER MULTIPLICATIVE PROPERTIES

§1. Multiplicative Transfer,

The result obtained in this section bears a superficial resemblance to Proposition 5.1 of [7], but neither seems to be a consequence of the other.

Lemma 7.1. If $|G| = p^{\alpha}$, then $\forall n \in \mathbb{Z}$: $\begin{cases} \dim_K c_{2n}(G) \equiv 1 (\mathrm{mod}\ p^{\alpha}), \\ \dim_K c_{2n+1}(G) \equiv -1 (\mathrm{mod}\ p^{\alpha}). \end{cases}$

Further, if A is a KG-module such that $c(A) \cong c_n(G)$ for some $n \in \mathbb{Z}$ and D is any direct summand of A, then either:

(i) $\dim_K D \equiv 0 (\mathrm{mod}\ p^{\alpha})$, in which case D is projective, or

(ii) $\dim_K D \equiv \dim_K c_n(G) (\mathrm{mod}\ p^{\alpha})$, in which case $c(D) \cong c_n(G)$.

Proof. First part by induction, second part by Krull-Schmidt Theorem.

Note. The lemma remains valid for arbitrary G, taking p^{α} to be the p-part of $|G|$, as is seen by restricting from G to a Sylow p-subgroup.

Lemma 7.2. Let G be a group of order n and $\sigma: G \longrightarrow S_n$ its right regular representation. Further, suppose $x_1, \ldots x_r$ is a given set of symbols, and make $S = \{(x_{k_1}, \ldots x_{k_n}) \mid 1 \leq k_i \leq r \text{ for all } i\}$ into a right G-set by: $(x_{k_1}, \ldots x_{k_n}).g = (x_{k_{1\sigma(g)}}, \ldots x_{k_{n\sigma(g)}})$, $g \in G$. Then,

if $(n,p) = 1$ (for some prime p) and $r \equiv 1 (\mathrm{mod}\ p^{\alpha})$ (some $\alpha \in \mathbb{Z}$), the total number of G-orbits in S is congruent to 1 modulo p^{α}.

Proof Let $H \leqslant G$ and s_H = number of elements of S stablised by H. Then, since H acts transitively on each of its left cosets in G, $S_H = r^{[G:H]}$. If we now let s'_H = the number of elements of S whose stablisers are precisely H, we have: $s_H = \sum_{H \leqslant D \leqslant G} s'_D = s'_H + \sum_{H < D < G} s'_D + s_G$.

It follows that $s'_H + \sum_{H<D<G} s'_D \equiv 0 \pmod{p^\alpha}$.

Hence, by induction on $[G:H]$, $H < G \Rightarrow s'_H \equiv 0 \pmod{p^\alpha}$.

$\therefore$ Total number of G-orbits in S =

$$\sum_{\{e\} \leqslant H \leqslant G} \{\text{number of orbits whose stablisers are } H\}$$

$$= \sum_{\{e\} \leqslant H < G} \frac{s'_H}{[G:H]} + r, \text{ (Orbit-Stabiliser Theorem)},$$

$$\equiv 1 \pmod{p^\alpha}, \text{ since } (p, [G:H]) = 1, \ \forall H \leqslant G.$$

Theorem 7.3 Let G be a group of order np^α, $(p,n) = 1$, with normal Sylow p-subgroup H, having a complement $M = \{g_i \mid 1 \leqslant i \leqslant n\}$ in G, and let $\chi \in \hat{H}^{2l}(H,K)$; then

$$\bigcup_{i=1}^{n} \chi g_i \in \operatorname{Im} res_{G \to H}.$$

Proof In view of 5.2, we can identify $c_k(G)$ and $c_k(H)$ as sets, $\forall k \in \mathbb{Z}$, and can thus regard χ as a KH-homomorphism: $c_{2l}(G) \to K$. Further, define KH-homomorphism

$$\psi = \bigotimes_{i=1}^{n} \chi g_i : c_{2l}(G)^{\otimes n} \longrightarrow K.$$ Let

$\{x_j \mid 1 \leqslant j \leqslant r\}$ be a K-basis for $c_{2l}(G)$ and define the

KG-idempotent,

$$\varepsilon : c_{21}(G)^{\otimes n} \longrightarrow c_{21}(G)^{\otimes n} \text{ by } \varepsilon\left(\bigotimes_{i=1}^{n} x_{k_i}\right) = \frac{1}{n}\sum_{g\in M}\left[\bigotimes_{i=1}^{n} x_{k_{i\sigma(g)}}\right],$$

σ being the right regular representation of M.

Then we claim that $\psi_o\varepsilon$ is a KG-homomorphism, it being sufficient to prove that $\psi_o\varepsilon$ commutes with the elements of M for arguments in a K-basis for $c_{21}(G)^{\otimes n}$.

Well $\forall m \in M$,

$$\psi_o\varepsilon\left[m.\bigotimes_{i=1}^{n} x_{k_i}\right] = \psi\left[m.\frac{1}{n}\sum_{g\in M}\left(\bigotimes_{i=1}^{n} x_{k_{i\sigma(g)}}\right)\right]$$

$$= \frac{1}{n}\sum_{g\in M} \psi\left[m.\bigotimes_{i=1}^{n} x_{k_{i\sigma(g)}}\right]$$

$$= \frac{1}{n}\sum_{g\in M} \psi\left[\bigotimes_{i=1}^{n} m.x_{k_{i\sigma(g)}}\right]$$

$$= \frac{1}{n}\sum_{g\in M}\left\{\prod_{i=1}^{n} \chi\left(g_i m.x_{k_{i\sigma(g)}}\right)\right\}$$

$$= \frac{1}{n}\sum_{g\in M}\left\{\prod_{i=1}^{n} \chi\left(g_{i\sigma(m)}.x_{k_{i\sigma(g)}}\right)\right\}$$

$$= \frac{1}{n}\sum_{g\in M}\left\{\prod_{i=1}^{n} \chi\left(g_i.x_{k_{i\sigma(m^{-1}g)}}\right)\right\}, \text{ (K commutative)},$$

$$= \frac{1}{n}\sum_{g\in M}\left\{\prod_{i=1}^{n} \chi\left(g_i.x_{k_{i\sigma(g)}}\right)\right\}, \text{ altering}$$

variable of summation.

By definition of ψ , ε, this is seen to be equal to $\psi \circ \varepsilon\left(\bigotimes_{i=1}^{n} x_{k_i}\right)$, as required. But, by Lemmas 7.1 and 7.2, $\dim_K \operatorname{Im} \varepsilon \equiv 1 \pmod{p^\alpha}$, and so, by Lemma 7.1 again $\psi(g-e)$ vanishes on a KG-direct summand of $c_{21}(G)^{\otimes n}$ isomorphic to $c_{21n}(G)$, $\forall g \in G$, and hence on a KH-direct summand of $[\,c_{21}(G)^{\otimes n}\,]_H$ isomorphic to $c_{21n}(H)$. By Lemma 6.2, $\psi(g-e)$ vanishes on all such direct summands, $\forall g \in G$. In particular, it vanishes on $[\operatorname{Im}(q_{21}^{\otimes n} \circ j_{21n})]_H$, $\forall g \in G$, (the j's and q's being as in Chapter VI). Thus, the composite η

$$c_{21n}(G) \xrightarrow{j_{21n}} \mathcal{U}(G)^{\otimes 21n} \xrightarrow{q_{21}^{\otimes n}} c_{21}(G)^{\otimes n} \xrightarrow{\psi} K$$

is a KG-homomorphism, and it is clear that the image of η under restriction from G to H is $\bigcup_{i=1}^{n} \chi g_i$, as required.

<u>Note</u> If we write $\eta = T(\chi)$-well-defined since restriction is (1-1) here by 3.6 - it is clear from the commutativity of "$\cup$" that $\forall \chi, \psi \in \hat{H}^*(H,K)$, involving no component of odd degree, $T(\chi \cup \psi) = T(\chi) \cup T(\psi)$, so that T is a multiplicative analogue of the Eckmann Transfer. Corresponding to 3.6, we have the result that, if χ is in the image of the restriction from G to H, then $T(\chi) = \chi^n$.

§2. Divisors of Zero

Lemma 7.4 Let G be a p-group, K = GF(p), A a non-trivial, non-projective, indecomposable KG-module. Let a K-basis for $B = \{a \in A \mid x.a = a, \forall x \in G\}$ be $\{b_i \mid 1 \leq i \leq m\}$ and suppose $\{a_j \mid 1 \leq j \leq n\}$ extends this to a K-basis for A. Suppose also that $\{u\}$ is a complementary K-basis for $\mathfrak{U}.\mathfrak{U}^*$ in $\mathfrak{U}^*$. Then a minimal set of generators for $\mathfrak{U}^* \otimes A$ is $B' \cup A'$, where $B' = \{u \otimes b_i \mid 1 \leq i \leq m\}$ and $A' = \{u \otimes a_j \mid 1 \leq j \leq n\}$. Further, $KG.A' \cong p(A)$, of KG-rank n, and a minimal set of generators for $\frac{A}{KG.A'} \cong c(A)$ is $\{b' + KG.A' \mid b' \in B'\}$.

Proof The first part of the theorem is an easy consequence of the results of Chapter II, while the third part is a consequence of the preceding two, and the fact that

$\sigma = \sum_{x \in G} x, \in KG,$ annihilates B'. Thus it will be sufficient to prove that the set $\{\sigma.a' \mid a' \in A'\}$ is linearly independent over K.

Well, let $G^\# = G \setminus \{e\}$ and $\lambda_1, \ldots \lambda_n \in K$; then:

$$0 = \sum_{j=1}^{n} \lambda_j \sigma.(u \otimes a_j)$$

$$\Rightarrow 0 = \sum_{j=1}^{n} \lambda_j \Big\{ \sum_{x \in G} xu \otimes x a_j \Big\}$$

$$\Rightarrow 0 = \sum_{j=1}^{n} \lambda_j \Big\{ \Big(\sum_{x \in G^\#} xu \otimes x a_j \Big) + u \otimes a_j \Big\}$$

$$\Rightarrow 0 = \sum_{j=1}^{n} \lambda_j \Big\{ \Big(\sum_{x \in G^*} xu \otimes x a_j \Big) + \Big(-\sum_{x \in G^*} xu \Big) \otimes a_j \Big\}, \text{ since } \sigma u = 0,$$

$$\Rightarrow 0 = \sum_{j=1}^{n} \lambda_j \Big\{ \sum_{x \in G^*} [xu \otimes (x-e) a_j] \Big\}$$

$$\Rightarrow 0 = \sum_{x \in G^*} xu \otimes \Big[\sum_{j=1}^{n} \lambda_j (x-e) a_j \Big].$$

But $\{xu \mid x \in G^*\}$ are linearly independent over K, and so:

$$\sum_{j=1}^{n} \lambda_j (x-e) a_j = 0, \ \forall x \in G^*,$$

i.e. $\sum_{j=1}^{n} \lambda_j a_j \in B$.

But $\{a_j \mid 1 \leq j \leq n\}$ are a complementary K-basis for B in A, and thus $\lambda_j = 0$, $1 \leqslant j \leqslant n$, as required.

<u>Theorem 7.5</u> Let $\{\omega\}$ be a K-basis for $\hat{H}^{-1}(G,K)$ and χ an element of $\hat{H}^*(G,K)$ whose n^{th} component is zero for all n such that $[c_n(G)]_H \sim K$, (H being a Sylow p-subgroup of G); then $\omega \cup \chi = 0$.

<u>Proof</u> By 3.6 and 6.5, it is sufficient to prove the result for G a p-group, and (by linearity) for χ homogeneous (of degree n, say). We use the notation of lemma 7.4, with $c_n(G) = A$. Since σ annihilates $c_{n-1}(G)$, it is sufficient to prove that:

$$\text{for } c \in \mathfrak{U}^* \otimes A, \ \sigma . c = 0 \Rightarrow c \in \operatorname{Ker} \omega \otimes \chi.$$

Assume $\sigma.c = 0$ and write $c = \sum_{i=1}^{n} \gamma_i(u \otimes a_i) + \sum_{j=1}^{m} \delta_j (u \otimes b_j)$, each γ_i, $\delta_j \in KG$, in accordance with Lemma 7.4. We have $\sum_{i=1}^{n} \sigma\gamma_i(u \otimes a_i) = 0$. Now put $\sigma\gamma_i = \lambda_i\sigma$, $\lambda_i \in K$, $1 \leqslant i \leqslant n$. Thus, as in the proof of 7.4, $\lambda_i = 0 \ \forall i$, $\Rightarrow \sigma\gamma_i = 0 \ \forall i$, $\Rightarrow \gamma_i \in \mathfrak{U} \ \forall i$.

Thus, $c \in \mathfrak{U}.(\mathfrak{U}^* \otimes A) + \mathfrak{U}^* \otimes B$. Now,

$c_{n-1}(G) \nleqq K \Rightarrow B \leqq \operatorname{Ker}\chi \Rightarrow \mathfrak{U}^* \otimes B \leqq \operatorname{Ker} \omega\otimes\chi$, and $\omega\otimes\chi$ a KG-homomorphism $\Rightarrow \mathfrak{U}.(\mathfrak{U}^* \otimes A) \leqslant \operatorname{Ker} \omega\otimes\chi$.

Thus, $(\omega\otimes\chi)(c) = 0$, as required.

<u>Corollary 7.6</u> Let G be a group whose Sylow p-subgroup is neither cyclic nor quaternionic, then if $\chi \in \hat{H}^*(G,K)$ has no component of degree zero and ω is as in the Theorem, then $\omega_\cup\chi = 0$.

<u>Corollary 7.7</u> If G is as in 7.6, the only homogeneous units in $\hat{H}^*(G,K)$ are the non-zero scalar multiples of the identity.

<u>Note</u> The results of 2.14 together with the above allow us to characterise the homogeneous units of $\hat{H}^*(G,K)$ in all cases.

§3. Non-nilpotent Elements

Theorem 7.8 Let G be a p-group possessing a minimal set of generators $x = x_1, x_2, \ldots x_n$ and let $X_i = \langle x_i \rangle$ and $Y_i = \langle x_1, \ldots \hat{x}_i, \ldots x_n \rangle$. If we suppose further that $X_1 \cap G'Y_1 = \{e\}$, then we can define, in terms of x, a non-nilpotent element of $\hat{H}^2(G,K)$.

Proof It is well-known that the set $\{(e-x_i) \mid 1 \leqslant i \leqslant n\}$ is a minimal set of generators for $\mathfrak{U}$ as a left (or a right) KG-module. Let F be a free KG-module of rank n, with KG-basis $t_1, \ldots t_n$, and define a minimal projective presentation, $\phi: F \twoheadrightarrow \mathfrak{U}$ by $\phi(t_i) = (e-x_i)$, $1 \leqslant i \leqslant n$. Then, regarding $F = KG.\{t_1\} \oplus \ldots \oplus KG.\{t_n\}$, we have:

$$c_2(G) \cong \operatorname{Ker}\phi = \left\{ (\gamma_1, \cdots \gamma_n) \in F \,\middle|\, \sum_{i=1}^{n} \gamma_i (e - x_i) = 0 \right\},$$

and regarding γ_i as an element of KG, $1 \leqslant i \leqslant n$, call this KG-module Q. If we let $\sigma = \sum_{g \in X_1} g$, ε KG, then it is clear that $\tilde{\sigma} = (\sigma, 0, \ldots 0) \varepsilon Q$; we claim that under the hypothesis $X_1 \cap G'Y_1 = \{e\}$, $\tilde{\sigma} \notin \mathfrak{U}.Q$. Now a typical element of $\mathfrak{U}.Q$ has the form $\sum_{i=1}^{n} (e-x_i)q_i$, $q_i \varepsilon Q$ all i, so suppose (for a contradiction) that

$$\tilde{\sigma} = \sum_{i=1}^{n} (e-x_i)q_i, \text{ where } Q \ni q_i = (\gamma_{i1}, ---- \gamma_{in}), \gamma_{ij} \varepsilon \text{ KG}, \forall i,j,$$

so that: $$\sum_{j=1}^{n} \gamma_{ij}(e-x_j) = 0, \quad i = 1, \ldots n.$$

The situation can best be expressed in terms of matrices and vectors over KG - let $\Gamma = (\gamma_{ij})$, $\alpha = (e-x_1, \dots e-x_n)$. Then we have: $\alpha\Gamma = \tilde{\sigma}$ and $\Gamma\alpha' = 0$. We use only the first components of these equations viz.:

$$\sum_{i=1}^{n} (e - x_i)\gamma_{i1} = \sigma, \text{ - - - - } (1),$$

$$\sum_{j=1}^{n} \gamma_{1j}(e - x_j) = 0, \text{ - - - - } (2).$$

We first prove the result for G abelian, in which case the hypothesis states that $G = X_1 \times Y_1$. If $\pi = \sum_{g \in Y_1} g$, it is clear that $\pi(e-x_i) = 0$ for $i = 2, \dots n$. Multiplying (1) and (2) by π, we find that:

$$\left.\begin{aligned} \pi(e-x)\gamma_{11} &= \sum_{g \in G} g \;, \\ \pi(e-x)\gamma_{11} &= 0 \;, \end{aligned}\right\} \quad \text{respectively,}$$

which is a contradiction.

Now if G is non-abelian, consider the natural map, $\nu : KG \rightarrow K\frac{G}{G'}$, given by $\nu(g) = gG' (\forall g \in G)$, and write $\nu(\gamma) = \bar{\gamma}, \forall \gamma \in KG$. $x_1G', \dots x_nG'$ form a minimal generating set for $\frac{G}{G'}$, and the hypothesis implies that $\frac{G}{G'} = \frac{X_1G'}{G'} \times \frac{Y_1G'}{G'}$. If we let $\pi' = \sum_{g \in \frac{Y_1G'}{G'}} g$ and replace equations (1) and (2) by their images under ν, the above proof carries over, (since $\pi'\bar{\sigma} \neq 0$).

Now let $\frac{M}{\mathfrak{U}.Q}$ be a complement for $\frac{K.\{\tilde{\sigma}\}+\mathfrak{U}.Q}{\mathfrak{U}.Q}$, and define $\chi \in \hat{H}^2(G,K)$ by $\chi(m) = 0$ if $m \in M$ and $\chi(\tilde{\sigma}) = 1 \in K$. Since $\tilde{\sigma}$ is X_1-trivial, the short exact sequence:

$$0 \longrightarrow M_{X_1} \longrightarrow Q_{X_1} \longrightarrow K.\{\tilde{\sigma}\} \longrightarrow 0$$

splits, so that M_{X_1} is projective by the Krull-Schmidt Theorem and Theorem 3.1. Thus in the definition of the restriction homomorphism (Chapter III, §2, (1)), we can define $i_2 : K \longrightarrow Q$ by $i_2(1) = \tilde{\sigma}$, (since $c_2(X_1) \cong K$). Thus, the image of χ under restriction to X_1 is the identity on K, which is non-nilpotent. Thus, since restriction is a ring homomorphism, χ is itself non-nilpotent.

<u>Note</u> Since $G'Y_1 < G$ in all cases, the above conditions can certainly be realised whenever G contains a generator of order p. In the case of abelian $G = X_1 \times \cdots \times X_n$, we can define in the above fashion a set of n linearly independent χ's, (one for each $i = 1, \ldots n$). This result, together with those of the previous section can be applied to the generalised quaternions to deduce that for these groups, no minimal set of generators with the property hypothesised in the theorem can exist.

CHAPTER VIII: A PROPERTY OF THE RESTRICTION HOMOMORPHISM IN CHARACTERISTIC 2

§1. Notation

The notation given in this section will stand throughout the chapter. Let G be a 2-group and K any field of characteristic 2. Let H be a maximal subgroup of G, and choose (by the Burnside Basis Theorem) a minimal set $x = x_1, \ldots x_n$ of generators for G in such a way that $x_i \in H$, $2 \leqslant i \leqslant n$. Further, write $\mathfrak{U} = \mathfrak{U}(G)$, $\mathfrak{U}' = \mathfrak{U}(H)$, $\sigma = \sum_{g \in G} g \in KG$, $\sigma' = \sum_{h \in H} h \in KH$ and regard KH as a subset of KG. Denote the cohomological restriction from G to H by r. Since induced modules will not concern us here, we revise the use of the superscript notation as follows: if A is a KG-module, set $A^G = \{a \in A \mid g.a = a, \forall g \in G\}$, which in the present context is just the socle $\mathfrak{S}(A)$ of A. (We retain, however, the corresponding subscript notation for restriction of modules).

We now observe that $KG.\mathfrak{U}'$ is a maximal KG-submodule of $\mathfrak{U}$ (with complementary K-basis $\{e+x\}$), so that $\mathfrak{U}^2 \leqslant KG.\mathfrak{U}' \leqslant \mathfrak{U}$, and define $\phi : \mathfrak{U} \to K$ by $\phi(v) = 0$ if $v \varepsilon KG.\mathfrak{U}'$ and $\phi(e+x) = 1$. Thus, $\phi \varepsilon \hat{H}^{1}(G,K)$ and, since $(\mathrm{Ker}\ \phi)_H \cong \mathfrak{U}' \dotplus \mathfrak{U}'$, $r(\phi) = 0$. Since r is a ring homomorphism, it is clear that $\mathrm{Ker}\ r \geqslant \{\chi \cup \phi \mid \chi \in \hat{H}^*(G,K)\}$; our purpose here is to demonstrate the reverse inequality, thus showing that Ker r is a principal ideal of $\hat{H}^*(G,K)$, viz., that generated by ϕ.

Now let $\begin{tikzcd} c_n(G) \arrow[r, shift right, "j_n"'] & \mathfrak{U}^{\otimes n} \arrow[l, shift right, "q_n"'] \end{tikzcd}$ be as in Chapter VI and define $p = p_0 : \mathfrak{U} \otimes \mathfrak{U}^* \longrightarrow K$ as follows. Let $\{u\}$ be a complementary basis for $\mathfrak{U}.\mathfrak{U}^*$ in $\mathfrak{U}^*$ and let $\mathfrak{B}$ be a K-basis for $\mathfrak{U}$ which contains σ. Thus, by Lemma 7.4, $\{b \otimes u \mid b \in \mathfrak{B}\}$ is a minimal set of KG-generators for $\mathfrak{U} \otimes \mathfrak{U}^*$ and we can define a split onto KG-homomorphism, $p_0 : \mathfrak{U} \otimes \mathfrak{U}^* \longrightarrow K$ by specifying $p_0(b \otimes u) = 0$ for $b \in \mathfrak{B} \setminus \{\sigma\}$, and $p_0(\sigma \otimes u) = 1$. Further, define p_{n-1} $(n > 1)$ by the commutative diagram:

$$\begin{tikzcd} c_n(G) \otimes \mathfrak{U}^* \arrow[r, "p_{n-1}"] \arrow[d, "j_n \otimes 1"'] & c_{n-1}(G) \\ \mathfrak{U}^{\otimes n} \otimes \mathfrak{U}^* \arrow[r, "1 \otimes p_0"'] & \mathfrak{U}^{\otimes(n-1)}\ (\otimes K). \arrow[u, "q_{n-1}"'] \end{tikzcd}$$

§2. Preliminary Lemmas

Lemma 8.1 With the notation of §1, p_{n-1} is a split onto mapping.

Proof If $i_{n-1} : c_{n-1}(G) \longrightarrow c_n(G) \otimes \mathcal{U}^*$ is any split embedding,

$$\mathrm{Im}(j_n \otimes 1)_\circ\, i_{n-1} \cap \mathrm{Ker}\, q_{n-1} \circ (1 \otimes p_o) = (0),$$

by Lemma 7.

Thus, $p_{n-1} \circ i_{n-1}$ is (1-1), and thus (by dimensions) is a KG-isomorphism. The lemma follows.

Lemma 8.2 Let A be a KG-module and consider the submodule $A \otimes KG.\mathcal{U}'$ of $A \otimes \mathcal{U}$; then for any a ε A,

$$\sigma.[a \otimes (e+x)] \in A \otimes KG.\mathcal{U}' \Rightarrow \sigma.a = 0.$$

Proof We work with congruences modulo $A \otimes KG.\mathcal{U}'$.

$$\begin{aligned} 0 &\equiv \sigma.[a \otimes (e+x)] \\ &\equiv \sum_{g \in G} ga \otimes g(e+x) \\ &\equiv \sum_{g \in G} ga \otimes (e+g)(e+x) + \sum_{g \in G} [g.a \otimes (e+x)] \\ &\equiv \qquad 0 \qquad + \sum_{g \in G} [g.a \otimes (e+x)], \text{ since } \mathcal{U}^2 \leq KG.\mathcal{U}', \\ &\equiv (\sigma.a) \otimes (e+x), \end{aligned}$$

and since $(e+x) \notin KG.\mathfrak{U}'$, $\sigma.a = 0$, as required.

The following two lemmas are dual to one another.

<u>Lemma 8.3</u> Let A be a KG-module with direct summand $C \cong c(A)$, and let $A' = \{\sigma.a \mid a \in A\}$; then $A^G \leqslant A' + C$.

<u>Proof</u> Let $A = C \oplus P$, so that P is projective; then the lemma follows from the following two identities:

$$A^G = C^G \oplus P^G, \qquad P^G = A'.$$

<u>Lemma 8.4</u> Let A be a KG-module with direct summand $P \cong p(A)$; then if $a \in P'$ and $\sigma.a = 0$, $a \in \mathfrak{U}.A$.

<u>Proof</u> This follows at once from the fact that the annihilator of σ in KG is precisely $\mathfrak{U}$.

§3. <u>The Endomorphism π</u>

Let the mapping $s_n : c_n(G) \longrightarrow c_n(G) \otimes \mathfrak{U}^*$ be given by $s_n(c) = \sigma'.c \otimes u$, $\forall n \in \mathbb{Z}$. We then define a mapping $\pi : \hat{H}^*(G,K) \longrightarrow \hat{H}^*(G,K)$ as follows; let $\chi \in \hat{H}^{n-1}(G,K)$, $(n \geqslant 1)$ and set $\pi(\chi) = \chi \circ p_{n-1} \circ s_n$, and dually for homogeneous elements of negative degree. [Note: We restrict ourselves to a discussion of homogeneous elements of non-negative degree, since dualisation is not difficult to effect. However, all results are proved explicitly for the ring $H^*(G,K)$.]

Although s_n is not, in general, a KG-homomorphism, $\pi(\chi)$ is, for let $g \in G$ and $c \in c_n(G)$. Then

$$\chi \circ p_{n-1} \circ s_n\big((e+g).c\big) = \chi \circ p_{n-1}\big(\sigma'(e+g).c\big),$$

and $\sigma'(e+g) = 0$ or σ (according as g is, or is not, in H). Thus, since both 0 and σ annihilate $c_n(G)$, $\pi(\chi)$ annihilates $\mathfrak{U}.c_n(G)$. Finally, since $\pi(\chi)$ is K-linear and its image is G-trivial, $\pi(\chi)$ is a KG-homomorphism. It can be checked that π is independent of the j's and q's, (cf. Chapter VI).

<u>Theorem 8.5</u> π is an $\hat{H}^*(G,K)$-endomorphism.

<u>Proof</u> We must show that, $\forall \chi, \psi \in \hat{H}^*(G,K)$,

$\pi(\psi \cup \chi) = \psi \cup \pi(\chi)$. (It is sufficient to take $\chi = 1$, but the general homogeneous case is no harder, so we do this). So, let ψ, χ be homogeneous of degrees m, $n-1$ (respectively), with $m, n > 0$. Applying the definition of π, we have to establish the equality of the two 'outside' maps in the following diagram, (writing $c_n(G) = C_n$, $\forall n \in \mathbb{Z}$).

$$\begin{tikzcd}
C_{m+n} \arrow[r, "j_{m+n}"] \arrow[d, "s_{m+n}"'] \arrow[dr, phantom, "1"] & \mathcal{U}^{\otimes(m+n)} \arrow[r, "q_m \otimes q_n"] \arrow[d, "\mathcal{J}_{m+n}"'] \arrow[dr, phantom, "2"] & C_m \otimes C_n \arrow[d, "1 \otimes s_n"] \\
C_{m+n} \otimes \mathcal{U}^{*} \arrow[r, "j_{m+n} \otimes 1"] \arrow[d, "p_{m+n-1}"'] \arrow[dr, phantom, "4"] & \mathcal{U}^{\otimes(m+n)} \otimes \mathcal{U}^{*} \arrow[r, "q_m \otimes q_n \otimes 1"] \arrow[d, "1 \otimes p"'] \arrow[dr, phantom, "3"] & C_m \otimes (C_n \otimes \mathcal{U}^{*}) \arrow[l, bend left, "j_m \otimes j_n \otimes 1"] \arrow[d, "1 \otimes p_{n-1}"] \\
C_{m+n-1} \arrow[r, "j_{m+n-1}"] & \mathcal{U}^{\otimes(m+n-1)} \arrow[r, "q_m \otimes q_{n-1}"'] \arrow[l, bend left, "q_{m+n-1}"] & C_m \otimes C_{n-1} \arrow[r, "\psi \otimes \chi"] & K,
\end{tikzcd} \quad ----(1).$$

In the diagram, $\mathcal{J}_{m+n}(v) = \sigma'.v \otimes u$, $\forall v \in \mathcal{U}^{\otimes(m+n)}$.

In accordance with the notation in the diagram, the proof falls naturally into four sections.

1 Let $c \in C_{m+n}$; then

$$\begin{aligned} \mathcal{J}_{m+n} \circ j_{m+n}(c) &= \sigma'.\, j_{m+n}(c) \otimes u \\ &= j_{m+n}(\sigma'.c) \otimes u \\ &= (j_{m+n} \otimes 1)_\circ\, s_{m+n}(c). \end{aligned}$$

2 The two edge maps in this square are equalised by $(\psi \otimes \chi)_\circ (1 \otimes p_{n-1})$:

$$(\psi \otimes \chi)_\circ (1 \otimes p_{n-1})_\circ \left[(1 \otimes s_n)_\circ (q_m \otimes q_n) + (q_m \otimes q_n \otimes 1)_\circ \mathcal{J}_{m+n} \right] \left(\bigotimes_{i=1}^{m+n} u_i \right)$$

$$= \psi \otimes (\chi \circ p_{n-1})\Big\{ q_m\big(\bigotimes_{i=1}^{m} u_i\big) \otimes \Big[\sum_{h\in H} h.q_n\big(\bigotimes_{i=m+1}^{m+n} u_i\big)\Big] \otimes u$$

$$+ \sum_{h\in H}\Big[q_m\big(\bigotimes_{i=1}^{m} hu_i\big) \otimes q_n\big(\bigotimes_{i=m+1}^{m+n} h.u_i\big) \otimes u\Big]\Big\}$$

$$= \psi \otimes (\chi \circ p_{n-1})\Big\{\sum_{h\in H}\Big(\Big[(e+h).q_m\big(\bigotimes_{i=1}^{m} u_i\big)\Big] \otimes \Big[h.q_n\big(\bigotimes_{i=m+1}^{m+n} u_i\big)\Big] \otimes u\Big)\Big\}$$

$$= \sum_{h\in H}\Big\{\Big[(e+h).\,\psi \circ q_m\big(\bigotimes_{i=1}^{m} u_i\big)\Big] . \chi \circ p_{n-1}\Big[\Big(h.q_n\big(\bigotimes_{i=m+1}^{m+n} u_i\big)\Big) \otimes u\Big]\Big\}$$

$$= 0,$$

since the former factor in each summand is zero.

③ $\operatorname{Im} j_{m+n} \circ j_{m+n} \leq \big[\mathcal{U}^{\otimes(m+n)}\big]^G \otimes u$, so by Lemma 8.3, it is sufficient to prove that $(\psi \otimes \chi)\circ\big[(1\otimes p_{n-1})\circ(q_m \otimes q_n \otimes 1) + (q_m \otimes q_{n-1})\circ(1\otimes p)\big]$ vanishes on

a) $\big[\operatorname{Im}(j_m \otimes j_n)\big] \otimes u$ and b) $\{\sigma.v \otimes u \mid v \in \mathcal{U}^{\otimes(m+n)}\}$.

a) Since $p_{n-1} = q_{n-1} \circ (1\otimes p) \circ (j_n \otimes 1)$, this part is obvious.

b) Here, again using the definition of p_{n-1}, it is sufficient to show that $(\psi \otimes \chi)\circ(q_m \otimes q_{n-1})\circ(1\otimes p)$ vanishes on all elements of the form

$\Big[\sum_{g\in G} g.\bigotimes_{i=1}^{m+n} u_i\Big] \otimes u$, $u_i \in \mathcal{U}$, (all i).

Well, $(\psi \otimes \chi)\circ(q_m \otimes q_{n-1})\circ(1\otimes p)\Big[\Big(\sum_{g\in G} g.\bigotimes_{i=1}^{m+n} u_i\Big) \otimes u\Big]$

$$= g.(\psi \otimes \chi)\circ(q_m \otimes q_{n-1})\circ(1\otimes p)\Big[\sum_{g\in G}\Big(\bigotimes_{i=1}^{m+n} u_i \otimes g^{-1}u\Big)\Big]$$

$= 0$, since $\sigma.u = 0$.

④ Since $j_{m+n-1} \circ p_{m+n-1} = j_{m+n-1} \circ q_{m+n-1} \circ (1 \otimes p) \circ (j_{m+n} \otimes 1)$, it is sufficient to show that:

$$(\psi \otimes \chi) \circ (q_m \otimes q_{n-1}) \circ (1 + j_{m+n-1} \circ q_{m+n-1}) \circ (1 \otimes p) \circ j_{m+n}$$

vanishes on Im j_{m+n}. The proof falls into a number of steps.

(i) First, we claim that any element in Im j_{m+n} is expressible in the form:

$$z \otimes (e+x) + y\,, \qquad \text{------------ (2)}$$

where $y \in \mathfrak{U}^{\otimes(m+n-1)} \otimes KG.\mathfrak{U}'$ and $\sigma.z = 0$. The first condition is automatic and the second follows from Lemma 8.2.

(ii) The next step is to show that, if $u_{m+n} \in KG.\mathfrak{U}'$, then:

$$(1 \otimes p) \circ j_{m+n}\Big[\bigotimes_{i=1}^{m+n} u_i\Big] \equiv 0 \left(\bmod\ \mathfrak{U}.[\mathfrak{U}^{\otimes(m+n-1)}]\right) \text{---} (3).$$

Well, $(1 \otimes p) \circ j_{m+n}\Big[\bigotimes_{i=1}^{m+n} u_i\Big] = (1 \otimes p)\Big(\Big[\sum_{h \in H} \bigotimes_{i=1}^{m+n} h.u_i\Big] \otimes u\Big)$

$$= \sum_{h \in H} h.\Big(\bigotimes_{i=1}^{m+n-1} u_i\Big).\lambda_h\,,$$

where $\lambda_h = p(h.u_{m+n} \otimes u) \in K$, $\forall h \in H$.

Thus we must show that $\gamma = \sum_{h \in H} \lambda_h h \in \mathfrak{U}$.

To do this, observe that:

$$\sum_{h \in H} \lambda_h = p[\sigma'.u_{m+n} \otimes u] = 0\,, \text{ since } u_{m+n} \in KG.\mathfrak{U}'.$$

(iii) For any $z \in \mathcal{U}^{\otimes(m+n-1)}$, we have that:

$$(1\otimes p)\circ \mathfrak{f}_{m+n}[z\otimes(e+x)] \equiv z \ \left(\mathrm{mod}\ \mathcal{U}.[\mathcal{U}^{\otimes(m+n-1)}]\right), \text{---}(4).$$

As in (ii), we have that $(1\otimes p)\circ \mathfrak{f}_{m+n}(z\otimes(e+x)) = \gamma.z$,

where $\gamma = \sum_{h\in H} p[h.(e+x)\otimes u].h$, $\in KG$.

Now by definition of p, $\sum_{h\in H} p[(h+hx)\otimes u] = p(\sigma\otimes u) = 1$,

so that $\gamma \equiv 1$ (mod $\mathcal{U}$), and (4) is established.

(iv) It follows from (2), (3) and (4) that, for any

$c \in C_{m+n}$, we can write:

$(1\otimes p)\circ \mathfrak{f}_{m+n}\circ j_{m+n}(c) = z + t$,

where $\sigma.z = 0$ and $t \in \mathcal{U}.[\mathcal{U}^{\otimes(m+n-1)}]$.

Thus we have only to show that

$(\psi\otimes\chi)\circ(q_m\otimes q_{n-1})\circ(1 + j_{m+n-1}\circ q_{m+n-1})(z+t) = 0$,

and we are finished. Well, for any $v \in \mathcal{U}^{\otimes(m+n-1)}$

write $v = v_p + v_c$ for a decomposition into components

into Ker q_{m+n-1}, Im j_{m+n-1} (respectively), so that

$$(1 + j_{m+n-1}\circ q_{m+n-1})(z+t) = z_p + t_p \ .$$

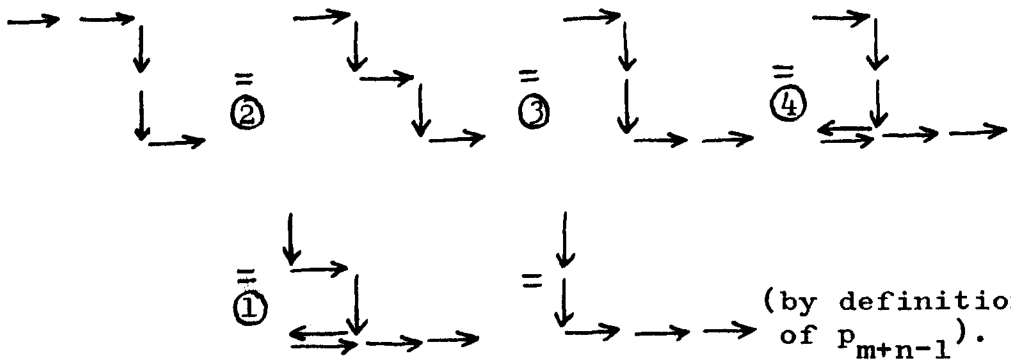
Now $t \in \mathcal{U}.[\mathcal{U}^{\otimes(m+n-1)}]$, and therefore so is t_p.

Also, $\sigma z = 0$ and $\sigma.z_c = 0$, and so $\sigma.z_p = 0$, and it

follows from Lemma 8.4 that $z_p \in \mathcal{U}.[\mathcal{U}^{\otimes(m+n-1)}]$also.

Thus, since $(\psi \otimes \chi)\circ(q_m \otimes q_{n-1})$ is a KG-homomorphism with G-trivial image, it sends $z_p + t_p$ to zero, as required.

We now complete the proof of the theorem, the symbols below referring in an obvious way to the diagram (1).



(by definition of p_{m+n-1}).

<u>Corollary 8.6</u> $\phi = \pi(1)$, so that $\forall \chi \in \hat{H}^*(G,K)$, $\pi(\chi) = \phi \cup \chi$.

§4. <u>Relevance to the Restriction Homomorphism</u>

<u>Theorem 8.7</u> $\operatorname{Im} \pi = \operatorname{Ker} r$.

<u>Proof</u> As mentioned in §1, $\operatorname{Im} \pi \leqslant \operatorname{Ker} r$, so suppose $\chi \in \hat{H}^n(G,K)$, for some fixed $n \in \mathbb{Z}$, and $r(\chi) = 0$. It will be sufficient to show that $\chi \in \operatorname{Im} \pi$.

To this end, let $X \cong c_n(H)$ be a direct summand of $c_n(G)_H$, so that $X \leqslant \operatorname{Ker} \chi$, by hypothesis and definition of r. Thus, $\operatorname{Ker} \chi \geqslant \mathfrak{U}.c_n(G) + X = \operatorname{Ker} s_n$.

Let $y_1, \ldots y_r$ be a complementary K-basis for Ker s_n in Ker χ and let $y_{r+1} \varepsilon\, c_n(G)$ be such that $\chi(y_{r+1}) = 1$. Thus, $\{\sigma' y_i = z_i \mid 1 \leqslant i \leqslant r+1\}$ is a K-basis for $\{\sigma'.c \mid c \,\varepsilon\, c_n(G)\}$, which can be extended to a K-basis $\{z_i \mid 1 \leqslant i \leqslant s\}$ for $c_n(G)^G$, ($s = \dim_K c_n(G)^G \geqslant r+1$).

We now claim that the set $\{p_{n-1}(z_i \otimes u) \mid 1 \leqslant i \leqslant s\}$ forms a minimal set of KG-generators for Im $p_{n-1} = c_{n-1}(G)$. Since the set has the correct cardinality, it is sufficient to prove that, for λ's in K,

$$\sum_{i=1}^{s} \lambda_i \cdot p_{n-1}(z_i \otimes u) \;\varepsilon\; \mathfrak{u}.c_{n-1}(G) \Rightarrow \lambda_i = 0, \text{ all } i.$$

Well, suppose that $\sum_{i=1}^{s} \lambda_i \cdot p_{n-1}(z_i \otimes u) \;\varepsilon\; \mathfrak{u}.c_{n-1}(G)$, so that, if $\{w_k \mid 1 \leqslant k \leqslant t\}$ is a KG-basis for Ker $p_{n-1} \cong \mathfrak{p}[c_n(G) \otimes \mathfrak{u}^*]$, we have:

$$\sum_{i=1}^{s} \lambda_i \cdot (z_i \otimes u) + \sum_{k=1}^{t} \mu_k w_k \;\varepsilon\; \mathfrak{u}.[c_n(G) \otimes \mathfrak{u}^*] \text{ ---- } (5)$$

where the μ's are scalars. Multiplying (5) by σ, we obtain:

$$\sum_{k=1}^{t} \mu_k \sigma w_k = 0,$$ so that (since the w_k are free generators), we have $\mu_k = 0$, $1 \leqslant k \leqslant t$. On the other hand, the elements $z_i \otimes u$, $1 \leqslant i \leqslant s$ form a subset of a

minimal set of KG-generators for $c_n(G) \otimes \mathfrak{U}^*$, (by Lemma 7.4), so that the λ's are all zero, as required. Thus we can define $\psi : c_{n-1}(G) \longrightarrow K$ by: $\psi(\mathfrak{U}.c_{n-1}(G)) = (0)$, $\psi[p_{n-1}(z_i \otimes u)] = \begin{Bmatrix} 1, & \text{if } i = r+1, \\ 0, & \text{otherwise}, \end{Bmatrix}$ whence it is clear that $\pi(\psi) = \chi$, as required.

The following remarks are corollaries of the spectral sequence of [16].

Corollary 8.8 If $g_n = \dim_K \hat{H}^n(G,K)$, $h_n = \dim_K \hat{H}^n(H,K)$, we have $\forall n \in \mathbb{Z}$,

$$g_n \leqslant h_n + g_{n-1}, \quad - - - - - - - - (6).$$

Corollary 8.9 If $|G| = 2^\alpha > 1$, $\dim_K \hat{H}^n(G,K) \leqslant \binom{n+\alpha-1}{n}, \forall n \geqslant 0$.

Proof We induct on α, the case $\alpha = 1$ being obvious, **(by Remark 2.14)**. An inductive argument allows us to deduce from (6) that:

$$g_n \leqslant \sum_{k=0}^{n} h_k .$$

Assuming $\alpha \geqslant 2$, we have:

$$\begin{aligned} g_n &\leqslant \sum_{k=0}^{n} h_k \\ &\leqslant \sum_{k=0}^{n} \binom{k+\alpha-2}{k}, \text{ by induction,} \\ &= \binom{n+\alpha-1}{n}, \end{aligned}$$

by a well-known combinatoric identity.

§5. Remarks

8.10 The bounds in 8.8 and 8.9 are achieved when G is elementary abelian, as is shown in [25].

8.11 Corollary 8.8 (and hence also 8.9) can be proved for general p by the above methods. The method is as follows for 8.8.

Let $C_i = \{c \in c_n(G) \mid (e-x)^i.\ \sigma'.c = 0\} + \mathfrak{U}.c_n(G)$,

(with notation corresponding to the above in the obvious way), $0 \leqslant i \leqslant p-1$.

Thus we have a chain: $\mathfrak{U}.c_n(G) \leqslant C_o \leqslant C_1 \leqslant \dots \leqslant C_{p-1} = c_n(G)$.

Let $\{a_j^{(i)} \mid 1 \leqslant j \leqslant r_i\}$ be a complementary basis for C_i in C_{i+1}, $0 \leqslant i \leqslant p-2$, so that $S = \{(e-x)^i\sigma' a_j^{(i)} \mid 1 \leqslant j \leqslant r_i, 0 \leqslant i \leqslant p-2\}$ is a subset of $c_n(G)^G$. We claim that the elements of S are linearly independent over K, for if

$$\sum_{i=o}^{p-2} \sum_{j=1}^{r_i} \lambda_j^{(i)}.(e-x)^i.\sigma'.a_j^{(i)} = 0,$$

(all λ's in K), we have that

$$\sum_{i=o}^{p-2} \sum_{j=1}^{r_i} \lambda_j^{(i)}(e-x)^i.a_j^{(i)} \in C_o,$$

and hence

$$\sum_{j=1}^{r_o} \lambda_j^{(o)} . a_j^{(o)} \in C_o,$$

implying that $\lambda_j^{(o)} = 0$, $1 \leqslant j \leqslant r_o$. Thus, we see that $(e-x)\sigma'$ annihilates

$$\sum_{i=1}^{p-2} \sum_{j=1}^{r_i} \lambda_j^{(i)} . (e-x)^{i-1} . a_j^{(i)},$$

implying that $\sum_{j=1}^{r_1} \lambda_j^{(1)} a_j^{(1)} \in C_1$.

Hence, $\lambda_j^{(1)} = 0$, $1 \leqslant j \leqslant r_1$. Continuing in this way, we find that all the λ's are zero.

Further, as in the proof of Theorem 8.7, we have that $\chi(C_o) = (0) \Leftrightarrow r(\chi) = 0$, $\forall \chi \in \hat{H}^n(G,K)$, and since $\dim_K c_n(G)^G = \dim_K \hat{H}^{n-1}(G,K)$, (by 2.6), we have $g_n \leqslant h_n + g_{n-1}$, as claimed.

It seems very likely that 8.5 and 8.7 are true for general p, the proofs being essentially the same as those given for p = 2, but with slight alterations in one or two places.

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