MARKOV CHAIN APPROXIMATIONS TO SCALE FUNCTIONS OF LÉVY PROCESSES

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Abstract. We introduce a general algorithm for the computation of the scale functions of a spectrally negative Lévy process X, based on a natural weak approximation of X via upwards skip-free continuous-time Markov chains with stationary independent increments. The algorithm consists of evaluating a finite linear recursion with its (nonnegative) coefficients given explicitly in terms of the Lévy triplet of X. Thus it is easy to implement and numerically stable. Our main result establishes sharp rates of convergence of this algorithm providing an explicit link between the semimartingale characteristics of X and its scale functions, not unlike the one-dimensional Itô diffusion setting, where scale functions are expressed in terms of certain integrals of the coefficients of the governing SDE.

1. Introduction

It is well-known that, for a spectrally negative Lévy process X [5, Chapter VII] [33, Section 9.46], fluctuation theory in terms of the two families of scale functions, $(W^{(q)})_{q\in[0,\infty)}$ and $(Z^{(q)})_{q\in[0,\infty)}$, has been developed [22, Section 8.2]. Of particular importance is the function $W := W^{(0)}$, in terms of which the others may be defined, and which features in the solution of many important problems of applied probability [21, Section 1.2]. It is central to these applications to be able to evaluate scale functions for any spectrally negative Lévy process X.

The goal of the present paper is to define and analyse a very simple novel algorithm for computing W. Specifically, to compute $W(x)$ for some $x > 0$, choose small $h > 0$ such that x/h is an integer. Then the approximation $W_h(x)$ to $W(x)$ is given by the recursion:

$$W_h(y+h) = W_h(0) + \sum_{k=1}^{y/h+1} W_h(y+h-kh)\frac{\gamma_{-kh}}{\gamma_h}, \qquad W_h(0) = (\gamma_h h)^{-1} \tag{1.1}$$

for $y = 0, h, 2h, \ldots, x-h$, where the coefficients γ_h and $(\gamma_{-kh})_{k\geq 1}$ are expressible directly in terms of the Lévy measure λ, (possibly vanishing) Gaussian component σ^2 and drift μ of the Lévy process X, as follows. Let:

$$\tilde{\sigma}_h^2 := \frac{1}{2h^2}\left(\sigma^2 + \int_{[-h/2,0)} y^2 \mathbb{1}_{[-V,0)}(y)\lambda(dy)\right), \quad \tilde{\mu}^h := \frac{1}{2h}\left(\mu + h\sum_{k\in\mathbb{N}} k\lambda\left(\left[\left(-k-\frac{1}{2}\right)h, \left(-k+\frac{1}{2}\right)h\right)\cap[-V,0)\right)\right),$$

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where V equals 0 or 1 according as to whether λ is finite or infinite, and the drift μ is relative to the cut-off function $\tilde{c}(y) := y\mathbb{1}_{[-V,0)}(y)$ (see Eq. 2.1 for the Laplace exponent of X); remark $\tilde{\sigma}_h^2 = \sigma^2/2h^2$ and $\tilde{\mu}^h = \mu/2h$, when $V = 0$. Then the coefficients in (1.1) are given by:

$$\gamma_h := \tilde{\sigma}_h^2 + \mathbb{1}_{(0,\infty)}(\sigma^2)\tilde{\mu}^h + \mathbb{1}_{\{0\}}(\sigma^2)2\tilde{\mu}^h, \qquad \gamma_{-h} := \tilde{\sigma}_h^2 - \mathbb{1}_{(0,\infty)}(\sigma^2)\tilde{\mu}^h + \lambda(-\infty, -h/2] \tag{1.2}$$

$$\gamma_{-kh} := \lambda(-\infty, -kh + h/2], \qquad \text{where } k \geq 2. \tag{1.3}$$

Indeed, the algorithm just described is based on a purely probabilistic idea of weak approximation: for small positive h, X is approximated by what is a random walk X^h on a lattice with spacing h, skip-free to the right, and embedded into continuous time as a compound Poisson process (see Definition 3.1). Then, in recursion (1.1), W_h is the scale function associated to X^h — it plays a probabilistically analogous rôle for the process X^h, as does W for the process X. Thus W_h is computed as an approximation to W (see Proposition 3.5).

When it comes to existing methods for the evaluation of W, note that analytically W is characterized via its Laplace transform $\widehat{W}$, $\widehat{W}$ in turn being a certain rational function of the Laplace exponent ψ of X. However, already ψ need not be given directly in terms of elementary/special functions, and less often still is it possible to obtain closed-form expressions for W itself. The user is then faced with a Laplace inversion algorithm [13] [21, Chapter 5], which (i) necessarily involves the evaluation of ψ, typically at complex values of its argument and requiring high-precision arithmetic due to numerical instabilities; (ii) says little about the dependence of the scale function on the Lévy triplet of X (recall that ψ depends on a parametric complex integral of the Lévy measure, making it hard to discern how a perturbation in the Lévy measure influences the values taken by the scale function); and (iii) being a numerical approximation, fails *a priori* to ensure that the computed values of the scale function are probabilistically meaningful (e.g. given an output of a numerical Laplace inversion, it is not necessary that the formulae for, say, exit probabilities, involving W, should yield values in the interval $[0, 1]$).

By contrast, it follows from (1.1) and the discussion following, that our proposed algorithm (i) requires no evaluations of the Laplace exponent of X and is numerically stable, as it operates in nonnegative real arithmetic [31, Theorem 7]; (ii) provides an explicit link between the deterministic semimartingale characteristics of X, in particular its Lévy measure, and the scale function W; and (iii) yields probabilistically consistent outputs. Further, the values of W_h are so computed by a simple finite linear recursion and, as a by-product of the evaluation of $W_h(x)$, values $W_h(y)$ for all the grid-points $y = 0, h, 2h, \ldots, x - h, x$, are obtained (see Matlab code for the algorithm in [28]), which is useful in applications (see Section 6 below).

Our main results will (I) show that W_h converges to W pointwise, and uniformly on the grid with spacing h (if bounded away from 0 and $+\infty$), for any spectrally negative Lévy process, and (II) establish sharp rates for this convergence under a mild assumption on the Lévy measure.

Due to the explicit connection between the coefficients appearing in (1.1) and the Lévy triplet of X, (1.1) also has the spirit of its one-dimensional Itô diffusion analogue, wherein the computation of the scale function requires numerical evaluation of certain integrals of the coefficients of the SDE driving said diffusion (for the explicit formulae of the integrals see e.g. [9, Chapters 2 and 3]). Indeed, we express W as a single limit, as $h \downarrow 0$, of nonnegative terms explicitly given in terms of the Lévy triplet. This is more direct than the Laplace inversion of a rational transform of the Laplace exponent, and hence may be of purely theoretical significance (e.g. [30, Remark 3.3]).

Finally, note that an algorithm, completely analogous to (1.1), for the computation of the scale functions $W^{(q)}$, and also $Z^{(q)}$, $q \geq 0$, follows from our results (see Proposition 3.5, Eq. (3.1) and (3.2)) and presents no further difficulty for the analysis of convergence (see Theorem 1.2 below). Indeed, our discretization allows naturally to approximate other quantities involving scale functions, which arise in application: the derivatives of $W^{(q)}$ by difference quotients of $W_h^{(q)}$; the integrals of a continuous (locally bounded) function against $dW^{(q)}$ by its integrals against $dW_h^{(q)}$; expressions of the form $\int_0^x F(y, W^{(q)}(y))dy$, where F is continuous locally bounded, by the sums $\sum_{k=0}^{\lfloor x/h \rfloor -1} F(kh, W_h^{(q)}(kh))h$ etc. (See Section 6 for examples.)

1.1. **Overview of main results.** The key idea leading to the algorithm in (1.1) is best described by the following two steps: (i) approximate the spectrally negative Lévy process X by a continuous-time Markov chain (CTMC) X^h with state space $\mathbb{Z}_h := \{hk\colon k \in \mathbb{Z}\}$ ($h \in (0, h_\star)$ for some $h_\star > 0$), as described in Subsection 2.1; (ii) find an algorithm for computing the scale functions of the chain X^h. The approximation in Subsection 2.1 implies that X^h is a compound Poisson (CP) process, which is not spectrally negative. However, since the corresponding jump chain of X^h is a skip-free to the right $\mathbb{Z}_h$-valued random walk, it is possible to introduce (right-continuous, nondecreasing) scale functions $(W_h^{(q)})_{q\geq 0}$ and $(Z_h^{(q)})_{q\geq 0}$ (with measures $dW_h^{(q)}$ and $dZ_h^{(q)}$ supported in $\mathbb{Z}_h$), in analogy to the spectrally negative case. Moreover, as described in Proposition 3.5, a straightforward recursive algorithm is readily available for evaluating *exactly* any function in the families $(W_h^{(q)})_{q\geq 0}$ and $(Z_h^{(q)})_{q\geq 0}$ at any point. More precisely, it emerges, that for each $x \in \mathbb{Z}_h$, $W_h^{(q)}(x)$ (resp. $Z_h^{(q)}(x)$) obtains as a *finite* linear combination of the preceding values $W_h^{(q)}(y)$ (resp. $Z_h^{(q)}(y)$) for $y \in \{0, h, \ldots, x-h\}$; with the starting value $W_h^{(q)}(0)$ (resp. $Z^{(q)}(0)$) being known explicitly. This is in spite of the fact that the state space of the Lévy process X^h is in fact the *infinite* lattice $\mathbb{Z}_h$.

In order to precisely describe the rates of convergence of the algorithm in (1.1), we introduce some notation. Fix $q \geq 0$ and define for K, G bounded subset of $(0, \infty)$:

$$\Delta_W^K(h) := \sup_{x\in\mathbb{Z}_h\cap K} \left| W_h^{(q)}(x - \delta^0 h) - W^{(q)}(x) \right| \text{ and } \Delta_Z^G(h) := \sup_{x\in\mathbb{Z}_h\cap G} \left| Z_h^{(q)}(x) - Z^{(q)}(x) \right|,$$

where δ^0 equals 0 if X has sample paths of finite variation and 1 otherwise. We further introduce:

$$\kappa(\delta) := \int_{[-1,-\delta)} |y|\lambda(dy), \qquad \text{for any } \delta \geq 0.$$

If the jump part of X has paths of infinite variation, i.e. in the case the equality $\kappa(0) = \infty$ holds, we assume (throughout the paper we shall make it explicit when this assumption is in effect):

Assumption 1.1. *There exists $\epsilon \in (1,2)$ with:*

(1) $\limsup_{\delta\downarrow 0} \delta^\epsilon \lambda(-1,-\delta) < \infty$ *and*
(2) $\liminf_{\delta\downarrow 0} \int_{[-\delta,0)} x^2\lambda(dx)/\delta^{2-\epsilon} > 0$.

Note that this is a fairly mild condition, fulfilled if e.g. $\lambda(-1,-\delta)$ "behaves as" $\delta^{-\epsilon}$, as $\delta \downarrow 0$; for a precise statement see Remark 5.10.

Here is now our main result:

Theorem 1.2. *Let K and G be bounded subsets of $(0,\infty)$, K bounded away from zero when $\sigma^2 = 0$. If $\kappa(0) = \infty$, suppose further that Assumption 1.1 is fulfilled. Then the rates of convergence of the scale functions are summarized by the following table:*

$\lambda(\mathbb{R}) = 0$	$\Delta_W^K(h) = O(h^2)$ *and* $\Delta_Z^G(h) = O(h)$
$0 < \lambda(\mathbb{R})$ *&* $\kappa(0) < \infty$	$\Delta_W^K(h) + \Delta_Z^G(h) = O(h)$
$\kappa(0) = \infty$	$\Delta_W^K(h) + \Delta_Z^G(h) = O(h^{2-\epsilon})$

Moreover, the rates so established are sharp in the sense that for each of the three entries in the table above, examples of spectrally negative Lévy processes are constructed for which the rate of convergence is no better than stipulated.

Remark 1.3. (1) The rates of convergence depend on the behaviour of the tail of the Lévy measure at the origin; by contrast behaviour of Laplace inversion algorithms tends to be susceptible to the degree of smoothness of the scale function (for which see [12]) itself [1].
(2) More exhaustive and at times general statements are to be found in Propositions 5.8, 5.9 and 5.11. In particular, the case $\sigma^2 > 0$ and $\kappa(0) = \infty$ does not require Assumption 1.1 to be fulfilled, although the statement of the convergence rate is more succinct under its proviso. Furthermore, in the interest of space, we present sharpness of the rates for the functions $W^{(q)}$ in the case when $\sigma^2 > 0$ only. For additional examples in this direction, please see the extended arXiv version [30].
(3) The proof of Theorem 1.2 consists of studying the differences of the integral representations of the scale functions. The integrands decaying only according to some power law makes the analysis much more involved than was the case in [29], where the corresponding decay was exponential. In particular, one cannot, in the pure-jump case, directly apply the integral triangle inequality. The structure of the proof is explained in detail in Subsection 5.1.
(4) Since scale functions often appear in applications (see Section 1.2 below) in the form $W^{(q)}(x)/W^{(q)}(y)$ $(x, y > 0,\ q \geq 0)$, we note that the rates from Theorem 1.2 transfer directly to such quotients, essentially because $W_h^{(q)}(y) \to W^{(q)}(y) \in (0,\infty)$, as $h \downarrow 0$, and since for all $h \in (0, h_\star)$, $\frac{1}{W^{(q)}(y)} - \frac{1}{W_h^{(q)}(y)} = \frac{W_h^{(q)}(y) - W^{(q)}(y)}{W^{(q)}(y)W_h^{(q)}(y)}$.
(5) For a result concerning the derivatives of $W^{(q)}$ see Proposition 5.13.

1.2. **Overview of the literature and of the applications of scale functions.** For the general theory of spectrally negative Lévy processes and their scale functions we refer to [22, Chapter 8] and [5, Chapter VII], while an excellent account of available numerical methods for computing them can be found in [21, Chapter 5]. Examples, few, but important, of processes when the scale functions *can* be given analytically, appear e.g. in [17]; and in certain cases it is possible to construct them *indirectly* [21, Chapter 4] (i.e. not starting from the basic datum, which we consider here to be the characteristic triplet of X). Finally, in the special case when X is a positive drift minus a compound Poisson subordinator, we note that numerical schemes for (finite time) ruin/survival probabilities (expressible in terms of scale functions), based on discrete-time Markov chain approximations of one sort or another, have been proposed in the literature (see [36, 16, 11, 15] and the references therein).

In terms of applications of scale functions in applied probability, there are numerous identities concerning boundary crossing problems and related path decompositions in which scale functions feature [21, p. 100]. They do so either (a) indirectly (usually as Laplace transforms of quantities which are ultimately of interest), or even (b) directly (then typically, but not always, as probabilities in the form of quotients $W(x)/W(y)$). For examples of the latter see the two-sided exit problem [5, Chapter VII, Theorem 8]; ruin probabilities [22, p. 217, Eq. (8.15)] and the Gerber-Shiu measure [23, Section 5.4] in the insurance/ruin theory context; laws of suprema of continuous-state branching processes [8, Proposition 3.1]; Lévy measures of limits of continuous-state branching processes with immigration (CBI processes) [20, Eq. (3.7)]; laws of branch lengths in population biology [24, Eq. (7)]; the Shepp-Shiryaev optimal stopping problem (solved for the spectrally negative case in [4, Theorem 2, Eq. (30)]); [26, Proposition 1] for an optimal dividend control problem. A further overview of these and other applications of scale functions (together with their derivatives and the integrals $Z^{(q)}$), e.g. in queuing theory and fragmentation processes, may be found in [21, Section 1.2], see also the references therein. A suite of identities involving Laplace transforms of quantities pertaining to the reflected process of X appears in [27].

1.3. **Organisation of the remainder of the paper.** Section 2 provides the setting, gives some preliminary observations/comments and fixes general notation. Section 3 introduces upwards skip-free Lévy chains (they being the continuous-time analogues of random walks, which are skip-free to the right), describes their scale functions and how to compute them. In Section 4 we demonstrate pointwise convergence of the approximating scale functions to those of the spectrally negative Lévy process. Then Section 5 goes on to study the rate at which this convergence transpires. Finally, Section 6 provides some numerical illustrations and further discusses the computational side of the proposed algorithm.

2. Setting, preliminary observations/comments, and general notation

Throughout this paper we let X be a spectrally negative Lévy process (i.e. X has stationary independent increments, is càdlàg, $X_0 = 0$ a.s., the Lévy measure λ of X is concentrated on $(-\infty, 0)$ and X does not have a.s. monotone paths). The Laplace exponent ψ of X, defined via

$\psi(\beta) := \log \mathsf{E}[e^{\beta X_1}]$ ($\beta \in \{\gamma \in \mathbb{C} : \Re\gamma \geq 0\} =: \overline{\mathbb{C}^{\rightarrow}}$), can be expressed as (see e.g. [5, p. 188]):

$$\psi(\beta) = \frac{1}{2}\sigma^2\beta^2 + \mu\beta + \int_{(-\infty,0)} \left(e^{\beta y} - \beta\tilde{c}(y) - 1\right)\lambda(dy), \quad \beta \in \overline{\mathbb{C}^{\rightarrow}}. \tag{2.1}$$

The Lévy triplet of X is thus given by $(\sigma^2, \lambda, \mu)_{\tilde{c}}$, $\tilde{c} := \mathrm{id}_{\mathbb{R}}\mathbb{1}_{[-V,0)}$ with V equal to either 0 or 1, the former only if $\int_{[-1,0)} |x|\lambda(dx) < \infty$ (where $\mathrm{id}_{\mathbb{R}}$ is the identity on $\mathbb{R}$). Further, when the Lévy measure satisfies $\int_{[-1,0)} |x|\lambda(dx) < \infty$, we may always express ψ in the form $\psi(\beta) = \frac{1}{2}\sigma^2\beta^2 + \mu_0\beta + \int_{(-\infty,0)} \left(e^{\beta y} - 1\right)\lambda(dy)$ for $\beta \in \overline{\mathbb{C}^{\rightarrow}}$. If in addition $\sigma^2 = 0$, then necessarily the drift μ_0 must be strictly positive, $\mu_0 > 0$ [22, p. 212].

2.1. **The approximation.** We now recall from [29], specializing to the spectrally negative setting, the spatial discretisation of X by the family of CTMCs $(X^h)_{h\in(0,h_\star)}$ (where $h_\star \in (0, +\infty]$). This family weakly approximates X as $h \downarrow 0$. As in [29] we will use two approximating schemes, scheme 1 and 2, according as $\sigma^2 > 0$ or $\sigma^2 = 0$. Recall that two different schemes are introduced since the case $\sigma^2 > 0$ allows for a better (i.e. a faster converging) discretization of the drift term, but the case $\sigma^2 = 0$ (in general) does not [29, Paragraph 2.2.1]. Let also $V = 0$, if λ is finite and $V = 1$, if λ is infinite. Notation-wise, define for $h > 0$, $c_y^h := \lambda(A_y^h)$ with $A_y^h := [y - h/2, y + h/2)$ ($y \in \mathbb{Z}_h^{--} := \mathbb{Z}_h \cap (-\infty, 0)$); $A_0^h := [-h/2, 0)$;

$$c_0^h := \int_{A_0^h} y^2 \mathbb{1}_{[-V,0)}(y)\lambda(dy) \quad \text{and} \quad \mu^h := \sum_{y\in\mathbb{Z}_h^{--}} y \int_{A_y^h} \mathbb{1}_{[-V,0)}(z)\lambda(dz).$$

We now specify the law of the approximating chain X^h by insisting that (i) X^h is a compound Poisson (CP) process, with $X_0^h = 0$, a.s., and whose positive jumps do not exceed h – hence admits a Laplace exponent $\psi^h(\beta) := \log \mathsf{E}[e^{\beta X_1^h}]$ ($\beta \in \overline{\mathbb{C}^{\rightarrow}}$) –; and (ii) specifying ψ^h under scheme 1, as:

$$\psi^h(\beta) = (\mu - \mu^h)\frac{e^{\beta h} - e^{-\beta h}}{2h} + (\sigma^2 + c_0^h)\frac{e^{\beta h} + e^{-\beta h} - 2}{2h^2} + \sum_{y\in\mathbb{Z}_h^{--}} c_y^h \left(e^{\beta y} - 1\right), \tag{2.2}$$

and under scheme 2, as:

$$\psi^h(\beta) = (\mu - \mu^h)\frac{e^{\beta h} - 1}{h} + c_0^h\frac{e^{\beta h} + e^{-\beta h} - 2}{2h^2} + \sum_{y\in\mathbb{Z}_h^{--}} c_y^h \left(e^{\beta y} - 1\right). \tag{2.3}$$

This is consistent with the approximation of [29]: the above Laplace exponents follow from the forms of the characteristic exponents [29, Eq. (3.1) and (3.2)] via analytic continuation, and to properly appreciate where the different terms appearing in (2.2)-(2.3) come from, we refer the reader to our paper [29], especially Section 2.1 therein.

Indeed, note that, starting directly from [29, Eq. (3.2)], the term $(\mu - \mu^h)\frac{e^{\beta h}-1}{h}$ in (2.3) should actually read as $(\mu - \mu^h)\left(\frac{e^{\beta h}-1}{h}\mathbb{1}_{[0,\infty)}(\mu - \mu^h) + \frac{1-e^{-\beta h}}{h}\mathbb{1}_{(-\infty,0]}(\mu - \mu^h)\right)$. However, when X is a spectrally negative Lévy process with $\sigma^2 = 0$, we have $\mu - \mu^h \geq 0$, at least for all sufficiently small h. Indeed, if $\int_{[-1,0)} |y|\lambda(dy) < \infty$, then $\mu_0 > 0$ and by dominated convergence $\mu - \mu^h \to \mu_0$ as $h \downarrow 0$. On the other hand, if $\int_{[-1,0)} |y|\lambda(dy) = \infty$, then we deduce by monotone convergence

Lévy measure/diffusion part	$\sigma^2>0$	$\sigma^2=0$
$\lambda(\mathbb{R})<\infty$	$V=0$, scheme 1	$V=0$, scheme 2
$\lambda(\mathbb{R})=\infty$	$V=1$, scheme 1	$V=1$, scheme 2

TABLE 1. The usage of schemes 1 and 2 and of V depends on the nature of σ^2 and λ.

$-\mu^h\geq\frac{1}{2}\int_{[-1,-h/2)}|y|\lambda(dy)\to\infty$ as $h\downarrow 0$. We shall assume throughout that $h_\star$ is already chosen small enough, so that $\mu-\mu^h\geq 0$ holds for all $h\in(0,h_\star)$.

In summary, then, $h_\star$ is chosen so small as to guarantee that, for all $h\in(0,h_\star)$: (i) $\mu-\mu^h\geq 0$ and (ii) ψ^h is the Laplace exponent of some CP process X^h, which is also a CTMC with state space $\mathbb{Z}_h$ (note that in [29, Proposition 3.9] it is shown $h_\star$ can indeed be so chosen, viz. point (ii)). Eq. (2.2) and (2.3) then determine the *weak* approximation $(X^h)_{h\in(0,h_\star)}$ precisely. Finally, for $h\in(0,h_\star)$, let λ^h denote the Lévy measure of X^h. In particular, $\psi^h(\beta)=\int\left(e^{\beta y}-1\right)\lambda^h(dy)$, $\beta\in\overline{\mathbb{C}^{\rightarrow}}$, $h\in(0,h_\star)$, so that the jump intensities, equivalently the Lévy measure, of X^h can be read off directly from (2.2)-(2.3). It will also be convenient to define $\psi^0:=\psi$.

2.2. **Connection with integro-differential equations.** An alternative form of (1.1) (as generalized to the case of arbitrary $q\geq 0$) [35, p. 18, Eq. (4.10)] is the analogue of the relation $(L-q)W^{(q)}=0$ in the spectrally negative case [35, p. 20, Remark 4.16], the latter holding true under sufficient regularity conditions on $W^{(q)}$ (see e.g. [7, Eq. (12)]). Here L is the infinitesimal generator of X [33, p. 208, Theorem 31.5]:

$$Lf(x)=\frac{\sigma^2}{2}f''(x)+\mu f'(x)+\int_{(-\infty,0)}\left(f(x+y)-f(x)-yf'(x)\mathbb{1}_{[-V,0)}(y)\right)\lambda(dy)$$

($f\in C_0^2(\mathbb{R})$, $x\in\mathbb{R}$). This suggests there might be a link between our probabilistic approximation and solutions to integro-differential equations.

Indeed, on the one hand, as follows by taking Laplace transforms, the function $W^{(q)}$ ($q\geq 0$) satisfies the following integro-differential equation (cf. [3, Corollary IV.3.3] for the case $\sigma^2=0$, $\lambda(\mathbb{R})<+\infty$, and survival probabilities):

$$\frac{1}{2}\sigma^2\frac{dW^{(q)}}{dx}=1-\mu W^{(q)}(x)+\int_0^\infty\left(W^{(q)}(x-y)(\lambda(-\infty,-y)+q)-W^{(q)}(x)\lambda[-V,-y)\mathbb{1}_{(0,V]}(y)\right)dy \tag{2.4}$$

(for the value of $W^{(q)}(0)$ see [21, p. 127, Lemma 3.1]).

On the other hand, (1.1) as generalized to arbitrary $q\geq 0$ (see Proposition 3.5, Eq. 3.1) can be rewritten, when $\sigma^2=0$, as (for $x\in\mathbb{Z}_h^{++}$):

$$c_0^h\frac{W_h^{(q)}(x)-W_h^{(q)}(x-h)}{2h}=1+(\mu^h-\mu)W_h^{(q)}(x)+h\sum_{k=1}^{x/h}W_h^{(q)}(x-kh)\left(\lambda(-\infty,-(k-1/2)h)+q\right), \tag{2.5}$$

with $W_h^{(q)}(0)=\frac{1}{\frac{1}{2}c_0^h/h+\mu-\mu^h}$, and when $\sigma^2>0$, as (again for $x\in\mathbb{Z}_h^{++}$):

$$(\sigma^2+c_0^h)\frac{W_h^{(q)}(x)-W_h^{(q)}(x-h)}{2h}=1+(\mu^h-\mu)\frac{W_h^{(q)}(x)+W_h^{(q)}(x-h)}{2}+h\sum_{k=1}^{x/h}W_h^{(q)}(x-kh)\left(\lambda(-\infty,-(k-1/2)h)+q\right), \tag{2.6}$$

with $W_h^{(q)}(0) = \frac{2h}{\sigma^2+c_0^h+(\mu-\mu^h)h}$.

Thus (2.5) and (2.6) can be seen as (simple) approximation schemes for the integro-differential equation (2.4). Note, however, that from this viewpoint alone, it would be very difficult indeed to "guess" the correct discretization, which would also yield meaningful generalized scale functions of approximating chains — the latter being our starting point and precisely the aspect of our schemes, which we wish to emphasize. Indeed, higher-order schemes for (2.4), if and should they exist, would (likely) no longer be connected with Lévy chains.

2.3. **General notation.** Number sets (some of which recalled): $\mathbb{R}^+ := (0,\infty)$; $\mathbb{N}_0 := \{0,1,2,\ldots\}$, $\mathbb{N} := \{1,2,\ldots\}$; $\mathbb{Z}_h := \{hk : k \in \mathbb{Z}\}$, $\mathbb{Z}_h^+ := \mathbb{Z}_h \cap [0,\infty)$; $\mathbb{Z}_h^{++} := \mathbb{Z}_h \cap (0,\infty)$, $\mathbb{Z}_h^{--} := \mathbb{Z}_h \cap (-\infty,0)$; $\mathbb{C}^{\rightarrow} := \{z \in \mathbb{C}\colon \Re z > 0\}$, $\overline{\mathbb{C}^{\rightarrow}} = \{z \in \mathbb{C}\colon \Re z \geq 0\}$. A real-valued function f is of M-exponential order, if $|f|(x) \leq Ce^{Mx}$ for all x in the domain of f, for some $C < \infty$. Further, for functions $g \geq 0$ and $h > 0$ defined on some right neighborhood of 0, $g \sim h$ (resp. $g = O(h)$, $g = o(h)$) means $\lim_{0+} g/h \in (0,\infty)$ (resp. $\limsup_{0+} g/h < \infty$, $\lim_{0+} g/h = 0$). Next, the Laplace transform of a measurable function $f : \mathbb{R} \to \mathbb{R}$ of M-exponential order, with $f|_{(-\infty,0)}$ constant (resp. measure μ on $\mathbb{R}$, concentrated on $[0,\infty)$) is denoted $\hat{f}$ (resp. $\hat{\mu}$): $\hat{f}(\beta) = \int_0^\infty e^{-\beta x} f(x)dx$ for $\Re\beta > M$ (resp. $\hat{\mu}(\beta) = \int_{[0,\infty)} e^{-\beta x}\mu(dx)$ for all $\beta \geq 0$ such that this integral is finite). A sequence $(h_n)_{n\in\mathbb{N}}$ of non-zero real numbers is said to be nested, if $h_n/h_{n+1} \in \mathbb{N}$ for all $n \in \mathbb{N}$. DCT stands for the Dominated Convergence Theorem; and we interpret $\pm a/0 = \pm\infty$ for $a > 0$.

3. Upwards skip-free Lévy chains and their scale functions

In the sequel, we will require a fluctuation theory (and, in particular, a theory of scale functions) for random walks, which are skip-free to the right, once these have been embedded into continuous-time as CP processes (see next definition and remark). Indeed, this theory has been developed in full detail in [35] and we recall here for the readers convenience the pertinent results.

Definition 3.1. A Lévy process Y with Lévy measure ν is said to be an *upwards skip-free Lévy chain*, if it is compound Poisson, and for some $h > 0$, $\mathrm{supp}(\nu) \subset \mathbb{Z}_h$, while $\mathrm{supp}(\nu|_{\mathcal{B}((0,\infty))}) = \{h\}$.

Remark 3.2. For all $h \in (0, h_\star)$, X^h is an upwards skip-free Lévy chain.

For the remainder of this section we let Y be an upwards skip-free Lévy chain with Lévy measure ν, such that $\nu(\{h\}) > 0$ ($h > 0$).

The following is either clear or else can be found in [35, Subsection 3.1]. First, one can introduce the Laplace exponent $\varphi : \overline{\mathbb{C}^{\rightarrow}} \to \mathbb{C}$, given by $\varphi(\beta) := \int_{\mathbb{R}}(e^{\beta x} - 1)\nu(dx)$ ($\beta \in \overline{\mathbb{C}^{\rightarrow}}$), for which $\mathsf{E}[e^{\beta Y_t}] = \exp\{t\varphi(\beta)\}$ ($\beta \in \overline{\mathbb{C}^{\rightarrow}}$, $t \geq 0$). φ is continuous in $\overline{\mathbb{C}^{\rightarrow}}$, analytic in $\mathbb{C}^{\rightarrow}$, $\lim_{+\infty} \varphi|_{[0,\infty)} = +\infty$ with $\varphi|_{[0,\infty)}$ strictly convex. Second, letting $\Phi(0) \in [0,\infty)$ be the largest root of φ on $[0,\infty)$, $\varphi|_{[\Phi(0),\infty)} : [\Phi(0),\infty) \to [0,\infty)$ is an increasing bijection with inverse $\Phi := (\varphi|_{[\Phi(0),\infty)})^{-1} : [0,\infty) \to [\Phi(0),\infty)$.

We introduce in the next proposition two families of scale functions for Y, which play analogous roles in the solution of exit problems, as they do in the case of spectrally negative Lévy processes, see [35, Subsections 4.1-4-3]:

Proposition 3.3 (Scale functions). *There exists a family of functions $W^{(q)}:\mathbb{R}\to[0,\infty)$ and their integrals $Z^{(q)}(x)=1+q\int_0^{\lfloor x/h\rfloor h}W^{(q)}(y)dy$, $x\in\mathbb{R}$, defined for each $q\geq 0$ such that for any $q\geq 0$, we have $W^{(q)}(x)=0$ for $x<0$ and $W^{(q)}$ is characterised on $[0,\infty)$ as the unique right-continuous and piecewise constant function of exponential order whose Laplace transform satisfies:*

$$\widehat{W^{(q)}}(\beta)=\frac{e^{\beta h}-1}{\beta h(\varphi(\beta)-q)}\text{ for }\beta>\Phi(q).$$

Remark 3.4.

(i) The functions $W^{(q)}$ are nondecreasing and the corresponding measures $dW^{(q)}$ are supported in $\mathbb{Z}_h$ for each $q\geq 0$.

(ii) The Laplace transform of the functions $Z^{(q)}$ is given by:

$$\widehat{Z^{(q)}}(\beta)=\frac{1}{\beta}\left(1+\frac{q}{\varphi(\beta)-q}\right)\text{ for }\beta>\Phi(q),\ q\geq 0.$$

(iii) For all $q\geq 0$: $W^{(q)}(0)=1/(h\nu(\{h\}))$.

Finally, the following proposition gives rise to a method for calculating the values of the scale functions associated to Y (see [35, Subsection 4.4]).

Proposition 3.5. *Assume $h=1$. We have for all $n\in\mathbb{N}\cup\{0\}$:*

$$W^{(q)}(n+1)=W^{(q)}(0)+\sum_{k=1}^{n+1}W^{(q)}(n+1-k)\frac{q+\nu(-\infty,-k]}{\nu(\{1\})},\quad W^{(q)}(0)=1/\nu(\{1\}),\tag{3.1}$$

and for $\widetilde{Z^{(q)}}:=Z^{(q)}-1$,

$$\widetilde{Z^{(q)}}(n+1)=(n+1)\frac{q}{\nu\{1\}}+\sum_{k=1}^{n}\widetilde{Z^{(q)}}(n+1-k)\frac{q+\nu(-\infty,-k]}{\nu(\{1\})},\quad \widetilde{Z^{(q)}}(0)=0.\tag{3.2}$$

4. Convergence of scale functions

First we fix some notation. Pursuant to [22, Subsections 8.1 & 8.2] (resp. Section 3) we associate henceforth with X (resp. X^h) two families of scale functions $(W^{(q)})_{q\geq 0}$ and $(Z^{(q)})_{q\geq 0}$ (resp. $(W_h^{(q)})_{q\geq 0}$ and $(Z_h^{(q)})_{q\geq 0}$, $h\in(0,h_\star)$). Note that these functions are defined on the whole of $\mathbb{R}$, are nondecreasing, càdlàg, with $W^{(q)}(x)=W_h^{(q)}(x)=0$ and $Z^{(q)}(x)=Z_h^{(q)}(x)=1$ for $x\in(-\infty,0)$. We also let $\Phi(0)$ (resp. $\Phi^h(0)$) be the largest root of $\psi|_{[0,\infty)}$ (resp. $\psi^h|_{[0,\infty)}$) and denote by Φ (resp. Φ^h) the inverse of $\psi|_{[\Phi(0),\infty)}$ (resp. $\psi^h|_{[\Phi^h(0),\infty)}$, $h\in(0,h_\star)$). As usual W (resp. W_h) denotes $W^{(0)}$ (resp. $W_h^{(0)}$, $h\in(0,h_\star)$).

Next, for $q\geq 0$, recall the Laplace transforms of the functions $W^{(q)}$ and $Z^{(q)}$ [22, p. 214, Theorem 8.1] (for $\beta>\Phi(q)$): $\int_0^\infty e^{-\beta x}W^{(q)}(x)dx=1/(\psi(\beta)-q)$ and $\int_0^\infty e^{-\beta x}Z^{(q)}(x)dx=$

$\frac{1}{\beta}\left(1+\frac{q}{\psi(\beta)-q}\right)$ (where the latter formula follows using e.g. integration by parts). The Laplace transforms of $W_h^{(q)}$ and $Z_h^{(q)}$, $h \in (0, h_\star)$, follow from Proposition 3.3 and Remark 3.4 (ii).

Proposition 4.1 (Pointwise convergence)**.** *Suppose $\psi^h \to \psi$ and $\Phi^h \to \Phi$ pointwise as $h \downarrow 0$. Then, for each $q \geq 0$, $W_h^{(q)} \to W^{(q)}$ and $Z_h^{(q)} \to Z^{(q)}$ pointwise, as $h \downarrow 0$.*

Remark 4.2. We will see in (Δ_1) of Paragraph 5.2.1 that, in fact, $\psi^h \to \psi$ locally uniformly in $[0, \infty)$ as $h \downarrow 0$, which implies that $\Phi^h \to \Phi$ pointwise as $h \downarrow 0$. In particular, given any $q \geq 0$, $\gamma > \Phi(q)$ implies $\gamma > \Phi^h(q)$ for all $h \in (0, h_0)$, for some $h_0 > 0$.

Proof. Since $\Phi^h(q) \to \Phi(q)$ as $h \downarrow 0$, it follows via integration by parts ($\int_{[0,\infty)} e^{-\beta x} dF(x) = \beta \int_{(0,\infty)} e^{-\beta x} F(x) dx$ for any $\beta \geq 0$ and any nondecreasing right-continuous $F : \mathbb{R} \to \mathbb{R}$ vanishing on $(-\infty, 0)$ [32, Chapter 0, Proposition (4.5)]) that, for some $h_0 > 0$, the Laplace transforms of $dW^{(q)}$, $dZ^{(q)}$, $(dW_h^{(q)})_{h\in(0,h_0)}$ and $(dZ_h^{(q)})_{h\in(0,h_0)}$, are defined (i.e. finite) on a common halfline. These measures are furthermore concentrated on $[0, \infty)$ and since $\psi^h \to \psi$ pointwise as $h \downarrow 0$, then $\widehat{dW_h^{(q)}} \to \widehat{dW^{(q)}}$ and $\widehat{dZ_h^{(q)}} \to \widehat{dZ^{(q)}}$ pointwise as $h \downarrow 0$. By [6, p. 110, Theorem 8.5], it follows that $dW_{h_n}^{(q)} \to dW^{(q)}$ and $dZ_{h_n}^{(q)} \to dZ^{(q)}$ vaguely as $n \to \infty$, for any sequence $(h_n)_{n\geq 1} \downarrow 0$. This implies that, as $h \downarrow 0$, $W_h^{(q)} \to W^{(q)}$ (resp. $Z_h^{(q)} \to Z^{(q)}$) pointwise at all points of continuity of $W^{(q)}$ (resp. $Z^{(q)}$). Now, the functions $Z^{(q)}$ are continuous everywhere, whereas $W^{(q)}$ is continuous on $\mathbb{R}\backslash\{0\}$ and has a jump at 0, if and only if X has sample paths of finite variation [22, p. 222, Lemma 8.6]. In the latter case, however, we necessarily have $\sigma^2 = 0$ and $\int_{[-1,0)} |y|\lambda(dy) < \infty$ [33, p. 140, Theorem 21.9] and the jump size is $W^{(q)}(0) = 1/\mu_0$ (see Section 2 for the definition of μ_0). By Remark 3.4 (iii) and (2.3), $W_h^{(q)}(0) = 1/(h\lambda^h(\{h\})) = 1/(\mu - \mu^h + c_0^h/h)$. The latter quotient, however, converges to $1/\mu_0$, as $h \downarrow 0$, by the DCT (since $\int_{[-1,0)} |y|\lambda(dy) < \infty$ in this case, and $c_0^h \leq (h/2)\int_{[-V\wedge(h/2),0)} |y|\lambda(dy)$). □

5. Rates of convergence

5.1. **Method of proof, integral representations of scale functions, notation.** The key step in the proof of Theorem 1.2 consists of a detailed analysis of the relevant differences (see Paragraph 5.1.2) arising in the integral representations (see Paragraph 5.1.1) of the scale functions. A more exhaustive explanation of the method of proof follows in Paragraph 5.1.3.

5.1.1. *Integral representations of scale functions.*

Proposition 5.1. *Let $q \geq 0$. For all $\beta \in \mathbb{C}$ with $\Re\beta > \Phi(q)$, $\psi(\beta) - q \neq 0$ (resp. $\psi^h(\beta) - q \neq 0$) and one has $(\psi(\beta) - q)\widehat{W^{(q)}}(\beta) = 1$ (resp. $\beta h(\psi^h(\beta) - q)\widehat{W_h^{(q)}}(\beta) = e^{\beta h} - 1$), and $\beta\widehat{Z^{(q)}}(\beta) = 1 + \frac{q}{\psi(\beta)-q}$ (resp. $\beta\widehat{Z_h^{(q)}}(\beta) = 1 + \frac{q}{\psi^h(\beta)-q}$) for the scale functions of X (resp. X^h, $h \in (0, h_\star)$).*

Proof. The stipulated equalities extend from $\beta > \Phi(q)$ real, to complex β with $\Re\beta > \Phi(q)$, via analytic continuation, using expressions for the Laplace transforms of the scale functions (the latter having been noted in Section 4). In particular, so extended, they then imply $\psi(\beta) - q \neq 0$ for the range of β as given. □

Corollary 5.2 (Integral representation of scale functions)**.** *Let $q \geq 0$. For any $\gamma > \Phi(q)$, we have, for all $x > 0$ (with $\beta := \gamma + is$):*

$$W^{(q)}(x) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{\beta x}}{\psi(\beta) - q} ds \qquad \textit{and} \qquad Z^{(q)}(x) = \frac{1}{2\pi} \lim_{T\to\infty} \int_{-T}^{T} \frac{e^{\beta x}}{\beta} \left(1 + \frac{q}{\psi(\beta) - q}\right) ds. \tag{5.1}$$

Likewise, for any $h \in (0, h_\star)$ and then any $\gamma > \Phi^h(q)$, we have, for all $x \in \mathbb{Z}_h^+$ (again with $\beta := \gamma + is$):

$$W_h^{(q)}(x) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} \frac{e^{\beta(x+h)}}{\psi^h(\beta) - q} ds \qquad \textit{and} \qquad Z_h^{(q)}(x) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} \frac{e^{\beta x}}{\beta} \frac{\beta h}{1 - e^{-\beta h}} \left(1 + \frac{q}{\psi^h(\beta) - q}\right) ds. \tag{5.2}$$

Proof. First note that $W^{(q)}$ and $Z^{(q)}$ (resp. $W_h^{(q)}$ and $Z_h^{(q)}$) are of γ-exponential order for all $\gamma > \Phi(q)$ (resp. $\gamma > \Phi^h(q)$, $h \in (0, h_\star)$). Then use the inverse Laplace [14, Section 3.3] (resp. Z [18, p. 11]) transform. □

5.1.2. *The differences $\Delta_W^{(q)}$ and $\Delta_Z^{(q)}$.* For $x \geq 0$, let T_x (resp. T_x^h) denote the first entrance time of X (resp. X^h) to $[x, \infty)$, let $\underline{X}_t := \inf\{X_s : s \in [0, t]\}$ (resp. $\underline{X^h}_t := \inf\{X_s^h : s \in [0, t]\}$), $t \geq 0$, be the running infimum process and $\underline{X}_\infty := \inf\{X_s : s \in [0, \infty)\}$ (resp. $\underline{X^h}_\infty := \inf\{X_s^h : s \in [0, \infty)\}$) the overall infimum of X (resp. X^h, $h \in (0, h_\star)$).

In the case of the spectrally negative process X, it follows from [22, Theorem 8.1(iii)], regularity of 0 for $(0, \infty)$ [22, p. 212], dominated convergence and continuity of $W^{(q)}|_{(0,\infty)}$ that, for $q \geq 0$ and $\{x, y\} \subset \mathbb{R}^+$:

$$\mathsf{E}[e^{-qT_y} \mathbb{1}(\underline{X}_{T_y} \geq -x)] = \frac{W^{(q)}(x)}{W^{(q)}(x+y)} \text{ and } \mathsf{E}[e^{-qT_y} \mathbb{1}(\underline{X}_{T_y} > -x)] = \frac{W^{(q)}(x)}{W^{(q)}(x+y)}. \tag{5.3}$$

On the other hand, we find [35, Theorem 4.6] the direct analogues of these two formulae in the case of the approximating processes X^h, $h \in (0, h_\star)$ as being ($q \geq 0$, $\{x, y\} \subset \mathbb{Z}_h^{++}$):

$$\mathsf{E}[e^{-qT_y^h} \mathbb{1}(\underline{X^h}_{T_y^h} \geq -x)] = \frac{W_h^{(q)}(x)}{W_h^{(q)}(x+y)} \text{ and } \mathsf{E}[e^{-qT_y^h} \mathbb{1}(\underline{X}_{T_y^h} > -x)] = \frac{W_h^{(q)}(x-h)}{W_h^{(q)}(x-h+y)}. \tag{5.4}$$

We conclude by comparing (5.3) with (5.4) that there is *no a priori probabilistic reason* to favour either $W_h^{(q)}$ or $W_h^{(q)}(\cdot - h)$ in the choice of which of these two quantities to compare to $W^{(q)}$. Nevertheless, this choice is not completely arbitrary:

(a) In view of (5.1) and (5.2), the quantity $W_h^{(q)}(\cdot - h)$ seems more favourable (cf. also the findings of Proposition 5.8, especially when $q = |\mu| = 0$). In addition, when X has sample paths of infinite variation, a.s., $W^{(q)}(0)$ is equal to zero [21, p. 33, Lemma 3.1] and so is $W_h^{(q)}(-h)$, whereas $W_h^{(q)}(0)$ is always strictly positive ($h \in (0, h_\star)$).

(b) On the other hand, when X has sample paths of finite variation, a.s., then $W^{(q)}(0) = 1/\mu_0 > 0$ [21, p. 33, Lemma 3.1] and if in addition the Lévy measure is finite, then in fact also $W_h^{(q)}(0) = 1/\mu_0$ for all $h \in (0, h_\star)$.

Remark 5.3. (i) It follows from the above discussion that it is reasonable to approximate $W^{(q)}$ by $W_h^{(q)}(\cdot - h)$ (resp. $W_h^{(q)}$), when X has sample paths of infinite (resp. finite) variation. Indeed, in the Brownian motion with drift case, approximating $W^{(q)}$ by $W_h^{(q)}$, or even the average $(W_h^{(q)} + W_h^{(q)}(\cdot - h))/2$, rather than by $W_h^{(q)}(\cdot - h)$, would lower the order of convergence from quadratic to linear (see Proposition 5.8).

(ii) When $q = 0$ or $x = 0$, $Z^{(q)}(x) = Z_h^{(q)}(x) = 1$, $h \in (0, h_\star)$. Thus, when comparing these functions, $q \wedge x > 0$ is the only interesting case.

In view of Remark 5.3 (i) we define δ^0 to be equal to 0 or 1 according as the sample paths of X are of finite or infinite variation (a.s.). Fix $q \geq 0$. For $h \in (0, h_\star)$ we then define the differences:

$$\Delta_W^{(q)}(x,h) := W^{(q)}(x) - W_h^{(q)}(x - \delta^0 h), \quad x \in \mathbb{Z}_h^{++} \cup \{\delta_0 h\} \tag{5.5}$$

and

$$\Delta_Z^{(q)}(x,h) := Z^{(q)}(x) - Z_h^{(q)}(x), \quad x \in \mathbb{Z}_h^{++}. \tag{5.6}$$

Fix further any $\gamma > \Phi(q)$. Let $h \in (0, h_\star)$ be such that also $\gamma > \Phi^h(q)$, Then Corollary 5.2 implies, for any $x \in \mathbb{Z}_h^{++} \cup \{\delta_0 h\}$ (we always let, here and in the sequel, $\beta := \gamma + is$ to shorten notation):

$$\begin{aligned} e^{-\gamma x} 2\pi \Delta_W^{(q)}(x,h) \quad = \quad & \underbrace{\lim_{T\to\infty} \int_{(-T,T)\backslash(-\pi/h,\pi/h)} e^{isx} \frac{ds}{\psi(\beta) - q}}_{\text{(a)}} + \underbrace{\int_{[-\pi/h,\pi/h]} e^{isx} \left[\frac{\psi^h - \psi}{(\psi - q)(\psi^h - q)}\right](\beta) ds}_{\text{(b)}} + \\ & \underbrace{(1 - \delta^0) \int_{[-\pi/h,\pi/h]} e^{isx} \left(1 - e^{ish}\right) \frac{ds}{\psi^h(\beta) - q}}_{\text{(c)}} \end{aligned} \tag{5.7}$$

whereas for $x \in \mathbb{Z}_h^{++}$:

$$\begin{aligned} e^{-\gamma x} 2\pi \Delta_Z^{(q)}(x,h) = & \underbrace{\lim_{T\to\infty} \int_{(-T,T)\backslash(-\pi/h,\pi/h)} \frac{e^{isx}}{\beta} \left(\frac{q}{\psi(\beta) - q}\right) ds}_{\text{(a)}} + \\ & \underbrace{\int_{[-\pi/h,\pi/h]} \frac{e^{isx}}{\beta} \left(1 - \frac{\beta h}{1 - e^{-\beta h}}\right) \left(\frac{q}{\psi^h(\beta) - q}\right) ds}_{\text{(b)}} + \underbrace{q \int_{[-\pi/h,\pi/h]} \frac{e^{isx}}{\beta} \left[\frac{\psi^h - \psi}{(\psi^h - q)(\psi - q)}\right](\beta) ds}_{\text{(c)}}. \end{aligned} \tag{5.8}$$

Note that in (5.8) we have taken into account that the difference between the inverse Laplace and inverse Z transform, for the function, which is identically equal to 1, vanishes identically.

5.1.3. *Method for obtaining the rates of convergence in* (5.7) *and* (5.8). Apart from the Brownian motion with drift case, which can be treated explicitly, the method for obtaining the rates of convergence for the differences (5.7) and (5.8) is as follows (recall $\beta = \gamma + is$):

(1) First we estimate $|\psi^h - \psi|(\beta)$ to control the numerators. In particular, we are able to conclude $\psi^h \to \psi$, uniformly in bounded subsets of $\overline{\mathbb{C}^{\rightarrow}}$. See Paragraph 5.2.1.

(2) Then we show $|\psi^h - q|(\beta)$ is suitably bounded from below on $s \in (-\pi/h, \pi/h)$, uniformly in $h \in [0, h_0)$, for some $h_0 > 0$. See Paragraph 5.2.3. This property, referred to as *coercivity*, controls the denominators. The faster that these increase, the easier, *ceteris paribus*, it is

to prove the convergence rates. In particular, proofs for the functions $Z^{(q)}$ are always less difficult than for $W^{(q)}$, due to the presence of the extra $1/\beta$-factor in the integrands of (5.8).

(3) Finally, using (1) and (2), one can estimate the integrals appearing in (5.7) and (5.8) either by a direct $|\int \cdot ds| \leq \int |\cdot| ds$ argument, or else by first applying a combination of integrations by parts (see (5.9) below) and Fubini's Theorem. In the latter case, the estimates of $|\frac{d(\psi(\beta)-\psi^h(\beta))}{ds}|$ and the growth in s, as $|s| \to \infty$, of $\frac{d(\psi^h-q)(\beta)}{ds}$, $h \in [0, h_\star)$, also become relevant, and we provide these in Paragraph 5.2.2.

Remark 5.4. (i) Note that the integral representation of the scale functions is crucial for our programme to yield results. The formulae (5.7) and (5.8) suffice to give a precise rate locally uniformly in $x \in (0, \infty)$.

(ii) The integration by parts in (3) is applied as follows (f differentiable, $x > 0$):

$$\frac{1}{x}\frac{d}{ds}\left(e^{isx}f(s)\right) = ie^{isx}f(s) + \frac{1}{x}e^{isx}f'(s), \tag{5.9}$$

and with the integral $\int e^{isx}f(s)ds$ (over a relevant domain) in mind. Then, upon integration against ds, the left-hand side and the second term on the right-hand side of (5.9) admit for an estimate, which could not be made for $\int e^{isx}f(s)ds$ directly, but in turn a factor of $1/x$ emerges, implying (as we will see) that the final bound is locally uniform in $(0, \infty)$ (in the estimates there is always also present a factor of $e^{\gamma x}$ which from the perspective of the relative error, and in view of the growth properties of $W^{(q)}$ and $Z^{(q)}$ at $+\infty$ [21, p. 129, Lemma 3.3], is perhaps not so bad). In fact, the convergence rate obtained via (1)-(3) is uniform in bounded subsets of $(0, \infty)$, if (3) does not involve integration by parts.

(iii) Now, it is usually the case that the estimates from (3) may be made by a direct application of the integral triangle inequality. We were not able to avoid integration by parts, however, in the case of the convergence for the functions $W^{(q)}$, $q \geq 0$, when $\sigma^2 = 0$.

(iv) Even when $\sigma^2 = 0$, however, numerical experiments (see Section 6) seem to suggest that, at least in some further subcases, one should be able to establish convergence for the functions $W^{(q)}$, $q \geq 0$, which is uniform in bounded (rather than just compact) subsets of $(0, \infty)$. This remains open for future research.

Remark 5.5. Sharpness of the rates is obtained by constructing specific examples of Lévy processes, for which convergence is no better than stipulated (cf. the statement of Theorem 1.2). The key observation here is the following principle of *reduction by domination*:

> Suppose we seek to prove that $f \geq 0$ converges to 0 no faster than $g > 0$, i.e. that $\limsup_{h\downarrow 0} f(h)/g(h) \geq C > 0$ for some C. If one can show $f(h) \geq A(h) - B(h)$ and $B = o(g)$, then to show $\limsup_{h\downarrow 0} f(h)/g(h) \geq C$, it is sufficient to establish $\limsup_{h\downarrow 0} A(h)/g(h) \geq C$.

Remark 5.6. With reference to Subsection 2.2, there is of course extensive literature on numerical solutions to integro-differential equations (IDE) of the relevant (Volterra) type (viz. Eq. (2.4)).

This literature will, however, typically assume at least the continuity of the kernel appearing in the integral of the IDE, to even pose the problem, and obtain rates of convergence under additional smoothness conditions thereon (and the solution to the IDE) [10, Chapters 2 and 3] [25, Chapters 7 and 11]. In our case the kernel appearing in (2.4) is of course not (necessarily) even continuous (let alone possessing higher degrees of smoothness). Further, discounting for a moment the continuity requirement on the kernel, which may appear technical, some relevant *general* results on convergence do exist, e.g. [25, p. 102, Theorem 7.2] for the case $\sigma^2 = 0$ & $\lambda(\mathbb{R}) < +\infty$, but are not really (directly) applicable, since one would need to *a priori* establish (at least) a rate of convergence for the difference between the integral appearing in (2.4) and its discretization (local consistency error; see [25, p. 101, Eq. (7.12)]). This does not appear possible in general without a knowledge of the (sufficient) smoothness properties of the target function $W^{(q)}$ (the latter not always being clear; see [12]) and indeed, it would seem, those of the tail $(y \mapsto \lambda(-\infty, -y))$. Such an error analysis would be further complicated when $\sigma^2 > 0$ (resp. $\kappa(0) = \infty$), since then we are dealing with the discretization also of the derivative of $W^{(q)}$ (resp. the integral in (2.4) cannot be split up as the difference of the integrals of each individual term of the integrand). It is then not very likely that looking at this problem from the integro-differential perspective alone would allow us to obtain, moreover sharp, rates of convergence (at least not in general).

By contrast, the method for obtaining the sharp rates of convergence just described, allows to handle all the cases within a single framework.

5.1.4. *Further notation.* Notation-wise, we let (where $\delta \in [0, 1]$):

$$\xi(\delta) := \int_{[-\delta,0)} u^2\lambda(du),\ \kappa(\delta) := \int_{[-1,-\delta)} |y|\lambda(dy),\ \zeta(\delta) := \delta\kappa(\delta) \text{ and } \gamma(\delta) := \delta^2\lambda([-1, -\delta))$$

and remark that, by the findings of [29, Lemma 3.8], $\gamma(\delta) + \zeta(\delta) + \xi(\delta) \to 0$ as $\delta \downarrow 0$.

Finally, note that, unless otherwise indicated, we consider henceforth as having fixed:

$$X, \quad (X^h)_{h\in(0,h_\star)}, \quad q \geq 0 \quad \text{and a} \quad \gamma > \Phi(q).$$

We insist that the dependence on x of the error estimates will be kept explicit throughout, whereas the dependence on the Lévy triplet, q and γ will be subsumed in the capital (or small) O (o) notation. In particular, the notation $f(x, h) = g(x, h) + l(x)O(h)$, $x \in A$, means that $l(x) > 0$ for $x \in A$ and:

$$\sup_{x\in A} |(f(x, h) - g(x, h))/l(x)| = O(h)$$

and analogously when $O(h)$ is replaced by $o(h)$ etc. Further, we shall sometimes resort to the notation:

$$A(s) := A^0(s) := \psi(\gamma + is) - q \text{ (for } s \in \mathbb{R}) \text{ and } A^h(s) := \psi^h(\gamma + is) - q \text{ (for } s \in [-\pi/h, \pi/h]),$$

$h \in (0, h_\star)$, where reference to q and γ has been suppressed.

5.2. **Auxiliary results.** In this subsection we shall have throughout:

$$\beta := \gamma + is.$$

5.2.1. *Estimating the absolute difference* $|\psi^h - \psi|$.

(Δ_1) $\psi^h \to \psi$ as $h \downarrow 0$, uniformly in bounded subsets of $\overline{\mathbb{C}^{\rightarrow}}$.

(Δ_2) There exists $A_0 \in (0,\infty)$, such that for all $h \in (0, h_\star \wedge 2)$ and then all $s \in [-\pi/h, \pi/h]$, the following holds (see Table 1 for values of the parameter V):

(i) When $\sigma^2 > 0$, $|\psi^h - \psi|(\beta) \leq A_0 \left[h^2|\beta|^4 + h\xi(h/2)|\beta|^3 + h|\beta| + V\zeta(h/2)|\beta|^2\right]$. In particular, if in addition $\kappa(0) < \infty$, we have $|\psi^h - \psi|(\beta) \leq A_0[h^2|\beta|^4 + h|\beta|^2]$. If, moreover, $\lambda(\mathbb{R}) < \infty$, then $|\psi^h - \psi|(\beta) \leq A_0[h^2|\beta|^4 + h|\beta|]$.

(ii) When $\sigma^2 = 0$, $|\psi^h - \psi|(\beta) \leq A_0 \left[h\xi(h/2)|\beta|^3 + (h + \zeta(h/2))|\beta|^2\right]$. If in addition $\kappa(0) < \infty$, then$|\psi^h - \psi|(\beta) \leq A_0 h|\beta|^2$.

Sketch proof of (Δ_1) and (Δ_2). Decompose, referring to (2.1), and to (2.2) when $\sigma^2 > 0$ (resp. to (2.3) when $\sigma^2 = 0$), the difference $\psi^h - \psi$ into terms, which allow for elementary estimates. To wit, one considers, for any fixed $h_0 \in (0, 2]$ and $\rho_0 > 0$, the following expressions in $h \in (0, h_0)$, $s \in [-\pi/h, \pi/h]$, $\rho \in [0, \rho_0]$ (with $\alpha := \rho + is$): $\sigma^2\left(\frac{e^{\alpha h}+e^{-\alpha h}-2}{2h^2} - \frac{\alpha^2}{2}\right)$; $c_0^h\left(\frac{e^{\alpha h}+e^{-\alpha h}-2}{2h^2} - \frac{\alpha^2}{2}\right)$; $V\int_{[-h/2,0)} y^2\frac{\alpha^2}{2}\lambda(dy) - \int_{[-h/2,0)}\left(e^{\alpha y} - V\alpha y - 1\right)\lambda(dy)$; $\mu\left(\frac{e^{\alpha h}-e^{-\alpha h}}{2h} - \alpha\right)$ (resp. $\mu\left(\frac{e^{\alpha h}-1}{h} - \alpha\right)$); $(-\mu^h)\left(\frac{e^{\alpha h}-e^{-\alpha h}}{2h} - \alpha\right)$ (resp. $(-\mu^h)\left(\frac{e^{\alpha h}-1}{h} - \alpha\right)$); $\sum_{y\in\mathbb{Z}_h^{--}}\int_{A_y^h\cap(-\infty,-V)}\left(e^{\alpha y} - e^{\alpha z}\right)\lambda(dz)$; $\sum_{y\in\mathbb{Z}_h^{--}}\int_{A_y^h\cap[-V,0)}\left(e^{\alpha y} - e^{\alpha z} - V\alpha(y-z)\right)\lambda(dz)$. Note these expressions sum to $(\psi^h - \psi)(\alpha)$. The relevant trigonometric estimates, allowing to suitably bound these terms, may then be found in [30, Paragraph 5.2.1]; see [30, Subsection 5.3] for full detail. □

Remark 5.7. Pursuant to (Δ_1) above and Remark 4.2, we assume henceforth that $h_\star$ has already been chosen small enough, so that in addition $\gamma > \Phi^h(q)$ for all $h \in (0, h_\star)$

5.2.2. *Estimating the absolute difference* $|A^{h\prime} - A'|$ *and growth of* $A^{h\prime}$ *at infinity.*

(Δ_1') For any finite $h_0 \in (0, h_\star]$, there exists an $A_0 \in (0,\infty)$ such that for all $h \in [0, h_0)$ and then all $s \in (-\pi/h, \pi/h)$, $|A^{h\prime}(s)| \leq A_0|\beta|^{\epsilon-1}$, where $\epsilon = 2$, if $\sigma^2 > 0$; $\epsilon = 1$, if $\sigma^2 = 0$ and $\kappa(0) < \infty$; finally, if $\sigma^2 = 0$ and $\kappa(0) = \infty$, then ϵ must satisfy Assumption 1.1(1) from the Introduction.

(Δ_2') There is an $A_0 \in (0,\infty)$, such that for all $h \in (0, h_\star \wedge 2)$ and then all $s \in [-\pi/h, \pi/h]$, the following holds:

(i) When $\sigma^2 > 0$, $|A^{h\prime}(s) - A'(s)| \leq A_0(h^2|\beta|^3 + h\xi(h/2)|\beta|^2 + (h + \zeta(h/2))|\beta|)$.

(ii) When $\sigma^2 = 0$, $|A'(s) - A^{h\prime}(s)| \leq A_0\left[h + \zeta(h/2) + \xi(h/2)\right]|\beta|$. If in addition $\kappa(0) < \infty$, then $|\psi^h - \psi|(\beta) \leq A_0 h|\beta|$.

Sketch proof of (Δ_1') and (Δ_2'). Using differentiation under the integral sign, (2.1), and (2.2) when $\sigma^2 > 0$ (resp. (2.3) when $\sigma^2 = 0$), yields the following expression for A': $A'(s) = i\sigma^2\beta + i\mu + i\int_{(-\infty,0)} z\left(e^{\beta z} - \mathbb{1}_{[-V,0)}(z)\right)\lambda(dz)$; and $A^{h\prime}$: $A^{h\prime}(s) = i(\mu - \mu^h)\frac{e^{\beta h}+e^{-\beta h}}{2} +$

$(\sigma^2 + c_0^h)i\frac{e^{\beta h}-e^{-\beta h}}{2h} + i\sum_{y\in\mathbb{Z}_h^{--}} c_y^h y e^{\beta y}$, resp. $A^{h\prime}(s) = \frac{ic_0^h}{2h}\left(e^{\beta h} - e^{-\beta h}\right) + i(\mu - \mu^h)e^{\beta h} + i\sum_{y\in\mathbb{Z}_h^{--}} c_y^h y e^{\beta y}$. The conclusions of (Δ_1') then follow readily in the case when either $\sigma^2 > 0$ or $\kappa(0) < \infty$. The remaining instance uses the fact that, as a consequence of Assumption 1.1(1), each of the quantities $\limsup_{\delta\downarrow 0}\delta^{\epsilon-1}\int_{[-1,1]\backslash[-\delta,\delta]}|u|\lambda(du)$, $\limsup_{\delta\downarrow 0}\delta^{\epsilon-2}\int_{[-\delta,\delta]}u^2\lambda(du)$ and $\sup_{s\in\mathbb{R}\backslash\{0\}}\frac{1}{|s|^{\epsilon-1}}\left|\int_{[-1,1]}(e^{isy}-1)|y|\lambda(dy)\right|$ is finite [30, Paragraph 5.2.2, Proposition 5.4]. Finally, to show (Δ_2'), one employs a decomposition similar to the one used in Paragraph 5.2.1 (which allowed to establish (Δ_2)). See [30, Subsection 5.4] for full detail. □

5.2.3. *Coercivity of* $|\psi^h - q|$.

(C) There exists an $h_0 \in (0, h_\star]$ and a $B_0 \in (0,\infty)$, such that for all $h \in [0, h_0)$ and then all $s \in (-\pi/h, \pi/h)$ (recall $\psi^0 = \psi$, $\beta = \gamma + is$) $|\psi^h(\beta) - q| \geq B_0|\beta|^\epsilon$, where $\epsilon = 2$, if $\sigma^2 > 0$; $\epsilon = 1$, if $\sigma^2 = 0$ and $\kappa(0) < \infty$; finally, if $\sigma^2 = 0$ and $\kappa(0) = \infty$, then ϵ must satisfy Assumption 1.1(2) from the Introduction.

Sketch proof of (C). Again refer to expressions (2.1), (2.2) and (2.3). There are two key observations: (i) $(s \mapsto (\psi - q)(\beta))$ is bounded away from zero on bounded subsets of $\mathbb{R}$, by continuity and Proposition 5.1; (ii) as $h \downarrow 0$, $\psi^h(\beta) - q \to \psi(\beta) - q$ uniformly in s belonging to bounded sets, hence, on any given bounded set, the maps $(s \mapsto (\psi^h - q)(\beta))$ are also bounded away from zero, uniformly in all h small enough. Thus it is sufficient to establish the relevant coercivity appearing in (C) outside a (sufficiently large) bounded set. In particular, when $\sigma^2 > 0$, it can be shown via elementary means, that the terms involving σ^2 (suitably estimated) yield a quadratic term in s, whilst the others have strictly subquadratic growth in s. When $\sigma^2 = 0$, but $\kappa(0) < \infty$, the terms involving μ_0 ($\mu - \mu^h \to \mu_0$ as $h \downarrow 0$) grow linearly with s, whilst the others have strictly sublinear growth in s. Finally, one proceeds similarly in the most intricate instance $\sigma^2 = 0$ & $\kappa(0) < \infty$, where, in particular, one uses the ideas which appear in the proofs of [33, p. 190, Proposition 28.3] and [29, p. 17, Proposition 4.1], in order to get the $|s|^\epsilon$-growth in s. See [30, Subsection 5.5] for full detail. □

5.3. **Convergence rates.** Now the main results of this paper follow.

Proposition 5.8 ($\sigma^2 > 0 = \lambda(\mathbb{R})$)**.** *Suppose $\sigma^2 > 0 = \lambda(\mathbb{R})$ and let $q \geq 0$. If $q \vee |\mu| = 0$, then for all $h \in (0, h_\star)$ and all $x \in \mathbb{Z}_h^{++}$: $\Delta_W^{(q)}(x,h) = 0$. If, however, $q \vee |\mu| > 0$, then:*

(i) There exist $A_0, h_0 \subset (0,\infty)$ such that for $h \in (0, h_0)$ and then $x \in \mathbb{Z}_h^{++}$ with $xh^2 \leq 1$:

$$\left|\Delta_W^{(q)}(x,h)\right| \leq A_0 h^2(1+x)e^{\alpha_+ x} \text{ and } \left|\Delta_Z^{(q)}(x,h)\right| \leq A_0\left[h^2(1+x)e^{\alpha_+ x} + h(e^{\alpha_+ x} - e^{\alpha_- x})\right].$$

(ii) For any nested sequence $(h_n)_{n\geq 1} \downarrow 0$ and then any $x \in \cup_{n\geq 1}\mathbb{Z}_{h_n}^{++}$:

$$\lim_{n\to\infty}\frac{\Delta_W^{(q)}(x,h_n)}{h_n^2} = \frac{q^2}{2(\mu^2 + 2\sigma^2 q)}W^{(q)}(x) + \frac{x}{\sqrt{\mu^2 + 2\sigma^2 q}}\left(e^{\alpha_+ x}\theta_+ - e^{\alpha_- x}\theta_-\right).$$

(In particular, when $q = 0$, this limit is $-\frac{2}{3}\frac{\mu^2 x}{(\sigma^2)^3}e^{-2\mu x/\sigma^2}$.)

Here $\alpha_\pm := \frac{-\mu\pm\sqrt{\mu^2+2q\sigma^2}}{\sigma^2}$; $\theta_\pm := \frac{\mu^3\sqrt{2q\sigma^2+\mu^2}\pm(\frac{1}{2}q^2(\sigma^2)^2-\mu^4-\mu^2\sigma^2 q)}{3(\sigma^2)^3\sqrt{2q\sigma^2+\mu^2}}$.

Proof. The scale functions can be calculated explicitly here. The claim then follows by Taylor expansions. □

Proposition 5.9 ($\sigma^2>0$)**.** *Suppose $\sigma^2>0$ and let $q\geq 0$.*

(i) For any $\gamma>\Phi(q)$, there are $A_0,h_0\in(0,\infty)$, such that for $h\in(0,h_0)$ and then $x\in\mathbb{Z}_h^{++}$:

$$\left|\Delta_W^{(q)}(x,h)\right|\leq A_0\left(h+\zeta(h/2)+\xi(h/2)h\log(1/h)\right)e^{\gamma x}\ \text{and}\ \left|\Delta_Z^{(q)}(x,h)\right|\leq A_0\left(h+\zeta(h/2)\right)e^{\gamma x}.$$

In particular, if $\kappa(0)<\infty$, then $\left|\Delta_W^{(q)}(x,h)\right|+\left|\Delta_Z^{(q)}(x,h)\right|\leq A_0he^{\gamma x}$ and under Assumption 1.1, $\left|\Delta_W^{(q)}(x,h)\right|+\left|\Delta_Z^{(q)}(x,h)\right|\leq A_0h^{2-\epsilon}e^{\gamma x}$.

(ii) There exist:

(a) a Lévy triplet (σ^2,λ,μ) with $\sigma^2\neq 0$ and $0<\kappa(0)<\infty$;

(b) for each $\epsilon\in(1,2)$, a Lévy triplet (σ^2,λ,μ) with $\sigma^2\neq 0$ and $\lambda(-1,-\delta)\sim 1/\delta^\epsilon$ as $\delta\downarrow 0$;

and then in each of the cases (a)-(b), a nested sequence $(h_n)_{n\geq 1}\downarrow 0$, such that for each $q\geq 0$, there is an $x\in\cup_{n\geq 1}\mathbb{Z}_{h_n}^{++}$, with:

$$\liminf_{n\to\infty}\frac{\left|\Delta_W^{(q)}(x,h_n)\right|}{h_n\vee\zeta(h_n)}>0,$$

where $h_n\vee\zeta(h_n)\sim h_n$, if $\kappa(0)<\infty$ and $\sim h_n^{2-\epsilon}$, if $\kappa(0)=\infty$, as $n\to\infty$.

Remark 5.10. Note that if $\lambda(-1,\delta)\sim 1/\delta^\epsilon$ as $\delta\downarrow 0$, with $\epsilon\in(1,2)$, then (as $h\downarrow 0$) $\xi(h/2)\sim h^{2-\epsilon}$, and $\zeta(h/2)\sim h^{2-\epsilon}$, so that $h\xi(h/2)\log(1/h)=o(h\kappa(h/2))$. More generally, Assumption 1.1 is fulfilled if $\lambda(-1,-\delta)\sim\delta^{-\epsilon}l(\delta)$ where $0<\liminf_{0+}l<\limsup_{0+}l<+\infty$. Finally, under Assumption 1.1, always, $\zeta(h/2)+\xi(h/2)=O(h^{2-\epsilon})$ as $h\downarrow 0$.

Proof. First, with respect to (i) and the functions $W^{(q)}$, we have as follows. (5.7)(a) is seen immediately to be of order $O(h)$ by coercivity (C); whereas (5.7)(b) is of order $O(h+h\xi(h/2)\log(1/h)+V\zeta(h/2))$ by coercivity (C) and the estimate of the absolute difference $|\psi^h-\psi|$ (Δ_2). Since $\delta^0=1$, (5.7)(c) is void. The argument for the functions $Z^{(q)}$ is similar, via the decomposition (5.8) (but uses also the fact that $\sup_{s_0\in[-\pi,\pi],0<|\gamma_0|\leq K}|(1-\frac{\gamma_0+is_0}{1-e^{-\gamma_0-is_0}})\frac{1}{\gamma_0+is_0}|<\infty$ for any $K<\infty$, in order to estimate (5.8)(b)).

Second we prove (ii)(a). Take $\lambda=\delta_{-1/2}$, $h_n=1/3^n$ ($n\geq 1$), $\mu=0$, $\sigma^2=1$, $x\in\cup_{n\geq 1}\mathbb{Z}_{h_n}^{++}$ (x is now fixed!). The goal is to establish no better than linear convergence in this case. Indeed (5.7)(a) is actually of order $O(h^2)$. We can thus focus on (5.7)(b). Further reduction by domination (using e.g. the fact that the scale function $W^{(q)}$ converge at a quadratic rate for Brownian motion with drift) then allows to concentrate on $\frac{1}{2\pi i}\int_{[-\pi/h_n,\pi/h_n]}e^{\beta x}\frac{e^{-\beta/2}\beta}{(\psi-q)(\beta)(\psi^{h_n}-q)(\beta)}ds$, which we would like bounded away from 0, as $n\to\infty$. Now, by coercivity (C), and the DCT, this expression in fact converges to $\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{\beta(x-1/2)}\beta}{(\psi-q)^2(\beta)}ds$ which clearly cannot vanish identically on $x\in\cup_{n\geq 1}\mathbb{Z}_{h_n}^{++}$.

Consider finally (ii)(b). We take $\sigma^2=1$, $\mu=0$, $h_n=1/3^n$ ($n\geq 1$), $\lambda=\sum_{k=1}^\infty w_k\delta_{-x_k}$, $x_k=\frac{3}{2}h_k$ and $w_k=1/x_k^\epsilon$ ($k\geq 1$), and we are seeking to establish strictly worse than linear convergence here, since $\kappa(0)=\infty$. For sure (5.7)(a) is of order $O(h)$. When it comes to (5.7)(b), consider its decomposition, in the numerator of the integrand, according to the items appearing in the proof of (Δ_2) of Paragraph 5.2.1. Then, apart from the part corresponding to $\mathbb{1}_{[-1,0)}\cdot\lambda$, these are seen to contribute terms of order $o(\zeta(h/2))$ to (5.7)(b). Further reduction by domination, allows to focus on $(\sigma_2-\sigma_1)\int_{[-\pi/h_n,\pi/h_n]}e^{isx}\frac{\beta^2}{[(\psi-q)(\psi^{h_n}-q)](\beta)]}ds$, where $\sigma_1:=\int_{[-1,-h_n/2)}u^2\lambda(du)=\sum_{k=1}^n x_k^2w_k$ and $\sigma_2:=\sum_{k=1}^n(x_k-h_n/2)^2w_k$, so $\sigma_1-\sigma_2=2\zeta(h_n/2)-\gamma(h_n/2)\geq\zeta(h_n/2)$. Moreover, $\int_{[-\pi/h_n,\pi/h_n]}e^{isx}\frac{\beta^2(\sigma_2-\sigma_1)}{[(\psi-q)(\psi^{h_n}-q)](\beta)]}ds\to\int_{\mathbb{R}}e^{isx}\frac{\beta^2}{(\psi-q)^2(\beta)}ds$ as $n\to\infty$ by the DCT. This integral does not vanish simultaneously in all $x\in\cup_{n\geq 1}\mathbb{Z}_{h_n}^{++}$, whence tightness obtains. □

Proposition 5.11 ($\sigma^2=0$)**.** *Suppose $\sigma^2=0$, $q\geq 0$, $\gamma>\Phi(q)$. There are $A_0,h_0\in(0,\infty)$ such that for $h\in(0,h_0)$ and then $x\in\mathbb{Z}_h^+$:*

$$\left|\Delta_W^{(q)}(x,h)\right|\leq A_0\frac{h}{x}e^{\gamma x}\text{ and }\left|\Delta_Z^{(q)}(x,h)\right|\leq A_0he^{\gamma x},$$

if $\kappa(0)<\infty$, while under Assumption 1.1, for all $x\in\mathbb{Z}_h^{++}$:

$$\left|\Delta_W^{(q)}(x,h)\right|\leq A_0\frac{h^{2-\epsilon}}{x}e^{\gamma x}\text{ and }\left|\Delta_Z^{(q)}(x,h)\right|\leq A_0h^{2-\epsilon}e^{\gamma x}.$$

Proof. For the finite variation claims, the key lemma is:

Lemma 5.12. *Suppose $\sigma^2=0$, $\kappa(0)<\infty$. Let $\{l,a,b,M\}\subset\mathbb{N}_0$ and let $h_0\in(0,h_\star]$ be given by the coercivity condition (C).*

(1) If $a+b+l\geq M+1$, then $\sup_{(h,z)\in(0,h_0)\times\mathbb{R}}\left|\int_{[-\frac{\pi}{h},\frac{\pi}{h}]}e^{isz}\frac{(h\wedge 1)^{l-l\wedge(M+1)}h^{l\wedge(M+1)}s^M}{A(s)^aA^h(s)^b}ds\right|<\infty$.

(2) If even $a+b+l\geq M+2$, then

$$\sup_{(h,x,z)\in(0,h_0)\times\mathbb{R}\times(\mathbb{R}\backslash\{0\})}\left|\frac{1}{z}\int_{[-\frac{\pi}{h},\frac{\pi}{h}]}e^{isx}\frac{(e^{isz}-1)(h\wedge 1)^{l-l\wedge(M+2)}h^{l\wedge(M+2)}s^M}{A(s)^aA^h(s)^b}ds\right|<\infty.$$

Proof. Suppose $l=b=0$ and $a=M+1$ (resp. $a=M+2$) for simplicity (this is without loss of generality, due to (C) and (Δ_2)). Then for large $|s|$, the integrand in (1) (resp. (2)) behaves as $\sim e^{isz}/s$ (resp. $e^{isx}(e^{isz}-1)/s^2$) in the variable s. The proof then consists of a modification of the argument implying that (in the sense of Cauchy's principal values, as appropriate) $\int_{[-\pi/h,\pi/h]}(e^{isz}/s)ds$ and $\int_{[-\pi/h,\pi/h]}(e^{isx}(e^{isz}-1)/s^2)ds/z$ are bounded in the relevant suprema (as they are). A key observation in this programme is that, by decomposing:

$$A(s)=\psi(\beta)-q=\overbrace{\mu_0\gamma+\int(e^{\gamma y}\cos(sy)-1)\lambda(dy)-q}^{=:A_e(s)}+\overbrace{is\mu_0+i\int e^{\gamma y}\sin(sy)\lambda(dy)}^{=:A_o(s)}$$

into its even and odd part, crucially, $\int_1^\infty\frac{ds}{s^2}|A_e(s)|<\infty$. We avoid making explicit the remaining, tedious, but ultimately elementary, details (they may be found in the proof of [30, Lemma 5.6]). □

Consider now the convergence of the functions $W^{(q)}$. First, (5.7)(a) is easily seen to be of order $\frac{1}{x}O(h)$ by an obvious integration by parts argument, using coercivity and the fact that A' is bounded (Δ_1'). Second, when it comes to (5.7)(b), an integration by parts is performed immediately:

$$\frac{d}{ds}\left(e^{isx}\left(\frac{1}{A(s)}-\frac{1}{A^h(s)}\right)\right)=ixe^{isx}\left(\frac{1}{A(s)}-\frac{1}{A^h(s)}\right)+e^{isx}\left(-\frac{A'(s)}{A(s)^2}+\frac{A^{h\prime}(s)}{A^h(s)^2}\right).$$

Upon integration on $[-\pi/h,\pi/h]$, by coercivity, the left-hand side is of order $O(h)$ and hence will contribute $\frac{1}{x}O(h)$ to the right-hand side of (5.7). It is less easy to see that the second term the right-hand side is of order $O(h)$ also. Nevertheless, it can be done, via elementary means (decompositions, estimates of trigonometric/hyperbolic functions, Fubini's theorem), exploiting (Δ_2), (Δ_2'), (C) and Lemma 5.12 in the process (see proof of [30, Proposition 5.9] for the detailed steps). Third, with respect to of (5.7)(c), again an integration by parts is made outright, thus:

$$\frac{d}{ds}\left(e^{isx}\frac{1-e^{ish}}{A^h(s)}\right)=ixe^{isx}\frac{1-e^{ish}}{A^h(s)}+e^{isx}\frac{-ihe^{ish}}{A^h(s)}-e^{isx}\frac{(1-e^{ish})A^{h\prime}(s)}{A^h(s)^2}.$$

Now the left-hand side is handled using coercivity (C). On the right-hand side, we apply Lemma 5.12(1) to the second term. Finally, in the third term on the right-hand side, we may replace $A^{h\prime}(s)$ by $A'(s)$, followed by Fubini for A' and an application of Lemma 5.12(2). All in all, a term of order $\frac{1}{x}O(h)$ thus emerges on the right-hand side of (5.7).

Proving the convergence result for the functions $Z^{(q)}$ in the finite variation case employs similar techniques (but is easier).

Now assume $\kappa(0)=\infty$.

With respect to the functions $W^{(q)}$, we have as follows. (5.7)(a) is $\frac{1}{x}O(h^\epsilon)$ by an integration by parts argument and (Δ_1'). We do the same for (5.7)(b):

$$\frac{d}{ds}\left(e^{isx}\left(\frac{1}{A(s)}-\frac{1}{A^h(s)}\right)\right)=ixe^{isx}\left(\frac{1}{A(s)}-\frac{1}{A^h(s)}\right)+e^{isx}\left(\frac{A^2(s)(A^{h\prime}(s)-A'(s))+A'(s)(A(s)-A^h(s))(A(s)+A^h(s))}{A^2(s)A^h(s)^2}\right).$$

Upon integration, one gets on the left-hand side a contribution of $\frac{1}{x}O(h^\epsilon)$ to (5.7), by coercivity (C). With regard to the rightmost quotient on the right-hand side, we obtain a contribution of order $\frac{1}{x}O(h^{2-\epsilon})$, as follows again by coercivity (C), Remark 5.10, (Δ_1') and the estimates (Δ_2) and (Δ_2'). Remark that $\epsilon > 2-\epsilon$.

Finally we consider the functions $Z^{(q)}$. (5.8)(a) is of order $O(h^\epsilon)$ by coercivity (C). Also, (5.8)(b) is of order $O(h)$. Finally (5.8)(c) is of order $O(h^{2-\epsilon})$ immediately, with no need for an integration by parts. □

Proposition 5.13 (Convergence of $W^{(q)\prime}$ $(\sigma^2>0)$)**.** *Let $q\geq 0$, $\sigma^2>0$. Note that $W^{(q)}$ is then differentiable on* $(0,\infty)$ [21, Lemma 2.4]. *Moreover, for any $\gamma>\Phi(q)$, there are $A_0, h_0\subset(0,\infty)$, such that for $h\in(0,h_0)$ and $x\in\mathbb{Z}_h^{++}\backslash\{h\}$:*

$$\left|W^{(q)\prime}(x)-\frac{W_h^{(q)}(x)-W_h^{(q)}(x-2h)}{2h}\right|\leq A_0\frac{e^{\gamma x}}{x}\left(h+\zeta(h/2)+\xi(h/2)h\log(1/h)\right).$$

Proof. An integral representation may be got for $W^{(q)\prime}$ from its Laplace transform. Then one employs a combination of the techniques developed in the proofs of the previous two propositions, in order to estimate the difference between $W^{(q)\prime}$ and the apposite difference quotient of $W_h^{(q)}$. $\square$

6. Numerical illustrations and concluding remarks

6.1. **Numerical examples.** We illustrate our algorithm for computing W, described in Eq. (1.1) of the Introduction, in two concrete examples, applying it to determine some relevant quantities arising in applied probability. The examples are chosen with two criteria in mind:
(1) They are natural from the modeling perspective (computation of (Example 6.1) ruin parameters in the classical Cramér-Lundberg model with log-normal jumps and (Example 6.2) the Lévy-Kintchine triplet of the limit law of a CBI process).
(2) They do not posses a closed form formula for the Laplace exponent of the spectrally negative Lévy processes. Such examples arise often in practice, making it difficult to apply the standard algorithms for scale functions based on Laplace inversion. Our algorithm is well-suited for such applications.

Example 6.1. A popular choice for the claim-size modeling in the Cramér-Lundberg surplus process is the log-normal distribution [3, Paragraph I.2.b, Example 2.8]. Fixing the values of the various parameters, consider the spectrally negative Lévy process X having $\sigma^2 = 0$; $\lambda(dy) = \mathbb{1}_{(-\infty,0)}(y)\exp(-(\log(-y))^2/2)/(\sqrt{2\pi}(-y))dy$; and (with $V = 0$) $\mu = 5$ (this satisfies the security loading condition [23, Section 1.2]). Remark that the log-normal density has fat tails and is not completely monotone.

We complement the computation of W by applying it to the calculation of the density k of the deficit at ruin, on the event that X goes strictly above the level $a = 5$, before venturing strictly below 0, conditioned on $X_0 = x = 2$, $\mathsf{E}_x[-X_{\tau_0^-} \in dy, \tau_0^- < \tau_a^+] = k(y)dy$ (τ_0^-, respectively τ_a^+, being the first entrance time of X to $(-\infty, 0)$, respectively (a,∞)). Indeed, $k(y)$ may be expressed as [23, Theorem 5.5] $k(y) = \int_0^a f(z+y)\frac{W(x)W(a-z)-W(a)W(x-z)}{W(a)}dz$, where $f(y) := \exp(-(\log(y))^2/2)/(\sqrt{2\pi}y)$, $y \in (0,+\infty)$. We approximate the integral k by the discrete sum k_h, given for $y \in (0,\infty)$ as follows:

$$k_h(y) := h\left[f(y+a)\frac{W_h(x)W_h(0)}{2W_h(a)} + \sum_{k=1}^{a/h-1} f(kh+y)W_h(a-kh)\frac{W_h(x)}{W_h(a)} - \sum_{k=1}^{x/h-1} W_h(x-kh)f(kh+y) - \frac{W_h(0)f(x+y)}{2}\right].$$

Results are reported in Figure 1.

Example 6.2. We take $\sigma^2 = 0$; the Lévy measure $\lambda = \lambda_a + \lambda_c$ has atomic part $\lambda_a = \frac{1}{2}(\delta_{-1} + \delta_{-2})$, whilst the density of its absolutely continuous part $\lambda_c(dy) = l(y)dy$ is given by:

$$l(y) = \frac{3}{2(-y)^{5/2}}\mathbb{1}_{[-1,0)}(y) + \frac{1}{2(-y-1)^{1/2}}\mathbb{1}_{[-2,-1)}(y) + \left(\frac{e^{\cos(y)}(3+y\sin(y))}{(-y)^4} + \frac{e}{(-y)^3}\right)\mathbb{1}_{(-\infty,-1)}(y),\ y \in \mathbb{R};$$

and (with $V = 1$) $\mu = 15$. Remark the case is extreme: there are two atoms, while the density is stable-like at 0, has a fat tail at $-\infty$, and a discontinuity (indeed, a pole; in particular, it is not completely monotone). Furthermore, there is no Gaussian component, and the sample paths of the process have infinite variation.

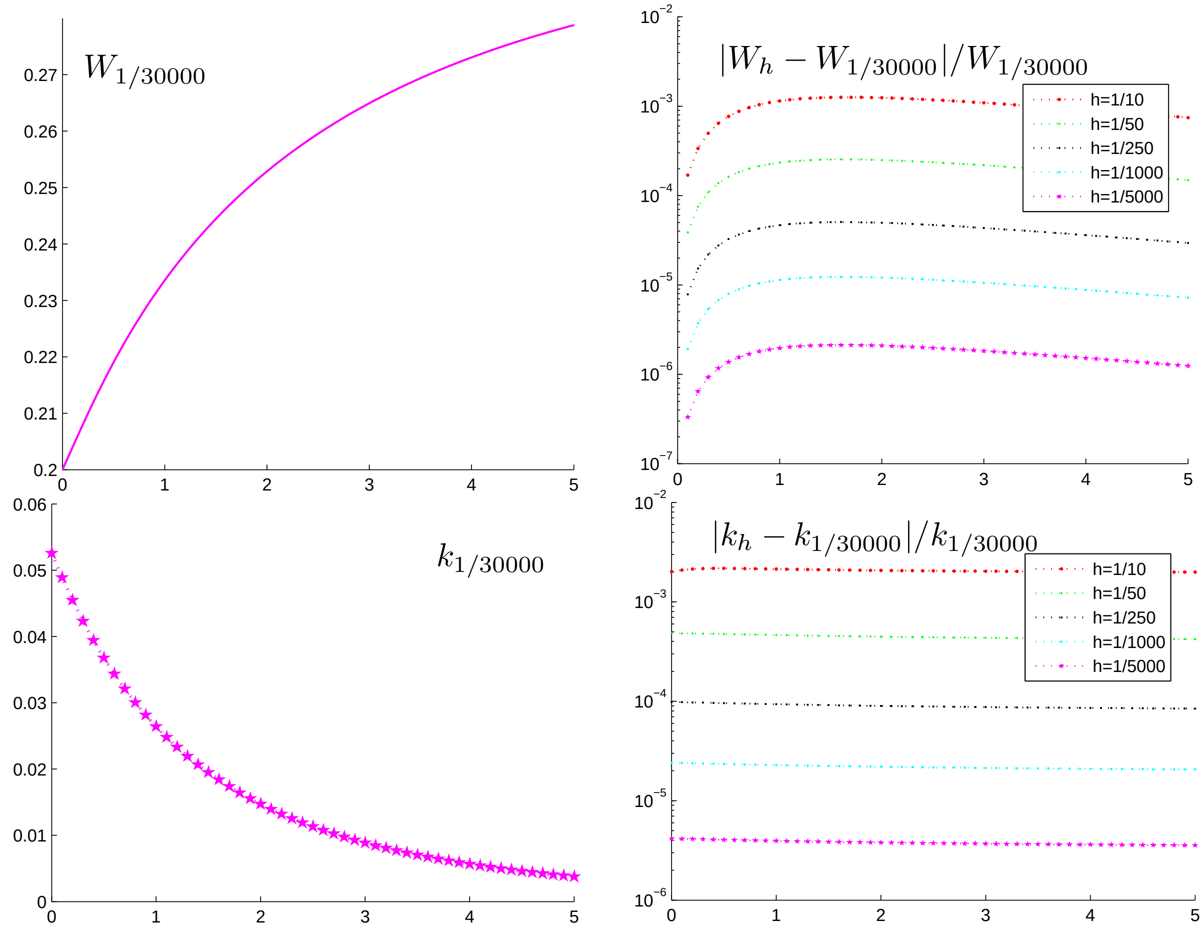

FIGURE 1. The scale function W and the density k of the deficit at ruin on the event $\{\tau_0^- < \tau_5^+\}$, for the log-normal Cramér-Lundberg process. The relative errors are consistent with the linear order of convergence predicted by Theorem 1.2. See Example 6.1 for details.

We compute W for the Lévy process X having the above characteristic triplet, and complement this with the following application. Let furthermore X^F be an independent Lévy subordinator, given by $X_t^F = t + Z_t$, $t \in [0, \infty)$, where Z is a compound Poisson process with Lévy measure $m(dy) = e^{-y}\mathbb{1}_{(0,\infty)}(y)dy$. Denote the dual $-X$ of X by X^R. To the pair (X^F, X^R) there is associated, in a canonical way, (the law of) a (conservative) CBI process [19]. The latter process converges to a limit distribution L, as time goes to infinity, since $\psi'(0+) > 0$ [20, Theorem 2.6(c)] and since further the log-moment of m away from zero is finite [20, Corollary 2.8]. Moreover, the limit L is infinitely divisible; specifically (for $u \geq 0$) $-\log \int_0^\infty e^{-ux} dL(x) = u\gamma - \int_{(0,\infty)} (e^{-ux} - 1)\frac{k(x)}{x}dx$, with $\gamma = bW(0)$ vanishing, whilst $k(x) = bW'(x+) + \int_{(0,\infty)}[W(x) - W(x - \xi)]m(d\xi)$, where $b = 1$ and m are the drift, respectively the Lévy measure, of X^F [20, Theorem 3.1]. We

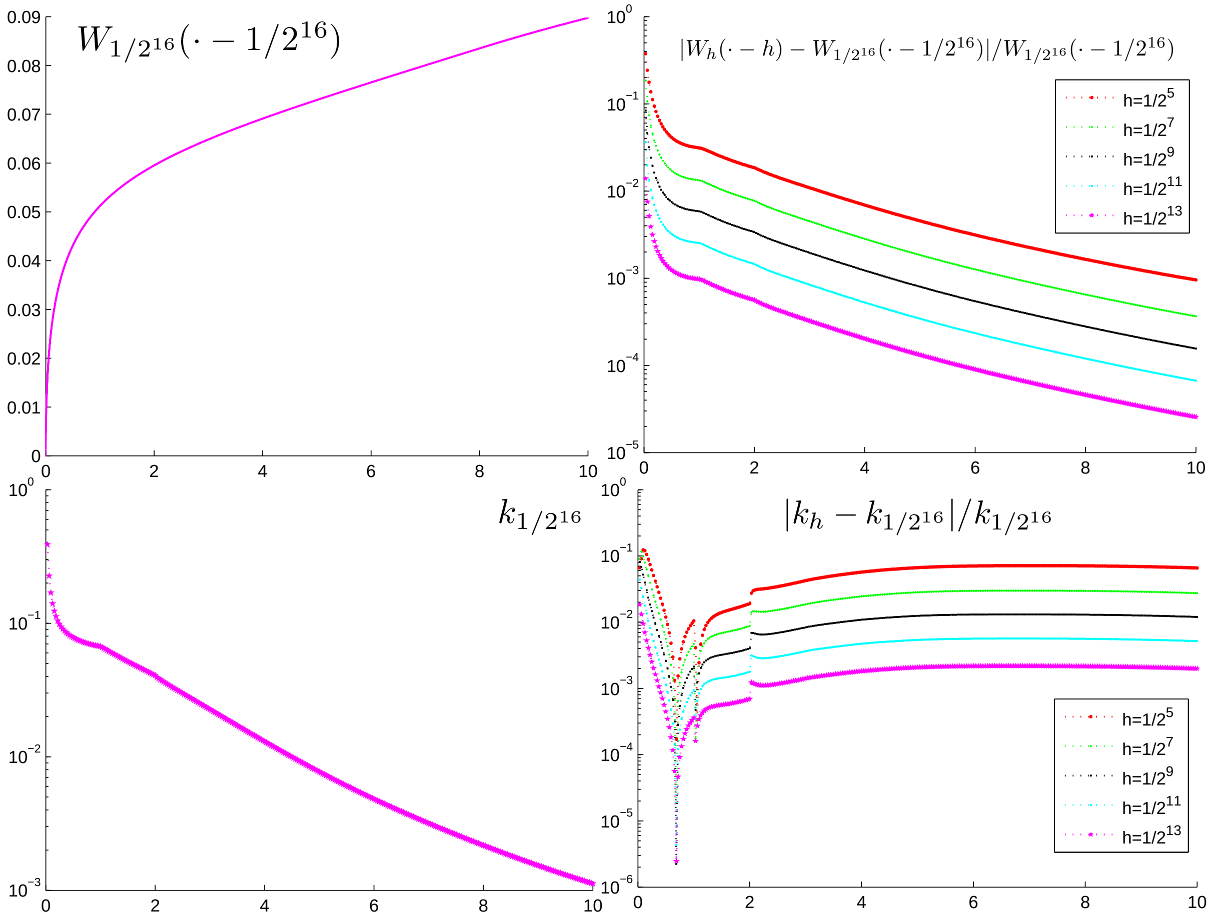

FIGURE 2. The scale function W for the spectrally negative Lévy process X, as described in Example 6.2; followed by the k-function k of the weak limit (as time goes to infinity) of the CBI process, whose spectrally positive component is the dual $X^R = -X$ of X, whilst the Lévy subordinator part is the sum of a unit drift and a compound Poisson process of unit intensity and mean one exponential jumps. The relative errors are consistent with the $O(\sqrt{h})$ order of convergence predicted by Theorem 1.2. See Example 6.2 for details.

compute k via approximating, for $x \in \mathbb{Z}_h^{++}$, $k(x)$ by $k_h(x)$:

$$k_h(x) := b\frac{W_h(x) - W_h(x-h)}{h} + W_h(x-h)m[h/2,\infty) - \sum_{k=1}^{x/h-1} W_h(x-kh-h)m[kh-h/2, kh+h/2).$$

Results are reported in Figure 2.

Let us also mention that we have tested our algorithm on simple processes with completely monotone Lévy densities [2] (Brownian motion with drift; positive drift minus a compound Poisson subordinator with exponential jumps; spectrally negative stable Lévy process — see the arXiv version of this paper [30, Appendix C]), and the results were in nice agreement with the explicit formulae which are available for the scale functions in the latter cases.

6.2. **Concluding remarks.** (1) Computational cost. To compute $W_h^{(q)}(x)$ or $Z_h^{(q)}(x)$ for some $x \in \mathbb{Z}_h$ one effects recursions (3.1) and (3.2) (as applied to $Y = X^h/h$, $h \in (0, h_\star)$), at a cost of $O((x/h)^2)$ operations (assuming given the parameters of X^h).
(2) Quantity $Z_h^{(q)}(x)$ may be obtained from the values of $W_h^{(q)}$ on $[0, x] \cap \mathbb{Z}_h$ at a cost of order $O(x/h)$ operations by a nonnegative summation (see Proposition 3.3).
(3) The computation of the functions $W^{(q)}$, $q \geq 0$, can be reduced, under an exponential change of measure, to the computation of W [22, p. 222, Lemma 8.4] for a process having $\Phi(0) = 0$ [34]. Under such an exponential tilting $W_h(x)$ will have a temperate growth [35, Proposition 4.8(ii)], since $\Phi^h(0) \to \Phi(0)$.

Finally, in comparison to the Laplace inversion methods discussed in [21, Section 5.6], we note:
(1) Regarding *only* the efficiency of our algorithm (i.e. how costly it is, to achieve a given precision): Firstly, that Filon's method (with Fast Fourier Transform) appears to outperform ours when an explicit formula for the Laplace exponent ψ is known. Secondly, that our method is largely insensitive to the degree of smoothness of the target scale function – and can match or outperform Euler's, the Gaver-Stehfest and the fixed Talbot's method in regimes when the scale function is less smooth, even as ψ remains readily available (such, at least, was the case for Sets 3 and 4 of [21, pp. 177-178] — see arXiv version of this paper [30, Appendix C]). Thirdly, that when ψ is not given in terms of elementary/special function, any Laplace inversion algorithm, by its very nature, must resort to further numerical evaluations of ψ (at complex values of its argument), which hinders its efficiency and makes it hard to control the error. Indeed, such evaluations of ψ appear disadvantageous, as compared to the more innocuous operations required to compute the coefficients present in our recursion.
(2) In our method there is only one spatial discretization parameter h to vary. On the other hand, Filon's method (which, when coupled with Fast Fourier Transform, appears the most efficient of the Laplace inversion techniques), has additionally a cutoff parameter in the (complex) Bromwich integral.

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