Statistics of α - μ Random Variables and Their Applications in Wireless Multihop Relaying and Multiple Scattering Channel

Kezhi Wang, *Student Member, IEEE*, Tian Wang, Yunfei Chen, *Senior Member, IEEE*, Mohamed-Slim Alouini, *Fellow, IEEE*

Abstract

Exact results for the probability density function (PDF) and cumulative distribution function (CDF) of the sum of ratios of products (SRP) and the sum of products (SP) of independent α - μ random variables (RVs) are derived. They are in the form of one-dimensional integral. They are based on existing works on the products and ratios of α - μ RVs. Generalized Gamma ratio approximation (GGRA) is proposed to approximate SRP. Gamma ratio approximation (GRA) is proposed to approximate SRP and the ratio of sums of products (RSP). Generalized Gamma approximation (GGA) and Gamma approximation (GA) are presented to approximate SP. The proposed results of the SRP can be used to calculate the outage probability (OP) for wireless multihop relaying or multiple scattering systems with interferences. Also, the proposed results of the SP can be used to calculate the OP for these systems without interferences. Moreover, the proposed approximate result of the RSP can be used to calculate the OP of signal-to-interference ratio (SIR) in multiple scattering system with interference.

Index Terms

α - μ random variables, GGRA, interference, multiple scattering, wireless multihop relaying.

Kezhi Wang and Yunfei Chen are with the School of Engineering, University of Warwick, Coventry, U.K. CV4 7AL (e-mails: Kezhi.Wang@warwick.ac.uk, Yunfei.Chen@warwick.ac.uk)

Tian Wang and Mohamed-Slim Alouini are with the Computer, Electrical and Mathematical Sciences and Engineering (CEMSE) Division, King Abdullah University of Science and Technology (KAUST), Thuwal, Makkah Province, Saudi Arabia (e-mails: tian.wang@kaust.edu.sa, slim.alouini@kaust.edu.sa).

I. INTRODUCTION

Amplify-and-forward (AF) is widely used for cooperative/collaborative wireless system. Multihop communication has been extensively used in vehicular ad hoc networks (VANETs) to provide support for road management, traffic information, safety or other applications by enabling vehicle-to-vehicle communications [1]. AF can be divided into disintegrated channel (DC) and cascaded channel (CC). Instantaneous channel state information (ICSI) is often required in DC resulting in a variable-gain (or variable amplification factor). However, ICSI is not necessarily needed in CC leading to a fixed-gain (or fixed amplification factor). Several researchers have studied the performance of wireless multihop relaying with AF in CC using fixed amplification gain due to its lower complexity and high usage efficiency. Reference [2] has studied the lower bounds for the performance of multihop transmissions in CC over Nakagami-m fading channels, while [3] has derived the closed-form expression of the outage probability (OP) over N*Nakagami fading in terms of the Meijer's G-function. In [4], the closed-form expression of the cumulative distribution function (CDF) of CC with multihop relaying over generalized Gamma (GG) fading channels has been obtained and the closed-form expressions for bounds of equal gain combining (EGC) over GG has also been derived. But these works either use infinite series or special functions that are computationally complicated. Reference [5] has presented the approximations to the statistics of products of independent random variables but without considering relay combining. Reference [6] has proposed the approximations to EGC and maximal ratio combining (MRC) of GG variables but without considering multihop transmission. Also, none of these works has considered the multihop interferences from other transmitting sources in the network. In a multiple-access system or a frequency-reused cell, interferences from other transmitting sources may cause performance degradation and therefore cannot be ignored. Similar research has been done in multiple scattering (keyhole) channel [7], in which the electromagnetic wave propagates through several keyholes. Then the overall channel gain can be modeled as the linear combination of product of the random variables (RVs) but without considering interference.

To fill in this gap, we derive the exact CDF in terms of one-dimensional integral for the sum of ratios of products (SRP) and the sum of products (SP) of independent α - μ RVs. To reduce the computational complexity, the approximate CDF of SRP in closed-form expression using the generalized Gamma ratio approximation (GGRA) and Gamma ratio approximation (GRA) are proposed as well. Also, generalized Gamma approximation (GGA) and Gamma approximation (GA) are presented to approximate the SP. SRP can be used to calculate the OP of EGC or MRC receivers for wireless relaying system. In this case, statistically independent multiple hops are used with AF protocol and fixed amplification gain. SP can be used to calculate the OP of EGC or MRC receivers of the same channel without interferences. Also, SRP can be applied to model the channel gain in multiple scattering system with interferences, while SP can be applied to model the channel gain in multiple scattering system without interferences. Moreover, GRA is used to approximate the ratio of sums of products (RSP) of α-μ RVs that models the signal-to-interference ratio (SIR) and the corresponding OP in scattering channels.

II. System Model

Consider independent α - μ RVs $\{X_{jl}\}_{j=1,l=1}^{k,L}$, each having probability density function (PDF) of $f_{X_{jl}}(x) = \frac{\alpha_{jl}\mu_{jl}^{\mu_{jl}}}{\Gamma(\mu_{jl})\hat{\gamma_{jl}}^{\alpha_{jl}\mu_{jl}}} x^{\alpha_{jl}\mu_{jl}-1} e^{-\frac{\mu_{jl}x^{\alpha_{jl}}}{\hat{\gamma_{jl}}^{\alpha_{jl}}}}, x > 0$, where $\Gamma(\cdot)$ is the Gamma function, $\alpha_{jl} > 0$ is related to the non-linearity of the environment, $\hat{\gamma_{jl}} = E^{1/\alpha_{jl}}(X_{jl}^{\alpha_{jl}})$ is a α_{jl}-root mean value, $\mu_{jl} = \frac{E^2[X_{jl}^{\alpha_{jl}}]}{V[X_{jl}^{\alpha_{jl}}]}$ is the inverse of the normalized variance of $X_{jl}^{\alpha_{jl}}$, and $E(\cdot)$ and $V(\cdot)$ are the expectation and variance operators, respectively [8]. One significant character of the α - μ distribution is that it includes the Nakagami-m fading ($\alpha_{jl} = 1$), Weibull fading ($\mu_{jl} = 1$), and Rayleigh fading ($\alpha_{jl} = \mu_{jl} = 1$), as special cases.

Define a random variable R as the SRP of independent α-μ RVs $\{X_{jl}\}_{j=1,l=1}^{k,L}$, giving $R = \sum_{l=1}^{L} R_l$, where $R_l = \omega_l \frac{\prod_{j=1}^{m} X_{jl}}{\prod_{j=m+1}^{k} X_{jl}}$ and ω_l is constant. This SRP can be used to calculate the OP of EGC or MRC receivers for wireless multihop relaying system [2] considering multihop interferences. Also, in multiple scattering channels [7], if each scattering is affected by interferences, the overall channel gain can be described as a linear combination of the ratios of signal to interference and following that, OP in multiple scattering channels can also be obtained.

Define a random variable P as the SP of independent α-μ RVs $\{X_{jl}\}_{j=1,l=1}^{m,L}$, giving $P = \sum_{l=1}^{L} P_l$, where $P_l = \omega_l \prod_{j=1}^{m} X_{jl}$ and ω_l is constant. This SP can be used to calculate the OP of the EGC or MRC receivers for wireless multihop relaying system without interferences [2]. Also, it can be used to model the overall channel gain in multiple scattering system [7] and based on that, OP in multiple scattering channels can be obtained.

Finally, define a random variable Z as the RSP of independent α-μ RVs $\{X_{jl}\}_{j=1,l=1}^{k,L}$, giving $Z = \frac{P^{(1)}}{P^{(2)}}$, where $P^{(1)} = \sum_{l=1}^{L^{(1)}} \omega_l \prod_{j=1}^{m} X_{jl}$, $P^{(2)} = \sum_{l=1}^{L^{(2)}} \omega_l \prod_{j=m+1}^{k} X_{jl}$. If we consider $P^{(1)}$ as a linear combination of signal components and $P^{(2)}$ as a linear combination of interferer components, then the square of this RSP can be considered as the SIR in a multiple scattering system [7] with multiple scattering interferences. Thus, the OP for SIR in multiple scattering channels can be obtained. These simple expressions give the relationships between OP and important system parameters, such as number of hops, number of links, weighting factors and signal powers. They could be optimized for different applications.

III. Statistics analysis and applications

In the first subsection, we derive the exact PDF and CDF of R and P in terms of one-dimensional integral, while in the second subsection, we derive the approximate PDF and CDF of R, P and Z in closed-form. Applications are then given in the third subsection.

A. Exact result

If the α-μ RVs are independent, the MGF of R_l can be derived as

$$M_{R_l}(s) = \frac{C_l \alpha_l^{\frac{1}{2}}}{2\pi^{\frac{\alpha_l - 1}{2}}} \times$$
$$G_{n'+\alpha_l,m'}^{m',n'+\alpha_l}\left(\left(\frac{\alpha_l}{\omega_l}\right)^{\alpha_l} \frac{t_l}{s^{\alpha_l}} \Big| \begin{array}{c} \Delta(\alpha_l,1), 1-\frac{\mu_{jl}}{m_{jl}}-\frac{r_l}{m_{jl}}, r_l=0,1,\cdots,m_{jl}-1, j=m+1,\cdots,k \\ \frac{\mu_{jl}}{m_{jl}}+\frac{r_l}{m_{jl}}, r_l=0,1,\cdots,m_{jl}-1, j=1,\cdots,m \end{array} \right) \tag{1}$$

where $G_{a,b}^{c,d}(\cdot)$ is the Meijer's G-function [9], α_l and m_{jl} are positive integers with $\alpha_{jl} = \frac{\alpha_l}{m_{jl}}$, $j = 1, 2, \cdots, k$, $C_l = 1/\left(\prod_{j=1}^{k}\prod_{r_l=0}^{m_{jl}-1}\Gamma(\frac{\mu_{jl}}{m_{jl}} + \frac{r_l}{m_{jl}})\right)$, $t_l = \prod_{j=1}^{m}\left(\frac{\mu_{jl}}{\hat{\gamma}_{jl}^{\alpha_{jl}}}\right)^{m_{jl}} / \prod_{j=m+1}^{k}\left(\frac{\mu_{jl}}{\hat{\gamma}_{jl}^{\alpha_{jl}}}\right)^{m_{jl}}$, $m' = \sum_{j=1}^{m} m_{jl}$, $n' = \sum_{j=m+1}^{k} m_{jl}$ and $\Delta(a;b) \triangleq b/a, \cdots, (b+a-1)/a$. See Appendix A for proof.

If the α-μ RVs are independent and identically distributed (i.i.d), α_{jl}, μ_{jl} and $\hat{\gamma_{jl}}$ become α, μ and $\hat{\gamma}$, respectively. Therefore, the MGF of R_l in (1) is specialized to

$$\begin{aligned} M_{R_l}(s) = & \frac{\alpha e^{(m+n)(\mu-\frac{1}{2})+1}}{\Gamma^k(\mu) f^{\frac{1}{2}} 2\pi^{(e-1)(\frac{n+m}{2})+\frac{f-1}{2}}} \times \\ & G_{en+f,em}^{em,en+f}\left(\frac{f^f}{\omega_l^{\alpha e} s^l}\left(\frac{\mu}{\hat{\gamma}^{\alpha} e}\right)^{(m-n)e} \middle| \begin{array}{c} \Delta(f,1), \Delta(e, 1-\mu), \cdots, \Delta(e, 1-\mu) \\ \Delta(e,\mu), \cdots, \Delta(e,\mu) \end{array}\right) \end{aligned} \tag{2}$$

where $n = k - m$, f and e are relatively prime integers satisfying $\alpha = f/e$.

Also, when the α-μ RVs are independent, the MGF of P_l can be derived as

$$\begin{aligned} M_{P_l}(s) = & \frac{C_l \alpha_l^{\frac{1}{2}}}{2\pi^{\frac{\alpha_l - 1}{2}}} \times \\ & G_{\alpha_l, m'}^{m', \alpha_l}\left(\left(\frac{\alpha_l}{\omega_l}\right)^{\alpha_l} \frac{t_l}{s^{\alpha_l}} \middle| \begin{array}{c} \Delta(\alpha_l, 1) \\ \frac{\mu_{jl}}{m_{jl}} + \frac{r_l}{m_{jl}}, r_l = 0, 1, \cdots, m_{jl} - 1, j = 1, \cdots, m \end{array}\right) \end{aligned} \tag{3}$$

by setting $k = m$ in (1), where $C_l = 1/\prod_{j=1}^{m}\Gamma(\mu_{jl})$ and $t_l = \prod_{j=1}^{m}\left(\frac{\mu_{jl}}{\hat{\gamma}_{jl}^{\alpha_{jl}}}\right)^{m_{jl}}$.

If the α-μ RVs in P_l are i.i.d, the MGF of P_l is specialized to

$$M_{P_l}(s) = \frac{\alpha e^{m(\mu-\frac{1}{2})+1}}{\Gamma^m(\mu) f^{\frac{1}{2}} 2\pi^{(e-1)\frac{m}{2}+\frac{f-1}{2}}} G_{f,em}^{em,f}\left(\frac{f^f}{\omega_l^{\alpha e} s^l}\left(\frac{\mu}{\hat{\gamma}^{\alpha} e}\right)^{me} \middle| \begin{array}{c} \Delta(f,1) \\ \Delta(e,\mu), \cdots, \Delta(e,\mu) \end{array}\right). \tag{4}$$

Therefore, the PDF and CDF of R in terms of one-dimensional integral can be expressed as $f_R(x) = \frac{1}{2\pi i}\oint_C \prod_{l=1}^{L} M_{R_l}(s) e^{sx} ds$ and $F_R(y) = \frac{1}{2\pi i}\oint_C \frac{\prod_{l=1}^{L} M_{R_l}(s) e^{sy}}{s} ds$, respectively. Note that $f_R(x)$ and $F_R(y)$ can be calculated numerically by using popular mathematical software packages, such as MATLAB, MATHEMATICA and MAPLE. Also, $f_R(x)$ and $F_R(y)$ can be evaluated using a finite series in closed-form as [10, eq. (9) and eq. (11)]. The PDF $f_P(x)$ and CDF $F_P(y)$ of P can be obtained by simply replacing $M_{R_l}(s)$ with $M_{P_l}(s)$ in $f_R(x)$ and $F_R(y)$, respectively. No exact results for Z are available due to the difficulty in obtaining them.

Note also that [11] provides the PDF of products, quotients and powers of H-function variates. Although H-function variates include α - μ variates, the result in this reference is only a special case of ours when summation is not considered.

B. Approximate result

The exact PDF and CDF expressions in integral or series in the previous subsection are complicated. Thus, in the following, we present new approximation methods GGRA and GRA to approximate R and GRA to approximate Z. Also, we propose to use the conventional approximation methods GGA and GA to approximate P.

1) GGRA: The PDF and CDF of GGRA are given by

$$f_{X_{GGRA}}(x) = \frac{pk^{-\frac{d_1}{p}} x^{d_1-1}(1+k^{-1}\ x^p)^{-\frac{d_1+d_2}{p}}}{B(\frac{d_1}{p},\frac{d_2}{p})}, x>0, \tag{5}$$

$$F_{X_{GGRA}}(y) = \frac{k^{-\frac{d_1}{p}} y^{d_1}\ {}_2F_1\left(\frac{d_1}{p},\frac{d_1}{p}+\frac{d_2}{p};1+\frac{d_1}{p};-k^{-1}y^p\right)}{\frac{d_1}{p}B(\frac{d_1}{p},\frac{d_2}{p})} \tag{6}$$

respectively, where k, d_1, d_2, p, >0 are the parameters to be determined. See Appendix B for proof. One can calculate d_1, d_2 and p in (5) and (6) by solving the following equations

$$\begin{cases} \frac{E\{R^2\}}{E^2\{R\}} = \frac{\Gamma(\frac{d_1}{p})\Gamma(\frac{d_2}{p})\Gamma\left(\frac{d_1}{p}+\frac{2}{p}\right)\Gamma\left(\frac{d_2}{p}-\frac{2}{p}\right)}{\Gamma\left(\frac{d_1}{p}+\frac{1}{p}\right)^2\Gamma\left(\frac{d_2}{p}-\frac{1}{p}\right)^2} \\ \frac{E\{R^3\}}{E^3\{R\}} = \frac{\Gamma(\frac{d_1}{p})^2\Gamma(\frac{d_2}{p})^2\Gamma\left(\frac{d_1}{p}+\frac{3}{p}\right)\Gamma\left(\frac{d_2}{p}-\frac{3}{p}\right)}{\Gamma\left(\frac{d_1}{p}+\frac{1}{p}\right)^3\Gamma\left(\frac{d_2}{p}-\frac{1}{p}\right)^3} \\ \frac{E\{R^4\}}{E^4\{R\}} = \frac{\Gamma(\frac{d_1}{p})^3\Gamma(\frac{d_2}{p})^3\Gamma\left(\frac{d_1}{p}+\frac{4}{p}\right)\Gamma\left(\frac{d_2}{p}-\frac{4}{p}\right)}{\Gamma\left(\frac{d_1}{p}+\frac{1}{p}\right)^4\Gamma\left(\frac{d_2}{p}-\frac{1}{p}\right)^4} \end{cases} \tag{7}$$

using popular mathematical software packages, where the n-th order moment $E\{R^n\}$ of R is given in (13). Then, the value of k can be obtained by inserting the solutions of d_1, d_2 and p into one of the equations in (14). See Appendix C for proof.

2) GRA: To simplify the parameter calculation, GRA is proposed by setting $p=1$ in GGRA. Therefore, the PDF and CDF of GRA are given by

$$f_{X_{GRA}}(x) = \frac{k^{-d_1} x^{d_1-1}(1+xk^{-1})^{-d_1-d_2}}{B(d_1,d_2)}, x>0, \tag{8}$$

$$F_{X_{GRA}}(y) = \frac{y^{d_1} k^{-d_1} \, {}_2F_1\left(d_1, d_1+d_2; 1+d_1; -yk^{-1}\right)}{d_1 B(d_1, d_2)} \tag{9}$$

respectively. GRA can also be used to approximate R for a simplified form. Fortunately, unlike numerical solutions in (7), one can get the closed-form expressions of d_1, d_2 and k by setting $p=1$ and ignoring the fourth equations in (14), as

$$\begin{cases} d_1 = \frac{2E\{R\}\left(E^2\{R^2\}-E\{R\}E\{R^3\}\right)}{2E^2\{R\}E\{R^3\}-E\{R\}E^2\{R^2\}-E\{R^2\}E\{R^3\}} \\ d_2 = \frac{E^2\{R\}E\{R^2\}+3E\{R\}E\{R^3\}-4E^2\{R^2\}}{E^2\{R\}E\{R^2\}+E\{R\}E\{R^3\}-2E^2\{R^2\}} \\ k = \frac{-2E\{R\}^2E\{R^3\}+E\{R\}E\{R^2\}^2+E\{R^2\}E\{R^3\}}{E\{R\}^2E\{R^2\}+E\{R\}E\{R^3\}-2E\{R^2\}^2} \end{cases} \tag{10}$$

Moreover, GRA is proposed to approximate Z. It is not easy to get the closed form of n-th order moment for Z. However, it is shown in Appendix B that GRA can be considered as the ratio of two GAs. Therefore the values of d_1 and d_2 can be decided by $d_1 = \frac{E^2(P^{(1)})}{E(P^{(1)})-E^2(P^{(1)})}$ and $d_2 = \frac{E^2(P^{(2)})}{E(P^{(2)})-E^2(P^{(2)})}$, respectively, and k can be solved by $k = \frac{a_1}{a_2}$, where $a_1 = \frac{E(P^{(1)})-E^2(P^{(1)})}{E(P^{(1)})}$ and $a_2 = \frac{E(P^{(2)})-E^2(P^{(2)})}{E(P^{(2)})}$. Numerical results in Section IV will show that GRA has a slightly worse performance than GGRA when approximating R but has a very good match for Z.

3) GGA and GA: One can have the PDF and CDF of the conventional approximation method GGA as $f_{X_{GGA}}(x) = \frac{p\, a^{-d} x^{d-1} e^{-\left(\frac{x}{a}\right)^p}}{\Gamma\left(\frac{d}{p}\right)}, x > 0$, and $F_{X_{GGA}}(y) = \frac{\gamma(d/p, (y/a)^p)}{\Gamma(d/p)}$, respectively, and one can also have the PDF and CDF of GA as $f_{X_{GA}}(x) = \frac{x^{d-1} e^{-\frac{x}{a}}}{a^d \Gamma(d)}, x > 0$, and $F_{X_{GA}}(y) = \frac{\gamma(d, y/a)}{\Gamma(d)}$, respectively. The unknown values of a, d and p in GGA and GA can be decided by moment-matching method which is available in [12]. Although GGA and GA are well-known approximation methods, to the best of the authors' knowledge, none of the works in the literature has considered using GGA and GA to approximate the SP of independent α-μ RVs before. Therefore, in this paper, we propose to use GGA and GA to approximate P. Also, the results in Section IV show that GGA has a very good match with P while GA has a slightly worse performance than GGA but with a simplified form. More importantly, both of them have much simpler form than the exact results in Section II, which shows their usefulness.

C. Applications

For L-branch EGC receivers of AF relaying system through statistically independent multihop with fixed amplification gain and interferences, the instantaneous end-to-end SNR is given by [4] $\gamma_R = \frac{1}{L}\left(\sum_{l=1}^{L}\sqrt{\gamma_l}\right)^2$, where $\gamma_l = \frac{E_s}{N_0}R_l^2$, $R_l = \omega_l \frac{\prod_{j=1}^{m} X_{jl}}{\prod_{j=m+1}^{k} X_{jl}}$, $\omega_l = 1$, E_s is the transmitted energy, N_0 is the single sided AWGN power spectral density, $\prod_{j=1}^{m} X_{jl}$ represents the product of fading coefficients of signals through m hop in the lth path [2] , $\prod_{j=m+1}^{k} X_{jl}$ represents the product of fading coefficients of the interferers through $k-m$ hop in the lth path. On the other hand, one can rewrite γ_R as $\gamma_R' = \frac{1}{L}\frac{E_s}{N_0}R^2$, which can be seen as the instantaneous SNR in multiple scattering radio channel. In this case, R can be seen as the overall channel gain considering interferences in each scattering, R^2 can be seen as the fading power, ω_l can be seen as the weight which is a nonnegative real-valued constant that determine the mixture weight of the multiple scattering component [7]. Define the corresponding average SNR in wireless multihop relaying system or multiple scattering channel as $\bar{\gamma_R} = \frac{1}{L}\frac{E_s}{N_0}E\left(\left(\sum_{l=1}^{L} R_l\right)^2\right)$. By normalizing γ_R or γ_R' with respect to the average end-to-end SNR $\bar{\gamma_R}$, one has $\hat{\gamma_R} = \frac{R^2}{E(R^2)}$. The OP is defined as the probability that the SNR is below a certain threshold as $P_o = Pr\{\gamma_R < \gamma_0\}$. By normalizing both sides of the inequality with the positive value of $\bar{\gamma_R}$ and letting $\gamma_{th} = (\gamma_0/\bar{\gamma_R})$, one has $P_o = Pr\{\hat{\gamma_R} < \gamma_{th}\}$. Then, using the exact CDF of R in Section III.A, one has $P_o = F_R\left(\sqrt{\gamma_{th}}E^{\frac{1}{2}}\left(R^2\right)\right)$. Also, using the approximate CDF of GGRA in (6) and the approximate CDF of GRA in (9), one can get approximate OP of $\hat{\gamma_R}$ as $P_o = F_{GGRA}\left(\sqrt{\gamma_{th}}E^{\frac{1}{2}}\left(R^2\right)\right)$ and $P_o = F_{GRA}\left(\sqrt{\gamma_{th}}E^{\frac{1}{2}}\left(R^2\right)\right)$, respectively.

Similarly, for L-branch EGC receivers of wireless multihop relaying system without interferences [2] or for transmitting signal in multiple scattering radio channel without interference [7], the exact results of OP are given by $P_o = F_P\left(\sqrt{\gamma_{th}}\cdot E^{\frac{1}{2}}\{P^2\}\right)$. Also, using the approximate CDF of GGA and the approximate CDF of GA, one can get the approximate OP as $P_o = F_{GGA}\left(\sqrt{\gamma_{th}}\cdot E^{\frac{1}{2}}\{P^2\}\right)$ and $P_o = F_{GA}\left(\sqrt{\gamma_{th}}\cdot E^{\frac{1}{2}}\{P^2\}\right)$, respectively. Note that the exact and approximate results of OP derived above can also be applied to L-branch MRC receivers of wireless multihop relaying system with proper modifications. For the exact results, a variable substitution of x by x^2 is needed in PDF expression of α - μ RVs and in the following MGF expressions. For the approximate results, moment matching should be done by considering $E(R^2)$, $E(R^4)$, $E(R^6)$, $E(R^8)$, $E(P^2)$, $E(P^4)$, $E(P^6)$, $E(P^8)$ with the corresponding moments of GGRA, GRA, GGA, GA. However, EGC is simpler to implement and its performance is very close to that of MRC although MRC is the optimal combining scheme [13]. Thus, we consider EGC from here on.

Next, we consider the OP for SIR in multiple scattering radio channel. In this case, the SIR I_Z is given by $I_Z = Z^2 = \frac{Z_1}{Z_2}$, where $Z_1 = (P^{(1)})^2 = (\sum_{l=1}^{L^{(1)}} \omega_l \prod_{j=1}^{m} X_{jl})^2$ represents the signal power in multiple scattering channel and $Z_2 = (P^{(2)})^2 = (\sum_{l=1}^{L^{(2)}} \omega_l \prod_{j=m+1}^{k} X_{jl})^2$ represents the interference power in the same channel. Due to the complexity of the exact OP expression of I_Z and the high accuracy of the approximate OP expression GRA of I_Z, we only provide the approximate OP for I_Z. Thus, the OP of SIR I_Z can be $P_o = F_{GRA}\left(\sqrt{\gamma_{th}} \cdot E^{\frac{1}{2}}\{Z^2\}\right)$.

IV. Numerical Results and Discussion

In this section, numerical results are presented to show the accuracy of the derived exact expressions and the derived approximate expressions. In these examples, the exact results in Section III.A are evaluated using the series form in [10, eq. (11)] with $A = 23.026$, $N = 15$ and $Q = 10$, whereas the approximate results are calculated using our proposed closed-form approximations GGRA, GRA, GGA and GA. Let $\alpha_{jl} = 2$ be the same for any j, $\mu_{jl} = 4$ be the same for any j and $\hat{\gamma_{jl}} = 5$ be the same for any j. Note that our results are general enough to include other cases but these settings are used here as examples.

Tables I and II compare the exact and approximate PDFs and CDFs of $R = 2\frac{X_{11}}{X_{21}X_{31}} + 3\frac{X_{12}X_{22}X_{32}}{X_{42}X_{52}}$, respectively. One can see that the PDF and CDF of the approximate GGRA match very well with the exact result while GRA match well with the exact result in most values examined but with a simplified form. Fig. 1 and Fig. 2 show the OP vs. γ_{th} using EGC receivers in wireless multihop relaying system over α - μ channels. Fig. 1 describes the signal model as

$R = \frac{X_{11}}{X_{21}} + \frac{X_{12}X_{22}}{X_{32}X_{42}}$ which assumes $L = 2$ and considering interferences while Fig. 2 describes the signal model as $R = X_{11} + X_{12}X_{22} + X_{13}X_{23}X_{33}$ which assumes $L = 3$ and does not consider interferences. Fig. 3 and Fig. 4 show the OP vs. γ_{th} in multiple scattering channels, in which Fig. 3 describes the signal model considering interferences as $R = 2\frac{X_{11}}{X_{21}X_{31}} + 3\frac{X_{12}X_{22}X_{32}}{X_{42}X_{52}}$ while Fig. 4 describes the signal model without interferences as $R = X_{11} + 2X_{12}X_{22} + 3X_{13}X_{23}X_{33}$. Moreover, Fig. 5 shows the OP vs. γ_{th} in terms of SIR in multiple scattering channel. In this case, the SIR can be described as $I_z = Z^2$, where $Z = \frac{X_{11}+X_{21}X_{31}}{X_{42}+X_{52}X_{62}+X_{72}X_{82}X_{92}}$.

From Figs. 1 - 5, one can see that the OP increases when the SNR threshold increases, as expected. In the extreme case when the SNR threshold goes to infinity, the OP will approach one. Furthermore, the exact results of OP agree very well with the simulations in Figs. 1 - 4. The approximate results based on GGRA match very well with simulations in Fig. 1 and Fig. 3 while GGA matches very well with simulations in Fig. 2 and Fig. 4, showing the usefulness of our approximate expressions. The approximate result based on GRA matches well with the simulations in Fig. 1 and Fig. 3 and GA matches well with the simulations in Fig. 2 and Fig. 4, both between 10^{-1} to 1 but with simple structures. Moreover, GRA has a very good match with the simulation in Fig. 5.

V. CONCLUSIONS

The exact results of CDF of the sum of ratios of products and the sum of products of independent α - μ RVs in the form of one-dimensional integral have been derived. Also, the closed-form expressions for the CDF using GGRA, GRA, GGA and GA have been given. Applications in obtaining the outage probability of EGC receivers for wireless multihop relaying system considering interferences and without considering interferences have also been discussed. Moreover, applications in obtaining the outage probability of signal transmitting in multiple scattering channel considering interferences and without considering interferences are provided. Numerical examples have shown the exact result derived agree very well with the simulations and the approximate results GGRA and GGA have a good match with the simulations while

GRA and GA have a considerable match in some cases but with a simple structure.

APPENDIX

A. Derivation of the PDF of $\Gamma_{1,j}$

If α-μ RVs $\{X_{jl}\}_{j=1,l=1}^{k,L}$ are independent, from [14] and using variable transformation, one can get the PDF of R_l as

$$\begin{aligned} & f_{R_l}(x) = C_l x^{-1} \alpha_l \times \\ & G_{n',m'}^{m',n'}\left(t_l \left(\frac{x}{\omega_l}\right)^{\alpha_l} \middle| \begin{array}{c} 1-\frac{\mu_{jl}}{m_{jl}}-\frac{r_l}{m_{jl}}, r_l=0,1,\cdots,m_{jl}-1, j=m+1,\cdots,k \\ \frac{\mu_{jl}}{m_{jl}}+\frac{r_l}{m_{jl}}, r_l=0,1,\cdots,m_{jl}-1, j=1,\cdots,m \end{array} \right). \end{aligned} \tag{11}$$

The moment generating function (MGF) of R_l is defined as $M_{R_l}(s) = \int_0^\infty e^{xs} f_{R_l}(x)dx$. With the help of the Meijer's G-function identity [15], [16, 07.34.22.0003.01], [16, 07.34.02.0004.01] and [16, 07.34.02.0005.01], one can get MGF of R_l as (1).

B. Derivation of GGRA and GRA

One can see that R is a sum of ratios of products. Therefore, we propose to use the ratio of GG distribution to approximate R. Let $x = \frac{r_1}{r_2}$, where r_1 and r_2 follow GG distribution with the PDFs of $f_{r_1}(r; a_1, d_1, p) = \frac{p\, a_1^{-d_1} r^{d_1-1} e^{-\left(\frac{r}{a_1}\right)^p}}{\Gamma\left(\frac{d_1}{p}\right)}, r > 0$, and $f_{r_2}(r; a_2, d_2, p) = \frac{p\, a_2^{-d_2} r^{d_2-1} e^{-\left(\frac{r}{a_2}\right)^p}}{\Gamma\left(\frac{d_2}{p}\right)}, r > 0$, respectively. Then, one has the PDF of x as $f_{GGRA}(x) = \int_0^\infty |r|\, f_1(rx)\, f_2(r)\, dr$. After simplification and using [17], one can get $f_{GGRA}(x) = \frac{p a_1^{-d_1} a_2^{d_1} x^{d_1-1} (1+(\frac{a_2}{a_1})^p\, x^p)^{-\frac{d_1+d_2}{p}}}{B(\frac{d_1}{p}, \frac{d_2}{p})}$. Using $k = (\frac{a_1}{a_2})^p$, one can get the PDF of GGRA as (5). The CDF of GGRA is given by $F_{GGRA}(y) = \int_0^y f_{GGRA}(x)dx$. Using (5), the CDF expression above and with the help of [9, (3.259)], one can get the CDF of GGRA as (6).

GRA have a simplified form which can be seen as the ratio of Gamma distribution. When $p = 1$ in $f_{r_1}(r; a_1, d_1, p)$ and $f_{r_2}(r; a_2, d_2, p)$, they become $f_{r_1}(r; a_1, d_1) = \frac{r^{d_1-1} e^{-r/a_1}}{a_1^{d_1} \Gamma(d_1)}, r > 0$, and $f_{r_2}(r; a_2, d_2) = \frac{r^{d_2-1} e^{-r/a_2}}{a_2^{d_2} \Gamma(d_2)}, r > 0$, respectively. Similarly, using $f_{GRA}(x) = \int_0^\infty |r|\, f_1(rx)\, f_2(r)\, dr$ and $k = \frac{a_1}{a_2}$, one can get the PDF and CDF of GRA as (8) and (9), respectively.

C. Moment-matching approximations

With the help of [17], one has the w-th order moment of GGRA as

$$E\left(X_{GGRA}^{w}\right) = k^{w/p}\frac{\Gamma(\frac{d_1}{p}+\frac{w}{p})}{\Gamma(\frac{d_1}{p})}\frac{\Gamma(\frac{d_2}{p}-\frac{w}{p})}{\Gamma(\frac{d_2}{p})}. \tag{12}$$

Also, with the help of the [18], n-th order moment of R can be given as

$$E\{R^n\} = \sum_{n_1=0}^{n}\sum_{n_2=0}^{n_1}\cdots\sum_{n_{L-1}=0}^{n_{L-2}}\binom{n}{n_1}\binom{n_1}{n_2}\cdots\binom{n_{L-2}}{n_{L-1}}\omega_1^{n-n_1}\omega_2^{n_1-n_2}\cdots\omega_L^{n_{L-1}}\prod_{j=1}^{m}E\{X_{j1}^{n-n_1}\}$$
$$\prod_{j=1}^{m}E\{X_{j2}^{n_1-n_2}\}\cdots\prod_{j=1}^{m}E\{X_{jL}^{n_{L-1}}\}\prod_{j=m+1}^{k}E\{\frac{1}{X_{j1}^{n-n_1}}\}\prod_{j=m+1}^{k}E\{\frac{1}{X_{j2}^{n_1-n_2}}\}\cdots\prod_{j=m+1}^{k}E\{\frac{1}{X_{jL}^{n_{L-1}}}\} \tag{13}$$

where the n-th order moment for X_{jl} is $E\{X_{jl}^n\} = \frac{\hat{\gamma_{jl}}^n\Gamma(\mu_{jl}+n/\alpha_{jl})}{\mu_{jl}^{n/\gamma_{jl}}\Gamma(\mu_{jl})}$ [8] and the n-th order moment for $\frac{1}{X_{jl}}$ is $E\{\frac{1}{X_{jl}^n}\} = \frac{\hat{\gamma_{jl}}^{-n}\Gamma(\mu_{jl}-n/\alpha_{jl})}{\mu_{jl}^{-n/\gamma_{jl}}\Gamma(\mu_{jl})}$. Similarly, one can get the n-th order moment of P by setting $k = m$ and ignoring $E\{\frac{1}{X_{jl}^n}\}$ in (13) but they are omitted here to save space. Then d_1, d_2, p and k in (5) and (6) can be determined by matching the 1-st, 2-nd, 3-rd and 4-th order moment of R with the 1-st, 2-nd, 3-rd and 4-th order moment of GGRA, respectively as

$$\begin{cases} E\{R\} = k^{1/p}\frac{\Gamma(\frac{d_1}{p}+\frac{1}{p})}{\Gamma(\frac{d_1}{p})}\frac{\Gamma(\frac{d_2}{p}-\frac{1}{p})}{\Gamma(\frac{d_2}{p})} \\ E\{R^2\} = k^{2/p}\frac{\Gamma(\frac{d_1}{p}+\frac{2}{p})}{\Gamma(\frac{d_1}{p})}\frac{\Gamma(\frac{d_2}{p}-\frac{2}{p})}{\Gamma(\frac{d_2}{p})} \\ E\{R^3\} = k^{3/p}\frac{\Gamma(\frac{d_1}{p}+\frac{3}{p})}{\Gamma(\frac{d_1}{p})}\frac{\Gamma(\frac{d_2}{p}-\frac{3}{p})}{\Gamma(\frac{d_2}{p})} \\ E\{R^4\} = k^{4/p}\frac{\Gamma(\frac{d_1}{p}+\frac{4}{p})}{\Gamma(\frac{d_1}{p})}\frac{\Gamma(\frac{d_2}{p}-\frac{4}{p})}{\Gamma(\frac{d_2}{p})} \end{cases}. \tag{14}$$

Further, one can simplify (14) by getting rid of k as (7).

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TABLE I

COMPARISON OF THE EXACT AND APPROXIMATE PDFS OF $R = 2\frac{X_{11}}{X_{21}X_{31}} + 3\frac{X_{12}X_{22}X_{32}}{X_{42}X_{52}}$

x	Exact	GGRA	GRA
3	0.005123366578	0.00553831	0.00657853
6	0.03437761985	0.0340766	0.0339683
9	0.05414392525	0.0538482	0.0528716
12	0.055066531	0.0550564	0.0544886
15	0.04670142108	0.0468074	0.0467473
18	0.03634886	0.0364498	0.0366334
21	0.027117528	0.0271797	0.0274094
24	0.019823972	0.0198519	0.0200418
27	0.014372777	0.014379	0.0145091
30	0.01040560788	0.0104009	0.0104784
33	0.007553075	0.00754393	0.0075832
36	0.00550956	0.00549977	0.00551409
39	0.0040442581	0.00403557	0.00403502
42	0.002989825	0.00298263	0.00297411
45	0.00222678	0.00222117	0.00220904
48	0.001670996	0.00166687	0.00165373
51	0.00126327	0.00126047	0.00124778
54	0.0009624243	0.00096032	0.000948797
57	0.0007384826	0.00073699	0.000726922
60	0.00057033	0.000569594	0.000561023

TABLE II

COMPARISON OF THE EXACT AND APPROXIMATE CDFS OF $R = 2\frac{X_{11}}{X_{21}X_{31}} + 3\frac{X_{12}X_{22}X_{32}}{X_{42}X_{52}}$

y	Exact	GGRA	GRA
3	0.002918777902	0.00343157	0.00456575
6	0.05963044177	0.0603741	0.0635416
9	0.1977381439	0.197387	0.198466
12	0.3651634794	0.36438	0.363
15	0.5189965633	0.518401	0.516134
18	0.6435316511	0.643263	0.641246
21	0.7382844348	0.738262	0.736898
24	0.8082033983	0.808312	0.807589
27	0.8590801037	0.859235	0.858993
30	0.8959262807	0.89608	0.896146
33	0.9226282412	0.922759	0.922997
36	0.9420521725	0.942154	0.942469
39	0.9562621779	0.956336	0.956669
42	0.9667270005	0.966777	0.967096
45	0.9744897491	0.974522	0.974809
48	0.9802920019	0.980311	0.980559
51	0.9846609351	0.984671	0.98488
54	0.9879744607	0.987979	0.988152
57	0.9905062535	0.990509	0.990649
60	0.9924555580	0.992457	0.992569

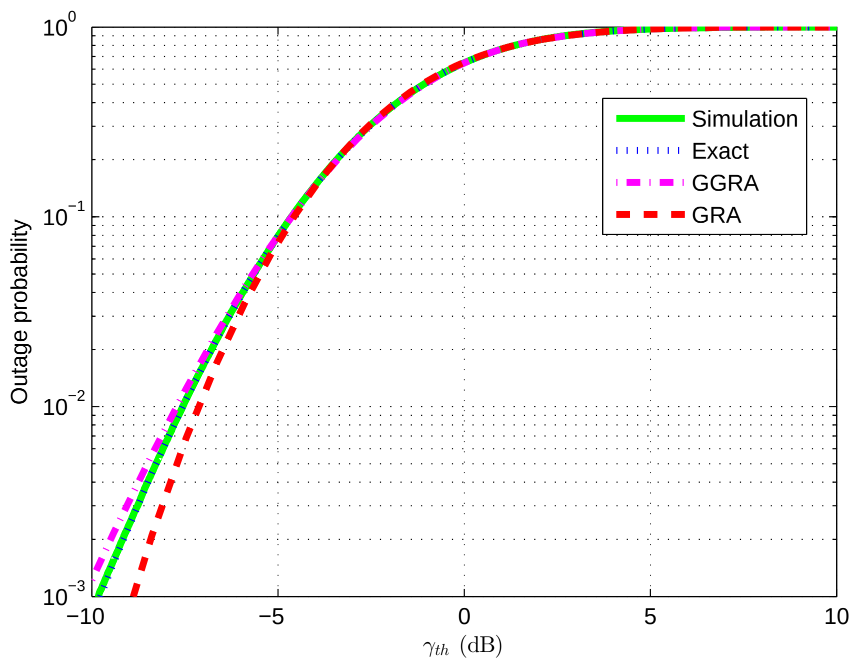

Fig. 1. Outage probability vs. γ_{th} using EGC receivers in wireless multihop relaying system considering interferences.

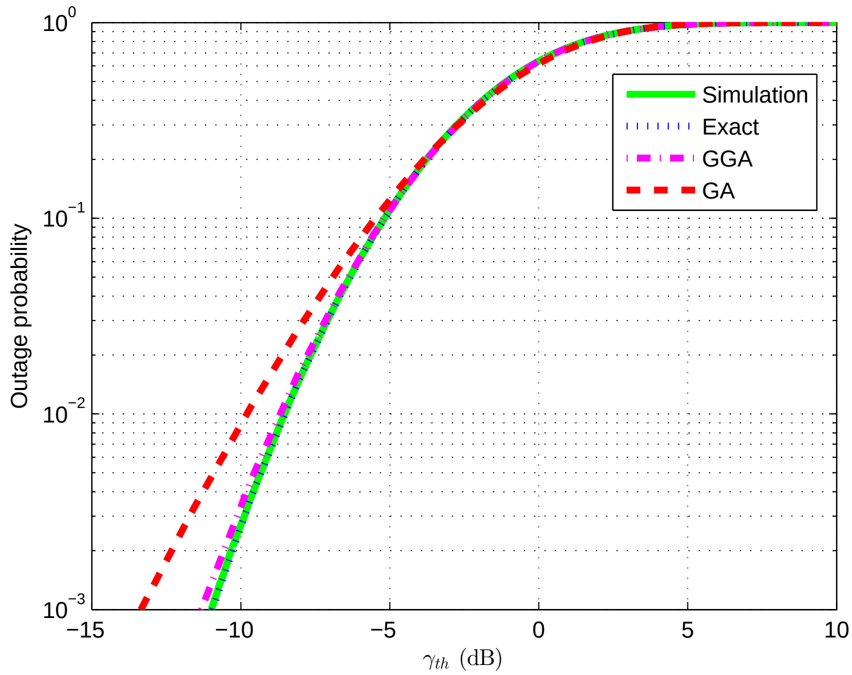

Fig. 2. Outage probability vs. γ_{th} using EGC receivers in wireless multihop relaying system without interferences.

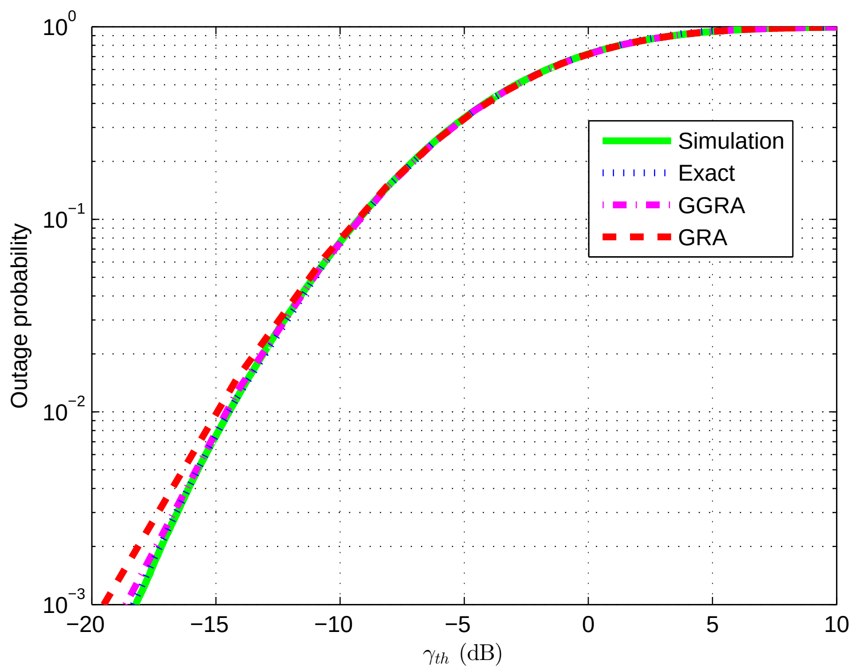

Fig. 3. Outage probability vs. γ_{th} in multiple scattering channel considering interferences.

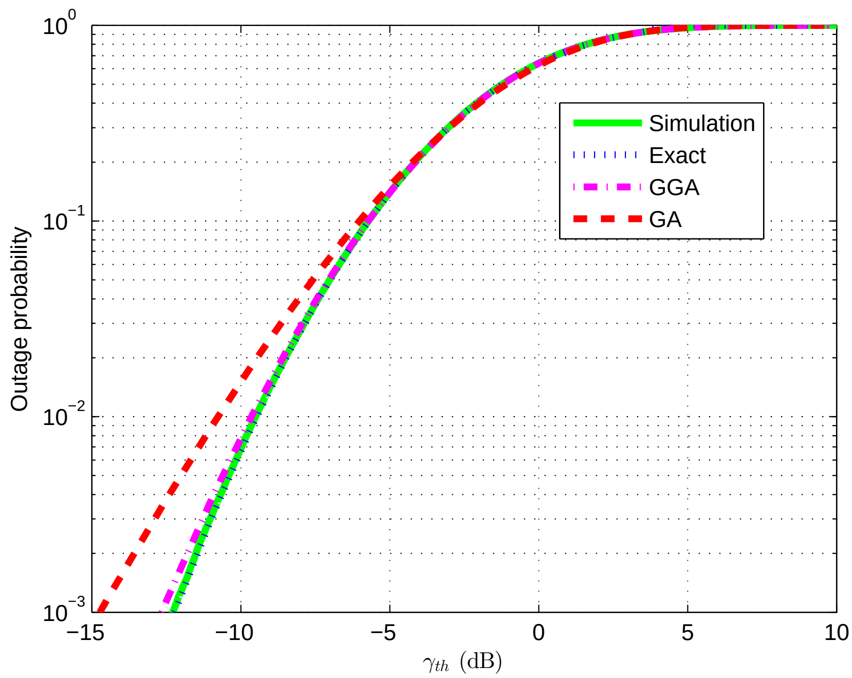

Fig. 4. Outage probability vs. γ_{th} in multiple scattering channel without interferences.

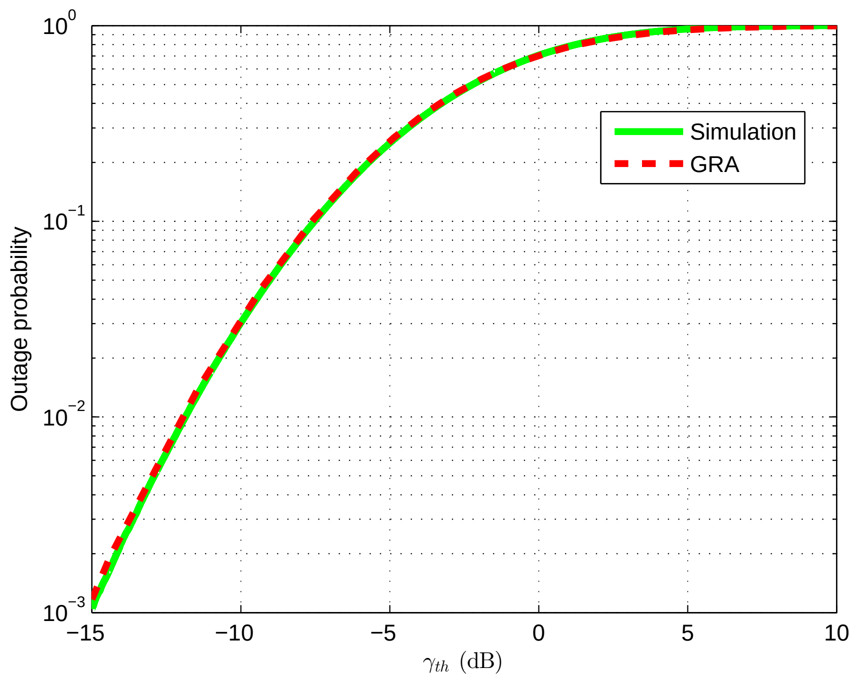

Fig. 5. Outage probability vs. γ_{th} in terms of SIR in multiple scattering channel considering interferences.