Departamento de Matemática
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CLUSTER ALGEBRAS

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Preface

The theory of cluster algebras, which has recently made its appearance on the mathematical scene, is developing rapidly and finding application in many different areas of mathematics. In view of the potential importance of this theory, developed primarily by S. Fomin and A. Zelevinsky, it may be useful to give a rather straightforward exposition of some of the basic ideas on cluster algebras. This is the aim of the present volume. Although the foundational paper on cluster algebras appeared as recently as 2002 the literature on this subject is now becoming extensive. We have provided only a very basic introduction to this literature in the present article.

In July 2003 I was invited to give a short course of three lectures in the Mathematics Department of the University of Coimbra on the subject of cluster algebras. The present exposition is an expanded version of these lectures.

I would like to thank the Centre of Mathematics of the University of Coimbra for their financial support, and to express my appreciation of the kind hospitality and assistance given by Dr. Ana Paula Santana and other colleagues at Coimbra.

R. W. Carter

Chapter 1

Clusters of finite type

1.1 Some background on root systems

We begin by recalling some basic properties of the root system of a finite dimensional semisimple Lie algebra over the complex field, as this will be relevant to the understanding of clusters of finite type. Let $\mathfrak{g}$ be such a Lie algebra and Φ be the root system of $\mathfrak{g}$. Φ is a set of vectors in a Euclidean space V which span V but which are not linearly independent. A subset Π of fundamental roots may be chosen in Φ which form a basis for V and which has the property that, if $\Pi = \{\alpha_1, ..., \alpha_l\}$, then each $\alpha \in \Phi$ can be written in the form $\alpha = n_1\alpha_1 + ... + n_l\alpha_l$ with each $n_i \in \mathbb{Z}$ satisfying $n_i \geq 0$ for all i or $n_i \leq 0$ for all i. Elements of Φ satisfying the former condition are called positive roots and those satisfying the latter condition are negative roots. We have

$$\Phi = \Phi^+ \cup \Phi^-$$

where Φ^+, Φ^- are the positive and negative roots respectively.

Let $s_i : V \to V$ be the reflection in the hyperplane orthogonal to α_i. Then we have

$$s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i$$

where

$$A_{ij} = \frac{2\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle}.$$

The numbers A_{ij} lie in $\mathbb{Z}$ and satisfy $A_{ii} = 2$ for all i and $A_{ij} \leq 0$ for all $i \neq j$. The matrix

$$A = (A_{ij})$$

is an $l \times l$ matrix over $\mathbb{Z}$ called the *Cartan matrix* of $\mathfrak{g}$. Each of the fundamental reflections s_i satisfies $s_i(\Phi) = \Phi$.

The group W of orthogonal transformations of V generated by $s_1, ..., s_l$ is called the *Weyl group*. W is a finite group which is generated by $s_1, ..., s_l$ as a Coxeter group. This means that, if n_{ij} is the order of $s_i s_j$ when $i \neq j$, W can be described as an abstract group by generators and relations

$$W = \langle s_1, ..., s_l;\ s_i^2 = 1,\ (s_i s_j)^{n_{ij}} = 1 \text{ for } i \neq j \rangle.$$

We have $w(\Phi) = \Phi$ for each $w \in W$. Let $n(w)$ be the number of $\alpha \in \Phi^+$ such that $w(\alpha) \in \Phi^-$. Then it is known that $n(w) = l(w)$, where $l(w)$ is the shortest length of any expression of w as a product of fundamental reflections s_i.

These basic properties of root systems can be found in any systematic exposition of the theory of semisimple Lie algebras. We shall introduce further properties of the root system and Weyl group as we need them in connection with the properties of clusters.

1.2 The PL-reflections σ_i

Fomin and Zelevinsky modified the classical theory of roots, reflections and the Weyl group by replacing the linear map s_i by a piecewise-linear map σ_i. We define the subset $\Phi_{\geq -1}$ of Φ by

$$\Phi_{\geq -1} = \Phi^+ \cup \{-\Pi\}.$$

We then define $\sigma_i : \Phi_{\geq -1} \to \Phi_{\geq -1}$ as follows. For each $\alpha \in \Phi_{\geq -1}$ we define $\sigma_i(\alpha)$ by

$$\sigma_i(\alpha) = \begin{cases} s_i(\alpha) & \text{if } s_i(\alpha) \in \Phi_{\geq -1} \\ \alpha & \text{otherwise.} \end{cases}$$

We note that if $\alpha \in \Phi^+$ then $s_i(\alpha) \in \Phi_{\geq -1}$. On the other hand

$$\begin{array}{rcll} s_i(-\alpha_i) & = & \alpha_i \in \Phi_{\geq -1} & \\ s_i(-\alpha_j) & = & -\alpha_j \in \Phi_{\geq -1} & \text{if } A_{ij} = 0 \\ s_i(-\alpha_j) & \notin & \Phi_{\geq -1} & \text{if } j \neq i \text{ and } A_{ij} \neq 0. \end{array}$$

Now the map $\sigma_i : \Phi_{\geq -1} \to \Phi_{\geq -1}$ is an involution. To see this we check $\sigma_i(\sigma_i(\alpha)) = \alpha$ for all $\alpha \in \Phi_{\geq -1}$. If $s_i(\alpha) \in \Phi_{\geq -1}$ we have

$$\sigma_i(\sigma_i(\alpha)) = \sigma_i(s_i(\alpha)) = s_i(s_i(\alpha)) = \alpha.$$

On the other hand, if $s_i(\alpha) \notin \Phi_{\geq -1}$ then

$$\sigma_i(\sigma_i(\alpha)) = \sigma_i(\alpha) = \alpha.$$

Thus we have $\sigma_i^2 = 1$.

We also note that if $A_{ij} = 0$ then $\sigma_i\sigma_j = \sigma_j\sigma_i$. This follows from the relations

$$\begin{aligned} \sigma_i\sigma_j(\alpha) &= s_is_j(\alpha) \quad \text{if } \alpha \in \Phi^+ \\ \sigma_i\sigma_j(-\alpha_i) &= \alpha_i \\ \sigma_i\sigma_j(-\alpha_j) &= \alpha_j \\ \sigma_i\sigma_j(-\alpha_k) &= -\alpha_k \quad \text{if } k \neq i, j. \end{aligned}$$

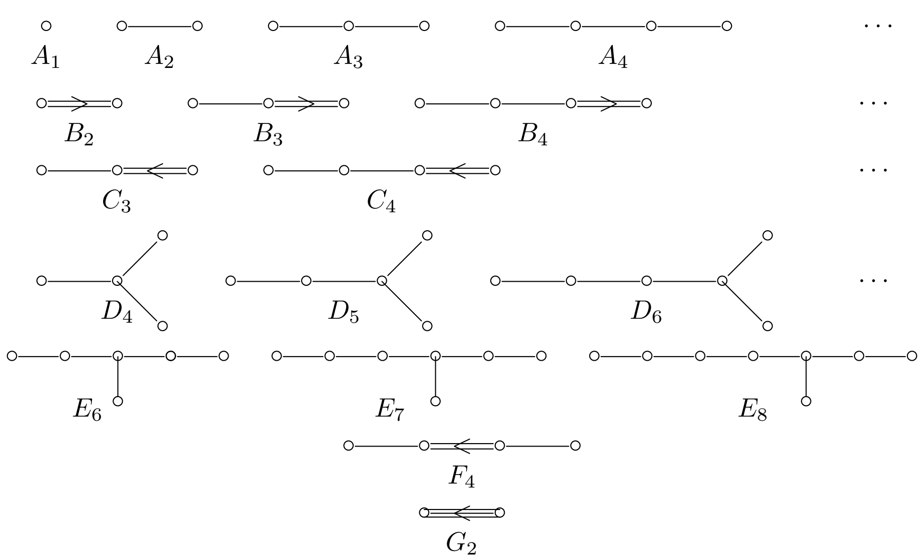
1.3 A dihedral group of PL-transformations

We now recall the definition of the Dynkin diagram attached to the Cartan matrix $A = (A_{ij})$. This is a graph with vertices $1, ..., l$ corresponding to the simple roots $\Pi = \{\alpha_1, ..., \alpha_l\}$. For $i \neq j$ the vertices i, j are joined by n_{ij} edges where n_{ij} is the non-negative integer defined by

$$n_{ij} = A_{ij}A_{ji}.$$

In fact n_{ij} always takes one of the values $0, 1, 2, 3$.

Let Δ be the Dynkin diagram of the Cartan matrix A. Then the semisimple Lie algebra $\mathfrak{g}$ with Cartan matrix A is simple if and only if Δ is connected. Moreover there is a bijection between simple non-trivial Lie algebras of finite dimension over $\mathbb{C}$ and the possible connected Dynkin diagrams. This is a very useful parametrisation of the simple Lie algebras. The theory of simple Lie algebras, due to E. Cartan and W. Killing, shows that the possible connected Dynkin diagrams are the following:



The meaning of the arrows on the double and triple edges is as follows. If $n_{ij} = 2$ or 3 we have $\{A_{ij}, A_{ji}\} = \{-1, -2\}$ or $\{-1, -3\}$ respectively. Suppose we are in the former case. Then we either have $A_{ij} = -1$, $A_{ji} = -2$ or $A_{ij} = -2$, $A_{ji} = -1$. Since

$$A_{ij} = 2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle}$$

the former case gives

$$2\frac{\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_i, \alpha_i \rangle} = -1 \qquad 2\frac{\langle \alpha_j, \alpha_i \rangle}{\langle \alpha_j, \alpha_j \rangle} = -2$$

and so

$$\langle \alpha_i, \alpha_i \rangle = 2\langle \alpha_j, \alpha_j \rangle.$$

The latter case gives

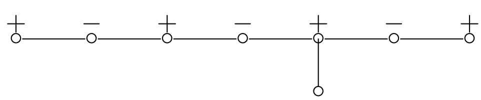
$$\langle \alpha_j, \alpha_j \rangle = 2\langle \alpha_i, \alpha_i \rangle.$$

We place an arrow on the edge joining i, j pointing from the longer root toward the shorter root. Thus in the former case above we have an arrow $\underset{i}{\circ}\!\Rightarrow\!\underset{j}{\circ}$ and in the latter case we have $\underset{i}{\circ}\!\Leftarrow\!\underset{j}{\circ}$ The arrow can then be considered as an inequality sign relating the lengths of the roots α_i, α_j, viz $|\alpha_i| > |\alpha_j|$ or $|\alpha_i| < |\alpha_j|$ where $|\alpha_i| = \sqrt{\langle \alpha_i, \alpha_i \rangle}$. In general the arrow points from i towards j when $|A_{ij}| < |A_{ji}|$.

Now each connected Dynkin diagram Δ can be decomposed into the disjoint union of two subsets

$$\Delta = I_+ \cup I_-$$

with the property that $A_{ij} = 0$ for all i, $j \in I_+$ with $i \neq j$ and $A_{ij} = 0$ for all i, $j \in I_-$ with $i \neq j$. For example the Dynkin diagram of type E_8 has such a decomposition



where I_+ is the set of vertices marked $+$ and I_- the set marked $-$.

We have $s_i s_j = s_j s_i$ for all i, $j \in I_+$ and so we may define an element $t_+ \in W$ by

$$t_+ = \prod_{i \in I_+} s_i.$$

The fact that the s_i for $i \in I_+$ commute shows that t_+ is uniquely determined by I_+. Also we have $(t_+)^2 = 1$. Similarly we may define $t_- \in W$ by

$$t_- = \prod_{i \in I_-} s_i.$$

t_- is uniquely determined by I_-, and satisfies $(t_-)^2 = 1$. Thus the subgroup $\langle t_+, t_- \rangle$ of W generated by t_+ and t_- is a dihedral group, being generated by two involutions. The product t_+t_- is an example of a Coxeter element of the Weyl group W, i.e a product of a complete set of fundamental reflections. All Coxeter elements of W have the same order h, given by

$$h = |\Phi|/|\Pi|.$$

h is called the Coxeter number of W. The dihedral group $\langle t_+, t_- \rangle = \langle t_+t_-, t_- \rangle$ has order $2h$ since

$$(t_-)^{-1}(t_+t_-)(t_-) = t_-t_+ = (t_+t_-)^{-1}.$$

Now Fomin and Zelevinsky introduced a PL-analogue of the linear transformations t_+ and t_-. Let

$$\tau_+ : \Phi_{\geq -1} \to \Phi_{\geq -1} \quad , \quad \tau_- : \Phi_{\geq -1} \to \Phi_{\geq -1}$$

be defined by

$$\tau_+ = \prod_{i \in I_+} \sigma_i \quad , \quad \tau_- = \prod_{i \in I_-} \sigma_i.$$

Since the factors σ_i of τ_+ all commute with one another we have $\tau_+^2 = 1$. Similarly we have $\tau_-^2 = 1$. Thus τ_+ and τ_- are permutations of $\Phi_{\geq -1}$ which are both involutions. Let

$$D = \langle \tau_+, \tau_- \rangle$$

be the subgroup of the group of permutations of $\Phi_{\geq -1}$ generated by τ_+ and τ_-. D is a dihedral group, being generated by two involutions. The order of the group D is twice the order of the element $\tau_+\tau_-$.

In order to describe the order of this element we recall a further idea from Lie theory. The Weyl group W contains a unique element w_0 with the property that $w_0(\Phi^+) = \Phi^-$. We have $l(w_0) = |\Phi^+|$ and w_0 is the unique element of W of maximal length. We have $w_0^2 = 1$. In fact $w_0(\Pi) = -\Pi$ and so there is a map $\Pi \to \Pi$ given by $\alpha_i \to -w_0(\alpha_i)$ which induces a graph automorphism on the Dynkin diagram Δ. This automorphism has order either 1 or 2, and is called the *opposition involution* on Δ.

It was shown by Fomin and Zelevinsky that the order of $\tau_+\tau_-$ is

$$\begin{array}{ll} h+2 & \text{if } -w_0 \neq 1 \\ (h+2)/2 & \text{if } -w_0 = 1. \end{array}$$

Here h is the Coxeter number of W defined above. Fomin and Zelevinsky also show that each D-orbit of $\Phi_{\geq -1}$ intersects $-\Pi$ and that the intersection of a D-orbit with $-\Pi$ is a $(-w_0)$-orbit on $-\Pi$, i.e an orbit on $-\Pi$ of the opposition involution $-w_0$.

1.4 The compatibility degree

Following Fomin and Zelevinsky we now define a map

$$\begin{array}{ccccc} \Phi_{\geq -1} & \times & \Phi_{\geq -1} & \longrightarrow & \mathbb{Z}_{\geq 0} \\ \alpha & , & \beta & \longrightarrow & (\alpha\|\beta) \end{array}$$

called the *compatibility degree.* Let $Q = \mathbb{Z}\Pi$ be the root lattice, i.e the set of linear combinations $\sum m_i\alpha_i$ with $m_i \in \mathbb{Z}$. If $\alpha = \sum m_i\alpha_i$ we write $[\alpha : \alpha_i] = m_i$. This is the multiplicity of α_i in α.

It is possible, given α, $\beta \in \Phi_{\geq -1}$, to define $(\alpha\|\beta)$ by the following rules.

(i) $(-\alpha_i\|\beta) = \begin{cases} [\beta : \alpha_i] & \text{if } [\beta : \alpha_i] \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

In fact the only element $\beta \in \Phi_{\geq 0}$ for which $[\beta : \alpha_i]$ is negative is $\beta = -\alpha_i$.

(ii) $(\tau_+\alpha\|\tau_+\beta) = (\alpha\|\beta)$ for all α, $\beta \in \Phi_{\geq -1}$.

(iii) $(\tau_{-\alpha}\|\tau_{-\beta}) = (\alpha\|\beta)$ for all α, $\beta \in \Phi_{\geq -1}$.

Rules (i), (ii), (iii) will determine $(\alpha\|\beta)$ uniquely since each $\alpha \in \Phi_{\geq -1}$ lies in the same D-orbit as some $-\alpha_i \in -\Pi$. Moreover if $-\alpha_i$ and $-\alpha_{\bar{i}}$ both lie in this D-orbit then we have $-w_0(\alpha_i) = \alpha_{\bar{i}}$, as above. Thus α_i, $\alpha_{\bar{i}}$ are images under the opposition involution and the values of $(\alpha\|\beta)$ obtained by using these alternatives will agree. Thus the compatibility degree $(\alpha\|\beta)$ of α and β is well defined. It is shown by Fomin and Zelevinsky, but is not obvious, that

$$(\alpha\|\beta) = (\beta^\vee\|\alpha^\vee) \text{ for all } \alpha, \beta \in \Phi_{\geq -1}\,,$$

where $\alpha^\vee$ denotes the root $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ in the dual root system.

1.5 Clusters in the set $\Phi_{\geq -1}$

A subset of $\Phi_{\geq -1}$ is called *compatible* if any pair α, β of its elements satisfy $(\alpha\|\beta) = 0$. A maximal compatible subset of $\Phi_{\geq -1}$ is called a *cluster.* We shall state a number of basic properties on clusters proved by Fomin and Zelevinsky.

<u>Cluster property 1</u>. Any two clusters in $\Phi_{\geq -1}$ have the same number of elements. This number is $l = |\Pi|$. Each cluster is a $\mathbb{Z}$-basis for the root lattice $Q = \mathbb{Z}\Pi$. $\{-\alpha_1, ..., -\alpha_l\}$ is an example of a cluster.

<u>Cluster property 2</u>. There is a bijective correspondence between clusters for $\Phi_{\geq -1}$ containing $-\alpha_i$ and clusters for $\Phi_{\geq -1}(\Delta - \{i\})$.

In the latter set we are considering clusters for the root system whose Dynkin diagram is obtained from Δ by omitting vertex i. The remaining graph may be disconnected, but all the concepts we have used based on connected Dynkin diagrams can be extended in a rather obvious way to disconnected diagrams. The bijection in the above result removes $-\alpha_i$ from C, i.e maps C to $C - \{-\alpha_i\}$.

The next result gives us the total number of clusters in $\Phi_{\geq -1}$. In order to describe this we need some further ideas from the theory of Weyl groups and root systems. Suppose W is a Weyl group acting on the Euclidean space V spanned by the root system Φ. Thus W acts on the space $P(V)$ of all polynomial functions on V, and the subspace $P(V)^W$ of W-invariant polynomials turns out to be isomorphic, by a theorem of Chevalley, to a polynomial ring $\mathbb{R}[I_1, ..., I_l]$ in l variables. The generators $I_1, ..., I_l$ may all be chosen as homogeneous polynomial invariants of degrees $d_1, d_2, ..., d_l$ respectively. Although the basic polynomial invariants $I_1, ..., I_l$ are not uniquely determined, their degrees $d_1, ..., d_l$ are unique. They are called the degrees of the basic polynomial invariants of W.

Cluster property 3. The number of clusters in $\Phi_{\geq -1}$ is

$$\frac{\prod_{i=1}^{l}(d_i + h)}{|W|}.$$

Note. It is known that $\displaystyle\prod_{i=1}^{l} d_i = |W|$, thus the above formula could also be written

$$\prod_{i=1}^{l} \frac{d_i + h}{d_i}.$$

In the papers published so far by Fomin and Zelevinsky only a case by case proof of this result is available.

Cluster property 4. Given any cluster C in $\Phi_{\geq -1}$ and any $\alpha \in C$ there exists a unique cluster C' with $C \cap C' = C - \{\alpha\}$.

This property is called the *replacement property* of clusters. It enables us to define a graph called the *exchange graph* for clusters. The vertices are the clusters and two vertices are joined by an edge if and only if their corresponding clusters C, C' satisfy

$$|C \cap C'| = |C| - 1.$$

Finally we describe a result giving rise to what is called the *cluster expansion* of any element of the root lattice $Q = \mathbb{Z}\Pi$.

Cluster property 5. Each element y of the root lattice Q has a unique expansion

$$y = \sum_{\alpha \in \Phi_{\geq -1}} m_\alpha \alpha$$

with $m_\alpha \geq 0$ in $\mathbb{Z}$, such that all α with $m_\alpha > 0$ are compatible.

Thus the clusters in $\Phi_{\geq 0}$ have some very striking properties. In the next section we shall illustrate these results by considering the example of a root system of type A_2.

1.6 Clusters in type A_2

The Cartan matrix of type A_2 is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

and the Dynkin diagram is $\overset{1}{\circ}\!\!\text{———}\!\!\overset{2}{\circ}$. We shall write $\Delta = I_+ \cup I_-$ where $I_+ = \{1\}$, $I_- = \{2\}$. Then τ_+ and τ_- give the following involutary permutations of $\Phi_{\geq -1}$:

$$\begin{aligned} \tau_+ &= (-\alpha_1 \quad \alpha_1)(-\alpha_2)(\alpha_2 \quad \alpha_1+\alpha_2) \\ \tau_- &= (-\alpha_1)(-\alpha_2 \quad \alpha_2)(\alpha_1 \quad \alpha_1+\alpha_2) \end{aligned}$$

where

$$\Phi_{\geq -1} = \{-\alpha_1, -\alpha_2, \alpha_1, \alpha_2, \alpha_1+\alpha_2\}.$$

The product $\tau_+\tau_-$ (in which τ_+ is performed first) is given by

$$\tau_+\tau_- = (-\alpha_1 \quad \alpha_1+\alpha_2 \quad -\alpha_2 \quad \alpha_2 \quad \alpha_1).$$

Thus $\tau_+\tau_-$ has order 5 and the group $D = \langle \tau_+, \tau_- \rangle$ is a dihedral group of order 10.

The compatibility degree $(\alpha \| \beta)$ is given by the following symmetric matrix, where α describes the row and β the column of the matrix.

	$-\alpha_1$	$-\alpha_2$	α_1	α_2	$\alpha_1+\alpha_2$
$-\alpha_1$	0	0	1	0	1
$-\alpha_2$	0	0	0	1	1
α_1	1	0	0	1	0
α_2	0	1	1	0	0
$\alpha_1+\alpha_2$	1	1	0	0	0

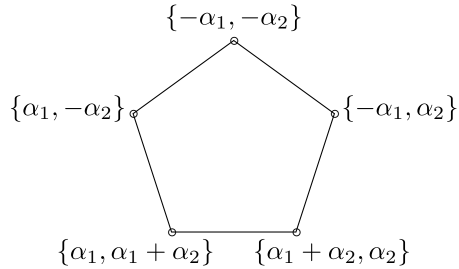
The clusters may be identified by inspecting this matrix. They are

$$\{-\alpha_1, -\alpha_2\} \quad \{-\alpha_1, \alpha_2\} \quad \{-\alpha_2, \alpha_1\} \quad \{\alpha_1, \alpha_1+\alpha_2\} \quad \{\alpha_2, \alpha_1+\alpha_2\}.$$

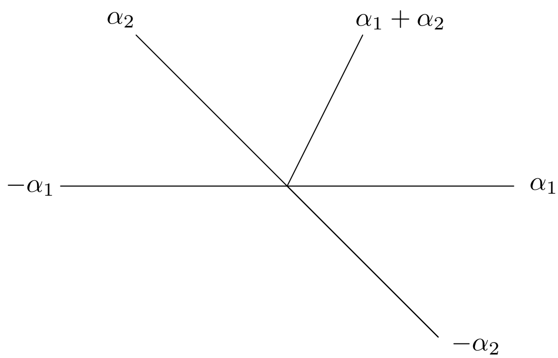
The Weyl group W is isomorphic to S_3, and its basic polynomial invariants have degrees 2 and 3. The Coxeter number is given by $h = 3$. Thus the Fomin-Zelevinsky formula for the number of clusters is

$$\frac{(2+3)(3+3)}{2.3} = 5.$$

The exchange graph for clusters is a pentagon, given below.



Finally the cluster expansion in the root lattice can be described with the help of the following figure.

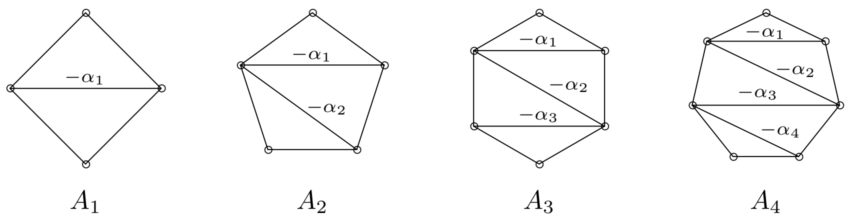


These five half-lines through the origin divide the set of points not lying on any half-line into five chambers. Points in such a chamber can be expressed as positive combinations of the roots on the two walls of the chamber concerned. Points on a half-line are positive multiples of the root along that half-line. Finally the zero vector gives the empty sum. Thus any $y \in Q$ is a non-zero positive combination of one of the following sets of vectors:

$$\{-\alpha_1, -\alpha_2\}$$
$$\{-\alpha_1, \alpha_2\}$$
$$\{\alpha_2, \alpha_1 + \alpha_2\}$$
$$\{\alpha_1 + \alpha_2, \alpha_1\}$$
$$\{\alpha_1, -\alpha_2\}$$
$$\{-\alpha_1\}$$
$$\{-\alpha_2\}$$
$$\{\alpha_1\}$$
$$\{\alpha_1 + \alpha_2\}$$
$$\{\alpha_2\}$$
$$\emptyset$$

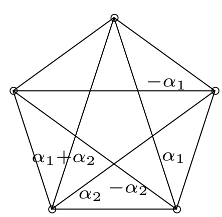
1.7 Clusters in type A_l

There is a pleasant geometrical description of the clusters of type A_l. We begin with a regular $(l+3)$-gon. We draw a chord in this figure joining a pair of vertices which have a common neighbouring vertex, and label this chord by the root $-\alpha_1$. We then draw a succession of chords labelled by $-\alpha_2, -\alpha_3, ..., -\alpha_l$ such that consecutive chords have a common vertex, as shown in the figures.



This set of chords $\{-\alpha_1, ..., -\alpha_l\}$ of the regular $(l+3)$-gon is called the *snake*.

We now consider the additional chords of the $(l+3)$-gon, not in the snake. Each such chord will cross certain chords in the snake. In fact each such chord can be labelled by one of the positive roots $\alpha_i + \alpha_{i+1} + ... + \alpha_j$ of A_l, where $i \leq j$, such that the given chord crosses the chord $-\alpha_k$ in the snake if and only if $i \leq k \leq j$. In this way the chords of the $(l+3)$-gon can be parametrised by elements of $\Phi_{\geq -1} = \Phi^+ \cup (-\Pi)$. For example the chords of a regular pentagon can be parametrised by elements of $\Phi_{\geq -1}$ of type A_2 as shown.



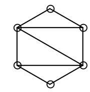
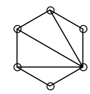
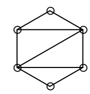
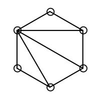
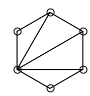
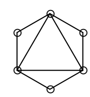
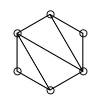
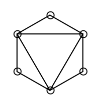
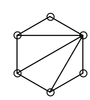
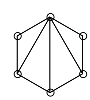
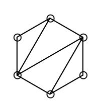
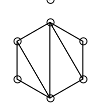
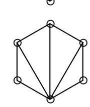
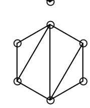
The clusters then correspond to the maximal sets of non-crossing chords. For example the 5 clusters in A_2 correspond to the 5 non-crossing pairs of chords

$$\{-\alpha_1, -\alpha_2\} \quad \{-\alpha_1, \alpha_2\} \quad \{\alpha_1 + \alpha_2, \alpha_2\} \quad \{\alpha_1 + \alpha_2, \alpha_1\} \quad \{-\alpha_2, \alpha_1\}.$$

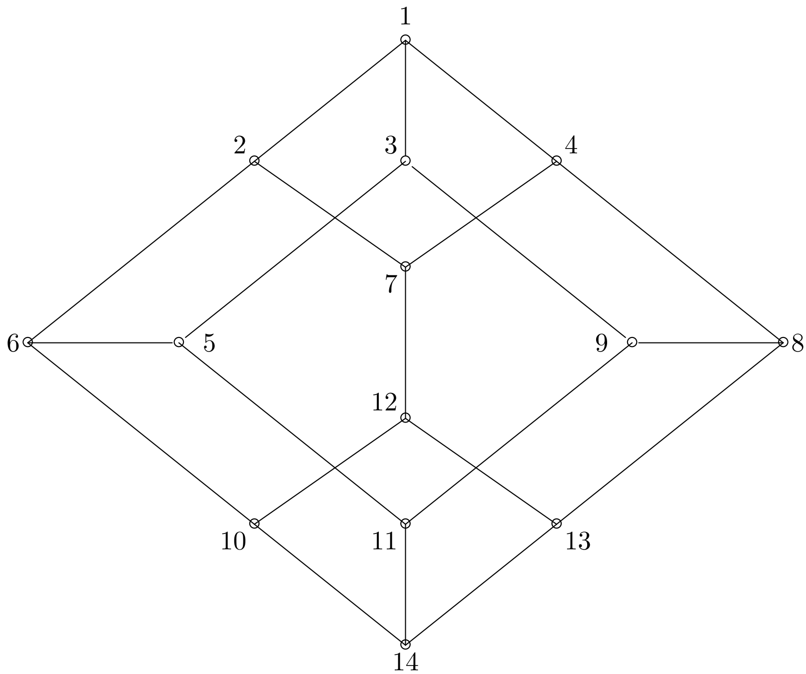
We shall now describe how the clusters in A_3 can be obtained in this way. The Weyl group of A_3 is isomorphic to the symmetric group S_4, and the degrees of the 3 basic polynomial invariants are $2, 3, 4$. The Coxeter number of A_3 is 4. Thus the number of clusters is

$$\frac{6.7.8}{2.3.4} = 14.$$

For each of the 14 clusters we shall describe the cluster in the form $w(-\Pi)$ for some w which is a word in $\sigma_1, \sigma_2, \sigma_3$. We shall also give the geometrical figure consisting of 3 non-crossing chords of a regular hexagon corresponding to the cluster.

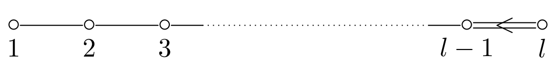
	Cluster	$w(-\Pi)$	Hexagon
1.	$\{-\alpha_1, -\alpha_2, -\alpha_3\}$	$-\Pi$	
2.	$\{\alpha_1, -\alpha_2, -\alpha_3\}$	$\sigma_1(-\Pi)$	
3.	$\{-\alpha_1, \alpha_2, -\alpha_3\}$	$\sigma_2(-\Pi)$	
4.	$\{-\alpha_1, -\alpha_2, \alpha_3\}$	$\sigma_3(-\Pi)$	
5.	$\{\alpha_1 + \alpha_2, \alpha_2, -\alpha_3\}$	$\sigma_2\sigma_1(-\Pi)$	
6.	$\{\alpha_1, \alpha_1 + \alpha_2, -\alpha_3\}$	$\sigma_1\sigma_2(-\Pi)$	
7.	$\{\alpha_1, -\alpha_2, \alpha_3\}$	$\sigma_3\sigma_1(-\Pi)$	
8.	$\{-\alpha_1, \alpha_2 + \alpha_3, \alpha_3\}$	$\sigma_3\sigma_2(-\Pi)$	
9.	$\{-\alpha_1, \alpha_2, \alpha_2 + \alpha_3\}$	$\sigma_2\sigma_3(-\Pi)$	
10.	$\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3\}$	$\sigma_1\sigma_2\sigma_3(-\Pi)$	
11.	$\{\alpha_1 + \alpha_2, \alpha_2, \alpha_2 + \alpha_3\}$	$\sigma_2\sigma_3\sigma_1(-\Pi)$	
12.	$\{\alpha_1, \alpha_1 + \alpha_2 + \alpha_3, \alpha_3\}$	$\sigma_3\sigma_1\sigma_2(-\Pi)$	
13.	$\{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3, \alpha_3\}$	$\sigma_3\sigma_2\sigma_1(-\Pi)$	
14.	$\{\alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3\}$	$\sigma_2\sigma_3\sigma_1\sigma_2(-\Pi)$	

The exchange graph for the clusters in A_3 is as shown below. The vertices are labelled $1-14$ as in the above table.

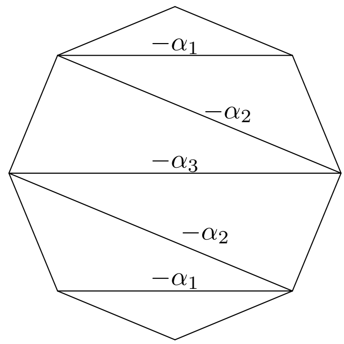


1.8 Clusters in type C_l

We recall that the Dynkin diagram of C_l is



In order to describe the clusters of type C_l we begin with a regular $(2l+2)$-gon. We distinguish between two different types of chord of this figure - those which are diameters and those which are not. We note that the chords which are non-diameters occur in symmetric pairs. We begin, as in type A_l, with a set of chords which form a snake. This involves $2l-1$ chords, one of which is a diameter and the remaining $2l-2$ give $l-1$ pairs of symmetric chords. We give the non-diameters in a symmetric pair the same label $-\alpha_i$, $i = 1, ..., l-1$, and label the diameter by $-\alpha_l$. We illustrate this in the diagram for the snake of type C_3.

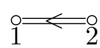


Each chord not in the snake is labelled by a positive root of C_l corresponding to the chords $-\alpha_i$ it crosses. Pairs of symmetric non-diameters will be labelled by the same positive root of C_l. For example the four diameters of the figure for C_3 are labelled by the roots $-\alpha_3,\ \alpha_3,\ 2\alpha_2+\alpha_3,\ 2\alpha_1+2\alpha_2+\alpha_3$ and the 8 pairs of symmetric non-diameters are labelled by the roots

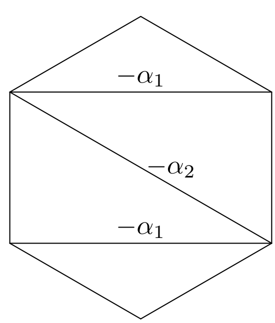
$$-\alpha_1,\ -\alpha_2,\ \alpha_1,\ \alpha_2,\ \alpha_1+\alpha_2,\ \alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+2\alpha_2+\alpha_3.$$

In this way we obtain a bijection between $\Phi_{\geq -1}$ and the set which is the union of the diameters and the pairs of symmetric non-diameters. Then each triangulation of the given figure by non-crossing chords gives rise to a cluster, just as in type A_l.

We illustrate this procedure in type C_2.

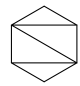
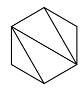
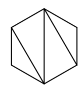
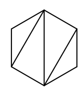
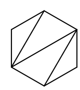
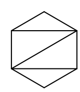


We begin with a regular hexagon, and have a snake



The remaining chords are labelled by the positive roots $\alpha_1,\ \alpha_2,\ \alpha_1+\alpha_2,\ 2\alpha_1+\alpha_2$.

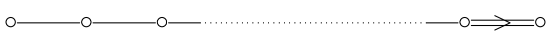
The clusters are given in the following table.

Cluster	Triangulation
$\{-\alpha_1, -\alpha_2\}$	
$\{\alpha_1, -\alpha_2\}$	
$\{\alpha_1, 2\alpha_1 + \alpha_2\}$	
$\{\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$	
$\{\alpha_1 + \alpha_2, \alpha_2\}$	
$\{-\alpha_1, \alpha_2\}$	

We note that the long roots in C_l correspond to the diameters in the $(2l+2)$-gon and the short roots in C_l correspond to the symmetric pairs of non-diameters.

1.9 Clusters in type B_l

The Dynkin diagram of B_l is



which differs from that of C_l only in the direction of the arrow joining $l-1$ and l. The clusters of type B_l are obtained from a regular $(2l+2)$-gon, as in type C_l. The difference is that the diameters of the $(2l+2)$-gon correspond to the short roots of B_l and the symmetric pairs of non-diameters correspond to the long roots of B_l. Every positive long root of C_l has form

$$\sum_{i=1}^{l-1} m_i\alpha_i + \alpha_l$$

where each m_i is divisible by 2. This gives rise to a positive short root of B_l

$$\sum_{i=1}^{l-1} \frac{1}{2} m_i\alpha_i + \alpha_l.$$

On the other hand each positive short root of C_l has form

$$\sum_{i=1}^{l-1} m_i\alpha_i + m_l\alpha_l,$$

and gives a positive long root of B_l

$$\sum_{i=1}^{l-1} m_i\alpha_i + 2m_l\alpha_l.$$

When we apply this map from long roots of C_l to short roots of B_l and short roots of C_l to long roots of B_l the clusters of type C_l are transformed into clusters of type B_l.

For example the clusters of B_2 which arise from the 6 clusters of C_2 listed in Section 1.8 are

$$\{-\alpha_1, -\alpha_2\} \qquad \{\alpha_1, -\alpha_2\} \qquad \{\alpha_1, \alpha_1+\alpha_2\} \qquad \{\alpha_1+2\alpha_2, \alpha_1+\alpha_2\}$$

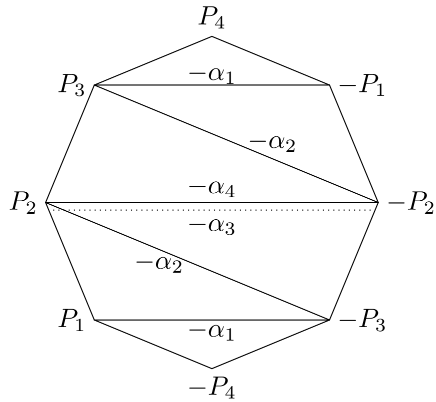
$$\{\alpha_1+2\alpha_2, \alpha_2\} \qquad \{-\alpha_1, \alpha_2\}.$$

1.10 Clusters in type D_l

We recall that the Dynkin diagram of D_l is

The clusters of type D_l are obtained from a regular $2l$-gon. We have diameters and symmetric pairs of non-diameters, just as in Sections 1.8 and 1.9, but this time we take two diameters joining each pair of opposite points. These diameters could be distinguished by drawing them in different colours. We shall find it convenient to represent one by a continuous line and the other by a dotted line. One diameter involves the fundamental root α_{l-1} and the other involves α_l.

We begin as before with a snake, which we illustrate in type D_4.

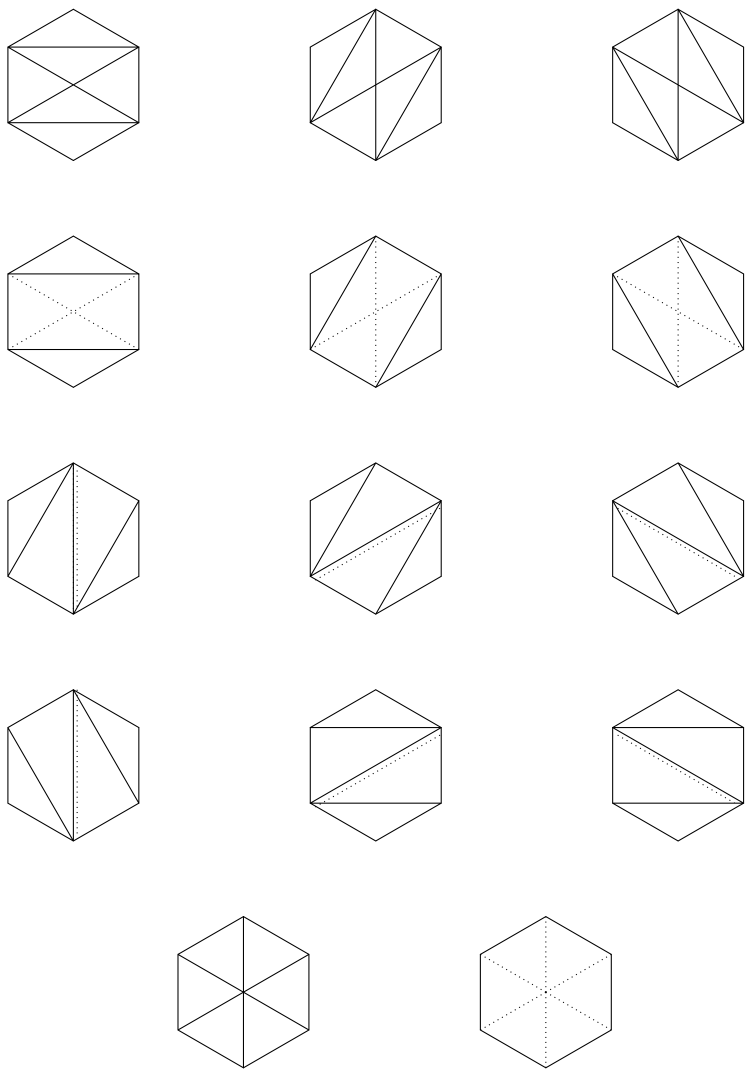


The snake contains both diameters joining P_2 and its opposite vertex $-P_2$. The positive roots corresponding to the remaining diameters and symmetric pairs of chords are as follows.

Diameter or symmetric pair		Root
$(P_1, -P_3)$	$(-P_1, P_3)$	$-\alpha_1$
$(P_2, -P_3)$	$(-P_2, P_3)$	$-\alpha_2$
$(P_2, -P_4)$	$(-P_2, P_4)$	α_1
$(P_1, -P_2)$	$(-P_1, P_2)$	α_2
(P_2, P_4)	$(-P_2, -P_4)$	$\alpha_1 + \alpha_2$
(P_1, P_3)	$(-P_1, -P_3)$	$\alpha_2 + \alpha_3 + \alpha_4$
$(P_3, -P_4)$	$(-P_3, P_4)$	$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$
(P_1, P_4)	$(-P_1, -P_4)$	$\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$
$(P_2, -P_2)$		$-\alpha_4$
$(P_2, -P_2)'$		$-\alpha_3$
$(P_3, -P_3)$		α_3
$(P_3, -P_3)'$		α_4
$(P_1, -P_1)$		$\alpha_2 + \alpha_3$
$(P_1, -P_1)'$		$\alpha_2 + \alpha_4$
$(P_4, -P_4)$		$\alpha_1 + \alpha_2 + \alpha_3$
$(P_4, -P_4)'$		$\alpha_1 + \alpha_2 + \alpha_4$

Diameters $(P_i, -P_i)$ are represented by a continuous line and $(P_i, -P_i)'$ by a dotted line. Two diameters of the same type are not regarded as crossing. This is why $(P_3, -P_3)$ corresponds to the root α_3 as it crosses diameter $(P_2, -P_2)'$ but not diameter $(P_2, -P_2)$. When a diameter crosses a symmetric pair of non-diameters the corresponding root is only taken once. This is why $(P_1, -P_1)$ corresponds to the root $\alpha_2 + \alpha_3$ rather than $2\alpha_2 + \alpha_3$.

In order to obtain the clusters we again take triangulations of the regular $2l$-gon by non-crossing chords. We recall that two diameters of the same type are not regarded as crossing, so can both be taken, and we may also take both diameters joining a given pair of opposite vertices, as these are not regarded as crossing. We give as an example the triangulations giving the 14 clusters of type D_3. (We must of course get 14 since A_3 has 14 clusters and $D_3 = A_3$).



14 triangulations of a hexagon, giving clusters of D_3

1.11 The number of clusters in each type

To conclude Chapter 1 we give the number of clusters in $\Phi_{\geq -1}$ for each type of simple root system. This may be derived from the general formula for the number of clusters given in Section 1.5, which depends on the degrees of the basic polynomial invariants of the Weyl group.

Type	Number of clusters
A_l	$\frac{1}{l+2}\binom{2l+2}{l+1}$
B_l	$\binom{2l}{l}$
C_l	$\binom{2l}{l}$
D_l	$\frac{3l-2}{l}\binom{2l-2}{l-1}$
G_2	8
F_4	105
E_6	833
E_7	4160
E_8	25080

We note that the numbers of clusters in the A_l series are the Catalan numbers

$$2,\ 5,\ 14,\ 42,\ 132,\ \ldots$$

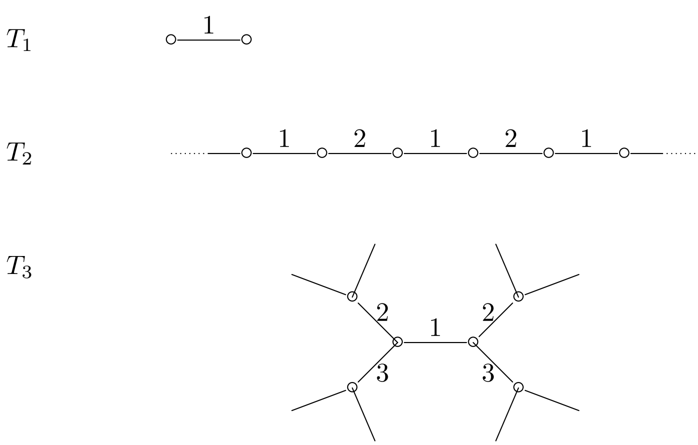
Chapter 2

Cluster algebras

In this chapter we shall describe how clusters can be defined in a more general context, following Fomin and Zelevinsky, and how one can define a corresponding family of commutative algebras called cluster algebras.

2.1 Exchange patterns

We begin with a graph T_l called the l-regular tree. This graph has l edges issuing from each vertex. We illustrate the examples T_1, T_2, T_3.



The edges of T_l will be labelled $1, 2, ..., l$ with one edge of each of these types coming from each vertex.

For each vertex t of T_l we suppose we are given indeterminates $x_1(t)$, $x_2(t)$, ..., $x_l(t)$. The l-tuple $x_1(t)$, ..., $x_l(t)$ is called a cluster. Clusters at neighbouring vertices are related. We suppose that for each $t \in T_l$ and each $j = 1, ..., l$ a monomial

$$M_j(t) = x_1(t)^{b_1} ... \, x_l(t)^{b_l}$$

is given, with $b_i \in \mathbb{Z}$ and $b_i \geq 0$ for each i. If $t,\ t'$ are vertices of T_l joined by an edge of type j the clusters at t and t' are related by the conditions

$$x_i(t) = x_i(t') \text{ if } i \neq j$$

$$x_j(t)x_j(t') = M_j(t) + M_j(t').$$

Thus if we know the monomials $M_j(t)$ for all pairs (t, j) we may obtain the cluster at each point of T_l starting from the cluster at the initial point t.

The family of monomials $M_j(t)$ is called an *exchange pattern* if the following four axioms are satisfied.

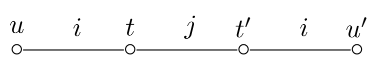
(i) $M_j(t)$ does not involve $x_j(t)$, i.e $b_j = 0$.

(ii) If $t,\ t'$ are vertices of T_l joined by an edge of type j then $M_j(t)$ and $M_j(t')$ cannot both involve x_i for any $i = 1, ..., l$.

(A consequence of this axiom is that $M_j(t)/M_j(t')$ determines both $M_j(t)$ and $M_j(t')$.)

(iii) Suppose $t,\ t',\ t''$ are vertices of T_l such that $t,\ t'$ are joined by an edge of type i and t', t'' are joined by an edge of type j where $i \neq j$. Then $M_i(t)$ involves x_j if and only if $M_j(t')$ involves x_i.

(iv) Suppose $t,\ t'$ are vertices of T_l joined by an edge of type j and let $i \in \{1, ..., l\}$ with $i \neq j$. Let u be the vertex of T_l joined to t by an edge of type i and u' be joined to t' by an edge of type i. Then $M_i(t')/M_i(u')$ is obtained from $M_i(t)/M_i(u)$ by replacing x_j by M_0/x_j, where M_0 is obtained from $M_j(t) + M_j(t')$ by replacing x_i by 0.



(The following comment on axiom (iv) turns out to be useful. We know that the monomials $M_j(t), M_j(t')$ do not both involve x_i. If one of them involves x_i then x_j is replaced by $M_j(v)/x_j$ where $M_j(v)$ is the other one. If neither of $M_j(t), M_j(t')$ involve x_i then neither of $M_i(t), M_i(u)$ involve x_j, by axiom (iii). So in this case we have

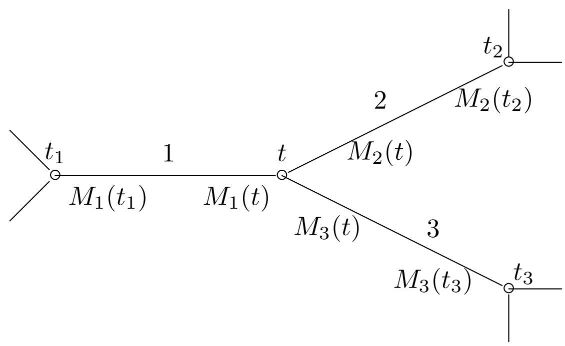
$$M_i(t')/M_i(u') = M_i(t)/M_i(u)$$

hence $M_i(t') = M_i(t)$ and $M_i(u') = M_i(u)$.)

The significance of these four axioms is as follows. Suppose we are given a single vertex t of T_l, the monomials $M_j(t)$ for $j = 1, ..., l$ and, for each j, the monomial $M_j(t')$ where t' is the vertex of T_l joined to t by an edge of type j. Thus we are given $2l$ monomials associated with the vertex t. For example in the case $l = 3$ we have the 6 monomials

$$M_1(t)\ ,\ M_2(t)\ ,\ M_3(t)\ ,\ M_1(t_1)\ ,\ M_2(t_2)\ ,\ M_3(t_3)$$

where t_1, t_2, t_3 are as shown in the figure



Then the axioms for an exchange pattern enable us to obtain these $2l$ monomials associated with any neighbouring vertex t'. For suppose t' is joined to t by an edge of type j. Then $M_j(t')$ and $M_j(t)$ are already known. Thus suppose $i \in \{1, ..., l\}$ satisfies $i \neq j$. Suppose u is the vertex joined to t by an edge of type i and u' is the vertex joined to t' by an edge of type i. Then $M_i(t')/M_i(u')$ can be obtained from $M_i(t)/M_i(u)$ by axiom (iv). This implies that $M_i(t')$ and $M_i(u')$ are known. Thus the $2l$ monomials associated with t' are determined.

Also the cluster at t' is determined by the cluster at t and the given set of $2l$ monomials associated with t. For we have

$$\begin{aligned} x_j(t') &= \frac{M_j(t) + M_j(t')}{x_j(t)} \\ x_i(t') &= x_i(t) \qquad \text{for } i \neq j. \end{aligned}$$

Thus information can be propagated around the graph T_l starting from information associated with just one vertex of T_l.

2.2 Matrix mutation

We now describe the propagation of information around the graph T_l in terms of matrices. We define an $l \times l$ matrix $B(t)$ associated with a vertex t of T_l. For each edge $t \underset{j}{\text{———}} t'$ involving t let

$$\frac{M_j(t)}{M_j(t')} = \prod_{i=1}^{l} x_i^{b_{ij}(t)}$$

Then $b_{jj}(t) = 0$ and we have

$$\begin{aligned} M_j(t) &= \prod_{\substack{i \\ b_{ij}(t)>0}} x_i^{b_{ij}(t)} \\ M_j(t') &= \prod_{\substack{i \\ b_{ij}(t)<0}} x_i^{-b_{ij}(t)} \end{aligned}$$

Let $B(t)$ be the $l \times l$ matrix over $\mathbb{Z}$ given by $B(t) = (b_{ij}(t))$. Then we have

$$\begin{array}{ll} b_{ii}(t) = 0 & \text{for all } i \\ b_{ij}(t) > 0 & \text{if and only if } b_{ji}(t) < 0. \end{array}$$

These conditions are given by axioms (i) and (iii) for an exchange pattern. A matrix satisfying these two conditions will be called *sign skew-symmetric*. Thus for each vertex t of T_l we have a sign skew-symmetric matrix $B(t)$. We consider the relation between $B(t)$ and $B(t')$ where t' is a neighbouring vertex of T_l joined to t by an edge of type j. We shall write, for convenience,

$$B(t) = (b_{ij}) \qquad B(t') = (b'_{ij}).$$

Suppose $i \in \{1, ..., l\}$ satisfies $i \neq j$ and let u be the vertex joined to t by an edge of type i and u' be the vertex joined to t' by an edge of type i. Thus we have a diagram

$$u \ \overset{i}{\text{———}} \ t \ \overset{j}{\text{———}} \ t' \ \overset{i}{\text{———}} \ u'$$

We have

$$\begin{aligned} \frac{M_i(t')}{M_i(u')} &= \prod_k x_k^{b'_{ki}} \\ \frac{M_i(t)}{M_i(u)} &= \prod_k x_k^{b_{ki}} \end{aligned}$$

In axiom (iv) for an exchange pattern we replace x_j in $M_i(t)/M_i(u)$. We have

$$x_j(t)x_j(t') = \prod_{\substack{k \\ b_{kj}>0}} x_k^{b_{kj}} + \prod_{\substack{k \\ b_{kj}<0}} x_k^{-b_{kj}}$$

Suppose $b_{ij} \neq 0$. Then x_j is replaced in axiom (iv) by M/x_j where M is the monomial not involving x_i. If $b_{ij} > 0$ then

$$M = \prod_{\substack{k \\ b_{kj}<0}} x_k^{-b_{kj}}$$

and if $b_{ij} < 0$ then

$$M = \prod_{\substack{k \\ b_{kj}>0}} x_k^{b_{kj}}.$$

In either case we have

$$M = \prod_{\substack{k \\ b_{kj}b_{ij}<0}} x_k^{|b_{kj}|}$$

We now apply axiom (iv) and replace x_j by M/x_j in $M_i(t)/M_i(u)$. The result is equal to $M_i(t')/M_i(u')$. Hence

$$\prod_{\substack{k \\ k\neq i,j}} x_k^{b'_{ki}} \cdot x_j^{b'_{ji}} = \prod_{\substack{k \\ k\neq i,j}} x_k^{b_{ki}} \frac{\prod_{\substack{k \\ b_{kj}b_{ij}<0}} x_k^{|b_{kj}|b_{ji}}}{x_j^{b_{ji}}}$$

Comparing exponents we get

$$b'_{ji} = -b_{ji}.$$

If $k \neq i, j$, then

$$b'_{ki} = \begin{cases} b_{ki} & \text{if } b_{kj}b_{ij} \geq 0 \\ b_{ki} + |b_{kj}|b_{ji} & \text{if } b_{kj}b_{ij} < 0 \end{cases}$$

The latter equation can be expressed as follows without a split into two cases:

$$b'_{ki} = b_{ki} + \frac{|b_{kj}|b_{ji} + b_{kj}|b_{ji}|}{2} \text{ if } k \neq i, j.$$

For if $b_{kj}b_{ij} \geq 0$ then b_{kj} and b_{ij} have the same sign so b_{kj} and b_{ji} have opposite signs and the terms $|b_{kj}|b_{ji}$ and $b_{kj}|b_{ji}|$ cancel. If $b_{kj}b_{ij} < 0$ then b_{kj}, b_{ji} have the same sign and so the terms $|b_{kj}|b_{ji}$ and $b_{kj}|b_{ji}|$ are equal.

Thus we have obtained the following rule for matrix mutation.

$$\begin{aligned} b'_{ki} &= -b_{ki} && \text{if } k = j \text{ or } i = j \\ b'_{ki} &= b_{ki} + \frac{|b_{kj}|b_{ji} + b_{kj}|b_{ji}|}{2} && \text{if } k \neq j \text{ and } i \neq j. \end{aligned}$$

Although we have assumed $b_{ij} \neq 0$ we note that this relation holds when $b_{ij} = 0$ also. For axiom (iv) becomes particularly simple in this case and gives

$$\frac{M_i(t')}{M_i(u')} = \frac{M_i(t)}{M_i(u)}.$$

This implies that $b'_{ki} = b_{ki}$ if $k \neq j$ and $i \neq j$. We say that $B(t')$ is obtained from $B(t)$ by matrix mutation.

It is evident that if, conversely, we are given $l \times l$ matrices $B(t)$ over $\mathbb{Z}$ for each $t \in T_l$ satisfying:

(a) $B(t)$ is sign skew-symmetric for each $t \in T_l$,

(b) the $B(t)$ satisfy the rule for matrix mutation,

then these matrices determine an exchange pattern.

2.3 Some examples

Let Φ be a root system with Cartan matrix $A = (a_{ij})$. Let $B = (b_{ij})$ be any matrix satisfying:

B is sign skew-symmetric,

$|b_{ij}| = -a_{ij}$ if $i \neq j$.

If we put $B(t) = B$ for some vertex $t \in T_l$ and obtain $B(t')$ for all other vertices by matrix mutation then the $B(t')$ are sign skew-symmetric also, and so we have an exchange pattern. This follows from the fact that the Cartan matrix A is symmetrisable, i.e there exists a diagonal matrix D with positive coefficients such that DA is symmetric. Then DB is skew-symmetric. It follows from the mutation rules that $DB(t')$ is skew-symmetric for all $t' \in T_l$, and so $B(t')$ is sign skew-symmetric.

Type A_2.

The Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

We choose

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then B is sign skew-symmetric with $|b_{ij}| = -a_{ij}$ for $i \neq j$. We begin with a vertex $t = t_0$ with cluster (x_1, x_2). Let t_1 be the vertex joined to t by an edge of type 1. Then the cluster at t_1 is (x_3, x_2) where $x_1 x_3 = 1 + x_2$. The mutated matrix is

$$B(t_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Let t_2 be the vertex joined to t_1 by an edge of type 2. Then the cluster at t_2 is (x_3, x_4) where $x_2 x_4 = 1 + x_3$. Continuing in this way, using vertices

$$t = t_0, t_1, t_2, t_3, ...$$

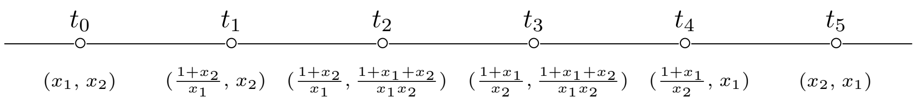
related by

$$t = t_0 \overset{}{\underset{1}{\text{———}}} t_1 \underset{2}{\text{———}} t_2 \underset{1}{\text{———}} t_3 \underset{2}{\text{———}} t_4 \text{——}$$

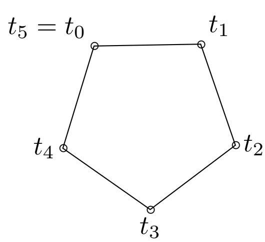
we have clusters

$$
\begin{aligned}
\underline{x}(t_0) &= (x_1, x_2)\\
\underline{x}(t_1) &= \left(\frac{1+x_2}{x_1}, x_2 \right)\\
\underline{x}(t_2) &= \left(\frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1 x_2} \right)\\
\underline{x}(t_3) &= \left(\frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1 x_2} \right)\\
\underline{x}(t_4) &= \left(\frac{1+x_1}{x_2}, x_1 \right)\\
\underline{x}(t_5) &= (x_2, x_1)
\end{aligned}
$$

Thus the cluster (x_1, x_2), regarded as an unordered set, is the same at t_5 as at t_0.



The clusters may therefore be regarded as sets defined on the quotient graph of T_2 which is a pentagon.



The mutated matrices are

$$
B(t_0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B(t_1) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B(t_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
$$

$$
B(t_3) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B(t_4) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B(t_5) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
$$

etc.

Type B_2

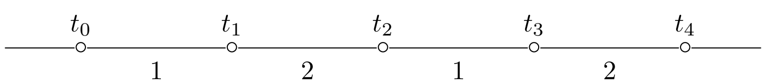
The Cartan matrix in this case is

$$A = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

We choose

$$B = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}.$$

This has the required properties. We take vertices $t = t_0, t_1, t_2, t_3, t_4, ...$ joined by edges 1 and 2 alternately.



We begin with a cluster (x_1, x_2) at t_0. Then the clusters at subsequent vertices are

$$\begin{aligned}
\underline{x}(t_0) &= (x_1, x_2) \\
\underline{x}(t_1) &= (x_3, x_2) \\
\underline{x}(t_2) &= (x_3, x_4) \\
\underline{x}(t_3) &= (x_5, x_4) \\
\underline{x}(t_4) &= (x_5, x_6) \\
\underline{x}(t_5) &= (x_7, x_6) \\
\underline{x}(t_6) &= (x_7, x_8)
\end{aligned}$$

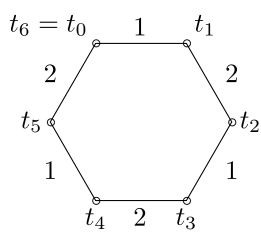
where

$$\begin{aligned}
x_1 x_3 &= 1 + x_2^2 \\
x_2 x_4 &= 1 + x_3 \\
x_3 x_5 &= 1 + x_4^2 \\
x_4 x_6 &= 1 + x_5 \\
x_5 x_7 &= 1 + x_6^2 \\
x_6 x_8 &= 1 + x_7.
\end{aligned}$$

Thus we obtain

$$
\begin{aligned}
\underline{x}(t_0) &= (x_1, x_2)\\
\underline{x}(t_1) &= \left(\frac{1+x_2^2}{x_1} \;,\; x_2 \right)\\
\underline{x}(t_2) &= \left(\frac{1+x_2^2}{x_1} \;,\; \frac{1+x_2^2+x_1}{x_1x_2} \right)\\
\underline{x}(t_3) &= \left(\frac{1+x_1^2+2x_1+x_2^2}{x_1x_2^2} \;,\; \frac{1+x_2^2+x_1}{x_1x_2} \right)\\
\underline{x}(t_4) &= \left(\frac{1+x_1^2+2x_1+x_2^2}{x_1x_2^2} \;,\; \frac{1+x_1}{x_2} \right)\\
\underline{x}(t_5) &= \left(x_1 \;,\; \frac{1+x_1}{x_2} \right)\\
\underline{x}(t_6) &= (x_1, x_2)
\end{aligned}
$$

Thus the cluster at t_6 is the same as the cluster at t_0. The clusters may thus be regarded as being defined on the quotient graph of T_2 which is a hexagon.



The mutated matrices are

$$
\begin{aligned}
B(t_0) = B(t_2) = B(t_4) &= \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}\\
B(t_1) = B(t_3) = B(t_5) &= \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}.
\end{aligned}
$$

<u>Type C_2</u>

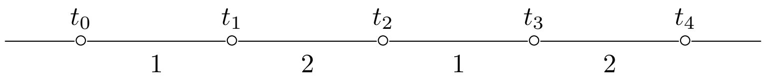
The Cartan matrix is

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

We choose

$$B = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}.$$

We take vertices $t = t_0, t_1, t_2, t_3, t_4, \ldots$ joined by edges 1 and 2 alternately.



We begin with a cluster (x_1, x_2) at t_0. Then the clusters at subsequent vertices are

$$\begin{aligned}
\underline{x}(t_0) &= (x_1, x_2) \\
\underline{x}(t_1) &= (x_3, x_2) \\
\underline{x}(t_2) &= (x_3, x_4) \\
\underline{x}(t_3) &= (x_5, x_4) \\
\underline{x}(t_4) &= (x_5, x_6) \\
\underline{x}(t_5) &= (x_7, x_6) \\
\underline{x}(t_6) &= (x_7, x_8)
\end{aligned}$$

where

$$\begin{aligned}
x_1 x_3 &= 1 + x_2 \\
x_2 x_4 &= 1 + x_3^2 \\
x_3 x_5 &= 1 + x_4 \\
x_4 x_6 &= 1 + x_5^2 \\
x_5 x_7 &= 1 + x_6 \\
x_6 x_8 &= 1 + x_7^2
\end{aligned}$$

Thus we obtain

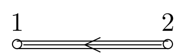
$$\begin{aligned}
\underline{x}(t_0) &= (x_1, x_2) \\
\underline{x}(t_1) &= \left(\frac{1 + x_2}{x_1} \quad , \quad x_2 \right) \\
\underline{x}(t_2) &= \left(\frac{1 + x_2}{x_1} \quad , \quad \frac{1 + 2x_2 + x_2^2 + x_1^2}{x_1^2 x_2} \right) \\
\underline{x}(t_3) &= \left(\frac{1 + x_2 + x_1^2}{x_1 x_2} \quad , \quad \frac{1 + 2x_2 + x_2^2 + x_1^2}{x_1^2 x_2} \right) \\
\underline{x}(t_4) &= \left(\frac{1 + x_2 + x_1^2}{x_1 x_2} \quad , \quad \frac{1 + x_1^2}{x_2} \right) \\
\underline{x}(t_5) &= \left(x_1 \quad , \quad \frac{1 + x_1^2}{x_2} \right) \\
\underline{x}(t_6) &= (x_1, x_2)
\end{aligned}$$

Thus the cluster at t_6 is the same as that at t_0. The clusters may therefore be regarded as being defined on the quotient graph of T_2 in which t_0 and t_6 are identified, which is a hexagon.

The mutated matrices are

$$\begin{aligned} B(t_0) &= B(t_2) = B(t_4) = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix} \\ B(t_1) &= B(t_3) = B(t_5) = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

<u>Type G_2</u>



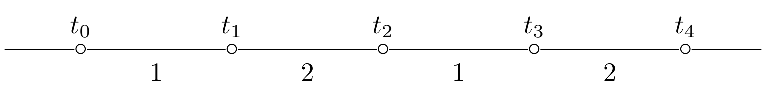
The Cartan matrix is

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

We choose

$$B = \begin{pmatrix} 0 & 3 \\ -1 & 0 \end{pmatrix}.$$

We take vertices $t = t_0, t_1, t_2, t_3, t_4, \ldots$ joined by edges 1 and 2 alternately.



We begin with a cluster (x_1, x_2) at t_0. Then the clusters at subsequent vertices are

$$\begin{aligned} \underline{x}(t_0) &= (x_1, x_2) \\ \underline{x}(t_1) &= (x_3, x_2) \\ \underline{x}(t_2) &= (x_3, x_4) \\ \underline{x}(t_3) &= (x_5, x_4) \\ \underline{x}(t_4) &= (x_5, x_6) \\ \underline{x}(t_5) &= (x_7, x_6) \\ \underline{x}(t_6) &= (x_7, x_8) \\ \underline{x}(t_7) &= (x_9, x_8) \\ \underline{x}(t_8) &= (x_9, x_{10}) \end{aligned}$$

where

$$\begin{aligned}
x_1x_3 &= 1+x_2\\
x_2x_4 &= 1+x_3^3\\
x_3x_5 &= 1+x_4\\
x_4x_6 &= 1+x_5^3\\
x_5x_7 &= 1+x_6\\
x_6x_8 &= 1+x_7^3\\
x_7x_9 &= 1+x_8\\
x_8x_{10} &= 1+x_9^3
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
\underline{x}(t_0) &= (x_1,x_2)\\
\underline{x}(t_1) &= \left(\frac{x_2+1}{x_1}\ ,\ x_2\right)\\
\underline{x}(t_2) &= \left(\frac{x_2+1}{x_1}\ ,\ \frac{(x_2+1)^3+x_1^3}{x_1^3x_2}\right)\\
\underline{x}(t_3) &= \left(\frac{(x_2+1)^2+x_1^3}{x_1^2x_2}\ ,\ \frac{(x_2+1)^3+x_1^3}{x_1^3x_2}\right)\\
\underline{x}(t_4) &= \left(\frac{(x_2+1)^2+x_1^3}{x_1^2x_2}\ ,\ \frac{(x_2+1)^3+x_1^3(x_1^3+3x_2+2)}{x_1^3x_2^2}\right)\\
\underline{x}(t_5) &= \left(\frac{x_1^3+1+x_2}{x_1x_2}\ ,\ \frac{(x_2+1)^3+x_1^3(x_1^3+3x_2+2)}{x_1^3x_2^2}\right)\\
\underline{x}(t_6) &= \left(\frac{x_1^3+1+x_2}{x_1x_2}\ ,\ \frac{x_1^3+1}{x_2}\right)\\
\underline{x}(t_7) &= \left(x_1\ ,\ \frac{x_1^3+1}{x_2}\right)\\
\underline{x}(t_8) &= (x_1,x_2)
\end{aligned}$$

Thus the cluster at t_8 is the same as that at t_0. The clusters may therefore be regarded as being defined on the quotient graph of T_2 in which t_0 and t_8 are identified, which is an octagon.

The mutated matrices are

$$\begin{aligned}
B(t_0) &= B(t_2) = B(t_4) = B(t_6) = \begin{pmatrix} 0 & 3\\ -1 & 0\end{pmatrix}\\
B(t_1) &= B(t_3) = B(t_5) = B(t_7) = \begin{pmatrix} 0 & -3\\ 1 & 0\end{pmatrix}.
\end{aligned}$$

2.4 The Laurent phenomenon

Let $\{M_j(t);\ t \in T_l,\ j = 1, ..., l\}$ be an exchange pattern. We pick an initial point $t \in T_l$. Then each point in T_l can be joined to t by a unique sequence of edges. Thus for any $t' \in T_l$ and any $j = 1, ..., l$ $x_j(t')$ can be expressed as a natural function of $x_1, ..., x_l$ where $x_i = x_i(t)$, using the relations described earlier. Now it was proved by Fomin and Zelevinsky that each $x_j(t')$ is in fact a *Laurent polynomial* in $x_1, ..., x_l$, i.e its denominator is a monomial in $x_1, ..., x_l$. This is clearly illustrated by the examples A_2, B_2, C_2, G_2 which we described in Section 2.3. We call this property the *Laurent phenomenon.* The subring of the ring of all Laurent polynomials in $x_1, ..., x_l$ over $\mathbb{Z}$ generated by the cluster variables $x_j(t')$ for all $t' \in T_l$ and all $j = 1, ..., l$ is called the *cluster algebra* associated with the given exchange pattern.

Now there may be finitely or infinitely many distinct cluster variables $x_j(t')$. For example in the cases A_2, B_2, C_2, G_2 there are only finitely many distinct elements $x_j(t')$. This is a special case of the following more general result. Suppose we have an exchange pattern described by a sign skew-symmetric matrix $B = B(t)$. Let $B = (b_{ij})$ and let $A = (a_{ij})$ be the $l \times l$ matrix defined by

$$\begin{array}{rcll} a_{ii} & = & 2 & \text{for all } i, \\ a_{ij} & = & -|b_{ij}| & \text{when } i \neq j. \end{array}$$

Then A is a generalized Cartan matrix in the sense of the theory of Kac-Moody algebras. Suppose that A is in fact the Cartan matrix of a finite dimensional semisimple Lie algebra. It was then shown by Fomin and Zelevinsky that there are only finitely many distinct cluster variables $x_j(t')$. Cluster algebras with only finitely many cluster variables are said to be of *finite type*. Fomin and Zelevinsky also proved conversely that if the cluster algebra arising from the matrix $B = B(t)$ has finite type then the matrix A associated to $B(t')$ for some point t' is the Cartan matrix of a finite dimensional semisimple Lie algebra. Thus the classification of the cluster algebras of finite type is the same as the Cartan-Killing classification of Cartan matrices.

Given a cluster algebra of finite type its distinct cluster variables are in bijective correspondence with $\Phi^+ \cup (-\Pi)$ where Φ is the root system of the Cartan matrix and Π is the fundamental system contained in Φ^+. The clusters arising in this way are those described in Part 1 of this article. The correspondence between cluster variables and the set $\Phi^+ \cup (-\Pi)$ is obtained by considering the monomial in the denominator when a cluster variable is expressed as a Laurent polynomial in $x_1, ..., x_l$. We illustrate this correspondence in types A_2, B_2, C_2, G_2.

Type A_2

Cluster variable	Root in $\Phi^+ \cup (-\Pi)$
x_1	$-\alpha_1$
x_2	$-\alpha_2$
$\frac{1+x_2}{x_1}$	α_1
$\frac{1+x_1}{x_2}$	α_2
$\frac{1+x_1+x_2}{x_1x_2}$	$\alpha_1+\alpha_2$

Type B_2

Cluster variable	Root in $\Phi^+ \cup (-\Pi)$
x_1	$-\alpha_1$
x_2	$-\alpha_2$
$\dfrac{1+x_2^2}{x_1}$	α_1
$\dfrac{1+x_1}{x_2}$	α_2
$\dfrac{1+x_2^2+x_1}{x_1x_2}$	$\alpha_1+\alpha_2$
$\dfrac{1+x_1^2+2x_1+x_2^2}{x_1x_2^2}$	$\alpha_1+2\alpha_2$

Type C_2

Cluster variable	Root in $\Phi^+ \cup (-\Pi)$
x_1	$-\alpha_1$
x_2	$-\alpha_2$
$\dfrac{1+x_2}{x_1}$	α_1
$\dfrac{1+x_1^2}{x_2}$	α_2
$\dfrac{1+x_2+x_1^2}{x_1x_2}$	$\alpha_1+\alpha_2$
$\dfrac{1+2x_2+x_2^2+x_1^2}{x_1^2x_2}$	$2\alpha_1+\alpha_2$

Type G_2

Cluster variable	Root in $\Phi^+ \cup (-\Pi)$
x_1	$-\alpha_1$
x_2	$-\alpha_2$
$\frac{x_2+1}{x_1}$	α_1
$\frac{x_1^3+1}{x_2}$	α_2
$\frac{x_1^3+1+x_2}{x_1x_2}$	$\alpha_1+\alpha_2$
$\frac{(x_2+1)^2+x_1^3}{x_1^2x_2}$	$2\alpha_1+\alpha_2$
$\frac{(x_2+1)^3+x_1^3}{x_1^3x_2}$	$3\alpha_1+\alpha_2$
$\frac{(x_2+1)^3+x_1^3(x_1^3+3x_2+2)}{x_1^3x_2^2}$	$3\alpha_1+2\alpha_2$

In general a positive root $\alpha = m_1\alpha_1 + ... + m_l\alpha_l$ corresponds to a Laurent polynomial of form

$$\frac{f(x_1,...,x_l)}{x_1^{m_1}x_2^{m_2}...x_l^{m_l}}, \text{ where } f(x_1,...,x_l) \in \mathbb{Z}[x_1,...,x_l]$$

and the roots $-\alpha_1,...,-\alpha_l$ correspond to x_1, ..., x_l respectively (x_i may be regarded as $1/x_i^{-1}$).

Although we have concentrated on cluster algebras of finite type it should be pointed out that cluster algebras of infinite type are also important and appear frequently in applications.

2.5 Cluster algebras with constants

There is a more general theory of cluster algebras than the one we have so far outlined, in which various constants make an appearance. Such cluster algebras with constants appear frequently in applications to other areas of mathematics.

We shall develop the theory of exchange patterns, outlined in Section 2.1, in a more general context.

Let P be a free abelian group, written multiplicatively, with generators p_i, $i \in I'$, for some finite index set I'. Suppose we are given, for each $t \in T_l$, a

$|I'| \times l$ matrix $C(t)$ over $\mathbb{Z}$ with

$$C(t) = (c_{ij}(t)), \qquad i \in I',\ j \in I,$$

where $I = \{1, ..., l\}$ and $c_{ij}(t) \in \mathbb{Z}$.

For each $j \in I$ we define $p_j(t) \in P$ by

$$p_j(t) = \prod_{\substack{i \in I' \\ c_{ij}(t) > 0}} p_i^{c_{ij}(t)}.$$

We now define

$$M_j(t) = p_j(t) \prod_{\substack{i \\ b_{ij}(t) > 0}} x_i(t)^{b_{ij}(t)}.$$

This is the analogue of the monomial $M_j(t)$ defined in Section 2.1. $M_j(t)$ originally depended on a matrix $B(t)$. It now depends on the matrices $B(t)$ and $C(t)$.

If $t,\ t'$ are neighbouring vertices of T_l joined by an edge of type j then their cluster variables are related by

$$\begin{aligned} &x_i(t) = x_i(t') \text{ if } i \neq j \\ &x_j(t)x_j(t') = M_j(t) + M_j(t'). \end{aligned}$$

We assume that the exchange axioms (i), (ii), (iii), (iv) hold just as in Section 2.1. This implies that, if $C(t) = C,\ C(t') = C'$ with

$$C = (c_{ij}),\ C' = (c'_{ij}),\ B(t) = B = (b_{ij})$$

then

$$c'_{ki} = \begin{cases} -c_{ki} & \text{if} \quad i = j\,, \\ c_{ki} + \dfrac{|c_{kj}|b_{ji} + c_{kj}|b_{ji}|}{2} & \text{if} \quad i \neq j\,. \end{cases}$$

Thus the rules for matrix mutation of the C's look very similar to those we previously obtained for the B's.

In fact, if we define $\tilde{B}(t)$ to be the $(l + |I'|) \times l$ matrix

$$\tilde{B}(t) = \begin{pmatrix} B(t) \\ C(t) \end{pmatrix}$$

then the rules for matrix mutation of the $\tilde{B}(t)$ look precisely the same as those previously obtained for the $B(t)$.

The square matrix $B(t)$ is called the *principal part* of $\tilde{B}(t)$.

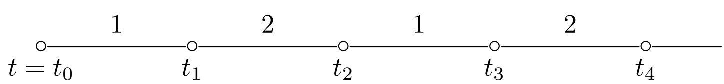
2.6 The example $Gr_{2,5}$

We illustrate these general ideas by taking the example

$$B(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C(t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\tilde{B}(t) = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \\ \hline 0 & -1 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \end{array}\right).$

We begin with vertex $t = t_0$ and consider vertices $t_1, t_2, t_3, ...$ joined to t_0 by edges of type $1, 2$ alternately. Thus we have



We begin with a cluster (x_1, x_2) at t_0. Then the clusters at subsequent vertices are

$$\begin{aligned} \underline{x}(t_0) &= (x_1, x_2) \\ \underline{x}(t_1) &= (x_3, x_2) \\ \underline{x}(t_2) &= (x_3, x_4) \\ \underline{x}(t_3) &= (x_5, x_4) \\ \underline{x}(t_4) &= (x_5, x_6) \\ \underline{x}(t_5) &= (x_7, x_6) \end{aligned}$$

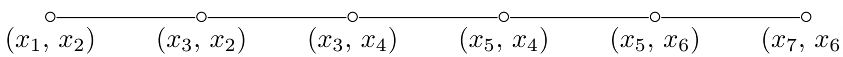
where

$$\begin{aligned} x_1 x_3 &= p_2 x_2 + p_4 p_5 \\ x_2 x_4 &= p_3 x_3 + p_5 p_1 \\ x_3 x_5 &= p_4 x_4 + p_1 p_2 \\ x_4 x_6 &= p_5 x_5 + p_2 p_3 \\ x_5 x_7 &= p_1 x_6 + p_3 p_4 \end{aligned}$$

Thus

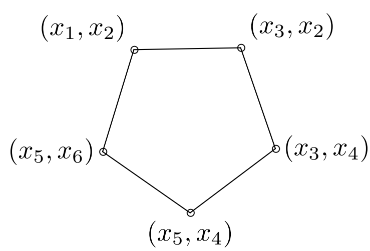
$$\begin{aligned}
x_3 &= \frac{p_2x_2+p_4p_5}{x_1}\\
x_4 &= \frac{p_1p_5x_1+p_2p_3x_2+p_3p_4p_5}{x_1x_2}\\
x_5 &= \frac{p_1x_1+p_3p_4}{x_2}\\
x_6 &= x_1\\
x_7 &= x_2.
\end{aligned}$$

By interchanging the order $(x_7,x_6)\to(x_6,x_7)$ we obtain $(x_6,x_7)=(x_1,x_2)$ with the same matrix $\tilde{B}$ as originally. The mutation of matrices is as shown:



$$\left(\begin{array}{cc} 0 & -1\\ 1 & 0\\ \hline 0 & -1\\ 1 & 0\\ 0 & 1\\ -1 & 1\\ -1 & 0 \end{array}\right)\quad
\left(\begin{array}{cc} 0 & 1\\ -1 & 0\\ \hline 0 & -1\\ -1 & 0\\ 0 & 1\\ 1 & 0\\ 1 & -1 \end{array}\right)\quad
\left(\begin{array}{cc} 0 & -1\\ 1 & 0\\ \hline -1 & 1\\ -1 & 0\\ 0 & -1\\ 1 & 0\\ 0 & 1 \end{array}\right)\quad
\left(\begin{array}{cc} 0 & 1\\ -1 & 0\\ \hline 1 & 0\\ 1 & -1\\ 0 & -1\\ -1 & 0\\ 0 & 1 \end{array}\right)\quad
\left(\begin{array}{cc} 0 & -1\\ 1 & 0\\ \hline 1 & 0\\ 0 & 1\\ -1 & 1\\ -1 & 0\\ 0 & -1 \end{array}\right)\quad
\left(\begin{array}{cc} 0 & 1\\ -1 & 0\\ \hline -1 & 0\\ 0 & 1\\ 1 & 0\\ 1 & -1\\ 0 & -1 \end{array}\right)$$

Thus we have periodicity and the clusters are defined on the quotient graph which is finite. The quotient graph of T_l is the exchange graph of the cluster, and is a pentagon.



This example is related to the Grassmann variety $Gr_{2,5}$. The set of all 2-dimensional subspaces in a 5-dimensional space $\mathbb{C}^5$ forms a projective variety $Gr_{2,5}$. Such a 2-dimensional subspace may be described by a 2×5 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{pmatrix}.$$

For $i<j$ let Δ_{ij} be defined by

$$\Delta_{ij} = \begin{vmatrix} a_{1i} & a_{1j}\\ a_{2i} & a_{2j} \end{vmatrix}.$$

Δ_{ij} lies in the homogeneous coordinate ring $\mathbb{C}[Gr_{2,5}]$ of the Grassmann variety. These 2×2 minors are connected by certain quadratic relations. If $i<j<k<l$ we have

$$\Delta_{ik}\Delta_{jl} = \Delta_{ij}\Delta_{kl} + \Delta_{il}\Delta_{jk}$$

In particular we have

$$\begin{aligned}
\Delta_{24}\Delta_{35} &= \Delta_{23}\Delta_{45} + \Delta_{25}\Delta_{34}\\
\Delta_{14}\Delta_{25} &= \Delta_{12}\Delta_{45} + \Delta_{15}\Delta_{24}\\
\Delta_{13}\Delta_{24} &= \Delta_{12}\Delta_{34} + \Delta_{14}\Delta_{23}\\
\Delta_{14}\Delta_{35} &= \Delta_{13}\Delta_{45} + \Delta_{15}\Delta_{34}\\
\Delta_{13}\Delta_{25} &= \Delta_{12}\Delta_{35} + \Delta_{15}\Delta_{23}
\end{aligned}$$

Suppose we write

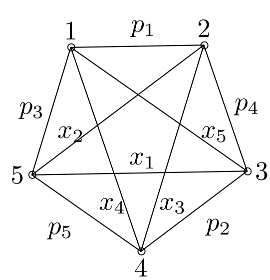
$$\begin{array}{lllll}
x_1 = \Delta_{35}, & x_2 = \Delta_{25}, & x_3 = \Delta_{24}, & x_4 = \Delta_{14}, & x_5 = \Delta_{13}\\
p_1 = \Delta_{12}, & p_2 = \Delta_{34}, & p_3 = \Delta_{15}, & p_4 = \Delta_{23}, & p_5 = \Delta_{45}.
\end{array}$$

Then the quadratic relations become

$$\begin{aligned}
x_1x_3 &= p_2x_2 + p_4p_5\\
x_2x_4 &= p_3x_3 + p_5p_1\\
x_3x_5 &= p_4x_4 + p_1p_2\\
x_4x_1 &= p_5x_5 + p_2p_3\\
x_5x_2 &= p_1x_1 + p_3p_4
\end{aligned}$$

These are precisely the relations we had earlier! Thus the propagation relations become the Plücker relations between the 2×2 minors.

These 2×2 minors correspond to the chords of the pentagon giving the cluster variables.



The minors corresponding to the chords of the pentagon give the cluster variables x_1, x_2, x_3, x_4, x_5 and the minors corresponding to the boundary edges give the constants p_1, p_2, p_3, p_4, p_5.

Let F be the field of rational functions in $x_1(t), ..., x_l(t)$ with coefficients in the group ring $\mathbb{C}P$. The propagation relations show that F is independent of the choice of vertex $t \in T_l$. The $\mathbb{C}P$-subalgebra of F generated by $x_1(t), ..., x_l(t)$ for all $t \in T_l$ is called the cluster algebra associated with the matrix $\tilde{B}$.

In the above example the cluster algebra is the coordinate ring $\mathbb{C}[Gr_{2,5}]$ of the Grassmannian $Gr_{2,5}$.

This is only one of a number of similar examples. It has been shown by Fomin and Zelevinsky that

$\mathbb{C}[Gr_{2,6}]$ is a cluster algebra, with constants, of type A_3 ;
$\mathbb{C}[Gr_{2,l+3}]$ is a cluster algebra, with constants, of type A_l ;
$\mathbb{C}[Gr_{3,6}]$ is a cluster algebra, with constants, of type D_4 ;
$\mathbb{C}[Gr_{3,7}]$ is a cluster algebra, with constants, of type E_6 ;
$\mathbb{C}[Gr_{3,8}]$ is a cluster algebra, with constants, of type E_8 .

Chapter 3

Applications of clusters and cluster algebras

3.1 Canonical bases of quantized enveloping algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra over $\mathbb{C}$ and $\mathfrak{U}(\mathfrak{g})$ be its universal enveloping algebra. We recall that $\mathfrak{g}$ has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{n}_+$ is the sum of the root spaces for a set of positive roots and $\mathfrak{n}_-$ the sum of the root spaces for the negative roots. This triangular decomposition gives rise to a triangular tensor product decomposition of $\mathfrak{U}(\mathfrak{g})$

$$\mathfrak{U}(\mathfrak{g}) = \mathfrak{U}(\mathfrak{n}_-) \otimes \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{n}_+).$$

Now let $U(\mathfrak{g})$ be the corresponding quantised enveloping algebra. $U(\mathfrak{g})$ has a corresponding tensor product decomposition

$$U(\mathfrak{g}) = U^- \otimes U^0 \otimes U^+.$$

G. Lusztig discovered a basis B of U^- called the *canonical basis* which has remarkable and important properties. For example we have finite dimensional irreducible $U(\mathfrak{g})$-modules $V(\lambda)$ corresponding to the dominant integral weights λ. $V(\lambda)$ has a highest weight vector v_λ. The canonical basis has the property that the vectors $bv_\lambda \in V(\lambda)$ for $b \in B$ which are non-zero form a basis for $V(\lambda)$. Thus the canonical basis for U^- gives rise to bases for all finite dimensional highest weight modules $V(\lambda)$ simultaneously. For this and other reasons the canonical basis has been the topic of much recent investigation.

The Lie algebra $\mathfrak{g}$ has a natural system of generators $e_1, ..., e_l$, $h_1, ..., h_l$, $f_1, ..., f_l$ where

$$\mathfrak{n}_- = \langle f_1, ..., f_l \rangle \qquad \mathfrak{h} = \langle h_1, ..., h_l \rangle \qquad \mathfrak{n}_+ = \langle e_1, ..., e_l \rangle$$

and the quantised enveloping algebra has a corresponding set of generators. In particular we have

$$U^- = \langle F_1, ..., F_l \rangle .$$

It is therefore natural to ask how the canonical basis elements are expressed in terms of the generators $F_1, ..., F_l$ of U^-. The answer to this question turns out to be difficult but very intriguing.

The elements of the canonical basis are parametrised by $(\mathbb{Z}_{\geq 0})^N$ where $N = |\Phi^+| = |\Phi^-|$. If $\mathfrak{g}$ has type A_1 the canonical basis is given by

$$B = \{F_1^n/[n]! \quad ; \quad n \in \mathbb{Z}, n \geq 0\}$$

where $[n]! = [1][2]...[n]$ and

$$[i] = \frac{q^i - q^{-i}}{q - q^{-1}} = q^{i-1} + q^{i-3} + ... + q^{-(i-3)} + q^{-(i-1)}$$

is the quantum integer corresponding to $i \in \mathbb{Z}$.

However if $\mathfrak{g}$ has type A_2 we have

$$B = \left\{ \frac{F_1^{c_1}}{[c_1]!} \frac{F_2^{c_2}}{[c_2]!} \frac{F_1^{c_3}}{[c_3]!} \text{ , } c_2 \geq c_1 + c_3; \frac{F_2^{c_1}}{[c_1]!} \frac{F_1^{c_2}}{[c_2]!} \frac{F_2^{c_3}}{[c_3]!} \text{ , } c_2 \geq c_1 + c_3 \right\}$$

Thus in this case there are two different types of canonical basis element. This means that the parameter space $(\mathbb{Z}_{\geq 0})^3$ is divided into two by its intersection with a hyperplane, such that the canonical basis elements on one side are those of the first type above, and those on the other side are those of the second type.

We now suppose that $\mathfrak{g}$ has type A_3. There then turn out to be 14 different types of canonical basis element as regards the way in which the basis elements are expressed in terms of the generators F_1, F_2, F_3. The parameters describing each of these 14 subsets of canonical basis elements are those which are $\mathbb{Z}_{\geq 0}$-combinations of certain primitive parameters, and these primitive parameters have a cluster structure of type A_3. Thus the existence of such a cluster structure explains how the canonical basis splits into 14 subsets in the required way. This result showing that there are 14 types of canonical basis vectors in type A_3 is due to N. H. Xi and an analogous result for vectors in the dual canonical basis is due to A. Berenstein and A. Zelevinsky.

When the Lie algebra $\mathfrak{g}$ is simple of type A_4 there again appears to be a cluster structure which determines the behaviour of the canonical basis. The situation here has been investigated by various authors and is very interesting. Let Δ be a Dynkin diagram and Q be a quiver of type Δ and arbitrary orientation. A representation of Q is given by a finite dimensional vector space V_i at each vertex i together with a homomorphism $\phi_\mu : V_i \to V_j$ for each edge μ with an arrow from i to j. Let $\mathbb{C}Q$ be the path algebra of Q over $\mathbb{C}$. Let $\tilde{Q}$ be the quiver obtained from Q by adding a new edge $\overline{\mu}$ with an arrow from j to i for each edge μ in Q with an arrow from i to j.

We consider the preprojective algebra $\overline{\mathbb{C}Q}$ of Q. This is the quotient of $\mathbb{C}\tilde{Q}$ with additional relations

$$\sum_{\substack{\tau \\ \tau \text{ begins at } i}} \overline{\tau}\tau = \sum_{\substack{\tau \\ \tau \text{ ends at } i}} \tau\overline{\tau}$$

one for each vertex i, summed over all edges τ of the quiver Q. By Gabriel's theorem the $\mathbb{C}Q$-indecomposable modules correspond to the positive roots Φ^+. However $\overline{\mathbb{C}Q}$ may have more indecomposables than $\mathbb{C}Q$ since a $\mathbb{C}Q$-module may decompose on restriction to $\overline{\mathbb{C}Q}$.

For example if Δ has type A_4 then $\mathbb{C}Q$ has 10 indecomposable modules, one for each positive root, but $\overline{\mathbb{C}Q}$ has 40 indecomposable modules, of which 4 are projective. We define a *clique* to be a set of indecomposable modules which is maximal with respect to

$$\mathrm{Ext}(M, N) = 0\ , \quad \mathrm{Ext}(N, M) = 0$$

for all modules in the given set.

Now Marsh and Reineke have conjectured that there is a bijective correspondence between types of canonical basis elements in $U^-(\mathfrak{g})$ for types $A_1 - A_4$ and cliques of indecomposable modules for $\overline{\mathbb{C}Q}$. This is so for the 14 types of canonical basis element in type A_3, as there are 14 cliques of indecomposable modules for $\overline{\mathbb{C}Q}$ in this case.

If Δ has type A_4 there are 4 indecomposable projective $\overline{\mathbb{C}Q}$-modules and these lie in all the cliques. There are 672 cliques altogether, each containing 10 indecomposable modules. These are the 4 projective indecomposable modules together with 6 others. Thus Marsh and Reineke's conjecture would imply that there are 672 types of canonical basis elements in type A_4. The way in which subsets of 6 non-projective indecomposable modules are chosen from the 36 such modules which exist is in accordance with the cluster structure of type D_6, in which $|\Phi^+| = 30, |\Pi| = 6, |\Phi^+ \cup \{-\Pi\}| = 36$. Thus it appears that the behaviour of the canonical basis in type A_4 is governed by a cluster structure of type D_6! (Zelevinsky has indicated that this is the case.) See also [5,2.24] and recent work of Geiss, Leclerc and Schröer for information on research in this direction.

It seems quite likely that in type A_n for $n \geq 5$ the behaviour of the canonical basis is governed by a cluster structure of infinite type.

3.2 The cluster category

Let Q be a Dynkin quiver with an alternating orientation. (This means that each vertex of Q is either a source or a sink). Let $\mathbb{C}Q$ be the path algebra of Q over $\mathbb{C}$ and consider the category of finite dimensional $\mathbb{C}Q$-modules. The indecomposable modules are in bijective correspondence with the set Φ^+ of positive roots, by Gabriel's theorem.

Let $\mathcal{D} = \mathcal{D}^b(\mathbb{C}Q)$ be the bounded derived category of this category of finite dimensional $\mathbb{C}Q$-modules. The objects of $\mathcal{D}$ are bounded complexes of finite dimensional $\mathbb{C}Q$-modules modulo the equivalence relation of quasi-isomorphism. Each $\mathbb{C}Q$-module M determines a complex $\underline{M}$ in which M appears in degree 0 and 0 appears elsewhere, and $\underline{M}$ can be regarded as an object in $\mathcal{D}$. The indecomposable objects of $\mathcal{D}$ then have the form $\underline{M}[i]$ where M is an indecomposable $\mathbb{C}Q$-module, $i \in \mathbb{Z}$, and $\underline{M}[i]$ is $\underline{M}$ with the i^{th} degree shift applied.

The Auslander-Reiten quiver of $\mathcal{D}$ is a graph whose objects are indecomposable modules for $\mathcal{D}$. This graph admits a well-known map τ called the Auslander-Reiten translate.

We define the cluster category $\mathcal{C}$ by $\mathcal{C} = \mathcal{D}/F$ where $F : \mathcal{D} \to \mathcal{D}$ is the auto equivalence $\tau^{-1} \circ [1]$. Then the objects of $\mathcal{C}$ are the objects of $\mathcal{D}$ and the morphisms of $\mathcal{C}$ are given by

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}\ (F^i X, Y).$$

The indecomposable modules in the category $\mathcal{C}$ are given by

$$\begin{aligned} \mathrm{Ind}\mathcal{C} = &\{\underline{M}\,;\ M \text{ an indecomposable } \mathbb{C}Q\text{-module}\} \\ \bigcup\ &\{\underline{P_i}[1]\,;\ P_i \text{ is the projective indecomposable module at vertex } i \text{ of } Q\}\,. \end{aligned}$$

There is thus a bijection

$$\Phi_{\geq -1} \underset{\phi}{\longrightarrow} \mathrm{Ind}\mathcal{C}$$

given by

$$\begin{aligned} \alpha \in \Phi &\longrightarrow \underline{M_\alpha}\,, \\ -\alpha_i &\longrightarrow \underline{P_i}[1]\,, \end{aligned}$$

between cluster variables and indecomposable modules for $\mathcal{C}$.

We obtain a natural interpretation of the compatibility degree in this context. Given $\alpha, \beta \in \Phi_{\geq -1}$ we have

$$(\alpha \| \beta) = \dim\ \mathrm{Ext}^1_{\mathcal{C}}(\phi(\alpha), \phi(\beta)).$$

The clusters in $\Phi_{\geq -1}$ correspond to what are called *tilting objects* in $\mathcal{C}$. An object T of $\mathcal{C}$ is called a tilting object if it satisfies the conditions

$$\mathrm{Ext}^1_{\mathcal{C}}(T, T) = 0$$

$$\text{and} \quad T = \bigoplus_{i=1}^{l} T_i$$

is a decomposition into non-isomorphic indecomposables, where l is maximal (in the sense that no further component could be added to preserve the conditions). In fact l is the number of vertices of Q.

We then have a bijection between clusters in $\Phi_{\geq -1}$ and tilting objects in the cluster category $\mathcal{C}$. It seems therefore that there is a fundamental connection between clusters and the concepts of tilting. This work on the cluster category is due to a group of mathematicians Buan, Marsh, Reineke, Reiten, Todorov. A graphical approach has also been developed by Caldero, Chapoton and Schiffler in type A. This whole area of work is undergoing a rapid development.

3.3 Geometry associated to algebraic groups

It has been conjectured by Zelevinsky that the coordinate rings of a number of algebraic varieties which arise naturally in the study of algebraic groups have a cluster algebra structure. For example, if G is a semisimple algebraic group with Borel subgroup B, U the unipotent radical of B, B^- an opposite Borel subgroup to B, and Weyl group W, Zelevinsky has conjectured that the coordinate rings $\mathbb{C}[G]$, $\mathbb{C}[B]$, $\mathbb{C}[U]$, $\mathbb{C}[G/U]$ might all have cluster algebra structures. For example it is known that the coordinate ring $\mathbb{C}[SL_3/U]$ has a cluster algebra structure of type A_1, $\mathbb{C}[SL_4/U]$ has such a structure of type A_3, $\mathbb{C}[SL_5/U]$ has such a structure of type D_6, and $\mathbb{C}[Sp_4/U]$ has such a structure of type B_2.

In addition, Berenstein, Fomin and Zelevinsky have studied the coordinate ring

$$\mathbb{C}[BuB \cap B^-vB^-]$$

of a double Bruhat cell $BuB \cap B^-vB^-$ where u, v are arbitrary elements of W. A great deal of information has been obtained about this coordinate ring, and it is conjectured that it might have the structure of a cluster algebra for arbitrary $u, v \in W$. If this is so it would be of considerable interest to know for which pairs $u, v \in W$ this cluster algebra has finite type.

It appears then that the theory of cluster algebras, still at quite an early stage of development but advancing rapidly, may give a powerful new technique for investigating the geometry and representation theory of algebraic groups.

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