Estimating Singularity Dimension

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Abstract

In this article we address an interesting question on the computation of the dimension of self-affine sets in Euclidean space. A well known result of Falconer showed that under mild assumptions the Hausdorff dimension of typical self-affine sets is equal to its Singularity dimension. Heuter and Lalley subsequently presented a smaller open family of non-trivial examples for which there is an equality of these two dimensions. In this article we analyse the size of this family and present an efficient algorithm for estimating the dimension.

1 Introduction

Despite the prominent role played by Hausdorff dimension in both analysis and dynamical systems, there remain very few non-trivial examples for which the value of dimension can be explicitly stated, or even effectively computed. There has been some success in the case of conformal iterated function schemes [7] and [4], but considerably less in the case of non-conformal maps.

On the other hand there is a very elegant result of Falconer which shows that the Hausdorff dimension of the limit set for a typical finite set of (non-conformal) affine contractions is equal to the singularity dimension, whose presentation is more suggestive of allowing estimation of the value. However, even this doesn't necessarily lend itself to numerical evaluation. In this note we want to consider a particular setting, introduced by Hueter and Lalley, where they showed that the singularity dimension is always equal to the Hausdorff dimension. In this case we shall describe a very effective method for its rapid numerical evaluation.

We begin by recalling the general setting in which we will be working.

Notation 1.1. *Let $A_1, \cdots, A_k \in GL(2, \mathbb{R})$ be 2×2 invertible matrices and assume that $\|A_1\|, \cdots, \|A_k\| < \frac{1}{2}$. Given vectors $b_1, \cdots, b_k \in \mathbb{R}^2$ we can consider affine maps $T_i : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_i(x) = A_i x + b_i$ $(i = 1, \cdots, k)$.*

We next give a basic definition.

Definition 1.2. *The* limit set *$\Lambda \subset \mathbb{R}^2$ is the unique smallest closed non-empty set such that $\Lambda = T_1\Lambda \cup \cdots \cup T_k\Lambda$.*

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Falconer introduced the notion of the singularity dimension $\dim_S(\Lambda)$, which is typically equal to the Hausdorff dimension, but while having a better implicit characterization is still remarkably difficult to estimate numerically.

Notation 1.3. *If $n \geq 1$ and $\underline{i} = (i_1, \cdots, i_n) \in \{1, \cdots, k\}^n$, then we write $|\underline{i}| = n$. We can then associate to $\underline{i}$ the product of matrices $A_{\underline{i}} = A_{i_1} A_{i_2} \cdots A_{i_n}$.*

We can associate to the 2×2 matrix $A_{\underline{i}}$ the two singular values $\alpha_1(A_{\underline{i}}) \geq \alpha_2(A_{\underline{i}})$. These are the major and minor axes of the ellipse which is the image of the unit circle under $A_{\underline{i}}$. Equivalently, these are the eigenvalues of the 2×2-matrix $\sqrt{A_{\underline{i}}^* A_{\underline{i}}}$ [1].

Definition 1.4. *We denote*

$$\phi^s(A_{\underline{i}}) = \begin{cases} \alpha_1(A_{\underline{i}})^s & \text{if } 0 < s \leq 1 \\ \alpha_1(A_{\underline{i}}) \alpha_2(A_{\underline{i}})^{1-s} & \text{if } 1 \leq s < 2. \end{cases}$$

This leads to the following definition of singularity dimension due to Falconer.

Definition 1.5. *We define the* singularity dimension *by*

$$\dim_S(\Lambda) \colon = \inf \left\{ s > 0 : \sum_{n=1}^{\infty} \sum_{|\underline{i}|=n} \phi^s(A_{\underline{i}}) < +\infty \right\}.$$

We now recall the following fundamental theorem of Falconer.

Theorem 1.6 (Falconer [1])**.** *Assume that $\|A_1\|, \cdots, \|A_k\| < \frac{1}{2}$. Then for a.e. $(b_1, \cdots, b_k) \in \mathbb{R}^{2k}$, we have $\dim_H(\Lambda) = \dim_S(\Lambda)$.*

In fact, Falconer proved the result under some slightly more restrictive assumptions, which were removed by Solomyak [11]. The significance of this result is that the formula holds quite generally: for any contractions with Euclidean norm less than $\frac{1}{2}$; and almost all translational parts. On the other hand, except in very special cases it is not always easy to give explicit examples to which the formula applies. Heuter and Lalley showed that under more restrictive hypotheses on the maps it is possible to remove the almost everywhere hypothesis.

Let $Q_2 = \{(x, y) : x \leq 0, y \geq 0\}$ denote the closed second quadrant

Hypotheses 1.7 (Heuter–Lalley conditions)**.** *We want to assume the following technical conditions:*

1. *$\|A_i\| < 1$ for $i = 1, \cdots, k$;*
2. *$\alpha_1(A_i)^2 < \alpha_2(A_i)$ for $i = 1, \cdots, k$;*
3. *$A_1^{-1} Q_2, \cdots, A_k^{-1} Q_2$ are pairwise disjoint subsets of $\mathrm{int}(Q_2)$;*
4. *there is a bounded open set V such that $\overline{T_i V}$ are disjoint, $i = 1, \cdots, k$.*

Conditions (1)–(3) depend only on the A_i, but condition (4) also depends on the b_i. (An additional simplifying assumption would be that $A_1, \cdots, A_k$ have positive determinants.) Condition (1) is a weaker contraction hypothesis than in Theorem 1.6. Condition (2) is a bunching condition; condition (3) a separation condition. Condition (4) is a closed set condition. Next, we recall the result of Heuter–Lalley.

Theorem 1.8 (Heuter–Lalley [3])**.** *Under Hypotheses 1.7 we have that*

$$0 < \dim_H(\Lambda) = \dim_S(\Lambda) < 1.$$

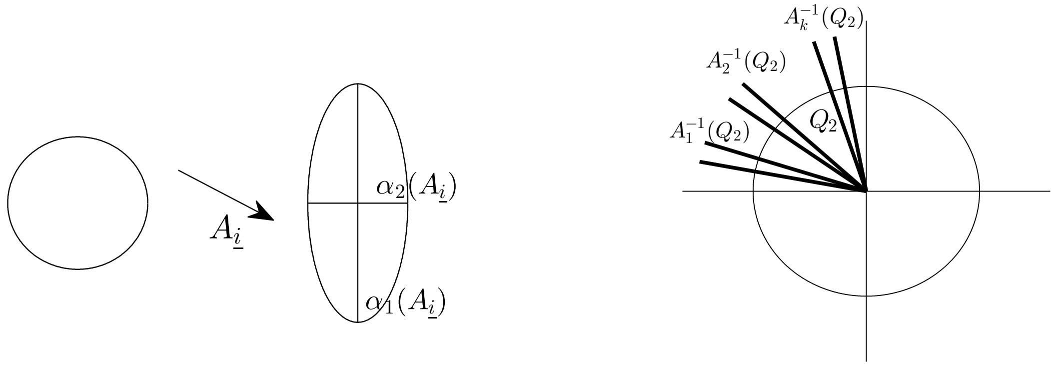

Figure 1: The singularities of the matrix $A_{\underline{i}}$ and a few images of the second quadrant.

However, it remains to address the question of effectively estimating the dimension. Our main result is the following

Theorem 1.9 (Main Theorem). *Under Hypotheses 1.7 there exists* $0 < \theta < 1$ *such that we can define a sequence* d_n *using the* 2^{n+1} *singularities* $\{\alpha_1(A_{\underline{i}}) : |\underline{i}| \leq n\}$ *so that*

$$|\dim_S(\Lambda) - d_n| = O\left(\theta^{n^2}\right) \text{ for } n \geq 1.$$

In particular, we see that the rate of convergence of the n'th approximation to the dimension is super exponential, whereas the number of terms needed to compute is exponential.

Example 1.10. *Heuter and Lalley proposed the matrices*

$$A_1 = \begin{pmatrix} \frac{1}{30} & \frac{1}{120} \\ \frac{1}{30} & \frac{1}{60} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{1}{30} & \frac{1}{40} \\ \frac{1}{30} & \frac{1}{30} \end{pmatrix}, A_3 = \begin{pmatrix} \frac{1}{40} & \frac{1}{30} \\ \frac{1}{60} & \frac{1}{30} \end{pmatrix}.$$

It is easy to show that for suitable translations Heuter–Lalley conditions hold. Moreover, we estimate that the dimension of the limit set is

$$\dim_S(\Lambda) = 0.375797704495199\ldots$$

using products of the length 5*, so we need to calculate only* $3^5 = 243$ *matrices overall.*

It can be difficult to find explicit examples of matrices satisfying the Heuter–Lalley conditions (Hypotheses 1.7). It is an interesting question to ask how likely it is that a family randomly chosen matrices $A_1, \ldots, A_k$ satisfy them. We will discuss this in the next section.

In §2 we consider how restrictive hypothesis 1.7 are. In particular, we consider the probability that a k-tuple of matrices chosen at random with respect to a natural measure has these properties. In §3 we relate to the matrices projective maps, which allows us to use dynamical techniques. In §4 we use this formulation to describe the singularity dimension in terms of thermodynamic formalism. In §5 we describe the main mechanism in the proof, the determinant of the transfer operator and in §6 we complete the proof of the main theorem 1.9. Finally, in §7 we present examples illustrating the rapid convergence in the main theorem.

We would like to thank Professor Karoly Simon for posing the question that motivated this work, that of whether it was possible to estimate the dimension of the limit set for a family of affine contractions satisfying Hypotheses 1.7 using these methods.

2 The likelihood of the hypotheses

In this section we address the following question: *What is the probability that k matrices chosen randomly satisfy conditions (1)–(3) of Hypotheses 1.7?* In fact, we will see, both empirically and rigorously, that these conditions can be difficult to satisfy, particularly when the number of matrices k is large. On the one hand, if the singular values of two or more of the matrices are sufficiently large then the image of the positive quadrant will be a large sector and part 3 of Hypotheses 1.7 may be impossible to satisfy. We will quantify this in this section. On the other hand, if the singular values of the matrices are all small, it is relatively easy to estimate the probability that a k-tuple of such matrices satisfy Hypotheses 1.7. In particular, in this case the images of the positive quadrant are very narrow sectors and we could consider the product of their independent distributions. It remains to understand the general case, which we can approach by estimating the number of k-tuples where the singular values have a common lower bound τ, say. We will present formulae for the density of the k-tuples which satisfy the hypotheses, and consider their asymptotic behaviour as the lower bound on the singular values tends to zero. In particular, we will show that there is a simple asymptotic formula (Proposition 2.7) which fits with the empirical results for $k = 1, 2$.

We begin by presenting a natural parametrization of matrices which is useful for both interpreting Hypotheses 1.7 (1)–(3) and quantifying the probability they are satisfied.

Using the Singular Value Decomposition for matrices we can write the inverse of each matrix as

$$A_i^{-1} = R_{\theta_1(A_i)} \begin{pmatrix} 1/\alpha_1(A_i) & 0 \\ 0 & 1/\alpha_2(A_i) \end{pmatrix} R_{\theta_2(A_i)}$$

where R_θ is rotation by an angle θ. Provided $\theta_2(A_i) \in (0, \frac{\pi}{2}) \cup (\pi, \frac{3\pi}{2}) =: I_{2,i}$ the image of $R_{\theta_2}(Q_2)$ contains the real axis. More precisely, the image cone $R_{\theta_2}(Q_2)$ is bounded by lines containing the vectors

$$\begin{pmatrix} -\sin\theta_2(A_i) \\ \cos\theta_2(A_i) \end{pmatrix} \text{ and } \begin{pmatrix} -\cos\theta_2(A_i) \\ -\sin\theta_2(A_i) \end{pmatrix}.$$

The action of the diagonal matrix then squeezes the cone $R_{\theta_2(A_i)}(Q_2)$ inside itself. The new cone is bounded by lines containing the vectors

$$\begin{pmatrix} -\sin\theta_2(A_i)/\alpha_1(A_i) \\ \cos\theta_2(A_i)/\alpha_2(A_i) \end{pmatrix} \text{ and } \begin{pmatrix} -\cos\theta_2(A_i)/\alpha_1(A_i) \\ \sin\theta_2(A_i)/\alpha_2(A_i) \end{pmatrix}.$$

In particular, these lines make angles with the horizontal axes equal to

$$\phi_1 := \tan^{-1}\left(\frac{\alpha_2(A_i)}{\alpha_1(A_i)}\tan\theta_2(A_i)\right) \text{ and } \phi_2 := \tan^{-1}\left(\frac{\alpha_2(A_i)}{\alpha_1(A_i)}\cot\theta_2(A_i)\right),$$

respectively, and the angle for the image cone is given by $\phi = \phi_1 + \phi_2$. Therefore we may write

$$\tan\phi = \frac{\alpha_2(A_i)}{\alpha_1(A_i)} \cdot \frac{\tan\theta_2(A_i) + \cot\theta_2(A_i)}{1 - \left(\frac{\alpha_2(A_i)}{\alpha_1(A_i)}\right)^2}. \tag{1}$$

Finally, the map $R_{\theta_1(A_i)}$ maps this cone back into the second quadrant Q_2 under the condition that $\theta_1 \in [\frac{\pi}{2}, \pi] \cup [\frac{3\pi}{2}, 2\pi] =: I_{1,i}$.

The following result is now very easy to establish and gives preliminary restrictions on the matrices to satisfy the Hypotheses 1.7.

Lemma 2.1. *Given k matrices such that preimages of Q_2 are disjoint, at least one of them satisfies*

$$\frac{\alpha_2(A_i)}{\alpha_1(A_i)} \leq \frac{\sqrt{1+\tan^2(\pi/2k)}-1}{\tan(\pi/2k)}.$$

Proof. This is an explicit computation. By (1) the image of Q_2 under any A_i^{-1} is a cone with angle ϕ that satisfies

$$\tan\phi \geq \frac{\alpha_2(A_i)}{\alpha_1(A_i)} \cdot \frac{2}{1-\left(\frac{\alpha_2(A_i)}{\alpha_1(A_i)}\right)^2}.$$

In particular, if there are k disjoint cones then we require that for at least one choice $1 \leq i \leq k$ we have that

$$\tan\left(\frac{\pi}{2k}\right) \geq \tan\phi \geq \frac{\alpha_2(A_i)}{\alpha_1(A_i)} \cdot \frac{2}{1-\left(\frac{\alpha_2(A_i)}{\alpha_1(A_i)}\right)^2},$$

which implies the result. □

Example 2.2. *For example, in order to have $k = 2$ matrices with disjoint preimages, we need one of them to satisfy $\frac{\alpha_2(A_i)}{\alpha_1(A_i)} \leq \sqrt{2}-1$.*

We now turn our attention to the likelihood that the hypotheses hold. We assume that parameters defining matrices A_i^{-1} are uniformly distributed on the corresponding intervals:

$$X_i\colon = \left\{(\alpha_1, \alpha_2, \theta_1, \theta_2) \subset (0,1)^2 \times I_{1,i} \times I_{2,i}\right\}. \tag{2}$$

The probability space is defined by $X\colon = \prod_{i=1}^{k} X_i$, where we assume a uniform distribution.

We observe that if $\alpha_1 \ll 1$, the contraction is strong and the image of Q_2 under A_i^{-1} is a very narrow cone. Thus, at least quantatively, the probability that k matrices satisfy Hypotheses 1.7 is high. We are therefore interested in the probability that k matrices chosen at random satisfy Hypotheses 1.7 when the singular values are assumed not to be too small. In particular, we want to add an additional condition on singularities $0 < \tau \leq \alpha_2 \leq \alpha_1 < 1$ and study the probability that Hypotheses 1.7 (1)–(3) hold true as a function of τ.

Definition 2.3. *We denote by $\nu_k(\tau)$ the emprically observed proportion of k-tuples of matrices satisfying the hypotheses, whereas we denote by $P_k(\tau)$ the theoretically predicted value.*

The graphs in Fig. 1 below show the proportion $\nu_k(\tau)$ of families of k matrices that satisfy Hypotheses 1.7 among all possible families of k matrices. They are obtained by a routine straightforward computer calculation. More precisely, we take 200 values of τ between 0 and 0.125, and for every τ we consider 50 values of angles $\theta_1 \in I_{1,i}$, $\theta_2 \in I_{2,i}$ and 30 values of singularities α_1, α_2 on the interval $(\tau, 1)$. Afterwards, we consider all possible matrices and calculate $\nu_1(\tau)$, the proportion of matrices that satisfy Hypotheses 1.7 (1)–(3). Then we look for pairs, triples, quartets and quintets.

We would like to explain the shape of these empirically observed plots by rigorously estimating the asymptotic behaviour of $\nu_k(\tau)$ as $\tau \to 0$ and to find the probability $P_k(\tau)$ that k matrices, chosen randomly with respect to uniform distribution satisfy

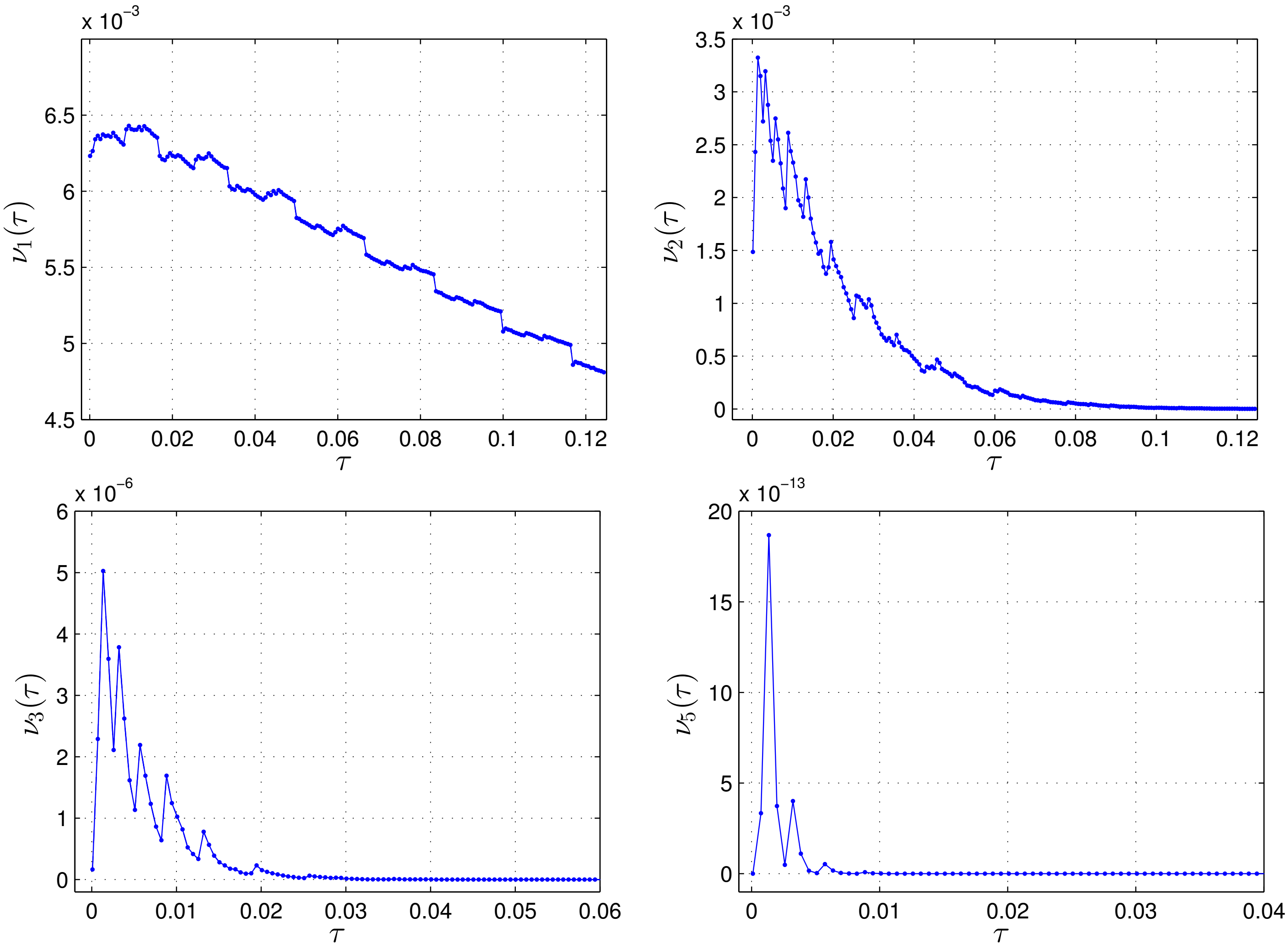

Figure 2: There are plots of the proportion of (a) Heuter–Lalley matrices A; (b) Heuter–Lalley pairs of matrices (A_1, A_2); (c) Heuter–Lalley triplets of matrices (A_1, A_2, A_3); and (d) Heuter–Lalley quintets of matrices $(A_1,\dots, A_5)$ with singularities $\alpha_1 > \alpha_2 > \tau$.

Hypotheses (1.7). We start with $P_1(\tau)$. Let $\phi(\theta_1, \theta_2, \alpha_1, \alpha_2)$ be the angle of the cone $A^{-1}(Q_2)$. Then

$$P_1(\tau) = \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - \phi\right) \rho_\phi(x) dx \tag{3}$$

where $\rho_\phi(x)$ is the probability density function for ϕ. The following lemma gives us the density of the distribution for $\tan(\phi)$.

Lemma 2.4. *The random variable* $\tan\phi$ *is distributed with density*

$$\rho_{\tan\phi}(x, \tau) = \int_{\frac{2\sqrt{\tau}}{1-\tau}}^{x/2} \frac{1}{\sqrt{2\pi}} \frac{y}{x\sqrt{x^2 - 4y^2}} \Big(\frac{\tau^2}{u^2(y)} + \frac{u^2(y)}{4}\Big) u'(y) dy, \tag{4}$$

where $u(x) = \frac{\sqrt{4x^2+1}-1}{2x}$.

Proof. We can write $\tan\phi$ as a product of independent variables using (1)

$$\tan\phi = \frac{2}{\sin(2\theta_2)} \cdot \frac{\alpha_1\alpha_2}{\alpha_1^2 - \alpha_2^2}.$$

To simplify the calculations, we introduce $\tilde{\alpha}\colon = \frac{\alpha_1\alpha_2}{\alpha_1^2-\alpha_2^2}$ and $\tilde{\theta}\colon = \frac{2}{\sin(2\theta_2)}$. We can now use standard formulae for the density of the product distribution to approach the density of $\tan\phi$. We obtain the density $\rho_{\tilde{\theta}}(x)$ by straightforward calculation.

$$\rho_{\tilde{\theta}}(x) = \begin{cases} \frac{1}{2\pi}\frac{1}{x\sqrt{x^2-4}}, & \text{if } x > 2; \\ 0, & \text{otherwise.} \end{cases}$$

To calculate the density of $\tilde{\alpha}$ we introduce new a function $u(x) = \frac{\sqrt{1+4x^2}-1}{2x}$. Then the probability $\mathbb{P}(\tilde{\alpha} < x)$ is given by the area in the (α_1, α_2)-plane bounded by the lines $\alpha_2 = \tau$, $\alpha_2 = u(x)\alpha_1$, and parabola $\alpha_2 = \alpha_1^2$. Hence by definition

$$\rho_{\widetilde{\alpha}}(x) = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(\mathbb{P}(\widetilde{\alpha} \le x) - \mathbb{P}(\widetilde{\alpha} \le x+\epsilon)\right) = \begin{cases} \left(\frac{\tau^2}{u^2(x)} + \frac{u^2(x)}{4}\right)u'(x) & \text{if } x > \frac{\sqrt{\tau}}{1-\tau}, \\ 0 & \text{otherwise .} \end{cases}$$

Convolving $\rho_{\tilde{\theta}}$ and $\rho_{\tilde{\alpha}}$ together, we conclude

$$\rho_{\tan(\phi)}(x,\tau) = \int_{\mathbb{R}} \frac{1}{y}\cdot\rho_{\tilde{\theta}}\Big(\frac{x}{y}\Big)\rho_{\tilde{\alpha}}(y)dy = \int_{\frac{\sqrt{\tau}}{1-\tau}}^{x/2} \frac{1}{\sqrt{2\pi}}\frac{y}{x\sqrt{x^2-4y^2}}\Big(\frac{\tau^2}{u^2(y)} + \frac{u^2(y)}{4}\Big)u'(y)dy,$$

provided $x > \frac{2\sqrt{\tau}}{1-\tau}$. □

As a corollary to Lemma 2.4 we have the following asymptotic expansion for $\rho_{\tan\phi}(x,\tau)$ in powers of $\tau^{1/2}$.

Corollary 2.5. *There is a power series expansion*

$$\rho_{\tan\phi}(x,\tau) = \begin{cases} \sum_{j=0}^{\infty} a_j(x)\tau^{j/2} & \textit{if } x > \frac{2\sqrt{\tau}}{1-\tau}, \\ 0 & \textit{otherwise;} \end{cases}$$

which converges uniformly for any $x > \frac{2\sqrt{\tau}}{1-\tau}$, *and all* $a_j(x)$ *are analytic on the same domain.*

Proof. It suffices to observe that the integrand, and thus its anti-derivative, is analytic at 0 as a function of x. □

Combining 2.1 with 2.4 we get an upper bound for $\tau_k\colon = \inf\{\tau : P_k(x,\tau) = 0\}$:

Corollary 2.6.

$$\tau_k \le \frac{2 + \tan^2(\pi/2k) - 2\sqrt{1+\tan^2(\pi/2k)}}{\tan^2(\pi/2k)}$$

Heuristically, as k increases we see that at least one of the matrices in the k-tuple must correspond to a small value of α. In particular, this imposes conditions which suggest this is a relatively rare event.

We now use corollary 2.5 to show that $P(\tau)$ has an asymptotic formula in terms of $\tau^{1/2}$.

Lemma 2.7. *The probability $P(\tau)$ that $A(Q_2) \subset Q_2$, $\|A\| \leq 1$, and $\alpha_1(A)^2 < \alpha_2(A)$ subject to $\alpha_1(A), \alpha_2(A) \geq \tau > 0$ can be expanded as a power series in $\sqrt{\tau}$ (where the coefficients for $\sqrt{\tau}$ and τ vanish).*

Proof. The probability that the image $A(Q_2)$ lies back in the quadrant Q_2 is

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right) \rho_\phi(x,\tau)dx &= \int_0^\infty \left(\frac{\pi}{2} - \tan^{-1}(x)\right) \rho_{\tan\phi}(x,\tau)dx \\ &= \int_{\frac{2\sqrt{\tau}}{1-\tau}}^\infty \left(\frac{\pi}{2} - \tan^{-1}(x)\right) \rho_{\tan\phi}(x,\tau)dx.\end{aligned}$$

We observe that for small $\tau > 0$ the integrand is analytic at $x = 2$, with radius of convergence 2. In particular, it is analytic at the value of the lower value in the range of integration. Therefore we can expand it in a power series in x at $x = 2$, and integrate term by term. This gives the result. □

Similarly, the probability $P_2(\tau)$ that the images of two sectors are in the quadrant is given by

$$P_2(\tau) = \frac{2}{\pi}\int_0^{\pi/2} \left(\int_0^y (y - x_1)\,\rho_\phi(x_1,\tau)dx_1\right)\left(\int_0^{\frac{\pi}{2}-y} \left(\frac{\pi}{2} - y - x_2\right)\rho_\phi(x_2,\tau)dx_2\right) dy$$

can be expanded as a power series in $\sqrt{\tau}$. (We make sure that two sectors are disjoint by putting one of them between the real axis and a ray $(0, y)$ and another one between $(0, y)$ and the imaginary axis). This brings us to an asymptotic expansion for $P_k(\tau)$.

Continuing inductively one can show that the probability $P_k(\tau)$ that k images of the quadrant are disjoint takes the following form:

Proposition 2.8. *The probability $P_k(\tau)$ to have k matrices satisfying Hypotheses 1.7 (1)–(3) and with singularities at least τ, may be expanded as a power series in $\sqrt{\tau}$.*

In view of Proposition 2.8 we can attempt to fit polynomials in $\sqrt{\tau}$ to the plots in Figure 2. In Figure 2 we illustrate this for the representative cases of single matrix and triples of matrices.

In summary, the proportion of k-tuples of matrices satisfying Hypotheses 1.7 is sensitive to the restrictions on the singular values (2.6). We can explicitly compute $P_k(\tau)$ and compare these with the numerical estimates $\nu_k(\tau)$. In the case that we allow a lower bound on the singular values to tend to zero, we get asymptotic estimates $P_k(\tau)$ (in Proposition 2.8).

3 Projective maps

We can introduce a dynamical viewpoint by looking at the projective action of the matrices on $\mathbb{R}P^1$. This technique is classical in the study of the ergodic theory of random matrix products, but will prove to be particularly useful in the present context. The idea is that the linear action of the matrices on $\mathbb{R}^2$ induces a map on the real projective space $\mathbb{R}P^1$. We recall that $\mathbb{R}P^1$ corresponds to $\mathbb{R}^2 - \{(0,0)\}/\sim$ where we use define the equivalence relation $v \sim w$ if there exists $\lambda \in \mathbb{R} - \{0\}$ such that $\lambda v = w$. Equivalently, we can identify $\mathbb{R}P^1$ with the unit circle in $\mathbb{R}^2$ with the antipodal points identified. It is well known that one can naturally parameterise $\mathbb{R}P^1$ locally by using the arc distance on the unit circle. In particular, these matrices correspond to an iterated function scheme of projective maps.

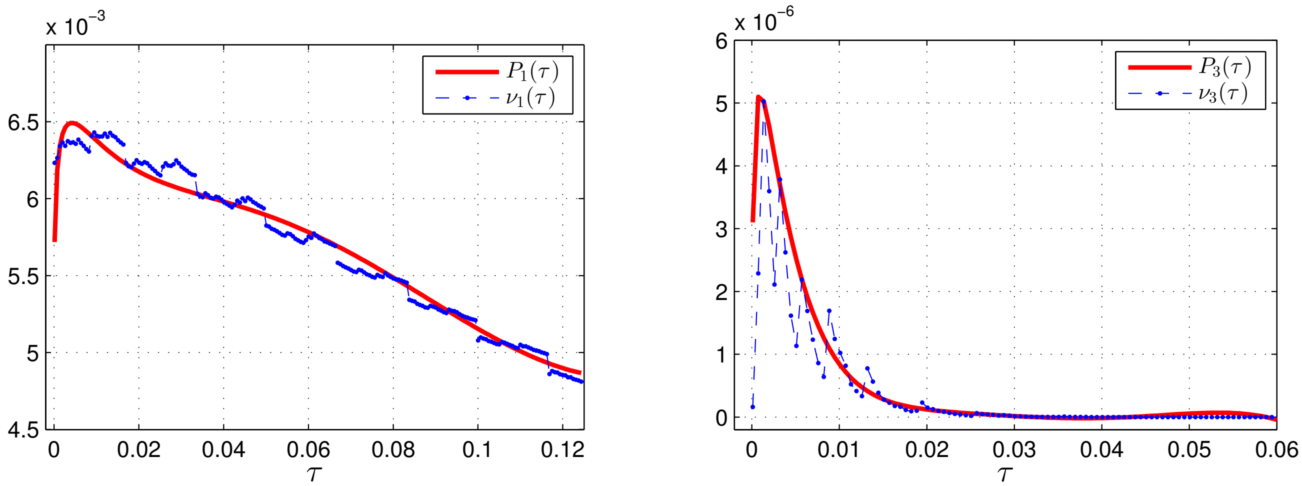

Figure 3: The first and the third plots from Figure 1 with approximating polynomial curves of degree 6 in $\sqrt{\tau}$ superimposed.

Let us write each matrix $A_{\underline{i}}$ in the form

$$A_{\underline{i}} = \begin{pmatrix} a_{\underline{i}} & b_{\underline{i}} \\ c_{\underline{i}} & d_{\underline{i}} \end{pmatrix},$$

where $a_{\underline{i}}, b_{\underline{i}}, c_{\underline{i}}, d_{\underline{i}} \in \mathbb{R}$, then we can associate the linear maps $\check{A}_{\underline{i}} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\check{A}_{\underline{i}}(x, y) = (a_{\underline{i}}x + b_{\underline{i}}y, c_{\underline{i}}x + d_{\underline{i}}y).$$

The assumption of positivity of the matrices ensures that the first quadrant $\mathcal{Q}_1 = \{(x, y) \in \mathbb{R}^2 : x, y \geq 0\}$ is preserved by the linear maps. Consider the one dimensional simplex $\Delta = \{(x, 1-x) : 0 \leq x \leq 1\}$ then the linear maps naturally give rise to projective maps $\widehat{A}_{\underline{i}} : \Delta \to \Delta$ given by

$$\widehat{A}_{\underline{i}}(x, 1-x) = \left(\frac{a_{\underline{i}}x + b_{\underline{i}}(1-x)}{a_{\underline{i}}x + b_{\underline{i}}(1-x) + c_{\underline{i}}x + d_{\underline{i}}(1-x)}, \frac{c_{\underline{i}}x + d_{\underline{i}}(1-x)}{a_{\underline{i}}x + b_{\underline{i}}(1-x) + c_{\underline{i}}x + d_{\underline{i}}(1-x)} \right).$$

In particular, the first component is a linear fractional map $\overline{A}_{\underline{i}} : [0, 1] \to [0, 1]$ given by

$$\overline{A}_{\underline{i}}(x) = \frac{(a_{\underline{i}} - b_{\underline{i}})x + b_{\underline{i}}}{(a_{\underline{i}} + c_{\underline{i}} - b_{\underline{i}} - d_{\underline{i}})x + (b_{\underline{i}} + d_{\underline{i}})}.$$

Each of these maps is merely the same action on (part of) $\mathbb{R}P^1$ viewed using different coordinates. The strict positivity of the matrices ensures that the images $\overline{A_{\underline{i}}}([0, 1])$ lie inside the open interval $(0, 1)$. Although the fact will not be necessary in our analysis, this is sufficient to ensure that these maps are contracting with respect to a suitable metric. Given the product of 2×2 matrices $A_{\underline{i}}$ we can denote its eigenvalues by $\lambda_1(A_{\underline{i}}) \geq \lambda_2(A_{\underline{i}})$ and then we can write its determinant as their product $\det(A_{\underline{i}}) = \lambda_1(A_{\underline{i}})\lambda_2(A_{\underline{i}})$. The rest of this section is devoted to relating the fixed point of the projective map $\overline{A}_{\underline{i}}$ to the eigenvalues of the matrix $A_{\underline{i}}$.

We collect together in the following lemmas some simple estimates which will prove useful in the next section.

Lemma 3.1. *If* $v_{\underline{i}} = (x_{\underline{i}}, 1 - x_{\underline{i}})$ *is an eigenvector for* $A_{\underline{i}}$ *then* $x_{\underline{i}}$ *is a fixed point for* $\overline{A}_{\underline{i}}$. *Moreover, we can write the derivative by*

$$D\overline{A}_{\underline{i}}(x_{\underline{i}}) = \frac{\det(A_{\underline{i}})}{\lambda_1(A_{\underline{i}})^2}.$$

Proof. This could be deduced indirectly from equation (41) of [3]. However, it can also be seen easily directly by a simple geometric argument. Consider a small ϵ-ball $B(v_{\underline{i}}, \epsilon)$ around $v_{\underline{i}}$ in $\mathbb{R}^2$. The ratio of the areas of the original ball $B(v_{\underline{i}}, \epsilon)$ to its image $A_{\underline{i}}(B(v_{\underline{i}}, \epsilon))$ under the linear map $A_{\underline{i}} : \mathbb{R}^2 \to \mathbb{R}^2$ is $\det A_{\underline{i}}$. On the other hand, we see that $A_{\underline{i}}$ maps $v_{\underline{i}}$ to $A_{\underline{i}}(v_{\underline{i}}) = \lambda_1(A_{\underline{i}})v_{\underline{i}}$. Thus, since the point $v_{\underline{i}}$ is fixed by $\overline{A}_{\underline{i}}$ a consideration of the area of $A_{\underline{i}}(B(v_{\underline{i}}, \epsilon))$ shows the contraction in the projective distance in a neighbourhood of this fixed point must be approximately

$$\frac{\det(A_{\underline{i}})}{\lambda_1(A_{\underline{i}})^2}.$$

Letting ϵ tend to zero gives the result. □

The next lemma shows that the largest singular value and the largest eigenvalue are of a comparable size.

Lemma 3.2. *We can estimate:*

$$\alpha_1(A_{\underline{i}}) \asymp \lambda_1(A_{\underline{i}})$$

i.e., there exists $C \geq 1$ *such that* $\frac{1}{C}\lambda_1(A_{\underline{i}}) \leq \alpha_1(A_{\underline{i}}) \leq C\lambda_1(A_{\underline{i}})$, *for all strings* $\underline{i}$.

Proof. This is suggested by comparing equations (23) and (29) in [3]. However, it can be seen directly by considering cones $\pm Q_1$ associated to the first and third quadrant. It is immediate to see that the action of the positive matrices is such that the direction of the longest axes of the ellipse image of the unit disk must lie in the image come $A_{\underline{i}}(Q_1)$. Moreover, this also contains the eigenvector associated to the largest eigenvalue. One then sees easily the result by considering the contracting maps associated to the projective versions.

□

4 Singularity dimension and transfer operators

The use of positive matrices and the contracting nature of their projective action has been used by several authors in different contexts, including [9], [8]. This has the advantage that it places us in the context of an expanding real analytic map and thus allows us to employ in §5 the powerful theory of nuclear transfer operators associated to this setting.

In the present context, the hypotheses imply that $0 < \dim_S(\Lambda) < 1$. Thus we have by definition that $\phi^s(A_{\underline{i}}) = \alpha_1(A_{\underline{i}})^s$ and

$$\dim_S(\Lambda) = \inf\left\{ s > 0 : \sum_{n=1}^{\infty} \sum_{|\underline{i}|=n} \alpha_1(A_{\underline{i}})^s < +\infty \right\}.$$

In order to understand the convergence and divergence of the above series we can define a pressure-type function in a natural way as follows.

Definition 4.1. *Let $P : \mathbb{R} \to \mathbb{R}$ be defined by*

$$P(s) := \lim_{n\to+\infty} \frac{1}{n} \log \left(\sum_{|\underline{i}|=n} \alpha_1(A_{\underline{i}})^s \right).$$

The function $P(s)$ can be viewed as a generalisation of the more familiar pressure function in thermodynamical formalism, which has proved so useful in analysing the Hausdorff dimension of conformal attractors. The map $P : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and its significance is that the singularity dimension $\dim_S(\Lambda)$ is then given by the following result:

Proposition 4.2 ([3], corollary 4.4)**.** *The value $\delta = \dim_S(\Lambda)$ satisfies $P(\delta) = 0$.*

By the estimates in the previous section we can see that this is equivalent to the following.

Lemma 4.3. *We can write*

$$P(t) := \lim_{n\to+\infty} \frac{1}{n} \log \left(\sum_{|\underline{i}|=n} \Psi_n(\underline{i})^t \right)$$

where for convenience we denote $\Psi_n(\underline{i}) := \left(\frac{\det(A_{\underline{i}})}{D\overline{A}_{\underline{i}}(x_{\underline{i}})}\right)^{\frac{1}{2}}$.

Proof. By Lemma 3.1 we have that $\lambda_1(A_{\underline{i}}) = \left(\frac{\det(A_{\underline{i}})}{D\overline{A}_{\underline{i}}(x_{\underline{i}})}\right)^{1/2}$ and by Lemma 3.2 (1) we have that $\alpha_1(A_{\underline{i}}) \asymp \lambda_1(A_{\underline{i}})$. The result then follows by taking limits. □

We want to relate the pressure to a transfer operator on the Banach space of bounded analytic functions. As with the classical theory of Thermodynamical Formalism, the value $P(s)$ will be characterised in terms of the maximal positive eigenvalue of a corresponding linear operator. We first introduce a suitable Banach space upon which the operator acts.

Definition 4.4. *Let $[0,1] \subset U \subset \mathbb{C}$ be an open neighbourhood of the unit interval. Let $B \subset C^\omega(U)$ be the Banach space of bounded analytic functions on U, with respect to the supremum norm $\|w\| = \sup_{z\in U} |w(z)|$.*

We could equally well work with the Banach space of square integral analytic functions on U, but we use this particular space to be consistent with [10].

We can next associate the bounded operator which we will use to characterise $P(t)$.

Definition 4.5. *Given $s \in \mathbb{C}$, we can consider the transfer operator $\mathcal{L}_s : B \to B$ defined by*

$$\mathcal{L}_s w(z) = \sum_{i=1}^{k} \psi_i(z)^s w(\overline{A}_i z)$$

where $\psi_i(z) = \left(\frac{\det A_i}{D\overline{A}_i(\overline{A}_i z)}\right)^{1/2}$.

The above operator is well defined provided the open set U is chosen sufficiently small that the closures of its images are contained in U (i.e., $\overline{A_i(U)} \subset U$ for $i = 1, \cdots, k$) and also sufficiently small that $\psi_0, \psi_1 : U \to \mathbb{C}$ and analytic (i.e., the square root is applied away from the negative real axis).

To understand the relationship of the transfer operator to the singularity dimension we require the following standard result.

Lemma 4.6. *If $t \in \mathbb{R}$ then $e^{P(t)}$ is the spectral radius of $\mathcal{L}_t$.*

Proof. Observe that from the definitions, we have that for any finite string $\underline{i} = (i_0, i_1, \cdots, i_{n-1})$ we have that $\Psi_n(\underline{i}) = \prod_{j=0}^{n-1} \psi_{i_j}(A_{i_0} \cdots A_{i_{n-1}} x_{\underline{i}})$. The result then follows from [10]. □

5 Traces and Determinants

In this section we will recall some classical results on operators. In many other applications of transfer operators, it is sufficient to consider the transfer operator acting on the Banach space of Hölder continuous functions. In that context, the operators are quasi-compact. However, for our purposes it is important that we are considering a small Banach space of analytic functions for which the transfer operators have smaller spectrum.

We begin with a general definition due to Grothendieck [2] and apply these to the particular case of the transfer operator $\mathcal{L}_t$.

Definition 5.1. *We say that an operator $T : B \to B$ on a Banach space is* nuclear *if:*

1. *There exist vectors $v_n \in B$ with $\|v_n\| = 1$;*
2. *There exist linear functionals $l_n \in B^*$ with $\|l_n^*\| = 1$;*
3. *There exists an absolutely summable sequence $\lambda_n \in \mathbb{C}$,*

such that we can write $T(v) = \sum_{n=1}^{\infty} \lambda_n v_n l_n(v)$, for all $v \in B$.

Such nuclear operators are automatically compact operates and thus consequently have only countably many non-zero eigenvalues, whose only possible accumulation value is 0. In particular, nuclear operators are trace class and we can define their traces in terms of the sum of their eigenvalues. For our present purposes we can also assume that $(\lambda_n)_{n=1}^{\infty} \in l^p$ for all $p > 0$.

We recall some properties of these operators that can be easily deduced from more general results of Ruelle [10] (and we refer the reader to Appendices A, B and C of [6] for a useful summary). These are contained in the next two propositions [2].

Proposition 5.2 (Grothendieck, Ruelle)**.** *For each $t > 0$ the operators $\mathcal{L}_t : B \to B$ are nuclear and we can explicitly write the trace of the n'th power $\mathcal{L}_t^n$ as:*

$$\mathrm{trace}(\mathcal{L}_t^n) = \sum_{|\underline{i}|=n} \frac{(\Psi_n(\underline{i}))^t}{1 - D\overline{A}_{\underline{i}}(x_{\underline{i}})}$$

for each $n \geq 1$.

The proof of this explicit form for the trace $\mathrm{trace}(\mathcal{L}_t^n)$ is quite explicit. It involves calculating the eigenvalues, and thus traces, of each of the composition operators associated to the individual contractions $\overline{A}_{\underline{i}}$ and then summing.

We next introduce a family of complex functions $D(z, t)$ of the complex variable $z \in \mathbb{C}$, parameterized by a real variable $t \in \mathbb{R}$.

Definition 5.3. *We can define the* determinant

$$D(z, t) := \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{trace}(\mathcal{L}_t^n) \right),$$

which converges for t sufficiently large and $|z|$ sufficiently small.

Finally, we recall the following useful result on the analytic domain and expansion of $D(z,t)$.

Proposition 5.4 (Grothendieck, Ruelle)**.** *The function $D(z,t)$ is entire in $\mathbb{C}^2$. Moreover, there exists $0<\theta<1$ such that*

$$D(z,t) = 1 + \sum_{k=1}^{\infty} a_k(t) z^k$$

where $|a_k(t)| = O(\theta^{k^2})$.

In order to exploit the nuclearity of the transfer operators it is crucial that we work with the Banach space of analytic functions rather than, say, the more familiar setting of Hölder continuous or continuously differentiable functions. In particular, it is a crucial (but trivial) observation that an analytic function in $\mathcal{B}$ remains analytic under composition with the linear maps $\overline{A_i}$. This is a simple observation based on such maps being linear fractional maps, which arises automatically from their construction.

Remark 5.5. We recall that the constant $0<\theta<1$ is related to the minimal contraction of the $\overline{A}_i$, for $i=1,\cdots,n$. In particular, it can be readily estimated.

6 Proof of Theorem 1.9

The proof of Theorem 1.9 depends on the results in the previous section. The use of determinants to compute Hausdorff dimension appeared in [4] in the context of general hyperbolic repellers (e.g. hyperbolic Julia sets and limit sets of suitable Fuchasian groups). The general principle is the same here, although the application is somewhat different. Let us set $z=1$ and then denoting $\eta(t) := D(1,t)$ and using Proposition 5.4 we can expand

$$\eta(t) := D(1,t) = \sum_{k=1}^{\infty} a_k(t)$$

where $|a_k(t)| = O(\theta^{k^2})$. The significance of this function is the following simple result.

Lemma 6.1. *The value $\delta = \dim_S(\Lambda)$ is the abscissa of convergence of $\eta(s)$ (i.e., the least value for which $\eta(s)$ converges to an analytic function for $Re(s) > \delta$)*

Proof. By definition, the convergence (or divergence) of the function $\eta(t)$, for $t \in \mathbb{R}$, depends on the growth of the terms trace$(\mathcal{L}_t^n)$, whose explicit form is given in Proposition 5.4. However, we can then deduce from Definition 4.1 that for $t>0$ the series converges (since the terms tend to zero exponentially fast) and for $t<0$ the series diverges (since the terms grow exponentially fast). Finally, the lemma follows by Proposition 4.2. □

Next, we can write $\eta_N(t)$ for the truncation of this series of the form:

$$\eta_N(t) := \sum_{k=1}^{N} a_k(t)$$

for $N \geq 1$. Let $\delta_N > 0$ denote the smallest zero, i.e., $\eta_N(\delta_N) = 0$. Thus since for each t we have $|\eta_N(t) - \eta(t)| = O(\theta^{N^2})$ and δ is a simple zero for $\eta(t)$ we deduce that $\delta - \delta_N = O(\theta^{N^2})$.

Remark 6.2. The value of $0 < \theta < 1$ in the bound $|\eta_N(t) - \eta(t)| = O(\theta^{N^2})$ depends on the hyperbolicity of the projective maps associated to the matrices. It is not simply a bound on the derivatives, since it also reflects the complexification of the maps, but it can be assumed close to this value. The implied constant can also be effectively estimated.

7 The Numerical Algorithm

It remains to show empirically that this method gives an efficient way to estimate δ. In this section we present a basic numerical algorithm resulting from theorem 1.9 and illustrate its efficiency using two examples 7.2 and 7.3.

Consider matrices

$$A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \text{ and } i = 1, \dots, k$$

satisfying the hypotheses (1) – (3) of Hypotheses 1.7.

Step 1. For each $n \geq 1$ we can consider a string $\underline{i} = (i_0, \cdots, i_{n-1}) \in \{1, \cdots, k\}^n$. We associate the product matrix

$$A_{\underline{i}} = A_{i_0} A_{i_1} \cdots A_{i_{n-1}}(x_{\underline{i}}) = \begin{pmatrix} a_{\underline{i}} & b_{\underline{i}} \\ c_{\underline{i}} & d_{\underline{i}} \end{pmatrix},$$

say, and the linear fractional maps $\overline{A}_{\underline{i}} : [0,1] \to [0,1]$ given by

$$\overline{A}_{\underline{i}}(x) = \frac{(a_{\underline{i}} - b_{\underline{i}})x + b_{\underline{i}}}{(a_{\underline{i}} + c_{\underline{i}} - b_{\underline{i}} - d_{\underline{i}})x + (b_{\underline{i}} + d_{\underline{i}})}.$$

Step 2. We can then associate to each string $\underline{i} = (i_0, \cdots, i_{n-1}) \in \{1, \cdots, k\}^n$:

1. the determinant $\det A_{\underline{i}}$;
2. the unique fixed point $\overline{A_{\underline{i}}}(x_{\underline{i}}) = x_{\underline{i}}$;
3. the derivative $D\overline{A_{\underline{i}}}(x_{\underline{i}})$ of the map at the fixed point;
4. the weight

$$\Phi_n(\underline{i}, t) = \left(\frac{\det(A_{\underline{i}})}{D\overline{A_{\underline{i}}}(x_{\underline{i}})} \right)^{t/2} \frac{1}{1 - D\overline{A_{\underline{i}}}(x_{\underline{i}})}.$$

Step 3. We can write

$$D_N(z,t) := \exp\left(-\sum_{n=1}^{N} \frac{z^n}{n} \sum_{|\underline{i}|=n} \Phi_n(\underline{i}, t) \right)$$

and expanding the exponential as $\exp(y) = 1 + y + y^2/2 + \cdots + y^N/N! + O(y^{N+1})$ with

$$y = -\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{|\underline{i}|=n} \Phi_n(\underline{i}, t)$$

we rewrite this as

$$D_N(z,t) = 1 + \sum_{k=1}^{N} a_k(t) z^k + O(z^{N+1}).$$

Step 4. Setting $z = 1$ we can define

$$\eta_N(t) := 1 + \sum_{k=1}^{N} a_k(t).$$

Let $\delta_N > 0$ be the smallest positive solution to $\eta_N(\delta_N) = 0$.

Remark 7.1 (Comparing with the matrix approach)*.* A more standard approach is to associate to each N a matrix whose entries are approximations to the derivatives raised to the power t_N. In particular, for each $N \geq 1$ we can solve for $t_N > 0$ such that

$$\sum_{|\underline{i}|=n} D\overline{A}_{\underline{i}}(x_{\underline{i}})^{t_N} = 1,$$

say. It then follows, as in [7] that $t_N \to \dim_H(\Lambda)$ at an exponential rate, i.e., there exists $0 < \theta < 1$ such that $\dim_H(\Lambda) = t_N + O(\theta^N)$.

Example 7.2. *With the matrices considered in Example 1.10, one can consider the approximations to the dimension using determinants (Theorem 1.9) and compare it with the matrix approximation method (Remark 7.1).*

N	δ_N	t_N
1	0.410717582765210	0.373123313880933
2	0.375211732460593	0.375566771742160
3	0.375799107164494	0.375775898884967
4	0.375797703892749	0.375795619644123
5	0.375797704495199	0.375797504758157
6	0.375797704495199	0.375797685359066
7	0.375797704495199	0.375797702683667
8	0.375797704495199	0.375797704340403
9	0.375797704495199	0.375797704507750
10	0.375797704495199	0.375797704514025

Table 1: Approximations for Example 7.2

In particular, we see that for $N = 5$ the determinant method gives a solution

$$\delta = 0.375797704495199\cdots$$

which is accurate to 15 *decimal places. However, even when $N = 10$ the matrix method is only accurate to* 9 *decimal places.*

Example 7.3. *With the matrices*

$$A_1 = \frac{1}{2^6}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}, A_2 = \frac{1}{2^6}\begin{pmatrix} 5 & 3 \\ 5 & 6 \end{pmatrix} \text{ and } A_3 = \frac{1}{2^6}\begin{pmatrix} 4 & 5 \\ 2 & 9 \end{pmatrix}$$

N	δ_N	t_N
1	0.609325221387553	0.514374159566069
2	0.502335263611167	0.508602279690240
3	0.507406976235507	0.507597431583781
4	0.507371544351918	0.507413527612153
5	0.507371616545424	0.507379412950468
6	0.507371616478486	0.507373067887602
7	0.507371616478486	0.507371886819237
8	0.507371616478486	0.507371666879226
9	0.507371616478486	0.507371625895939
10	0.507371616478486	0.507371618256548

Table 2: Approximations for Example 7.3

we can consider the approximations to the dimension using determinants, and compare it with the matrix approximation method.

In particular, we see that for $N = 6$ the determinant method gives a solution

$$\delta = 0.507371616478486\cdots$$

which is accurate to 15 decimal places. However, even when $N = 10$ the matrix method is only accurate to 8 decimal places.

To construct examples satisfying Hypotheses 1.7, part (1) is easy to check. For part (3), we can first consider inverse matrices

$$A_i^{-1} = \begin{pmatrix} c_i & -a_i \\ -d_i & b_i \end{pmatrix}$$

with $a_i, b_i, c_i, d_i > 0$, for $i = 1, \cdots, k$, since then we have that $A_i(Q_2) \subset Q_2$. If we also assume that $\det(A_i^{-1}) > 0$, then in order to have these images disjoint it suffices to arrange that $\frac{a_{i+1}}{b_{i+1}} > \frac{c_i}{d_i}$. Part (2) can be confirmed by explicit computation. Part (4) can be satisfied for any choice of matrices.

Remark 7.4. A nonlinear extension of the work of Hueter and Lalley was stated by Luzia [5]. Let $f_1, \cdots, f_k : \mathbb{R}^2 \to \mathbb{R}^2$ be C^2 diffeomorphisms such that:

1. $\sup_{x\in\mathbb{R}^2} \|D_x f_i\| < 1$ for $i = 1, \cdots, k$;
2. there is a convex bounded open set U such that $f_1(\overline{U}), \ldots, f_k(\overline{U})$ are pairwise disjoint subsets of U;
3. $D_x f_i(P) \subset \mathrm{int}(P)$ for every $x \in U$, where P is the union of the closed first and third quadrants; and
4. $\|D_x f_i v\|^3/|\det(D_x f_i)| < 1$ for every $x \in U$ and $v \in P$ with $\|v\| = 1$;

We can consider the function $\Phi : \{1, \ldots, k\}^{\mathbb{N}} \to \mathbb{R}$ defined by $\Phi(\underline{i}) = \log \|D_{\pi(\underline{i})} f_{i_1}|V\|$ where $\underline{i} = (i_1, i_2, i_3, \cdots)$ and

$$\pi(\underline{i}) = \lim_{n\to+\infty} (f_{i_2} \circ \cdots \circ f_{i_n})(\overline{U})$$

and $V = V(\underline{i})$ is a line given by

$$V = \lim_{n\to+\infty} D_{f_{i_1}\circ\cdots\circ f_{i_n}\pi(\underline{i})}(f_{i_1} \circ \cdots \circ f_{i_n})P$$

Then we have $\dim_S J = \dim_H J = s$, where s is the unique root of the equation $P(s\Phi) = 0$, where P is the pressure function. A variant of the method of this note should also apply in this case when the maps are real analytic.

References

[1] K. Falconer, The Hausdorff dimension of self-affine fractals, *Math. Camb. Phil. Soc.*, **103** (1988), 339–350

[2] A. Grothendieck, Produits tensoriels topologiques et espaces nucleaires, *Mem. Amer. Math. Soc.*, 16 (1955), 1–140.

[3] I. Heuter and S. Lalley, Falconer's formula for the Hausdorff dimension of a self-affine set in $\mathbb{R}^2$, *Ergod. Th. and Dynam. Sys.*, 15 (1995), 77–97.

[4] O. Jenkinson and M. Pollicott, Calculating Hausdorff dimensions of Julia sets and Kleinian limit sets, *Amer. J. Math.*, 124 (2002), 495–545.

[5] N. Luzia, Hausdorff dimension for an open class of repellers in $\mathbb{R}^2$, *Nonlinearity*, 19 (2006), 2895–2908.

[6] D. Mayer, *The Ruelle-Araki Transfer Operator in Classical Statistical Mechanics*, Lecture Notes in Physics, 123, Springer, Berlin, 1980.

[7] C. McMullen, Hausdorff dimension and conformal dynamics. III. Computation of dimension, *Amer. J. Math.*, 120 (1998), 691–721.

[8] Yu. Peres, Domains of analytic continuation for the top Lyapunov exponent Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques (1992) Vol. 28(1), 131–148.

[9] M. Pollicott, Maximal Lyapunov exponents for random matrix products, Invent. Math., Volume 181 (2010), 209–226.

[10] D. Ruelle, Zeta-functions for expanding maps and Anosov flows, *Invent. Math.*, 34 (1976), 231–242.

[11] B. Solomyak, Measure and dimension for some fractal families, Math. Proc. Camb. Phil. Soc., 124 (1998), no. 3, 531–546.