Endogenous Entry To Security-Bid Auctions

By Takeharu Sogo and Dan Bernhardt and Tingjun Liu*

We endogenize entry to a security-bid auction, where participation is costly, and bidders must decide given their private valuations whether to participate. We first consider any minimum reserve security-bid of a fixed expected value that weakly exceeds the asset's value when retained by the seller. Demarzo, Kremer and Skrzypacz (2005) establish that with a fixed number of bidders, auctions with steeper securities yield the seller more revenues. Counterintuitively, we find that auctions with steeper securities also attract more entry, further enhancing the revenues from such auctions. We then establish that with optimal reserve securities, auctions with steeper securities always yield higher expected revenues.
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DeMarzo, Kremer and Skrzypacz (2005) (hereafter, DKS) characterize expected seller revenues for security-bid auctions—auctions whose payouts depend on both the security bid paid by the auction winner, and the ultimate (stochastic) payoff of the asset won by the bidder. DKS consider a setting where ex-ante symmetric bidders receive i.i.d. signals about their private value of the asset if they win the auction. DKS show that auctions using steeper securities—those whose payments to the seller are tied more tightly to the winning bidder's private valuation—yield the seller greater expected revenues.

We extend their analysis to a setting where it is costly to participate in the security-bid auction and potential bidders know their private valuations when deciding whether to enter. A natural conjecture is that because auctions that use steeper securities for payments yield the seller greater expected revenues, they must also attract fewer bidders—as more revenues for the seller would seemingly imply less for bidders. Indeed, the analysis in Gorbenko and Malenko (2011) reveals that this is what happens when bidders make entry decisions before learning their valuations. Our paper shows that the *opposite* is true when bidders *know*

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their valuations when deciding whether to enter: not only do steeper securities extract more revenues from any given set of bidders, but they also attract more bidders, *and* this increased entry enhances revenue extraction from bidders.

We consider a seller seeking to sell an asset in an open outcry security-bid auction. Potential bidders receive their signals, and must then weigh whether it is worthwhile to participate in the auction. Participation is costly, reflecting bid preparation costs, time costs, etc. In addition, the expected net value of the asset with the potential bidder could be less than its value when it is retained by the seller; i.e., "synergies" could be negative.

The seller specifies a reserve security, i.e., a minimum acceptable security-bid. In our benchmark setting, the reserve security is set so that the seller's expected revenue given a single bidder—the only case in which it binds—has a fixed value that is at least as high as the asset's retained value: ex ante, the seller does not regret selling at the reserve. For example, if an asset is being sold due to bankruptcy, the seller, perhaps the firm's trustee, cannot reject a bid whose value is expected to exceed the asset's scrap value. We also consider the possibilities that (a) the reserve security across auction designs generates the same expected revenue from the *marginal* bidder, and (b) reserves are set optimally for each class of securities.

We first establish that in our benchmark setting, steeper securities attract *more entry* and extract more from winning bidders, yielding the seller *higher expected revenues*. The reasoning behind the seemingly paradoxical result is as follows. Conditional on a single bidder, a seller expects the same revenues regardless of the security design. Further, *because* steeper securities extract more revenues from bidders with higher valuations, the steeper reserve security must extract less from bidders with lower valuations, making them *more* willing to enter.

One might conjecture that this extra entry could harm a seller; i.e., steeper security designs could reduce expected revenues, e.g., because a marginal entrant has a low private valuation.[1] This conjecture is false. Because an entrant expects to earn enough to cover its sunk entry costs, it must be willing to bid above the reserve security; else, it should not enter. Hence, with multiple entrants, all losing bidders drop out at bids above the reserve security. Thus, the greater entry with steeper securities *increases* a seller's expected revenues whenever there are multiple bidders. Further, the greater entry raises the probability of (profitable) sale, i.e., of having at least one bidder. We extend this result to show that an auction using steeper securities and associated optimal reserve generates higher expected revenues than an auction using less steep securities and its optimal reserve. We then establish further properties of auctions with optimally-chosen reserve securities.

[1]Indeed, results in Narayanan (1988) suggest that steeper securities could have a negative impact. He considers a firm with private information about the value of a project that requires external funding from risk-neutral investors who break even in expectation. He shows that in the signaling equilibrium, more bad types seek funding with equity than debt. As a result, firm value is reduced by the steeper security.

Our paper contributes to the security-bid auction literature (see a review by Skrzypacz (2013)). Hansen (1985) shows that equity auctions yield higher expected revenues than cash auctions. DKS extend this result to a general class of security-bid auctions, establishing that a greater linkage between a bidder's private information and his expected payment upon winning raises a seller's expected revenue (Milgrom and Weber, 1982). Other security-bid auction papers include Rhodes-Kropf and Viswanathan (2000), Che and Kim (2010), Kogan and Morgan (2010), Abhishek, Hajek and Williams (2015) and Liu (2016). Fishman (1989) builds a sequential entry model in which low-valuation bidders use securities while high-valuation bidders use cash, signaling high values to preempt competition.

Our analysis is closest to Gorbenko and Malenko (2011). They endogenize competition between sellers in the design of second-price security-bid auctions. Potential bidders can only enter one auction, making entry decisions based on auction design, *not* knowing their valuations of the goods being auctioned. Steeper securities extract more from a given number of bidders, but *because* steeper securities extract more, flatter securities draw more bidders. We extend this analysis to show that when bidders do not learn their valuations prior to entry, steeper security designs draw less entry even with a reserve of a fixed value to the seller. Thus, the equilibrium security design typically trades off between entry and rent extraction. In sharp contrast, if bidders know their valuations when making entry decisions, a seller does not need to trade off between extracting more from a winning bidder and attracting more entry: steeper security designs do both.

I. Model

We modify the framework of DKS by introducing an entry decision by risk-neutral bidders to an open outcry security-bid auction held by a risk-neutral seller.[2] The asset being auctioned has a value of $v \geq 0$ if the seller retains it. There are n ex-ante identical potential bidders. A bidder incurs cost $\phi > 0$ if it participates in the auction. The auction winner must make a non-contractible investment of $X > 0$ for the asset to pay off. If bidder i acquires the asset and invests X, then it will yield a (contractible) stochastic payoff of Z_i.

At date 0, each potential bidder i receives a private signal Θ_i of the incremental value of the asset if bidder i wins it. Thus, conditional on the asset being acquired by bidder i of type $\Theta_i = \theta$, the expected value of Z_i is

$$E(Z_i|\Theta_i = \theta) = X + v + \theta.$$

That is, the expected value added if bidder i wins is $E(Z_i|\Theta_i = \theta) - (X + v) = \theta$. We assume that Z_i is distributed i.i.d. according to a density $h\left(\cdot|\theta\right)$ with full

[2] We focus on an open outcry auction largely to deal with semantics following a single entrant. In an open outcry auction, a single entrant bids the reserve security. Our analysis extends to a second-price auction as long as either a bidder knows whether it is the sole entrant, or the reserve security is fixed.

support on $[0, \infty)$, and that the family $\{h(\cdot|\theta)\}$ has the *strict monotone likelihood ratio property (sMLRP)*: $h(z|\theta)/h(z|\theta')$ is increasing in z for $\theta > \theta'$. That is, higher signals are good news.

Signals are distributed i.i.d. according to a distribution $F(\cdot)$ with full support over $[\underline{\theta}, \overline{\theta}]$. We assume that $\underline{\theta} < \phi < \bar{\theta}$. Thus, the net value of the asset to a bidder may or may not exceed the retained value to the seller. In particular, $\bar{\theta} > \phi$ means that it is efficient to allocate the asset to a potential bidder with a high valuation. Conversely, it is not efficient for potential bidders with low valuations $\theta < \phi$ to enter the auction—it would be better for the seller to retain the asset. Our model allows for $\underline{\theta} < 0$: not only may "synergies" fail to cover entry costs, they may be negative in nature.

At date 1, after receiving their signals, potential bidders simultaneously decide whether to enter a security-bid auction $(\mathcal{S}, \underline{s}(\mathcal{S}))$ for the asset. $(\mathcal{S}, \underline{s}(\mathcal{S}))$ specifies a set of bids $\mathcal{S}$ and a reserve security $\underline{s}(\mathcal{S})$. Bids are made in the form of securities that are contingent on the stochastic payoff Z_i, which is realized at date 2. The reserve security $\underline{s}(\mathcal{S})$ is the minimum bid accepted under $\mathcal{S}$. Let $S(s, z)$ denote the payment to the seller from security s when $Z_i = z$. Bids are restricted to an *ordered set of securities*, $\mathcal{S} = \{S(s, \cdot) : s \in [\underline{s}(\mathcal{S}), \overline{s}]\}$ such that (i) for all s, $S(s, z)$ and $z - S(s, z)$ are weakly increasing in z; and (ii) $ES(s, \theta) \equiv E(S(s, Z_i)|\Theta_i = \theta)$, the expected value of security $S(s, \cdot)$ conditional on $\Theta_i = \theta$, is differentiable and strictly increasing in both arguments with $ES(\overline{s}, \overline{\theta}) \geq v + \overline{\theta}$.

At date 2, asset payoff $Z_i = z$ is realized. If bidder i is the sole entrant, i wins the auction if and only if it submits a feasible bid; i.e., its bid s_i weakly exceeds the reserve security $\underline{s}(\mathcal{S})$, paying $S(s_i, z)$. If multiple bidders submit feasible bids, the winning bidder i pays with the security bid s^2 of the last bidder to drop out of the auction, paying $S(s^2, z)$.

We use the notion of steepness introduced in DKS: an ordered set of securities $\mathcal{S}_A$ is *steeper* than $\mathcal{S}_B$ if for all $s_A \in [\underline{s}(\mathcal{S}_A), \overline{s}_A]$ and $s_B \in [\underline{s}(\mathcal{S}_B), \overline{s}_B]$, $ES_A(s_A, \theta^*) = ES_B(s_B, \theta^*)$ implies $\partial ES_A(s_A, \theta^*)/\partial\theta > \partial ES_B(s_B, \theta^*)/\partial\theta$. Thus, if a bidder with a private valuation θ^* expects to pay the same amount with a steeper security as with a flatter security, then any bidder with a higher private valuation $\theta > \theta^*$ expects to pay strictly more with the steeper security than with the flatter security. That is, the payment of the steeper security is tied more tightly to the winning bidder's valuation.

We first consider bidding decisions conditional on entry, i.e., on paying the participation cost ϕ. The logic in Proposition 1 of DKS yields the following results.

- If a bidder i with type $\Theta_i = \theta$ is the sole entrant, it has a dominant strategy to bid $\underline{s}(\mathcal{S})$ if $ES(\underline{s}(\mathcal{S}), \theta) \leq v + \theta$; and not to bid if $ES(\underline{s}(\mathcal{S}), \theta) > v + \theta$.

- With multiple bidders, a bidder i of type $\Theta_i = \theta$ has a weakly dominant strategy to drop out at the bid $s^*(\theta)$ where $ES(s^*(\theta), \theta) = v + \theta$. Further, $s^*(\cdot)$ increases in θ.

- If the ordered set of securities $\mathcal{S}_A$ is steeper than $\mathcal{S}_B$, then conditional on entry of the two highest types, the seller's expected revenue is greater under $\mathcal{S}_A$ than $\mathcal{S}_B$.

Next, we consider the entry decisions of bidders à la Samuelson (1985). With positive entry costs, not all potential bidders may enter. Since the equilibrium expected payoff upon entry increases in θ for a given auction $(\mathcal{S}, \underline{s}(\mathcal{S}))$, there must be some cutoff $\theta(\mathcal{S})$ such that only bidders with $\theta \geq \theta(\mathcal{S})$ enter the auction. For the marginal bidder $\theta(\mathcal{S})$, the expected payoff from participation just cover the entry costs; i.e., $\theta(\mathcal{S})$ solves:

$$[v + \theta(\mathcal{S}) - ES(\underline{s}(\mathcal{S}), \theta(\mathcal{S}))] F^{n-1}(\theta(\mathcal{S})) = \phi. \tag{1}$$

We assume that the seller rejects bids with expected values below some *fixed* $\hat{v} \in [v, v + \bar{\theta} - \phi)$. Thus, the seller rejects bids below the asset's value if retained, but $\hat{v}$ is not so high that it precludes all entry. In equilibrium, a single entrant bids $\underline{s}(\mathcal{S})$, and the seller only learns that the entrant's type θ is at least $\theta(\mathcal{S})$. Thus, $\underline{s}(\mathcal{S})$ solves:[3]

$$\int_{\theta(\mathcal{S})}^{\overline{\theta}} ES(\underline{s}(\mathcal{S}), \theta) F(d\theta | \theta \geq \theta(\mathcal{S})) = \hat{v}. \tag{2}$$

The left-hand side of (1) increases in $\theta(\mathcal{S})$, and from (2) it would become $v + \overline{\theta} - \hat{v}$ by substituting $\overline{\theta}$ for $\theta(\mathcal{S})$. Thus, the assumption that $\hat{v} < v + \bar{\theta} - \phi$ ensures that $\theta(\mathcal{S}) < \bar{\theta}$ holds for all $\mathcal{S}$, i.e., entry occurs with strictly positive probability.

LEMMA 1: *$s^*(\theta(\mathcal{S})) > \underline{s}(\mathcal{S})$.*

Proof. The left-hand side of (1) is decreasing in $\underline{s}(\mathcal{S})$ and would become $0 < \phi$ if we replaced $\underline{s}(\mathcal{S})$ with $s^*(\theta(\mathcal{S}))$. □

PROPOSITION 1: *If the ordered set of securities $\mathcal{S}_A$ is steeper than $\mathcal{S}_B$, then $\theta(\mathcal{S}_A) < \theta(\mathcal{S}_B)$: auction $(\mathcal{S}_A, \underline{s}(\mathcal{S}_A))$ attracts at least as many entrants as $(\mathcal{S}_B, \underline{s}(\mathcal{S}_B))$. This greater entry leads to higher expected seller revenues. In particular,*

- *When auction $(\mathcal{S}_A, \underline{s}(\mathcal{S}_A))$ either attracts multiple entrants or more entrants than $(\mathcal{S}_B, \underline{s}(\mathcal{S}_B))$, it yields the seller higher expected revenue.*
- *When auctions $(\mathcal{S}_A, \underline{s}(\mathcal{S}_A))$ and $(\mathcal{S}_B, \underline{s}(\mathcal{S}_B))$ both attract zero entrants or both attract one entrant, they yield the seller the same expected revenue.*

Proof. See Appendix.

[3] We assume that the lowest security is low enough that it does not constrain $\underline{s}(\mathcal{S})$. In the proof of Proposition 1, we provide a sufficient condition, $ES(s_0, \bar{\theta}) \leq v$, where s_0 denotes the lowest security, for equation (2) to have a solution.

To see why auction $(\mathcal{S}_A, \underline{s}(\mathcal{S}_A))$ attracts more entry, observe that regardless of the class of securities, the seller's expected revenues given a single entrant are $\hat{v}$. However, steeper securities extract a greater share of its revenues from bidders with higher valuations. It follows that they extract less from bidders with lower valuations. Thus, the steeper is the security design, the more willing are bidders with lower valuations to enter.

The seller may expect to suffer a loss when the marginal auction participant is the sole bidder; indeed, the marginal entrant's private valuation could be negative. Nonetheless, even though steeper security designs draw more entrants with low valuations, this greater entry always generates higher expected seller revenues for any realization of bidders' types. First, the greater entry raises the probability of sale, and $\hat{v} \geq v$ ensures that, in expectation, this sale is profitable. Second, the greater entry enhances competition: with multiple entrants, the marginal entrant bids above the reserve, i.e., the winning bidder pays with a security bid that is at least $s^*(\theta(\mathcal{S})) > \underline{s}(\mathcal{S})$. Thus, the greater entry to auctions with steeper securities reinforces their revenue-enhancing advantages.

Proposition 1 reveals that with reserves of a fixed value to the seller, steeper security-bid designs attract more entry and generate higher expected revenue. Corollary 1 shows that the revenue advantage of steeper securities extends to optimal reserves.

COROLLARY 1: *With the optimal reserve security for a given class of securities, the steeper is the security-bid design, the greater is the seller's expected revenue.*

Corollary 1 follows from Proposition 1 if the reserve for the flatter security design delivers expected revenue that exceeds the asset's value if retained. We prove a stronger result in the appendix: given *any* reserve for the flatter design, including the optimal one, the steeper design delivers higher revenues when its reserve induces the same *marginal* entrant as the reserve for the flatter design.

This result does *not* imply that a steeper design attracts more entry when its reserve is set optimally. To gain insights into when it does, we specialize to securities that consist of a fixed royalty rate $\alpha \in [0, 1)$ plus a cash payment: the higher is α, the steeper is the design. Such securities are common in oil and gas lease auctions, where the cash payment is determined in the auction and the royalty payment is set by state or federal law (Gorbenko and Malenko 2011; Skrzypacz 2013). As in Myerson (1981), we impose the regularity condition on the distribution of valuations that $\hat{\theta} - (1 - F(\hat{\theta}))/f(\hat{\theta})$ strictly increases in $\hat{\theta}$. Given α, if c is the cash reserve price, the marginal entrant $\hat{\theta}$ solves

$$F^{n-1}(\hat{\theta})((1-\alpha)E(Z_i|\Theta_i = \hat{\theta}) - c - X) = \phi$$

$$\Rightarrow \; c = (1-\alpha)(X + v + \hat{\theta}) - X - \frac{\phi}{F^{n-1}(\hat{\theta})}. \tag{3}$$

The right-hand side increases in $\hat{\theta}$, so setting a cash reserve price c (which can be negative) amounts to choosing $\hat{\theta}$.

PROPOSITION 2: *For the class of securities in which the winner pays with a combination of cash and a fixed royalty rate $\alpha \in [0, 1)$, if the reserve price maximizes expected revenues, then steeper securities attract more entry: the marginal entrant $\hat{\theta}$ induced by the optimal reserve $c^*(\alpha)$ decreases in α, solving*

$$\hat{\theta} - \frac{\phi}{F^{n-1}(\hat{\theta})} - (1-\alpha)\frac{1-F(\hat{\theta})}{f(\hat{\theta})} = 0. \tag{4}$$

Proof. See Appendix.

Steeper securities allow a seller to keep more of the social surplus, and attracting more entry increases the number of positive-NPV trades ($\hat{\theta}$ is positive by (4)), increasing social surplus. With fixed royalty rates, there is no cost to attracting worse types: the reserve does not affect the sensitivity of payouts to underlying cash flows. However, with other securities, lowering the reserve can weaken the tying of the security's value to bidder types. Liu (2016) shows that the gain from restricting entry of worse types can dominate the competition effect: if the distribution F of valuations has a fat upper tail, it can be optimal to set a higher reserve in equity auctions than in cash auctions.

Contrasting Known and Unknown Values. We next contrast our result that steeper securities attract more entry when the reserve has a fixed value $\hat{v}$ to the seller with the analogous setting in which bidders do not know their values before making entry decisions. Two types of equilibria exist: a pure strategy equilibrium (see McAfee and McMillan, 1987) in which an entrant expects a non-negative profit, but with greater entry, expected profits would become negative; and a mixed strategy equilibrium (see Levin and Smith, 1994) in which each of N potential bidders enters with probability q. We now show that regardless of the nature of the equilibrium, steeper auctions attract less entry.

A seller's expected equilibrium net profit conditional on a single bidder submitting a valid bid is $\hat{v} - v \geq 0$. Given a set $\mathcal{S}$ of securities, if $\underline{s}_{un}(\mathcal{S})$ is the reserve security and $\theta_{un}(\mathcal{S})$ is the cutoff bidder type such that those with higher valuations submit valid bids, then (i) $\underline{s}_{un}(\mathcal{S})$ and $\theta_{un}(\mathcal{S})$ solve (2), and (ii) the marginal bidder is indifferent between winning at $\underline{s}_{un}(\mathcal{S})$ and losing; i.e.,

$$\theta_{un}(\mathcal{S}) + v = ES\left(\underline{s}_{un}(\mathcal{S}), \theta_{un}(\mathcal{S})\right). \tag{5}$$

PROPOSITION 3: *With unknown valuations, steeper securities attract fewer entrants. If the ordered set of securities $\mathcal{S}_A$ is steeper than $\mathcal{S}_B$, then auction $(\mathcal{S}_B, \underline{s}_{un}(\mathcal{S}_B))$ draws at least as many entrants as auction $(\mathcal{S}_A, \underline{s}_{un}(\mathcal{S}_A))$.*

Proof. See Appendix.

To see the logic, suppose that $\mathcal{S}_A$ draws the same number of entrants as $\mathcal{S}_B$. As with known valuations, trade is more likely with steeper securities: $\theta_{un}(\mathcal{S}_A) <$

$\theta_{un}(\mathcal{S}_B)$. Letting Δ be the difference in the probability of trade, we show that this additional trading probability arises for private valuations below $\hat{v}-v$. Thus, welfare is higher by less than $(\hat{v}-v)\Delta$ for the steeper design. With a single entrant, a seller's expected net profit *conditional on trade occurring* is always $\hat{v}-v$. Thus, expected seller profit is higher for the steeper design by $(\hat{v}-v)\Delta$, implying that entry is less profitable. With multiple entrants, stronger results hold: steeper securities extract more rent from bidders, so expected seller profit for the steeper design is higher by more than $(\hat{v}-v)\Delta$, implying that entry is even less profitable. Hence, steeper securities attract less entry, in stark contrast to settings where bidders know their valuations before making entry decisions.

To further emphasize the contrast between known and unknown valuations, we consider the possibility that the reserve is set so that the seller expects the same profit conditional on the *marginal entrant*, i.e., $ES(\underline{s}(\mathcal{S}),\theta(\mathcal{S}))=\hat{v}$, rather than the same profit conditional on a *single entrant*, as in (2).[4] Then with known valuations, the marginal bidder $\theta(\mathcal{S})$ is not affected by the security design—because the expected payment to the seller given a single entrant is always $\hat{v}$, $\theta(\mathcal{S})$ solves

$$[v+\theta(\mathcal{S})-\hat{v}]\,F^{n-1}(\theta(\mathcal{S}))=\phi.$$

In contrast, when potential bidders do not know their types prior to entry, the marginal bidder's type and profit do not hinge on the security design, but steeper securities reduce the expected profits of all higher types, which reduces entry.

II. Conclusion

We endogenize entry to security-bid auctions, when entry is costly and bidders know their valuations prior to making entry decisions. We first consider *any* reserve security of a *fixed* value that exceeds the asset's retained value to the seller. Counter-intuitively, we show that security-bid auctions that use steeper securities for payment, which generate greater expected revenues for the seller for a fixed number of bidders, also make bidders with worse signals more willing to enter. This increased entry reinforces the revenue superiority of such auctions even when the marginal entrant has a very low private valuation. We extend this analysis to optimally-chosen reserves; and we contrast our findings with those that obtain when bidders do not know their valuations prior to entry.

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[4]For example, if the seller is a government that is concerned with social efficiency, it might want to set the reserve in this way, making the marginal bidders type invariant across auction design.

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Appendix

Proof of Proposition 1. Let $\underline{s}(\mathcal{S}_A)$ and $\underline{s}(\mathcal{S}_B)$ be the reserve securities under $\mathcal{S}_A$ and $\mathcal{S}_B$, respectively. By way of contradiction, first suppose $\theta(\mathcal{S}_A) = \theta(\mathcal{S}_B) = \tilde{\theta}$. Then,

$$\int_{\tilde{\theta}}^{\overline{\theta}} ES_A(\underline{s}(\mathcal{S}_A), \theta) F\left(d\theta | \theta \geq \tilde{\theta}\right) = \int_{\tilde{\theta}}^{\overline{\theta}} ES_B(\underline{s}(\mathcal{S}_B), \theta) F\left(d\theta | \theta \geq \tilde{\theta}\right) \tag{A1}$$

must hold to satisfy (2). Also, using (1) yields $ES_A\left(\underline{s}(\mathcal{S}_A), \tilde{\theta}\right) = ES_B\left(\underline{s}(\mathcal{S}_B), \tilde{\theta}\right)$, which, together with the definition of steepness, implies that

$$\int_{\tilde{\theta}}^{\overline{\theta}} ES_A(\underline{s}(\mathcal{S}_A), \theta) F\left(d\theta | \theta \geq \tilde{\theta}\right) > \int_{\tilde{\theta}}^{\overline{\theta}} ES_B(\underline{s}(\mathcal{S}_B), \theta) F\left(d\theta | \theta \geq \tilde{\theta}\right),$$

a contradiction to (A1). Next suppose $\theta(\mathcal{S}_A) > \theta(\mathcal{S}_B)$. Then, it follows that

$$\begin{aligned} &\int_{\theta(\mathcal{S}_A)}^{\overline{\theta}} ES_B(\underline{s}(\mathcal{S}_B), \theta) F(d\theta | \theta \geq \theta(\mathcal{S}_A)) > \int_{\theta(\mathcal{S}_B)}^{\overline{\theta}} ES_B(\underline{s}(\mathcal{S}_B), \theta) F(d\theta | \theta \geq \theta(\mathcal{S}_B)) \\ &\qquad = \int_{\theta(\mathcal{S}_A)}^{\overline{\theta}} ES_A(\underline{s}(\mathcal{S}_A), \theta) F(d\theta | \theta \geq \theta(\mathcal{S}_A)), \end{aligned} \tag{A2}$$

where the equality holds by (2). This, together with the definition of steepness, implies $ES_B(\underline{s}(\mathcal{S}_B), \theta(\mathcal{S}_A)) > ES_A(\underline{s}(\mathcal{S}_A), \theta(\mathcal{S}_A))$; else, the left-hand side of (A2) would become smaller, a contradiction. Let $U(\underline{s}(\mathcal{S}_j), \theta)$ denote type θ's expected payoff given reserve securities $\underline{s}(\mathcal{S}_j)$ for $j = A, B$. Then, since $ES_B(\underline{s}(\mathcal{S}_B), \theta(\mathcal{S}_A)) > ES_A(\underline{s}(\mathcal{S}_A), \theta(\mathcal{S}_A))$,

$$U(\underline{s}(\mathcal{S}_A), \theta(\mathcal{S}_A)) > U(\underline{s}(\mathcal{S}_B), \theta(\mathcal{S}_A)) > U(\underline{s}(\mathcal{S}_B), \theta(\mathcal{S}_B)) = U(\underline{s}(\mathcal{S}_A), \theta(\mathcal{S}_A)),$$

a contradiction, where the second inequality holds by $\theta(\mathcal{S}_A) > \theta(\mathcal{S}_B)$ and the equality holds by (1). Therefore, $\theta(\mathcal{S}_A) < \theta(\mathcal{S}_B)$.

To establish existence and uniqueness of equilibrium outcomes, we assume that the set of securities has sufficient range on the low side that $ES\left(s_0, \bar{\theta}\right) \leq v$, where s_0 is the lowest security. We now show that this ensures a unique $\underline{s} \in [s_0, \bar{s}]$ and $\theta \in \left[\underline{\theta}, \overline{\theta}\right]$ that solve equations (1) and (2). First, by $ES\left(s_0, \bar{\theta}\right) \leq v$, $ES(\bar{s}, \overline{\theta}) \geq v + \overline{\theta}$ and $\bar{\theta} > \phi$, a unique $s^* \in (s_0, \bar{s})$ exists such that $ES\left(s^*, \bar{\theta}\right) = v + \bar{\theta} - \phi$. Second, define $\hat{\theta}(s)$ to be the cutoff type associated with a reserve $s \in [s_0, s^*]$; i.e.,

$$\left[v + \hat{\theta}(s) - ES\left(s, \hat{\theta}(s)\right)\right] F^{n-1}\left(\hat{\theta}(s)\right) = \phi. \tag{A3}$$

Because $\hat{\theta} - ES(s,\hat{\theta})$ weakly rises in $\hat{\theta}$ (see Lemma 1 of DKS) and $F(\hat{\theta})$ strictly rises in $\hat{\theta}$, the left-hand side of (A3) strictly rises in $\hat{\theta}$. For all $s \in [s_0, s^*]$, the left-hand side of (A3) is no less than ϕ at $\hat{\theta} = \bar{\theta}$, and is zero at $\hat{\theta} = \underline{\theta}$; hence (A3) has a unique solution. Because $ES(s,\hat{\theta})$ strictly increases in s, $\hat{\theta}(s)$ strictly increases in s. Third, for $s \in [s_0, s^*]$, denote by $L(s)$ the left-hand side of (2) when plugging s for $\underline{s}(\mathcal{S})$ and $\hat{\theta}(s)$ for $\theta(\mathcal{S})$; i.e.,

$$L(s) = \int_{\hat{\theta}(s)}^{\bar{\theta}} ES(s,\theta) F(d\theta | \theta \geq \hat{\theta}(s)).$$

Because $\hat{\theta}(s^*) = \bar{\theta}$, $L(s^*) = ES\left(s^*, \bar{\theta}\right) = v + \bar{\theta} - \phi > \hat{v}$. As $L(s_0) < v \leq \hat{v}$ and $L(s)$ is strictly increasing, a unique $s^{**} \in (s_0, s^*)$ exists such that $L(s^{**}) = \hat{v}$. Thus, there exists a unique solution ($\underline{s} = s^{**}$, $\theta = \hat{\theta}(s^{**})$) that describes the equilibrium.

Let θ^2 be the second-highest type. When $\theta^2 \leq \theta(\mathcal{S}_A)$, a seller expects v if there is no entry and $\hat{v} \geq v$ if there is entry; and entry is more likely for $(\mathcal{S}_A, \underline{s}(\mathcal{S}_A))$ than $(\mathcal{S}_B, \underline{s}(\mathcal{S}_B))$. When $\theta^2 \in (\theta(\mathcal{S}_A), \theta(\mathcal{S}_B)]$, at least two bidders enter for $(\mathcal{S}_A, \underline{s}(\mathcal{S}_A))$ so expected revenue exceeds $\hat{v}$ by Lemma 1, while at most one bidder enters for $(\mathcal{S}_B, \underline{s}(\mathcal{S}_B))$. When $\theta^2 > \theta(\mathcal{S}_B)$, multiple bidders enter both auctions; and from DKS a seller expects higher revenue from $(\mathcal{S}_A, \underline{s}(\mathcal{S}_A))$. □

Proof of Corollary 1: We prove a stronger revenue dominance result. Let an ordered set of securities $\mathcal{S}_A$ be steeper than $\mathcal{S}_B$. Set the reserve for $\mathcal{S}_B$ (possibly optimally, but possibly sub-optimally delivering expected revenue below v) so that a potential bidder enters if and only if his type exceeds some θ^*. Now set the reserve for $\mathcal{S}_A$ so that a type θ also enters if and only if $\theta \geq \theta^*$. Thus, the probability of getting any $m \in \{1, ..., n\}$ entrants is the same in the two auctions. Conditional on any $m \geq 1$ entrants, auction $\mathcal{S}_A$ generates higher expected revenue than $\mathcal{S}_B$: $m > 1$ follows from DKS, and when $m = 1$, $ES(\underline{s}(\mathcal{S}_A), \theta^*) = ES(\underline{s}(\mathcal{S}_B), \theta^*) = v + \theta^* - \frac{\phi}{F^{n-1}(\theta^*)}$ by the participation condition (1), and $ES(\underline{s}(\mathcal{S}_A), \theta) > ES(\underline{s}(\mathcal{S}_B), \theta)$ for $\theta > \theta^*$ by the definition of steepness. Setting the optimal reserve for $\mathcal{S}_A$ would only further raise its expected revenue. □

Proof of Proposition 2. For $\theta \in [\hat{\theta}, \bar{\theta}]$, let $\pi(\theta)$ be a bidder's expected profit conditional on θ. Then from an envelope condition (as in Myerson (1981)), we have $\frac{d}{d\theta}\pi(\theta) = (1-\alpha) F^{n-1}(\theta)$. Because $\pi(\hat{\theta}) = 0$,

$$\pi(\theta) = \int_{\hat{\theta}}^{\theta} (1-\alpha) F^{n-1}(t)\, dt. \tag{A4}$$

Then a bidder's unconditional expected profit is

$$\int_{\hat{\theta}}^{\bar{\theta}} \pi(\theta)\, dF(\theta) = \int_{\hat{\theta}}^{\bar{\theta}} \int_{\hat{\theta}}^{\theta} (1-\alpha) F^{n-1}(t)\, dt dF(\theta)$$
$$= \int_{\hat{\theta}}^{\bar{\theta}} \left((1-\alpha)(1-F(\theta)) F^{n-1}(\theta)\right) d\theta,$$

where we use integration by parts. Thus, the seller's expected profits are

$$\Pi = \int_{\hat{\theta}}^{\bar{\theta}} \theta d(F^n(\theta)) - n\phi\left(1 - F\left(\hat{\theta}\right)\right) - n\int_{\hat{\theta}}^{\bar{\theta}} \left((1-\alpha)(1-F(\theta)) F^{n-1}(\theta)\right) d\theta$$

Maximizing over $\hat{\theta}$ (via c) yields

$$\frac{d\Pi}{d\hat{\theta}} = n(1-\alpha)(1-F(\hat{\theta}))F^{n-1}(\hat{\theta}) - n\hat{\theta}F^{n-1}(\hat{\theta})f(\hat{\theta}) + n\phi f(\hat{\theta})$$
$$= -nf(\hat{\theta})F^{n-1}(\hat{\theta})\left(\hat{\theta} - \frac{\phi}{F^{n-1}(\hat{\theta})} - (1-\alpha)\frac{1-F(\hat{\theta})}{f(\hat{\theta})}\right) = 0, \tag{A5}$$

yielding (4). From the regularity condition, the left-hand side of (4) strictly increases in $\hat{\theta}$, implying a unique solution. For (4) to hold, if α increases, $\hat{\theta}$ falls, so c^* must fall. □

Proof of Proposition 3: First consider pure strategy equilibria. For $k \in \{A, B\}$, let π_k be bidders' total expected profits (excluding entry costs), $\tilde{\Pi}_k$ be the seller's profit, and $\Pi_k \equiv E\left[\tilde{\Pi}_k\right]$ be the seller's expected profit in auction $(\mathcal{S}_k, \underline{s}_{un}(\mathcal{S}_k))$. Let θ^j be the j highest type among m bidders. Denote the distribution of θ^1 by $G\left(\theta^1\right) \equiv F^m\left(\theta^1\right)$.

To show Proposition 3, it suffices to show $\pi_A < \pi_B$ whenever both auctions draw the same number $m \geq 1$ of entrants. First, since $\theta_{un}(\mathcal{S}_k)$ satisfies (2), $\theta_{un}(\mathcal{S}_A) < \theta_{un}(\mathcal{S}_B)$ holds by a similar logic as that for Proposition 1. Further, (2) and (5), together with the property that $ES\left(\underline{s}_{un}(\mathcal{S}_k), \theta\right)$ increases in θ, yield

$$\theta_{un}(\mathcal{S}_k) < \hat{v} - v, \tag{A6}$$

for $k = A, B$. Note that π_k is the difference between the increase in expected social welfare and the seller's expected profit Π_k: $\pi_k = \int_{\theta_{un}(\mathcal{S}_k)}^{\bar{\theta}} \theta dG(\theta) - \Pi_k$. Thus,

$$\pi_A - \pi_B = \int_{\theta_{un}(\mathcal{S}_A)}^{\theta_{un}(\mathcal{S}_B)} \theta dG(\theta) - (\Pi_A - \Pi_B). \tag{A7}$$

Suppose $m = 1$. Then, $\Pi_k = (\hat{v} - v)(1 - F(\theta_{un}(\mathcal{S}_k)))$. Plugging this into (A7) yields

$$\pi_A - \pi_B = \int_{\theta_{un}(\mathcal{S}_A)}^{\theta_{un}(\mathcal{S}_B)} \theta dG(\theta) - (\hat{v} - v)\left(F\left(\theta_{un}\left(\mathcal{S}_B\right)\right) - F\left(\theta_{un}\left(\mathcal{S}_A\right)\right)\right)$$
$$= \int_{\theta_{un}(\mathcal{S}_A)}^{\theta_{un}(\mathcal{S}_B)} \left(\theta - (\hat{v} - v)\right) dF(\theta) < 0,$$

where the inequality holds by (A6). This completes the proof for $m = 1$.

Now consider $m \geq 2$. We first show that given any $\theta^2 \in \left[\underline{\theta}, \bar{\theta}\right]$, the seller's expected profit when the asset is sold under $\mathcal{S}_A$ exceeds that under $\mathcal{S}_B$, which exceeds $\hat{v} - v$:

$$\text{(A8)} \qquad E\left[\tilde{\Pi}_A | \theta^1 \geq \theta_{un}\left(\mathcal{S}_A\right), \theta^2\right] \geq E\left[\tilde{\Pi}_B | \theta^1 \geq \theta_{un}\left(\mathcal{S}_B\right), \theta^2\right] \geq \hat{v} - v.$$

We prove (A8) in 3 cases. Case 1: $\theta^2 \leq \theta_{un}\left(\mathcal{S}_A\right)$. Then (A8) holds trivially: the two weak inequalities hold as equalities. Case 2: $\theta^2 \in \left(\theta_{un}\left(\mathcal{S}_A\right), \theta_{un}\left(\mathcal{S}_B\right)\right]$. Then

$$E\left[\tilde{\Pi}_A | \theta^1 \geq \theta_{un}\left(\mathcal{S}_A\right), \theta^2\right] > \int_{\theta^2}^{\bar{\theta}} ES\left(\underline{s}_{un}\left(\mathcal{S}_A\right), \theta\right) F(d\theta | \theta \geq \theta^2) - v$$
$$> \int_{\theta_{un}(\mathcal{S}_A)}^{\bar{\theta}} ES\left(\underline{s}_{un}\left(\mathcal{S}_A\right), \theta\right) F(d\theta | \theta \geq \theta_{un}\left(\mathcal{S}_A\right)) - v$$
$$= \hat{v} - v.$$

The first inequality holds because the second-highest bid exceeds $\underline{s}_{un}\left(\mathcal{S}_A\right)$; the second holds because $ES\left(\underline{s}_{un}\left(\mathcal{S}_A\right), \theta\right)$ increases in θ over $\left[\theta_{un}\left(\mathcal{S}_A\right), \bar{\theta}\right]$ and $F(\theta | \theta \geq \theta_2)$ first-order stochastically dominates $F(\theta | \theta \geq \theta_{un}\left(\mathcal{S}_A\right))$. Further, because $\theta^2 \leq \theta_{un}\left(\mathcal{S}_B\right)$, $E\left[\tilde{\Pi}_B | \theta^1 \geq \theta_{un}\left(\mathcal{S}_B\right), \theta^2\right] = \hat{v} - v$, yielding (A8). Case 3: $\theta^2 > \theta_{un}\left(\mathcal{S}_B\right)$. With multiple bidders, the logic of DKS gives $E\left[\tilde{\Pi}_A | \theta^1 \geq \theta_{un}\left(\mathcal{S}_A\right), \theta^2\right] > E\left[\tilde{\Pi}_B | \theta^1 \geq \theta_{un}\left(\mathcal{S}_B\right), \theta^2\right]$. Since $ES(\cdot, \cdot)$ increases in its arguments, $E\left[\tilde{\Pi}_B | \theta^1 \geq \theta_{un}\left(\mathcal{S}_B\right), \theta^2\right] > \hat{v} - v$, yielding (A8).

The seller's expected profit can be written as the probability the asset is sold multiplied by the expected profit conditional on it being sold:

$$E\left[\tilde{\Pi}_k | \theta^2\right] = E[\mathbf{1}_{\theta^1 \geq \theta_{un}(\mathcal{S}_k)} | \theta^2] E\left[\tilde{\Pi}_k | \theta^1 \geq \theta_{un}\left(\mathcal{S}_k\right), \theta^2\right],$$

where $\mathbf{1}_{\theta^1 \geq \theta_{un}(\mathcal{S}_k)}$ is 1 if $\theta^1 \geq \theta_{un}\left(\mathcal{S}_k\right)$ and 0 otherwise. By the law of iterated expectations

$$\Pi_k = E\left[E[\mathbf{1}_{\theta^1 \geq \theta_{un}(\mathcal{S}_k)} | \theta^2] E\left[\tilde{\Pi}_k | \theta^1 \geq \theta_{un}\left(\mathcal{S}_k\right), \theta^2\right]\right].$$

Thus it follows that

$$
\begin{aligned}
\Pi_A - \Pi_B &> E\left[E\left[\tilde{\Pi}_B|\theta^1 \geq \theta_{un}\left(\mathcal{S}_B\right), \theta^2\right]\left(E[\mathbf{1}_{\theta^1 \geq \theta_{un}(\mathcal{S}_A)}|\theta^2] - E[\mathbf{1}_{\theta^1 \geq \theta_{un}(\mathcal{S}_B)}|\theta^2]\right)\right] \\
&\geq \left(\hat{v} - v\right) E\left[\left(E[\mathbf{1}_{\theta^1 \geq \theta_{un}(\mathcal{S}_A)}|\theta^2] - E[\mathbf{1}_{\theta^1 \geq \theta_{un}(\mathcal{S}_B)}|\theta^2]\right)\right] \\
&= \left(\hat{v} - v\right)\left(E[\mathbf{1}_{\theta^1 \geq \theta_{un}(\mathcal{S}_A)}] - E[\mathbf{1}_{\theta^1 \geq \theta_{un}(\mathcal{S}_B)}]\right) \\
&= \left(\hat{v} - v\right)\left(G\left(\theta_{un}\left(\mathcal{S}_B\right)\right) - G\left(\theta_{un}\left(\mathcal{S}_A\right)\right)\right). \qquad \text{(A9)}
\end{aligned}
$$

The first inequality follows from the first inequality in (A8) and the strict inequality in Case 3 above. The second inequality follows from the second inequality in (A8) and $E[1_{\theta^1 \geq \theta_{un}(S_A)}|\theta^2] - E[1_{\theta^1 \geq \theta_{un}(S_B)}|\theta^2] \geq 0$. The first equality follows from the law of iterated expectations. Substituting (A9) into (A7) yields

$$
\begin{aligned}
\pi_A - \pi_B &= \int_{\theta_{un}(\mathcal{S}_A)}^{\theta_{un}(\mathcal{S}_B)} \theta dG\left(\theta\right) - \left(\Pi_A - \Pi_B\right) \\
&< \int_{\theta_{un}(\mathcal{S}_A)}^{\theta_{un}(\mathcal{S}_B)} \theta dG\left(\theta\right) - \left(\hat{v} - v\right)\left(G\left(\theta_{un}\left(\mathcal{S}_B\right)\right) - G\left(\theta_{un}\left(\mathcal{S}_A\right)\right)\right) \\
&= \int_{\theta_{un}(\mathcal{S}_A)}^{\theta_{un}(\mathcal{S}_B)} \left(\theta - \left(\hat{v} - v\right)\right) dG\left(\theta\right) < 0,
\end{aligned}
$$

where the last inequality holds by (A6). This completes the proof for $m \geq 2$.

Now consider mixed-strategy equilibria in which each of N potential bidders enters with probability q_k for $k \in \{A, B\}$. To show that $q_A < q_B$, let $\pi_k\left(m\right)$ be bidders' total expected profits (excluding entry costs) given m entrants, for $k \in \{A, B\}$. Each bidder must be indifferent between entering and not entering, so q_k solves

$$
\sum_{m=1}^{N} \binom{N-1}{m-1} q_k^{m-1}\left(1 - q_k\right)^{N-m} \frac{\pi_k\left(m\right)}{m} = \phi. \qquad \text{(A10)}
$$

Since $\frac{\pi_k(m)}{m}$ decreases in m (a bidder's payoff falls as competition rises), the left-hand side decreases in q_k. Then, since $\pi_A\left(m\right) < \pi_B\left(m\right)$ for any m (from our argument for pure strategy equilibria), (A10) implies that $q_A < q_B$. $\square$