Algebraic Covers

by

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Thesis

Submitted to the University of Warwick

for the degree of

Doctor of Philosophy

Mathematics Institute

February 2016

THE UNIVERSITY OF
WARWICK

This thesis was accomplished with financial support from the Fundação para a Ciência e Tecnologia, Portugal, through the Ph.D. Grant SFRH/BD/61216/2009, under the POPH/FSE programmes.

Contents

Acknowledgments

I wish to thank my advisor Miles Reid for all the mathematics I learned from him and through all the events he organized at the University of Warwick. I am specially grateful to him for having accepted me back as a PhD student after my accident and treating me as if nothing happened.

I would also like to thank my examiners Christian Böhning and Gregory Sankaran for the time spent reading my thesis and their helpful suggestions.

I thank everyone from the Department of mathematics in Coimbra where part of the thesis was written and I started learning mathematics. A special thanks to Jorge Neves for introducing me to Algebraic Geometry and all the help throughout my PhD and to Alexander Kovačec whose enthusiasm for mathematics infected me.

Warm thanks to my friends and colleges from Warwick. Seung-Jo, Andrew, Sara, Enrico, Lorenzo, Sina, Rosemberg, thanks for all the discussions around the department, nights at the duck and all the time we spent playing guitar. To Simon and Mirna, thanks for all the great times, travels and life lessons.

A big thank you to the *titanium* without which I would never have written this thesis and a bigger one to my parents who had to deal with my non-mathematical problems. I would also like to express my gratitude to my uncles António and Vitor Carreira and their families who were extraordinary with me and to the Silvas who were a great help during my time in the clinic. João Nuno, Diana, Álvaro, Domingos, Marco, Camélia, Márcia, Joana Capitão and Dana Balibanu, thanks for your support and kindness. I would like to thank the speech therapists in the Royal Leamington Spa Rehabilitation Hospital, Marta Domingues and Michel Santos for all the help they gave me.

Being a member of *Filhos da Fruta* made the last years awesome and for this I dedicate this thesis to the other members, my brothers José and David Dias.

Declarations

I hereby declare that the material in this thesis is my own work and has not been previously submitted to this or any other institution for any degree, diploma or other qualification.

Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature.

Abstract

The main goal of this thesis is the description of the section ring of a surface $R(S,\mathcal{L}) = \bigoplus_{n=0}^{\infty} H^0(S, n\mathcal{L})$ where $\mathcal{L}$ is an ample base point free divisor defining a covering map $\varphi_{\mathcal{L}}\colon S \to \mathbb{P}^2$ such that $\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$. This is an abelian surface with a polarization of type $(1,3)$ which was studied before in [BL94, Cas99, Cas12].

Given a covering map $\varphi\colon X \to Y$, following the methods introduced by Miranda for general d covers, in chapter 3 we will define a *cover homomorphism* that will induce a commutative and associative multiplication in $\varphi_*\mathcal{O}_X$.

Chapter 4 focuses in the $\mathcal{O}_{\mathbb{P}^2}$-modules $\operatorname{Hom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$ that will be used to define a commutative multiplication for our surface. Chapter 5 is about the associative condition. It is a computational method based on the paper [Rei90].

In the last chapter we use the ring $R(S,\mathcal{L})$ to prove that the moduli space of abelian surfaces with a polarization of type $(1,3)$ and canonical level structure is rational. We will also show how to use the same method to find models for covering maps such that $\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m_1) \oplus \Omega^1_{\mathbb{P}^2}(-m_2) \oplus \mathcal{O}_{\mathbb{P}^2}(-m_1 - m_2 - 3)$.

The last section contains new problems whose goal is to construct and study algebraic varieties given by the vanishing of a high codimensional Gorenstein ideal.

Chapter 1

Introduction

Given a smooth complex projective variety X, its *irregularity* is $q(X) = h^0(\Omega^1_X) = h^1(\mathcal{O}_X)$, and X is said to be *irregular* if $q(X) > 0$. Explicitly describing the equations of an irregular variety has always been considered a challenge, mainly because the *section ring* of $\mathcal{L}$

$$R(X, \mathcal{L}) = \bigoplus_{n=0}^{\infty} H^0(X, n\mathcal{L})$$

where $\mathcal{L}$ is an ample base point free divisor, is not Cohen–Macaulay. Our main goal is describing the section ring of an irregular surfaces family.

Problem 1.0.1. Describe the section ring $R(S, \mathcal{L})$ where S is a surface and $\mathcal{L}$ an ample base point free divisor that defines a morphism $\varphi_{\mathcal{L}} \colon S \to \mathbb{P}^2$ such that

$$\varphi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m_1) \oplus \Omega^1_{\mathbb{P}^2}(-m_2) \oplus \mathcal{O}_{\mathbb{P}^2}(-m_1 - m_2 - 3) \tag{1.1}$$

for non negative integers m_1, m_2.

Assume that $m_1 \leq m_2$. Then the surfaces are only irregular if $m_1 = 0$ but the method we will present works for a general pair (m_1, m_2). The case we will give more emphasis to is $m_1 = m_2 = 0$ which is an abelian surface with a polarization of type $(1, 3)$.

Birkenhake and Lange in [BL94] considered a one dimensional family of $(1, 3)$-abelian surfaces, $S = E \times E$ and $\mathcal{L} = \mathcal{O}_X(E \times \{0\} + \{0\} \times E + A)$, E an elliptic curve E, A the antidiagonal in S. Casnati described a two dimensional family of such surfaces in [Cas99], the surfaces for which φ can be decomposed as $S \xrightarrow{\rho} S' \xrightarrow{\varsigma} \mathbb{P}^2$ where ρ and ς are a double and triple cover respectively.

The dimension of the moduli space of abelian surfaces has dimension three. By describing the section ring of abelian surfaces with a polarization of type $(1, 3)$

we will find the "third dimension" missing from the papers above. Furthermore we will prove that the moduli space of $(1, 3)$-polarized abelian surfaces with canonical level structure is rational.

Our method to describe the section ring of a covering map is strongly based on the methods developed in the last years to construct the section ring of regular varieties. [BK$^+$09] is a database of section rings for these that includes classification of toric varieties, polarized K3 surfaces, Fano 3-folds and 4-folds, (see [ABR02] for an introduction to the methods used).

Given a variety X over a field k, the constructions in the database start with the use of *Hilbert Series* method to describe an ambient space $\mathbb{P}(a_0, \ldots, a_n)$, $a_i \geq 1$, and a free resolution of the ideal defining an embedding $X \hookrightarrow \mathbb{P}(a_i)$. Such method does not work for irregular varieties as the section ring is not Cohen–Macaulay.

Nonetheless, we mimic the method as we start by describing the ambient space. This will be a relative affine space over the projective space that we will define in chapter 2 - *Preliminaries.* Having an embedding $X \hookrightarrow \mathbb{A}$, we will study the ideal $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{A}}$ and, as X is a finite cover over $\mathbb{P}^n$, the ideal is locally Cohen–Macaulay. Notice that this does not contradict the failure of the Cohen–Macaulay condition for the section ring of an $(1, 3)$-abelian surface as the non CM point is the vertex of the affine cone over $\mathbb{P}^2$.

Studying this local structure will be the connection with the methods for regular varieties. In particular, we will determine local generators for the initial ideal $\mathrm{in}(\mathcal{I}_X)$ and study under which conditions its resolution lifts to one for $\mathcal{I}_X$ following a method of Reid [Rei90], i.e. a deformation–theoretic / infinitesimal view–point that we get by considering $\mathrm{in}(\mathcal{I}_X)$ an hyperplane section in a larger ambient variety. The theory will be explained with detail in chapter 3 - *Algebraic Covers* and the computations for the abelian surface carried out in chapter 5 - *Associative Multiplication* (with the computer code associated with it shown in Appendix).

Given $\varphi_*\mathcal{O}_X$ and an embedding $X \hookrightarrow \mathbb{A}(\mathcal{F})$ for a covering map the initial ideal $\mathrm{in}(\mathcal{I}_X)$ might not be unique. As an example take a general regular surfaces S_d such that $h^0(S_d) = 3, K_{S_d}^2 = d$. Then the initial ideal will be given by a Gorenstein ideal with codimension $d - 2$ and there is no structure theorem for such ideals if $d \geq 6$. This is where irregularity comes as an advantage.

In section 3.3 - *Gorenstein Covers* we will prove that the term $\mathcal{O}_{\mathbb{P}^2}(-3)$ in the equation describing $\varphi_*\mathcal{O}_S$ as an $\mathcal{O}_{\mathbb{P}^2}$-module for the abelian surface appears as a direct summand of $S^2\left(\Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}\right)$ (equation (1.1) for $m_1 = m_2 = 0$).

Chapter 4 - *Commutative Multiplication*, will be fundamentally about the $\mathcal{O}_{\mathbb{P}^2}$-modules $\mathrm{Hom}\left(S^2\Omega^1_{\mathbb{P}^2}, -\right)$. There we prove that $\mathrm{Hom}\left(S^2\Omega^1_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(-3)\right) = 0$

which implies $\text{in}(\mathcal{I}_X) \cong \text{Hom}\left((S^2\Omega^1_{\mathbb{P}^2})^{\oplus 3}, \mathcal{O}_{\mathbb{P}^2}\right)$ for $(1,3)$-polarized abelian surface. Moreover, in this chapter we also determine the structure of the morphisms in $\text{Hom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$ which will provide the gluing of the local structures. To make it clear, take into account that all the results about extensions of the ideal $\text{in}(\mathcal{I}_X)$ we will arrive at will be independent of a choice of basis and, for any open sets $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{P}^2$, the transition morphisms

$$\text{Hom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)\big|_{\mathcal{U}_1} \mapsto \text{Hom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)\big|_{\mathcal{U}_2}$$

are given by a $\mathcal{O}_{\mathbb{P}^2}$-linear change of basis.

The last chapter of the thesis, chapter 6 - *Constructing varieties as algebraic covers of the projective plane*, will contain models that are solutions to Problem 1.0.1 for general pairs (m_1, m_2). For (m_1, m_2) equal to $(0,0)$ and $(0,1)$ we can completely determine the section ring, being the second an irregular surface S with invariants $p_g(S) = 3, K_S^2 = 6, q(S) = 1$.

Covering Maps

The constructions described above are obtained via study of algebraic covering maps over the projective space. A surjective morphism between schemes is called a *covering map* if it is finite and flat.

In section 2.2.1 - *Trace free basis* will start with the notion of *trace free* module introduced in the celebrated paper of Miranda - *Triple Covers in Algebraic Geometry*, [Mir85]. For a covering map $\varphi\colon X \to Y$, the trace free lemma states that $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$ where $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module called the trace free module. Picking $(z_i)_{1\leq i\leq d-1}$ a local basis for $\mathcal{E}$, the (local) equations defining the covering map can be written as

$$\forall_{i\leq j}, z_i z_j = \sum_k c_{ijk} z_k + d_{ij}$$

with $c_{ijk}, d_{ij} \in \mathcal{O}_Y$ such that this multiplication is associative. Assuming Y is integral, by Nakayama lemma this is the structure of a fibre, i.e. the equations of d points in $\mathbb{A}^{d-1} = \text{Spec}\left(k[z_1, \ldots, z_{d-1}]\right)$.

Besides the trace free conditions, Miranda found that for (local) triple covers we can pick any c_{ijk} and write the d_{ij} as quadratic forms in the c_{ij}. Miranda and Hahn proved that the c_{ijk} defining a quadruple cover have to satisfy the equations defining the Plücker embedding of $\text{Gr}(2,6)$ in $\mathbb{P}^{14}$ and the d_{ij} can be written as quadratic forms in the c_{ijk}.

In section 3.4 we present our point of view on trace free basis. We show that for $\mathcal{Q} = \{q_1, \ldots, q_d\}$ a set of d points in $\mathrm{Spec}(k[z_i])$, the trace free condition can be restated as having $\sum_j z_i(q_j) = 0$. This brings a relation between the equations defining $\mathcal{Q}$ as a subset in $\mathbb{A}^{d-1}$ and the equations defining $\overline{\mathcal{Q}}$ as a subset of $\mathbb{P}^{d-2}$ obtained by projection from the origin.

In particular, the projection of four points from $\mathbb{A}^3$ to $\mathbb{P}^2$ is given by the vanishing of two quadratic polynomials in $k[z_1, z_2, z_3]$. As $\dim\left(k[z_1, z_2, z_3]^{(2)}\right) = 6$, these two polynomials can be considered a point in the Grassmannian $\mathrm{Gr}(2, 6)$.

We have done the change of affine equations to projective ones only for triple covers and using computer. Four quadruple covers such relation is too heavy to get it via computer so we leave it as an open problem to explored in the future. Furthermore, for the family of surfaces in Problem 1.0.1 we get a fibre defined by nine equations

$$q_i = \sum_{j=1}^{4} c_{ij} z_j + d_i$$

for $1 \leq i \leq 9$, q_i quadratic equations in the z_i and $c_{ij}, d_i \in k$. The d_i can be written as quadratic forms in the c_{ij} which satisfy the equations of the spinor embedding of the affine orthogonal Grassmannian $\mathrm{aOGr}(5, 10)$. Unfortunately we can see no reason for this relation and leave it as an open problem as well.

High Codimension Gorenstein Ideals

One of the major difficulties behind describing the section ring of a smooth variety is the lack of structure theorems for Gorenstein ideals with codimension bigger or equal to four. Miles Reid describes the projective resolution of these in [Rei15]. Unfortunately it still is far from bringing an explicit description of such ideals. The $(1, 3)$-polarized abelian surface section ring, which is (locally) given by a codimension four Gorenstein ideal, is obtained in a natural way and is described by three parameters.

This should be a common feature of the section ring for $(1, d)$-polarized abelian surfaces as their moduli space is three dimensional. On the other hand the codimension of the ideal defining each section ring will grow with d which makes the study of these section rings a good starting point to find and study high codimension Gorenstein ideals.

As we only work with an abelian surface which is a cover of the $\mathbb{P}^2$, we leave throughout the thesis open problems related with the study of irregular varieties which are covers of $\mathbb{P}^n$. Problem 2.1.13 relates with the method of unprojection initiated by Miles Reid ([Pap04, PR04]) that allows to construct higher codimension

Gorenstein ideals from existing ones. In section 6.3 - *Future Work*, we will leave as open problems the construction of the section ring for irregular varieties which can be achieved using similar methods to the ones used to solve the main problem in this thesis.

As said in the beginning, the goal of this thesis is the description of a ring so it is strongly commutative algebra flavored and we assume the reader familiar with the standard results from the field. When needed we will recall results as stated in [Eis95, Mat80, BH93]. In section 2.2.3 we will go over the results about abelian surfaces with a polarization of type $(1, 3)$ we will need from [BL04, BL94, Cas99] and recall some definitions from [GP98, GP01] concerning their moduli space.

The computations in the Appendix are done via Sage or Magma.

Notation 1.0.2.

1. k will denote an algebraically closed field, $\mathrm{char}(k) = 0$,
2. X, Y for Noetherian schemes, separated of finite type over k,
3. For a covering map $\varphi\colon X \to Y$, Y is always considered integral.

Chapter 2

Preliminaries

In this section we present the tools and background material that will be used throughout the thesis. We start by presenting one example of the *Hilbert Series method* for constructing the canonical ring of a regular variety.

First Problem. *Let S be a surface with ample base point free canonical divisor K_S such that $K_S^2 = 5, p_g = 3, q = 0$. Construct its canonical ring.*

With a simple application of Riemann-Roch one can compute its Hilbert series

$$\mathrm{HS}_S(t) = \frac{1 - 5t^4 + 5t^6 - t^{10}}{(1-t)^3(1-t^2)^3}$$

which suggests the weighted projective space $\mathbb{P}(1^3, 2^3) = \mathrm{Proj}(k[x_i, y_i])$, $0 \le i \le 2$, $\deg(x_i) = 1$, $\deg(y_i) = 2$, as an ambient space from the denominator. The numerator suggests the resolution

$$R(S, K_S) \leftarrow T \leftarrow T(-4)^{\oplus 5} \xleftarrow{M} T(-6)^{\oplus 5} \leftarrow T(-10) \leftarrow 0,$$

$T = k[x_i, y_i]$, as a way of constructing examples. As it is symmetric, the Buchsbaum–Eisenbud symmetrizer trick allows you to change bases in the two modules to make M a skew 5×5 matrix with entries of weight 2 in S. Then the ideal is given by the 4×4 Pfaffians of M.

This is one idea that we will use for the definition of the ambient space and to describe the structure of the fibres. For the ambient space, given an $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$ we will use a set of generators and the relations between them to define $\mathbb{A}(\mathcal{F})$. To study the equations defining a fibre, i.e. the relations between the coefficients involved in them, we will use an extension method based on the resolution of the ideal. As for *First Problem* the first sygyzy matrix plays a fundamental role.

Notice that K_S is an ample base point free divisor and $h^0(S, K_S) = 3$ so K_S determines a morphism $\varphi\colon S \to \mathbb{P}^2$. Furthermore, assuming that M is general enough, as a $k[x_0, x_1, x_2]$ vector space, $R(S, K_S)$ is generated by $\langle 1, y_0, y_1, y_2, q\rangle$, where q is a quadratic form in the variables y_i. We get then the following equality

$$\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-4).$$

This brings the definition that will be our main object of study.

Definition 2.0.1. Let X, Y be schemes over a field k, Y irreducible and integral. A covering map of degree d is a flat and finite morphism $\varphi\colon X \to Y$, such that the locally free $\mathcal{O}_Y$-module $\varphi_*\mathcal{O}_X$ has rank d.

A covering map $\varphi\colon X \to Y$ is called a *Gorenstein covering map* if all its fibres are Gorenstein. In section 3.3 we will recall the main results presented in [CE96] about them by Casnati and Ekedahl. Gorenstein covering maps will be of importance for us as our main problem is the description of one.

We recall now a family of surfaces studied by Casnati which includes the family of surfaces from *First Problem.*

Proposition 2.0.2. *[Cas96, Proposition 6.1] Let X_d be the canonical model of a minimal and regular surface S_d of general type with $p_g(S_d) = 3, K^2_{S_d} = d$ and assume that $|K_{S_d}|$ is base point free and not composed with a pencil. Then the canonical map of S_d induces a Gorenstein cover $\varphi\colon X_d \to \mathbb{P}^2$ of degree d such that*

$$\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus d-2} \oplus \mathcal{O}_{\mathbb{P}^2}(-4).$$

Notice that for $d = 5$ it is the surface we described before. For $d \in \{3, \dots, 9\}$, Casnati shows they can be constructed as bidouble covers of a Del Pezzo surface $Y_d \in \mathbb{P}^d$ of degree d.

To explicitly describe the equations defining the canonical ring $R(S_d, K_{S_d})$ we need to describe a Gorenstein ideal of codimension $d-2$ in the ring $T_d = k[x_i, y_j]$, $\deg(x_i) = 1$, $\deg(y_i) = 2$, $0 \le i \le 2, 0 \le j \le d-3$. For $d = 5$ the ideal has codimension three, the highest codimension for which there exists a structure theorem for Gorenstein ideals. For codimension one and two we have complete intersections and for codimension three we have Buschbaum–Eisenbud theorem, that we used in the solution above. Casnati also goes over the explicit construction for this case but for $d - 2 \ge 4$ this is not possible anymore.

2.1 Ambient Space

2.1.1 Definition of $\mathbb{P}(\mathcal{F})$

This section is based on the notes about scrolls by Miles Reid, [Rei97]. For a locally free $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$ of rank d, fix a free resolution

$$\mathcal{F} \xleftarrow{\bar{\xi}} \mathcal{P}_0 \xleftarrow{M} \mathcal{P}_1 \leftarrow \cdots \leftarrow \mathcal{P}_m \leftarrow 0 \tag{2.1}$$

where $\mathcal{P}_i = \bigoplus_{j=0}^{k_i} \mathcal{O}_{\mathbb{P}^n}(-a_{ij})$, $\bar{\xi} = (\xi_0, \dots, \xi_{k_0})$ and M is a $(k_0+1)\times(k_1+1)$ matrix with entries in $k[x_0, \dots, x_n]$. Assume that for all i, j, $a_{i,j} \geq 0$ and for simplicity sake denote k_0 by k and a_{0j} by a_j.

Denoting by $(x_0, \dots, x_n)$, $(\xi_0, \dots, \xi_k)$ and (λ, μ) a choice of coordinates for $\mathbb{A}^{n+1}$, $\mathbb{A}^{k+1}$ and $\mathbb{G}_m^2$, we define $\mathbb{P}(\mathcal{F})$ as the quotient of $\mathcal{Z}_M$ by an action of $\mathbb{G}_m^2$,

$$\mathcal{Z}_M := \{\bar{\xi}M = 0\} \subset (\mathbb{A}^{n+1}\backslash 0) \times (\mathbb{A}^{k+1}\backslash 0)\ ,\ \bar{\xi} = (\xi_0, \dots, \xi_k)$$

and the action of $(\lambda, \mu) \in \mathbb{G}_m^2$ is given by

$$(\lambda, \mu) \colon (x_0, \dots, x_n; \xi_0, \dots, \xi_k) \mapsto (\lambda x_0, \dots, \lambda x_n; \mu\lambda^{a_0}\xi_0, \dots, \mu\lambda^{a_k}\xi_k) \tag{2.2}$$

The action is well defined as all polynomials in $\bar{\xi}M$ are homogeneous for μ and λ. Furthermore, the ratio $(x_0 : x_1 : \cdots : x_n)$ is preserved by the action of $\mathbb{G}_m^2$ so the projection to the first factor defines a morphism $\pi \colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}^n$.

$$\begin{array}{ccc} (\mathbb{A}^{n+1}\backslash 0) \times (\mathbb{A}^{k+1}\backslash 0) & \longrightarrow & \mathbb{P}(\mathcal{F}) \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ (\mathbb{A}^{n+1}\backslash 0) & \longrightarrow & \mathbb{P}^n \end{array}$$

Definition 2.1.1. Given a locally free $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$ of rank d, and $\mathcal{P}_\bullet$ a free resolution starting with the terms

$$\mathcal{F} \xleftarrow{(\xi_0,\dots,\xi_k)} \bigoplus_{i=0}^{k} \mathcal{O}_{\mathbb{P}^n}(-a_i) \xleftarrow{M} \bigoplus_{i=0}^{k_1} \mathcal{O}_{\mathbb{P}^n}(-a_{1i}) \leftarrow \dots$$

we denote by $R(\mathcal{P}_\bullet)$ the ring $k[x_0, \dots, x_n; \xi_0, \dots, \xi_k]/(\bar{\xi}M)$ and denote the scheme $\operatorname{Proj}(R(\mathcal{P}_\bullet))$ as

$$\pi \colon \mathbb{P}(\mathcal{F}) \to \mathbb{P}^n.$$

Notice that we defined $\mathbb{P}(\mathcal{F})$ for a locally free sheaf $\mathcal{F}$ but for the construction we use a free resolution which is not unique as we see in the next example.

Example 2.1.2. Let $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a)$, $a \geq 0$. As is it a free sheaf,

$$\mathbb{P}(\mathcal{F}) = \operatorname{Proj}(k[x_0, x_1; \xi_0, \xi_1])$$

where $\deg x_i = (1,0)$, $\deg \xi_0 = (0,1)$, $\deg \xi_1 = (a,1)$. On the other hand we can write the following exact sequence

$$\mathcal{F} \xleftarrow{d_0} \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-a-1)^{\oplus 2} \xleftarrow{d_1} \mathcal{O}_{\mathbb{P}^1}(-a-2) \leftarrow 0$$

where $d_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x_0 & x_1 \end{bmatrix}$, $d_1 = \begin{bmatrix} 0 \\ x_1 \\ -x_0 \end{bmatrix}$ which gives us

$$\mathbb{P}(\mathcal{F}) = \operatorname{Proj}\left(k\left[x_0, x_1; \xi'_0, \xi'_1, \xi'_2\right] / \left(x_1\xi'_1 - x_0\xi'_2\right)\right)$$

with $\deg x_i = (1,0)$, $\deg \xi'_0 = (0,1)$ and $\deg \xi'_1 = \deg \xi_2 = (a+1, 1)$.

Notice that we have a homomorphism

$$h\colon k[x_0, x_1; \xi'_0, \xi'_1, \xi'_2]/(x_1\xi'_1 - x_0\xi'_2)) \to k[x_0, x_1; \xi_0, \xi_1]$$

where $h(x_i) = x_i$, $h(\xi'_0) = \xi_0$, $h(\xi'_1) = x_0\xi_1$ and $h(\xi'_2) = x_1\xi_1$. This homomorphism shows that $\mathbb{P}(\mathcal{F})$ is well defined as the constructions are isomorphic as schemes (any homogeneous ratio in $k[x_0, x_1; \xi_0, \xi_1]$ has an inverse image, $\xi_1 \mapsto \xi'_1/x_0$ or ξ'_2/x_1, over the open sets in $\mathbb{P}^1$, $\mathcal{U}_0 = \{x_0 \neq 0\}$ and $\mathcal{U}_1 = \{x_0 \neq 0\}$, respectively).

We wish to prove that as in the example above $\mathbb{P}(\mathcal{F})$ is independent of the choice of free resolution for $\mathcal{F}$. To do so, given $\mathcal{P}_\bullet$ a free resolution of an $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$, denote by $\mathbb{P}(\mathcal{P}_\bullet)$ the scheme $\operatorname{Proj}(R(\mathcal{P}_\bullet))$ (as in definition 2.1.1).

Theorem 2.1.3. *Let $\mathcal{P}_\bullet$ and $\mathcal{Q}_\bullet$ be two free resolutions of the locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank d, $\mathcal{F}$. Then, as schemes over $\mathbb{P}^n$,*

$$\left(\pi\colon \mathbb{P}(\mathcal{P}_\bullet) \to \mathbb{P}^n\right) \cong \left(\pi'\colon \mathbb{P}(\mathcal{Q}_\bullet) \to \mathbb{P}^n\right).$$

Proof. Assume $\mathcal{P}_\bullet$ is the minimal free resolution of $\mathcal{F}$. Then we have a morphism of chain complexes,

$$\begin{array}{ccccccc}
\mathcal{F} & \xleftarrow{\bar{\xi}'} & \mathcal{Q}_0 & \xleftarrow{M_{\mathcal{Q}}} & \mathcal{Q}_1 & \longleftarrow & \cdots \\
\| & & \downarrow f_0 & & \downarrow f_1 & & \\
\mathcal{F} & \xleftarrow{\bar{\xi}} & \mathcal{P}_0 & \xleftarrow{M_{\mathcal{Q}}} & \mathcal{P}_1 & \longleftarrow & \cdots
\end{array}$$

that induces a morphism between the bi-graded rings $R(\mathcal{Q}_\bullet)$ and $R(\mathcal{P}_\bullet)$,

$$\begin{array}{rccc}
f\colon & R(\mathcal{Q}_\bullet) & \longrightarrow & R(\mathcal{P}_\bullet) \\
& x_i & \mapsto & x_i \\
& \xi'_i & \mapsto & f_0(\xi'_i)
\end{array}$$

which gives us a morphism $\mathbb{P}(\mathcal{P}_\bullet) \to \mathbb{P}(\mathcal{Q}_\bullet)$.

As $\mathcal{F}$ is locally free, take $\mathcal{U} \subset \mathbb{P}^n$ an open set such that

$$\mathbb{P}(\mathcal{Q}_\bullet)|_{\mathcal{U}} \cong \operatorname{Proj}\left(\mathcal{O}_{\mathbb{P}^n}(\mathcal{U})[\xi'_{(0)}, \ldots, \xi'_{(d-1)}]\right).$$

Then $\bar{\xi}'|_{\mathcal{U}}$ is an isomorphism and, by the commutativity of the first square in the diagram, $f_0|_{\mathcal{U}}$ is also an isomorphism.

Then for any two resolutions of $\mathcal{F}$, $\mathcal{Q}^1_\bullet$, $\mathcal{Q}^2_\bullet$, we have isomorphisms

$$\mathbb{P}(\mathcal{Q}^1_\bullet) \xleftarrow{d_1} \mathbb{P}(\mathcal{P}_\bullet) \xrightarrow{d_2} \mathbb{P}(\mathcal{Q}^2_\bullet),$$

where $\mathcal{P}_\bullet$ is a minimal free resolution, so even without a morphism in between them, they are isomorphic through $\mathbb{P}(\mathcal{P}_\bullet)$. □

With this theorem we can indeed write $\mathbb{P}(\mathcal{F})$. Most of the time we will be using the construction with respect to the minimal free resolution of $\mathcal{F}$. When not, we will be working locally so there will be no harm done as we can see by the following proposition.

Proposition 2.1.4. *Let $R(\mathcal{P}_\bullet)$ be the ring associated with a free resolution of a locally free $\mathcal{O}_{\mathbb{P}^n}$-module $\mathcal{F}$ of rank d. Then there is a set of elements $\gamma_i \in k[x_i]$ such that for any $i \in I$, I a index set, we have*

$$R(\mathcal{P}_\bullet)_{\gamma_i} \cong k[\gamma_i^{-1}, x_0, \ldots, x_n; \xi_{(0,i)}, \xi_{(1,i)} \ldots, \xi_{(d-1,i)}]$$

where each $\xi_{(j,i)}$ is a $\mathcal{O}_{\mathbb{P}^n}$-linear combination of the ξ_j.

Proof. Just take γ_i as the element such that $\mathcal{U}_i = D(\gamma_i)$, where $\mathcal{U}_i$ is an open covering of $\mathbb{P}^n$ and $\mathcal{E}|_{\mathcal{U}_i}$ is a free $\mathcal{O}_{\mathbb{P}^n}(\mathcal{U}_i)$-sheaf.

Over the ring $k[\gamma_i^{-1}, x_0, \ldots, x_n]$ we can run Gaussian elimination in M to its echelon form with leading coefficients equal to 1. As the rank of M is $k-d$, the cokernel of M is generated by d elements that we denote by $\xi_{(j,i)}$, for $0 \le j \le d$. □

Proposition 2.1.4 shows that definition 2.1.1 agrees with the one presented in [Har77, §II.7], $\mathbb{P}(\mathcal{F}) = \operatorname{Proj}(\mathcal{F})$. For each open affine subset $\mathcal{U} = \operatorname{Spec} A$ of $\mathbb{P}^n$, we are choosing as generators for $\Gamma(\mathcal{U}, \mathcal{F}|_{\mathcal{U}})$, $(\xi_{(0,i)}, \ldots, \xi_{(d-1,i)})$, from the first term of the resolution $\mathcal{P}_\bullet$, with the glueing of the patches

$$\pi\colon \operatorname{Proj}\left(A[\xi_{(0,i)}, \ldots, \xi_{(d-1,i)}]\right) \to \mathcal{U}$$

given by the relations M.

Proposition 2.1.5. *For a locally free sheaf $\mathcal{F}$,*

1. $\mathbb{P}(\mathcal{F}) \cong \mathbb{P}(\mathcal{F}(m))$*, for any* $m \in \mathbb{Z}$*,*
2. *the divisor class group of* $\mathbb{P}(\mathcal{F})$ *is the free abelian group*

$$\operatorname{Pic} \mathbb{P}(\mathcal{F}) = \mathbb{Z}L \oplus \mathbb{Z}F$$

with $nF + mL$ *the divisor class attached to a polynomial* $p(x_i, \xi_j)$ *such that*

$$p((\lambda, \mu)(x_i, \xi_j)) = \lambda^n \mu^m p(x_i, \xi_j)$$

Proof. Item (1) comes from the geometric definition of $\mathbb{P}(\mathcal{F})$, by

$$(\lambda, \mu) \mapsto (\lambda, \mu + m\lambda)$$

and item (2) comes directly from the definition of Proj of a bi-graded ring. □

Notice that the divisor class L is the class corresponding to $\pi^{-1}(l)$ where l is a hyperplane in $\mathbb{P}^n$. If $\mathcal{F} = \varphi_* \mathcal{O}_X$ for a covering map over $\mathbb{P}^n$, then ξ_0 will be the variable associated with the term $\mathcal{O}_{\mathbb{P}^n}$, from where we have $\deg(\xi_0) = (0, 1)$ and the hyperplane $\{\xi_0 = 0\}$ is an element in the divisor class F.

2.1.2 Definition of $\mathbb{A}(\mathcal{E})$

Definition 2.1.6. Let $\mathcal{F}$ be a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank d such that $\mathcal{F} = \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{E}$. We define

$$\mathbb{A}(\mathcal{E}) = \mathbb{P}(\mathcal{F})|_{\xi_0 \neq 0} = \operatorname{Spec}\left(\bigoplus_{n=0}^{\infty} \operatorname{Sym}^n \mathcal{E}\right).$$

We will keep the notation π for the morphism $\pi\colon \mathbb{A}(\mathcal{E}) \to \mathbb{P}^n$.

Exactly as for the definition of $\mathrm{Proj}(\mathcal{F})$, in [Har77, §II.5] we can find the definition of $\mathrm{Spec}\left(\bigoplus_n S^n\mathcal{E}\right)$ and our description just shows how to chose the local generators and relations between them, i.e. given a free resolution of $\mathcal{E}$,

$$\mathcal{E} \xleftarrow{(\xi_1,\dots,\xi_k)} \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^n}(-a_i) \xleftarrow{M} \bigoplus_{i=0}^{k_1} \mathcal{O}_{\mathbb{P}^n}(-a_{1i}) \leftarrow \dots$$

$\mathbb{A}(\mathcal{E})$ can be written as

$$\mathrm{Proj}\left(k[x_0,\dots,x_n;\xi_1,\dots,\xi_k]/(\bar{\xi}M)\right)$$

where $\deg(x_i) = 1$, $\deg(\xi_j) = a_j$. We use the semi-colon just to denote that we have a morphism to $\mathbb{P}^n$ so we only consider prime ideals not containing the irrelevant ideal of $k[x_i]$.

Remark 2.1.7. We could consider a larger ambient space as $\mathbb{A}(\mathcal{E})$ is contained in the weighted projective space

$$\widetilde{\mathbb{A}(\mathcal{E})} = V\left(\bar{\xi}M\right) \subset \mathbb{P}(1^{n+1}, a_1, \dots, a_k)$$

and the polynomials defining X guarantee that if $x_i = 0$ for all i, then $\xi_j = 0$ for all j which is not a point in $\mathbb{P}(1^{n+1}, a_j)$ and although we the object is different this is always the idea we have in our minds.

We started with the definition of $\mathbb{P}(\mathcal{F})$ and not of $\mathbb{A}(\mathcal{F})$ as it allows to put together with the arguments in section 3.2. It was also a way to explain better our constructions with the description of Gorenstein covers as in [CE96], section 2.2.2.

2.1.3 Examples

The first example we want to look at is the one obtain in the construction of the section ring in *First Problem* and Casnati theorem 2.0.2.

Example 2.1.8. Let S_d be a surface such that $K(S_d) = d, p_g(S_d) = 3, q(S_s) = 0$. Then by Casnati theorem 2.0.2 the canonical map of S_d induces a Gorenstein cover of degree d $\varphi\colon X_d \to \mathbb{P}^2$ such that

$$\varphi_*\mathcal{O}_{X_d} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{d-2} \oplus \mathcal{O}_{\mathbb{P}^2}(-4).$$

As $\varphi_* \mathcal{O}_{X_d}$ is a free $\mathcal{O}_{\mathbb{P}^2}$-module we have the following bigraded ring defining $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^{d-2} \oplus \mathcal{O}_{\mathbb{P}^2}(-4)\right)$

$$R\left(\varphi_* \mathcal{O}_{X_d}\right) = k[x_0, x_1, x_2; \xi_0, \xi_1, \ldots, \xi_{d-2}, \xi_{d-1}]$$

$\deg(x_i) = (1,0), \deg(\xi_0) = (0,1), \deg(\xi_j) = (2,1)$, $\deg(\xi_{d-1}) = (4,1)$, $1 \leq j \leq d-2$. And the graded ring defining $\mathbb{A}\left(\mathcal{O}_{\mathbb{P}^2}(-2)^{d-2} \oplus \mathcal{O}_{\mathbb{P}^2}(-4)\right)$

$$R\left(\varphi_* \mathcal{O}_{X_d}\right)_{\xi_0} = k[x_0, x_1, x_2, y_1, \ldots, y_{d-2}, y_{d-1}]$$

where $y_j = \xi_j/\xi_0$, $\deg(x_i) = 1, \deg(y_k) = 2, \deg(y_{d-1}) = 4$, $1 \leq k \leq d-2$.

Recall that for the case $d = 5$ the solution we presented does not have the variable y_{d-1} of degree four. Instead, the term $\mathcal{O}_{\mathbb{P}^2}(-4)$ corresponds to a quadratic form in the variables y_i not killed by the equations defining the ideal $\mathcal{I}_{X_5} \subset k[x_i, y_j]$. As we will discuss in section 3.3 the same will apply for all d as φ is a Gorenstein covering map. So we have the embedding

$$X_d \hookrightarrow \mathbb{A}\left(\mathcal{O}_{\mathbb{P}^2}(-2)^{d-2}\right).$$

Example 2.1.9. Let $\mathcal{E} = \Omega^1_{\mathbb{P}^2}(-m)$, $m \geq 0$. Using the Euler sequence

$$\Omega^1(-m) \leftarrow \mathcal{O}_{\mathbb{P}^2}(-m-2)^{\oplus 3} \xleftarrow{M} \mathcal{O}_{\mathbb{P}^2}(-m-3) \leftarrow 0$$

$M = \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}^t$, we get

$$\mathbb{A}\left(\Omega^1_{\mathbb{P}^2}(-m)\right) = \operatorname{Proj}\left(k[x_0, x_1, x_2; y_0, y_1, y_2] / \left(\sum x_i y_i\right)\right)$$

$\deg x_i = 1$, $\deg y_i = m+2$. Let $\mathcal{U}_i = \{x_i \neq 0\}$, then over $\mathcal{U}_2$ we have

$$\mathbb{A}\left(\Omega^1_{\mathbb{P}^2}(-m)\right)\big|_{\pi^{-1}(\mathcal{U}_2)} = \operatorname{Proj}\left(k[x_0, x_1, x_2^{\pm}; y_0, y_1]\right)$$

and the transition map from $\mathcal{U}_2$ to $\mathcal{U}_1$ is given by the change of coordinates

$$\begin{pmatrix} y_0 & y_1 \end{pmatrix} \mapsto \begin{pmatrix} y_0 & -\frac{x_0}{x_2} y_0 - \frac{x_1}{x_2} y_1 \end{pmatrix} = \begin{pmatrix} y_0 & y_2 \end{pmatrix}$$

The computation of the section ring in our main problem is based on the example above. Now we will look for a relation like the one found in *First Problem*, i.e. as we had the term $\mathcal{O}_{\mathbb{P}^2}(-4)$ a quadratic term in the three generators $\mathcal{O}_{\mathbb{P}^2}(-2)$, how can we write $\mathcal{O}_{\mathbb{P}^2}(-m_1 - m_2 - 3)$ as a summand of $S^2\left(\Omega^1_{\mathbb{P}^2}(-m_1) \oplus \Omega^1_{\mathbb{P}^2}(-m_2)\right)$.

Example 2.1.10. In this example we want to see what the following inclusion

$$\mathcal{O}_{\mathbb{P}^2}(-3) \subset \bigoplus_{n \geq 0} \mathrm{Sym}^n \left(\Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}\right)$$

implies in terms of the graded ring defining $\mathbb{A}\left(\Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}\right)$. Recall that the inclusion above is expected

$$\begin{array}{rcl} S^2\left(\Omega^1_{\mathbb{P}^2}\right)^{\oplus 2} & = & S^2\left(\Omega^1_{\mathbb{P}^2}\right) \oplus \Omega^1_{\mathbb{P}^2} \otimes \Omega^1_{\mathbb{P}^2} \oplus S^2\left(\Omega^1_{\mathbb{P}^2}\right)^{\oplus 2} \\ & = & \left(S^2\left(\Omega^1_{\mathbb{P}^2}\right)\right)^{\oplus 3} \oplus \left(\Omega^1_{\mathbb{P}^2} \bigwedge \Omega^1_{\mathbb{P}^2}\right) \\ & = & \left(S^2\left(\Omega^1_{\mathbb{P}^2}\right)\right)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-3). \end{array}$$

As for the example 2.1.9 we have the following equality

$$\mathbb{A}\left(\Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}\right) = \mathrm{Proj}\left(k[x_i; y_i, z_i] / \left(\sum x_i y_i, \sum x_i z_i\right)\right)$$

where $0 \leq i \leq 2$, $\deg(x_i) = 1$, $\deg(y_i) = \deg(z_i) = 2$. The interesting fact is that the summand $\mathcal{O}_{\mathbb{P}^2}(-3)$ should correspond to a variable of degree three and at the same time to the multiplication of variables of degree two.

The first explanation for this came from using unprojection methods (see [Pap04, PR04]), i.e. the unprojection of the following pair of ideals

$$\left(\sum x_i y_i, \sum x_i z_i\right) \subset (x_i).$$

This example is described in [Pap04, §4] but for completeness sake we will recall the result. From the equations

$$\begin{pmatrix} y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} = 0$$

we know that the first matrix has rank 1 and by Cramer's rule also the following vector is in its kernel

$$\bigwedge^2 \begin{pmatrix} y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{pmatrix} = \begin{pmatrix} y_1 z_2 - y_2 z_1 \\ y_2 z_0 - y_0 z_2 \\ y_0 z_1 - y_1 z_0 \end{pmatrix}$$

from where we conclude that there exists an element t such that

$$\bigwedge^2 \begin{pmatrix} y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{pmatrix} = t \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$$

$\deg(t) = 3$ which is the quadratic term corresponding to the summand $\mathcal{O}_{\mathbb{P}^2}(-3)$.

A different reasoning comes from applying theorem 2.1.3 to the following free resolutions of $\mathcal{O}_{\mathbb{P}^2}(-3)$

$$\begin{array}{ccccccc} \Omega^1_{\mathbb{P}^2} \bigwedge \Omega^1_{\mathbb{P}^2} & \longleftarrow & \mathcal{O}_{\mathbb{P}^2}(-4)^{\oplus 3} & \longleftarrow & \mathcal{O}_{\mathbb{P}^2}(-5)^{\oplus 3} & \longleftarrow & \cdots \\ \| & & \downarrow f_0 & & & & \\ \mathcal{O}_{\mathbb{P}^n}(-3) & \longleftarrow & \mathcal{O}_{\mathbb{P}^n}(-3) & \longleftarrow & 0 & & \end{array}$$

The map f_0 is exactly the map $\begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}$.

Besides being an important part in the description of the section ring of our abelian surface, the example above shows a relation between the unprojection methods and the description of a ring $R(\mathcal{F})$ for $\mathcal{F}$ a locally free $\mathcal{O}_{\mathbb{P}^n}$-module.

Based on this example, we believe that the study of irregular covers $\mathbb{P}^n$ might bring new types of unprojection and in this direction we finish with the following examples.

Example 2.1.11. Let $\mathcal{E} = \Omega^1_{\mathbb{P}^3}(-m)$, $m \geq 0$. Using that $\Omega^1_{\mathbb{P}^3} \cong \mathrm{T}^2_{\mathbb{P}^3}(-4)$ we have the resolution

$$\Omega^1(-m) \leftarrow \mathcal{O}_{\mathbb{P}^3}(-m-2)^{\oplus 6} \stackrel{M}{\longleftarrow} \mathcal{O}_{\mathbb{P}^3}(-m-3)^{\oplus 4} \leftarrow \mathcal{O}_{\mathbb{P}^3}(-m-4) \leftarrow 0.$$

We get then 6 variables, y_{ij}, $0 \leq i < j \leq 3$, with $\deg y_{ij} = m + 2$, satisfying

$$\begin{pmatrix} 0 & y_{01} & y_{02} & y_{03} \\ & 0 & y_{12} & y_{13} \\ & & 0 & y_{23} \\ -\,\mathrm{sym} & & & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

so we can write

$$\mathbb{A}\left(\Omega^1_{\mathbb{P}^3}\right) = \operatorname{Proj}\left(k[x_0, \ldots, x_3; y_{ij}]/(\text{equations above})\right).$$

In the description above we are not using the matrix M but the isomorphism

$\Omega^1_{\mathbb{P}^3} \cong \mathrm{T}^2_{\mathbb{P}^3}(-4)$ and the Koszul complex of a complete intersection (see [Eis95, §17] for further reading).

Over the open set $\mathcal{U}_3 = \{x_3 \neq 0\}$ we have

$$\mathbb{A}\left(\Omega^1_{\mathbb{P}^3}\right)\big|_{\pi^{-1}(\mathcal{U}_3)} = \mathrm{Proj}\left(k[x_0, x_1, x_2, x_3^{\pm}; y_{01}, y_{02}, y_{12}]\right)$$

with transition morphism to $\mathcal{U}_2 = \{x_2 \neq 0\}$ given by

$$\begin{pmatrix} y_{01} \\ y_{02} \\ y_{12} \end{pmatrix} \mapsto \begin{pmatrix} y_{01} \\ -\frac{x_1}{x_3} y_{01} - \frac{x_2}{x_3} y_{02} \\ \frac{x_0}{x_3} y_{01} - \frac{x_2}{x_3} y_{12} \end{pmatrix} = \begin{pmatrix} y_{01} \\ y_{03} \\ y_{13} \end{pmatrix}.$$

Example 2.1.12. Let $\mathcal{E} = \Omega^2_{\mathbb{P}^3}(-m)$, $m \geq 0$. From $\Omega^2_{\mathbb{P}^3} \cong \mathrm{T}^1_{\mathbb{P}^3}(-4)$ we can use the Euler sequence

$$\Omega^2_{\mathbb{P}^3}(-m) \leftarrow \mathcal{O}_{\mathbb{P}^3}(-m-3)^{\oplus 4} \xleftarrow{M} \mathcal{O}_{\mathbb{P}^3}(-m-4) \leftarrow 0$$

to get the equality

$$\mathbb{A}\left(\Omega^2_{\mathbb{P}^3}(-m)\right) = \mathrm{Proj}\left(k[x_i; y_i] / \left(\sum x_i y_i\right)\right)$$

$0 \leq i \leq 3$, $\deg(x_i) = 1$, $\deg(y_i) = m + 3$.

Although the case $\Omega^2_{\mathbb{P}^3}$ seems less complex than $\Omega^1_{\mathbb{P}^3}$ as it as fewer equations, it is the opposite. More conditions bring more obstructions which in the end makes everything easier to compute as we will see in chapter 4.

Problem 2.1.13. Let $X = \mathbb{A}\left(\Omega^1_{\mathbb{P}^3} \oplus \Omega^1_{\mathbb{P}^3} \oplus \Omega^1_{\mathbb{P}^3}\right)$. Determine the equations defining the following inclusions

$$\begin{aligned} \mathbb{A}\left(\Omega^2_{\mathbb{P}^3}\right) &\subset X \\ \mathbb{A}\left(\mathcal{O}_{\mathbb{P}^3}(-4)\right) &\subset X. \end{aligned}$$

The solution of the problem above involves again the study of an ideal derived from an inclusion $I \subset J = (x_i)$, $\mathrm{codim}(J) = 4$. It suggests again some type of unprojection method but not necessarily the Kustin–Miller type as the result is not globally Gorenstein (about different types of unprojection [Pap06a, Pap06b, Pap07]).

2.2 Background

2.2.1 Trace free basis

Lemma 2.2.1. *[HM99, 2.2] Let $\varphi\colon X \to Y$ be a covering map. Then $\varphi_* \mathcal{O}_X$ splits* *as $\mathcal{O}_Y \oplus \mathcal{E}$, where $\mathcal{E}$ is the sub-module of trace-zero elements.*

Proof. For an open set $\mathcal{U} \subset Y$, the multiplication by $x \in \mathcal{O}_X\left(\varphi^{-1}(\mathcal{U})\right)$ defines a morphism

$$\varphi_*\mathcal{O}_X(\mathcal{U}) \xrightarrow{x} \varphi_*\mathcal{O}_X(\mathcal{U}).$$

After a choice of basis for $\varphi_*\mathcal{O}_X(\mathcal{U})$, this morphism is represented by a matrix, M_x, with entries in $\mathcal{O}_Y(\mathcal{U})$. If $\operatorname{char} k \nmid d$, we can define the map $\frac{1}{d}\operatorname{tr}\colon \varphi_*\mathcal{O}_X(\mathcal{U}) \to \mathcal{O}_Y(\mathcal{U})$ by $x \mapsto \frac{1}{d}\operatorname{tr}(M_x)$. This map is well defined as the trace of a matrix is unchanged by a change of basis and so we get the following short exact sequence,

$$0 \to \mathcal{E} \to \varphi_*\mathcal{O}_X \xrightarrow{\frac{1}{d}tr} \mathcal{O}_Y \to 0.$$

This sequence splits as for $y \in \mathcal{U}$ the matrix is $\operatorname{diag}(y, y, \ldots, y)$ and so $\frac{1}{d}\operatorname{tr}(y) = y$ and $\frac{1}{d}\operatorname{tr} \circ \varphi^* = \operatorname{id}$. ☐

The sheaf $\mathcal{E}$ is called the *Tschirnhausen*, or trace-free, module of the covering map φ. Let $(z_1, \ldots, z_{d-1})$ be a local basis for $\mathcal{E}$. Then lemma 2.2.1 allows us to write the local equations defining a multiplication in $\varphi_*\mathcal{O}_X$ as

$$z_i z_j = \sum_{k=1}^{d-1} c_{ijk} z_k + d_{ij}$$

$i \leq j$, $c_{ijk}, d_{ij} \in \mathcal{O}_Y$, and $\operatorname{tr}(y_i) = \sum_{j=1}^{d-1} c_{ijj} = 0$ for all i.

Such multiplication needs to be commutative and associative. The commutative condition is directly satisfied as it is, but the associative condition

$$(z_i z_j) z_k = z_i (z_j z_k)$$

will bring the relations between the coefficients c_{ijk}, d_{ij} that we need to detect.

From [Mir85, Lemma 2.4, Lemma 2.6], for triple covers these relations are

$$\begin{aligned} z_1^2 &= c_1 z_1 + c_0 z_2 &&+ 2(c_1^2 - c_0 c_2) \\ z_1 z_2 &= -c_2 z_1 - c_1 z_2 &&- (c_0 c_3 - c_1 c_2) \\ z_2^2 &= c_3 z_1 + c_2 z_2 &&+ 2(c_2^2 - c_1 c_3) \end{aligned} \tag{2.3}$$

where $c_i \in \mathcal{O}_Y(\mathcal{U})$[1]. Such multiplication is just a form of writing a morphism $\Phi \in \operatorname{Hom}(S^2\mathcal{E}, \mathcal{O}_Y \oplus \mathcal{E})$ that we can decompose as $\Phi_1 \oplus \Phi_2$, Φ_1 the map to $\mathcal{O}_Y$

[1]The notation z_i, c_j was not the one used by Miranda but it gives index homogeneity that is easier to remember.

and Φ_2 the map to $\mathcal{E}$. The structure above shows that Φ_1 is locally determined by Φ_2 and furthermore, as Y is irreducible and the transition functions are given by $\mathcal{O}_Y$-automorphisms (that keep a basis trace-free), Φ_1 is determined by Φ_2 globally.

Theorem 2.2.2. *[Mir85, Theorem 1.1] A triple cover of Y is determined by a locally free rank 2 $\mathcal{O}_Y$-module $\mathcal{E}$ and a map $\Phi\colon S^3\mathcal{E} \to \bigwedge^2 \mathcal{E}$, and conversely.*

The equations (2.3) already show that the triple cover is determined by Φ_2. Miranda then proved the existence of a natural isomorphism between $\mathrm{CHom}(S^2\mathcal{E}, \mathcal{E})$ and $\mathrm{Hom}(S^3\mathcal{E}, \bigwedge \mathcal{E})$, where $\mathrm{CHom}(S^2\mathcal{E}, \mathcal{E})$ is the set of locally trace-free morphisms.

Locally, this isomorphism is given by

$$\begin{aligned} \Phi(z_1^3) &= -c_0(z_1 \wedge z_2) \\ \Phi(z_1^2 z_2) &= c_1(z_1 \wedge z_2) \\ \Phi(z_1 z_2^2) &= -c_2(z_1 \wedge z_2) \\ \Phi(z_2^2) &= c_3(z_1 \wedge z_2). \end{aligned} \tag{2.4}$$

In section 3.4 we will show a different way in which a triple cover is birational to a hyperplane defined by a cubic polynomial.

The fact that a triple cover is defined by a morphism $\Phi_2 \in \mathrm{CHom}(S^2\mathcal{E}, \mathcal{E})$ was proven for quadruple covers as well in [HM99] where $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module. For this case the coefficients c_{ijk} will satisfy more relations than the trace free ones as we can see in the next theorem.

Theorem 2.2.3. *[HM99, Theorem 3.1] Under a linear change of coordinates, the relations which must be satisfied for the multiplication in the trace zero module $\mathcal{E}$ to be associative, along with the trace zero relations, are the equations of an affine cone over a $\mathrm{Gr}(2, 6)$, the Grassmanian of the two dimensional subspaces of a six dimensional space, under its natural Plücker embedding in $\mathbb{P}^{14}$.*

The equations of the Plücker embedding of an affine cone over a $\mathrm{Gr}(2, 6)$ are quadratic equations. We will generalise this result for covers of any degree in chapter 3 where we present our method for computing these relations, and prove that a cover homomorphism is always determined by a morphism in $\mathrm{Hom}(S^2\mathcal{E}, \mathcal{E})$.

2.2.2 Gorenstein covers

Definition 2.2.4. A covering map $\varphi\colon X \to Y$ is called a Gorenstein covering map if for every $y \in Y$ the fibre $\varphi^{-1}(y)$ is a Gorenstein scheme.

This was the definition Casnati and Ekedahl used in the paper [CE96] from which we will present the main theorem. Further papers from Casnati are also a very interesting read on the topic, some of which about the polarization of type $(1,3)$ of an abelian surface ([Cas96, Cas12, Cas99] are some we will cite later on).

Before going to Casnati and Ekedahl theorem we recall the definition and some results about Gorenstein rings.

Definition 2.2.5. Let (R, m, k) be a local Noetherian Cohen-Macaulay ring. A R-module ω_R is called canonical module if

$$\operatorname{Ext}^i_R(k, \omega_R) = \begin{cases} 0, \text{ if } i \neq d \\ k, \text{ if } i = d \end{cases}$$

where $d = \dim R$. R is Gorenstein if $R \cong \omega_R$.

Notice that for $d = 0$, the injective hull is its canonical module. More than that we have the following proposition.

Proposition 2.2.6. *[Eis95, 21.5] A zero dimensional ring R is Gorenstein if and only if the socle of R is simple.*

In particular, a zero dimensional k-algebra R is Gorenstein if as a k-vector space it is generated by

$$\begin{array}{cccc} & q & & \\ z_1 & z_1 & \dots & z_{d-2} \\ & 1 & & \end{array}$$

where q is a quadratic polynomial in the z_i which are the generators of R as a k-algebra. The upper element q is called the *socle* of R. Note that we always have such decomposition but the z_i do not need all to be generators of the ring R, in example 3.3.4 we will show a different case.

One can also define Gorenstein rings by their minimal free resolution as we see in the next proposition.

Proposition 2.2.7. *[Eis95, 21.16] Let R be a regular local ring, suppose that I is an ideal of codimension c in R, and suppose that $S = R/I$ is Cohen-Macaulay. If $F_\bullet$ is the minimal free resolution of S as an R-module, then the length of F is c, and $F^\vee$ is the minimal free resolution of ω_S. Then the following statements are equivalent:*

1. *S is Gorenstein.*

2. $F_\bullet$ *is symmetric,* $F_i \cong F_{c-i}^\vee$.

3. $F_c \cong R$.

Below we have the *Factorization theorem* by Casnati and Ekedahl. It is a general structure theorem for Gorenstein covers of degree d and it has been used for characterizing covers of degree $3, 4$ and 5 (in [CE96] and [Cas96] respectively) of algebraic varieties.

Theorem 2.2.8. *[CE96, Theorem 2.1] Let X and Y be schemes, Y integral and let $\varphi\colon X \to Y$ be a Gorenstein cover of degree $d \geq 3$. There exists a unique $\mathbb{P}^{d-2}$-bundle $\pi\colon \mathbb{P} \to Y$ and an embedding $i\colon X \hookrightarrow \mathbb{P}$ such that $\varphi = \pi \circ i$. Moreover the following hold.*

1. *$\mathbb{P} \cong \mathbb{P}(\mathcal{E})$ where $\mathcal{E}^\vee \cong \operatorname{coker} \varphi^\#$, $\varphi^\# \colon \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ being the trace map.*

2. *The composition $\varphi\colon \varphi^*\mathcal{E} \to \varphi^*\varphi_*\omega_{X|Y} \to \omega_{X|Y}$ is surjective and the ramification divisor R satisfies $\mathcal{O}_X(R) \cong \omega_{X|Y} \cong \mathcal{O}_X(1) := i^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$.*

3. *There exists an exact sequence $\mathcal{N}_*$ of locally free $\mathcal{O}_{\mathbb{P}}$-sheaves*
$$0 \leftarrow \mathcal{O}_X \leftarrow \mathcal{O}_{\mathbb{P}} \xleftarrow{\alpha_1} \mathcal{N}_1(-2) \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_{d-3}} \mathcal{N}_{d-3}(-d+2) \xleftarrow{\alpha_{d-2}} \mathcal{N}_{d-2}(-d) \leftarrow 0 \tag{2.5}$$
unique up to isomorphisms and whose restriction to the fibre $\mathbb{P}_y := \pi^{-1}(y)$ over y is a minimal free resolution of the structure sheaf of $X_y := \varphi^{-1}(y)$, in particular $\mathcal{N}_i$ is fibrewise trivial. $\mathcal{N}_{d-2}$ is invertible and, for $i = 1, \dots, d-3$, one has
$$\operatorname{rk} \mathcal{N}_i = \beta_i = \frac{i(d-2-i)}{d-1}\binom{d}{i+1},$$
hence $X_y \subset \mathbb{P}_y$ is an arithmetically Gorenstein subscheme. Moreover $\pi^*\pi_*\mathcal{N}_* \cong \mathcal{N}_*$ and $\mathcal{H}om\,(\mathcal{N}_*, \mathcal{N}_{d-2}(-d)) \cong \mathcal{N}_*$.

4. *If $\mathbb{P} \cong \mathbb{P}(\mathcal{E}')$, then $\mathcal{E}' \cong \mathcal{E}$ if and only if $\mathcal{N}_{d-2} \cong \pi^* \det \mathcal{E}'$ in the resolution (2.5) computed with respect to the polarization $\mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1)$.*

As we are following Miles Reid notation introduce in [Rei97] we stumble in a terminology difference for the ambient space. We can see it explicitly for the surfaces S_d of example 2.1.8 where we got
$$S_d \hookrightarrow \mathbb{A}\left(\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus(d-2)}\right)$$

and from the theorem above we have

$$S_d \hookrightarrow \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^2}(2)^{\oplus(d-2)} \oplus \mathcal{O}_{\mathbb{P}^2}(4)\right).$$

It is just a different perspective. In our way we look at the generators of the graded ring, Casnati and Ekedahl look at the sections of the sheaf.

Furthermore, in the proof of lemma 2.2.1 we saw that for a covering map $\varphi\colon X \to Y$ the (local) basis element of $\varphi_*\mathcal{O}_X$ defined by $\mathcal{O}_Y$ behaves as the identity when multiplying by elements in $\mathcal{O}_Y$. This is what allows us to consider $\xi_0 \neq 0$.

2.2.3 Abelian Surface with a polarization of type $(1,3)$

Let S be an abelian surface over k with a polarization of type $(1,3)$ defined by a symmetric line bundle $\mathcal{L}$. From [BL04, Example 10.1.5] the morphism $\varphi_{\mathcal{L}}\colon S \to \mathbb{P}^2$ is a 6-fold covering map ramified on a curve of degree 18 in $\mathbb{P}^2$.

The group $K(\mathcal{L}) := \{x \in S | t_x^*\mathcal{L} \cong \mathcal{L}\}$ is the kernel of $\varphi_{\mathcal{L}}$ and is isomorphic to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Choosing coordinates (x_0, x_1, x_2) for $\mathbb{P}^2$, the extended Heisenberg group $H(1,3)^e$ associated to $\mathcal{L}$ is just the Heisenberg group $H(3)^e$ generated by $\sigma_1', \sigma_2', \tau' \in \mathrm{PGL}(3,\mathbb{C})$

$$\sigma_1' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \sigma_2' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tau' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi^2 \end{pmatrix} \tag{2.6}$$

where ξ is a cube root of unity, and the morphism $\varphi_{\mathcal{L}}$ is equivariant for $H(3)^e$.

Birkenhake and Lange in [BL94] use the kernel of $\varphi_{\mathcal{L}}$ to determine the branch locus of a one dimensional family of $(1,3)$ polarized abelian surfaces $S = E \times E$, where E is an elliptic curve, and

$$\mathcal{L} = \{E \times \{0\} + \{0\} \times E + A\} \tag{2.7}$$

where A is the antidiagonal.

Proposition 2.2.9. *[BL94, Proposition 3.3] Let $(S,\mathcal{L})$ be an abelian surface with a polarization of type $(1,3)$ such that $S = E \times E$, E an elliptic curve, and $\mathcal{L}$ is given by equation (2.7). Then its branch locus in $\mathbb{P}^2$ is $3D$ where D is given by the vanishing of the polynomial*

$$\begin{aligned}(x_0^6 + x_1^6 + x_2^6) + 2(2\lambda^3 - 1)(x_0^3x_1^3 + x_0^3x_2^3 + x_1^3x_2^3) \\ -6\lambda^2(x_0^4x_1x_2 + x_0x_1^4x_2 + x_0x_1x_2^4) - 3\lambda(\lambda^3 - 4)(x_0^2x_1^2x_2^2)\end{aligned}. \tag{2.8}$$

$\lambda \in \mathbb{C} - \{1, \xi, \xi^2\}$.

In [Cas99] Casnati studied a bigger family of abelian surfaces considering the covering maps $\varphi\colon S \to \mathbb{P}^2$ that decompose as $S \xrightarrow{\rho} S' \xrightarrow{\varsigma} \mathbb{P}^2$ where ρ and ς are a double and triple cover respectively. Such surfaces are called *bielliptic* as they have a nontrivial involution $j\colon S \to S$.

S' is a ruled surface with invariant $e(S') = -1$ over an elliptic surface E and as a triple cover $\varsigma\colon S' \to \mathbb{P}^2$ it satisfies $\varsigma_* \mathcal{O}_{S'} = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}$.

Proposition 2.2.10. *Let $(S, \mathcal{L})$ be a bielliptic abelian surface with a polarization of type $(1,3)$. Then its branch locus can be decomposed as $2D + D'$ where D is given by the vanishing of the polynomial (2.8) and $D' := \varsigma_* B_\rho$, $B_\rho \subset S'$ the branch locus of ρ. Furthermore, D and D' are $K(\mathcal{L})$-invariant.*

Connecting this work with the theory of Gorenstein covers developed by him and Ekedahl in [CE96], Casnati proves the following theorem.

Theorem 2.2.11. *[Cas99, Theorem 0.4] Let $(S, \mathcal{L})$ be a $(1,3)$-polarized surface. Then $\mathcal{L}$ defines a morphism $\varphi_{\mathcal{L}}\colon S \to \mathbb{P}^2$ such that*

$$\varphi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3).$$

Following [GP98, GP01]. For every $x \in K(\mathcal{L})$ there is an isomorphism $t_x^* \mathcal{L} \cong \mathcal{L}$. This induces a projective representation $K(\mathcal{L}) \to PGL\left(H^0(\mathcal{L})\right)$, which lifts uniquely to a linear representation of $K(\mathcal{L})$ after taking a central extension of $K(\mathcal{L})$

$$1 \to \mathbb{C}^* \to \mathcal{G}(\mathcal{L}) \to K(\mathcal{L}) \to 0,$$

whose Schur commutator map is the Weil pairing. $K(\mathcal{L})$ is the theta group of $\mathcal{L}$ and is isomorphic to the abstract Heisenberg group $H(D)$, while the above linear representation is isomorphic to the Schrödinger representation of $H(D)$ on $V = \mathbb{C}(\mathbb{Z}/\mathbb{Z}_d)$, the vector space of complex-valued functions on $\mathbb{Z}/\mathbb{Z}_d$. An isomorphism between $G(\mathcal{L})$ and $H(D)$, which restricts to the identity on centers induces a symplectic isomorphism between $K(\mathcal{L})$ and $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$. Such an isomorphism is called a level structure of canonical type on $(S, \mathcal{L})$.

Denote by $\mathcal{A}_D^{lev}$ the moduli space of g dimensional abelian varieties with a polarization of type D and with canonical level structure, where $D = (d_1, \ldots, d_g)$, $d_1 | d_2 | \ldots | d_g$.

Using theorem 2.2.11 we will construct the section ring $R(S, \mathcal{L})$ for a abelian surface with a polarization of type $(1,3)$. The section ring $R(S, \mathcal{L})$ is generated as a k-algebra by a basis $(x_i, y_i, z_i)_{0\leq i\leq 2}$ where the basis (x_i) is the image of a basis for $\mathcal{O}_{\mathbb{P}^2}(1)$. Choosing this basis so that $\varphi_{\mathcal{L}}$ is equivariant for σ_1, σ_2, τ will force by construction the $(y_i), (z_i)$ to be equivariant for $\sigma_1, \sigma_2, \tau^{-1}$.

This gives us that a general section ring is determined by three parameters from which we conclude unirationality of $\mathcal{A}_3^{lev}$ and more than that we will prove that it is rational. Notice that Birkenhake and Lange proved that $\mathcal{A}_3$, the moduli space of abelian surfaces with a polarization of type $(1,3)$, is rational in [BL95]. The rationality of $\mathcal{A}_3^{lev}$, as far as we know, was still an open question (see [GP01]).

Chapter 3

Algebraic Covers

Let $\varphi\colon X \to Y$ be a covering map of degree d and assume that $\mathcal{F}$ is an $\mathcal{O}_Y$-submodule of $\varphi_*\mathcal{O}_X$ such that $\varphi_*\mathcal{O}_X$ a direct summand of the graded $\mathcal{O}_Y$-algebra $\mathcal{R}(\mathcal{F}) := \bigoplus_{n=0}^{\infty} \operatorname{Sym}^n \mathcal{F}$, and $\mathcal{R}(\mathcal{F})\backslash\varphi_*\mathcal{O}_X$ a $\mathcal{R}(\mathcal{F})$-ideal. Then we have the diagram

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \mathbb{A}(\mathcal{F}) \\ & \underset{\varphi}{\searrow} & \downarrow{\scriptstyle \pi} \\ & & Y. \end{array} \tag{3.1}$$

The existence of such an $\mathcal{F}$ is always guaranteed by lemma 2.2.1, as we can take $\mathcal{F} = \mathcal{E}$ where $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$. Throughout this chapter whenever we take an $\mathcal{O}_Y$-module $\mathcal{F}$ such that $X \hookrightarrow \mathbb{A}(\mathcal{F}) \twoheadrightarrow Y$ we assume it satisfies the conditions written above.

3.1 Local analysis

Assume in this section that X, Y are local schemes over k and let $\varphi\colon X \to Y$ be a covering map. Furthermore, consider $\mathcal{F}$ an $\mathcal{O}_Y$-module with rank r such that we have an embedding $i\colon X \to \mathbb{A}(\mathcal{F})$. Then we have the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{O}_{\mathbb{A}(\mathcal{F})} \xrightarrow{i^\sharp} i_*\mathcal{O}_X \to 0. \tag{3.2}$$

Let $(z_1, \dots, z_r)$ be a basis for $\mathcal{F}$, then $\mathcal{O}_{\mathbb{A}(\mathcal{F})} \cong \mathcal{O}_Y[z_i]$. Given an element $f \in \mathcal{O}_{\mathbb{A}(\mathcal{F})}$ we use this isomorphism to define $\operatorname{in}(f)$ as the sum of the terms with maximal degree in the variables z_i of f and the initial ideal of $\mathcal{I}_X$ as

$$\operatorname{in}(\mathcal{I}_X) = \{\operatorname{in}(f)\colon f \in \mathcal{I}_X\}.$$

Lemma 3.1.1. *Let $\mathcal{O}_Y$ be a local k-algebra which is an integral domain and $\mathcal{O}_X$ a finite flat $\mathcal{O}_Y$-algebra such that $\mathcal{O}_X \cong (\bigoplus_{n=0}^{\infty} \operatorname{Sym}^n \mathcal{F}) / \mathcal{I}_X$, where $\mathcal{F}$ is a free $\mathcal{O}_Y$-module with rank r. Then*

1. *the ideal sheaf $\mathcal{I}_X$ is Cohen-Macaulay,*
2. *the ideal $\operatorname{in}(\mathcal{I}_X)$ is independent of the choice of basis for $\mathcal{F}$,*
3. *the free resolution of the ideal $\operatorname{in}(\mathcal{I}_X)$ has the same format as the one for $\mathcal{I}_X$, in particular $\operatorname{in}(\mathcal{I}_X)$ is also CM.*

Proof. 1: As $\mathcal{O}_X$ is a finite flat $\mathcal{O}_Y$-algebra, $\dim \mathcal{O}_X = \dim \mathcal{O}_Y \Rightarrow \operatorname{codim}(\mathcal{I}_X) = r$. Take $n \in \mathbb{N}$ such that $\operatorname{Sym}^n \mathcal{F} \cap \varphi_* \mathcal{O}_X = \emptyset$. Then the sequence $(z_j^n - i^\sharp(z_j^n))$, $1 \le j \le r$, is a $\mathcal{O}_{\mathbb{A}(\mathcal{F})}$-sequence in the ideal $\operatorname{in}(\mathcal{I}_X)$ from where get

$$r \le \operatorname{depth}(\mathcal{I}_X) \le \operatorname{codim}(\mathcal{I}_X) = r,$$

which asserts that $\mathcal{I}_X$ is CM.

2: Denote by $\operatorname{in}(\mathcal{I}_X)_{(z_i)}$ the initial ideal when defined with the basis (z_i) for $\mathcal{F}$. A change of basis for $\operatorname{Sym} \mathcal{F}$ is given by a $\mathcal{O}_Y$-linear automorphism, $\Psi \colon \mathcal{F} \to \mathcal{O}_Y \oplus \mathcal{F}$. Decomposing Ψ as $\Psi_1 \oplus \Psi_2$, where $\Psi_1 \colon \mathcal{F} \to \mathcal{O}_Y$, $\Psi_2 \colon \mathcal{F} \to \mathcal{F}$, we have that Ψ_2 is an isomorphism and sends $\operatorname{in}(\mathcal{I}_X)_{(z_i)}$ into $\operatorname{in}(\mathcal{I}_X)_{\Psi(z_i)}$ so we are done.

3: Use a change of variables $z_i \mapsto z_i/z_0$ to homogenize $\mathcal{I}_X$ and take the quotient by (z_0). By proposition 5.1.2, as z_0 is a non-zero divisor taking the tensor of a minimal free resolution of $\mathcal{I}_X$ with $\mathcal{O}_Y[z_i]/(z_0)$ gives a resolution of the ideal $\operatorname{in}(\mathcal{I}_X)$ so we have the right depth. As its codimension is smaller or equal to r we are done. ☐

For a covering map $\varphi \colon X \to Y$, there are occasions where $\mathcal{F}$ and $\operatorname{in}(\mathcal{I}_X)$ can be determined. E.g. 1) if we want to assume that X and Y are Gorenstein as we will see in section 3.3, 2) for a general covering map of degree d such that $\varphi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$, then $\mathcal{F} = \mathcal{E}$ and $\operatorname{in}(\mathcal{I}_X) = (z_i z_j)$, for all $1 \le i \le j \le d-1$.

In such case denote by q_i the generators of the ideal $\operatorname{in}(\mathcal{I}_X)$ for a fixed basis $(z_1, \ldots, z_r)$ of $\mathcal{F}$. We want to determine the relations between coefficients $c_{ij} \in \mathcal{O}_Y$, $(i,j) \in \mathbb{N} \times \mathbb{N}^r$, such that the ideal generated by the polynomials

$$f_i := q_i - \sum_j c_{ij} \overline{z}^j$$

$\overline{z}^j = z_1^{j_1} \cdots z_r^{j_r}$, has the same resolution as $\operatorname{in}(\mathcal{I}_X)$. By definition q_i is the sum of the monomials with maximal degree in f_i so $c_{ij} = 0$ if $\deg(j) \ge \deg(q_i)$. We have then a finite number of coefficients c_{ij}.

By Nakayama lemma, the image of the generators of $\mathcal{I}_X$ in $\mathcal{I}_X \otimes k$ are the generators of $\mathcal{I}_X \otimes k$. As Y is integral, the relations between the c_{ij} are also kept, so in chapter 5 we study these relations in the case $\mathcal{O}_Y = k$.

The method uses the same idea behind the proof of item 3 in Lemma 3.1.1. Consider a new variable z_0 and homogenize the f_i to get the polynomials

$$F_i := q_i + z_0 f_i' + z_0^2 f_i'' + \cdots + z_0^{m_i} f_i^{(m_i)}$$

where $m_i = \deg(q_i)$ and $f_i^{(j)} = \sum_l c_{il} z^l$ with $\deg(l) = m_i - j$.

Let $R^{(k)} = k[z_0, z_1, \ldots, z_r]/(z_0^{k+1})$ and $\mathcal{I}_X^{(j)} \subset R^{(k)}$ the ideal generated by

$$F_i := q_i + z_0 f_i' + z_0^2 f_i'' + \cdots + z_0^k f_i^{(k)}.$$

With the use of a computer we check degree by degree the extension

$$R^{(j)} \leftarrow R^{(j+1)}$$

that keeps the resolution of $\mathcal{I}_X^{(j)}$ with the same format as the one of $\text{in}(\mathcal{I}_X)$. Each step is determined by checking a set of linear equations so, although it might be a quite large number of equations, they will not be heavy computations.

Notation 3.1.2. Let $\mathcal{O}_Y$ be a local k-algebra which is an integral domain, $\mathcal{O}_X$ a flat $\mathcal{O}_Y$-algebra and $\mathcal{F}$ a free $\mathcal{O}_Y$-module of rank r such that $\mathcal{O}_X$ is a direct summand of $R(\mathcal{F})$. For $q = (q_1, \ldots, q_m)$ a minimal set of generators for the ideal $\text{in}(\mathcal{I}_X) \subset R(\mathcal{F})$ denote by $\mathcal{I}_q \subset k[c_{ij}]$, $(i,j) \in \mathbb{N} \times \mathbb{N}^r$, $\deg(j) < \deg(q_i)$, the ideal with the relations between the c_{ij} such that the polynomials $q_i - \sum_j c_{ij} \overline{z}^j$ generate an ideal with the same resolution as $\text{in}(\mathcal{I}_X)$.

The two cases of main interest to us are,

1. Fat point deformation, $q = \left(z_1^2, z_1 z_2, \ldots, z_i z_j, \ldots, z_r^2\right)$,
2. $\text{in}(\mathcal{I}_X)$ a Gorenstein ideal generated by polynomials of degree two, i.e., for a cover of degree d, $q = \left(q_1, \ldots, q_{\binom{d-1}{2}-1}\right)$, $q_i \in k[z_1, \ldots, z_{d-2}]_{(2)}$.

Although the method we will present can be used for any generating set q, for our main cases we will get the following results.

Theorem 3.1.3. *Let $\mathcal{O}_Y$ be an integral local k-algebra and $\mathcal{O}_X$ a flat $\mathcal{O}_Y$-algebra of rank d. Given a basis $(z_i)_{1 \le i \le d-1}$ for $\mathcal{E}$, where $\mathcal{O}_X$ decomposes as $\mathcal{O}_Y \oplus \mathcal{E}$, then* *the multiplication in $\mathcal{O}_X$ can be written as*

$$z_i z_j - \sum_{k=1}^{d-1} c_{ijk} z_k - d_{ij}$$

with $c_{ijk}, d_{ij} \in \mathcal{O}_Y$ satisfying the relations in the ideal $\mathcal{I}_q$, $q = (z_i z_j)_{1 \le i \le j \le d-1}$. The generators of $\mathcal{I}_q$ have degree smaller or equal to two and all the d_{ij} can be written as quadratic forms in the c_{ijk}.

For general covering maps of degree d, the fat point deformation, we will use the notation $\mathcal{I}_d$ for $\mathcal{I}_q +$ (trace free relations) and $\mathcal{E}$ is the trace free module. Often we will ignore the relations involving the d_{ij}.

The fat point deformation case was studied by Miranda when the relations between the c_{ijk} and d_{ij} were determined by explicitly checking the associativity condition. Notice that, if $z_i z_j = \sum_k c_{ijk} z_k + d_{ij}$ is an associative multiplication for $i \le j$, then these polynomials define an ideal $\mathcal{I}_X$. By Lemma 3.1.1 the free resolution of $\mathcal{I}_X$ has the same format as the one for $\mathrm{in}(\mathcal{I}_X)$ hence by finding all extensions of $\mathrm{in}(\mathcal{I}_X)$ one gets the same relations between the c_{ijk}.

This is the reason for the name of chapter 5 - *Associative multiplication.*

Theorem 3.1.4. *Let $\mathcal{O}_Y$ be an integral local k-algebra and $\mathcal{O}_X$ a flat $\mathcal{O}_Y$-algebra of rank d. Given a basis $(z_i)_{1 \le i \le d-2}$ for $\mathcal{F}$, a free summand of $\mathcal{O}_X$ of rank $d-2$, suppose that the multiplication in $\mathcal{O}_X$ can be written as*

$$q_i - \sum_{j=1}^{d-2} c_{ij} z_j - d_i$$

$c_{ij}, d_i \in \mathcal{O}_Y$, q_i being $\binom{d-1}{2} - 1$ quadratic forms in the z_i such that $\mathrm{in}(\mathcal{I}_X) = (q_i)$ is a Gorenstein ideal. Then the c_{ijk}, d_{ij} satisfy the relations in $\mathcal{I}_q$, $q = (q_i)_{1 \le i \le j \le d-1}$, whose generators have degree smaller or equal to two and all the d_{ij} can be written as quadratic forms in the c_{ijk}.

Furthermore, if q, q' are two sets of generators for $\mathrm{in}(\mathcal{I}_X)$, then $\mathcal{I}_q \cong \mathcal{I}_{q'}$.

Proof. The first sentence in both theorems follows from the discussion above. The statement about the ideal $\mathcal{I}_q$ is a corollary of proposition 5.2.1 and 5.3.1 respectively. □

The theorems above seem a bit vague and out of context but they will be useful as the process of extension to determine $\mathcal{I}_q$ finishes for a k large enough such that

$$k \ge \max_i \{\deg(d_i \circ d_{i+1})\}$$

where d_i is the i-syzygy matrix on a free resolution of the ideal $\mathrm{in}(\mathcal{I}_X)$. In both cases $k = 3$ and if one checks associativity directly from the equations there will be linear, quadratic and cubic equations. Theorem 3.1.3 and 3.1.4 prove that they are generated by the linear and quadratic ones.

3.2 Global analysis

Given a covering map $\varphi\colon X \to Y$ of degree d, as topological spaces we can write the following short exact sequence

$$F_d \hookrightarrow X \twoheadrightarrow Y$$

where F_d is just a set of d points. If for all $y \in Y$ the fibres $\varphi^{-1}(y) \subset X$ are homeomorphic the sequence is called a fibre bundle (see [Hat02, p.375]). Important to notice that X is not determined by F_d and Y, e.g. just take a section in the Möbius strip and a trivial double cover over a circle.

For covering maps, although it is still not enough to know the base and the fibre structure to determine X, we want to show it is enough to know the base and the stalk over any point in Y.

Proposition 3.2.1. *Given a covering map $\varphi\colon X \to Y$ and $\mathcal{F}$ an $\mathcal{O}_Y$-module such that $X \hookrightarrow \mathbb{A}(\mathcal{F}) \twoheadrightarrow Y$. Then $\mathcal{O}_X$ is determined by the stalk $\mathcal{I}_{X,y} := \mathcal{I}_X \otimes \mathcal{O}_{Y,y} \subset \mathcal{O}_{\mathbb{A}(\mathcal{F}_y)}$ for any point $y \in Y$.*

Proof. Given a covering map $\varphi\colon X \to Y$, taking the stalk at a point $y \in Y$ we have the short exact sequence

$$0 \to \mathcal{I}_{X,y} \to \mathcal{O}_{\mathbb{A}(\mathcal{F}_y)} \to i_*\mathcal{O}_{X,y} \to 0.$$

Let $(z_1, \ldots, z_r)$ be a basis for $\mathcal{F}_y$ and $(f_1, \ldots, f_m)$ a set of generators for $\mathcal{I}_{X,y}$

$$f_i \equiv q_i - \sum_j c_{ij} \overline{z}^j$$

with $\mathbb{N}^r \ni j$, $\deg(j) < \deg(q_i)$, q_i generators of $\mathrm{in}(\mathcal{I}_{X,y})$ and $c_{ij} \in \mathcal{O}_{Y,y}$ satisfying the relations in $\mathcal{I}_q$. If we take an open set $\mathcal{U} \subset Y$ such that all $c_{ij} \in \mathcal{O}_Y(\mathcal{U})$, then for any $y' \in \mathcal{U}$, the image of the c_{ij} in $\mathcal{O}_{Y,y'}$ also satisfy the relations in $\mathcal{I}_q$ (as $\mathcal{O}_Y(\mathcal{U}) \hookrightarrow \mathcal{O}_{Y,y}$ because Y is integral by assumption).

Furthermore, given any other point $y' \in \mathcal{U}' \subset Y$, as Y is irreducible and the transition morphisms in $\mathbb{A}(\mathcal{F})$ are given by $\mathcal{O}_Y$-linear automorphisms, we can compute $i_*\mathcal{O}_X|_{\mathcal{U}'}$. Note that if $\mathcal{I}_{X,y'}$ is generated by f_i', $\text{in}(\mathcal{I}_X|_{\mathcal{U}'}) \cong \text{in}(\mathcal{I}_X|_{\mathcal{U}})$ and $\mathcal{I}_q \cong \mathcal{I}_{q'}$. □

Important to notice that in the proof above we used a local basis of $\mathcal{F}$ to define the structure of $\mathcal{I}_q$. As we will see in Proposition 4.4.2 a general triple cover $\varphi\colon S \to \mathbb{P}^2$ such that $\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}$ is a smooth surface which implies that each fibre is Gorenstein but by construction each fibre is defined by three equations of the form $z_iz_j = \sum_{k=1}^2 c_{ijk}z_k + d_{ij}$, for $1 \le i \le j \le 2$, which could be Cohen–Macaulay but not Gorenstein. As described in [CE96] such triple cover is Gorenstein as it is generated by a cubic polynomial obtained by eliminating one of the variables z_i but such construction does not appear naturally from the structure of $\varphi_*\mathcal{O}_S$.

Having so, the proposition above shows that the ideal $\mathcal{I}_q$ is fundamental as the structure of a stalk give us the structure of $\mathcal{O}_X$. On the other hand, remember that the relations in $\mathcal{I}_q$ treat the entries c_{ij} as token variables and in the proof above we start with coefficients c_{ij} for a given stalk $\mathcal{I}_{X,y}$.

If $\mathcal{O}_{Y,y}$ is a regular k-algebra with maximal ideal $m_y \subset \mathcal{O}_{Y,y}$ generated by $(x_1, \ldots, x_n)$, one way to choose the c_{ij} would be to write them as

$$c_{ij} = \sum_k \alpha_{ijk}\overline{x}^k$$

$\deg(k) = \deg(q_i) - \deg(j)$, $\alpha_{ijk} \in k$ satisfying the relations in $\mathcal{I}_q$. Then, picking α_{ijk} for each open set of an open cover of Y, one could check impositions on them brought by the gluing.

Example 3.2.2. Let $\varphi\colon S \to \mathbb{P}^2$ be a triple cover such that $\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m)$. Then following example 2.1.9 we have the embedding

$$S \hookrightarrow \mathbb{A}\left(\Omega^1_{\mathbb{P}^2}\right) \cong \text{Proj}\left(k[x_0, x_1, x_2; y_0, y_1, y_2]/\left(\sum x_iy_i\right)\right)$$

where $\deg(x_i) = 1, \deg(y_j) = 2 + m$. Following the results by Miranda for triple covers, over the open set $\mathcal{U} = \{x_2 \neq 0\}$, $\mathcal{O}_S(\mathcal{U})$ is given by the equations

$$\begin{array}{rcrcl} y_0^2 & = & c_1y_0 + c_0y_1 & + & 2(c_1^2 - c_0c_2) \\ y_0y_1 & = & -c_2y_0 - c_1y_1 & - & (c_0c_3 - c_1c_2) \\ y_1^2 & = & c_3y_0 + c_2y_1 & + & 2(c_2^2 - c_1c_3) \end{array}$$

where $c_i \in \mathcal{O}_{\mathbb{P}^2}(\mathcal{U})$. As each y_i has degree 2 and we are over the open set $\{x_2 \neq 0\}$, we could take each c_i as a sum of monomials in $k[x_0, x_1, x_2^{\pm}]^{(2+m)}$ with token coefficients.

The problem with such approach is the computational effort needed to do so. For a general cover one would have to carry a large number of variables α_{ijk} satisfying quadratic equations between themselves on which we would impose further relations. So instead of starting with the stalk $\mathcal{I}_{Y,y}$, we will start with the global structure.

Recall that $\varphi_*\mathcal{O}_X$ is a direct summand of $\mathcal{R}(\mathcal{F}) = \bigoplus_{n=0}^{\infty} \operatorname{Sym}^n \mathcal{F}$. Let $\mathcal{Q}$ be an $\mathcal{O}_Y$-module generating $(\mathcal{R}(\mathcal{F}) \backslash \varphi_*\mathcal{O}_X)$, $(z_1, \ldots, z_r)$ a basis for $\mathcal{F}_y$ and $q = (q_1, \ldots, q_n)$ a homogeneous one for $\mathcal{Q}_y$. We can write a general element in $\operatorname{Hom}(\mathcal{Q}, \varphi_*\mathcal{O}_X) \otimes \mathcal{O}_{Y,y}$ as

$$\Phi|_y(q_i) = \sum_j c_{ij}\overline{z}^j$$

$c_{ij} \in \mathcal{O}_{Y,y}(\mathcal{U})$ for an open set $\mathcal{U}$ containing y.

Essentially, what we are doing is treating Φ as a general global section of $(\mathcal{Q}^\vee \otimes \varphi_*\mathcal{O}_X)\big|_{\mathcal{U}}$. The advantage of doing so is that now the gluing given by the transition morphisms in $\mathbb{A}(\mathcal{F})$ is guaranteed which was what we used in the proof of Proposition 3.2.1.

Definition 3.2.3. Given a covering map $\varphi \colon X \to Y$ and $\mathcal{F}, \mathcal{Q}$ locally free $\mathcal{O}_Y$-modules such that $X \hookrightarrow \mathbb{A}(\mathcal{F}) \twoheadrightarrow Y$ and $\mathcal{Q}$ is a direct summand of $\mathcal{R}(\mathcal{F})$ for which $\mathcal{R}(\mathcal{F}) = \mathcal{Q} \cdot \mathcal{R}(\mathcal{F}) \oplus \varphi_*\mathcal{O}_X$ and $\mathcal{Q} \cap \varphi_*\mathcal{O}_X = \emptyset$. Then a morphism $\Phi \in \operatorname{Hom}(\mathcal{Q}, \mathcal{F})$ is called a **cover homomorphism** if it induces a morphism $\Phi' \in \operatorname{Hom}(\mathcal{Q}, \varphi_*\mathcal{O}_X)$ that can be written as

$$\Phi'|_{\mathcal{U}}(q_i) = \sum_{\deg(j) < \deg(q_i)} c_{ij}\overline{z}^j$$

for any open $\mathcal{U} \subset Y$ where the $c_{ij} \in \mathcal{O}_Y(\mathcal{U})$ satisfy the relations in $\mathcal{I}_q$ for $(z_i)_{\leq i \leq r}$, $q = (q_i)_{\leq i \leq m}$ local basis of $\mathcal{F}$ and $\mathcal{Q}$.

The set of cover homomorphisms is denoted by $\operatorname{CHom}(\mathcal{Q}, \mathcal{F})$.

As said before, the two cases we are most interested in are

1. $\varphi \colon X \to Y$ a general cover of degree d, $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$. Then $\mathcal{F} = \mathcal{E}$, $\mathcal{Q} = S^2\mathcal{E}$.

   By Theorem 3.1.3 and Proposition 3.2.1, a multiplication in $\varphi_*\mathcal{O}_X$ defines and is defined by $\Phi \in \operatorname{Hom}\left(S^2\mathcal{E}, \mathcal{E}\right)$ that locally satisfies the relations in $\mathcal{I}_d$. So we are just keeping the notation introduced by Miranda.

2. $\varphi \colon X \to Y$ a covering map such that $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{F} \oplus \mathcal{L}$, for a line bundle $\mathcal{L}$ over Y which is a direct summand of $S^2\mathcal{F}$.

   Theorem 3.1.4 and Proposition 3.2.1 guarantee that if the multiplication in $\varphi_*\mathcal{O}_X$ is given by an ideal $\mathcal{I}_X$ such that $\operatorname{in}(\mathcal{I}_X) = \mathcal{Q} \subset S^2\mathcal{F}$ is locally a Gorenstein ideal, then from $\Phi \in \operatorname{Hom}(\mathcal{Q}, \mathcal{F})$ we can construct $\Phi' \in \operatorname{Hom}(\mathcal{Q}, \mathcal{O}_Y)$ that together define a Gorenstein covering map.

Important to notice that there is no reason to have a single $\mathcal{Q}$, e.g. for *First Problem*

$$(\mathcal{R}(\mathcal{F}) \backslash \varphi_* \mathcal{O}_S) = \mathcal{O}_{\mathbb{P}^2}(-4)^{\oplus 5} \subset \mathcal{O}_{\mathbb{P}^2}(-4)^{\oplus 6}$$

and there is not an unique way up to change of basis to choose 5 polynomials from $(z_i z_j)_{1 \leq i \leq j \leq 3}$ generating a Gorenstein ideal.

On the plus side, the obstructions brought by working with an irregular surface will be of help. For the abelian surface with a polarization of type $(1,3)$ we will have an unique $\mathcal{Q}$ as we are working with a Gorenstein cover.

3.3 Gorenstein Covers

A covering map $\varphi\colon X \to Y$ is called a Gorenstein covering map if all its fibres are Gorenstein. For the method we want to use, we need a stronger assumption which brings the next definition.

Definition 3.3.1. A covering map $\varphi\colon X \to Y$ is called a Gorenstein covering map if there is a line bundle $\mathcal{L}$ in Y such that $\omega_Y \cong \mathcal{L}^{\otimes k_Y}$, $\omega_X \cong \varphi^* \mathcal{L}^{\otimes k_X}$, $k_X, k_Y \in \mathbb{Z}$.

This will be a method for us to be able to use the ideal $\operatorname{in}(\mathcal{I}_X)$ as the structure of $\varphi_* \mathcal{O}_X$ will be the same as the structure of the fibre as we prove now.

Proposition 3.3.2. *Let $\varphi\colon X \to Y$ be a Gorenstein covering map of degree d, and $\mathcal{L}$ the line bundle in Y such that $\omega_Y \cong \mathcal{L}^{\otimes k_Y}$, $\omega_X \cong \varphi^* \mathcal{L}^{\otimes k_X}$, $k_X, k_Y \in \mathbb{Z}$. Then*

$$\varphi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{F} \oplus \mathcal{L}^{\otimes k_Y - k_X}, \; m \in \mathbb{N}.$$

Furthermore, X is contained in $\mathbb{A}(\mathcal{F})$.

Proof. Denote $\mathcal{O}_Y(\mathcal{L}^{\otimes n}), \varphi^*(\mathcal{L}^{\otimes m})$ by $\mathcal{O}_Y(n)$, $\mathcal{O}_X(m)$ respectively. By lemma 2.2.1, $\varphi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$ for a locally free $\mathcal{O}_Y$-module $\mathcal{E}$, from where we get

$$\varphi_* \omega_X \cong \varphi_*(\mathcal{O}_X(k_X)) = \mathcal{O}_Y(k_X) \oplus \mathcal{E}(k_X).$$

At the same time, by adjunction formula, we have

$$\varphi_* \omega_X = \mathcal{H}om(\varphi_* \mathcal{O}_X, \omega_Y) = (\varphi_* \mathcal{O}_X)^\vee \otimes \omega_Y = \mathcal{O}_Y(k_Y) \oplus \mathcal{E}^\vee(k_Y),$$

which implies that $\varphi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{F} \oplus \mathcal{O}_Y(k_Y - k_X)$ and $\mathcal{F}^\vee(k_Y - k_X) \cong \mathcal{F}$.

As X is Gorenstein, $\mathcal{O}_{X,y}$ is a Gorenstein local ring for all $y \in Y$, and the socle of $\mathcal{O}_{X,y} \otimes k(y)$ is the term $\mathcal{O}_Y(k_Y - k_X) \otimes k(y)$ in $\varphi^{-1}(y)$. By Nakayama's lemma we can drop the $- \otimes k(y)$ and we get a surjective morphism on the stalks

$$S^2\mathcal{F}_y \to \mathcal{O}_{Y,y}(k_Y - k_X).$$

As $\mathcal{F}$ is a locally free sheaf, there is a surjective morphism $S^2\mathcal{F} \to \mathcal{O}_Y(k_Y - k_X)$. $\square$

Corollary 3.3.3. *Let* $(S, \mathcal{L})$ *be an abelian surface with a polarization of type* $(1,3)$*. Then* $S \hookrightarrow \mathbb{A}\left(\left(\Omega^1_{\mathbb{P}^2}\right)^{\oplus 2}\right)$*.*

Proof. For an abelian surface with a polarization of type $(1,3)$, $\omega_S \cong \varphi^*\mathcal{O}_{\mathbb{P}^2}$ and $\omega_{\mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}\left(H^{-3}\right)$, where H is a hyperplane section. $\square$

We want now to show a continuation of Problem 2.1.13.

Example 3.3.4 (Future Work)**.** Let $\varphi\colon X \to \mathbb{P}^3$ be a Gorenstein covering map of degree 20 such that

$$\varphi_*\mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \left(\Omega^1_{\mathbb{P}^3}\right)^{\oplus 3} \oplus \left(\Omega^2_{\mathbb{P}^3}\right)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-4).$$

From the inclusions $\Omega^2_{\mathbb{P}^3} \subset S^2\left(\left(\Omega^1_{\mathbb{P}^3}\right)^{\oplus 3}\right)$, $\mathcal{O}_{\mathbb{P}^3}(-4) \subset S^3\left(\left(\Omega^1_{\mathbb{P}^3}\right)^{\oplus 3}\right)$ we can obtain model for X such that

$$X \subset \mathbb{A}\left(\left(\Omega^1_{\mathbb{P}^3}\right)^{\oplus 3}\right).$$

These are not all X with $\varphi_*\mathcal{O}_X$ satisfying the splitting above as we are imposing some structure on X. Without doing so we get a Gorenstein ideal of codimension 18 that should be more complex to study.

This is an interesting example as the fibre structure as a k-vector space is not the traditional one described in proposition 2.2.6 generated by the identity, $d-2$ linear generators and a quadratic form as socle.

As $\operatorname{rank}(\Omega^1_{\mathbb{P}^3}) = 3$, the fibre as nine generators as an algebra that we denote by y_i, z_j and w_k, $0 \leq i,j,k \leq 2$. The terms $\Omega^2_{\mathbb{P}^3}$ will be generated by the nine polynomials $y_i \wedge z_j, z_j \wedge w_k, w_k \wedge y_i$ (where $y_i \wedge z_j$ denotes $y_iz_j - y_jz_i$ for $0 \leq i < j \leq 2$), and the socle will be the term $y_i \wedge z_j \wedge w_k$ (which represents the polynomial $\sum(-1)^{\sigma(i,j,k)}y_iz_jw_k$ for $\sigma(i,j,k)$ the sign of the permutation $(i \quad j \quad k)$). The diagram is the following

	$y_i \wedge z_j \wedge w_k$	
$z_j \wedge w_k$	$w_k \wedge y_i$	$y_i \wedge z_k$
y_i	z_j	w_k
	1	

$0 \leq i, j, k \leq 2$.

3.4 Trace free basis

As described in section 2.2.1 in [Mir85] Miranda proves that given a locally free $\mathcal{O}_Y$-module of rank 2, $\mathcal{E}$, any $\mathcal{O}_Y$-linear map $\Phi : S^3\mathcal{E} \to \bigwedge^2 \mathcal{E}$ determines a unique triple cover homomorphism Φ_2 via the natural isomorphism from $\mathrm{Hom}(S^3\mathcal{E}, \bigwedge^2 \mathcal{E})$ to $\mathrm{CHom}(S^2\mathcal{E}, \mathcal{E})$ and vice-versa.

Recall that by the equations (2.4) this correspondence is given by

$$(z_1^3, z_1^2 z_2, z_1 z_2^2, z_2^3) \mapsto (-c_0, c_1, -c_2, c_3)$$

where (z_1, z_2) is a local basis for $\mathcal{E}$ and $c_i \in \mathcal{O}_Y$. To a certain level is just giving us that a triple cover is defined by the four elements c_i which is not a geometric result.

We want to address that question, what does it mean geometrically to choose a trace free basis to write d points in $\mathbb{A}^{d-1}$? Notice that the theorem seems to show that given a triple cover on two variables, we can write it as a single cubic polynomial. This is what we want to show.

Let $\mathcal{Q} = \{q_1, \ldots, q_d\}$ be d general points in $\mathbb{A}^{d-1} = \mathrm{Spec}\,(k[z_1, \ldots, z_{d-1}])$. By lemma 2.2.1 we can assume that the equations defining the points $\mathcal{Q}$ are trace free. Denote by $I_{\mathcal{Q}} \subset k[z_i]$ the ideal defined by such equations

$$\mathcal{I}_{\mathcal{Q}} = \left(\forall_{i \leq j}, z_i z_j = \sum_k c_{ijk} z_k + d_{ij} \right).$$

By a linear change of variables, $z_i' = \sum_j \alpha_{ij} z_j$, we can assume $z_j'(q_i) = \delta_{ij}$, for $1 \leq i, j \leq d-1$, and, as by lemma 2.2.1 a trace free basis is trace free after a change of basis, we can evaluate $z_i'(q_d)$.

Evaluating the polynomial $(z_1')^2 = \sum c_{11j} z_j' + d_{11}$ in the d points we get,

$$\begin{cases} 1 & = & c_{111} + d_{11} \\ 0 & = & c_{11i} + d_{11} \qquad \text{for } 2 \leq i \leq d-1 \\ (z_1'(q_d))^2 & = & \sum c_{11i} z_i'(q_d) + d_{11} \end{cases}$$

from which we get, $c_{111} = 1 - \frac{(z_1'(q_d))^2 - z_1'(q_d)}{1-(\sum_j z_j'(q_d))}$, $1 - (\sum_j z_j(q_d)) \neq 0$ as equality would imply that the d points to be in the hyperplane $\sum z_i = 1$. Evaluating the polynomial $z_1' z_i' = \sum c_{1ij} z_j' + d_{1i}$, for $i \neq 1$,

$$\begin{cases} 0 & = & c_{1ij} + d_{1i} & \text{for } 1 \leq j \leq d-1 \\ (z_1' z_i')(q_d) & = & \sum c_{1ij} z_j'(q_d) + d_{1i} & \end{cases}$$

we find $c_{1ii} = -\frac{(z_1' z_i')(q_d)}{1-(\sum_j z_j'(q_d))}$. $\mathrm{tr}(z_1') = \sum_i c_{1ii} = 0$ which implies that $z_1'(q_d) = -1$ and, by the same argument, for all i, $z_i'(q_d) = -1$. In particular, we get the following proposition.

Proposition 3.4.1. *Given $\mathcal{Q} = \{q_1, \ldots, q_d\} \subset \mathbb{A}^{d-1}$ a set of d generic points and $(z_1 \ldots, z_{d-1})$ a trace free basis for the zero ideal $I_{\mathcal{Q}} \subset k[z_1, \ldots, z_{d-1}]$. Then*

$$\sum_j z_i(q_j) = 0$$

The following lemmas will be of use to prove rationality of $\mathcal{A}^{lev}_{(1,3)}$.

Lemma 3.4.2. *Let $I_{\mathcal{Q}} \subset k[z_1, \ldots, z_{d-1}]$ be an ideal generated by the equations*

$$\forall_{i \leq j}, z_i z_j = \sum_k c_{ijk} z_k + d_{ij}$$

where the c_{ijk}, d_{ij} satisfy the relations in $\mathcal{I}_d$. Then $\mathcal{Q}$ generates $\mathbb{A}^{d-1}$ or there are two or more points that coincide.

Proof. Assume the d points in $\mathcal{Q}$ to be distinct and not generating $\mathbb{A}^{d-1}$. Then by change of variables we can assume $z_{d-1}(q) = 0$ for all $q \in \mathcal{Q}$. This implies

$$\mathrm{Rad}\,(\mathcal{I}_{\mathcal{Q}}) = (z_{d-1}) \cdot \mathcal{I}_{\overline{\mathcal{Q}}},$$

where $\overline{\mathcal{Q}}$ is the set of points in $\mathbb{A}^{d-2}$ obtained by projecting $\mathcal{Q}$ from $(0, \ldots, 0, 1)$. As $z_{d-1}(q) = 0$ for all $q \in \mathcal{Q}$, the ideal $\mathcal{I}_{\overline{\mathcal{Q}}}$ contains the equations

$$\forall_{1 \leq i \leq j \leq d-2}, z_i z_j = \sum_k c_{ijk} z_k + d_{ij}.$$

Hence $\overline{\mathcal{Q}}$ is a set with $d-1$ points which is a contradiction. $\square$

Remark 3.4.3. Let $c_{ijk}, c'_{ijk} \in k$ coefficients of the equations defining covering maps of degree d over a point. The lemma above shows that if the c_{ijk} and c'_{ijk} are not in the branch locus then, via a $\mathrm{GL}(d-1,k)$ change of variables in $k[z_i]$, we can send the c_{ijk} into c'_{ijk}.

Lemma 3.4.4. *Let $I_Q \subset k[z_1, z_2]$ be an ideal generated by the equations*

$$z_1^2 = c_1 z_1 + c_0 z_2, z_1 z_2 = -c_1 z_1 - c_2 z_2, z_2^2 = c_3 z_1 + c_2 z_2.$$

If the c_i are not in the branch locus of a triple cover then the 2×2 matrices that fix these equations are a representation of the symmetric group S^3 in $\mathrm{GL}(2,k)$.

If $(c_0, c_1, c_2, c_3) = (0, 1, 1, 0)$ then these matrices are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}.$$

Proof. By Lemma 3.4.6 with a change of variables we can choose the (c_i) to be $(0, 1, 1, 0)$ and the result comes from the second statement.

The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,k)$ in the equations can be written as

$$\begin{pmatrix} z_1^2 \\ z_1 z_2 \\ z_2^2 \end{pmatrix} = \begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}.$$

The system of equations

$$\begin{pmatrix} a^2 & 2ab & b^2 \\ ac & ad+bc & bd \\ c^2 & 2cd & d^2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -1 \\ 0 & 1 \end{pmatrix}$$

have as solutions the set of matrices claimed above. □

3.4.1 Affine vs. Projective equations

Given d general points in $\mathbb{A}^{d-1}$, picking a trace free basis allows us to treat them as d points in k^{d-1} and as a trace free basis keeps being trace free after a change of basis, we can also consider them as d points in $\mathbb{P}^{d-2}$. The passage from one to the other is simply a projection from the origin.

For a general covering map of degree d, $\varphi\colon X \to Y$, we can homogenize the equations defining $X \subset \mathbb{A}(\mathcal{E})$ by adding a variable t (corresponding to $\mathcal{O}_Y \subset \varphi_*\mathcal{O}_X$), and then eliminate this variable from the ideal.

$$\begin{array}{ccc} X & \overset{p|_X}{\dashrightarrow} & \widetilde{X} \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ \mathbb{A}(\mathcal{E}) \subset \mathbb{P}(\varphi_*\mathcal{O}_X) & \overset{p}{\dashrightarrow} & \mathbb{P}(\mathcal{E}) \\ & \searrow{\scriptstyle \pi} & \downarrow{\scriptstyle \pi'} \\ & & Y \end{array}$$

We get $\widetilde{X}$ birational to X (birational as it is not defined when $z_i = 0$ for all i). It is well defined as all transition morphisms keep a basis trace free.

In theory such a change could make us lose information (as we should end up with less equations) but given d points in $\mathbb{P}^{d-2}_y$, $y \in Y$, we can choose a basis $(z_1, \dots, z_{d-1})$ and write them as $q_i = \lambda_i(q_{i1}, \dots, q_{id-1})$. If we solve the linear system in the λ_i

$$\sum_{j=1}^{d-1} \lambda_j q_{ji} = 0$$

and fix the points $\widetilde{q}_i = \lambda_i q_i$, we can get the equations in $\mathbb{A}^{d-1}$ for a trace free basis by Proposition 3.4.1.

Example 3.4.5. For $d = 3$, over an open $\mathcal{U} \subset Y$ and with a local basis (z, w), $X|_{\mathcal{U}}$ is given by the polynomials

$$\begin{aligned} z_1^2 &= (c_1 z_1 + c_0 z_2)t + 2(c_1^2 - c_0 c_2)t^2 \\ z_1 z_2 &= (-c_2 z_1 - c_1 z_2)t - (c_1 c_2 - c_0 c_3)t^2 \\ z_1^2 &= (c_3 z_1 + c_2 z_2)t + 2(c_2^2 - c_1 c_3)t^2 \end{aligned}$$

where $c_i \in \mathcal{O}_Y(\mathcal{U})$. By p that is just the elimination of the variable t in the ideal generated by the three equations (see A.1.1) we get

$$\begin{aligned} F_X|_{\mathcal{U}} := {} & (c_1 c_2 c_3 - \tfrac{1}{3} c_0 c_3^2 - \tfrac{2}{3} c_2^3) z_1^3 + (2c_1^2 c_3 - c_1 c_2^2 - c_0 c_2 c_3) z_1^2 z_2 \\ & + (c_1^2 c_2 + c_0 c_1 c_3 - 2c_0 c_2^2) z_1 z_2^2 + (\tfrac{2}{3} c_1^3 - c_0 c_1 c_2 + \tfrac{1}{3} c_0^2 c_3) z_2^3 \end{aligned} \tag{3.3}$$

Proposition 3.4.6. *The branch locus of a triple cover in Y is defined locally by*

$$4c_0 c_2^3 + 4c_1^3 c_3 + c_0^2 c_3^2 - 3c_1^2 c_2^2 - 6c_0 c_1 c_2 c_3 = 0$$

This is not a new result as we can find it in [Mir85, Lemma 4.2] but now we can compute it directly.

Proof. The branch locus is given by the discriminant of the cubic $F_X|_{\mathcal{U}}$. In appendix A.1.2 we have the computations that show,

$$\Delta(F_X|_U) = (4c_0c_2^3 + 4c_1^3c_3 + c_0^2c_3^2 - 3c_1^2c_2^2 - 6c_0c_1c_2c_3)^3$$

□

The big question becomes how to go get the c_i back from the polynomial $F_X|_{\mathcal{U}}$. Denoting by C the column matrix with the coefficients of $F_X|_{\mathcal{U}}$ in the order $(z_1^3, z_1^2z_2, z_1z_2^2, z_2^3)$, its left kernel is generated by the rows of the matrix

$$M_X|_{\mathcal{U}} := \begin{pmatrix} 3c_1 & -2c_2 & c_3 & 0 \\ 3c_0 & -c_1 & -c_2 & 3c_3 \\ 0 & c_0 & -2c_1 & 3c_2 \end{pmatrix}$$

(see appendix A.1.3). If (c_0, c_1) or (c_2, c_3) are complete intersections then, using a set of generators for the module $\ker(C)$, we can determine the quadruple (c_0, c_1, c_2, c_3) by arranging them in a 3×4 matrix like $M_X|_{\mathcal{U}}$.

Notice that the coefficients of $F_X|_{\mathcal{U}}$ are in fact the minors of $M_X|_{\mathcal{U}}$.

For $d \geq 4$ it is not so simple. The c_{ij} have relations between them so the computation of the elimination ideal is not so easy. On the other hand, we know that d generic points in $\mathbb{P}^{d-2}$ are given as zeros of a Gorenstein ideal generated by $d(d-3)/2$ quadratic homogeneous polynomials, for $d = 4$ it is generated by two quadratic polynomials in three variables that we can see as a point in $\mathrm{Gr}(2, 6)$. By [HM99, Theorem 3.1], the relations between the coefficients c_{ijk} defining four points in $\mathbb{A}^3$ are the equations of the Plücker embedding of $\mathrm{Gr}(2, 6)$ in $\mathbb{P}^{14}$.

We have not explored this relation further but we finish by proposing a method to do so. We can say that d points in $\mathbb{A}^{d-1}$ written using a trace free basis are the GL_{d-1} orbits of the $d-1$ vertex points and the point $(-1, -1, -1, \ldots, -1)$ by proposition 3.4.1. These are given by the zeros of the equations

$$\begin{cases} z_i^2 &= z_i - \frac{2}{d}\left(\sum_{j=1}^{d-1} z_j\right) + \frac{2}{d} \\ z_iz_j &= -\frac{1}{d}\left(\sum_{j=1}^{d-1} z_j\right) + \frac{1}{d}, \quad i \neq j. \end{cases}$$

Their projection in $\mathbb{P}^{d-2}$ is given by the zeros of the equations

$$z_i z_j = z_k z_l$$

for all (i, j, k, l) where $i \neq j$, $k \neq l$. So a starting point would be to compute the equations obtained by the action of a general element of $GL(d-1, k)$ in each set of such equations. This is just a computational approach to the problem. Of greater value would be to predict the relations between the c_{ijk} for the equations defining d points in $\mathbb{A}^{d-1}$ for $d > 4$.

Chapter 4

Commutative multiplication

As discussed in chapter 3, part of constructing a general cover is to find a cover homomorphism $\Phi_2 \in \mathrm{CHom}\left(S^2\mathcal{E}, \mathcal{E}\right)$. If $\mathcal{E}$ is a sum of line bundles the commutative condition is trivial as we can pick a global basis for $\mathcal{E}$, $(z_1, \ldots, z_{d-1})$, and any map of the form

$$z_i z_j \mapsto \sum c_{ijk} z_k$$

$i \leq j$, $c_{ijk} \in \mathcal{O}_Y$. The goal of this chapter is to discuss the structure of the $\mathcal{O}_{\mathbb{P}^2}$-module

$$\mathrm{CHom}\left(S^2\left(\Omega^1_{\mathbb{P}^2}(-m)\right), \Omega^1_{\mathbb{P}^2}(-m)\right)$$

for a non-negative integer m. Recall the free resolution of $S^2(\Omega^1_{\mathbb{P}^2}(-m))$

$$S^2(\Omega^1_{\mathbb{P}^2}(-m)) \xleftarrow{p} \mathcal{O}_{\mathbb{P}^2}(-2m-4)^{\oplus 6} \xleftarrow{M} \mathcal{O}_{\mathbb{P}^2}(-2m-5)^{\oplus 3} \leftarrow 0. \tag{4.1}$$

As $S^2(\Omega^1_{\mathbb{P}^2}(-m))$ is the cokernel of M, for an $\mathcal{O}_{\mathbb{P}^2}$-module $\mathcal{F}$, we treat a morphism $\Phi \in \mathrm{Hom}(S^2(\Omega^1_{\mathbb{P}^2}(-m)), \mathcal{F})$ as a morphism $\widetilde{\Phi} \in \mathrm{Hom}(6\mathcal{O}_{\mathbb{P}^2}(-2m-4), \mathcal{F})$ such that $\widetilde{\Phi} \circ M = 0$. As we will always be interested in locally free modules, the starting point is the case $\mathcal{F} = \mathcal{O}_{\mathbb{P}^2}$.

4.1 $\mathrm{Hom}\left(S^2\left(\Omega^1_{\mathbb{P}^2}(-m)\right), \mathcal{O}_{\mathbb{P}^2}\right)$

Using the notation introduced in example 2.1.9, let (y_0, y_1, y_2) be a basis for $\Omega^1_{\mathbb{P}^2}(-m)$ and write a basis for $S^2(\Omega^1_{\mathbb{P}^2}(-m))$ as the symmetric matrix

$$Y^2 := \begin{pmatrix} y_0^2 & y_0y_1 & y_0y_2 \\ y_0y_1 & y_1^2 & y_1y_2 \\ y_0y_2 & y_1y_2 & y_2^2 \end{pmatrix}$$

The relation given by M translates as $Y^2\overline{x} = 0$, where $\overline{x} = \begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}^t$. So we need to find a 3×3 symmetric matrix, L, such that $L\overline{x} = 0$.

Lemma 4.1.1 (Kovačec Lemma)**.** *If L is a 3×3 symmetric matrix in $\mathbb{C}[x_0, x_1, x_2]$ such that $L\overline{x} = 0$ then $L = M_3 N M_3$ where N is a symmetric matrix and M_3 is the syzygy matrix of the ideal (x_0, x_1, x_2),*

$$M_3 = \begin{pmatrix} 0 & x_2 & -x_1 \\ -x_2 & 0 & x_0 \\ x_1 & -x_0 & 0 \end{pmatrix}$$

Proof. It follows from the Koszul complex that $L\overline{x} = 0$ implies the existence of a 3×3 matrix P such that $L = PM_3$,

$$\begin{aligned} 0 = L\overline{x} = L^t\overline{x} = -M_3 P^t \overline{x} \quad &\Longrightarrow P^t\overline{x} = \lambda\overline{x}, \\ &\Longrightarrow (P^t - \lambda I) = QM_3 \\ &\Longrightarrow L = -M_3 Q M_3 - \lambda M_3 \\ &\Longleftrightarrow L^t = -M_3 Q^t M_3 + \lambda M_3, \quad \lambda \in \mathbb{C}[x_0, x_1, x_2] \end{aligned}$$

taking $N = -\frac{(Q+Q^t)}{2}$ we get the result □

Corollary 4.1.2. *A morphism $\Phi \in \mathrm{Hom}(S^2(\Omega^1_{\mathbb{P}^2}(-m_1)), \mathcal{O}_{\mathbb{P}^2}(-m_2))$ can be written as a 3×3 matrix,*

$$D = M_3 \begin{pmatrix} d_1 & d_2 & d_3 \\ & d_4 & d_5 \\ & & d_6 \end{pmatrix} M_3$$

$d_i \in k[x_0, x_1, x_2]$, $\deg(d_i) = 2m_1 - m_2 + 2$. *In particular,* $\mathrm{Hom}(S^2\Omega^1_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(-3)) = 0$.

4.1.1 $\mathrm{Hom}\left(S^2\left(\Omega^{n-1}_{\mathbb{P}^n}(-m)\right), \mathcal{O}_{\mathbb{P}^n}\right)$

We can generalize Lemma 4.1.1 to $(n+1) \times (n+1)$ symmetric matrices. From the resolution of $\Omega^{n-1}_{\mathbb{P}^n}(-m)$

$$\Omega^{n-1}_{\mathbb{P}^n}(-m) \leftarrow \mathcal{O}_{\mathbb{P}^n}(-n-m)^{\oplus(n+1)} \leftarrow \mathcal{O}_{\mathbb{P}^n}(-n-1-m) \leftarrow 0,$$

it will give us morphisms in $\mathrm{Hom}\left(S^2\left(\Omega^{n-1}_{\mathbb{P}^n}(-m)\right), \mathcal{O}_{\mathbb{P}^n}\right)$.

Denote the vector $\begin{pmatrix} x_0 & x_1 & \dots & x_n \end{pmatrix}^t$ by $\overline{x}_n$.

Corollary 4.1.3. *A $(n+1)\times(n+1)$ symmetric matrix L such that $L\overline{x}_n = 0$ can be decomposed as $L = M_{n+1}NM_{n+1}^t$ where M_{n+1} is the first syzygy matrix of the ideal $(x_0,\dots,x_n)$ and N is a symmetric matrix.*

Proof. Let's assume the result true for $k \le n-1$. If $L\overline{x}_n = 0$, setting $x_n = 0$,

$$L|_{x_n=0} = \begin{pmatrix} M_nN_1M_n^t & M_nN_2 \\ N_3M_n^t & N_4 \end{pmatrix}$$

where N_1 is a $\binom{n+1}{2}\times\binom{n+1}{2}$ symmetric matrix, N_2, N_3 and N_4 are $\binom{n+1}{2}\times 1$, $1\times\binom{n+1}{2}$ and 1×1 matrices.

From the symmetry of L we have that $N_3M_n^t = (M_nN_2)^t$ so we can choose $N_3 = N_2^t$. By the homogeneity of the entries in L, the entries in N_2 have degrees ≥ 1 which allows us to write $N_2 = \widetilde{N}_2\overline{x}_{n-1}$.

As $\deg(N_4) \ge 2$ we can write $N_4 = \overline{x}_{n-1}^t\widetilde{N}_4\overline{x}_{n-1}$, with $\widetilde{N}_4$ symmetric, which gives us

$$L|_{x_n=0} = \begin{pmatrix} M_nN_1M_n^t & M_n\widetilde{N}_2\overline{x}_{n-1} \\ \overline{x}_{n-1}^t\widetilde{N}_2^tM_n^t & \overline{x}_{n-1}^t\widetilde{N}_4\overline{x}_{n-1} \end{pmatrix}$$

If $x_n \neq 0$, to keep $L\overline{x}_n^t = 0$, L will be of the form

$$L = \begin{pmatrix} M_nN_1M_n^t - x_n(M_n\widetilde{N}_2 + \widetilde{N}_2^tM_n^t) + x_n^2\widetilde{N}_4 & M_n\widetilde{N}_2\overline{x}_{n-1} - x_n\widetilde{N}_4\overline{x}_{n-1} \\ \overline{x}_{n-1}^t\widetilde{N}_2^tM_n^t - x_n\overline{x}_{n-1}^t\widetilde{N}_4 & \overline{x}_{n-1}^t\widetilde{N}_4\overline{x}_{n-1} \end{pmatrix}$$

which can be decomposed as

$$L = \begin{pmatrix} M_n & -x_nI_n \\ 0 & \overline{x}_{n-1}^t \end{pmatrix}\begin{pmatrix} N_1 & \widetilde{N}_2 \\ \widetilde{N}_2^t & \widetilde{N}_4 \end{pmatrix}\begin{pmatrix} M_n^t & 0 \\ -x_nI_n & \overline{x}_{n-1} \end{pmatrix} = M_{n+1}NM_{n+1}^t$$

which is exactly the decomposition we were looking for. □

Corollary 4.1.4. *Given a basis for $\mathbb{P}^n$, $\Phi \in \mathrm{Hom}(S^2(\Omega_{\mathbb{P}^n}^{n-1}(-m_1)), \mathcal{O}_{\mathbb{P}^n}(-m_2))$ can be written as a $(n+1)\times(n+1)$ matrix,*

$$D = M_{n+1}NM_{n+1}^t,$$

N a symmetric matrix with entries $n_{ij} \in k[x_0,\dots,x_n]$, $\deg(n_{ij}) = 2n+2m_1-m_2-2$.

Looking at Problem 2.1.13 and Example 3.3.4 we should be looking for the $\mathcal{O}_{\mathbb{P}^n}$-module $\mathrm{Hom}(S^2\Omega_{\mathbb{P}^n}^1, \Omega_{\mathbb{P}^n}^1)$. In the next section we will say some words about this case for $\mathbb{P}^3$.

4.1.2 Hom $\left(S^2\left(\Omega^1_{\mathbb{P}^3}(-m)\right), \mathcal{O}_{\mathbb{P}^3}\right)$

Recall example 2.1.11, the generators for $\Omega^1_{\mathbb{P}^3}$ are y_{ij}, $0 \le i < j \le 3$, with relations,

$$\begin{pmatrix} 0 & y_{01} & y_{02} & y_{03} \\ & 0 & y_{12} & y_{13} \\ & & 0 & y_{23} \\ -\operatorname{sym} & & & 0. \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Locally $\Omega^1_{\mathbb{P}^3}$ is given by 3 elements, i.e. over the set $\mathcal{U}_i = \{x_i \neq 0\}$ we can eliminate the variables y_{jk} when one of j, k is i. More than that, from the relation

$$\begin{pmatrix} y_{01} & y_{02} & y_{03} \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^t = 0,$$

by lemma 4.1.1, we have that

$$\begin{pmatrix} y_{01}^2 & y_{01}y_{02} & y_{01}y_{03} \\ y_{01}y_{02} & y_{02}^2 & y_{02}y_{03} \\ y_{01}y_{03} & y_{02}y_{03} & y_{03}^2 \end{pmatrix} = M_3 \begin{pmatrix} d_1 & d_2 & d_3 \\ & d_4 & d_5 \\ & & d_6 \end{pmatrix} M_3$$

with d_i homogeneous polynomials in $k[x_0, \dots, x_3]$.

Now we have to do the same computations for all the rows of the matrix defining the relations between the y_{ij} and make the d_i agree globally. Notice that not all pairs y_{ij}, y_{kl} appear in a common line but all the y_{ij} generating $\Omega^1_{\mathbb{P}^3}$ locally do and the local structure will define a global one.

We will not show the final result as it is a simple computation that does not bring much yet, but this was to show that the structure of $\Omega^1_{\mathbb{P}^n}$ has so many restrictions that all these constructions are simpler than expected and can be a link to connect purely algebraic constructions with geometric objects.

4.2 $\mathrm{Hom}(S^2(\Omega^1_{\mathbb{P}^2}(-m)), \Omega^1_{\mathbb{P}^2}(-m))$

We want to describe the module $\mathrm{Hom}(S^2(\Omega^1_{\mathbb{P}^2}(-m)), \Omega^1_{\mathbb{P}^2}(-m))$ and the trace free morphisms inside. By theorem 3.1.3 for $d = 3$ and the discussion in section 3.2, this will be enough to completely describe the triple covering maps of degree 3, $\varphi\colon S \to \mathbb{P}^2$, such that

$$\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m).$$

Let $\Phi \in \mathrm{Hom}\left(S^2(\Omega^1_{\mathbb{P}^2}(-m)), \Omega^1_{\mathbb{P}^2}(-m)\right)$ and $\widetilde{\Phi} \in \mathrm{Hom}\left(\mathcal{O}_{\mathbb{P}^2}(-2m-4)^{\oplus 6}, \Omega^1_{\mathbb{P}^2}(-m)\right)$

such that $\widetilde{\Phi} \circ M = 0$.

$$\begin{array}{ccccccc}
S^2(\Omega^1_{\mathbb{P}^2}(-m)) & \xleftarrow{p} & \mathcal{O}_{\mathbb{P}^2}(-2m-4)^{\oplus 6} & \xleftarrow{M} & \mathcal{O}_{\mathbb{P}^2}(-2m-5)^{\oplus 3} & \longleftarrow & 0 \\
\downarrow \Phi & \swarrow \widetilde{\Phi} & \downarrow \Phi' & & & & \\
\Omega^1_{\mathbb{P}^2}(-m) & \xleftarrow{q} & \mathcal{O}_{\mathbb{P}^2}(-m-2)^{\oplus 3} & \xleftarrow{N} & \mathcal{O}_{\mathbb{P}^2}(-m-3) & \longleftarrow & 0
\end{array}$$

Applying the functor Hom $\left(\mathcal{O}_{\mathbb{P}^2}(-2m-4)^{\oplus 6}, -\right)$ to the lower exact sequence we get

$$0 = H^1\left(\mathcal{O}_{\mathbb{P}^2}(m+1)^{\oplus 6}\right) \leftarrow H^0\left(\Omega^1_{\mathbb{P}^2}(m+4)^{\oplus 6}\right) \leftarrow H^0\left(\mathcal{O}_{\mathbb{P}^2}(m+2)^{\oplus 18}\right) \leftarrow \dots$$

so for any $\widetilde{\Phi}$ there exists $\Phi' \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-2m-4)^{\oplus 6}, \mathcal{O}_{\mathbb{P}^2}(-m-2)^{\oplus 3})$ such that $\widetilde{\Phi} = q \circ \Phi'$. Given a basis (x_i, y_i) for $\mathbb{P}^2$, $\Omega^1_{\mathbb{P}^2}$, we write Φ' as

$$Y^2 = C_0 y_0 + C_1 y_1 + C_2 y_2$$

where C_0, C_1, C_2 are symmetric matrices in $k[x_i]$. The ideal case would be to have $C_i = M_3 N_i M_3$ but this is not necessarily true, just consider $C_i = x_i C$ for some symmetric matrix C. On the other hand for this case we have $Y^2 = C\left(\sum x_i y_i\right) = 0$ and this comes from the fact that $\widetilde{\Phi} \circ M = 0 \Leftrightarrow q \circ \Phi' \circ M = 0 \Rightarrow \Phi' \circ M \in \mathrm{Im}\, N$.

Although we have this ambiguity in global terms, if $\Phi'_1 = \sum C_i y_i$ and $\Phi'_2 = \sum C'_i y_i$ differ by an element $\sum (x_i C) y_i \in \mathrm{Im}\, N$, then locally they are the same. E.g. take the open set $\mathcal{U} = \{x_2 \neq 0\}$ then

$$\Phi'_1|_{\mathcal{U}} = \sum_{i=0}^{1} \left(C_i - \frac{x_i}{x_2} C_n\right) y_i = \sum_{i=0}^{1} \left(C'_i - \frac{x_i}{x_2} C'_n\right) y_i = \Phi'_2|_{\mathcal{U}}$$

and the morphisms to any other open set are just given by an $\mathcal{O}_{\mathbb{P}^2}$-automorphism.

All this discussion is still valid for morphisms between $S^2\Omega^{n-1}_{\mathbb{P}^n}$ and $\Omega^{n-1}_{\mathbb{P}^n}$ so we prove now that by a change of basis we can write all the C_i in the form $M_{n+1} N_i M_{n+1}$.

Proposition 4.2.1. *Given a morphism $\Phi \in \mathrm{Hom}\left(S^2(\Omega^{n-1}_{\mathbb{P}^n}(-m)), \Omega^{n-1}_{\mathbb{P}^n}(-m)\right)$ and any basis (x_i) for $\mathbb{P}^n$, there is a basis (y_i) for $\Omega^{n-1}_{\mathbb{P}^n}$ such that Φ can be written as*

$$\Phi = C_0 y_0 + \dots + C_n y_n,$$

$C_i \overline{x}_n = 0$ for all i.

Proof. Assume Φ can be written as $C'_0 y_0 + \dots + C'_n y_n$ for a basis (x_i, y_i) of $\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}$.

If $C'_n\overline{x}_n = 0$ then

$$\Phi|_{\{x_n\neq 0\}} \circ M = 0 \Leftrightarrow (x_n C'_i - x_i C'_n)\overline{x}_n = 0 \Rightarrow C'_i\overline{x}_n = 0$$

so we just need to make a change of variables in the y_i such that $C'_n\overline{x}_n = 0$.

Decomposing $C'_n = C_n + x_n C$, where the entries of C_n are in $k[x_0, \ldots, x_{n-1}]$, define $C_i = C'_i - x_i C$. Let $\widehat{C}_{n-1}$ be the submatrix of C_n with its first n columns and n rows. Then $\widehat{C}_{n-1}$ satisfies $\widehat{C}_{n-1}\overline{x}_{n-1} = 0$ as none of its entries has a multiple of x_n and $(C_i - x_i/x_n C_n)\overline{x}_n = 0$.

We only have to deal with the last column and row of C_n. Choosing a decomposition of $C_n[n+1, n+1] = 2\left(\sum_{i=0}^{n-1} a_i x_i\right)$ and changing variables $y_n \mapsto y_n - \sum a_i x_i$ and $y_i \mapsto y_i + x_n a_i$ for $i \neq n$, only the last row and column of C_n change and $C_n[n+1, n+1] = 0$. C_n has the form

$$\begin{pmatrix} \widehat{C}_{n-1} & v^t \\ v & 0 \end{pmatrix}$$

where $\widehat{C}_{n-1}\overline{x}_{n-1} = 0$. Using the same trick as in the beginning, we can get rid of all the multiples of x_n in v. With all the entries of v in $k[x_0, \ldots, x_{n-1}]$, a change of variables $y_i \mapsto y_i - v[i]$ completes the proof. □

Remember that being trace-free is a local property but the form Φ is written above is a global decomposition that might not be trace-free.

Corollary 4.2.2. *Given $\Phi \in \operatorname{Hom}\left(S^2(\Omega_{\mathbb{P}^n}^{n-1}(-m)), \Omega_{\mathbb{P}^n}^{n-1}(-m)\right)$ and any basis (x_i) for $\mathbb{P}^n$, there is a basis (y_i) for $\Omega_{\mathbb{P}^n}^{n-1}$ such that we can write Φ as $\sum C_i y_i$ and for all open sets $\{x_i \neq 0\}$ the variables y_j, $j \neq i$, are trace zero elements.*

Proof. Using proposition 4.2.1, write Φ as $\sum C_i y_i$ where $C_i\overline{x}_n = 0$ for all i. If $x_n \neq 0$, we have that the trace of y_i, $i \neq n$, is given by

$$\operatorname{tr}(y_i) = \sum_{j\neq n}\left(C_j - \frac{x_j}{x_n}C_n\right)[i,j] = \sum_{j=0}^{n}\left(C_j - \frac{x_j}{x_n}C_n\right)[i,j] = \sum_{j=0}^{n} C_j[i,j]$$

In particular, $\operatorname{tr}(y_i)$ does not depend on x_i and we can define the row v as $v[i] = \operatorname{tr}(y_i)$. From the following equalities

$$\sum_i x_i v[i] = \sum_i x_i \sum_j C_j[i, i+j] = \sum_j \sum_i x_i C_j[i, i+j] = 0$$

one see that v can be written as $v' M_n$ which allows to use the change of variables

$y_i \mapsto y_i - \frac{v[i]}{n+1}$ and we are done. $\square$

Notice that the proof of corollary 4.2.2 is similar to the change of basis to make a polynomial of degree n trace free, i.e., if we take the polynomial $p(x) \in R[x]$, $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then the change of variables $x \mapsto x - \frac{a_{n-1}}{n}$ sends $p(x)$ to a trace free polynomial.

In the case $n = 2$ we can write $\Phi \in \operatorname{Hom}\left(S^2(\Omega^1_{\mathbb{P}^2}(-m)), \Omega^1_{\mathbb{P}^2}(-m)\right)$ as

$$Y^2 = C_0 y_0 + C_1 y_1 + C_2 y_2, \quad C_i = M_3 \begin{pmatrix} c_{i1} & c_{i2} & c_{i3} \\ & c_{i4} & c_{i5} \\ sym & & c_{i6} \end{pmatrix} M_3$$

where each c_{ij} is a polynomial of degree m. Using the change of variables described in Corollary 4.2.2, $\Phi|_{\{x_2 \neq 0\}}$ can be written as

$$\begin{aligned} y_0^2 &= c_1 y_0 + c_0 y_1 \\ y_0 y_1 &= -c_2 y_0 - c_1 y_1 \\ y_1^2 &= c_3 y_2 + c_2 y_1 \end{aligned} \tag{4.2}$$

where each c_i is of the form

$$\begin{aligned} c_0 =& x_2^2(c_{14}) - x_1 x_2(2c_{15} + c_{24}) + x_1^2(c_{16} + 2c_{25}) - \frac{x_1^3}{x_2}(c_{26}) \\ c_1 =& \tfrac{1}{3}x_2^2(c_{04} + 2c_{12}) - \tfrac{2}{3}x_1 x_2(c_{05} + c_{13} + c_{22}) + \tfrac{1}{3}x_1^2(c_{06} + 2c_{23}) \\ & - \tfrac{1}{3}x_0 x_2(2c_{15} + c_{24}) + \tfrac{2}{3}x_0 x_1(c_{16} + 2c_{25}) - \frac{x_0 x_1^2}{x_2}(c_{26}) \\ c_2 =& \tfrac{1}{3}x_2^2(2c_{02} + c_{11}) - \tfrac{1}{3}x_1 x_2(2c_{03} + c_{21}) - \tfrac{2}{3}x_0 x_2(c_{05} + c_{13} + c_{22}) \\ & + \tfrac{2}{3}x_0 x_1(c_{06} + 2c_{23}) + \tfrac{1}{3}x_0^2(c_{16} + 2c_{25}) - \frac{x_0^2 x_1}{x_2}(c_{26}) \\ c_3 =& x_2^2(c_{01}) - x_0 x_2(2c_{03} + c_{21}) + x_0^2(c_{06} + 2c_{23}) - \frac{x_0^3}{x_2}(c_{26}) \end{aligned} \tag{4.3}$$

From the open set $\{x_2 \neq 0\}$ to any other $\{x_1 \neq 0\}$ it is just a $\mathcal{O}_{\mathbb{P}^2}$–linear transformation, so the 10 parameters

$$\begin{aligned} c_{01}, 2c_{02} + c_{11}, 2c_{03} + c_{21}, c_{04} + 2c_{12}, c_{05} + c_{13} + c_{22}, \\ c_{06} + 2c_{23}, c_{14}, 2c_{15} + c_{24}, c_{16} + 2c_{25}, c_{26} \end{aligned}$$

describe all the morphisms in $\operatorname{CHom}(S^2(\Omega^1_{\mathbb{P}^2}(-m)), \Omega^1_{\mathbb{P}^2}(-m))$.

To simplify notation we will denote the parameters above as

c_{01}	β_0	$2c_{02}+c_{11}$	β_{01}	$c_{05}+c_{13}+c_{22}$	β_{012}
c_{14}	β_1	$2c_{03}+c_{21}$	β_{02}		
c_{26}	β_2	$c_{04}+2c_{12}$	β_{10}		
		$2c_{15}+c_{24}$	β_{12}		
		$c_{06}+2c_{23}$	β_{20}		
		$c_{16}+2c_{25}$	β_{21}		

so the equations over $\{x_2 \neq 0\}$ look like,

$$\begin{aligned}
c_0 =&\beta_1 x_2^2 - \beta_{12}x_1x_2 + \beta_{21}x_1^2 - \beta_2\frac{x_1^3}{x_2}\\
c_1 =&\tfrac{1}{3}\beta_{10}x_2^2 - \tfrac{2}{3}\beta_{012}x_1x_2 + \tfrac{1}{3}\beta_{20}x_1^2 - \tfrac{1}{3}\beta_{12}x_0x_2 + \tfrac{2}{3}\beta_{21}x_0x_1 - \beta_2\frac{x_0x_1^2}{x_2}\\
c_2 =&\tfrac{1}{3}\beta_{01}x_2^2 - \tfrac{1}{3}\beta_{02}x_1x_2 - \tfrac{2}{3}\beta_{012}x_0x_2 + \tfrac{2}{3}\beta_{20}x_0x_1 + \tfrac{1}{3}\beta_{21}x_0^2 - \beta_2\frac{x_0^2x_1}{x_2}\\
c_3 =&\beta_0 x_2^2 - \beta_{02}x_0x_2 + \beta_{20}x_0^2 - \beta_2\frac{x_0^3}{x_2}
\end{aligned} \tag{4.4}$$

Because we are always allowed to multiply any element in $\mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$ by an element in k^*, for each choice basis (x_i) and trace free basis (y_i), we have a correspondence between each morphism and a point in $\mathbb{P}^9$. In the next section we will see how to choose a better trace free basis and the relation with the group whose action fixes the abelian surface with a polarization of type $(1,3)$.

4.2.1 Group actions and change of basis

Recall that if $(S,\mathcal{L})$ is an abelian surface with a polarization of type $(1,3)$ the morphism $\varphi_{\mathcal{L}}$ is equivariant for the Heisenberg group $H(3)^e$. Choosing coordinates (x_0,x_1,x_2) a representation of this group in $\mathrm{PGL}(3,\mathbb{C})$ is generated by $\sigma_1', \sigma_2', \tau'$

$$\sigma_1' = \begin{pmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 1 & 0 & 0\end{pmatrix}, \sigma_2' = \begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}, \tau' = \begin{pmatrix} 1 & 0 & 0\\ 0 & \xi & 0\\ 0 & 0 & \xi^2\end{pmatrix} \tag{4.5}$$

$\xi \neq 1$ a cubic root of unity. Notice that σ_1', σ_2' generate the symmetric group S_3 and τ' generates the multiplicative group μ_3. These elements act on the graded ring

$$R = \mathbb{C}[x_0,x_1,x_2,;y_0,y_1,y_2]/\left(\sum x_iy_i\right)$$

$\deg(x_i) = 1, \deg(y_i) = 2$, as

$$\begin{aligned}
\sigma_1(x_i, y_i) &= (\sigma_1'(x_i), \sigma_1'(y_i))\\
\sigma_2(x_i, y_i) &= (\sigma_2'(x_i), \sigma_2'(y_i))\\
\tau(x_i, y_i) &= (\tau'(x_i), \tau'^{-1}(y_i))
\end{aligned}$$

These are actions of the groups in the scheme $\mathbb{A}\left(\Omega^1_{\mathbb{P}^2}\right) = \mathrm{Proj}(R)$, (see example 2.1.9). We will now find the conditions on the coefficients β_i so that an element $\Phi_2 \in \mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$ is an equivariant map for $H(3)^e$.

Let us start with σ_1. Over the open set $\mathcal{U}_1 = \{x_1 \neq 0\}$, $\Phi_2|_{\mathcal{U}_1}$ is of the form,

$$\begin{aligned}
y_0^2 &= c_1' y_0 + c_0' y_2\\
y_0 y_2 &= -c_2' y_0 - c_1' y_2\\
y_2^2 &= c_3' y_0 + c_2' y_2.
\end{aligned}$$

Applying σ_1 we get

$$\begin{aligned}
y_1^2 &= \sigma_1'(c_1') y_1 + \sigma_1'(c_0') y_0\\
y_1 y_0 &= -\sigma_1'(c_2') y_1 - \sigma_1'(c_1') y_0\\
y_0^2 &= \sigma_1'(c_3') y_1 + \sigma_1'(c_2') y_0,
\end{aligned}$$

which gives us that

$$c_0 = \sigma_1(c_3'), c_1 = \sigma_1(c_2'), c_2 = \sigma_1(c_1'), c_3 = \sigma_1(c_0'),$$

where the c_i appear in $\Phi_2|_{\mathcal{U}_2}$ (see 4.4), and the c_i' in $\Phi_2|_{\mathcal{U}_1}$,

$$\begin{aligned}
c_0' =& \beta_2 x_1^2 + \beta_{12} x_2^2 - \beta_{21} x_1 x_2 - \beta_1 \frac{x_2^3}{x_1}\\
c_1' =& \tfrac{1}{3}\beta_{10} x_2^2 - \tfrac{2}{3}\beta_{012} x_1 x_2 + \tfrac{1}{3}\beta_{20} x_1^2 + \tfrac{2}{3}\beta_{12} x_0 x_2 - \tfrac{1}{3}\beta_{21} x_0 x_1 - \beta_1 \frac{x_0 x_2^2}{x_1}\\
c_2' =& -\tfrac{1}{3}\beta_{01} x_1 x_2 + \tfrac{1}{3}\beta_{02} x_1^2 + \tfrac{2}{3}\beta_{10} x_0 x_2 - \tfrac{2}{3}\beta_{012} x_0 x_1 + \tfrac{1}{3}\beta_{12} x_0^2 - \beta_1 \frac{x_0^2 x_2}{x_1}\\
c_3' =& \beta_0 x_1^2 - \beta_{01} x_0 x_1 + \beta_{10} x_0^2 - \beta_1 \frac{x_0^3}{x_2}
\end{aligned} \tag{4.6}$$

As there is no relation between the x_i we get the following equalities

$$\begin{aligned}
&\beta_0 = \beta_1 = \beta_2\\
&\beta_{01} = \beta_{12} = \beta_{20}\\
&\beta_{02} = \beta_{10} = \beta_{21}.
\end{aligned}$$

To compute the relations between the coefficients for the action of σ_2 we just need to apply it to the equations 4.4

$$\begin{aligned} y_1^2 &= \sigma_2(c_1)y_1 + \sigma_2(c_0)y_0 \\ y_1y_0 &= -\sigma_2(c_2)y_1 - \sigma_2(c_1)y_0 \\ y_0^2 &= \sigma_2(c_3)y_1 + \sigma_2(c_2)y_0, \end{aligned}$$

from where we get

$$\sigma_2(c_0) = c_3, \sigma_2(c_1) = c_2, \sigma_2(c_2) = c_1, \sigma_2(c_3) = c_0,$$

which brings the equalities

$$\begin{aligned} \beta_0 &= \beta_1 \\ \beta_{01} &= \beta_{10} \\ \beta_{02} &= \beta_{12} \\ \beta_{20} &= \beta_{21}. \end{aligned}$$

For the action of $\tau \in \mu_3$, we get the relations

$$\tau(c_0) = \xi^2 c_0, \tau(c_1) = c_1, \tau(c_2) = \xi c_2, \tau(c_3) = \xi^2 c_3$$

from where we conclude

$$\beta_{01} = \beta_{02} = \beta_{10} = \beta_{12} = \beta_{20} = \beta_{21} = 0.$$

Proposition 4.2.3. *Given a morphism $\Phi_2 \in \mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$ and basis $(x_i), (y_i)$ for $\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}$ such that $\Phi_2|_{\{x_2 \neq 0\}}$ can be written as*

$$\begin{aligned} y_0^2 &= c_0y_0 + c_1y_1 \\ y_0y_1 &= -c_3y_3 - c_0y_1 \\ y_1^2 &= c_2y_2 + c_3y_1 \end{aligned}$$

with the c_i as in 4.4. Then Φ_2 is equivariant for the symmetric group S_3 if,

$$\begin{aligned} &\beta_0 = \beta_1 = \beta_2 \\ &\beta_{01} = \beta_{02} = \beta_{10} = \beta_{12} = \beta_{20} = \beta_{21}, \end{aligned}$$

equivariant for the group μ_3 if,

$$\beta_{01} = \beta_{02} = \beta_{10} = \beta_{12} = \beta_{20} = \beta_{21} = 0,$$

and equivariant for the actions of σ_1 and τ if,

$$\beta_0 = \beta_1 = \beta_2$$
$$\beta_{01} = \beta_{02} = \beta_{10} = \beta_{12} = \beta_{20} = \beta_{21} = 0.$$

Notice that if Φ_2 it is equivariant for σ_1 and τ, it also is for σ_2.

Remark 4.2.4. The choice of index in the coefficients β_* was not random. If we assign to the coefficient β_i the monomial x_i^3, to β_{ij} the monomial $x_i^2 x_j$ and to β_{012} the monomial $x_0 x_1 x_2$, each of the actions σ_1, σ_2, τ change them exactly as if they are acting on those monomials.

We have no explanation for this fact but it would be something worth further exploration, first studying if we have such a relation for $m > 0$ and if get some similar structure in the $\mathcal{O}_{\mathbb{P}^3}$-module $\mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^3}, \Omega^1_{\mathbb{P}^3}\right)$.

Theorem 4.2.5. *Given a general element $\Phi_2 \in \mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$, there is a choice of basis (x_i) for $\mathbb{P}^2$ such that $\Phi_2|_{\{x_2 \neq 0\}}$ can be written in the form 4.2 with the following c_i*

$$\begin{aligned}
c_0 &= \beta_0 x_2^2 - \beta_0 \frac{x_1^3}{x_2} \\
c_1 &= -\tfrac{2}{3}\beta_{012} x_1 x_2 - \beta_0 \frac{x_0 x_1^2}{x_2} \\
c_2 &= -\tfrac{2}{3}\beta_{012} x_0 x_2 - \beta_0 \frac{x_0^2 x_1}{x_2} \\
c_3 &= \beta_0 x_2^2 - \beta_0 \frac{x_0^3}{x_2}
\end{aligned}$$

In particular, Φ_2 is equivariant for the groups S_3, μ_3.

The basis choice is only for the (x_i) as the (y_i) are fixed for each such choice by the relation $\sum_i x_i y_i = 0$. The only change of basis left in the (y_i) would be a change

$$y_i \mapsto y_i + v[i]$$

with v a $3{\times}1$ matrix such that $\sum_i x_i v[i] = 0$, but such change would imply $\mathrm{tr}(y_i) \neq 0$.

Proof. In section 6.2.1 we see that a general morphism $\Phi_2 \in \mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$ generates a smooth model for an abelian surface with a polarization of type $(1, 3)$ and therefore, by a change of variables, we can make the basis (x_i) equivariant for $H(3)^e$. □

This is not a surprising result as a morphism in $\mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$ depends on 10 parameters and $\dim\left(\mathrm{PGL}(3, \mathbb{C})\right) = 8$.

4.3 Curve in the linear system $|\mathcal{L}|$

Given a curve C in the linear system $|\mathcal{L}|$, the linear system defining the covering map $\varphi\colon S \to \mathbb{P}^2$ such that $\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}$, we want to study $\varphi|_C$.

Proposition 4.3.1. *Given a general triple cover $\varphi\colon S \to \mathbb{P}^2$ such that*

$$\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}$$

then, for a general line $l \in \mathbb{P}^2$, $\varphi' := \varphi|_C\colon C \to l \cong \mathbb{P}^1$ satisfies

$$\varphi'_*\mathcal{O}_C = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2),$$

with $C \subset \mathbb{A}\left(\mathcal{O}_{\mathbb{P}^1}(-1)\right)$, in particular, C is a elliptic curve that can be seen as the zero of a cubic polynomial in $\mathbb{P}^2$.

Proof. Given a line $l \subset \mathbb{P}^2$,

$$\mathbb{A}\left(\Omega^1_{\mathbb{P}^2}(-m)\right)\Big|_l \cong \mathbb{A}\left(\mathcal{O}_{\mathbb{P}^1}(-m-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m-2)\right)$$

as from the free resolution of $\Omega^1_{\mathbb{P}^2}(-m)$

$$\Omega^1_{\mathbb{P}^2}(-m) \leftarrow \mathcal{O}_{\mathbb{P}^2}(-m-2)^{\oplus 3} \xleftarrow{M} \mathcal{O}(-m-3) \leftarrow 0,$$

by change of variables we can make the first entry of M to be l and denote the other 2 entries by x_1, x_2, from where we can decompose the resolution as the sum of the following resolutions

$$\mathcal{O}_{\mathbb{P}^1}(-m-2) \leftarrow 0 \text{ and } \mathcal{O}_{\mathbb{P}^1}(-m-2)^{\oplus 2} \leftarrow \mathcal{O}_{\mathbb{P}^1}(-m-3) \leftarrow 0.$$

Using 2.1.3 and the fact that the cokernel of the second one is $\mathcal{O}_{\mathbb{P}^1}(-m-1)$ we get the isomorphism as in example 2.1.2.

Taking a general morphism $\Phi_2 \in \operatorname{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$ we can assume $l = \{x_0 = 0\}$ and get $\Phi_2|_{\{x_2\neq 0, x_0=0\}}$ written with the equations in 4.2, where $\deg(y_0) = 2$, $\deg(y_1) = 1$, with the following c_i

$$\begin{aligned} c_0 =& \beta_1 x_2^3 - \beta_{12} x_1 x_2^2 + \beta_{21} x_1^2 x_2 - \beta_2 x_1^3 \\ c_1 =& \tfrac{1}{3}\beta_{10} x_2^2 - \tfrac{2}{3}\beta_{012} x_1 x_2 + \tfrac{1}{3}\beta_{20} x_1^2 \\ c_2 =& \tfrac{1}{3}\beta_{01} x_2 - \tfrac{1}{3}\beta_{02} x_1 \\ c_3 =& \beta_0 \end{aligned} \tag{4.7}$$

If Φ_2 is general enough we can rescale y_i and assume $\beta_0 = 1$. Setting $x_2 = 1$ we get the equation

$$\begin{aligned} y_1^2 &= y_0 + \left(\tfrac{1}{3}\beta_{01} - \tfrac{1}{3}\beta_{02}x_1\right) y_1 \\ &\quad + \tfrac{2}{9}\left((\beta_{01}^2 - 3\beta_{10}) - 2(\beta_{01}\beta_{02} - 3\beta_{012})x_1 + (\beta_{02}^2 - 3\beta_{20})x_1^2\right), \end{aligned}$$

which allows us to eliminate the variable y_0 and get the single equation

$$y_1^3 + 3\left(c_1c_3 - c_2^2\right) y_1 + \left(3c_1c_2c_3 - c_0c_3^2 - 2c_2^3\right) = 0.$$

The polynomial $(c_1c_3 - c_2^2)$ is quadratic and $(3c_1c_2c_3 - c_0c_3^2 - 2c_2^3)$ cubic in $\mathbb{C}[x_1, x_2]$. As the coefficients of the c_i are free we get any cubic polynomial in $\mathbb{C}[x_1, x_2, y_1]$ so in general it is smooth. □

4.4 Constructions

In this section we want to say some words about the polarized surfaces $(S, \mathcal{L})$, where $\mathcal{L}$ is a base point free ample divisor, such that

$$\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m).$$

Proposition 4.4.1. *[Mir85, Corollary 10.6] Assume that $\varphi\colon S \to \mathbb{P}^2$ is a triple cover with Tschirnhausen module $\Omega^1_{\mathbb{P}^2}(-m)$ with $m \geq 0$. Then*

1. *$q(S) = 0$ if $m \geq 1$; $q(S) = 1$ if $m = 0$.*
2. *$p_g(S) = m^2 - 1$ if $m \geq 1$; $p_g(S) = 0$ if $m = 0$.*
3. *$K_S^2 = m(5m - 9)$.*
4. *$e(S) = m(7m + 9)$.*

By theorem 3.1.3 for $d = 3$ and the discussion in section 3.2, we now have the explicit construction of such surfaces as we understand the $\mathcal{O}_{\mathbb{P}^2}$-module $\mathrm{CHom}\left(S^2\left(\Omega^1_{\mathbb{P}^2}(-m)\right), \Omega^1_{\mathbb{P}^2}(-m)\right)$. The structure of these modules gives us for free the following result.

Proposition 4.4.2. *Consider a covering map of degree three, $\varphi\colon S \to \mathbb{P}^2$, such that $\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m)$, $m \in \mathbb{N}$. If φ is determined by a general morphism $\Phi_2 \in \mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}(-m), \Omega^1_{\mathbb{P}^2}(-m)\right)$, then S is smooth.*

Proof. For $m = 0$ the magma code in appendix A.1.4 shows smoothness over the open set $\{x_i \neq 0\}$ for general β_*.

Singularities over the open set $\{x_2 \neq 0\}$ are given by the vanishing of all the 2×2 minors of the 3×4 Jacobian matrix of the local equations of S and this is an ideal in the variables x_0, x_1 that has no solution for $m=0$.

For $m>0$, we are changing the degree of the polynomials c_i from 3 to $3+m$. If the coefficients of the terms added to the polynomial are general, the 2×2 minors of the Jacobian matrix do not intersecting as well and the surface is smooth. $\square$

Note that by proposition 4.4.2 all the fibres are Gorenstein. If the fibre over a point $p \in \mathbb{P}^2$ is defined with $c_0 \neq 0$, one may solve for y_1 in terms of y_0 and S will be defined by

$$y_0^3 - 3(c_1^2 - c_0c_2) + (c_0(c_0c_3 - c_1c_2) - 2c_1(c_1^2 - c_0c_2)) = 0,$$

and similar for $c_3 \neq 0$. If $c_0 = c_3 = 0$ then, assuming Φ_2 is general, $c_1 \neq 0 \neq c_2$ and using a linear change of basis we can get $c_1' \neq 0$.

It does not agree with proposition 3.3.2 exactly because there is no line bundle on $\mathbb{P}^2$ such that $\omega_{S|\mathbb{P}^2} = \varphi^*\mathcal{L}$.

4.4.1 Main Problem

Now we can apply the results from the chapter to our main problem. Let $\varphi\colon S \to \mathbb{P}^2$ is a Gorenstein covering map of degree 6 such that

$$\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3).$$

By corollary 3.3.3 we have the embedding $S \hookrightarrow \mathbb{A}\left(\Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}\right)$. We want to find the initial ideal $\mathrm{in}(\mathcal{I}_S)$. As

$$S^2\left(\left(\Omega^1_{\mathbb{P}^2}\right)^{\oplus 2}\right) = \left(S^2\Omega^1_{\mathbb{P}^2}\right)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^2}(-3)$$

and by corollary 4.1.2

$$\mathrm{Hom}\left(S^2\Omega^1_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(-3)\right) = 0$$

we get

$$\mathcal{Q} = \left(S^2\Omega^1_{\mathbb{P}^2}\right)^{\oplus 3}.$$

We conclude that the Gorenstein covering maps solution we are searching for are given by sections of the $\mathcal{O}_{\mathbb{P}^2}$-module

$$\mathrm{CHom}\left(\left(S^2\Omega^1_{\mathbb{P}^2}\right)^{\oplus 3}, \left(\Omega^1_{\mathbb{P}^2}\right)^{\oplus 2}\right).$$

As $\mathbb{A}\left(\Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}\right) = \operatorname{Proj}\left(k[x_i; y_i, z_i]/\left(\sum x_i y_i, \sum x_i z_i\right)\right),\ 0 \leq i \leq 2$, for a point $p \in \{x_2 \neq 0\} \subset \mathbb{P}^2$ we have

$$q = \begin{pmatrix} y_0^2 & y_0 y_1 & y_1^2 & y_0 z_0 & \frac{1}{2}(y_0 z_1 + y_1 z_0) & y_1 z_1 & z_0^2 & z_0 z_1 & z_1^2 \end{pmatrix}.$$

By theorem 3.1.4 the associative condition will give us Φ_1 from Φ_2, i.e. the d_i will be given by quadratic equations on c_{ij}. In chapter 5 we will compute this equations and the relations that the entries c_{ij} have to satisfy between themselves.

Recall that for an $(1,3)$-polarized abelian surface $(S, \mathcal{L})$ the covering map $\varphi_{\mathcal{L}}$ is equivariant for the extended Heisenberg group $H^e(3)$. As, by Proposition 4.2.3, a $H^e(3)$-equivariant map in $\operatorname{CHom}(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2})$ depends on two parameters (β_0, β_{012}), we have a very small family of morphisms in $\operatorname{Hom}\left(\left(S^2\Omega^1_{\mathbb{P}^2}\right)^{\oplus 3}, \left(\Omega^1_{\mathbb{P}^2}\right)^{\oplus 2}\right)$ candidates for solution.

Chapter 5

Associative multiplication

Given a covering map $\varphi\colon X \to Y$, $\mathcal{O}_Y$ an integral local k-algebra, let $\mathcal{F}$ be an $\mathcal{O}_Y$-module, $\operatorname{rank}(\mathcal{F}) = r$, such that there is an embedding $i\colon X \hookrightarrow \mathbb{A}(\mathcal{F}) \to Y$.

Denote a basis for $\mathcal{F}$ by $(z_1, \ldots, z_r)$ so that we have the isomorphism $\mathcal{O}_{\mathbb{A}(\mathcal{F})} \cong \mathcal{O}_Y[z_i]$, and let $(f_1, \ldots, f_n)$ be generators of the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_Y[z_i]$, where

$$f_i = q_i - \sum_j c_{ij}\overline{z}^j,\ j \in \mathbb{N}^r,$$

q_i generators of the ideal $\operatorname{in}(\mathcal{I}_X)$. In this chapter we want to determine the relations between the c_{ij} so that the ideal $\mathcal{I}_X$ is Cohen-Macaulay. In Lemma 5.1.2 we prove that the minimal free resolution of $\mathcal{I}_X$ and $\operatorname{in}(\mathcal{I}_X)$ have the same format and by Nakayama's lemma, the relations between the c_{ij} are kept in $\mathcal{I}_X \otimes k$, so we can consider $c_{ij} \in k$.

Starting with the ideal $\operatorname{in}(\mathcal{I}_X)$, generated by the homogeneous polynomials $q \in \overline{R} = k[z_1, \ldots, z_r]$, where we are considering that $\deg(z_i) = 1$, we want to add a variable z_0 with degree 1 so that the ideal generated by the polynomials $q_i + z_0 q_i'$, $q_i' \in \overline{R}[z_0]$, has the same resolution.

This chapter is strongly based in the paper [Rei90] by Miles Reid. We will start by briefly recalling some results from the paper and apply them to prove theorems 3.1.3 and 3.1.4. In the end we use the theory to compute $\mathcal{I}_q$ for our main problem, q as shown in section 4.4.1.

5.1 Preliminaries

The (more) general problem attacked in paper [Rei90] is the following:

Problem 5.1.1. Given a graded ring $\overline{R}$ and $a_0 \in \mathbb{Z}$, $a_0 > 0$, describe the set of all pairs (z_0, R), where R is a graded ring and $z_0 \in R_{a_0}$ a homogeneous, non-zero divisor of degree a_0 such that

$$\overline{R} = R/(z_0).$$

In this chapter, we will keep the notation of the problem. R for a graded ring, $R = \bigoplus_{i=0}^{\infty} R_i$, $R_0 = k$, and $\overline{R} = R/(z_0)$, where z_0 is a non-zero divisor in R_{a_0}. If $\overline{R}$ is generated by $(z_1, \ldots, z_n)$, $\deg(z_i) = a_i$, then R is generated by $(z_0, z_1, \ldots, z_n)$. Let $S = k[z_0, z_1, \ldots, z_n]$ and $\overline{S} = S/(z_0)$, then we can write the free resolutions,

$$R \leftarrow \mathcal{P}_0 \xleftarrow{F} \mathcal{P}_1 \xleftarrow{L} \mathcal{P}_2 \leftarrow \cdots \leftarrow \mathcal{P}_c \leftarrow 0, \tag{5.1}$$

where $\mathcal{P}_i \cong \bigoplus_j S(-a_{ij})$,

$$\overline{R} \leftarrow \overline{\mathcal{P}}_0 \xleftarrow{f} \overline{\mathcal{P}}_1 \xleftarrow{l} \overline{\mathcal{P}}_2 \leftarrow \cdots \leftarrow \overline{\mathcal{P}}_c \leftarrow 0, \tag{5.2}$$

$\overline{\mathcal{P}}_i = \mathcal{P}_i \otimes S/(z_0)$. If we denote by I the ideal such that $R = S/I$, F is a vector with the generators for I, L its first syzygy matrix, $f = F|_{\{z_0=0\}}$ and $l = L|_{\{z_0=0\}}$.

In chapter 3 we already claimed that if z_0 is a non-zero divisor, then the resolutions have the same format without proof. For completeness sake, we add the result here.

Proposition 5.1.2. *If z_0 is a non-zero divisor in a ring $R = S/I$, then for all $i > 0$, $\operatorname{Tor}_i^S(R, S/(z_0)) = 0$.*

Proof. Using the resolutions above, by exactness of tensor product on the right, we have to prove first that $\overline{\mathcal{P}}_\bullet$ is exact at $\overline{\mathcal{P}}_1$. Denote by $^-$ residue classes modulo z_0.

If $f\overline{v} = 0$, then $Fv = z_0 w$, for some $w \in \mathcal{P}_0$ and $z_0 w \in I$. As z_0 is a non-zero divisor in R, $w \in I \Rightarrow w = Fv' \Rightarrow F(v - z_0 v') = 0$. By exactness at $\mathcal{P}_1$, there exists $v'' \in \mathcal{P}_2$ such that $v - z_0 v' = Lv''$ and $\overline{v} = l\overline{v}''$ as desired.

As z_0 is a non-zero divisor on all $\mathcal{P}_i$ it is also in the image of all morphisms $\mathcal{P}_i \xrightarrow{d_i} \mathcal{P}_{i-1}$ so we can apply the same argument to the sequence

$$0 \leftarrow \operatorname{im}(d_i) \leftarrow \mathcal{P}'_i \leftarrow \mathcal{P}'_{i+1} \leftarrow \mathcal{P}'_{i+2},$$

and by induction we are done. □

By the proposition above the resolution of R and $R/(z_0)$ have the same format. Now we will show conditions on the F_i such that the resolution of R has the same format as the one of $\overline{R}$.

Proposition 5.1.3. *Given a ring $\overline{R} = \overline{S}/\overline{I}$ with a free resolution as in 5.2, if every relation $\sum_i l_i f_i \equiv 0$, where $f_i \in \overline{I}$, $l_i \in \overline{S}$, lifts to a relation $\sum_i L_i F_i$, then the resolution lifts to a resolution of R and $\overline{R} = R/(z_0)$.*

Proof. Let $S = \overline{S}[z_0]$, I the ideal generated by F_i and let $\mathcal{P}_\bullet$ be its minimal free resolution,

$$0 \leftarrow I \overset{d_1}{\longleftarrow} \mathcal{P}_1 \overset{d_2}{\longleftarrow} \mathcal{P}_2 \leftarrow \cdots,$$

$\mathcal{P}_i \cong \bigoplus_j S(-a_{ij})$, d_1 is multiplication by $F = (F_i)$, d_2 by L.

By assumption, applying $- \otimes S/(z_0)$ to the exact sequence,

$$0 \leftarrow I \overset{F}{\longleftarrow} \bigoplus_j S(-a_{1j}) \overset{L}{\longleftarrow} \ker(d_1) \leftarrow 0,$$

we get that $\mathrm{Tor}_1^S(I, S/(z_0)) = 0$. As z_0 is a non-zero divisor for all $\mathcal{P}_i$, we can use proposition 5.1.2 and get $\mathrm{Tor}_i^S(I, S/(z_0)) = 0$, for all $i > 0$, so $\mathcal{P}_\bullet$ is the lift of $\overline{\mathcal{P}}_\bullet$, a minimal free resolution of $\overline{I}$. □

The proposition above solves the problem 5.1.1, the set of pairs (z_0, R) such that $z_0 \in R_{a_0}$ and $\overline{R} = R/(z_0)$ can be given as the set of polynomials F_i extending the relations f_i of $\overline{R}$ such that the syzygies σ_i extend to Σ_i.

Definition 5.1.4. We call the following set the big Hilbert scheme of extensions of $\overline{R}$ by a variable of degree a_0

$$\mathrm{BH}\left(a_0, \overline{R}\right) = \left\{ \begin{array}{c} F_i = f_i + z_0 g_i, \\ \Sigma_i : L_{ij} = l_{ij} + z_0 m_{ij} \end{array} \middle| \sum_i L_{ij} F_i \equiv 0 \right\}$$

where $\deg(z_0) = a_0$ and F_i, L_{ij} are homogeneous. $\mathrm{BH}(a_0, \overline{R})$ has the structure of an affine scheme with coordinates given by the coefficients of g_i, m_{ij}.

The coefficients of g_i, m_{ij} are finite in number and the conditions $\sum_i L_{ij} F_i$ are a finite set of polynomial relations on these coefficients. We will present the algorithm to compute the big Hilbert scheme of extensions that will give us the structure of a fibre but we also need to show that these constructions are independent of the choice of coordinates, which brings the next definition,

Notation 5.1.5. We call Hilbert scheme the following set,

$$\mathbb{H}\left(\overline{R}, a_0\right) = \left\{\forall R, z_0 | R/(z_0) \cong \overline{R}\right\}$$

The Hilbert scheme parametrises ideals $I \subset k[z_0, \ldots, z_n]$ such that $(z_0) \cap I = z_0 I$, $I/z_0 I = \overline{I}$ and it might be seen as dividing the big Hilbert scheme by an equivalence relation

$$F_i \sim F_i + z_0 \left(\sum a_{ij} F_j\right), \text{ and similarly for the } \Sigma_j.$$

Definition 5.1.6. Given a graded ring $\overline{R}$ and a degree a_0, we call a ring $R^{(k)}$ together with a homogeneous element $z_0 \in R^{(k)}_{a_0}$ such that $\overline{R} = R^{(k)}/(z_0)$, a k^{th}-order infinitesimal extension of $\overline{R}$ if $z_0^{k+1} = 0$ and $R^{(k)}$ is flat over the subring of $k[z_0]/(z_0^{k+1})$ generated by z_0.

Based on this definition we can introduce the big Hilbert scheme of k^{th}-order infinitesimal deformations of $\overline{R}$, $\mathrm{BH}^{(k)}(a_0, \overline{R})$ as the affine scheme parametrising the polynomials

$$F_i = f_i + z_0 f_i' + z_0^2 f_i'' + \cdots + z_0^k f_i^{(k)},$$
$$L_{ij} = l_{ij} + z_0 l_{ij}' + z_0^2 l_{ij}'' + \cdots + z_0^k l_{ij}^{(k)}$$

for which the syzygies are satisfied up to k^{th} order,

$$\sum_i L_{ij} F_i \equiv 0 \quad \mod x_0^{k+1}$$

The Hilbert scheme $\mathbb{H}^{(k)}(\overline{R}, a_0)$ parametrises ideals $I^{(k)} \subset k[z_0, \ldots, z_n]/(z_0^{k+1})$ with $(z_0) \cap I^{(k)} = x_0 I^{(k)}$ and $I^{(k)}/x_0 I^{(k)} = \overline{I}$.

For sufficiently large k, $\mathbb{H}(\overline{R}, a_0) = \mathbb{H}^{(k)}(\overline{R}, a_0)$, as the the syzygies $\sum_i l_{ij} f_i$ are identities in fixed degrees and F_i, L_{ij} have the same degrees as f_i, l_{ij}. As soon as

$$(k+1)a_0 = \deg z_0^{k+1} > \max\left(\deg \sum_i l_{ij} f_i\right)$$

we get that that $\mathbb{H}(\overline{R}, a_0) \subset \mathbb{H}^{(k)}(\overline{R}, a_0)$. We can now describe a tower of schemes,

$$\mathbb{H} \to \cdots \to \mathbb{H}^{(k)} \to \mathbb{H}^{(k-1)} \to \ldots \mathbb{H}^{(1)} \to \mathbb{H}^{(0)} = \mathrm{pt}$$

where the morphisms $\mathbb{H}^{(k)} \to \mathbb{H}^{(k-1)}$ correspond to the forgetful maps $R^{(k)} \to R^{(k-1)}$,

$$f_i + z_0 f_i' + z_0^2 f_i'' + \cdots + z_0^k f_i^{(k)} \mapsto f_i + z_0 f_i' + z_0^2 f_i'' + \cdots + z_0^{k-1} f_i^{(k-1)}$$
$$l_{ij} + z_0 l_{ij}' + z_0^2 l_{ij}'' + \cdots + z_0^k l_{ij}^{(k)} \mapsto l_{ij} + z_0 l_{ij}' + z_0^2 l_{ij}'' + \cdots + z_0^{k-1} l_{ij}^{(k-1)}$$

Given the $(k-1)^{st}$ order power series F_i and L_{ij} satisfying

$$\sum_i L_{ij} F_i \equiv 0 \mod z_0^k \tag{5.3}$$

an extension to the k^{th} order is equivalent to fixing up new terms, $f_i^{(k)}$, $l_{ij}^{(k)}$, that satisfy the same equation mod z_0^{k+1}. Writing down all the terms and, by 5.3, killing the terms divisible by z_0^k, we get

$$\sum_i l_{ij} f_i^{(k)} + \sum_{a=1}^{k-1} \sum_i l_{ij}^{(a)} f_i^{(k-a)} + \sum_i l_{ij}^{(k)} f_i \equiv 0 \mod z_0^{k+1}$$

This is a set of linear equations in the new unknowns $f_i^{(k)}$ and $l_{ij}^{(k)}$. As the last term in the left side is an element of $\overline{I}$, working mod $\overline{I}$

$$\sum_i l_{ij} f_i^{(k)} = -\sum_{a=1}^{k-1} \sum_i l_{ij}^{(a)} f_i^{(k-a)} = \psi_j \in \overline{R} \tag{5.4}$$

This is all one needs to carry the computations. The right-hand side of 5.4 is a given vector $\psi = \{\psi_j\} \in \bigoplus_j \overline{R}$. If the left hand side fails to hit ψ, then there's an obstruction to extending $R^{(k-1)}$, if it does hit ψ, the ambiguity in the choice of $f_i^{(k)}$ is the vector space $\{f_i^{(k)} | \sum l_{ij} f_i^{(k)} \in \overline{R}\}$, that depends only on $\overline{R}$ and not on $R^{(k-1)}$.

5.2 Fat point deformation

Our goal is to determine the relations that the c_{ijk}, d_{ij} have to satisfy so that the multiplication,

$$z_i z_j = \sum c_{ijk} z_k + d_{ij}$$

where $(z_1, \ldots, z_{d-1})$ is a basis for k^{d-1}, is associative. In section 3.1, we saw that this is equivalent to have the ideal $(z_i z_j - \sum c_{ijk} z_k - d_{ij})$ being Cohen-Macaulay with codimension $d-1$.

Fix the basis $(z_1, \ldots, z_{d-1})$. If we homogenize the ideal I with a non-zero divisor variable z_0, we get

$$I = \left(F_{ij} := z_i z_j - \left(\sum_k c_{ijk} z_k \right) z_0 - d_{ij} z_0^2 \right), 0 \leq i < j \leq d-1$$

Let $S = k[z_0, z_1, \dots, z_{d-1}]$ and $R = S/I$. Then we have the following exact sequence,

$$R \leftarrow S \xleftarrow{(F_{ij})} S(-2)^{\oplus n_1} \xleftarrow{L} S(-3)^{\oplus n_2} \leftarrow \cdots \leftarrow S(-d)^{\oplus n_{d-1}} \leftarrow 0 \qquad (5.5)$$

As z_0 is a non-zero divisor, the resolution of $\overline{R} = R/(z_0)$ has the same format and what we are looking for is the set of all pairs (R, z_0), $\deg z_0 = 1$, such that $\overline{R} = R/(z_0)$, the solution of problem 5.1.1, $\mathbb{H}\left(\overline{R}, 1\right)$.

We now have the generators for $\overline{I}$, $f_{ij} = z_i z_j$, with degree 2 and the syzygies are generated by the relations $z_j f_{ij} - z_k f_{ik}$, for any $i, j < k$, the entries in the matrix $l := L(0, z_1, \dots, z_{d-1})$ with degree 1. So we know that $\mathbb{H}\left(\overline{R}\right) \subset \mathbb{H}^{(3)}\left(\overline{R}\right)$.

We want to describe the algorithm that we implement in a computer to construct $\mathrm{BH}(\overline{R})$. Take the following matrices,

- $f := (z_1^2, z_1 z_2, \dots, z_i z_j, \dots, z_{d-1}^2) \in \mathrm{Mat}_{1\times\binom{d}{2}}(\mathcal{A})$,
- $\overline{z} := (z_1, \dots, z_{d-1}) \in \mathrm{Mat}_{(1\times(d-1))}(\mathcal{A})$,
- $C := [c_{ijk}] \in \mathrm{Mat}_{(d-1)\times\binom{d}{2}}(\mathcal{A})$,
- $D := [d_{ij}] \in \mathrm{Mat}_{1\times\binom{d}{2}}(\mathcal{A})$,
- $l \in \mathrm{Mat}_{\binom{d}{2}\times(d-1)\binom{d-1}{2}}(\mathcal{A})$, the first syzygy matrix of $\overline{I}$,
- $N := [n_{ij}] \in \mathrm{Mat}_{\binom{d}{2}\times(d-1)\binom{d-1}{2}}(\mathcal{A})$.

where $\mathcal{A} = k[z_i, c_{ijk}, d_{ij}, n_{ij}]$. The entries c_{ijk}, d_{ij}, n_{ij} are just token coordinates so that we have a morphism $\mathcal{A} \twoheadrightarrow \mathrm{BH}\left(\overline{R}\right)$ where the ideal is given by the relations that make the following equality true,

$$\left(f + \overline{z}Cz_0 + Dz_0^2\right)(l + Nz_0) = 0$$

Decomposing the equality in powers of z_0 we get the equations,

1. $fl = 0$

   Always true by definition of syzygy.

2. $z_0\left(fN + (\overline{z}C)\,l\right) = 0$

   Allows us to write all n_{ij} as linear combination of the c_{ijk} and get linear equations between the c_{ijk}.

3. $z_0^2\left(\overline{z}CN + Dl\right) = 0$

As we can write N with the entries of C, gives us identities $d_{ij} = q(c_{ijk})$, where the q are quadratic polynomials on the entries c_{ijk}, and quadratic equations in the c_{ijk}.

4. $z_0^3\,(DN) = 0$

   Should give us cubic equations in the c_{ijk} but they will always be contained in the ideal generated by the equations found before.

Proposition 5.2.1. *In the setting described above,* $\mathrm{BH}(\overline{R}) = \mathrm{BH}^{(2)}(\overline{R})$.

Proof. By proposition 5.1.3 after the second step we have the resolution of $R^{(2)}$,

$$S^{(2)} \xleftarrow{d_1} S^{(2)}(-2)^{\oplus n_1} \xleftarrow{d_2} S^{(2)}(-3)^{\oplus n_2} \xleftarrow{d_3} \cdots \leftarrow S^{(2)}(-d)^{\oplus n_{d-1}} \leftarrow 0$$

where $S^{(2)} = S/(z_0^3)$. For $i > 1$, the d_i are linear maps so $d_i \circ d_{i+1} = 0$ as maps in S and not only in $S^{(2)}$ from where we get that $\mathrm{rank}(d_2) = \binom{d}{2} - 1$. This implies that the left kernel of $d_2 = (l + Nz_0)$ is generated by a single vector, $v \in S^{\oplus n_1}$.

If $\deg(v) = 2$ then $v = d_1$. Assume that $\deg(v) > 2$. Then we can write

$$v = \alpha d_1 + \beta z_0^3$$

$\alpha \in \mathcal{A}$, $\beta \in \mathrm{Hom}(S(-1-a)^{\oplus n_1}, S)$, where $a = \deg_{(z_i)}(\alpha) \geq 1$. We have the free resolution

$$S \xleftarrow{v} S(-2-a)^{\oplus n_1} \xleftarrow{d_2} S(-3-a)^{\oplus n_2} \xleftarrow{d_3} \cdots \leftarrow S(-d-a)^{\oplus n_{d-1}} \leftarrow 0$$

with the following Hilbert series

$$\begin{aligned} \mathrm{HS}(t) &= \tfrac{1 - n_1 t^{2+a} + n_2 t^{3+a} - \cdots \pm n_{d-1} t^{d+a}}{(1-t)^d} \\ &= \tfrac{1 - t^a + t^a(1 - n_1 t^2 + n_2 t^3 - \cdots \pm n_{d-1} t^d)}{(1-t)^d} \\ &= t^a \left(\tfrac{1 + (d-1)t}{1-t} \right) + \tfrac{1 - t^a}{(1-t)^d} \end{aligned}$$

The Hilbert function is dominated by the terms from $\frac{1-t^a}{(1-t)^d}$. This implies that the projective dimension of $\mathrm{Proj}(S/I_v)$ is $d-2$ and we conclude that the v decomposes in two (or more) components. As

$$v \equiv \alpha d_1 \mod z_0$$

there is α', β' such that one of the components is generated by

$$v' = \alpha' d_1 + \beta' z_0^3$$

with $\deg(\alpha') < \deg(\alpha)$. By induction we get $\deg(v) = 2$ and we are done. $\square$

We just found that $\mathrm{BH}\left(\overline{R}\right)$ is determined by $\mathrm{BH}^{(2)}\left(\overline{R}\right)$. It's important to see that any linear change of coordinates in the basis $(z_1, \ldots, z_{d-1})$ gives a new ideal I that we can write again as a deformation of the same ideal $(z'_i z'_j)$ so the ideal $\mathcal{I}_d$ is fixed by a change of basis. Furthermore, fixing the basis $(z_1, \ldots, z_{d-1})$ to be a trace free basis is just adding a set of linear equations in A and, as proved in lemma 2.2.1, these equations are independent of a choice of basis.

Example 5.2.2 (Triple Covers)**.** Given a covering map of degree 3, $\varphi\colon X \to Y$, $\varphi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{E}$. Denoting by (z_1, z_2) a local trace-free basis for $\mathcal{E}|_{\mathcal{U}}$, i.e. the morphism $\Phi_2\colon S^2\mathcal{E}|_{\mathcal{U}} \to \mathcal{E}|_{\mathcal{U}}$ is of the form

$$\begin{aligned} z_1^2 &= \quad c_1 z_1 + c_0 z_2 \\ z_1 z_2 &= -c_2 z_1 - c_1 z_2 \\ z_2^2 &= \quad c_3 z_1 + c_2 z_2 \end{aligned}$$

then the syzygy matrix is

$$(l + N) = \begin{pmatrix} z_1 - 2c_1 & c_0 \\ z_2 + c_2 & z_1 + c_1 \\ c_3 & z_2 - 2c_2 \end{pmatrix}$$

that has left kernel generated by the 2×2 minors of $(l + N)$.

5.3 Gorenstein Fibre

For $d \geq 4$, consider a Gorenstein ideal, $I \subset k[z_1, \ldots, z_{d-2}]$, generated by $\binom{d-1}{2} - 1$ quadratic equations,

$$I = \left(f_i := q_i - \sum_j c_{ij} z_j - d_i \right)$$

where q_i are homogeneous quadratic polynomials in the z_i. As in the previous section, if we homogenize I by adding a non-zero divisor variable z_0, we get

$$I = \left(q_i - \left(\sum_j c_{ij} z_j \right) z_0 - d_i z_0^2 \right)$$

Let $S = k[z_0, \ldots, z_{d-2}]$ and $R = S/I$. As I is a Gorenstein ideal we have the resolution,

$$R \leftarrow S \xleftarrow{(f_i)} S(-2)^{\oplus n_1} \leftarrow S(-3)^{\oplus n_2} \leftarrow \cdots \leftarrow S(-d) \leftarrow 0$$

The quotient by z_0, $\overline{R} = R/(z_0)$, has the same resolution format, $\overline{I}$ has q_i as generators and we can run the same algorithm as before to find all pairs (R, z_0), $\deg z_0 = 1$, such that $\overline{R} = R/(z_0)$.

Remember $n_1 = \binom{d-1}{2} - 1$, $n_2 = d(d-2)(d-4)/3$, and take the matrices,

- $q := (q_1, \ldots, q_m) \in \mathrm{Mat}_{1\times n_1}(\mathcal{A})$,
- $\overline{z} := (z_1, \ldots, z_{d-1}) \in \mathrm{Mat}_{1\times(d-2)}(\mathcal{A})$,
- $C := [c_{ij}] \in \mathrm{Mat}_{(d-2)\times n_1}(\mathcal{A})$,
- $D := [d_i] \in \mathrm{Mat}_{1\times n_1}(\mathcal{A})$,
- $l \in \mathrm{Mat}_{n_1\times n_2}(\mathcal{A})$, the first syzygy matrix of $\overline{I}$,
- $N := [n_{ij}] \in \mathrm{Mat}_{n_1\times n_2}(\mathcal{A})$.

where $\mathcal{A} = k[z_i, c_{ij}, d_i, n_{ij}]$. The solution we are looking for is the ideal $J_q \subset k[c_{ij}, d_i, n_{ij}]$ such that

$$\left(q + \overline{z}Cz_0 + Dz_0^2\right)(l + Nz_0) \equiv 0 \mod J_q$$

For $d = 4$, I is given by a complete intersection of two quadratic polynomials so there are no obstructions to the deformation and for $d = 5$ we have a Gorenstein ideal of codimension 3 so by Auslander-Buchsbaum theorem we have the resolution,

$$R \leftarrow S \leftarrow 5S(-2) \xleftarrow{l} 5S(-3) \leftarrow S(-5) \leftarrow 0$$

where l is a 5×5 anti-symmetric matrix so N is also an anti-symmetric matrix and the 5 equations are given by the Pfaffians of $(l + N)$. For $d \geq 6$, we can mimic the argument of proposition 5.2.1

Proposition 5.3.1. *In the setting described above,* $\mathrm{BH}(\overline{R}) = \mathrm{BH}^{(2)}(\overline{R})$.

Proof. For $d \leq 5$ it is proven by the argument above. By proposition 5.1.3 after the second step we have the resolution of $R^{(2)}$. The maps d_i for $2 \leq i \leq d-3$ are linear maps so $d_{i+1} \circ d_i = 0$ as maps in S. As $I^{(2)}$ is Gorenstein, the resolution is symmetric, $\mathrm{rank}(d_2) = \mathrm{rank}(d_{d-3}) = \binom{d-1}{2} - 2$ as a matrix over S so there is a unique vector in the left kernel of d_2, $v \in S^{\oplus n_1}$, such that

$$v = \alpha d_1 + \beta z_0^3, \quad \alpha, \beta \in \mathcal{A}$$

and the Hilbert series,

$$\begin{aligned} \mathrm{HS}(t) &= \tfrac{1-n_1t^{2+a}+n_2t^{3+a}-\cdots\pm t^{d+2a}}{(1-t)^{d-1}} \\ &= \tfrac{1-t^a\mp t^{d+a}\pm t^{2d+a}+t^a(1-n_1t^2+n_2t^3-\cdots\pm t^d)}{(1-t)^{d-1}} \\ &= t^a\left(\tfrac{1+(d-2)t+t^2}{1-t}\right) + \tfrac{1-t^a\mp t^{d+a}\pm t^{d+2a}}{(1-t)^{d-1}}, \end{aligned}$$

$a = \deg(\alpha)$. If $\deg(\alpha) > 0$, then for d odd we get a component with projective dimension $d-3$, degree 0 and if d is even two components with dimension $d-3$ and degree a which means we can cancel α and $\mathrm{BH}(\overline{R}) = \mathrm{BH}^{(2)}(\overline{R})$. □

The proposition proves the first statement of theorem 3.1.4. The second statement comes from the construction, a change of basis (z_i) will give $\mathcal{I}'_q$ isomorphic to $\mathcal{I}_q$, just the relation between $\mathrm{BH}(\overline{R})$ and $\mathbb{H}\left(\overline{R}\right)$. For Gorenstein covers it is not so easy to talk about a trace-free basis but what we know is that for any cover of codimension r we will always have r change of variables to use. This will be visible in the main example that we will compute now.

5.3.1 Main Example

Following the discussion from section 4.4.1 and using the machinery described in this chapter we want to describe the ideal $\mathcal{I}_q$ where

$$q = \begin{pmatrix} y_0^2 & y_0y_1 & y_1^2 & y_0z_0 & \frac{1}{2}(y_0z_1 + y_1z_0) & y_1z_1 & z_0^2 & z_0z_1 & z_1^2 \end{pmatrix}. \tag{5.6}$$

In appendix A.2 we have the calculations using Sage. The first step shows us that

$$C^t = \begin{pmatrix} c_{32} + 2c_{43} & -c_{33} & -c_{13} & c_{03} \\ c_{53} & c_{32} & -c_{23} & c_{13} \\ -c_{52} & 2c_{42} + c_{53} & c_{22} & c_{23} \\ c_{73} & -c_{63} & c_{32} & c_{33} \\ -\frac{1}{2}c_{72} + \frac{1}{2}c_{83} & \frac{1}{2}c_{62} - \frac{1}{2}c_{73} & c_{42} & c_{43} \\ -c_{82} & c_{72} & c_{52} & c_{53} \\ -c_{71} & c_{61} & c_{62} & c_{63} \\ -c_{81} & c_{71} & c_{72} & c_{73} \\ c_{80} & c_{81} & c_{82} & c_{83} \end{pmatrix}$$

With a change of variables

$$\begin{pmatrix} y_0 \\ y_1 \\ z_0 \\ z_1 \end{pmatrix} \mapsto \begin{pmatrix} y_0 - (c_{32} + c_{43}) \\ y_1 - (c_{42} + c_{53}) \\ z_0 - (c_{62} + c_{73}) \\ z_1 - (c_{72} + c_{83}) \end{pmatrix}$$

we get the matrix

$$C^t = \begin{pmatrix} c_{43} & -c_{33} & -c_{13} & c_{03} \\ c_{53} & -c_{43} & -c_{23} & c_{13} \\ -c_{52} & -c_{53} & c_{22} & c_{23} \\ c_{73} & -c_{63} & -c_{43} & c_{33} \\ c_{83} & -c_{73} & -c_{53} & c_{43} \\ -c_{82} & -c_{83} & c_{52} & c_{53} \\ -c_{71} & c_{61} & -c_{73} & c_{63} \\ -c_{81} & c_{71} & -c_{83} & c_{73} \\ c_{80} & c_{81} & c_{82} & c_{83} \end{pmatrix} = \begin{pmatrix} c'_{11} & c'_{10} & c'_{01} & c'_{00} \\ -c'_{12} & -c'_{11} & -c'_{02} & -c'_{01} \\ c'_{13} & c'_{12} & c'_{03} & c'_{02} \\ -c'_{21} & -c'_{20} & -c'_{11} & -c'_{10} \\ c'_{22} & c'_{21} & c'_{12} & c'_{11} \\ -c'_{23} & -c'_{22} & -c'_{13} & -c'_{12} \\ c'_{31} & c'_{30} & c'_{21} & c'_{20} \\ -c'_{32} & -c'_{31} & -c'_{22} & -c'_{21} \\ c'_{33} & c'_{32} & c'_{23} & c'_{22} \end{pmatrix}$$

In the last matrix we renamed the variables to have a better look at the structure of C and we see that C has the same decomposition as a triple cover homomorphism where each of the entries is a triple cover homomorphism,

$$C^t = \begin{pmatrix} C_1 & C_0 \\ -C_2 & -C_1 \\ C_3 & C_2 \end{pmatrix}, C_i = \begin{pmatrix} c_{i1} & c_{i0} \\ -c_{i2} & -c_{i1} \\ c_{i3} & c_{i2} \end{pmatrix} \tag{5.7}$$

With the second step we get the vector D,

$$D^t = \begin{pmatrix} -2c_{11}^2 + 2c_{10}c_{12} + 2c_{01}c_{21} - c_{02}c_{20} - c_{00}c_{22} \\ -c_{10}c_{13} + c_{11}c_{12} - 2c_{02}c_{21} + c_{03}c_{20} + c_{01}c_{22} \\ 2c_{11}c_{13} - 2c_{12}^2 - c_{03}c_{21} - c_{01}c_{23} + 2c_{02}c_{22} \\ -c_{01}c_{31} + c_{00}c_{32} + c_{11}c_{21} + c_{12}c_{20} - 2c_{10}c_{22} \\ \frac{1}{2}(-c_{00}c_{33} + c_{01}c_{32} - 5c_{12}c_{21} + c_{13}c_{20} + 4c_{11}c_{22}) \\ c_{01}c_{33} - c_{02}c_{32} + c_{13}c_{21} - 2c_{11}c_{23} + c_{12}c_{22} \\ 2c_{11}c_{31} - c_{12}c_{30} - c_{10}c_{32} - 2c_{21}^2 + 2c_{20}c_{22} \\ c_{12}c_{31} + c_{10}c_{33} - 2c_{11}c_{32} - c_{20}c_{23} + c_{21}c_{22} \\ -c_{13}c_{31} - c_{11}c_{33} + 2c_{12}c_{32} + 2c_{21}c_{23} - 2c_{22}^2 \end{pmatrix}$$

and all the quadratics equations that the c_{ij} need to satisfy,

$$\mathcal{I}_q = \begin{cases} 3c_{12}c_{01} - 3c_{11}c_{02} - c_{13}c_{00} + c_{10}c_{03} \\ 3c_{02}c_{21} - 3c_{01}c_{22} - c_{03}c_{20} + c_{00}c_{23} \\ 3c_{12}c_{31} - 3c_{11}c_{32} - c_{13}c_{30} + c_{10}c_{33} \\ 3c_{32}c_{21} - 3c_{31}c_{22} - c_{33}c_{20} + c_{30}c_{23} \\ 3c_{10}c_{21} - 3c_{11}c_{20} - c_{00}c_{31} + c_{01}c_{30} \\ 3c_{12}c_{20} - 3c_{10}c_{22} - c_{02}c_{30} + c_{00}c_{32} \\ 3c_{13}c_{21} - 3c_{11}c_{23} - c_{03}c_{31} + c_{01}c_{33} \\ 3c_{12}c_{23} - 3c_{13}c_{22} - c_{02}c_{33} + c_{03}c_{32} \\ 9c_{12}c_{21} - 9c_{11}c_{22} - c_{03}c_{30} + c_{00}c_{33} \\ c_{02}c_{31} - c_{01}c_{32} - c_{13}c_{20} + c_{10}c_{23} \end{cases}$$

Theorem 5.3.2. *Given a basis (z_1, z_2, z_3, z_4) for k^4, the relations which must be satisfied by the coefficients c_{ij} for the ideal defined by the equations*

$$q_i = \sum c_{ij} z_j + d_i$$

q_i being the i^{th} entry of the vector q (5.6), to be Gorenstein are the equations of the spinor embedding of the affine orthogonal Grassmannian $\mathrm{aOGr}(5, 10)$ *in* $\mathbb{P}^{15}$.

Let us recall the definition of $\mathrm{OGr}(5, 10)$. Consider the vector space $V = \mathbb{C}^{10}$ with p a nondegenerate quadratic form. Using a change of basis we can put p in the form

$$p = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \text{ that is, } V = U \oplus U^{\vee}.$$

A vector space $F \subset V$ is isotropic if p is identically zero on F, for example U is an isotropic 5-space. The orthogonal Grassmannian variety $\mathrm{OGr}(5, 10)$ is defined as

$$\mathrm{OGr}(5, 10) = \left\{ F \in \mathrm{Gr}(5, V) \;\middle|\; \begin{array}{c} \text{F is isotropic for } p \\ \text{and } \dim F \cap U \text{ is odd} \end{array} \right\}.$$

The equations of $\mathrm{aOGr}(5, 10)$ can be written as the 10 equations centered at x,

$$xv - \mathrm{Pfaff}\, M = 0 \text{ and } Mv = 0$$

where

$$M = \begin{pmatrix} x_{12} & x_{13} & x_{14} & x_{15} \\ & x_{23} & x_{24} & x_{25} \\ & & x_{34} & x_{35} \\ -sym & & & x_{45} \end{pmatrix} \text{ and } v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

To read more about the orthogonal Grassmannian $\mathrm{OGr}(5,10)$ and its construction see [CR02].

Proof 5.3.2. Consider the matrices,

$$M = \begin{pmatrix} 3c_{21} & c_{30} & -c_{33} & -c_{31} \\ & -c_{00} & c_{03} & c_{01} \\ & & 3c_{12} & -c_{10} \\ \text{-sym} & & & c_{13} \end{pmatrix}, v = \begin{pmatrix} c_{02} \\ c_{32} \\ c_{23} \\ c_{20} \\ 3c_{22} \end{pmatrix}$$

then $\mathcal{I}_q$ is given by the equations

$$\begin{cases} 3c_{11}v - \mathrm{Pfaff}(M) = 0 \\ Mv = 0 \end{cases}$$

that up to rescaling are the equations of $\mathrm{aOGr}(5,10)$ centered at $3c_{11}$. □

Exactly as in the general covers case we end up finding the structure of the fibre being related with the structure of a Grassmannian but unfortunately we do not have a interpretation for this fact. We hope that with the computation of more examples some light will be shed in the topic.

A cover homomorphism in $\mathrm{CHom}\left(\left(S^2\Omega^1_{\mathbb{P}^2}\right)^{\oplus 3}, \left(\Omega^1_{\mathbb{P}^2}\right)^{\oplus 2}\right)$ can be written as described in (5.7) from where we get the following result.

Proposition 5.3.3. *Given a Gorenstein covering map, $\varphi\colon S \to \mathbb{P}^2$, such that*

$$\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-3),$$

then the associative multiplication $\Phi \in \mathrm{Hom}\left(S^2\left(\varphi_*\mathcal{O}_S\right), \varphi_*\mathcal{O}_S\right)$, is determined by

$$\Phi_2 \in \mathrm{Hom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)^{\bigoplus 6}$$

which is generated by $\mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)^{\bigoplus 4}$.

Proof. The first statement comes from the decomposition $\Phi = \Phi_1 \oplus \Phi_2$ where $\Phi_1 \in \mathrm{Hom}\left(S^2\left(\Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}\right), \mathcal{O}_{\mathbb{P}^2}\right)$ and Theorem 3.1.4.

The form of the matrix C shows that locally Φ_2 is generated by C_0, C_1, C_2, C_3 and so it is determined globally by a morphism in $\mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)^{\bigoplus 4}$. □

Additionally, we can determine $\mathrm{CHom}\left(\bigoplus_{i=1}^9 \mathcal{O}_{\mathbb{P}^2}(-4), \bigoplus_{i=1}^4 \mathcal{O}_{\mathbb{P}^2}(-2)\right)$ using $\mathcal{I}_q$ that will give a canonical model for S_6, the surface satisfying $p_g(S_6) = 3, q(S_6) = 0, K^2_{S_6} = 6$ (see Theorem 2.0.2). We will get a large number of equations in the coefficients of each $c_{ij} \in k[x_0, x_1, x_2]^{(2)}$ but it shows how the study of irregular varieties can bring new algebraic methods for regular varieties.

In the next sections we will show two simple models that end up being related with two families for the abelian surface that were studied before.

5.3.2 First Model

In a quick glance at the equations in $\mathcal{I}_q$, one can see index homogeneity (e.g. in $3c_{10}c_{21} - 3c_{11}c_{20} - c_{00}c_{31} + c_{01}c_{30}$ the indices add to the pair $(3,1)$) which suggests that as a first model we can use the case where all the C_i are all multiples of a common factor $\widetilde{C}$, i.e.

$$C_i = k_i\widetilde{C},$$

$$\widetilde{C} = \begin{pmatrix} c_1 & c_0 \\ -c_2 & -c_1 \\ c_3 & c_2 \end{pmatrix}.$$

In 5.8 below one can see the equations. Important to see that in this model we can use a use a $\mathrm{GL}(2) \times \mathrm{GL}(2)$ action on the basis (y_i, z_i) to change c_i, k_i, i.e., for $g, h \in \mathrm{Mat}_{2\times 2}$ consider the action,

$$\begin{array}{l} g\begin{pmatrix} y_0 & y_1 & z_0 & z_1 \end{pmatrix} \mapsto \begin{pmatrix} (y_0 \quad y_1)g & (z_0 \quad z_1)g \end{pmatrix} \\ h\begin{pmatrix} y_0 & z_0 & y_1 & z_1 \end{pmatrix} \mapsto \begin{pmatrix} (y_0 \quad z_0)h & (y_1 \quad z_1)h \end{pmatrix}. \end{array}$$

By lemma 3.4.2, if $(k_i, c_i) \in \left(k^4 \setminus \mathrm{Bl}_{tc}\right)^2$, then we can change them at will. In general we are not in such case but for this family of examples, for all i, $k_i \in k$ so we can choose them to be in $\left(k^4 \setminus \mathrm{Bl}_{tc}\right)$.

$$
\begin{cases}
y_0^2 + k_1(c_1y_0 + c_0y_1) + k_0(c_1z_0 + c_0z_1) - 2(k_1^2 - k_0k_2)(c_1^2 - c_0c_2) \\
y_0y_1 - k_1(c_2y_0 + c_1y_1) - k_0(c_2z_0 + c_1z_1) + (k_1^2 - k_0k_2)(c_1c_2 - c_0c_3) \\
y_1^2 + k_1(c_3y_0 + c_2y_1) + k_0(c_3z_0 + c_2z_1) - 2(k_1^2 - k_0k_2)(c_2^2 - c_1c_3) \\
y_0z_1 - k_2(c_1y_0 + c_0y_1) - k_1(c_1z_0 + c_0z_1) + (k_1k_2 - k_0k_3)(c_1^2 - c_0c_2) \\
\frac{1}{2}(y_0z_1 + y_1z_0) + k_2(c_2y_0 + c_1y_1) + k_1(c_2z_0 + c_1z_1) - \frac{1}{2}(k_1k_2 - k_0k_3)(c_1c_2 - c_0c_3) \\
y_1z_1 - k_2(c_3y_0 + c_2y_1) - k_1(c_3z_0 + c_2z_1) + (k_1k_2 - k_0k_3)(c_2^2 - c_1c_3) \\
z_0^2 + k_3(c_1y_0 + c_0y_1) + k_2(c_1z_0 + c_0z_1) - 2(k_2^2 - k_1k_3)(c_1^2 - c_0c_2) \\
z_0z_1 - k_3(c_2y_0 + c_1y_1) - k_2(c_2z_0 + c_1z_1) + (k_2^2 - k_1k_3)(c_1c_2 - c_0c_3) \\
z_1^2 + k_3(c_3y_0 + c_2y_1) + k_2(c_3z_0 + c_2z_1) - 2(k_2^2 - k_1k_3)(c_2^2 - c_1c_3).
\end{cases}
\tag{5.8}
$$

At this point we do not know exactly if this model brings six distinct points for any quadruples (k_i, c_i) but for that we only need to compute its branch locus.

Proposition 5.3.4. *The branch locus of a covering map which locally can be written in the format (5.8) is given by the vanishing of the polynomial*

$$
\begin{aligned}
&(4k_0k_2^3 + 4k_1^3k_3 + k_0^2k_3^2 - 3k_1^2k_2^2 - 6k_0k_1k_2k_3)\times \\
&\qquad (4c_0c_2^3 + 4c_1^3c_3 + c_0^2c_3^2 - 3c_1^2c_2^2 - 6c_0c_1c_2c_3).
\end{aligned}
\tag{5.9}
$$

The branch locus is given by the projection on $\mathcal{U} \subset Y$ of the Jacobian matrix $\begin{pmatrix} \frac{\partial}{\partial y_i} & \frac{\partial}{\partial z_j} \end{pmatrix}$ applied to the equations 5.8. We used Sage for the computation but just by the equations it was expected as we see that the branch locus is the product of branch locus of triple covers with coefficients c_i and k_i, see proposition 3.4.6.

Another relevant question is if this model will ever give us a smooth variety. As for our main case we proved smoothness of triple covers such that $\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m)$, the following proposition will be enough.

Proposition 5.3.5. *Let $\varphi\colon X \to Y$ over k be a covering map of degree 6. If for a general $y \in Y$*

$$\mathcal{O}_{X,y} \cong \mathcal{O}_{Y,y}[y_0, y_1, z_0, z_1]/I$$

where I is an ideal generated by polynomials with the format (5.8), then we can decompose φ as

$$X \xrightarrow{\rho} X' \xrightarrow{\sigma} Y,$$

where X' is a triple cover of Y, and X a double cover of X'. Furthermore, if X' is smooth then so is X.

Proof. If the k_i are not in the zero locus of a triple cover, by lemma 3.4.2, we can use a change of variables so that $k_i = (0, 1, 1, 0)$. Take the involution $i\colon X \to X$,

$z_i \leftrightarrow y_i$, then $X/i \cong X'$ and as this double covering map is unramified the last statement follows. $\square$

Note that for degree reasons we might not be allowed to have a decomposition like this, e.g., if $\deg(z_i) \neq \deg(y_i)$. This motivates our next model.

5.3.3 Second Model

Suppose that in the decomposition 5.7, $C_0 = 0$. Then we can compute the relations between the c_{ij} easily

$$\begin{cases} \operatorname{rk}\begin{pmatrix} c_{10} & c_{11} & c_{12} & c_{13} \\ c_{20} & c_{21} & c_{22} & c_{23} \end{pmatrix} = 0, \\ c_{13}c_{30} - 3c_{12}c_{31} + 3c_{11}c_{32} - c_{10}c_{33} = 0, \\ c_{23}c_{30} - 3c_{22}c_{31} + 3c_{21}c_{32} - c_{20}c_{33} = 0. \end{cases}$$

If there is no common divisor to the c_{0i}, there is $l \in \mathcal{O}_Y(\mathcal{U})$ such that $c_{2i} = lc_{1i}$ for all i. By change of coordinates $z_i \mapsto z_i + ly_i$, we get $C_2 = 0$ and the model is very simple

$$\begin{cases} y_0^2 & +c_{01}y_0 + c_{00}y_1 & -2c_{11}^2 + 2c_{10}c_{12} \\ y_0y_1 & -c_{02}y_0 + c_{01}y_1 & -c_{10}c_{13} + c_{11}c_{12} \\ y_1^2 & +c_{03}y_0 + c_{02}y_1 & +2c_{11}c_{13} - 2c_{12}^2 \\ y_0z_1 & -c_{01}z_0 + c_{00}z_1 & \\ \frac{1}{2}(y_0z_1 + y_1z_0) & +c_{02}z_0 + c_{01}z_1 & \\ y_1z_1 & -c_{03}z_0 + c_{02}z_1 & \\ z_0^2 & +c_{31}y_0 + c_{30}y_1 & +2c_{11}c_{31} - c_{12}c_{30} - c_{10}c_{32} \\ z_0z_1 & -c_{32}y_0 + c_{31}y_1 & +c_{12}c_{31} + c_{10}c_{33} - 2c_{11}c_{32} \\ z_1^2 & +c_{33}y_0 + c_{32}y_1 & -c_{13}c_{31} - c_{11}c_{33} + 2c_{12}c_{32}, \end{cases} \tag{5.10}$$

with the relation $c_{13}c_{30} - 3c_{12}c_{31} + 3c_{11}c_{32} - c_{10}c_{33} = 0$.

Taking $c_{3i} = c_{1i}$ for all i gives us the First Model. As smoothness is an open condition we can then argue that this model is smooth as well. If $\deg(c_{3i}) > \deg(c_{0i})$ this model is not a degeneration of the First Model and the argument does not work.

Chapter 6

Constructing varieties as algebraic covers of the projective plane

Recall that the main goal of this thesis is to solve Problem 1.0.1. Describe the section ring of a surface $(S, \mathcal{L})$ for which $\mathcal{L}$ is an ample base point free divisor inducing a covering map $\varphi_{\mathcal{L}} \colon S \to \mathbb{P}^2$ such that

$$\varphi_* \mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m_1) \oplus \Omega^1_{\mathbb{P}^2}(-m_2) \oplus \mathcal{O}_{\mathbb{P}^2}(-m_1 - m_2 - 3)$$

$m_1, m_2 \in \mathbb{Z}_{\geq 0}$.

We will start with the case $m_1 = m_2 = 0$, the section ring of an abelian surface with a polarization of type $(1, 3)$ and a canonical level structure. For this case we can completely describe the family of section rings which we will use to prove that $\mathcal{A}^{lev}_{(1,3)}$ is rational.

Using the structure of CHom $\left(S^2\Omega^1_{\mathbb{P}^2}(-m), \Omega^1_{\mathbb{P}^2}(-n)\right)$ for a pair $(m, n) \in \mathbb{Z}^2_{\geq 0}$ we show models for any pair (m_1, m_2) using the same fibre structure used for the abelian surface.

We finish this chapter with some words of conclusion for the *Future Work* examples and new open problems related with constructing section rings for irregular n-folds that are covers of $\mathbb{P}^n$.

6.1 $(1,3)$-polarized abelian surface

Let $(S,\mathcal{L})$ be an abelian surface with a polarization of type $(1,3)$. By Theorem 2.2.11, $\mathcal{L}$ defines a morphism $\varphi\colon S\to\mathbb{P}^2$ such that

$$\varphi_*\mathcal{O}_S=\mathcal{O}_{\mathbb{P}^2}\oplus\Omega^1_{\mathbb{P}^2}\oplus\Omega^1_{\mathbb{P}^2}\oplus\mathcal{O}_{\mathbb{P}^2}.$$

From Corollary 3.3.3, we have an embedding

$$S\hookrightarrow\mathbb{A}:=\mathbb{A}\left(\Omega^1_{\mathbb{P}^2}\oplus\Omega^1_{\mathbb{P}^2}\right)\cong\operatorname{Spec}\left(\bigoplus_{n\geq0}\operatorname{Sym}^n\left(\Omega^1_{\mathbb{P}^2}\oplus\Omega^1_{\mathbb{P}^2}\right)\right)$$

so we have to determine $\mathcal{I}_S\subset\mathcal{O}_{\mathbb{A}}$.

Corollary 4.1.2 states that $\operatorname{Hom}\left(S^2\Omega^1_{\mathbb{P}^2},\mathcal{O}_{\mathbb{P}^2}(-3)\right)=0$. From the equality

$$S^2\left(\Omega^1_{\mathbb{P}^2}\right)^{\oplus2}\cong\left(S^2\Omega^1_{\mathbb{P}^2}\right)^{\oplus3}\oplus\mathcal{O}_{\mathbb{P}^2}(-3)$$

we conclude that $\mathcal{I}_S$ is generated by a cover homomorphism in

$$\operatorname{CHom}\left(\left(S^2\Omega^1_{\mathbb{P}^2}\right)^{\oplus3},\left(\Omega^1_{\mathbb{P}^2}\right)^{\oplus2}\right)$$

(see Definition 3.2.3 and section 4.4.1). Recall from chapter 2 that

$$\mathbb{A}\cong\operatorname{Proj}\left(\mathbb{C}\left[x_i;y_i,z_i\right]/\left(\sum x_iy_i,\sum x_iz_i\right)\right)$$

$0\leq i\leq2$. Using this choice of basis, $\operatorname{in}(\mathcal{I}_{S,p})$ is locally generated by the polynomials in the following vector

$$q=\begin{pmatrix} y_0^2 & y_0y_1 & y_1^2 & y_0z_0 & \frac{1}{2}(y_0z_1+y_1z_0) & y_1z_1 & z_0^2 & z_0z_1 & z_1^2\end{pmatrix}$$

$p\in\{x_2\neq0\}=:\mathcal{U}$. $\mathcal{I}_{S,p}$ is generated by the entries of the vector

$$q+\begin{pmatrix} y_0 & y_1 & z_0 & z_1\end{pmatrix}C+D$$

where C and D are a 4×9 and 1×9 matrices with entries $c_{ij},d_i\in\mathcal{O}_{\mathbb{P}^2}(\mathcal{U})$ satisfying the relations in $\mathcal{I}_q$. By Theorem 3.1.4 the d_i can be written as quadratic forms in the c_{ij} which satisfy the relations given by the spinor embedding of the affine orthogonal Grassmannian $\operatorname{aOGr}(5,10)$ in $\mathbb{P}^{15}$, see Theorem 5.3.2.

The linear relations between the c_{ij} show that C can be decomposed as

$$\begin{pmatrix} C_1 & C_0 \\ -C_2 & -C_1 \\ C_3 & C_2 \end{pmatrix}$$

where is C_i is of the form

$$C_i = \begin{pmatrix} c_{i1} & c_{i0} \\ -c_{i2} & -c_{i1} \\ c_{i3} & c_{i2} \end{pmatrix}$$

Each C_i represents a morphism in $\mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)\big|_{\mathcal{U}}$. Assuming that the basis (x_i) of $\mathbb{P}^2$ is so that φ is equivariant for the extended Heisenberg group $H(3)^e$, then by Proposition 4.2.3 the c_{ij} can be written as

$$\begin{array}{rclcrcl} c_{i0} & = & \beta_i x_2^2 - \beta_i \frac{x_1^3}{x_2} & & c_{i2} & = & -\frac{2}{3}\alpha_i x_0 x_2 - \beta_i \frac{x_0^2 x_1}{x_2} \\ c_{i1} & = & -\frac{2}{3}\alpha_i x_1 x_2 - \beta_i \frac{x_0 x_1^2}{x_2} & & c_{i3} & = & \beta_i x_2^2 - \beta_i \frac{x_0^3}{x_2} \end{array}$$

The relation between the α_i, β_i so that the c_{ij} satisfy the relations in $\mathcal{I}_q$ is

$$3\alpha_0\beta_3 - \alpha_1\beta_2 + \alpha_2\beta_1 - 3\alpha_3\beta_0 = 0.$$

As we have the embedding $S \hookrightarrow \mathbb{A}\left(\Omega^1_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}\right) \twoheadrightarrow \mathbb{P}^2$ and for a $p \in \mathbb{P}^2$ the structure of the stalk $\mathcal{I}_{S,p}$, by Proposition 3.2.1 we have the global structure of $\mathcal{O}_S$.

Notice that the matrix C has the structure of a triple cover homomorphism. Furthermore, although we are not allowed a change of variables for the (x_i) as we picked a specific one so that φ is equivariant for $H(3)^e$, we can use a change of variables between the y_i and z_i, i.e. $g \in \mathrm{GL}(2, \mathbb{C})$ acts on (y_0, y_1, z_0, z_1) as

$$\begin{pmatrix} y_i & z_i \end{pmatrix} \mapsto \begin{pmatrix} y_i & z_i \end{pmatrix} g$$

for all i. If we assume that $(\beta_0, \beta_1, \beta_2, \beta_3) \notin \mathrm{Bl}_{tc}$, i.e. the β_i do not satisfy the equation defining the branch locus of a triple cover, then using Lemma 3.4.2 there is a $\mathrm{GL}(2, \mathbb{C})$-change of variables as above such that

$$(\beta_0, \beta_1, \beta_2, \beta_3) \mapsto (0, 1, 1, 0).$$

The relation between the α_i becomes $\alpha_1 = \alpha_2$ and a cover homomorphism for an abelian surface is determined by the three parameters $\alpha_0, \alpha_1 = \alpha_2, \alpha_3$ which proves that $\mathcal{A}^{lev}_{(1,3)}$ is unirational.

Theorem 6.1.1. *The moduli space of abelian surfaces with a polarization of type* $(1,3)$ *and canonical level structure* $\mathcal{A}^{lev}_{(1,3)}$ *is rational.*

Proof. Denote by $R(S,\mathcal{L})_{(\alpha_i)}$ the section ring of a $(1,3)$-polarized surface with a model defined by $(\alpha_0,\alpha_1,\alpha_3)$ and $\operatorname{Proj}\left(R(S,\mathcal{L})_{(\alpha_i)}\right)$ by $S_{(\alpha_i)}$.

Let $\gamma\colon S_{(\alpha_i)} \to S$ be an isomorphism. As $S_{(\alpha_i)}$ is a covering map it is defined by a stalk over a point in $\mathbb{P}^2$ so γ is defined by the images $\gamma(x_i), \gamma(y_i), \gamma(z_i)$, $0 \leq i \leq 2$. Recall from section 4.2.1 that the action of $H(3)^e$ on $R(S,\mathcal{L})_{(\alpha_i)}$ is linear which implies that γ is $H(3)^e$-equivariant and $(\gamma(x_i),\gamma(y_i),\gamma(z_i))$ is a basis for which $\varphi'\colon \gamma\left(S_{(\alpha_i)}\right) = S \to \mathbb{P}^2$ is $H(3)^e$-equivariant as well.

Using the basis $(\gamma(x_i),\gamma(y_i),\gamma(z_i))$, a projective model for S depends on the coefficients α'_i, β'_i. By a $(\gamma(y_i),\gamma(z_1))$ change of variables we can set $(\beta'_i) = (0,1,1,0)$ so that $S = S_{(\alpha'_i)}$ and we are looking for the automorphisms of $R(S,\mathcal{L})_{(\alpha_i)}$ that fix x_i and the coefficients β_i. Using the following equalities

$$\operatorname{Hom}\left(\Omega^1_{\mathbb{P}^2},\Omega^1_{\mathbb{P}^2}\right) \cong H^0\left(\mathbb{P}^2, T_{\mathbb{P}^2}\otimes\Omega^1_{\mathbb{P}^2}\right) \cong H^0\left(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}\oplus\left(S^2\Omega^1_{\mathbb{P}^2}\right)(-3)\right) \cong \mathbb{C}$$

these automorphisms are given by a (y_i,z_i) linear change of variables.

By Lemma 3.4.4 the morphisms that fix the β_i are given by a representation of S_3 generated by the matrices $I_{2\times 2}, s, s^2, r, sr, s^2r$ where

$$s = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, r = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These elements act on $(\alpha_0,\alpha_1,\alpha_3)$ as

$$\begin{aligned} s\colon &\ (-\alpha_0+\alpha_3, -\alpha_0+\alpha_1, -\alpha_0) \\ s^2\colon &\ (-\alpha_3, \alpha_1-\alpha_3, \alpha_0-\alpha_3) \\ r\colon &\ (\alpha_3,\alpha_1,\alpha_0) \\ sr\colon &\ (\alpha_0-\alpha_3, \alpha_1-\alpha_3, -\alpha_3) \\ s^2r\colon &\ (-\alpha_0, -\alpha_0+\alpha_1, -\alpha_0+\alpha_3). \end{aligned}$$

With a change of basis

$$\delta_0 = \alpha_0+\alpha_1+\alpha_3, \delta_1 = -3\alpha_0+\alpha_1+\alpha_3, \delta_2 = \alpha_0+\alpha_1+-3\alpha_3$$

the action of S_3 above is just the standard action generated by

$$\widetilde{s}(\delta_i) = \delta_{i+1}, \widetilde{r}(\delta_i) = \delta_{-i}.$$

As $\mathbb{C}[\delta_0, \delta_1, \delta_2]^{S_3} \cong \mathbb{C}[s_1, s_2, s_3]$ where s_i are the symmetric polynomials of degree i we get that the moduli space $\mathcal{A}^{lev}_{(1,3)}$ is rational. □

We want to connect our construction with the families studied in [BL94, Cas99]. Taking $\alpha_0 = \alpha_3 = 0$ and a general α_1, we get the First Model described in section 5.3.2. By proposition 5.3.4, the branch locus is given by $3C \in \mathbb{P}^2$, where C is defined by the vanishing of the polynomial

$$\begin{aligned}&(x_0^6 + x_1^6 + x_2^6) + 2(2\alpha^3 - 1)(x_0^3x_1^3 + x_0^3x_2^3 + x_1^3x_2^3)\\ &\quad -6\alpha^2(x_0^4x_1x_2 + x_0x_1^4x_2 + x_0x_1x_2^4) - 3\alpha(\alpha^3 - 4)(x_0^2x_1^2x_2^2).\end{aligned}$$

This was the case studied in [BL94].

Setting $\alpha_0 = 0$ and taking general α_1, α_3 we get the Second Model - section 5.3.3. This is the case of a bielliptic surface studied in [Cas99] as having $C_0 = 0$ we can project to the triple cover $\varsigma\colon S' \to \mathbb{P}^2$. S' is a ruled surface as its Kodaira dimension is $-\infty$ and $q = 1$.

We can then decompose $\varphi\colon S \xrightarrow{\rho} S' \xrightarrow{\varsigma} \mathbb{P}^2$, where ρ is a double cover. The branch locus is $2C + C'$, where C is the curve giving the branch locus above with the parameter α_1 and $C' := \varsigma_* B_\rho$, B_ρ the branch locus of the double cover ρ. Unfortunately we do not have a relation with the parameter α_3.

For the general model, [LS02, proposition 5.2] says that the branch locus $C \in \mathbb{P}^2$ is a reduced and irreducible curve of degree 18. We do not know how to relate the values of α_i to the equation defining C and leave it as an open problem.

6.2 General Case

Assume in this section that $(S, \mathcal{L})$ is a surface for which $\mathcal{L}$ is an ample base point free divisor inducing a Gorenstein covering map $\varphi\colon S \to \mathbb{P}^2$ such that

$$\varphi_*\mathcal{O}_S = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m_1) \oplus \Omega^1_{\mathbb{P}^2}(-m_2) \oplus \mathcal{O}_{\mathbb{P}^2}(-m_1 - m_2 - 3)$$

for general (m_1, m_2). Denote $\mathcal{L} := \mathcal{O}_S(1)$, then S satisfies

- $\mathcal{L}^2 = 6$, $K_S = (m_1 + m_2)\mathcal{L}$,
- if $m_1 = m_2 = m$, $h^1(S, \mathcal{O}_S(m)) = 2$, 0 otherwise,
- if $m_1 \neq m_2$, $h^1(S, \mathcal{O}_S(m_i)) = 1$, 0 otherwise,
- $p_g = h^0(\omega_S) = h^0(\mathcal{O}_{\mathbb{P}^2}(-3) \oplus \Omega^1_{\mathbb{P}^2}(m_1) \oplus \Omega^1_{\mathbb{P}^2}(m_2) \oplus \mathcal{O}_{\mathbb{P}^2}(m_1 + m_2))$.

In particular, for $m_1 = 0, m_2 = 1$ we have an irregular surface for which $K_S^2 = 6, p_g(S) = 3, q = 1$. We want to present algebraic models for all pairs (m_1, m_2).

6.2.1 $m_1 = m_2 = m$

If $m = m_1 = m_2$ then $\operatorname{Hom}\left(S^2\left(\Omega^1_{\mathbb{P}^2}(-m)\right), \mathcal{O}_{\mathbb{P}^2}(-2m-3)\right) = 0$ by Corollary 4.1.2. This implies that the covering maps are constructed by cover homomorphisms in

$$\operatorname{CHom}\left(\left(S^2\left(\Omega^1_{\mathbb{P}^2}(-m)\right)\right)^{\oplus 3}, \left(\Omega^1_{\mathbb{P}^2}(-m)\right)^{\oplus 2}\right)$$

and the ideal $\mathcal{I}_q$ is exactly the same as the one for the abelian case. The only difference is that each C_i in the decomposition of C is generated by maps in $\operatorname{CHom}\left(S^2\left(\Omega^1_{\mathbb{P}^2}(-m)\right), \Omega^1_{\mathbb{P}^2}(-m)\right)\big|_{\mathcal{U}}$.

Take $C_i = k_i C$ where $(k_0, k_1, k_2, k_3) \notin \mathrm{Bl}_{tc}$ and C generated by

$$\begin{aligned}
c_0 =& \beta_1 x_2^2 - \beta_{12} x_1 x_2 + \beta_{21} x_1^2 - \beta_2 \frac{x_1^3}{x_2}\\
c_1 =& \tfrac{1}{3}\beta_{10} x_2^2 - \tfrac{2}{3}\beta_{012} x_1 x_2 + \tfrac{1}{3}\beta_{20} x_1^2 - \tfrac{1}{3}\beta_{12} x_0 x_2 + \tfrac{2}{3}\beta_{21} x_0 x_1 - \beta_2 \frac{x_0 x_1^2}{x_2}\\
c_2 =& \tfrac{1}{3}\beta_{01} x_2^2 - \tfrac{1}{3}\beta_{02} x_1 x_2 - \tfrac{2}{3}\beta_{012} x_0 x_2 + \tfrac{2}{3}\beta_{20} x_0 x_1 + \tfrac{1}{3}\beta_{21} x_0^2 - \beta_2 \frac{x_0^2 x_1}{x_2}\\
c_3 =& \beta_0 x_2^2 - \beta_{02} x_0 x_2 + \beta_{20} x_0^2 - \beta_2 \frac{x_0^3}{x_2}
\end{aligned}$$

β_* homogeneous polynomials of degree m in $\mathbb{C}[x_0, x_1, x_2]$. Then by Proposition 5.3.5 φ decomposes as

$$S \xrightarrow{\rho} S' \xrightarrow{\sigma} \mathbb{P}^2,$$

where ρ is a double cover and σ a triple cover such that $\sigma_* \mathcal{O}_{S'} = \mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}(-m)$. By Proposition 4.4.2, S' is smooth and so S is smooth as well.

Notice that for $m = 0$, as the Kodaira dimension of S is zero, $q(S) = \dim(S)$ and S is smooth for a general morphism in $\operatorname{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}, \Omega^1_{\mathbb{P}^2}\right)$, S is an abelian surface with a polarization of type $(1,3)$. Then we can make a change of basis in $\mathbb{P}^2$ such that the morphism φ is equivariant for $H(3)^e$.

6.2.2 $m_1 < m_2$

If $m_1 < m_2$ then we no longer have necessarily $\mathrm{in}(\mathcal{I}_S)$ locally generated by the entries in the vector q. Nonetheless we will still use the ideal $\mathcal{I}_q$ and the cover homomorphisms in

$$\mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}(-2m_1, -m_1-m_2, -2m_2), \Omega^1_{\mathbb{P}^2}(-m_1, -m_2)\right)$$

where for an $\mathcal{O}_{\mathbb{P}^2}$-module $\mathcal{F}$, $\mathcal{F}(n_1, \ldots, n_k)$ denotes $\mathcal{F} \otimes \left(\bigoplus_{i=1}^k \mathcal{O}_{\mathbb{P}^2}(n_i)\right)$.

As $\mathrm{CHom}\left(S^2\Omega^1_{\mathbb{P}^2}(-2m_1), \Omega^1_{\mathbb{P}^2}(-m_2)\right) = 0$ for degree reasons, we have $C_0 = 0$ and we can only use the Second Model from section 5.3.3. Via change of variables we can also assume $C_2 = 0$ and the cover homomorphism is determined by C_1 and C_3 under the relation

$$c_{13}c_{30} - 3c_{12}c_{31} + 3c_{11}c_{32} - c_{10}c_{33} = 0. \tag{6.1}$$

For $m_1 = 0, m_2 = 1$, the irregular surface case, in Appendix A.3.2 we prove that for a general C_1, C_3 satisfying the relation above we get a smooth surface. We conjecture that this is true for general $m_1 < m_2$ but we do not have a proof.

6.3 Future Work

The biggest question this thesis leaves open is the following.

Problem 6.3.1. Let $\mathcal{Q} \subset \mathbb{A}^4 = \mathrm{Spec}(k[y_0, y_1, z_0, z_1])$ be a set of six points whose vanishing ideal $\mathcal{I}_\mathcal{Q}$ is generated by

$$q = \begin{pmatrix} y_0^2 & y_0y_1 & y_1^2 & y_0z_0 & \frac{1}{2}(y_0z_1 + y_1z_0) & y_1z_1 & z_0^2 & z_0z_1 & z_1^2 \end{pmatrix}.$$

Find a reason for the ideal $\mathcal{I}_q$ to be isomorphic the spinor embedding of the affine orthogonal Grassmannian $\mathrm{aOGr}(5, 10)$ in $\mathbb{P}^{15}$.

Recall that in [HM99] it was proven that for a general quadruple cover $\mathcal{I}_4$ is generated by the equations defining the Plüker embedding of the Grassmannian $\mathrm{Gr}(2, 6)$ in $\mathbb{P}^{14}$ and in section 3.4, using a projection, we found that they can be defined by 2 elements in the 6 dimensional space $k[z_1, z_2, z_3]^{(2)}$.

Besides Problem 6.3.1, further exploring this relations for covers of degree $d > 4$ to the point we can predict the ideals $\mathcal{I}_q$ for simple q would be worth pursuing, e.g. $q = (z_iz_j)$, q given by local generators for Y_n, Y_n', X_n described below.

The computational aspect of studying general covering maps of degree d can be done using the algorithm explained in chapter 5. The problem is that for large d the number of variables and equations defining $\mathcal{I}_d$ will be quite large. For this reason we propose the following problem.

Problem 6.3.2. Let $\varphi\colon Y_n \to \mathbb{P}^n$ be a covering map of degree $n+1$ such that

$$\varphi_*\mathcal{O}_{Y_n} = \mathcal{O}_{\mathbb{P}^n} \oplus \Omega^1_{\mathbb{P}^n}.$$

Describe its section ring.

In section 4.1.2 we gave some hints in how to describe the elements in $\operatorname{Hom}\left(S^2\Omega^1_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}\right)$. Getting their full description and similarly for other $\mathbb{P}^n$ will be the first step to solve problem 6.3.2. Exactly as for the case of $\mathbb{P}^2$, for $\mathbb{P}^n$ the number of coefficients defining such morphism is small (relative to its degree and number of variables) and these can be determined using Kovačec Lemma. Then, even if the ideal $\mathcal{I}_{n+1}$ is generated by a large number of polynomials, the obstructions brought by the structure of $\Omega^1_{\mathbb{P}^n}$ might make it a simple problem.

A continuation of the problem above is the following.

Problem 6.3.3. Describe the section ring of Y'_n where $\varphi\colon Y'_n \to \mathbb{P}^n$ is a covering map of degree $2n+2$ such that

$$\varphi_*\mathcal{O}_{Y'_n} = \mathcal{O}_{\mathbb{P}^n} \oplus \Omega^1_{\mathbb{P}^n} \oplus \Omega^{n-1}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

This problem is a generalization to $\mathbb{P}^n$ of the main problem solved in this thesis. In particular, notice that for a point $p \in \mathbb{P}^n$ the fibre $\varphi^{-1}(p)$ is given by the vanishing of a Gorenstein ideal. We believe that the ideal $\mathcal{I}_q$, where q is a set of local generators for the ideal $\operatorname{in}(\mathcal{I}_{Y'_n})$, is related with the one for our case from the similarities in the structure of $\varphi_*\mathcal{O}_{Y'_n}$ with the structure of $\varphi_*\mathcal{O}_S$ for the abelian surface with a polarization of type $(1,3)$. In specific, we have $Y'_n \subset \mathbb{A}\left(\Omega^1_{\mathbb{P}^n} \oplus \Omega^{n-1}_{\mathbb{P}^n}\right)$ and the socle term in $\varphi_*\mathcal{O}_{Y'_n}$ is the wedge product of the two summands defining the ambient space.

Problem 6.3.4. Describe the section ring of X_n where $\varphi\colon X_n \to \mathbb{P}^n$ is a covering map of degree $\binom{2n}{n}$ such that

$$\varphi_*\mathcal{O}_{X_n} = \bigoplus_{i=0}^{n} \left(\Omega^i_{\mathbb{P}^n}\right)^{\oplus\binom{n}{i}}.$$

The varieties X_n are a different generalization of our main problem. As explained for the case $n=3$ in Example 3.3.4, given a point $p \in \mathbb{P}^n$, the fibre $\varphi^{-1}(p)$ does not have the typical decomposition a Gorenstein fibre has, i.e.

$$\varphi^{-1}(p) \cong k\left\langle 1, z_1, \dots, z_{d-2}, q\right\rangle$$

where the z_i are generators of $\varphi^{-1}(p)$ as a k-algebra and the socle q is a quadratic term in the z_i. Instead, as a k-vector space, the fibre will have generators in $n+1$ degrees corresponding to the summands $\Omega^i_{\mathbb{P}^n}$ that, as a k-algebra, will be polynomials of degree i in the generators of $\Omega^1_{\mathbb{P}^n}$.

The interest in studying these varieties is mainly related with the study of high codimensional Gorenstein ideals but it is also interesting to study the geometry of so many points in a relatively small dimensional space.

Proposition 6.3.5. *If $n > 2$, a general Y_n, X_n is singular.*

Proof. (Sketch) Notice that $h^i\left(X_n, \mathcal{O}_{X_n}\right) = \binom{n}{i}$. Then, if X_n is smooth it would be a n-dimensional abelian variety. By Riemann–Roch for abelian varieties

$$\chi(\mathcal{L}) = \mathcal{L}^n/n!$$

where $\mathcal{L}$ is the line bundle defining the morphism $\varphi\colon X_n \to \mathbb{P}^n$ from where we have

$$\mathcal{L}^n = \deg(\varphi) = \sum_{i=0}^{n} \binom{n}{i} = \binom{2n}{n}.$$

Then $\chi(\mathcal{L}) = n+1$ only if $n = 2$.

We can construct X_n as an unramified cover of Y_n as for the First Model (see section 5.3.2). Smoothness of Y_n would then imply the existence of a smooth X_n which is only true for $n = 2$. □

This brings the last problem we leave in this thesis.

Problem 6.3.6. Let $n > 2$ be an integer. Determine the singularities of a general model for Y_n, Y'_n, X_n.

It does not fall in the construction type of problem that this thesis was all about but the question of understanding the singularities of high codimensional ideals is an interesting one. Recall that Beauville proved in [Bea79] that if $p_g(S) = 3, q(S) = 0$ for a surface S, then $K_S^2 \leq 36$. Together with Theorem 2.0.2 that says a regular surface such that $p_g(S) = 3, K_S^2 = d$ is given by the vanishing of a Gorenstein ideal with codimension $d-2$, we should expect singularities to be attached with high codimension ideals.

Appendix A

Computations

A.1 Triple cover

A.1.1 Elimination Ideal

```
 RR<t,z,w,a,b,c,d> := PolynomialRing(Rationals(), 7);
Mat := Matrix(3, [z+a*t, w-2*d*t, c*t, b*t, z-2*a*t, w+d*t]);
Mat;
II := Minors(Mat,2); Ideal(II);
EliminationIdeal(Ideal(II), 1);
```

A.1.2 Branch locus

```
S.<a,b,c,d>=QQ[]

f1=a*c*d-(1/3)*b*c*c-(2/3)*d*d*d
f2=2*a*a*c-a*d*d-b*c*d
f3=a*a*d+a*b*c-2*b*d^2
f4=(2/3)*a^3 - a*b*d + (1/3)*b^2*c

F=f2^2*f3^2-4*f1*f3^3-4*f2^3*f4-27*f1^2*f4^2+18*f1*f2*f3*f4

F.factor()
```

A.1.3 Kernel of C

```
RR<a,b,c,d> := PolynomialRing(Rationals(), 4);
```

```
C := Matrix(1,[
   a*c*d-(1/3)*b*c*c-(2/3)*d*d*d,
   2*a*a*c-a*d*d-b*c*d,
   a*a*d+a*b*c-2*b*d^2,
   (2/3)*a^3 - a*b*d + (1/3)*b^2*c
]);

Kernel(C);
```

A.1.4 Triple cover Smoothness

```
KK := FiniteField(997);
P<y0,y1,z0,z1,x0,x1> := AffineSpace(KK,6);

x2:=1;
b0:=Random(996); b1:=Random(996);
b2:=Random(996); b01:=Random(996);
b02:=Random(996); b10:=Random(996);
b12:=Random(996); b20:=Random(996);
b21:=Random(996); b012:=Random(996);

c00:=(1/3)*b10*x2^2 -(2/3)*b012*x1*x2 +(1/3)*b20*x1^2
          -(1/3)*b12*x0*x2 +(2/3)*b21*x0*x1 -b2*x0*x1^2;
c01:=b1*x2^2 -  b12*x1*x2 +  b21*x1^2 - b2*x1^3;
c02:=b0*x2^2 -  b02*x0*x2 +  b20*x0^2 - b2*x0^3;
c03:=(1/3)*b01*x2^2  -(1/3)*b02*x1*x2 -(2/3)*b012*x0*x2
          +(2/3)*b20*x0*x1 + (1/3)*b21*x0^2 - b2*x0^2*x1;

X := Scheme(P,[
y0*y0 + (c00*y0+c01*y1)  + (c10*z0+c11*z1) -2*c00^2 + 2*c01*c03,
y0*y1 - (c03*y0+c00*y1)  - (c13*z0+c10*z1) -c01*c02 + c00*c03,
y1^2  + (c02*y0+c03*y1)  + (c12*z0+c13*z1) +2*c00*c02 - 2*c03^2
]);
IsNonsingular(X);
```

A.2 Main Example

A.2.1 Fibre Structure

```
 S.<y0, y1, z0, z1, nij, di, cij>=QQ[]

from sage.libs.singular.function_factory import singular_function
minbase = singular_function('minbase')

I = S.ideal([y0**2, y0*y1, y1**2, y0*z0,
       (1/2)*(y0*z1 + y1*z0), y1*z1, z0**2, z1*z0, z1**2])
M = I.syzygy_module()
F = matrix(S, 9,1, lambda i,j: I.gens()[i]);
v=matrix(S,4,1,[y0,y1,z0,z1])
N = matrix(S, 16, 9, lambda i, j: 'n'+ str(i) + str(j));
C = matrix(S, 9, 4, lambda i, j: 'c'+ str(i) + str(j));
D = matrix(S, 9, 1, lambda i, j: 'd'+ str(i) + str(j));
V = [y0**2, y0*y1, y1**2,z0**2, z1*z0,
                 z1**2, y0*z0, y1*z1, y0*z1, y1*z0]

Z2 = M*C*v+N*F

a = [c32 + c43,c42 + c53, c72 + c83,  c62 + c73]
for i in range(Z2.nrows()):
for j in range(len(V)):
a.append(Z2[i][0].coefficient(V[j]))

J = S.ideal(a)

R=S.quotient_ring(J)
R.inject_variables()

N = matrix(R, 16, 9, lambda i, j: 'n'+ str(i) + str(j));
C = matrix(R, 9, 4, lambda i, j: 'c'+ str(i) + str(j));

Clift = matrix(S, C.nrows(), C.ncols(), lambda i, j: C[i][j].lift())
Nlift = matrix(S, N.nrows(), N.ncols(), lambda i, j: N[i][j].lift())
```

```
Z1 = Nlift*Clift*v + M*D

b=[]
for i in range(Z1.nrows()):
for j in range(v.nrows()):
b.append(Z1[i][0].coefficient(v[j][0]))

JJ = S.ideal(b)
```

A.3 Irregular Surface

A.3.1 Relations

```
KK := FiniteField(23);
P<x0,x1,x2,a0,b0,a2,b2> := AffineSpace(KK,7);

c00:=a0*x1*x2^2-b0*x0*x1*x1;
c01:=b0*(x2^3-x1^3);
c02:=b0*(x2^3-x0^3);
c03:=a0*x0*x2^2-b0*x0*x0*x1;

c20:=a2*x1*x2^2-b2*x0*x1*x1;
c21:=b2*(x2^3-x1^3);
c22:=b2*(x2^3-x0^3);
c23:=a2*x0*x2^2-b2*x0*x0*x1;

c02*c21-3*c03*c20+3*c00*c23-c01*c22;
```

A.3.2 Smoothness

```
KK := FiniteField(997);
P<y0,y1,z0,z1,x0,x1> := AffineSpace(KK,6);

x2:=1; c10:=0; c30:=0; c13:=0;
a0:=1; c11:=0; c31:=0; c33:=0;
b0:=3; c12:=0; c32:=0;
a2:=x0^2+x1^2+1;
b2:= 5*x0*x1+x1^2+x0+x1+1;
```

```
c00:=a0*x1*x2^2-b0*x0*x1*x1;
c01:=b0*(x2^3-x1^3);
c02:=b0*(x2^3-x0^3);
c03:=a0*x0*x2^2-b0*x0*x0*x1;

c20:=a2*x1*x2^2-b2*x0*x1*x1;
c21:=b2*(x2^3-x1^3);
c22:=b2*(x2^3-x0^3);
c23:=a2*x0*x2^2-b2*x0*x0*x1;

X := Scheme(P,[
y0*y0 + (c00*y0+c01*y1)  + (c10*z0+c11*z1)
   -2*c00^2 + 2*c01*c03 + 2*c10*c30 - c13*c31 - c11*c33,
y0*y1 - (c03*y0+c00*y1)  - (c13*z0+c10*z1)
   -c01*c02 + c00*c03 - 2*c13*c30 + c12*c31 + c10*c33,
y1^2  + (c02*y0+c03*y1)  + (c12*z0+c13*z1)
   +2*c00*c02 - 2*c03^2 - c12*c30 - c10*c32 + 2*c13*c33,
y0*z0    - (c30*y0+c31*y1)  - (c00*z0+c01*z1)
   -c10*c20 + c11*c23 + c00*c30 + c03*c31 - 2*c01*c33,
(1/2)*(y0*z1+y1*z0) + (c33*y0+c30*y1) + (c03*z0+c00*z1)
   +(1/2)*( -c11*c22 + c10*c23 - 5*c03*c30 + c02*c31 + 4*c00*c33),
y1*z1    - (c32*y0+c33*y1)  - (c02*z0+c03*z1)
   +c10*c22 - c13*c23 + c02*c30 - 2*c00*c32 + c03*c33,
z0^2  + (c20*y0+c21*y1)  + (c30*z0+c31*z1)
   +2*c00*c20 - c03*c21 - c01*c23 - 2*c30^2 + 2*c31*c33,
z0*z1 - (c23*y0+c20*y1)  - (c33*z0+c30*z1)
   +c03*c20 + c01*c22 - 2*c00*c23 - c31*c32 + c30*c33,
z1^2  + (c22*y0+c23*y1)  + (c32*z0+c33*z1)
   -c02*c20 - c00*c22 + 2*c03*c23 + 2*c30*c32 - 2*c33^2]);
IsNonsingular(X);
```

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